APPLICATION OF A VARIATIONAL METHOD TO THE CALCULATION OF TWO-DIMENSIONAL AND AXISYMMETRIC FLOW FIELDS.

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ABSTRACT

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APPLICATION OF A VARIATIONAL METHOD TO THE CALCULATION OF TWO-DIMENSIONAL AND AXISYMMETRIC COMPRESSIBLE FLOW FIELDS

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The classical problem of steady, inviscid irrotational flow past an aerofoil is formulated as a variational principle, the Bateman-Dirichlet principle. The maximization of the resulting integral is an infinitely dimensional problem, which is replaced by a finitely dimensional problem by means of finite differences and an approximate maximizing function is then found by the Newton-Raphson method. A conformal mapping is used to transform the body to the unit circle and all calculations are carried out in the circle plane. An iterative scheme is used to give the solution for compressible flow, using either the solution for incompressible flow or the solution for a lower free stream Mach number as the starting point.

Both two-dimensional and axisymmetric flows are considered. The shapes considered in two-dimensional flows are circles, ellipses and Karman-Trefftz profiles, while the corresponding bodies of revolution are considered in axisymmetric flows. The solutions, obtained, compare well with those obtained by other approximate methods, except for ellipses, near the stagnation points, where differences of up to 5% with Sells' method are encountered.

Attention is drawn to the fact that the critical Mach number is appreciably higher for axisymmetric flow than for two-dimensional flow past an equivalent shape. In all these cases, both plane and axisymmetric, results are obtained for flows up to and slightly beyond the critical Mach number. It was found that when the free stream Mach number was increased further the variational procedure would not converge.

§1 INTRODUCTION AND REVIEW

1.1 Introduction

The classical problem of steady, inviscid, irrotational flow past a body can be formulated in two ways. The usual formulation is as a set of non-linear partial differential equations satisfying a set of boundary conditions. However, it can also be formulated in terms of two complementary variational principles, as in Serrin (1959), which are related, since for the exact solution, the variational integrals have the same value.

Most attempts at finding approximate solutions have used the formulation in terms of partial differential equations with boundary conditions, but the variational formulation has also been attempted. A review of the major methods used in solving the partial differential equations, and approximations using the variational method is given in the second part of this Chapter.

In this thesis a variational method for obtaining numerical solutions to the complete equations of motion is described. The method consists of replacing the infinitely dimensional problem by a finitely dimensional problem, by means of finite differences, and an approximate maximizing function is then found by the Newton-Raphson method. A conformal mapping is used to transform the body to the unit circle and all calculations are made in the circle plane. The solution for incompressible flow, or the solution for a lower free stream Mach number, is used as the starting point and an iterative scheme is used to give the solution for compressible flow. Greenspan and Jain (1967) used a similar approach to study the plane flow past a circle. However, their results, near the stagnation points, differ greatly from other approximate solutions, obtained either by the variational method, Lush and Cherry (1956) and Wang (1948), or by formulations in terms of the differential equations, for example Imai (1941) and Sells (1968), so it was felt that it would be advantageous to reconsider their method. The method used here and in Rasmussen and Heys (1973) differs in some important aspects from that used by Greenspan and Jain (1967), mainly in the approximation of the derivatives of the potential by finite differences and in the treatment of the boundary conditions on the aerofoil.

The method is also extended to flows past shapes other than a circle, without the difficulties associated with the Rayleigh-Ritz and Galerkin's methods and to axisymmetric flows. Since all calculations are carried out in the circle plane the only changes required for a different shape are the requisite transform modulus and the series solution away from the body. In axisymmetric flows, the incompressible flow solution and the distance of points from the axis of symmetry are also different for different shapes.

As a result of the changes, listed above, the method gives far closer agreement with the results of Sells (1968), Lush and Cherry (1956) and Imai (1941) than was obtained by Greenspan and Jain (1967) for two-dimensional flow past a circular cylinder.

Results in good agreement with Sells (1968) are also obtained for the flow past ellipses, mostly with a thickness ratio of 10%, apart from a discrepancy of 5% near the stagnation point and for a Karman-Trefftz 'F' profile, for which results are also compared with those obtained by use of one of the programs of Bauer, Garabedian and Korn (1972).

In order to show that the method was practical for any symmetric aerofoil, results were obtained for a NACA 0012 profile. These results gave reasonable agreement with those listed by Lock (1970) for Sells' method.

Results were also obtained for axisymmetric flows past a sphere, where good agreement was obtained with those of Wang and de los Santos (1951) who used a variational technique and with the Rayleigh-Jansen method, used by Lamla (1939). Calculations were also made on ellipsoids of different thickness ratios and results in close agreement with Pidcock (1969), who applied Sells' method to axisymmetric flows, were obtained. Calculations were also carried out for bodies of revolution with a Karman-Trefftz profile as cross-section. Attention is drawn to the fact that the critical Mach number is appreciably higher for axisymmetric flow than for two-dimensional flow past an equivalent shape.

In most of these cases, both plane and axisymmetric, results are obtained for flows up to and slightly beyond the critical Mach number. It was found that when the free stream Mach number was pushed appreciably higher than the critical Mach number the variational procedure would not converge.

A review of the work on compressible inviscid flow past bodies by the variational method and other methods is given in Section 1.2. Section 2 deals with the formulation of the variational approach used in this thesis. The details of the numerical method for plane flow are given in Section 3, while the convergence is discussed in Section 4.

Results for two-dimensional flows are given in Section 5. For axisymmetric flows, the details of the numerical method are given in Section 6, the convergence is discussed in Section 7 and the results are given in Section 8. Section 9 is a discussion of the points arising from this thesis.

Appendix A deals with the derivation of the variational integrals as formulated by Lush and Cherry (1956) and Serrin (1959). Appendices B and C deal with the calculation of the transform modulus for the ellipse and the Karman-Trefftz profile, respectively. The modifications required in the derivation of the variational integral to cope with axisymmetric flows are discussed in Appendix D. The series solution of the equation of motion a large distance from the body for use as the far boundary condition for plane flow is derived in Appendix E. Various coefficients required in plane flow for finding the maximum of the variational integral are found in Appendix F. Appendix G deals with the far boundary condition for axisymmetric flows. The changes to the coefficients derived in Appendix F, required for axisymmetric flows, are listed in Appendix H and in Appendix I the series solution, for the potential at the trailing edge of the body of revolution with a Karman-Trefftz profile as cross section is derived.

1.2 Review

It was stated in the introduction that there are two ways of formulating the problem of steady, inviscid, irrotational flow past a body. Normally the formulation is as a set of non-linear partial differential equations with a corresponding set of boundary conditions of which no solution in closed form is known. Methods which have been used, in an attempt to solve the problem , include small perturbation methods, where the velocity potential is expanded in terms of the free stream Mach number, the Rayleigh-Janzen method, or a thickness parameter, for which the Frantl-Glauert method is a first approximation. This means that accurate solutions are restricted either to small velocities for a Mach number expansion or thin bodies for a thickness parameter expansion. Increased mathematical difficulty with each higher approximation normally prevents their application beyond second or third order. Another drawback is that these approximations break down near the stagnation points.

An improved method for calculating the pressure distribution on a thin cambered aerofoil at moderate incidence in compressible flow up to second order accuracy was derived by Gretler (1965). His complicated method consisted of reducing Green's formula, by means of integral transforms and application of Tricomi's convolution theorem, to the calculation of integrals in one variable.

An alternative course is to use the hodograph method where the non-linear, partial differential equation is transformed to an exact linear differential equation. However, the boundary conditions are now far more difficult to fulfil, since part of the boundary is closed in two sheets of a Riemann surface in the hodograph plane. Thus, application to actual problems requires simplifying results which are only approximately correct. This method has been applied to transonic flows using a finite difference scheme, by Bauer, Garabedian and Korn (1972), who also considered the inverse problem of wing design.

One of the best formulations for solving the differential equations is that of Sells (1968) who mapped the exterior of the body onto the interior of the unit circle and introduced the stream function, from which he removed the two singularities of dipole and vortex type at the centre of the circle, so that they could be considered separately. The differential equation, for this modified stream function, was approximated by a difference equation on an annular mesh inside the circle. This equation and Bernoulli's equation were then solved by a convergent iterative process. The drawbacks were that near the critical Mach number, under-relaxation had to be used and the program would not cope with transonic flow with or without a shock. However, Albone (1971) modified Sells' treatment of Bernoulli's equation and found that under-relaxation was no longer necessary when the free stream Mach number approached the critical Mach number. He discovered that over-relaxation could even be used on the density. These changes caused a significant drop in the number of iterations required for convergence. He also obtained results for slightly supercritical flows, where the flow was only supersonic on the aerofoil surface.

It is also possible to formulate the problem in terms of complementary variational principles, which do not require linearization, and provide a direct method of solution. The governing equations and appropriate boundary conditions are equivalent to the Euler equations for two variational integrals. In one case, the Bateman-Kelvin integral, the integral is to be minimized and the integrand consists of the sum of the pressure and the product of the density and the square of the velocity. In the other case, the Bateman-Dirichlet integral, the integral is to be maximized and the integrand is just the pressure, though it is necessary to add a surface integral to avoid prescribing the velocity potential on the boundary, which is not physically acceptable.

Despite the fact that most researchers have used the differential equation formulation a number of attempts have used the variational formulation. The variational integrals were first formulated by Bateman (1929), based on Hargreaves' (1908) kinetic potential but do not seem to have been applied until Braun (1932) used the Bateman-Dirichlet integral to obtain a linearized approximate solution for the plane subsonic flow past a circular cylinder using the Rayleigh-Ritz method to approximate the extremal. Wang (1948) pointed out that Braun had omitted to add terms to give a convergent integral and solved the same problem without linearization, though in his method it was necessary to have γ, the ratio of the specific heats so that $^{\gamma}/_{\gamma-1}$ is an integer. He took γ as 2, although 1.5 satisfies this criterion, to make the calculations simpler.

Wang and his associates later made numerous extensions to his work. Wang and Rao (1950) considered other shapes in the physical plane, giving non-linear equations for the Rayleigh-Ritz parameters. They lost one advantage of the variational method by finding it necessary to linearize these equations and again γ was taken as 2. They found that above the critical Mach number the solution may no longer be unique and above a certain limiting Mach number no physically possible solution exists. This Mach number is the point at which shock waves occur, so the flow behind the shock is no longer irrotational. They also showed that a non-symmetric flow pattern may occur at this Mach number and claimed close agreement between the linearized and non-linearized flows.

When considering transonic flows, it must be remembered that the Bateman-Kelvin and Bateman-Dirichlet principles can only be proved for purely subsonic flows as they can only be shown to have an extremum in this case. However, Courant and Friedrichs (1943) pointed out that the

Rayleigh-Ritz method is still applicable when the flow does not have an extremum. The differential equations and boundary conditions are not satisfied completely but in the mean, and a good approximation can be obtained.

Another extension was to axisymmetric flows by Wang and de los Santos (1951). They formulated the problem for an arbitrary body but only calculated the flow for a sphere and an ellipsoid of 80% thickness to length ratio as it was difficult to calculate other shapes despite taking γ as 2. They claimed close agreement between their linearised and non-linearised solutions.

In an effort to use a more accurate value of γ, attempts were made to replace the Rayleigh-Ritz method. Wang and Chou (1950) used the Biezeno-Koch method which did not require the formulation of any variational principle, but Courant in the discussion of Biezeno and Koch (1924) pointed out that the Biezeno-Koch method was a special case of the variational principle. Wang and Brodsky (1950) used Galerkin's method and found that as well as being able to take γ as 1.405 it was not necessary to formulate variational integrals for each body. The disadvantage was that despite the circle being easier, more work was required for arbitrary bodies than with the Rayleigh-Ritz procedure. Kantorovich and Krylov (1958) pointed out that the Rayleigh-Ritz method was really a special case of the Galerkin method and that for problems connected with variational methods, they are equivalent in that Galerkin's method leads to the same approximate solution as the Rayleigh-Ritz method, usually with simpler computations.

Yet another extension of Wang's work was to the calculation of transonic flows with shock waves. Wang and Chou (1951) found that Wang's method failed as soon as shock waves occur as the flow behind the shock wave becomes rotational and has variable entropy. Using the Bateman-Kelvin integral, which is expressed in terms of the stream function rather than the velocity potential modified to allow for rotational flows they showed that a variational principle which allowed for flows with rotation and variable entropy could be obtained. It was necessary to use the stream function rather than the velocity potential, as the latter does not exist for rotational flows. Their new principle was applied to regions of flow behind the shock waves and Bateman's original principle to the other regions. Shock equations whose solution exists were obtained. They used Galerkin's method as the Rayleigh-Ritz method was too complicated. Calculations were made for flow past a circle, but the absence of adequate computing facilities, made detailed calculations for this case and any calculations for arbitrary bodies impractical.

Other work on transonic flows was done by Lin and Rubinov (1948), who formulated a variational principle for rotational flows suitable for plane and axisymmetric calculations, and Hölder (1950/1), who gave a proof of a generalised variational principle for rotational flows, with shockwaves, in three dimensional space. The use of variational methods in transonic flows has been reviewed by Fiszdon (1964).

Another application of the variational integrals was found by Shiffman (1952), who used them as the basis of his proof of the existence and uniqueness of the solution for plane subsonic flow past an aerofoil, which had only previously been established for flows with a sufficiently small Mach number at infinite by Frankl and Keldysh (1934) and for boundary value problems concerning minimal surfaces analogous to problems for flows, by Bers (1951), both approaches using non-variational methods.

A big step forward in variational methods was the work of Lush and Cherry (1956). They improved on Wang's treatment of the convergence problems and cleared up some of the problems associated with the boundary conditions. They also found it unnecessary to have $\gamma/_{\gamma-1}$ as an integer and indicated the relation between the two formulations of the variational method. They obtained results for flow past a circular cylinder taking γ as 1.405.

Despite the popularity of the Rayleigh-Ritz method in these early variational calculations for fluid dynamics problems, certain objections to this method were listed by Courant (1943), in the application of variational methods to problems of equilibrium and vibration. His objections were that the selection of the coordinate functions is often very difficult and laborious computations are sometimes necessary. The accuracy of the approximations is also difficult to determine. He considered that the approximation of the derivatives by finite differences was preferable, except when suitable analytic expressions are available for the Rayleigh-Ritz method. Courant (1943) stated that as the mesh is made finer, not only does the approximate problem tend to the original problem, but the solutions of the difference equations approach the solution of the original problem exceedingly well. He had previously proved that all the relevant difference quotients of first and higher order converge to the corresponding derivatives of the original problem. Although Courant (1943) considered using finite differences in the integral instead of in the Euler equations (which many previous authors had done, such as Sokolnikoff and Specht (1948) for elasticity problems) he gave no worked examples of their use.

A more detailed investigation and worked examples of the use of finite difference techniques to find extremals of the integrals directly, rather than from the Euler equations, seem to have been given first by Greenspan (1965) and extended by Allen (1966) and Greenspan (1967), the last named paper also considering convergence.

Greenspan and Jain (1967) then applied these techniques to the minimization of Lush and Cherry's variational integral. They approximated the integral on a suitably bounded region by an appropriate algebraic function, using finite differences to replace the derivatives. They calculated the flow past a circular cylinder though their results were not in complete agreement with those obtained by other methods and are discussed later.

In recent years the problem of deducing variational principles for compressible inviscid flow directly from Hamilton's principle has been considered. Seliger and Whitham (1968) discussed the general problem of finding a variational principle for a given system of equations. They claim that in continuum mechanics the troubles appear when the Eulerian description is used while the extension of Hamilton's principle is straightforward in the Lagrangian description. However, they admit that in fluid dynamics the Eulerian description is preferable. Bretherton (1970) used a slightly different approach to derive the Eulerian equations of motion directly from the Lagrangian formulation of Hamilton's principle for a perfect fluid, and used them, in a new derivation of Kelvin's circulation theorem.

There are four stages to be considered in using either of the two variational principles. Firstly it is necessary to ensure that the variational integral is suitably formulated so that it is always convergent even when the region of integration is infinite. Ideally a rigorous proof of the existence of a well determined solution to the variational problem should be found. It is also necessary to derive a method for obtaining an approximation to the extremal of the variational problem. Finally the method should be shown to converge and some bounds on the error involved should be obtained.

Apart from the paper of Shiffman (1952), little work seems to have been done on the existence and uniqueness problem, especially on finite dimensional analogous of the variational integrals which give rise to non-linear partial differential equations. Schechter (1962), however, proved existence and uniqueness of solutions to the minimization of certain convex functions. Stepleman (1971) extended Schechter's work obtaining existence and uniqueness results for solutions to the minimization of

$$J\,|u| = \iint f(s, t, u, u_s, u_t)\, ds\, dt$$

over an open, unbounded, simply connected region in the plane, using either finite difference techniques or the Rayleigh-Ritz method, for obtaining finite dimensional approximations to the integral over a square mesh, subject to the restriction that f is a convex function of u_s and u_t for fixed values of u. Of course, the Bateman-Kelvin and Bateman-Dirichlet integrals do not satisfy this restriction, but Rasmussen (1972) suggests that Stepleman's work may indicate a method of obtaining similar results for these variational principles.

On the other hand, a great deal of work has been carried out on the formulation of the problem, notably by Serrin (1959) and Lush and Cherry (1956), whose work is given in Appendix A. Sewell (1963) also considered the properties of the Bateman integrals and showed that the integrands of the two principles could be related by a Legendre transformation.

He proved a free variational principle for steady flow within a given region, from which generalizations of Bateman's variational principles to non-homentropic and non-homoenergic three-dimensional flow were found. He also proved the associated uniqueness and extremum theorems for homentropic, homoenergic flow, when the integrands related by the Legendre transformation are convex, without requiring the flow to be subsonic at every point in the field, but in an overall sense. He also claimed his work applied equally well to plane flow as to three-dimensional flow unlike that of Lush and Cherry (1956) and Serrin (1959).

The most important methods of approximating the extremal are the Rayleigh-Ritz and finite difference methods, though Galerkin's method and the Biezeno-Koch method have also been used and Angel (1968) has applied dynamic programming to problems with two independent variables, though this method has not been used in fluid dynamics, despite the fact that the example used was Laplace's equation.

The convergence problem only seems to have been tackled rigorously for Rayleigh-Ritz approximations to the variational integrals. Lush (1963) and O'Carroll and Lush (1968) showed that these approximations converge uniformly in a finite subregion provided the boundary of the aerofoil is sufficiently smooth, but they only showed the Rayleigh-Ritz method converged, not that it converged to the exact solution. However, a more satisfactory proof was given by Rasmussen (1973), who proved that the Rayleigh-Ritz approximation converges to the exact solution provided the aerofoil is sufficiently smooth and the coordinate functions are properly chosen. Despite the restiction to subsonic flows, it is not necessary for all the approximate solutions to be subsonic. The proof also applies to certain other two-dimensional problems as well as those concerning flow fields. Convergence theorems in linear but not non-linear problems, for Galerkin's method, were given by Mikhlin and Smolitskiy (1967). No such clear cut results exist for finite difference approximations. However, Greenspan (1967) proved convergence of the numerical solution for minimizing

$$J(y) = \int_a^b F(x, y, y')\ dx$$

the approximate solutions being limited to those whose corners cannot become arbitrarily sharp for a large number of iterations. This proof was extended to three-dimensional problems and the extension to n dimensions was indicated. Greenspan admitted that it was not known whether his convergence theorem was applicable to the integral in Greenspan and Jain (1967).

A review of the applications of variational methods in compressible flow has been given by Rasmussen (1972).

2 FORMULATION

We shall now formulate the boundary value problem for plane, subsonic, irrotational, inviscid flow past an aerofoil. Let (x, y) be a cartesian coordinate system with velocity vector $\underset{\sim}{u} = (u_1, u_2)$. Far from the aerofoil C, $\underset{\sim}{u}$ has the form $\underset{\sim}{u} = (U, 0)$ where U is a given constant, see Figure 1.

The flow is supposed to be irrotational, so a velocity potential can be defined by

$$\underset{\sim}{u} = \nabla\phi .$$

The pressure and density are denoted by p and ρ, respectively. The speed of sound is defined by

$$C^2 = \frac{dp}{d\rho} .$$

For a perfect compressible fluid
We can write p and ρ in the form

$$p = p_o\left(1 - \frac{q^2}{2\beta C_o^{\,2}}\right)^{\alpha}$$

$$\rho = \rho_o\left(1 - \frac{q^2}{2\beta C_o^{\,2}}\right)^{\beta}$$

where

$$q^2 = \underset{\sim}{u} \cdot \underset{\sim}{u}, \quad \alpha = \frac{\gamma}{\gamma-1}, \quad \beta = \frac{1}{\gamma - 1}$$

the suffix o indicates stagnation values.

It is known that the boundary value problem of ϕ is equivalent to a variational principle. If we let V be the flow region and B the boundary, we have from Serrin (1959):

The Bateman-Dirichlet Principle.

Consider the variational principle of maximizing the integral

$$J|\phi| = \int_N p \, d V + \int_B \phi \, h \, d A \tag{2.1}$$

among all subsonic velocities $\underset{\sim}{u} = \nabla\phi$. Then $J|\phi|$ is a maximum $\nabla\cdot(\rho \underset{\sim}{u}) = 0$ and $\rho \, \underset{\sim}{u} \cdot \hat{n} = h$ on B i.e. the continuity equation holds and the mass flow across B is a constant. Here the normal mass-flux h is prescribed on B such that

$$\text{outflow} = \int_B h \, d A = 0.$$

It is easily seen that if the flow region V becomes infinite the variational integral (2.1) becomes unbounded. Lush and Cherry (1956) showed how the integral should be formulated in order to remove this difficulty, and later Lush (1963) wrote it in the form

$$J|\phi| = \iint_\infty \left[p - p_\infty + \rho_\infty \nabla \phi_0 \cdot \nabla(\phi - \phi_\infty)\right] \, dx \, dy \tag{2.2}$$

where

p_∞ = pressure at infinity,

ρ_∞ = density at infinity,

ϕ_∞ = potential for a uniform stream,

ϕ_0 = potential for incompressible flow past C.

The details of the derivation of this integral are given in Appendix A.

The class of admissible functions is restricted to functions for which

(i) $\frac{\partial\phi}{\partial n} = 0$ on C,

$$\text{(ii)} \quad \phi = \phi_\infty + U\chi \tag{2.3}$$

where $|\chi| \leqslant k\, r^{-1}$, $|\nabla\chi| \leqslant k\, r^{-2}$

as $r = (x^2 + y^2)^{\frac{1}{2}} \to \infty$ and

k is constant

In order to make it easier to treat a fairly general class of aerofoils, we shall use a conformal transformation to map the aerofoil C onto the unit circle. Let (r, θ) be a polar coordinate system in the transformed plane, the computation plane, with origin at the centre of the unit circle. Then if we write $z = x + i\,y$ and $\sigma = r(\cos\theta + i\sin\theta)$, the transform modulus becomes

$$T = \left|\frac{dz}{d\sigma}\right| = (x_r{}^2 + y_r{}^2)^{\frac{1}{2}}$$

The Jacobian of the transformation is

$$J = \frac{\partial\,(x,\, y)}{\partial\,(r,\, \theta)} = x_r\, y_\theta - x_\theta\, y_r \;.$$

Since the transformation is conformal

$$y_\theta = r\, x_r \quad \text{and} \quad y_r = -\frac{1}{r}\, x_\theta \,,$$

so the transform modulus is given by

$$T^2 = {x_r}^2 + \frac{1}{r^2}\, {x_\theta}^2$$

and

$$J = r({x_r}^2 + \frac{1}{r^2}\, {x_\theta}^2) \;.$$

Hence

$$J = r\, T^2 \;. \tag{2.4}$$

The coordinates r, θ are orthogonal so the element of length $ds = |dz|$ is given by

$$ds^2 = h_1^2\, dr^2 + h_2^2\, d\theta^2.$$

Also

$$\begin{aligned} ds^2 = |dz|^2 &= \left|\frac{dz}{d\sigma}\right|^2 |d\sigma|^2 \\ &= T^2(dr^2 + r^2\, d\theta^2) \end{aligned}$$

Therefore

$$h_1 = T \quad \text{and} \quad h_2 = r\,T$$

so the potential gradient is given by

$$\nabla\phi = \frac{1}{T}(\hat{r}\ \phi_r + \frac{\hat{\phi}}{r}\ \phi_\theta) \quad .$$

Since the speed q can be expressed by

$$q^2 = (\nabla\phi)^2$$

and the potential

$$\phi = U(r\ \cos\theta + \chi)\ ,$$

we have

$$q^2 = \frac{U^2}{T^2}\left[1 + 2\ \cos\theta\ \chi_r - \frac{2}{r}\ \sin\theta\ \chi_\theta + \chi_r{}^2 + \frac{1}{r^2}\ \chi_\theta{}^2\right] \tag{2.5}$$

Also, as previously stated

$$p = p_o\ (1 - \frac{q^2}{2\beta c_o{}^2})^\alpha$$

and by using the relation

$$p_o = \left(\frac{\rho_o}{\rho_\infty}\right)^\gamma p_\infty$$

we get after some manipulation

$$p = p_\infty \Big[1 + \frac{(\gamma - 1) M_\infty^2}{2T^2} (T^2 - 1 - 2 \cos\theta \, \chi_r + \frac{2}{r} \sin\theta \, \chi_\theta - \chi_r^2 - \frac{1}{r^2} \chi_\theta^2)\Big]^\alpha \tag{2.6}$$

where the free stream Mach number M_∞ is defined by

$$M_\infty^2 = \frac{2\beta U^2}{2\beta C_o^2 - U^2} .$$

Rearranging this and substituting for C_∞ we have

$$\frac{\gamma p_\infty}{\rho_\infty} = C_o^2 - \frac{U^2}{2\beta}$$

so

$$\rho_\infty = \frac{2\beta\gamma}{2\beta C_o^2 - U^2} p_\infty = \frac{\gamma M_\infty^2}{U^2} p_\infty \tag{2.7}$$

Thus since the incompressible potential

$$\phi_o = U(r + \frac{1}{r}) \cos\theta$$

we can write

$$\rho_\infty \nabla \phi_o \cdot \nabla(\phi-\phi_\infty) = p_\infty \frac{\gamma M_\infty^2}{T^2} \left[\frac{r^2-1}{r^2} \cos\theta\ \chi_r - \frac{r^2+1}{r^3} \sin\theta\ \chi_\theta\right] \tag{2.8}$$

When the expressions (2.4), (2.6) and (2.8) are used in (2.2), we see that the variational integral $J|\phi|$ becomes

$$\begin{aligned} J|\chi| = p_\infty \int_0^{2\pi} d\theta \int_1^\infty \Bigg\{ \Bigg[1 + \frac{(\gamma-1)\ M_\infty^2}{2T^2} & (T^2 - 1 - 2\cos\theta\ \chi_r \\ & + \frac{2}{r} \sin\theta\ \chi_\theta - \chi_r^{\ 2} - \frac{1}{r^2} \chi_\theta^{\ 2}) \Bigg]^\alpha \\ & - 1 + \frac{\gamma M_\infty^2}{T^2} (\frac{r^2-1}{r^2} \cos\theta\ \chi_r \\ & - \frac{r^2+1}{r^3} \sin\theta\ \chi_\theta) \Bigg\} r\ T^2 dr. \end{aligned} \tag{2.9}$$

The boundary conditions on χ are

$$\frac{\partial\chi}{\partial r} = -\cos\theta \qquad \text{at} \quad r = 1$$

$$\chi = 0(\frac{1}{r}) \qquad \text{as} \quad r \to \infty \tag{2.10}$$

The local Mach number M and the local nondimensional pressure p_L are given by

$$M = M_\infty q \left[1 + \frac{1}{2}(\gamma - 1) M_\infty^2 (1 - q^2)\right]^{-\frac{1}{2}} \tag{2.11}$$

$$p_L = \frac{p}{p_\infty} = \left[1 + \frac{1}{2}(\gamma - 1) M_\infty^2 (1 - q^2)\right]^{\gamma/\gamma-1} \tag{2.12}$$

where q is given by (2.5).

The transform modulus T cannot, in general be expressed analytically except for very special bodies such as ellipses and Karman-Trefftz profiles. Thus if the flow past a realistic aerofoil is desired T must be evaluated numerically. However, the analytic form for an ellipse is derived in Appendix B and for a Karman-Trefftz profile in Appendix C.

There are a number of methods available for evaluating the conformal mapping of an arbitrary body on to the unit circle. The most successful mapping programme developed so far seems to be that of Catherall, Foster and Sells (1968), though since this programme only gives the mapping on the body surface, it is necessary to find the transform modulus at exterior points, by a separate routine.

2.2 Axisymmetric Flow

In axisymmetric flows a form of the variational integral similar to (2.2) can be developed, but it is not always advantageous to use this form, since for bodies other than spheres and ellipsoids the incompressible potential cannot easily be evaluated.

If we take cylindrical polar coordinates (x, R, ξ) the form of the integral equivalent to (2.2) for flow past an axisymmetric body C_o is then

$$J|\phi| = \iint_{\infty} (p - p_{\infty} + \rho_{\infty} \nabla\phi_{o} \cdot \nabla(\phi - \phi_{\infty}))\ Rd\ Rdx \qquad (2.13)$$

while if it is difficult to find the incompressible potential ϕ_o the form

$$J|\phi| = \iint_{\infty} (p - p_{\infty} + U\rho_{\infty} \frac{\partial\chi'}{\partial x})\ Rd\ \ Rdx - U\rho_{\infty} \int_{B} \chi' \frac{\partial x}{\partial n}\ Rds \qquad (2.14)$$

is used instead. The notation used in these integrals which did not appear in the plane flow integral is as follows:

- U is the free stream velocity
- R is the distance from the axis of the body
- χ' is the perturbation potential i.e. $\phi - \phi_{\infty}$
- ds is the element of the contour B of the cross-section of body C.

The details of the derivation of these integrals from the Bateman-Dirichlet principle are given in Appendix D.

The same restrictions on the class of admissible functions are made as in two dimensional flows. These are

(i) $\frac{\partial\phi}{\partial n} = 0$ on C, and hence on B

(ii) $\phi = \phi_{\infty} + U\chi = \phi_{\infty} + \chi'$

where $|\chi| \leqslant Kr^{-1}$, $|\nabla\chi| \leqslant Kr^{-2}$

as $r = (x^2 + R^2)^{\frac{1}{2}} \to \infty$ and K is a constant.

As in two-dimensional flow we use a conformal transformation to map B onto the unit circle i.e. (x, R) is mapped into (r, θ) a polar coordinate system with origin at the centre of the unit circle.

The modulus of the transformation $(x, R) \to (r, \theta)$ in the cross sectional plane can be related to the Jacobian in exactly the same way as in two-dimensional flow. Thus

$$J = rT^2 \tag{2.4}$$

and the square of the velocity is given by

$$q^2 = \frac{U^2}{T^2}\left[1 + 2\cos\theta\,\chi_r - \frac{2}{r}\sin\theta\,\chi_\theta + \chi_r{}^2 + \frac{1}{r^2}\chi_\theta{}^2\right] \tag{2.5}$$

Therefore p and p_∞ are still related by (2.6). Substituting (2.4), (2.6) and (2.7) into the variational integral in the incompressible and surface integral forms, (2.13) and (2.14) respectively, gives

$$J|\chi| = p_\infty \int_0^{2\pi}\int_1^\infty \left\{\left[1 + \left(\frac{\gamma-1}{2T^2}\right) M_\infty{}^2 \left(T^2 - 1 - 2\cos\theta\chi_r + \frac{2}{r}\sin\theta\,\chi_\theta - \chi_r{}^2 - \frac{1}{r^2}\chi_\theta{}^2\right)\right]^\alpha - 1 + \frac{\gamma M_\infty{}^2}{2T^2}\left(\chi_{or}\,\chi_r + \frac{1}{r^2}\chi_{o\theta}\,\chi_\theta\right)\right\} rRT^2\,dr\,d\theta$$

where

$$U\chi_o = \phi_o \tag{2.15}$$

and

$$J|\chi| = p_\infty \int_0^{2\pi} \int_1^{\infty} \left\{ \left[1 + \frac{\gamma - 1}{2T^2} M_\infty^2 (T^2 - 1 - 2\cos\theta\, \chi_r + \frac{2}{r} \sin\theta\, \chi_\theta \right.\right.$$

$$\left.\left. - \chi_r^2 - \frac{1}{r^2} \chi_\theta^2) \right]^\alpha - 1 + \gamma M_\infty^2 (\chi_r \frac{\partial r}{\partial x} + \chi_\theta \frac{\partial \theta}{\partial x}) \right\}$$

$$rRT^2\, dr d\theta \quad + \quad \gamma M_\infty^2 \int_B \chi \frac{\partial x}{\partial r}\, R ds \tag{2.16}$$

subject to the boundary conditions (2.10) on χ.

In two-dimensional flow the incompressible potential in the transformed plane was always that for the circle, since we were solving Laplace's equation, which is invariant under transformation. However, for axisymmetric flows the equation of motion in the cross-sectional plane is no longer Laplace's equation and changes under transformation. Thus the incompressible potential must be calculated separately for each body and it is only practical to solve the partial differential equation for the sphere and the ellipsoid. For other shapes, it is better to use the integral (2.16). In this case, however, the functions $\partial r/\partial x$ and $\partial\theta/\partial x$ must be found at all points in the field. Once again, calculation of these quantities is fairly simple for the ellipsoid, but for a Karman-Trefftz profile the evaluation becomes rather complicated (see Appendix C). When the transform modulus is found numerically it is also necessary to determine these quantities numerically, since an analytic form of the transformation will not be known.

§3 NUMERICAL METHOD

The object of the calculation is to find for given M_∞ and aerofoil shape a function χ which maximizes $J|\chi|$ as given by (2.9) and satisfies the boundary conditions (2.10). If we only consider nonlifting bodies which are symmetric about the axis $y = 0$, it is only necessary to treat the interval $0 \lesssim \theta \lesssim \pi$. Since the derivatives in both directions are approximated by finite differences, it is necessary to have a finite computation region. This is obtained by replacing the infinite integration limit on r by a finite limit R_f and insisting that the reduced potential χ satisfies an appropriate condition at $r = R_f$. The manner in which R_f is determined is described later. The simplest condition to impose is that χ equals the reduced potential for incompressible flow at $r = R_f$. A more complicated procedure which involves an asymptotic solution that takes into account the shape of the body is developed in Appendix E and was generally used. When $R_f = 20$ or larger the two boundary conditions give results which are identical to within the accuracy of the method, but for smaller values of R_f the second condition is more accurate.

Thus the variational integral (2.8) reduces to

$$J|\chi| = p_\infty \int_0^\pi d\theta \int_1^{R_f} F(r, \theta, \chi_r, \chi_\theta) \; dr \tag{3.1}$$

where

$$F = \left\{\left[1 + \frac{(\gamma-1)\ M_\infty{}^2}{2T^2}\ \left(T^2 - 1 - 2\cos\theta\ \chi_r + \frac{2}{r}\ \sin\theta\ \chi_\theta\right.\right.\right.$$

$$\left.\left. - \chi_r{}^2 - \frac{1}{r^2}\ \chi_\theta{}^2\right)\right]^\alpha - 1 + \frac{\gamma M_\infty{}^2}{T^2}\ \left(\frac{r^2-1}{r^2}\ \cos\theta\ \chi_r\right. \tag{3.2}$$

$$\left.\left. - \frac{r^2+1}{r^3}\ \sin\theta\ \chi_\theta\right)\right\}\ r\ T^2\ .$$

The boundary conditions are now

$$\begin{aligned} \frac{\partial\chi}{\partial\theta} &= 0 \quad \text{at} \quad \theta = 0,\ \pi \\ \frac{\partial\chi}{\partial_r} &= -\cos\theta \ \text{at} \ r = 1 \\ \chi &= \frac{f(\theta)}{R_f} \quad \text{at} \quad r = R_f \end{aligned} \tag{3.3}$$

where $f(\theta)$ is derived in Appendix E, or can be taken as $\cos\theta$ if incompressible flow is acceptable at the far boundary.

If r and θ are measured along cartesian axes, we see that the integration domain is a rectangle (see fig. 2). The domain is divided into an irregular mesh given by the intersections of two sets of straight lines which are defined by

$$0 = \theta_1 < \theta_2 < \ldots < \theta_\ell = \pi$$

where

$$h_i = \theta_{i+1} - \theta_i$$

and

$$1 \leqslant r_1 < r_2 < \ldots < r_n = R_f$$

where

$$k_j = r_{j+1} - r_j .$$

We can now explain how a value for R_f is decided on. Let $\sigma = {}^1/_r$ so that the interval $1 \leqslant r < \infty$ is mapped onto $1 \geqslant \sigma \geqslant 0$, and divide $[0, 1]$ into n equal parts. By use of $r = {}^1/_\sigma$ the interval $1 \leqslant r < \infty$ is then divided into n unequal parts, and we set $R_f = r_n$. Since the grid lines in the σ plane are ${}^1/_n$ apart, the last line before the origin will be a distance ${}^1/_n$ from the origin. Hence $r_n = n$ and $R_f = n$. Thus the value of R_f depends on the number of mesh points in the radial direction. THis procedure is, of course, equivalent to mapping the outside of the unit circle onto the inside using the mapping $r = {}^1/_\sigma$ in order to obtain a finite computation region as was done by Sells (1968).

The grid lines of constant θ map into curves in the physical plane which are clustered around the areas of high curvature on the aerofoil, see Sells (1968), p. 381. Thus the grid used ensures that there are more points in the regions where the flow varies rapidly than elsewhere.

The infinitely dimensional variational problem is now replaced by a finitely-dimensional problem. Consider four neighbouring points as shown in figure 3, rectangle 1. The derivatives of χ in the rectangle formed by the grid lines $i, i+1$ and $j, j+1$ are approximated by finite differences

$$\frac{\partial \chi}{\partial \theta} = \frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1}}{2hi}$$

$$\frac{\partial \chi}{\partial r} = \frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2kj} .$$

With these expressions $J(\chi)$ can be approximated for the rectangle by

$$J|\chi| \simeq J_{i,j} = p_\infty \left\{ \left[1 + \frac{(\gamma-1)\, M_\infty^{\;2}}{2T^2} \quad (T^2 - 1 - 2 \cos\theta_1 , \right. \right.$$

$$\frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2kj}$$

$$+ \frac{2}{r_1} \sin \theta_1 \quad \frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1}}{2h_i}$$

$$- \left(\frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2kj} \right)^2$$

$$\left. - \frac{1}{r_1^{\;2}} \left(\frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1})^2}{2h_i} \right) \right]^\alpha$$

$$- 1 + \frac{\gamma M_\infty^{\;2}}{T^2} \quad \left(\frac{r_1^{\;2} - 1}{r_1^{\;2}} \quad \cos \theta_1 \right.$$

$$\frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2k_j}$$

$$- \frac{r_1^{\,2} + 1}{r_1^{\,3}} \quad \sin \theta_1$$

$$\left. \frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1}}{2h_i} \right) \Bigg\}$$

$$r_1 \quad T^2 \, h_i \, k_j \tag{3.4}$$

where $\theta_1 = \theta_i + 0.5\ h_i$, $r_1 = r_j + 0.5\ k_j$, and T is evaluated for θ_1 and r_1. Before we can sum the contributions for each rectangle, it is necessary to consider the treatment of the boundary conditions.

One of the four boundary conditions that χ must satisfy creates no problems. From (3.3) we see that, at the line $j = n$, χ is prescribed, so no modification is required to the procedure described above. However, at the lines $i = 1$, $i = \ell$ and $j = 1$ only the normal derivative of χ is prescribed. Let us first consider $i = \ell$. Here χ_θ must be zero, and in order to approximate this we add an extra line $i = \ell+1$ to the mesh such that $h_\ell = h_{\ell-1}$, and then set

$$\chi_{\ell+1,j} = \chi_{\ell-i,j}$$

Similarly at $i = 1$, we add an extra line $i = 0$ such that $h_1 = h_o$ and then set

$$\chi_{o,j} = \chi_{1,j}$$

The boundary condition at $j = 1$, i.e. at $r = 1$, is approximated in a different way. Here we use an interpolation between the three points $(i,1)$, $(i,2)$ and $(i,3)$ and find that

$$- \chi_r \Big|_{r=1} = \frac{k_2 + 2k_1}{k_1 (k_1 + k_2)} \chi_{i,1} - \frac{k_1 + k_2}{k_1 \, k_2} \chi_{i,2}$$

$$+ \frac{k_1}{k_2 \, (k_1 + k_2)} \chi_{i,3} \quad .$$

Since $\chi_r = - \cos \Theta$ at $r = 1$, we have that

$$\chi_{i,1} = \frac{1}{k2 + 2k_1} \left[\frac{1}{k_2} \left((k_1 + k_2)^2 \chi_{i,2} \right. \right.$$

$$\left. \left. - k_1^{\,2} \chi_{i,3} \right) + k_1 (k_1 + k_2) \cos \theta \right] . \qquad (3.5)$$

We can now sum the contribution (3.4) for each rectangle, and we see that $J|\chi|$ can be approximated by

$$J|\chi| \simeq \bar{J} = \sum_{i=1}^{\ell} \sum_{j=1}^{n-1} J_{i,j}$$

The values of $\chi_{i,j}$ which maximize this expression are given by the solutions to the equations

$$\frac{\partial \bar{J}}{\partial \chi_{i,j}} = \theta \qquad \begin{array}{l} i = 1, \ldots , \ell \\ j = 2, \ldots , n-1 \end{array} \qquad (3.6)$$

These equations were solved by the Newton-Raphson method in the following way. For a given (i,j) we can write (3.6) in the form

$$g(\chi_{i,j}) = \sum_{s=1}^{4} \Big[\alpha \, (A_s \; \chi_{i,j}^2 + B_s \, \chi_{i,j} + C_s)^{\alpha-1}$$

$$* \; (2 \, A_s \, \chi_{i,j} + B_s) + D_s \Big] \, H_s = 0$$

where A_s, B_s, C_s, D_s, H_s, are functions of χ at the neighbouring points derived in Appendix F and of h_i, h_{i-1}, k_j, k_{j-1}. Let $\chi_{i,j}^{(n)}$ be the n^{th} approximation to the solution. Then an improved estimate is given by

$$\chi_{i,j}^{(n+1)} = \chi_{i,j}^{(n)} - \frac{g(\chi_{i,j}^{(n)})}{g^1(\chi_{i,j}^{(n)})}$$

where $g'(z) = dg/dz$. When the difference $|\chi_{i,j}^{(n+1)} - \chi_{i,j}^{(n)}|$ was found to be less than 10^{-4} the iterative process was stopped. For a typical point this took about four iterations. A tolerance of 10^{-4} here, probably means that the local Mach numbers are only accurate to the third decimal place. If the Newton-Raphson method is continued until $|\chi_{i,j}^{(n+1)} - \chi_{i,j}^{(n)}| \leqslant 10^{-5}$ a large number of iterations are required for sweeps through the field, where the approximate solution is still close to incompressible flow. This process is carried out for each point in turn with the calculated values being used as soon as they are available. In this way χ can be calculated to the desired degree of accuracy. When equations (3.6) are solved, (3.5) is then used to evaluate χ on the surface.

Calculations were also carried out for the regions $^{\pi}/_2 \leq \theta \leq \pi$, $1 \leq r \leq R_f$ and $^{\pi}/_2 \leq \theta \leq 3^{\pi}/_2$, $1 \leq r \leq R_f$. The boundary conditions at $\theta = {}^{\pi}/_2$ and $^{3\pi}/_2$ for a body symmetric about $x = 0$ are that χ vanishes there.

The approach used by Greenspan and Jain (1967) differs in some important aspects from the one described above. Given an interior point (i,j) they approximate the derivatives of χ by

$$\left(\frac{\partial\chi}{\partial r}\right)_{i,j} = \frac{\chi_{i,j+1} - \chi_{i,j}}{\Delta r}$$

$$\left(\frac{\partial\chi}{\partial\theta}\right)_{i,j} = \frac{\chi_{i+1,j} - \chi_{i,j}}{\Delta\theta}$$

These expressions are then substituted into $J|\chi|$ to give an approximation $J_{i,j}$ for the rectangle (i,j), $(i+1,j)$, $(i+1,j+1)$, $(i,j+1)$. A summation over all the points gives a global approximation J' to J. The boundary conditions are then used to obtain approximations to χ on the boundaries in terms of the neighbouring interior points, and these boundary values of χ are substituted into J'. A maximizing expression is found for J' by solving the equation

$$\frac{\partial J'}{\partial\chi_{i,j}} = 0$$

for all interior points.

The main differences between the two approaches are the treatment of the boundary conditions on the surface of the body and the extent to which the values of χ at the neighbouring points appear in the equation for $\chi_{i,j}$. Greenspan and Jain approximate the condition $\chi_r = -\cos\theta$ at $r = 1$ by writing

$$\chi_{i,0} = \Delta r \cos\theta_i + \chi_{i,1}$$

where $(i,0)$ is on the surface, while we use a three points interpolation. Due to the way in which they approximate the derivatives of χ, their equation for $\chi_{i,j}$ depends only on χ at six of the eight neighbouring points, the points $(i-1, j-1)$ and $(i+1, j+1)$ being excluded. This is in contrast to the approach in this thesis where χ at all the neighbouring points are used. It is difficult to know if these differences account for the fact that our solutions for flow past a circle are closer to other approximations near the stagnation points than those obtained by Greenspan and Jain.

It was found that the most satisfactory way to sweep through the field was to start at the leading edge and, keeping the value of θ constant, to cover all the r values for this value of θ from the body to the far boundary. This was repeated for the other values of θ until the trailing edge was reached. In other words the first point, where the new value of the potential was found, was $r = 1$, $\theta = \pi$, then the other grid points where θ had the same value, had their potential values found working outwards from the body. Then the new value at $r = 1$, $\theta = \pi - h_{k-1}$ was found and the process repeated working outwards from the body. This method was continued until the new value of the perturbation potential had been found at all grid points.

This order was chosen, so that the errors in the process would tend to accumulate away from the body, while we were more interested in velocities on the body surface. If r had been fixed and θ varied, for each value of r some of the error would have tended to accumulate at the trailing edge.

For a Karman-Trefftz profile, it can be seen (see Appendix C) (C-6), that, T^2, the transform modulus squared is zero at the trailing edge, $r = 1$, $\theta = 0$ and for the rectangles close to the trailing edge, T^2 will be small, making the integral (3.4) small in these rectangles.

The singularity, however, is sufficiently weak not to affect the calculation of the potential at points near the trailing edge.

§ 4 CONVERGENCE FOR PLANE FLOW

Three different convergence criteria were used. At one stage the iterative scheme was continued until the Mach number on the surface of the body changed by less than 1.0×10^{-5} during one iteration. This gives a solution of similar accuracy to that by Sells (1968). It was found that this criterion was approximately fulfilled if the reduced potential did not change by more than the same amount during 100 iterations.

However, since the Newton-Raphson method was only iterated until a difference of less than 1.0×10^{-4} was achieved in successive iterations, the local Mach numbers are probably not accurate beyond the third decimal place, as they involve differentiation of the potential thus it was felt that an adequate convergence criterion was that the maximum difference in potential at any point in the field should not exceed 0.25×10^{-5} in two successive iterations.

In other words the iterative process was stopped when

$$\max_{\substack{1 \leq i \leq \ell \\ 1 \leq j \leq n}} \left| X_{ij}^{(n)} - X_{ij}^{(n-1)} \right| \leq 0.25 \times 10^{-5} \qquad (4.1)$$

The accuracy of the final results is obviously going to depend on the mesh size. The above convergence criterion (4.1) was usually used for a grid with 17 points around the upper half plane and 21 points radially outwards. This gives a step length of $11\frac{1}{4}^{\circ}$ around the body, which is probably only fine enough to give results accurate to the second decimal place. Therefore the iterative process could probably be terminated at an earlier stage, but waiting until (4.1) is satisfied ensures the results are as accurate as possible for a particular mesh. Results agreeing with Sells' (1968) to the second decimal place have been obtained, apart from near the stagnation point, signifying that our iterative process has converged to the right answer.

Usually, about 800 iterations were required, to satisfy (4.1), but this depended, of course, on the number of mesh points and on the value of the free stream Mach number. For the calculation of the flow past a 10% ellipse with a free stream Mach number of 0.8, on a 17 by 21 grid on the region $1 \leqslant r \leqslant 21$, $0 \leqslant \theta \leqslant \pi$, 800 iterations were required, and the computing time on an ICL 1907 was about 80 minutes.

The rate of convergence was improved by using over-relaxation at the end of each iteration. Thus if $\overline{\chi_{ij}}^{(q+1)}$ was obtained by solving (3.6), the new value of χ, $\chi_{ij}^{(q+1)}$ was defined by

$$\chi_{ij}^{(q+1)} = w\overline{\chi_{ij}} + (1 - w)\, \chi_{ij}^{(q)}$$

where w is a parameter greater than zero. If $w < 1$ we have under-relaxation and if $w > 1$ over-relaxation. It was found by trial and error that the best convergence for the ellipse was achieved with $w = 1.4$ and this value also seemed to give the best convergence for the Karman-Trefftz profiles, but no extensive search for an optimal value was carried out. Even when the local Mach number was close to unity, it was not necessary to use under-relaxation in order to obtain convergence. This is in contrast to Sells (1968), though Albone (1971) later modified Sells' method to make under-relaxation unnecessary.

A number of calculations were carried out for a 10% ellipse with a free stream Mach number of 0.8 on a quarter plane in order to test the importance of different grid sizes. Some results are given in table 4.1, and they seem to indicate that if the number of points in either the radial or angular direction is increased, the local Mach number on the surface converges.

Table 4.1

Local Mach numbers on the surface of a 10% ellipse with $M_\infty = 0.8$ for different grid sizes.

Grid

no. of points around the body x no. of points out from the body

θ	9 x 41	9 x 31	9 x 21	13 x 21	17 x 21
22.5°	0.8617	0.8615	0.8606	0.8615	0.8621
45.0°	0.9409	0.9413	0.9420	0.9447	0.9457
67.5°	0.9683	0.9689	0.9702	0.9739	0.9752
90.0°	0.9763	0.9769	0.9784	0.9825	0.9839

5 RESULTS

The main part of the calculations were carried out for nonlifting ellipses of different thickness ratios, usually 10% but a number were also obtained for Karman-Trefftz profiles. In most of the calculations the computation region was $0 \leqslant \theta \leqslant \pi$, $1 \leqslant r \leqslant R_\varepsilon$ and the solutions show that when the flow is subsonic the potentials for nonlifting ellipses are always symmetric about the line $\theta = \pi/2$. A few calculations were carried out for $\pi/2 \leq \theta \leq 3\pi/2$ in order to check the accuracy of the treatment of the boundary conditions at $\theta = 0$ and $\theta = \pi$. Since the results were identical with those obtained for $0 \leqslant \theta \leqslant \pi$ it was considered sufficient to only carry out the calculations for the upper half-plane. The results in this section for the different ellipses were obtained using either the reduced incompressible solutions as boundary conditions at $r = R_\varepsilon$ or the condition derived in Appendix E while for the Karman-Trefftz profile only the boundary condition derived in Appendix E was used.

One of the simplest cases to consider is that of a circle for a free stream Mach number just below the critical value. Table 5.1 shows results obtained with a 21 by 21 grid and gives also for comparison similar results by the program developed by Sells (1968). Very good agreement is achieved.

The non-dimensionalised speed on the surface of the circle at a free stream Mach number of 0.4 is compared with the results obtained by Lush and Cherry (1956), IMai (1941), Wang (1948), Greenspan and Jain (1967) and a linearized solution in Table 5.2 Good agreement is achieved, except with the results of Greenspan and Jain (1967) near the stagnation point where a discrepancy of 46% is found. The reason for this difference was discussed in Section 3.

TABLE 5.1.

Local Mach numbers on the surface of a circle, M = 0.39.

θ	Sells	Our results
0.0°	0.0	0.0
9.0°	0.1123	0.1123
18.0°	0.2246	0.2245
27.0°	0.3367	0.3366
36.0°	0.4483	0.4482
45.0°	0.5587	0.5585
54.0°	0.6665	0.6661
63.0°	0.7689	0.7682
72.0°	0.8604	0.8591
81.0°	0.9301	0.9276
90.0°	0.9582	0.9544

A more exacting test of the programme is to calculate soluətions for thin bodies at high speeds. In Table 5.3 results are given for a 10% ellipse with a free-stream Mach n mber of 0.8 and with the calculations done on a 17 by 21 grid. Again results obtained by Sells' programme are presented ———————————————————

TABLE 5.2

Speed q/U on the surface of the cylinder. A comparison of our method with those of Lush and Cherry, Imai, Wang, Greenspan and Jain and a linearized solution, for a free stream Mach number of 0.4.

Position on body surface	Speed q/U on Surface					
	Lush and Cherry	Imai M^6	Wang $\gamma = 2$	Greenspan and Jain	Linearized Soln.	Our Soln.
90°	2.3102	2.284	2.335	2.2844	2.285	2.2867
99°				2.4344		2.2339
100°	2.2492	2.227	2.271	2.2249*	2.229	2.2072 *
108°				2.0769		2.0937
110°	2.1074	2.068	2.097	2.0285 *	2.070	2.0502 *
117°				1.8590		1.8977
120°	1.8340	1.836	1.845	1.7774 *	1.836	1.8210 *
126°				1.6142		1.6676
130°	1.5568	1.561	1.552	1.4980 *	1.571	1.5554 *
135°				1.3527		1.4152
140°	1.2537	1.266	1.247	1.2011 *	1.260	1.2776 *
144°				1.0798		1.1475
150°	0.9536	0.959	0.941	0.8927 *	0.957	0.9651*
153°				0.7791		0.8690
160°	0.6464	0.644	0.635	0.5766 *	0.635	0.6467 *
162°				0.5130		0.5832
170°	0.3280	0.323	0.319	0.1952 *	0.319	0.3243 *
171°				0.1564		0.2926
180°	0	0	0	0	0	0

* results obtained by linear interpolotion.

TABLE 5.3

Local Mach numbers on the surface of a 10% ellipse, M = 0.8.

θ	Sells	Our results Incomp far b dy.	Series soln f.b
0.00°	0.0	0.0.	0.0
11.25°	0.7640	0.7250	0.7234
22.50°	0.8900	0.8606	0.8596
33.75°	0.9170	0.9132	0.9133
45.00°	0.9398	0.9419	0.9430
56.00°	0.9609	0.9590	0.9612
67.50°	0.9756	0.9702	0.9732
78.75°	0.9831	0.9762	0.9799
90.00°	0.9855	0.9783	0.9823

for comparison. The agreement away form the neighbourhoods the stagnation points are good, but near these points they differ by about 5%. It is not clear what the cuase is of this difference. Since in both procedures conformal mappings are used to transform the ellipse into the unit circle, it would seem that the treatments of the region of high curvature are identical. It is possible that it is the different treatments of the boundary condition on the surface that are the cause. In Sells' programme the streamfunction is used so the boundary condition is the simple one of setting it equal to zero on the surface. In our programme, however, it is the normal derivative of the velocity potention which is prescribed. Different ways of treating this condition were tried out, but no improvement over the results presented here was achieved.

The values of the free-stream Mach number were also increased until the numerical procedure ceased to converge. For a 10% ellipse with a 17 by 21 grid it converged for $M_\infty = 0.82$ but not for 0.83. In Table 5.1 the values of the local Mach number along the surface are given for different values of M_∞.

TABLE 5.4

Local Mach numbers on the surface of a 10% ellipse for different values of M_∞

	$M_\infty = 0.70$	$M_\infty = 0.80$	$M_\infty = 0.82$
0.0°	0.00	0.00	0.00
11.25°	0.6569	0.7250	0.7351
22.50°	0.7587	0.8606	0.8768
33.75°	0.7888	0.9132	0.9357
45.00°	0.8019	0.9419	0.9696
56.25°	0.8081	0.9590	0.9940
67.50°	0.8115	0.9702	1.0089
78.75°	0.8131	0.9762	1.0242
90.00°	0.8136	0.9783	1.0281
101.25°	0.8131	0.9762	1.0213
112.50°	0.8115	0.9702	1.0119
123.75°	0.8081	0.9590	0.9922
135.00°	0.8019	0.9419	0.9704
146.25°	0.7888	0.9132	0.9353
157.50°	0.7587	0.8606	0.8771
168.75°	0.6569	0.7251	0.7349
180.00°	0.00	0.00	0.00

In order to see if the method was feasible for more realistic shapes, the flows around two different Karman-Trefftz profiles were calculated.

Results for a profile of 10% thickness to chord ratio with a trailing edge angle of 6^o are shown in Table 8.5, for a range of free stream Mach numbers.

The flow around a Karman-Trefftz 'F' profile at various free stream Mach numbers was also calculated to compare the results with those provided by R.A.E. Farnborough, found using Sells' program and that of Bauer, Garabedian and Korn (1972). These are compared in Tables 5.6 and 5.7 and show a maximum difference of only 0.8% with Sells' program.

Since a Karman-Trefftz profile is not antisymmetric about $\theta = \frac{\pi}{2}$ all calculations were carried out on the half plane $0 \leq \theta \leq \pi$, $1 \leq r \leq R_f$.

Results were also obtained for a Karman-Trefftz 'F' profile at Mach 0.75 and the free stream Mach number was pushed above 0.76 to see if transonic flows could be obtained. A small supersonic bubble was found at Mach 0.78 but at 0.79 the iterative process failed to converge. The results are given in Table 5.8.

It was considered desirable to investigate the flow about a NACA 0012 profile to see if the method was still valid when a numerical transformation was used. Table 5.9 compares our results with those given by Lock (1970), obtained by Sells' (1968) method. Lock (1970) considered that a NACA 0012 profile at a free stream Mach number of 0.72, which is just subcritical, was a desirable test case for comparison of new and existing methods of calculating two-dimensional flows.

The greatest difference between Sells' results and ours (after using linear interpolation to obtain them at the same points) is one of 3.4% near the leading edge. The reason, that this difference is greater than in most other cases, is probably that the capacity of the ICL 1907 was insufficient to use enough points in finding the transform modulus at exterior points. However the results are sufficiently close to show that the variational method is valid for arbitrary profiles.

TABLE 5.5

Flow past a 10% thick Karman-Trefftz profile k = 0.9375, m = 1.9667 with a trailing edge angle of 6° and thickness ratio of 10%.

Position on the Aerofoil	Free Stream Mach No.			
	0.6	0.7	0.75	0.76
	Local Mach No.			
180°	0.0	0.0	0.0	0.0
$168\frac{3}{4}°$	0.5504	0.6250	0.6545	0.6594
$157\frac{1}{2}°$	0.6783	0.7938	0.8454	0.8543
$146\frac{1}{4}°$	0.7200	0.8620	0.9361	0.9490
135°	0.7328	0.8888	0.9852	1.0069
$123\frac{3}{4}°$	0.7305	0.8889	0.9943	1.0222
$112\frac{1}{2}°$	0.7193	0.8717	0.9688	0.9924
$101\frac{1}{4}°$	0.7021	0.8444	0.9280	0.9459
90°	0.6812	0.8124	0.8845	0.8993
$78\frac{3}{4}°$	0.6582	0.7787	0.8417	0.8542
$67\frac{1}{2}°$	0.6346	0.7454	0.8012	0.8121
$56\frac{1}{4}°$	0.6111	0.7134	0.7634	0.7731
45°	0.5885	0.6834	0.7289	0.7376
$33\frac{3}{4}°$	0.5670	0.6556	0.6973	0.7052
$22\frac{1}{2}°$	0.5463	0.6293	0.6678	0.6750
$11\frac{1}{4}°$	0.5235	0.6007	0.6361	0.6428
0°	0.0	0.0	0.0	0.0

At M = 0.77 the process failed to converge

TABLE 5.6

Comparison of Mach numbers along the surface of a Karman-Trefftz 'F' profile at free stream Mach number 0.6

$m = 1.94444 \qquad k = 0.95493$

Angle	Our Method	Sells Method
180°	0.0	0.0
174°	0.4267	0.4250
168°	0.5666	0.5765
162°	0.6254	0.6336
156°	0.6576	0.6638
150°	0.6780	0.6838
144°	0.6917	0.6973
138°	0.7009	0.7066
132°	0.7067	0.7123
126°	0.7097	0.7150
120°	0.7103	0.7153
114°	0.7088	0.7134
108°	0.7054	0.7095
102°	0.7003	0.7039
96°	0.6937	0.6968
90°	0.6859	0.6884
84°	0.6769	0.6789
78°	0.6671	0.6685
72°	0.6564	0.6574
66°	0.6451	0.6458
60°	0.6334	0.6336
54°	0.6212	0.6211
48°	0.6087	0.6084
42°	0.5960	0.5954

TABLE 5.6 (Continued)

Angle	Our Method	Sells Method
36^o	0.5829	0.5822
30^o	0.5694	0.5687
24^o	0.5552	0.5545
18^o	0.5396	0.5392
12^o	0.5211	0.5212
6^o	0.4945	0.4949
0^o	0.0	0.0

TABLE 5.7

Comparison between the results obtained by our program with those obtained by Bauer, Garabedian and Korn (1972) for a Karman-Trefftz 'F' profile.

Position on the profile	0.7 Our results	0.7 B.G.K's results	0.72 Our results	0.72 B.G.K's results	0.76 Our results	0.76 B.G.K's results
			Local Mach numbers			
180°	0.0	0.0002	0.0	0.0003	0.0	0.0002
$168\frac{3}{4}^{\circ}$	0.6353	0.6315	0.6494	0.6443	0.6746	0.6678
$157\frac{1}{2}^{\circ}$	0.7586	0.7632	0.7796	0.7835	0.8188	0.8219
$146\frac{1}{4}^{\circ}$	0.8125	0.8211	0.8388	0.8472	0.8909	0.8994
135°	0.8413	0.8514	0.8718	0.8822	0.9364	0.9482
$123\frac{3}{4}^{\circ}$	0.8527	0.8632	0.8857	0.8968	0.9597	0.9744
$112\frac{1}{2}^{\circ}$	0.8506	0.8600	0.8839	0.8939	0.9596	0.9735
$101\frac{1}{4}^{\circ}$	0.8376	0.8453	0.8692	0.8774	0.9394	0.9497
90°	0.8167	0.8225	0.8459	0.8518	0.9079	0.9147
$78\frac{3}{4}^{\circ}$	0.7907	0.7947	0.8171	0.8211	0.8714	0.8757
$67\frac{1}{2}^{\circ}$	0.7615	0.7642	0.7853	0.7879	0.8331	0.8356
$56\frac{1}{4}^{\circ}$	0.7308	0.7327	0.7523	0.7540	0.7946	0.7962
45°	0.6994	0.7006	0.7188	0.7199	0.7566	0.7575
$33\frac{3}{4}^{\circ}$	0.6673	0.6683	0.6849	0.6858	0.7188	0.7195
$22\frac{1}{2}^{\circ}$	0.6337	0.6345	0.6496	0.6503	0.6801	0.6805
$11\frac{1}{4}^{\circ}$	0.5928	0.5934	0.6069	0.6075	0.6336	0.6340
0°	0.0	0.5232	0.0	0.5349	0.0	0.5562

TABLE 5.8

Local Mach numbers on the surface of a Karman-Trefftz 'F' profile

Position on the aerofoil	Free stream Mach number 0.75	0.78
	Local Mach numbers	
180°	0.0	0.0
$168\frac{3}{4}^\circ$	0.6687	0.6855
$157\frac{1}{2}^\circ$	0.8094	0.8361
$146\frac{1}{4}^\circ$	0.8781	0.9154
135°	0.9198	0.9692
$123\frac{3}{4}^\circ$	0.9398	1.0052
$112\frac{1}{2}^\circ$	0.9389	1.0087
$101\frac{1}{4}^\circ$	0.9206	0.9804
90°	0.8918	0.9415
$78\frac{3}{4}^\circ$	0.8575	0.8991
$67\frac{1}{2}^\circ$	0.8212	0.8568
$56\frac{1}{4}^\circ$	0.7841	0.8151
45°	0.7473	0.7747
$33\frac{3}{4}^\circ$	0.7105	0.7349
$22\frac{1}{2}^\circ$	0.6726	0.6943
$11\frac{1}{4}^\circ$	0.6272	0.6461
0°	0.0	0.0

TABLE 5.9

Comparison of our results with those given by Lock (1970) for the flow past a NACA 0012 profile at a free stream Mach number of 0.72

Position on the profile	Local Mach Numbers Lock	Heys	Position on the profile	Local Mach numbers Lock	Heys
180°	0.0	0.0	87.13°	0.8392	
174.90°	0.3761		81.02°	0.8199	
168.75°	0.6302 *	0.6085	78.75°	0.8122 *	0.8196
168.52°	0.6397		74.93°	0.8018	
162.93°	0.7860		68.90°	0.7843	
157.50°	0.8610 *	0.8370	67.50°	0.7803 *	0.7859
157.22°	0.8647		62.73°	0.7675	
151.65°	0.9172		56.50°	0.7508	
146.25°	0.9497*	0.9361	56.25°	0.7501 *	0.7542
145.89°	0.9517		50.21°	0.7340	
140.00°	0.9748		45.00°	0.7195 *	0.7225
135.00°	0.9826	0.9774	43.95°	0.7166	
134.27°	0.9837		37.43°	0.6982	
128.32°	0.9807		33.75°	0.6872 *	0.6880
123.75°	0.9674 *	0.9721	30.92°	0.6784	
122.68°	0.9672		24.22°	0.6561	
116.75°	0.9475		22.50°	0.6501 *	0.6471
112.50°	0.9330*	0.9390	17.05°	0.6305	
110.85°	0.9257		11.25°	0.6098 *	0.5901
104.95°	0.9032		8.10°	0.5985	
101.25°	0.8890 *	0.8975	0°	0	0
99.12°	0.8809				
93.10°	0.8595				
90°	0.8490 *	0.8568			

* Found by linear interpolation from the other values.

6. NUMERICAL METHOD FOR AXISYMMETRIC FLOW

As in the two-dimensional case we can only consider the integral over a finite region, so we take the far boundary at $r = R_f$ and derive an asymptotic expansion for χ away from the body. In the cross-section plane we only consider bodies which are symmetric about $R = 0$ and only the non-lifting case is considered, so we need only treat the upper half plane $0 \leq \theta \leq \Pi$. The integrals (2.21) and (2.22) then become, after dividing by p_∞,

$$J\,|\phi| = \int_0^{\pi}\int_1^{R_f} \left\{\left[1 + \left(\frac{\gamma-1}{2}\right) M_\infty^2 \left(1 - \frac{1}{T^2}\right) - \left(\frac{\gamma-1}{2T^2}\right)^{o} M_\infty^2 \left(2\cos\theta\chi_r - \frac{2}{r}\sin\theta\chi_\theta + \chi_r^2 + \frac{1}{r^2}\chi_\theta^2\right)\right]^{\alpha} - 1 + \frac{\gamma M_\infty^2}{T^2}\left(\chi_{or}\chi_r + \frac{1}{r^2}\chi_{o\theta}\chi_\theta\right)\right\} rRT^2\,dr\,d\theta \tag{6.1}$$

and

$$J\,|\phi| = \int_0^{\Pi}\int_1^{R_f} \left\{\left[1 + \frac{\gamma-1}{2} M_\infty^2 \left(1 - \frac{1}{T^2}\right) - \frac{\gamma-1}{2T^2} M_\infty^2 \left(2\cos\theta\chi_r - \frac{2}{r}\sin\theta\chi_\theta + \chi_r^2 + \frac{\chi_\theta^2}{r^2}\right)\right]^{\alpha} - 1 + \gamma M_\infty^2\left(\chi_r\frac{\partial r}{\partial x} + \chi_\theta\frac{\partial\theta}{\partial x}\right)\right\} r\,RT^2 dr\,d\theta + \gamma M_\infty^2\int_B \chi\frac{\partial x}{\partial r} R\,ds \tag{6.2}$$

respectively, with the boundary conditions

$$\begin{aligned}
\chi_\theta &= 0 && \text{at } \theta = 0, \pi \\
\chi_r &= -\cos\theta && \text{at } r = 1 \\
\chi &= \frac{f(\theta)}{R_f} && \text{at } r = R_f
\end{aligned} \tag{6.3}$$

where $f(\theta)$ is derived in Appendix G.

The definition of the irregular mesh and the determination of s are exactly the same as for two dimensions, as are the expressions for $\frac{\partial \chi}{\partial \theta}$ and $\frac{\partial \chi}{\partial r}$ in finite differences.

Using these expressions we have the two following forms for the approximation of $J|\chi|$ on a rectangle of the mesh.

$$J|\chi| \triangleq J_{ij} = \Bigg\{\Bigg[1 + \frac{(\gamma - 1) Mas^2}{2T^2}\Bigg(T^2 - 1$$

$$- 2\cos\theta_1 \frac{\chi_{i\,j+1} + \chi_{i+1,j+1} - \chi_{ij} - \chi_{i+i,j.}}{2k_j}$$

$$+ \frac{2}{r_1}\sin\theta_1 \frac{\chi_{i+1,j} + \chi_{i+1,j+1}, - \chi_{i\,j} - \chi_{i,j+1}}{2h_i}$$

$$- \left(\frac{\chi_{i,i+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2k_j}\right)^2$$

$$- \frac{1}{r_1^2}\left(\frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1}}{2h_i}\right)^2\Bigg]^\alpha$$

$$- 1 + \frac{\gamma Mas^2}{T^2}\Bigg(\chi_{or} \frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{ij} - \chi_{i+1,j}}{2k_j}$$

$$+ \frac{\chi_{o\theta}}{r_1^2} \frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{ij} - \chi_{i,j+1}}{2h_i}\Bigg)\Bigg\}$$

$$r_1 R\, T^2 h_i k_j \tag{6.4.}$$

corresponding to (6.1.), and

$$J|\chi| \triangleq J_{ij} = \Bigg\{\Bigg[1 + \frac{(\gamma - 1) Mas^2}{2T^2}\Bigg(T^2 - 1 -$$

$$\frac{2\cos\theta_1\, \chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2k_j}$$

$$+ \frac{2}{r_1}\sin\theta_1 \frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1}}{2h_i}$$

$$-\left(\frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2k_j}\right)^2$$

$$- \frac{1}{r_1^2} \frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1}}{2h_i}^{2}\Bigg]^\alpha$$

$$- 1 + \gamma M_{\infty}^2 \left(\frac{\partial r}{\partial x} \; \frac{\chi_{i,j+1} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i+1,j}}{2 k_j} \right.$$

$$\left. + \frac{\partial \theta}{\partial x} \; \frac{\chi_{i+1,j} + \chi_{i+1,j+1} - \chi_{i,j} - \chi_{i,j+1}}{2 h_i} \right) \Bigg\}$$

$$\bar{r}_1 R T_2^2 h_i k_j$$

$$+ W \gamma M_{\infty}^2 \; \frac{\chi_{i+1,1} + \chi_{i,1}}{2} \; \frac{\partial x}{\partial r} \quad R h_i \qquad (6.5)$$

where $W = 0$ for $j \geqslant 2$ and $W = 1$ for $j = 2$. Also $\theta_1 = \theta i + 0.5hi$, $r_1 = rj + 0.5\,kj$. R, T, $\frac{\partial r}{\partial x}$, $\frac{\partial \theta}{\partial x}$, $\chi_{o\,r}$ and $\chi_{o\,\theta}$ are evaluated for r_1 and θ_1. The boundary conditions at $r = 1$ and $\theta = 0$ or π are treated in exactly the same way as for two-dimensional flow.

As for plane flow, the contributions (6.4) and (6.5) are summed over the field to give an approximation for $J|\chi|$

$$J|\chi| \simeq \bar{J} = \sum_{i=1}^{\ell} \sum_{j=1}^{n-1} J_{i,j}$$

Equation (3.6) is again solved by the Newton-Raphson method, where the expressions for As, Bs, Cs, Ds and Hs are given in Appendix H. The same process is carried out until the desired degree of accuracy is achieved.

In axisymmetric flows the factor R in (6.4) and (6.5) is small for rectangles bordering $\theta = 0$, so the contribution to the integral from rectangles near $\theta = 0$ is also small, but the singularity is sufficiently weak not to affect the calculation of the potential. Near the trailing edge, $r = 1$, $\theta = 0$, however, for a Karman-Trefftz profile of revolution, we have that T^2 is also small, as in the two-dimensional case.

In this case the factor RT^2 in (6.4) and (6.5) becomes very small, casting doubt on the validity of the results obtained by the variational method, near the trailing edge.

It was therefore decided to try replacing the variational method by the series expansion for a cone, of the same trailing edge angle, near the trailing edge. The flow past a cone was discussed by Mangler (1948) who had previously found, with Leuteritz that the potential was given by

$$\phi = -K \rho_c^{\nu} F(\Psi) \quad (6.6)$$

to within a constant, where

ρ_c is the distance from the cone vertex

K is a constant

ν is a constant related to the cone angle and lies between 1 and 2

and $\Psi = \pi - \Theta$ where Θ is the angle the line joining the cone vertex to the point being considered, makes with the positive x - axis (see fig. 5)

The constants are derivedin Appendix I.

§ 7 CONVERGENCE FOR AXISYMMETRIC FLOWS

The convergence criterion used for axisymmetric flows was the same as the one finally used for plane flow. Thus the iterative procedure was terminated when the maximum difference in the potential at any point in the field should not exceed 0.25×10^{-5} in two successive iterations.

The remarks, made in Section 4 for plane flow, that the mesh size probably restricts the accuracy to two decimal places, also apply to axisymmetric flows, though once again agreement with other approximate methods implies the iterative process converges to the correct values.

In general fewer iterations were required for convergence than in two-dimensional flow. For the calculation of the flow past a 10% ellipsoid with a free stream Mach number of 0.9 on a 17 by 21 grid on the region $1 \leqslant r \leqslant 21$, $0 \leqslant \theta \leqslant \pi$, 200 iterations were required and the computation took about 13 minutes on the ICL 1907 at Southampton University.

It was found, as in two-dimensional flows, that over-relaxation could be used at the end of each iteration to increase the rate of convergence. It was again found by trial and error that the best rate of convergence was achieved with a relaxation parameter of 1.4, though, again, no extensive search was carried out. Even for local and free stream Mach numbers close to sonic speed under-relaxation was unnecessary to obtain convergence, in contrast to Pidcock (1969) who adapted Sells' (1968) method to axisymmetric flows.

8. RESULTS

Calculations were carried out for a sphere, an ellipsoid, of thickness ratio 10%, and for bodies of revolution with Karman-Trefftz profiles as cross-section. In most of the calculations, as for two-dimensional flow, the computation region was $0 \leq \theta \leq \pi$ $I \leq r \leq R_f$ and, again for subsonic flow the potentials for a sphere and an ellipsoid are symmetric about $\theta = \frac{\pi}{2}$. Some calculations were carried out for $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$ in these cases to check the accuracy of the boundary conditions at $\theta = 0$ and $\theta = \pi$. Once again, as in plane flow the results were identical with those for $0 \leq \theta \leq \pi$, so it was only necessary to consider the upper half-plane. In all cases the assymptotic form of the boundary conditions derived in Appendix G was used.

The first case considered was the flow past a unit sphere, since no conformal mapping is required. The sphere has previously been calculated by Wang and de los Santos (1951) using a variational technique and Lamla (1939) and Kaplan (1940), among others, using a Rayleigh-Janzen method. Wang and de los Santos' integral was of the form,

$$J|\phi| = \int_V \left[\left(q_m^2 - \frac{\partial \phi}{\partial x_i}\frac{\partial \phi}{\partial x_i} \right)^\alpha - \left(q_m^2 - \frac{\partial \phi_0}{\partial x_i}\frac{\partial \phi_0}{\partial x_i} \right)^\alpha \right] d\omega + 2\alpha \left(q_m^2 - v^2 \right)^\beta \int_B \chi_A \frac{\partial \phi_0}{\partial n}\, ds$$

where the notation not previously used is q_m is the critical velocity and χ_A is given by $\phi = \phi_0 + \chi_A$

As in Wang's two-dimensional work γ was taken as 2 to simplify the calculation. However, they did obtain some results, for flow past a sphere, which are compared with our results in Table 8.I, for both their linearized and non-linearized theory. In the linearized theory they took fewer terms in the Rayleigh-Ritz series. The biggest discrepancy is at 20° where there is a difference of 0.4% between the non-linearized results and our results and 0.5% with the linearized theory.

It can be seen from Table 8.1. (b) that our method gives results closer to those obtained by the Rayleigh-Janzen method than those obtained by Wang and de los Santos (1951) at most points around the sphere. The reason for this is probably the use of $\gamma = 2$ instead of 1.405 by Wang and de los Santos (1951).

They also made some calculations for an 80% ellipsoid, but despite taking $\gamma = 2$, the calculation of the Rayleigh-Ritz parameters became formidable so they only took one parameter. Thus the accuracy of their results for the ellipsoid is doubtful and for higher Mach numbers the agreement with the Rayleigh-Janzen method was only in the first decimal place.

TABLE 8.1

Flow past a unit sphere at a free stream Mach number of 0.5. Comparison with results of Wang and de los Santos (1951)

Velocities compared to the free stream

Position on sphere	Heys' results	Wang and de los Santos' results Linearized	non-linearized
0°	0.0	0.0	0.0
10°	0.2481	0.2473	0.2473
20°	0.4862	0.4887	0.4881
30°	0.7192	0.7206	0.7187
40°	0.9411	0.9408	0.9376
50°	1.1464	1.146	1.143
60°	1.3282	1.331	1.328
70°	1.4765	1.481	1.479
80°	1.5770	1.580	1.580
90°	1.6132	1.615	1.616.

(a) Velocities compared to the free stream.

Position on sphere	Heys' results	Wang and de los Santos (1951)	Rayleigh-Janzen Lamla (1939)
	Pressure coefficients on the sphere		
0°	1.0640	1.0641	1.0641
10°	0.9947	0.9940	0.9968
20°	0.8007	0.7980	0.8018
30°	0.4974	0.4982	0.4997
40°	0.1151	0.1218	0.1144
50°	- 0.3081	- 0.3005	- 0.2891
60°	- 0.7283	- 0.7270	- 0.7339
70°	- 1.0955	- 1.1002	- 1.0971
80°	- 1.3537	- 1.3567	- 1.3435
90°	- 1.4481	- 1.4482	- 1.4330

(b) pressure coefficients on the sphere.

TABLE 8.2

Axisymmetric flow past a unit sphere for various free stream Mach numbers.

Position on sphere	Free stream Mach numbers 0.4	0.5	0.6
	Local Mach numbers		
180°	0.0.	0.0	0.0
$168\frac{3}{4}^\circ$	0.1109	0.1362	0.1574
$157\frac{1}{2}^\circ$	0.2198	0.2678	0.3097
$146\frac{1}{4}^\circ$	0.3245	0.3982	0.4640
135°	0.4220	0.5240	0.6182
$123\frac{3}{4}^\circ$	0.5082	0.6409	0.7713
$112\frac{1}{2}^\circ$	0.5774	0.7416	0.9157
$101\frac{1}{4}^\circ$	0.6230	0.8136	1.0534
90°	0.6305	0.8405	1.1245
$78\frac{3}{4}^\circ$			1.0533
$67\frac{1}{2}^\circ$			0.9159
$56\frac{1}{4}^\circ$			0.7714
45°			0.6183
$33\frac{3}{4}^\circ$			0.4641
$22\frac{1}{2}^\circ$			0.3098
$11\frac{1}{4}^\circ$			0.1575
0			0.0

There does not seem to have been a great deal of work done on axisymmetric flows. However, for the 10% ellipsoid, some calculations were made by Laitone (1947) using a first order Mach number correction in the linearized form of the potential equation of motion for a slender body of revolution. This correction considers terms of the order of $\log \beta_M$, added to the incompressible flow, where $\beta_M = 1 - M^2$.

Apart from the approximate form of the equation used, another drawback of the Laitone method is that it is not applicable near the stagnation points or for large values of $R' = \frac{dR(x)}{dx}$ where R and x are the cylindrical coordinates for the body of revolution. However, his results, read off a graph, are compared with our results for the 10% ellipsoid, below.

It can be seen from Table 8.3 that there is close agreement near the centre of the ellipsoid, but at points away from the centre, there is a large discrepancy, even at the points away from the stagnation points, which suggests that his approximation was not of a high enough order.

The results are compatible over a larger part of the ellipsoid for the lower free stream Mach number 0.6, as one would expect, since Laitone's approximation is exact when the free stream Mach number is zero, i.e. incompressible flow.

Results were also obtained by Pidcock (1969) for flow past ellipsoids of various thickness ratios, using Sells' method applied to axisymmetric flows. For an ellipsoid of 10% thickness ratio, he gave the maximum local Mach numbers on the surface which are compared with our results in Table 8.4. It can be seen that there is close agreement between our results at a free stream Mach number of 0.7 and 0.8 but the difference is larger, of the order of 0.6%, at a free stream Mach number of 0.9.

TABLE 8.3

Axisymmetric flow past a 10% ellipsoid compared with the results obtained by Laitone (1947) for subsonic flow.

Position	Free stream Mach number 0.6		0.9	
	Laitone's results	Heys' results	Laitone's results	Heys' results
	Pressure coefficients on the body surface			
180°		1.0932		1.2191
$168\frac{3}{4}^{\circ}$		0.1802		0.2273
$157\frac{1}{2}^{\circ}$	- 0.010	0.0200	- 0.023	0.0351
$146\frac{1}{4}^{\circ}$	- 0.024	- 0.0193	- 0.037	- 0.0232
135°	- 0.034	- 0.0330	- 0.047	- 0.0422
$123\frac{3}{4}^{\circ}$	- 0.040	- 0.0390	- 0.052	- 0.0503
$112\frac{1}{2}^{\circ}$	- 0.042	- 0.0420	- 0.055	- 0.0542
$101\frac{1}{4}^{\circ}$	- 0.044	- 0.0434	- 0.057	- 0.0560
90°	- 0.044	- 0.0438	- 0.057	- 0.0565
$78\frac{3}{4}^{\circ}$	- 0.044	- 0.0434	- 0.057	- 0.0560
$67\frac{1}{2}^{\circ}$	- 0.042	- 0.0420	- 0.055	- 0.0542
$56\frac{1}{4}^{\circ}$	- 0.040	- 0.0390	- 0.052	- 0.0504
45°	- 0.034	- 0.0330	- 0.046	- 0.0422
$33\frac{3}{4}^{\circ}$	- 0.024	- 0.0193	- 0.037	- 0.0232
$22\frac{1}{2}^{\circ}$	- 0.010	0.0199	- 0.024	0.0350
$11\frac{1}{4}^{\circ}$		0.1799		0.2271
0°		1.0932		1.2191

TABLE 8.4

A comparison of the maximum local Mach numbers found by Pidcock and Heys on the surface of a 10% ellipsoid for various free stream Mach numbers.

Free stream	Maximum local Mach number	
Mach number	Heys	Pidcock
0.7	0.7177	0.7168
0.8	0.8224	0.8223
0.9	0.9297	0.9355

The reason for this discrepancy is probably that, as in Sells' method for two-dimensional flow, Pidcock's method is slow to converge near the critical Mach number and under-relaxation must be used, when the free stream Mach number is above 0.8.

Pidcock (1969) gave detailed results for flow about 20%, 40%, and 60% ellipsoids at a free stream Mach number of 0.6 and these are compared with our results in Table 8.5.

Once again, it is found that the greatest discrepancy is near the stagnation point, as was the case for plane flow. As an example, for the 20% ellipsoid, there is a discrepancy of 3.7% at 6°.

However, for axisymmetric flow our local Mach number is higher than Pidcock's at this point, while in two-dimensional flow our results were 5% lower than Sells'.

In Table 8.6 the maximum velocities, obtained our our method, Pidcock's method, the linearized theory and the Prandtl-Glauert theory are compared. Results obtained by the two-different forms of the integral are compared in Table 8.7 for flows past a 10% ellipsoid. It can be seen that the two versions give virtually identical results near 90° while there is a greater difference near the stagnation point, 0.5% for a free stream Mach number of 0.9.

Other results found by using the incompressible form of the integral are given in Table 8.8 and flows with a supersonic region calculated from the surface integral form of the variational integral are given in Table 8.9.

At a free stream Mach number of 0.99 the iterative process failed to converge. At Mach 0.98 the flow is supersonic over most of the ellipsoid and at a distance of up to 0.15 times the chord, from the ellipsoid which has a maximum thickness of only 0.1 times the chord.

Calculations were also carried out for bodies of revolution with a Karman-Trefftz profile as cross.Section. Since there was some doubts whether the variational method would cope with the singularity at the trailing edge, it was decided to calculate the flow using both the standard variational method, as applied to other bodies and the series solution, for points near the trailing edge, as derived in Section 6.

The results are tabulated in Table 8.10 from which it can be seen that the difference in the local Mach number obtained by the two methods, for the point next to the trailing edge is only about 0.6%.

The corresponding calculations were also made for the Karman-Trefftz 'F' profile and are given in Table 8.11. In this case the maximum difference in the results, obtained by the two methods occurs at the second point away from the trailing edge and ranges from 0.40% for a free stream Mach number of 0.8 to 0.7% for a free stream Mach number of 0.96. The reason for this difference for the two profiles is not clear, but it is sufficiently small not to cause much concern.

TABLE 8.5

Surface Velocity against Angular Position $M_\infty = 0.6$

Comparison with Pidcock (1969)

Thickness ratio / Position on ellipsoid	20%		40%		60%	
	Pidcock	Heys	Pidcock	Heys	Pidcock	Heys
	Local Mach numbers					
0°	0.0	0.0	0.0	0.0	0.0	0.0
6°	0.2669	0.2767	0.1563	0.1620	0.1152	0.1193
12°	0.4400	0.4401	0.2974	0.2987	0.2274	0.2288
18°	0.5284	0.5277	0.4108	0.4104	0.3308	0.3313
24°	0.5730	0.5741	0.4964	0.4955	0.4232	0.4231
30°	0.5976	0.6000	0.5591	0.5588	0.5035	0.5033
36°	0.6125	0.6154	0.6047	0.6053	0.5721	0.5722
42°	0.6221	0.6251	0.6380	0.6396	0.6299	0.6307
48°	0.6286	0.6315	0.6626	0.6649	0.6781	0.6797
54°	0.6331	0.6358	0.6808	0.6836	0.7177	0.7201
60°	0.6363	0.6388	0.6944	0.6973	0.7498	0.7530
66°	0.6385	0.6409	0.7043	0.7074	0.7752	0.7790
72°	0.6401	0.6423	0.7113	0.7145	0.7944	0.7988
78°	0.6411	0.6433	0.7161	0.7193	0.8078	0.8126
84°	0.6417	0.6438	0.7188	0.7220	0.8158	0.8208
90°	0.6419	0.6440	0.7197	0.7229	0.8184	0.8235

TABLE 8.6

Comparison of maximum velocity compared to free stream with those obtained by Pidcock (1969), linearized theory and the Prandtl-Glauert theory.

Free stream Mach number	Maximum velocity compared to free stream δ			
	Heys	Pidcock	Linearized theory	Prandtl-Glauert
0.7	1.0229	1.021	1.025	1.030
0.8	1.0247	1.024	1.028	1.036
0.9	1.0281	1.034	1.034	1.050

TABLE 8.7

Comparison of the local Mach numbers on the surface of a 10% ellipsoid using the surface integral and incompressible forms of the variational integral.

Position on the ellipsoid	Free stream Mach numbers 0.7		0.9	
	Local Mach numbers			
	Surface integral	Incompressible integral	Surface integral	Incompressible integral
180°	0.0	0.0	0.0	0.0
$168\frac{3}{4}^{\circ}$	0.6263	0.6236	0.7823	0.7785
$157\frac{1}{2}^{\circ}$	0.6915	0.6910	0.8817	0.8807
$146\frac{1}{4}^{\circ}$	0.7078	0.7076	0.9122	0.9117
135°	0.7134	0.7132	0.9222	0.9219
$123\frac{3}{4}^{\circ}$	0.7158	0.7157	0.9265	0.9263
$112\frac{1}{2}^{\circ}$	0.7170	0.7169	0.9285	0.9283
$101\frac{1}{4}^{\circ}$	0.7175	0.7175	0.9294	0.9293
90°	0.7177	0.7177	0.9297	0.9297

TABLE 8.8

Other results using the incompressible form of the integral

Position on the ellipsoid	Free stream Mach numbers 0.8	0.95
	Local Mach numbers	
180°	0.0	0.0
$168\frac{3}{4}^\circ$	0.7049	0.8087
$157\frac{1}{2}^\circ$	0.7876	0.9231
$146\frac{1}{4}^\circ$	0.8096	0.9619
135°	0.8168	0.9756
$123\frac{3}{4}^\circ$	0.8199	0.9816
$112\frac{1}{2}^\circ$	0.8214	0.9843
$101\frac{1}{4}^\circ$	0.8222	0.9856
90°	0.8224	0.9860

TABLE 8.9.

Local Mach numbers on the surface of a 10% ellipsoid for free stream Mach numbers above the critical Mach number using the surface integral form of the variational integral.

Position on the ellipsoid	Free stream Mach numbers 0.97	0.98
	Local Mach numbers	
180°	0.0	0.0
$168\frac{3}{4}^\circ$	0.8221	0.8258
$157\frac{1}{2}^\circ$	0.9384	0.9443
$146\frac{1}{4}^\circ$	0.9811	0.9890
135°	0.9976	1.0078
$123\frac{3}{4}^\circ$	1.0054	1.0176
$112\frac{1}{2}^\circ$	1.0092	1.0228
$101\frac{1}{4}^\circ$	1.0109	1.0248
90°	1.0113	1.0254
$78\frac{3}{4}^\circ$	1.0109	1.0249
$67\frac{1}{2}^\circ$	1.0091	1.0224
$56\frac{1}{4}^\circ$	1.0054	1.0178
45°	0.9976	1.0076
$33\frac{3}{4}^\circ$	0.9811	0.9891
$22\frac{1}{2}^\circ$	0.9385	0.9444
$11\frac{1}{4}^\circ$	0.8222	0.8259
0°	0.0	0.0

TABLE 8.10

Comparison of the variational method and cone series solution at the trailing edge of a Karman-Trefftz profile of revolution.

Position on profile	Free stream Mach number 0.8 Variational	0.8 Cone	0.9 Variational	0.9 Cone	0.94 Variational	0.94 Cone
180°	0.0	0.0	0.0	0.0	0.0	0.0
$168\frac{3}{4}^\circ$	0.6576	0.6576	0.7197	0.7196	0.7385	0.7384
$157\frac{1}{2}^\circ$	0.7901	0.7901	0.8799	0.8800	0.9102	0.9103
$146\frac{1}{4}^\circ$	0.8361	0.8361	0.9454	0.9454	0.9869	0.9868
135°	0.8501	0.8501	0.9682	0.9682	1.0205	1.0206
$123\frac{3}{4}^\circ$	0.8508	0.8508	0.9702	0.9702	1.0259	1.0259
$112\frac{1}{2}^\circ$	0.8443	0.8443	0.9606	0.9606	1.0127	1.0127
$101\frac{1}{4}^\circ$	0.8337	0.8336	0.9452	0.9452	0.9929	0.9929
90°	0.8209	0.8209	0.9274	0.9275	0.9710	0.9710
$78\frac{3}{4}^\circ$	0.8076	0.8075	0.9095	0.9095	0.9501	0.9501
$67\frac{1}{2}^\circ$	0.7947	0.7948	0.8928	0.8929	0.9311	0.9311
$56\frac{1}{4}^\circ$	0.7833	0.7832	0.8784	0.8782	0.9151	0.9149
45°	0.7739	0.7740	0.8668	0.8670	0.9024	0.9026
$33\frac{3}{4}^\circ$	0.7664	0.7653	0.8580	0.8565	0.8930	0.8915
$22\frac{1}{2}^\circ$	0.7601	0.7575	0.8507	0.4870	0.8853	0.8808
$11\frac{1}{4}^\circ$	0.7490	0.7444	0.8388	0.8333	0.8734	0.8674
0°	0.0	0.0	0.0	0.0	0.0	0.0

Flow about a Karman-Trefftz 'F' profile of revolution at various free stream Mach numbers

Position on body	Free stream Mach numbers: 0.8.		0.9		0.95		0.96	
	Local Mach numbers							
	Variational	Cone	Variational	Cone	Variational	Cone	Variational	Cone
180°	0.0.	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$168\frac{3}{4}^\circ$	0.6837	0.6838	0.7531	0.7531	0.7807	0.7806	0.7852	0.7851
$157\frac{1}{2}^\circ$	0.7794	0.7794	0.8693	0.8693	0.9083	0.9084	0.9145	0.9145
$146\frac{1}{4}^\circ$	0.8153	0.8153	0.9186	0.9186	0.9678	0.9678	0.9770	0.9769
135°	0.8310	0.8310	0.9412	0.9413	0.9987	0.9988	1.0090	1.0090
$123\frac{3}{4}^\circ$	0.8370	0.8370	0.9503	0.9503	1.0132	1.0133	1.0284	1.0285
$112\frac{1}{2}^\circ$	0.8368	0.8368	0.9501	0.9501	1.0136	1.0138	1.0294	1.0294
$101\frac{1}{4}^\circ$	0.8319	0.8319	0.9432	0.9431	1.0038	1.0039	1.0171	1.0170
90°	0.8238	0.8238	0.9317	0.9317	0.9884	0.9884	1.0006	1.0007
$78\frac{3}{4}^\circ$	0.8134	0.8134	0.9176	0.9175	0.9704	0.9703	0.9807	0.9806
$67\frac{1}{2}^\circ$	0.8019	0.8019	0.9022	0.9023	0.9518	0.9518	0.9616	0.9617
$56\frac{1}{4}^\circ$	0.7900	0.7898	0.8869	0.8867	0.9339	0.9335	0.9427	0.9424
45°	0.7785	0.7788	0.8725	0.8728	0.9174	0.9176	0.9259	0.9262
$33\frac{3}{4}^\circ$	0.7676	0.7660	0.8591	0.8571	0.9025	0.9001	0.9106	0.9083
$22\frac{1}{2}^\circ$	0.7563	0.7532	0.8456	0.8412	0.8877	0.8818	0.8956	0.8891
$11\frac{1}{4}^\circ$	0.7369	0.7347	0.8240	0.8213	0.8652	0.8621	0.8730	0.8698
0°	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

9. DISCUSSION

There are two main areas for discussion which arise from the preceeding work. These are the differences between two dimensional and axisymmetric flows and the possible extensions of the method.

9.1 Comparison of Two-Dimensional and Axisymmetric Flows

A comparison of the results given in Section 5 for plane flows with those given in Section 8 for axisymmetric flows, shows immediately that the local Mach numbers are far closer to the free stream Mach number for axisymmetric flows than plane flows for the equivalent body, this is true for all the shapes considered. This difference means that the critical Mach numbers and the highest free stream Mach number for which convergence could be obtained are both higher for flow about the axisymmetric body than for its cross section. A comparison of this highest Mach number for which convergence could be obtained is given in Table 9.1 for various shapes, and the local Mach numbers around the surface of a 10% ellipse are compared with those for the corresponding ellipsoid in Table 9.2 at a free stream Mach number of 0.8. Table 9.3 compares the local Mach numbers on the surface of a Karman-Trefftz 'F' profile with those on the surface of the corresponding body of revolution at a free stream Mach number of 0.78.

TABLE 9.1

Highest free stream Mach number M for which the iterative scheme converged for plane and axisymmetric flows.

Two-Dimensional		Axisymmetric	
Body	Mach Number	Mach Number	Body
Circle	0.435	0.60	Sphere
10% ellipse	0.82	0.98	10% ellipsoid
Karman-Trefftz T.E. angle 6^o, thickness ratio 10%	0.76	0.94	Karman-Trefftz of reon. T.E. angle 6^o, thickness ratio 10%
Karman-Trefftz 'F'	0.78	0.96	Karman-Trefftz 'F' of revolution.

TABLE 9.2.

The flow past a 10% ellipse and the corresponding body of revolution at a free stream Mach number of 0.8.

Position on body	Local Mach Number	
	Plane Flow	Axisymmetric Flow
180^o	0.9783	0.8224
$168\frac{3}{4}^o$	0.9762	0.8222
$157\frac{1}{2}^o$	0.9702	0.8214
$146\frac{1}{4}^o$	0.9590	0.8199
135^o	0.9419	0.8168
$123\frac{3}{4}^o$	0.9132	0.8096
$112\frac{1}{2}^o$	0.8606	0.7876
$101\frac{1}{4}^o$	0.7250	0.7049
90^o	0.0	0.0

TABLE 9.3

The flow past a Karman-Trefftz 'F' profile and the corresponding body of revolution at a free stream Mach number of 0.78

Position on body	Local Mach Numbers: Plane Flow	Local Mach Numbers: Axisymmetric
180°	0.0	0.0
$168\frac{3}{4}^\circ$	0.6855	0.6687
$157\frac{1}{2}^\circ$	0.8361	0.7607
$146\frac{1}{4}^\circ$	0.9154	0.7946
135°	0.9692	0.8094
$123\frac{3}{4}^\circ$	1.0052	0.8151
$112\frac{1}{2}^\circ$	1.0087	0.8149
$101\frac{1}{4}^\circ$	0.9804	0.8103
90°	0.9415	0.8026
$78\frac{3}{4}^\circ$	0.8991	0.7927
$67\frac{1}{2}^\circ$	0.8568	0.7818
$56\frac{1}{4}^\circ$	0.8151	0.7703
45°	0.7747	0.7597
$33\frac{3}{4}^\circ$	0.7349	0.7474
$22\frac{1}{2}^\circ$	0.6943	0.7352
$11\frac{1}{4}^\circ$	0.6461	0.7170
0°	0.0	0.0

9.2 General Discussion

It has been shown in this thesis that the application of finite difference techniques to the modified form of the Bateman-Dirichlet principle can produce a satisfactory numerical solution for the plane and axisymmetric flow past non-lifting bodies. For a circle results in close agreement with those of Sells (1968) are obtained, which are much more accurate than those found by Greenspan and Jain (1967). For a 10% ellipse, the results agree well with Sells except near the stagnation points where for a free stream Mach number of 0.8 the discrepancy between the local Mach numbers is about 5%. No satisfactory explanation has been found, but one reason may be the different boundary condition on the surface since Sells worked with the stream function which vanishes at the surface, while we used the condition that the normal derivative of the velocity potential is zero at the surface. For a Karman-Trefftz 'F' profile the discrepancy between our results and Sells' results drops to 1.7% for a free stream Mach number of 0.6. When results were compared with those obtained by Bauer, Garabedian and Korn's (1972) program for a Karman-Trefftz 'F' profile close agreement was obtained except at the trailing edge, where their programme fails to give the required stagnation point. The maximum difference elsewhere is one of $1\frac{1}{2}$%, at the point of maximum thickness, for a free stream Mach Number of 0.76.

The discrepancy with Sells' results for a NACA 0012 profile, was about 3.4% near the stagnation point. This difference was probably caused by insufficient computer storage being available to take enough grid points in the numerical evaluation of the transform modulus away from the profile.

Results have also been compared with those obtained by other methods for axisymmetric flows. For the sphere fairly good agreement was found with Wang and de los Santos (1951) despite their use of $\gamma = 2$. Apart from a discrepancy near the stagnation point, which was smaller than that found for plane flow, close agreement with Pidcock's adaptation of Sells' (1968) method to axisymmetric flows was found for ellipsoids of various thickness ratios. Results have also been found for flow past the bodies of revolution with Karman-Trefftz profiles, as cross-section, though no results have been available for comparison purposes.

For nearly all the shapes considered the free stream Mach number has been pushed above the critical Mach number and a small supersonic region has been obtained. The iterative scheme fails to converge when the free stream Mach number is pushed even higher.

No attempt has yet been made to show that the numerical method converges. However, it may be possible to use a modified form of the technique used by Rasmussen (1973), for proving the convergence of the Rayleigh-Ritz method in plane subsonic flow, to prove the convergence of the reduced problem on the finite region $1 \leq r \leq R_f$.

No work seems to have been done on comparing the solution of the governing differential equations directly with the variational method, with respect to factors such as the computing time required to obtain a certain accuracy or ease of programming. It might be claimed the variational method should be more accurate, since only first derivatives need to be approximated and an integration carried out, while in the replacement of the differential equations by algebraic equations second derivatives also have to be approximated. However, when finite differences are used it may not be possible to evaluate the integrals to sufficient accuracy.

One advantage of the variational method is that a measure of the accuracy could, in theory, be obtained by using both variational principles, as the value of the two integrals should be the same for the exact solution, though this would double the amount of work required for a calculation. In practice, however, it was found that the value of the integral did not change over the last hundred or so interations, while the values of the perturbation potential were still changing. Thus, it does not appear that the value of the integral is a very sensitive test of convergence.

It has been shown in this thesis and Rasmussen and Heys (1973) that it is possible to treat a fairly general class of aerofoils using finite differences, without the difficulties associated with the Rayleigh-Ritz method.

The two-dimensional version of the programme could be used to investigate the flow past wings or tailplanes, while the axisymmetric version could have application in the study of flows past fuselages or missiles.

The reasons for the use of the Bateman-Dirichlet principle, rather than the Bateman-Kelvin one, are that the integrand is a fairly simple function of the velocity potential and some transonic flows can be obtained. The Bateman-Kelvin principle has the advantage that the boundary condition on the body is that the stream function vanishes there, instead of the normal derivative of the potential being zero, as required by the Bateman-Dirichlet principle. One is normally most interested in the velocities and pressure distribution on the body surface, so it is necessary to use a second order accurate formula for the normal derivative here, if the Bateman-Dirichlet principle is used. The disadvantage of using the Bateman-Kelvin principle is that the relationship between the stream function and the density is not one-to-one, so only subsonic flows can be considered.

It would be interesting to extend the method to the consideration of lifting bodies in plane flow. The main problem would be the need to evaluate the flow all round the body, instead of in the upper half-plane only. Another possible extension is to use finite difference techniques to consider rotational transonic flows with shock waves using the modified form of the Bateman-Kelvin principle derived by Wang and Chou (1951), to see if the difficulties they encountered in actually carrying out any calculations could be avoided.

The method, used in this thesis, has shown that the finite difference techniques, when used to approximate the derivatives in the Bateman-Dirichlet principle can give an accurate approximation to the compressible flow past various shapes in two-dimensional and axisymmetric flows. Since the difficulties in calculating flows past arbitrary shapes, using the Rayleigh-Ritz or Galerkin methods, are avoided the method used here could prove a useful design tool in compressible flow.

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APPENDIX A. THE DERIVATION OF THE VARIATIONAL INTEGRAL FOR PLANE FLOW

The formulation of the variational principles basically follows that in Rasmussen (1972).

The equations of motion are the continuity equation

$$\nabla \cdot (\rho \underline{u}) = 0 \qquad (A - 1a)$$

and Bernoulli's equation

$$\frac{1}{2} q^2 + \frac{\gamma}{\gamma - 1} \frac{p}{\rho} = \text{constant} \qquad (A - 2)$$

(A - 1a) can also be written in the form

$$\frac{\partial (\rho u_i)}{\partial x_i} = 0 \qquad (A - 1b)$$

where the summation convention is used with i summing through the values 1, 2, 3 and the xi form an orthogonal system with ui being the corresponsing velocity components.

We assume that p and ρ are related by

$$p = K \rho^{\gamma} \qquad (A - 3)$$

where K is a constant and γ is the ratio of the specific heats of the gas being considered. It is not necessary to make this restriction, Lush and Cherry (1956) and Serrin (1959) used more general relationships, but (A - 3) is valid for the cases considered in this thesis.

The local speed of sound was defined in Section 2 by $C^2 = \frac{dp}{d\rho}$ and we say the flow is subsonic if $q^2 < C^2$ everywhere in the flow field V. The boundary surface B of V is taken to be sufficiently regular to permit the application of Green's theorem. We suppose the normal moss flux h is prescribed on B such that

$$\text{Outflow} = \int_B h \, dA = 0.$$

We restrict the class of admissible functions to those for which (2,3) applies.

There are two variational principles. One is the Bateman-Kelvin principle which is as follows. Consider the variational problem of minimizing the integral

$$I\,[\underline{u}] = \int_V (p + \rho q^2)\, dv \qquad (A - 4)$$

among all subsonic velocity fields which satisfy the equation of continuity and have prescribed mass - flux on B. Then $I\,[\underline{u}]$ is a minimum if and only if the flow is irrotational.

The proof follows the treatment in Serrin (1959) since the proof in Lush and Cherry (1956) is only valid for two dimensions. Set $\underline{T} = \rho \underline{u}$ then from (A - 1a) $\nabla . \underline{T} = 0$. Hence by Bernoulli's equation (A - 2) it follows that $\underline{u}$ and therefore I may be considered as functions of $\underline{T}$. Set $\underline{T}^* = \underline{T} + \epsilon \underline{\eta}$ be any function of the field where $\underline{T}$ and $\underline{\eta}$ are regarded as fixed and ϵ as a small parameter. Writing $F(\underline{T}) = p + \rho q^2$, F can be expanded in a power series for small ϵ as

$$F\,(\underline{T} + \epsilon \underline{\eta}) = F\,(\underline{T}^*) = F\,(\underline{T}) + \epsilon \frac{dF}{dG}\bigg|_{\epsilon = 0} + \frac{1}{2}\epsilon^2 \frac{d^2F}{d\,\epsilon^2}\bigg|_{\epsilon = 0} + 0\,(\epsilon^3)$$

F can also be considered as a function of the three components of $\underline{T}$, i.e. of T_1, T_2, T_3 or of $\Lambda = T_i T_i$. Hence we can write

$$\frac{dF}{d\epsilon}\bigg|_{\epsilon = 0} = \eta_i \frac{\partial F}{\partial T_i}$$

and since

$$\frac{\partial F}{\partial T_i} = 2\, T_i \frac{dF}{d\Lambda}$$

it can be seen that

$$\frac{dF}{d\epsilon}\bigg|_{\epsilon = 0} = 2\, \eta_i\, T_i \frac{dF}{d\Lambda}$$

Also

$$\left.\frac{d^2F}{d\epsilon^2}\right|_{\epsilon=0} = 2\eta_i T_i \frac{d}{d\Lambda}\left[2\eta_i T_i \frac{dF}{d\Lambda}\right]$$
$$= 4(\eta_i T_i)^2 \frac{d^2F}{d\Lambda^2} + 2\eta_i\eta_i \frac{dF}{d\Lambda}$$

From (A - 2) and (A - 3) it can be shown that

$$\frac{dF}{d\Lambda} = \frac{1}{2\rho} \text{ and } \frac{d^2F}{d\Lambda^2} = \frac{1}{4\rho^3(C^2 - q^2)}$$

Hence $\left.\dfrac{dF}{d\epsilon}\right|_{\epsilon=0} = \eta.\underline{u}$

and $\left.\dfrac{d^2F}{d\epsilon^2}\right|_{\epsilon=0} = \dfrac{(\underline{u}.\eta)^2 + (C^2 - q^2)\eta^2}{(C^2 - q^2)}$

where $\eta^2 = \eta_i\eta_i$. Thus F can be expressed in the following form

$$F(\underline{T}') = F(\underline{T}) + \epsilon\,\eta.\underline{u}$$
$$+ \tfrac{1}{2}\epsilon^2 \frac{(\underline{u}.\eta)^2 + (C^2 - q^2)\eta^2}{(C^2 - q^2)} + 0(\epsilon^2)$$

When this expression is integrated over the region V we have

$$I[\underline{T}'] = I[\underline{T}] + \epsilon\delta I + \epsilon^2\delta^2 I + 0(\epsilon^3)$$

where

$$I = \int_V \underline{u}.\eta \; dv$$

and

$$\delta^2 I = \tfrac{1}{2}\int_V \frac{(\underline{u}.\eta)^2 + (C^2 - c^2)\eta^2}{(C^2 - q^2)} \, dv$$

$\delta^2 I$ is positive definite for subsonic flow so for an extremal which gives rise to subsonic flow will minimi-e I. Thus the proof of the Bateman-Kelvin principle involves showing that $\delta I = 0$ if and only if $\nabla \times \underline{u} = 0$, i.e. the flow is irrotational.

Suppose first the flow is irrotational, then we can write $\underline{u} = \nabla \phi$.

Since $\nabla . \underline{T} = 0$ for all variations, $\nabla . \eta = 0$, and

$$\int_V \underline{u} \cdot \eta \, dv = \int_V \nabla . (\phi \eta) \, dv$$

By the divergence theorem this becomes

$$\int_V \nabla . (\phi \eta) \, dv = \int_B \hat{n} . (\phi \eta) \, dA$$

Since the normal mass flux is prescribed on B, we can insist that $\hat{n} . \eta = 0$ on B. Hence the right hand integral equals zero, and so $\delta I = 0$.

Suppose on the other hand that $\delta I = 0$, but $\nabla \times \underline{u}$ is non-zero at some point P in V. Then a vector $\underline{z}$ can be found, which vanishes in V except in the neighbourhood of P, so that

$$\int_V \underline{z} \, . (\nabla \times \underline{u}) \, dv \neq 0$$

since

$$\nabla . (\underline{z} \times \underline{u}) = \underline{u} . (\nabla \times \underline{z}) - \underline{z} . (\nabla \times \underline{u})$$

and

$$\int_V . (\underline{z} \times \underline{u}) \, dv = \int_B \hat{n} . (\underline{z} \times \underline{u}) dA = 0.$$

we must have

$$\int_V \underline{u} . (\nabla \times \underline{z}) \, dv = 0$$

but since $\nabla \times \underline{z}$, is an admissible variation, i.e. $\nabla \times \underline{z} = 0$, this is a contradiction, so $\delta I = 0$ must imply that $\nabla \times \underline{u} = 0$ which completes the proof of the Bateman-Kelvin principle.

The other variational principle, the Bateman-Dirichlet principle is as follows.

Consider the variational problem of maximi-ing the integral

$$J \; |\emptyset| = \int_V p \, dv + \int_B \emptyset \; h dA \quad (A - 5)$$

among all subsonic velocities $\underline{u} = \nabla\emptyset$. Then $J \; |\emptyset|$ is a maximum if $\nabla.(\rho \underline{u}) = 0$ and $\rho \underline{u} . \hat{n} = h$ on B.

In other words $J \, |\emptyset|$ is a ma imum if the continuity equation holds and the mass flow normal to B is constant.

The proof follows that in Lush and Cherry (1956) as their's is valid for two or three dimensions.

Consider the integral

$$J_1 \; |\emptyset| = \int_V p \, dv$$

As before we expand in a power series in ϵ.

$$p(\emptyset + \epsilon \eta) = p(\emptyset) + \epsilon \left.\frac{dp}{d\epsilon}\right|_{\epsilon = 0} + \tfrac{1}{2} \epsilon^2 \left.\frac{d^2 p}{d \epsilon^2}\right|_{\epsilon = 0} + 0 \, (\epsilon^3)$$

Now (A - 2) can be written

$$\tfrac{1}{2} \emptyset xi \; \emptyset xi + \frac{\gamma}{\gamma - 1} K \rho^{\gamma - 1} = \text{constant} \quad (A - 6)$$

using (A - 3) and differentiating (A - 3) and (A - 6) with respect to $\emptyset xi$

$$\emptyset xi + \gamma K \rho^{\gamma - 2} \frac{\partial \rho}{\partial \emptyset xi} = 0$$

$$\frac{\partial \rho}{\partial \emptyset xi} = \gamma K \rho^{\gamma - 1} \frac{\partial \rho}{\partial \emptyset xi}$$

so

$$\frac{\partial p}{\partial \emptyset xi} = - \rho \, \emptyset xi$$

Also

$$\frac{\partial^2 p}{\partial \emptyset^2 xi} = - \frac{\rho (c^2 - q^2)}{c^2}$$

$$\frac{\partial^2 p}{\partial \emptyset xi \, \partial \emptyset xj} = \frac{\rho \, \emptyset xi \; \emptyset \, xj}{c^2}$$

where $\emptyset \, xi = \dfrac{\partial \emptyset}{\partial xi}$

By Taylor's theorem

$$p(\phi + \epsilon\eta) = p(\phi) + \epsilon \left.\frac{dF}{d\epsilon}\right|_{\epsilon=0} + \frac{1}{2}\epsilon^2 \left.\frac{d^2 F}{d\epsilon^2}\right|_{\epsilon=0} + 0(\epsilon^3)$$

$$= p(\phi) - \epsilon \rho U_i \eta_{xi} + \frac{1}{2}\epsilon^2 \rho \frac{Q}{C^2} + 0(\epsilon^3) \qquad (A-7)$$

where $Q = -C^2 \eta_{xi}\eta_{xi} + U_iU_j\eta_{xi}\eta_{xj}$

$$= -(C^2 - q^2)\eta_{xi}\eta_{xi}$$

$$- (U_1\eta_{x_2} - U_2\eta_{x_1})^2 + (U_1 X_3 - U_3 X_1)^2 + (U_2\eta_{x3} - U_3\eta_{x2})^2$$

For subsonic flow i.e. $q^2 < c^2$, Q is negative definite.

Integrating the expansion for p over V gives

$$J_1[\phi + \epsilon\eta] = J_1[\phi] - \epsilon\int_V \rho u_i \eta_{xi}\, dv + \frac{1}{2}\epsilon^2\int_V \frac{\rho Q}{C^2}\, dv + C(\epsilon$$

which by application of Green's theorem becomes

$$J_1[\phi+\epsilon\eta] = J_1[\phi] - \epsilon\int_B \rho\,\eta\,\hat{n}\cdot\underline{u}\, dA + \epsilon\int_V \eta(\nabla\cdot\rho\underline{u})\,dv + \frac{1}{2}\epsilon^2\int_V \frac{\rho Q}{C^2}\,dv + O(\epsilon^3) \qquad (A-8)$$

The surface integral vanishes if $\eta = 0$ or $\hat{n}\cdot\underline{u} = 0$ on B, but $\eta = 0$ corresponds to the assignment of ϕ on B which is physically unacceptable. It is therefore, necessary to modify $J_1[\phi]$ by the surface integral as in (A - 5) and the surface integral in (A - 7) will vanish if we restrict ourselves to functions ϕ of class C_3 for which $\hat{n}\cdot\underline{u}$ is given on B.

From (A - 7) it can be seen that the first variation of J is given by

$$\delta J = \int_V \eta(\nabla\cdot\rho\,\underline{u})\,dv$$

so J = 0 implies that $\nabla\cdot\underline{u} = 0$, i.e. the continuity equation holds. Since Q is negative - definite for subsonic flow, the second variation

$$\delta^2 J = \frac{1}{2}\int_V \frac{\rho Q}{C^2}\,dv$$

is negative - definite, hence, the integral $J[\phi]$ has a maximum value when the first variation vanishes, completing the proof of the Bateman-Dirichlet principle.

The extremals are unique for both principles. For the first principle δI is zero for an extremal flow $\underline{u}$, so

$$I[\underline{u}^*] = I[\underline{u}] + \epsilon^2 \delta^2 I$$

where $\underline{u}^*$ is any other flow. For subsonic flow $\delta^2 I$ is positive definite so $I[\underline{u}^*] > I[\underline{u}]$ unless $\underline{u}^* - \underline{u} = 0$, so the extremal must be unique. By a similar argument the Batemen-Dirichlet principle has a unique maximum.

In (A - 5) let ϕ be the extremal function ϕ extr. Since the function satisfies the continuity equation the boundary integral equals.

$$\int_V (\rho q^2) \text{ extr } d\tau$$

Hence

$$J[\underline{u} \text{ extr}] = I[\underline{u} \text{ extr}] = \int_V (p + \rho q^2) \, d\tau \qquad (A - 9)$$

In the subsonic case I is a minimum and J a maximum so for irrotational motion with zero normal mass flow we have two distinct formulations $\delta J_1 = 0$, $\delta J_2 = 0$. If $\underline{u}$ and $\underline{u}_2$ are two approximate solutions of this boundary value problem, then

$$J[\underline{u}_1] \leq J[\underline{u} \text{ extr}] \leq I[\underline{u}_2]$$

Thus for two solutions thus calculated

$$J[\underline{u}_1] - I[\underline{u}_2]$$

can be considered a"criterion of mean error".

Up to this point the analysis for two-dimensional and axisymmetric flows is the same, but it is now necessary to consider the two cases separately.

For a two dimensional flow without circulation past a body C in a region V bounded externally by a circle C_R, of large raduis R_1, the Bateman-Dirichlet problem is to maximize

$$J_1 [\phi] = \iint_V p \; dx\, dy + \int_{C_{R1}, C_o} \rho \phi \frac{\partial \phi}{\partial n} ds$$

and by (A - 9)

$J_1 [\phi \text{ extr}] = \iint_V (\,p + \rho q^2)\, dx\, dy.$ Thus for all functions ϕ which are close to the ϕ extr. $J_1 [\phi] \to \infty$ as $R_1 \to \infty$.

Assume the proposed extremal flow will be given by

$$\phi = \phi_\infty + \chi' = Ux + \chi' \qquad (A - 10)$$

where ϕ_∞ is the free stream potential and χ' is small when $r = \sqrt{x^2 + y^2}$ is large. Since we can subtract a part whose variation vanishes at ∞, we define

$$J_R [\phi] = \iint_V (p - p_\infty)\, dx\, dy + \int_{CR_1} \chi' \rho_\infty \frac{\partial (Ux)}{\partial n} ds + \int_C \phi \rho \frac{\partial \phi}{\partial n} ds \qquad (A - 11)$$

We can assume for large r that $\phi = Ux + \frac{f\,1\,(\theta)}{r} + \frac{f\,2\,(\theta)}{r^2} + \ldots = Ux + \chi'$

where f 1, f 2 are trignometric series in the polar angle θ, and the series converges uniformaly when $|\,\mathrm{Im}\,\theta\,|$ and r^{-1} are small. For a linear family $\phi = \phi_o + \epsilon\eta$ (or $\chi' = \chi'_o + \epsilon\eta$), will be involved linearly in the coefficients in the fn, so that differentation with respect to ϵ will not affect the order of a term when r is large.

(A - 3) can be rewritten $p \rho^{-\gamma} = \text{constant} = p_\infty \rho_\infty^{-\gamma}$ (A.- 12)

so by Bernoulli's equation

$$p = p_o \left(1 - \frac{q^2}{2\beta C_o^2}\right)^{\beta\gamma} \qquad \rho = \rho_o \left(1 - \frac{q^2}{2\beta C_o^2}\right)^{\beta} \text{ where } \beta = \frac{1}{\gamma - 1} \qquad (A. - 13)$$

the zero suffix denoting stagnation values. From (A - 10) for large r

$$q^2 = U^2 + 2U \frac{\partial \chi'}{\partial x} + (\nabla \chi')^2 = U^2 + 2U \frac{\partial \chi'}{\partial x} + O(\,r^{-4})$$

From (A - 11)

$$p_\infty = \rho_o \left[1 - \frac{U^2}{2\beta C_o^2}\right]^\alpha \qquad (\alpha = \beta\gamma)$$

so

$$p = p_\infty \left[\frac{2\beta C_o^2}{2\beta C_o^2 - U^2}\left(1 - \frac{q^2}{2\beta C_o^2}\right)\right]^\alpha .$$

and since $C_\infty^2 = C_o^2 - \frac{U^2}{2\beta}$

$$\begin{aligned} p &= p_\infty \left[1 + \frac{U^2}{2\beta C_\infty^2} - \frac{q^2}{2\beta C_\infty^2}\right] \\ &= p_\infty \left[1 - \frac{U}{\beta C_\infty^2}\frac{\partial \chi'}{\partial x} + 0\left(\frac{1}{r^4}\right)\right]^\alpha \\ &= p_\infty \left[1 - \frac{\gamma U}{C_\infty^2}\frac{\partial \chi'}{\partial x} + 0\left(\frac{1}{r^4}\right)\right]^c \\ &= p_\infty - \rho_\infty U \frac{\partial \chi'}{\partial x} + 0\left(\frac{1}{r^4}\right) \qquad (A - 14) \end{aligned}$$

From (A - 11)

$$J_{R_1}[\phi] = \iint_V (p - p_\infty)\, dx\, dy + \iint_V \frac{\partial \chi'}{\partial x} U \rho_\infty\, dx\, dy + W \qquad (A.- 15)$$

where

$$W = \int_C \left(\rho\phi \frac{\partial \phi}{\partial n} - U \rho_\infty \chi' \frac{\partial x}{\partial n}\right) ds \qquad (A - 16)$$

From (A - 14) therefore $J_{R_1}[\phi]$ converges as $R_1 \to \infty$ and

$$J_\infty[\phi] = \iint_\infty \left(p - p_\infty + U \rho_\infty \frac{\partial \chi'}{\partial x}\right) dxdy + W$$

Now let ϕ have the form $\phi_o + \epsilon\eta$, and hence χ' the form $\chi_o + \epsilon\eta$.

Then

$$\frac{\partial}{\partial \epsilon}\left(p - p_\infty + U \rho_\infty \frac{\partial \chi'}{\partial x}\right) = 0\left(\frac{1}{r^4}\right)$$

by (A - 14). Hence

$$\frac{\partial}{\partial \epsilon} J_\infty[\phi] = \lim_{R_1 \to \infty} \iint_V \left(\frac{\partial p}{\partial \epsilon} + U p_\infty \frac{\partial^2 \chi'}{\partial \epsilon \partial x}\right) dx\, dy + \frac{\partial W}{\partial \epsilon} \qquad (A - 16)$$

But by (A - 7) and Green's theorem

$$\iint_V \left(\frac{\partial p}{\partial \epsilon}\right)_{\epsilon=0} dx\,dy = -\int_{C, C_{R_1}} \rho\,\eta \frac{\partial \phi}{\partial n}\,dS + \iint_V \eta\,\mathrm{div}(\rho\nabla\phi)\,dx\,dy$$

and

$$\iint_V U\,\rho_\infty \left(\frac{\partial^2 \chi'}{\partial\epsilon\,\partial x}\right)_{\epsilon=0} dxdy = \iint_V U\,\rho_\infty \frac{\partial \eta}{\partial x}\,dxdy$$
$$= \int_{C, C_{R1}} U\,\rho_\infty\,\eta \frac{\partial x}{\partial n}\,ds$$

again by application of Green's theorem. On $C_{R1}, \rho = \rho_\infty \left\{ 1 + 0\,(r^{-2}) \right\}$

and

$$\frac{\partial \phi}{\partial n} = \frac{\partial (U\,x)}{\partial n} + \frac{\partial \chi'}{\partial n} = \frac{\partial (U\,x)}{\partial n} + 0(r^{-2})$$

so $\eta \left\{ \frac{\partial \phi}{\partial n} - \rho_\infty \frac{\partial (U\,x)}{\partial n} \right\} = 0\,(r^{-3})$

Substituting into (A - 16), the integrals over CR_1 vanish in the limit $R_1 \to \infty$ and for $\epsilon = 0$.

$$\frac{\partial}{\partial\epsilon} J_\infty[\phi] = \int_C \eta\left(U\,\rho_\infty \frac{\partial x}{\partial n} - \rho\frac{\partial \phi}{\partial n}\right) ds + \iint_\infty \eta\,\mathrm{div}\,(\rho\nabla\phi)dxdy + \frac{\partial W}{\partial\epsilon}$$

The order of $\eta\,\mathrm{div}\,(\rho\nabla\phi)$ is r^{-4}, so the integral converges. Substituting for $\frac{\partial K}{\partial\epsilon}$ from (A -16) with $\frac{\partial\phi}{\partial\epsilon} = \frac{\partial\chi'}{\partial\epsilon} = \eta$ and since $\rho\,\partial\phi/\partial n$ is prescribed on Co, so its ϵ derivative is zero then we have

$$\frac{\partial}{\partial\epsilon} J_\infty[\phi]\Big|_{\epsilon=0} = \iint_\infty \eta\,\mathrm{div}\,(\rho\nabla\phi)\,dx\,dy$$

Thus an extremal ϕ gives $\mathrm{div}\,(\rho\nabla\phi) = 0$ at all points outside C and for the infinite region we replace the hydrodynamic flow problem by one of maximizing $J_\infty[\phi]$

Since $\partial\phi/\partial n = 0$ on C, $J_\infty[\phi]$ can be written in the form

$$J_\infty[\phi] = \iint_\infty \left(p - p_\infty + U\,\rho_\infty \frac{\partial\chi'}{\partial x}\right) dx\,dy$$
$$- U\,\rho_\infty \int_C \chi' \frac{\partial x}{\partial n}\,ds$$

$$= \iint_\infty \left(p - p_\infty + U\,\rho_\infty \frac{\partial\chi'}{\partial x}\right) dx\,dy$$
$$+ U\,\rho_\infty \int_C \chi' \frac{\partial F}{\partial n}\,ds \qquad (A - 18)$$

since the incompressible potential

$$\phi_o = U(x + F) \text{ so}$$

$$U \frac{\partial x}{\partial n} = \frac{\partial \phi_o}{\partial n} - U \frac{\partial F}{\partial n}$$

and $\partial \phi_o / \partial n$ is zero on C

Thus

$$J_\infty [\phi] = \iint_\infty (p - p_\infty + U \rho_\infty \nabla x \,.\, \nabla \chi') \, dx\, dy$$

$$+ \; U \rho_\infty \int_C \chi' \; \nabla F . \hat{n} \; ds$$

where $\hat{n}$ is the unit normal to C.

By Green's theorem

$$\iint_V \nabla.(\chi' \nabla F) \, dx\, dy = \int_{C R_1 + C} \chi' \nabla F . \hat{n} \; ds$$

Also $\quad F = 0 \left(\frac{1}{r}\right), \quad \chi' = 0 \left(\frac{1}{r}\right)$

so $\quad \chi' \nabla F = 0 \left(\frac{1}{r^3}\right)$

Hence in the limit as $R_1 \rightarrow \infty$ we have

$$\iint_\infty \nabla.(\chi' \nabla F) \, dx\, dy = \int_C \chi' \nabla F . \hat{n} \; ds$$

Sine the incompressible potential satisfies Laplace's equation, so must F, so

$$\nabla.(\chi' \nabla F) = \nabla F \,.\, \nabla \chi'$$

and the integral becomes

$$J_\infty [\phi] = \iint_\infty (p - p_\infty + U \rho_\infty \; (\nabla x . \nabla \chi' + \nabla F . \nabla \chi')) dx dy$$

or

$$J_\infty [\phi] = \iint_\infty (p - p_\infty + \rho_\infty \; \nabla \phi_o . \nabla(\phi - \phi_\infty)) \, dx\, dy \quad (A - 19)$$

APPENDIX B THE TRANSFORM MODULUS FOR AN ELLIPSE

The conformal mapping

$$z = \sigma + \frac{\lambda^2}{\sigma}$$

converts a circle radius C_1, centre at the origin, in the σ - plane into the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

in the z - plane where

$$C_1 = \tfrac{1}{2} (a + b) \quad \text{and} \quad \lambda = \tfrac{1}{2} (a^2 - b^2)^{\frac{1}{2}}$$

Suppose the thickness ratio

$$b/a = \mu$$

Then to map the ellipse in the z - plane to the unit circle in the σ - plane, we require

$$1 = C_1 = \tfrac{1}{2} (\mu + 1) a$$

so

$$a = \frac{2}{\mu + 1}$$

and

$$b = \frac{2\mu}{\mu + 1}$$

giving

$$\lambda^2 = \frac{1 - \mu}{1 + \mu}$$

If $\sigma = \xi_1 + i\xi_2 = r \cos \theta + ir \sin \theta$

and

$$z = x + iy$$

then

$$x = \xi_1 + \frac{\lambda^2 \xi_1}{\xi_1^2 + \xi_2^2} = \left(r + \frac{\lambda^2}{r}\right) \cos \theta$$

and

$$y = \xi_2 - \frac{\lambda^2 \xi_2}{\xi_1^2 + \xi_2^2} = \left(r - \frac{\lambda^2}{r}\right) \sin \theta$$

Now the transform modulus, squared

$$T^2 = x_r^2 + \frac{1}{r^2} x_\theta^2$$

$$= \left(1 - \frac{\lambda^2}{r^2}\right)^2 \cos^2 \theta + \left(1 + \frac{\lambda^2}{r^2}\right)^2 \sin^2 \theta$$

$$= 1 - \frac{2\lambda^2}{r^2} \cos 2\theta + \frac{\lambda^4}{r^4}.$$

APPENDIX C THE TRANSFORM MODULUS FOR A KARMAN-TREFFTZ PROFILE

A symmetric Karman-Trefftz profile is transformed into a circle radius C centre ($-$ bk, 0) by the transformation

$$\frac{z - mk}{z + mk} = \left(\frac{\zeta - k}{\zeta + k}\right)^m \qquad (C - 1)$$

$$\text{where} \quad C = k + bk$$

We require the transformation of the symmetric Karman-Trefftz profile to the unit circle centre the origin.

Thus

$$C = 1$$

and we need another transformation

$$\sigma = \zeta + bk$$

to move the centre of the circle to the origin.

Also

$$m = 2 - \frac{\tau}{\pi}$$

where τ is the trailing edge angle of the profile.

Hence the full transformation is

$$\frac{z - mk}{z + mk} = \left(\frac{\sigma - bk - k}{\sigma - bk + k}\right)^m = \left(\frac{\sigma - 1}{\sigma - bk + k}\right)^m$$

Thus, if we rearrange this expression we have

$$z + mk = \frac{2\,mk\,(\sigma - b + k)^m}{\left((\sigma - bk + k)^m - (\sigma - 1)^m\right)} \qquad (C - 2)$$

Hence

$$\frac{dz}{d\sigma} = \frac{4m^2k^2\left[(\sigma - 1)(\sigma - bk + k)\right]^{(m-1)}}{\left\{(\sigma - bk + k)^m - (\sigma - 1)^m\right\}^2}$$

$$= \frac{4m^2k^2\,(\rho_1\rho_2)^{(m-1)}\,e^{i(m-1)\phi_2}\,e^{i(m-1)\phi_1}}{\left\{\rho_1^m e^{im\phi_1} - \rho_2^m e^{im\phi_2}\right\}^2}$$

where

$$\rho_1^2 = r^2 - 2\,(b_k - k)\, r\cos\theta + (b_k - k)^2 \qquad (C-3)$$

$$\rho_2^2 = r^2 - 2r\cos\theta + 1$$

$$\tan\phi_1 = \frac{r\sin\theta}{r\cos\theta - b_k + k} \qquad (C-4)$$

$$\tan\phi_2 = \frac{r\sin\theta}{r\cos\theta - 1} \qquad (C-5)$$

Now

$$\frac{dz}{d\sigma} = \frac{4m^2k^2\,(\rho_1\rho_2)^{(m-1)}\, e^{i(m-1)(\phi_1+\phi_2)}\left(\rho_1^{\,m} e^{-im\phi_1} - \rho_2^{\,m} e^{-im\phi_2}\right)^2}{\left\{\rho_1^{2m} + \rho_2^{2m} - 2\rho_1^m\rho_2^m\cos m(\phi_1-\phi_2)\right\}^2}$$

Hence

$$T^2 = \left|\frac{dz}{d\sigma}\right|^2 = 16\,m^4k^4\,(\rho_2\rho_1)^{2(m-1)}\Big[\rho_1^{4m} + \rho_2^{4m} + 2\rho_1^{2m}\rho_2^{2m}\big(2 + \cos 2m(\phi_1-\phi_2)\big) - 4\rho_1^m\rho_2^m\cos m(\phi_1-\phi_2)\,(\rho_1^{2m}+\rho_2^{2m})\Big] \Big/ \left\{\rho_1^{2m} + \rho_2^{2m} - 2\rho_1^m\rho_2^m\cos m(\phi_1-\phi_2)\right\}^4$$

$$= \frac{16\,m^4k^4\,(\rho_1\rho_2)^{2(m-1)}}{\left\{\rho_1^{2m} + \rho_2^{2m} - 2\rho_1^m\rho_2^m\cos m(\phi_1-\phi_2)\right\}^2} \qquad (C-6)$$

From (C - 4) and (C - 5), we can obtain

$$\tan(\phi_1 - \phi_2) = \frac{-2kr\sin\theta}{r^2 - 2b_k r\cos\theta + b_k - k}$$

so finally by (C - 3) we find

$$T^2 = \Big[(r^2 - 2r\cos\theta + 1)\,(r^2 - 2b_k r\cos\theta(b_k-k) + (b_k-k)^2\Big]^{(m-1)} \Big\{4m^2k^2 \Big/ \Big[(r^2 - 2b_k r\cos\theta(b_k-k) + (b_k-k)^2\Big)^{m} + (r^2 - 2r\cos\theta + 1)^m - 2\Big[\Big\{(r^2 - 2r\cos\theta(b_k-k) + (b_k-k)^2\Big)(r^2 - 2r\cos\theta + 1)\Big]^{\frac{m}{2}} \cos m\Big\{\tan^{-1}\left(\frac{-2kr\sin\theta}{r^2 - 2b_k r\cos\theta + b_k - k}\right)\Big\}\Big]\Big\}^2$$

Since the transformation contains non-integer powers we seek a simpler form for large values of $|\sigma|$. For large $|\sigma|$ we can write,

$$z = \sigma + a_0 + \frac{a_1}{\sigma} + \frac{a_2}{\sigma^2} + \ldots$$

so that

$$\frac{z - mk}{z + mk} = \frac{\sigma - mk + ao + a1/\sigma + a2/\sigma^2 + \ldots\ldots}{\sigma + mk + ao + a1/\sigma + a2/\sigma^2 + \ldots\ldots}$$

$$\simeq \left[1 + \frac{ao - mk}{\sigma} + \frac{a1}{\sigma^2} + \frac{a2}{\sigma^3}\right]\left[1 - \frac{ao+mk}{\sigma} - \frac{a1}{\sigma^2}\right.$$

$$\left. - \frac{a2}{\sigma^3} + \frac{(ao + mk)^2}{\sigma^2} + \frac{2a1}{\sigma^3}(ao + mk) - \frac{(ao + mk)^3}{\sigma^3}\right]$$

$$= 1 - \frac{2mk}{\sigma} + \frac{2mk\ (ao + mk)}{\sigma^2} + \frac{2mk\ (a1 - (ao + mk)^2}{\sigma^3}$$

Also we can express the term

$$\frac{\sigma - 1}{\sigma - b_k + k} \quad \text{as} \quad \frac{1 - \frac{1}{\sigma}}{1 + (k - b_k)/\sigma}$$

$$\simeq \left(1 - \frac{1}{\sigma}\right)\left(1 - \frac{k - b_k}{\sigma} + \frac{(k - b_k)^2}{\sigma^2} - \frac{(k - b_k)^3}{\sigma^3} + \ldots\right)$$

$$\simeq 1 - \frac{2k}{\sigma} + \frac{2k\ (k - b_k)}{\sigma^2} - \frac{2k\ (k - b_k)^2}{\sigma^3}$$

Hence

$$\left(\frac{1 - 1/\sigma}{1 + (k - b_k)/\sigma}\right)^m \simeq 1 - \frac{2mk}{\sigma} + \frac{2mk\ (mk - b_k)}{\sigma^2}$$

$$- \frac{2mk}{\sigma^3}\left\{\frac{1}{3}\ k^2\ (1 + 2m^2) + b_k\ (b_k - 2mk)\right\}$$

comparing coe ficients of powers of σ gives

$$ao = - b_k$$

and

$$a1 = \frac{1}{3} k^2\ (m^2 - 1)$$

Thus for large $|\sigma|$

$$z \simeq - b_k + \frac{\frac{1}{3} k^2(m^2-1)}{2}$$

so the transform modulus squared

$$T^2 \left|\frac{dz}{d\sigma}\right|^2 \simeq 1 - \frac{\frac{2}{3} k^2(m^2 - 1)}{r^2} \cos 2\theta$$

since

$$\frac{dz}{d\sigma} \simeq \frac{1 - \frac{1}{3}k^2(m^2-1)}{\sigma^2}$$

For axisymmetric flows $\frac{\partial r}{\partial x}$ and $\frac{\partial \theta}{\partial x}$ are also required.

Differentiating (C-2) with respect to r gives

$$\frac{\partial z}{\partial r} = 2m^2k\,\frac{\partial \sigma}{\partial r}\left[\frac{(\sigma-b_k+k)^{m-1}}{(\sigma-b_k+k)^m-(\sigma-1)^m} - (\sigma-b_k+k)^m \frac{\{(\sigma-b_k+k)^{m-1}-(\sigma-1)^{m-1}\}}{[(\sigma-b_k+k)^m-(\sigma-1)^m]^2}\right]$$

In terms of $\rho 1$, $\rho 2$, $\phi 1$ and $\phi 2$ and expressing σ as $r\,e^{i\theta}$ this formula becomes

$$\frac{\partial x}{\partial r} + i\frac{\partial R}{\partial r} = 4m^2k^2\frac{\partial \sigma}{\partial r}\frac{(\sigma-b_k-k)^{m-1}(\sigma-1)^{m-1}}{\{(\sigma-b_k+k)^m-(\sigma-1)^m\}^2}$$

$$= \frac{4m^2k^2e^{i\theta}(\rho 1\rho 2)^{m-1}\,e^{i(m-1)(\phi 1+\phi 2)}}{(\rho 1^m e^{im\phi 1}-\rho 2^m e^{im\phi 2})^2}$$

$$= \frac{4m^2k^2(\rho 1\rho 2)^{m-1}e^{i(\theta-\phi 1-\phi 2)}}{(\rho 1^{2m}+\rho 2^{2m}-2\rho 1^m\rho 2^m\cos m(\phi 1-\phi 2))^2}$$

$$\times\left(\rho 1^{2m}e^{-im(\phi 1-\phi 2)} - 2\rho_1^m\rho_2^m + \rho_2^m e^{im(\phi_1-\phi_2)}\right)$$

$$= \frac{T^2\,e^{i(\theta-\phi 1-\phi 2)}}{4m^2k^2(\rho 1\rho 2)^{m-1}}\left(\rho 1^{2m}e^{-im(\phi 1-\phi 2)} - 2\rho 1^m\rho 2^m + \rho 2^m e^{im(\phi 1-\phi 2)}\right)$$

by (C-6). Thus

$$\frac{\partial x}{\partial r} = \frac{T^2}{4m^2k^2(\rho 1\rho 2)^{m-1}}\Big\{\cos(\theta-\phi 1-\phi 2)\left[(\rho 1^{2m}+\rho 2^{2m})\cos m(\phi 1-\phi 2)-2\rho 1^m\rho 2^m\right] + (\rho 1^{2m}-\rho 2^{2m})\sin(\theta-\phi 1-\phi 2)\sin m(\phi 1-\phi 2)\Big\}$$

$$\frac{\partial R}{\partial r} = \frac{T^2}{4m^2k^2(\rho 1\rho 2)^{m-1}}\Big\{\sin(\theta-\phi 1-\phi 2)\;(\rho 1^{2m}+\rho 2^{2m})\cos m(\phi_1-\phi 2)$$

$$- 2\rho 1^m\rho 2^m - (\rho 1^{2m}-\rho 2^{2m})\cos(\theta-\phi 1-\phi 2)\sin m(\phi 1-\phi 2)\Big\}$$

Similarly we have

$$\frac{\partial z}{\partial \theta} = 4m^2k^2 \frac{\partial \sigma}{\partial \theta} \left[\frac{(\sigma - b_k + k)^{m-1} (\sigma - 1)^{m-1}}{\{(\sigma - b_k + k)^m - (\sigma - 1)^m\}^2} \right]$$

and since $\quad \dfrac{\partial \sigma}{\partial \theta} = i\, r\, e^{i\theta}$

this is equivalent to

$$\frac{\partial x}{\partial \theta} + i \frac{\partial R}{\partial \theta} = \frac{iT^2}{4m^2k^2(\rho 1 \rho 2)^{m-1}} r\, e^{i(\theta - \phi 1 - \phi 2)} \Big(\rho 1^{2m} e^{-im(\phi 1 - \phi 2)} - 2\rho 1^m \rho 2^m + \rho 2^{2m} e^{im(\phi 1 - \phi 2)} \Big)$$

Thus

$$\frac{\partial x}{\partial \theta} = \frac{-rT^2}{4m^2k^2(\ 1\ 2)^{m-1}} \Big[\sin(\theta - \phi 1 - \phi 2) \quad (\rho 1^{2m} + \rho 2^{2m}) \cos m(\phi 1 - \phi 2) - 2\rho 1^m \rho 2^m \Big] - (\rho 1^{2m} - \rho 2^{2m}) \cos(\theta - \phi 1 - \phi 2) \sin_m(\phi 1 - \phi 2) \Big]$$

$$\frac{\partial R}{\partial \theta} = \frac{r\, T^2}{4m^2k^2(\rho 1 \rho 2)^{m-1}} \Big[\cos(\theta - \phi 1 - \phi 2) \Big\{ (\rho 1^{2m} + \rho 2^{2m}) \cos m(\phi 1 - \phi 2) - 2\rho 1^m \rho 2^m \Big\} + (\rho 1^{2m} - \rho 2^{2m}) \sin(\theta - \phi 1 - \phi 2) \sin m(\phi 1 - \phi 2) \Big]$$

Thus

$$r \frac{\partial x}{\partial r} = \frac{\partial R}{\partial \theta}$$

and

$$\frac{\partial x}{\partial \theta} = - r \frac{\partial R}{\partial r}$$

Now $\quad J \dfrac{\partial r}{\partial x} = \dfrac{\partial R}{\partial \theta} \quad ; \quad J \dfrac{\partial \theta}{\partial x} = - \dfrac{\partial R}{\partial r}$

and $J = r\, T^2$

giving

$$\frac{\partial r}{\partial x} = \frac{1}{4\, m^2k^2(\rho 1 \rho 2)^{m-1}} \Big[\cos(\theta - \phi 1 - \phi 2) \Big\{ (\rho 1^{2m} + \rho 2^{2m}) \cos m(\phi 1 - \phi 2) - 2\rho 1^m \rho 2^m \Big\} + (\rho 1^{2m} - \rho 2^{2m}) \sin(\theta - \phi 1 - \phi 2) \sin m(\phi 1 - \phi 2) \Big]$$

and

$$r \frac{\partial \theta}{\partial x} = \frac{1}{4m^2k^2(\rho 1 \rho 2)^{m-1}} \Big[\cos(\theta - \phi 1 - \phi 2) \quad (\rho 1^{2m} - \rho 2^{2m}) \sin m(\phi 1 - \phi 2) - \sin(\theta - \phi 1 - \phi 2) \Big\{ (\rho 1^{2m} + \rho 2^{2m}) \cos m(\phi 1 - \phi 2) - 2\rho 1^m \rho 2^m \Big\} \Big]$$

Thus

$$\frac{\partial \chi}{\partial x} = \frac{1}{4m^2k^2(\rho_1\rho_2)^{m-1}} \Bigg[\frac{\partial \chi}{\partial r} \Big(\cos(\theta - \phi_1 - \phi_2) \left(\rho_1^{2m} + \rho_2^{2m} \right)$$

$$\cos m(\phi_1 - \phi_2) - 2\rho_1^m \rho_2^m + (\rho_1^{2m} - \rho_2^{2m}) \sin(\theta - \phi_1 - \phi_2)$$

$$\sin m(\phi_1 - \phi_2) \Big) + \frac{1}{r} \frac{\partial \chi}{\partial \theta} \cos(\theta - \phi_1 - \phi_2)(\rho_1^{2m} - \rho_2^{2m}) \sin m(\phi_1 - \phi_2)$$

$$- \sin(\theta - \phi_1 - \phi_2) \left\{ (\rho_1^{2m} + \rho_2^{2m}) \cos m(\phi_1 - \phi_2) - 2\rho_1^m \rho_2^m \right\} \Bigg] \quad (C-8)$$

APPENDIX D THE DERIVATION OF THE VARIATIONAL INTEGRAL FOR AXISYMMETRIC FLOW

The Bateman-Dirichlet principle for axisymmetric flows is considered below. For a region V bounded internally by Co and externally by the surface of a large sphere CR1 of radius R1, the Bateman-Dirichlet integral to be maximized is

$$J\left[\phi\right] = \iiint_V p \, dx\, dy\, dz + \int_{CR1,C} \rho\, \phi \frac{\partial \phi}{\partial n} \, dS$$

and by (A-9)

$$J\left[\phi \text{ extr}\right] = \iiint_V (p + \rho q^2)\, dx\, dy\, dz$$

As for two-dimensional flows, we assume the proposed extremal flow is given by

$$\phi = \phi_\infty + \chi'$$

$$= Ux + \chi' \qquad \text{(D-1)}$$

where ϕ_∞ is the free stream potential and χ' is small when $r = \sqrt{x^2+y^2+z^2}$ is large. It is also assumed that the principle part of $J\left[\phi\right]$ will be independent of χ so a part whose variation vanishes at ∞ can be subtracted.

We define

$$JR1\left[\phi\right] = \iiint_V (p - p_\infty)\, dx\, dy\, dz + \iint_{CR1} \chi' \rho_\infty \frac{\partial (Ux)}{\partial n}\, dS$$

$$+ \iint_C \phi\, \rho \frac{\partial \phi}{\partial n}\, dS \qquad \text{(D-2)}$$

In axisymmetric flow we can write

$$\phi = Ux + \frac{f1(\theta)}{r} + \frac{(f2(\theta)}{r^2} + \dots = Ux + \chi'$$

where r, θ are polar coordinates in the plane about which the flow is symmetric and f1, f2 ... are trignometric series which converge when (I m θ) and r^{-1} are sufficiently small. By the same process as in the two-dimensional case, we obtain

$$p = p_\infty - \rho_\infty U \frac{\partial \chi'}{\partial x} + 0\left(\frac{1}{r^4}\right) \qquad \text{(D-3)}$$

Now from (D-2) and Green's theorem

$$JR1[\phi] = \iiint_V (p - p_\infty)\, dx\, dy\, dz + \iiint_V U \rho_\infty \frac{\partial \chi'}{\partial x}\, dx\, dy\, dz + W \qquad (D-4)$$

where

$$W = \iint_C \left(\rho \phi \frac{\partial \phi}{\partial n} - U \rho_\infty \chi' \frac{\partial x}{\partial n} \right) dS \qquad (D-5)$$

From (D-3), therefore $JR1[\phi]$ converges as $R1 \rightarrow \infty$ and

$$J_\infty[\phi] = \iiint_\infty (p - p_\infty + U \rho_\infty \frac{\partial \chi'}{\partial x}\, dx\, dy\, dz + W$$

As before we let ϕ and χ' have the forms $\phi_o + \epsilon\eta$ and $\chi_o + \epsilon\eta$ respectively. Then

$$\frac{\partial}{\partial \epsilon}\left(p - p_\infty + U \rho_\infty \frac{\partial \chi'}{\partial x} \right) = 0 \ \frac{1}{r^4}$$

by (D-3). Hence

$$\frac{\partial}{\partial \epsilon} J_\infty[\phi] = \lim_{R1 \rightarrow \infty} \iint_V \left(\frac{\partial p}{\partial \epsilon} + U \rho_\infty \frac{\partial^2 \chi}{\partial \epsilon \partial x} \right) dx\, dy\, dz + \frac{\partial W}{\partial \epsilon} \qquad (D-6)$$

But by (A-7) and Green's theorem

$$\iiint_V \left(\frac{dp}{d\epsilon} \right)_{\epsilon=0} dx\, dy\, dz = - \iint_{C, C_{R1}} \rho \eta \frac{\partial \phi}{\partial n}\, dS + \iiint_V \operatorname{div}(\rho \nabla \phi) dx\, dy\, dz$$

and

$$\iiint_V U \rho_\infty \left(\frac{\partial^2 \chi'}{\partial \epsilon \partial x} \right)_{\epsilon=0} dx\, dy\, dz = \iiint_V U \rho_\infty \frac{\partial \eta}{\partial x}\, dx\, dy\, dz$$

$$= \iint_{C, C_{R1}} U \rho_\infty \eta \frac{\partial x}{\partial n}\, dS$$

again by Green's theorem

On. CR1 $\quad \rho = \rho_\infty \{1 + 0\, (r^{-2})\}$, and

$$\frac{\partial \phi}{\partial n} = \frac{\partial (Ux)}{\partial n} + \frac{\partial \chi'}{\partial n} = \frac{\partial (Ux)}{\partial n} + 0\, (r^{-2})$$

so

$$\eta \left\{ \rho \frac{\partial \phi}{\partial n} - \rho_\infty \frac{\partial (Ux)}{\partial n} \right\} = 0\, (r^{-3})$$

Substituting into (D-6), taking the limit $R1 \to \infty$ and for $\epsilon = 0$

$$\frac{\partial}{\partial \epsilon} J_\infty [\phi] = \iint_C \eta \left(U \rho_\infty \frac{\partial x}{\partial n} - \rho \frac{\partial \phi}{\partial n} \right) dS$$

$$+ \iiint_\infty \eta \, \text{div} \, (\rho \nabla \phi) \, dx \, dy \, dz + \frac{\partial W}{\partial \epsilon}$$

The order of η div $(\rho \nabla \phi)$ is r^{-4} so the integral converges. Substituting for $\partial W / \partial \epsilon$ from (D-5) with $\frac{\partial \psi}{\partial \epsilon} = \frac{\partial \chi}{\partial \epsilon} = \eta$ and since $\rho \partial \phi / \partial \eta$ is fixed on C so its ϵ derivative is zero, then we have

$$\frac{\partial}{\partial \epsilon} J_\infty [\phi] \Big|_{\epsilon 0} = \iiint_\infty \eta \, \text{div} \, (\rho \nabla \phi) \, dx \, dy \, dz$$

Thus an extremal ϕ gives div $(\rho \nabla \phi) = 0$ at all points outside C and as in two-dimensional flow the replacement of the hydrodynamic flow problem in the infinite region by one of making $J_\infty [\phi]$ stationary. The axisymmetric flow case uses the fact that all the remainders in the two-dimensional convergence discussion were all higher in order, by one power, than was needed. This spare power is cancelled by the third integral.

We now have

$$J_\infty [\phi] = \left(p - p_\infty + U \rho_\infty \frac{\partial \chi'}{\partial x} \right) dx \, dy \, dz$$

$$- U \rho_\infty \iint_\infty \chi' \frac{\partial x}{\partial n} dS \qquad \text{(D-7)}$$

There are now two possible ways of proceeding. We can proceed, as in the two-dimensional case, to a form of the integral only containing a volume integral with terms containing the incompressible flow potential. Alternatively, since it is difficult to calculate the incompressible potential for axisymmetric flows about most bodies, it is advantageous to retain the surface integral form, for bodies other than the ellipsoid and sphere.

Since the incompressible potential $\phi_0 = U(x + F)$ and $\frac{\partial \phi_0}{\partial n}$ is zero on C, we have

$$\frac{\partial x}{\partial n} = - \frac{\partial F}{\partial n}$$

so that

$$J_{\infty}\ \phi = \iiint_{\infty} \left(p - p_{\infty} + U\ \rho_{\infty} \nabla x . \nabla \chi' \right) dx\, dy\, dz$$
$$+ U\ \rho_{\infty} \iint_{C_1} \chi' \nabla F . \hat{n}\ \ dS$$

where $\hat{n}$ is the unit normal to the surface C. By Green's theorem

$$\iiint_{V} \nabla . (\chi' \nabla F)\ dx\, dy\, dz = \iint_{C_1+C} \chi' \nabla F . \hat{n}\ \ dS$$

Now $F = 0\ (\frac{1}{r})$ and $\chi' = 0\left(\frac{1}{r}\right)$

so

$$\chi' \nabla F = 0 \left(\frac{1}{r^3}\right)$$

Thus as $R1 \rightarrow \infty$

$$\iiint_{\infty} \nabla . (\chi' \nabla F)\ \ dx\, dy\, dz = \iint_{C} \chi' \nabla F .\ \hat{n}\ \ dS$$

For incompressible flow $\nabla^2\ \phi o$ and hence $\nabla^2 F$ are zero, therefore,

$$\nabla . (\chi' \nabla F) = \nabla F . \nabla \chi'$$

so the integral can be written

$$J_{\infty} [\phi] = \iiint_{\infty} (p - p_{\infty} + U\ \rho_{\infty} (\nabla x . \nabla \chi' + \nabla F . \nabla \chi') dx\, dy\, dz$$
$$= \iiint_{\infty} (p - p_{\infty} + \rho_{\infty} \nabla \phi o . \nabla (\phi - \phi_{\infty})) dx\, dy\, dz \quad (D\text{-}8)$$

If we change the coordinate system (x, y, z) in the volume i integral to cylindrical polars (x, R, ξ) we obtain for the form of the integral retaining the surface integral (D-7)

$$J1_{\infty}[\phi] = \frac{J_{\infty}[\phi]}{2\ \pi} = \iint_{\infty} \left(p - p_{\infty} + U \rho_{\infty} \frac{\partial \chi'}{\partial x}\right) R\ d\ R\ dx$$
$$- U\ \rho_{\infty} \int_{B} \chi' \frac{\partial\ x}{\partial\ n}\ \ R\ ds$$

where ds is an element of the contour of a cross-section about which the flow is axisymmetric i.e. $dS = R\ ds\ d\ \xi$ and B is the contour of C in the cross-section.

(D-8) can also be expressed in the form

$$J1_{\infty}[\phi] = \iint_{\infty} \left(p - p_{\infty} + \rho_{\infty} \nabla \phi\ o . \nabla (\phi - \phi_{\infty})\right) RdRdx$$
$$= \frac{J_{\infty}[\phi]}{2\ \pi} \quad (D\text{-}9)$$

APPENDIX E THE FAR BOUNDARY CONDITION IN PLANE FLOW

In order to find a series expansion for the potential at the far boundary it is first necessary to derive the equation of motion in terms of the potential.

The equations of motion are:

$$\frac{\partial}{\partial x}(\rho u_1) + \frac{\partial}{\partial y}(\rho u_2) = 0 \qquad \text{(E-1)}$$

$$u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{(E-2)}$$

$$p = f(\rho)$$

so we can define $C^2 = \dfrac{dp}{d\rho}$

giving $\dfrac{\partial p}{\partial x} = C^2 \dfrac{\partial \rho}{\partial x}$ and $\dfrac{\partial p}{\partial y} = C^2 \dfrac{\partial \rho}{\partial y}$

There is a velocity potential ϕ so that $u_1 = \phi x$ and $u_2 = \phi y$ giving for (A-2)

$$\phi x\, \phi xx + \phi y\, \phi xy = -\frac{C^2}{\rho}\frac{\partial \rho}{\partial x} \qquad \text{(E-3)}$$

$$\phi x\, \phi xy + \phi y\, \phi yy = -\frac{C^2}{\rho}\frac{\partial \rho}{\partial y}$$

Also (E-1) can be rewritten

$$\frac{\partial \rho}{\partial x}\phi x + \rho \phi xx + \frac{\partial \rho}{\partial y}\phi y + \rho \phi yy = 0$$

which is equivalent to

$$\frac{C^2}{\rho}\frac{\partial \rho}{\partial x}\phi x + C^2 \phi xx + \frac{C^2}{\rho}\frac{\partial \rho}{\partial y}\phi y + C^2 \phi yy = 0$$

which by (E-3) gives the equation of motion

$$(C^2 - \phi x^2)\,\phi xx - 2\,\phi x\,\phi y\,\phi xy + (C^2 - \phi y^2)\,\phi yy = 0 \qquad \text{(E-4)}$$

We need a conformal mapping from the body in the z - plane to the unit circle in the σ - plane.

If

$$z = x + iy$$

and

$$\sigma = \xi_1 + i\xi_2$$

The Jacobian

$$J = \left|\frac{\partial (x, y)}{\partial (\xi_1, \xi_2)}\right|$$

We can transform again to the σ' - plane so that

$\sigma' = r e^{i\theta} = r' e^{-i\theta}$ for a body $r = r'$. Thus, the coordinates in the σ - plane can be written

$$\xi_1 + i\xi_2 = \sigma = -\theta + i \log \frac{r}{r'} = -\log \frac{\sigma'}{r'}$$

Hence

$$\frac{d\sigma'}{\sigma'} = i\, d\sigma = d\xi_2 - d\xi_1 = \frac{dr}{r} + i\, d\theta$$

and the partial derivatives are

$$\frac{\partial}{\partial \xi_1} = -\frac{\partial}{\partial \theta} \quad \text{and} \quad \frac{\partial}{\partial \xi_2} = r\frac{\partial}{\partial r}$$

Let

$$\frac{\partial(x, y)}{\partial(\xi_1, \xi_2)} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}$$

Since the mapping is conformal $\rho_{11} = A_1^2$, $\rho_{22} = A_2^2$ and $\rho_{12} = \rho_{21} = 0$

where $A_1^2 A_2^2 = J$.

The equation of motion (E-4) can be rewritten as

$$c^2 \frac{\partial}{\partial x_n}\left(\frac{\partial \phi}{\partial x_n}\right) - \frac{\partial \phi}{\partial x_\ell}\frac{\partial}{\partial x_\ell}\left(\tfrac{1}{2}\left(\frac{\partial \phi}{\partial x_n}\right)^2\right) = 0$$

Since

$$q^2 = \phi_x^2 + \phi_y^2$$

$$c^2 \frac{\partial}{\partial x_n}\left(\frac{\partial \phi}{\partial x_n}\right) - \frac{\partial \phi}{\partial x_\ell}\,\delta_{n\ell}\frac{\partial}{\partial x_n}\left(\tfrac{1}{2}q^2\right) = 0$$

Transforming coordinates gives

$$\frac{c^2}{J}\frac{\partial}{\partial \xi_n}\left(J \rho_{n\ell}\frac{\partial \phi}{\partial \xi_\ell}\right) - \frac{\partial \phi}{\partial \xi_\ell}\,\delta_{n\ell}\frac{\partial}{\partial \xi_n}\left(\tfrac{1}{2}q^2\right) = 0$$

or

$$c^2 \frac{\partial}{\partial \xi_n}\left(\delta_{n\ell}\frac{\partial \phi}{\partial \xi_\ell}\right) - \delta_{n\ell}\frac{\partial \phi}{\partial \xi_\ell}\frac{\partial}{\partial \xi_n}\left(\tfrac{1}{2}q^2\right) = 0 \qquad \text{(E-5)}$$

Now in the transformed plane

$$q^2 = \frac{1}{J}\left(\left(\frac{\partial \phi}{\partial \xi_1}\right)^2 + \left(\frac{\partial \phi}{\partial \xi_2}\right)^2\right)$$

Thus

$$\frac{\partial}{\partial \xi_n}\left(\tfrac{1}{2}q^2\right) = \tfrac{1}{2}\left[\left(\frac{\partial \phi}{\partial \xi_1}\right)^2 + \left(\frac{\partial \phi}{\partial \xi_2}\right)^2\right]\frac{\partial J^{-1}}{\partial \xi_n} + \frac{1}{J}\frac{\partial \phi}{\partial \xi_1}\frac{\partial^2 \phi}{\partial \xi_1} + \frac{1}{J}\frac{\partial \phi}{\partial \xi_2}\frac{\partial^2 \phi}{\partial \xi_2 \partial \xi_1}$$

Hence (E-5) becomes

$$c^2\left(\frac{\partial^2\phi}{\partial\xi_1^2}+\frac{\partial^2\phi}{\partial\xi_2^2}\right)-\frac{1}{2}\left(\phi_{\xi_1}^{\,2}+\phi_{\xi_2}^{\,2}\right)\frac{\partial J^{-1}}{\partial\xi_1}\frac{\partial\phi}{\partial\xi_1}+\frac{\partial J^{-1}}{\partial\xi_2}\frac{\partial\phi}{\partial\xi_2}$$

$$-\frac{1}{J}\,\phi_{\xi_1}\frac{\partial^2\phi}{\partial\xi_1^2}-\frac{2}{J}\,\phi_{\xi_1}\phi_{\xi_2}\frac{\partial^2\phi}{\partial\xi_1\partial\xi_2}-\frac{1}{J}\,\phi_{\xi_2}^{\,2}\frac{\partial^2\phi}{\partial\xi_2^2}=0$$

or alternatively replacing ξ_1, ξ_2 by r and θ, and replacing C^2 by

$$U^2\left\{\frac{1}{M_\infty^2}+\frac{(\gamma-1)}{2}\left(1-\left(\frac{q}{U}\right)^2\right)\right\}=U^2\left\{\frac{1}{M_\infty^2}+\frac{\gamma-1}{2}\left(1-\frac{r^2\phi r^2+\phi_0{}^2}{U^2J}\right)\right\}$$

we have

$$U^2(\phi_{\theta\theta}+r^2\,\phi rr+r\phi r)\left\{\frac{1}{M_\infty^2}+\frac{\gamma-1}{2}\left(1-\frac{r^2\phi_r+\phi_\theta{}^2}{U^2J}\right)\right\}-\frac{1}{J}\,\phi_\theta{}^2\phi_{\theta\theta}$$

$$-\frac{2}{J}\;r^2\phi\,r\,\phi_\theta\,\phi r\theta-\frac{r^2}{J}\;\phi r^2\left(r^2\,\phi rr+r\,\phi\,r\right)$$

$$+\frac{1}{2}\,(r^2\phi\,r^2+\phi_0{}^2)\left(\frac{\phi_\theta}{J^2}\,\frac{\partial J}{\partial\theta}+\frac{r^2\phi r}{J^2}\,\frac{\partial J}{\partial r}\right)=0 \qquad \text{(E-6)}$$

Now the Jacobian

$$J=\left|\frac{\partial(x,y)}{\partial(\xi_1,\xi_2)}\right|=\left|x_{\xi_1}\,y_{\xi_2}-x_{\xi_2}\,y_{\xi_1}\right|$$

Since the mapping is conformal

$$x_{\xi_1}=y_{\xi_2}\quad\text{and}\quad x_{\xi_2}=y_{\xi_1}$$

Thus

$$\begin{aligned}J&=x_{\xi_1}^{\,2}+y_{\xi_2}^{\,2}\\&=\left(\frac{\partial x}{\partial\theta}\right)^2+r^2\left(\frac{\partial x}{\partial r}\right)^2\\&=r^2\;T^2\end{aligned}$$

Thus (E-6) can be rewritten

$$U^2\left\{\frac{1}{M_\infty^2}+\frac{\gamma-1}{2}\left(1-\frac{\phi r^2+\phi_\theta{}^2/r^2}{U^2\,T^2}\right)\right\}\left(\frac{\phi_{\theta\theta}}{r^2}+\phi rr+\frac{\phi r}{r}\right)$$

$$-\frac{1}{r^4T^2}\;\phi_\theta{}^2\;\phi_{\theta\theta}-\frac{2}{r^2T^2}\;\phi r\,\phi_\theta\,\phi r\theta-\frac{1}{T^2}\;\phi r^2\left(\phi rr+\frac{\phi r}{r}\right)$$

$$+\left(\phi r^2+\frac{1}{r^2}\;\phi_\theta{}^2\right)\left(\frac{\phi_\theta}{r^2\,T^3}\,\frac{\partial T}{\partial\theta}+\frac{\phi r}{rT^3}\left(T+\frac{r\partial T}{\partial r}\right)\right)=0 \qquad \text{(E-7)}$$

For large $|\sigma|$ we can assume that the transformation can be written

$$z = \sigma + a_0 + \frac{a_1}{\sigma} + \frac{a_2}{\sigma^2} + \ldots .$$

giving

$$\frac{dz}{d\sigma} = 1 - \frac{a_1}{\sigma^2} - \frac{2a_2}{\sigma^3} - \ldots$$

$$= 1 - \frac{a_1 e^{-2i\theta}}{r^2} - \frac{2a_2 e^{-3i\theta}}{r^3}$$

where $\sigma = r e^{i\theta}$

Thus for large r ($|\sigma|$)

$$T^2 = \left|\frac{dz}{d\sigma}\right|^2 \simeq \left(1 - \frac{a_1}{r^2}\cos 2\theta - \frac{2a_2\cos 3\theta}{r^3}\right)^2 + \frac{a_1^2\sin^2 2\theta}{r^4}$$

$$\simeq 1 - 2\frac{a_1}{r^2}\cos 2\theta \qquad \text{(E-8)}$$

Differentiating with respect to r and θ we have

$$2T\frac{\partial T}{\partial r} \simeq 4\frac{a_1}{r^3}\cos 2\theta$$

$$\text{(E-9)}$$

$$2T\frac{\partial T}{\partial \theta} \simeq 4\frac{a_1}{r^2}\sin 2\theta$$

Thus (E-7) can be written

$$U^2\left\{\frac{1}{M_\infty^2} + \frac{\gamma-1}{2}\left(1 - \frac{\phi_r^2 + \phi_\theta^2/r^2}{U^2T^2}\right)\right\}\left\{\frac{\phi_{\theta\theta}}{r^2} + \phi_{rr} + \frac{\phi_r}{r}\right\}$$

$$-\frac{1}{r^4T^2}\phi_\theta^2\phi_{\theta\theta} - \frac{2}{r^2T^2}\phi_r\phi_\theta\phi_{r\theta} - \frac{1}{T^2}\phi_r^2\phi_{rr} + \frac{\phi_r}{r}$$

$$+\left(\phi_r^2 + \frac{1}{r^2}\phi_\theta^2\right)\left(\frac{2\phi_\theta}{r^4T^4}a_1\sin 2\theta + \frac{\phi_r}{rT^2} + \frac{2\phi_r a_1}{r^3T^4}\cos 2\theta\right) = 0$$

$$\text{(E-10)}$$

If we try a solution of the form

$$\phi = U\left(r\cos\theta + \frac{1}{r}f(\theta)\right) \qquad \text{(E-11)}$$

we obtain

$$\phi_r = U\left(\cos\theta - \frac{1}{r^2}f\right)$$

$$\phi_{rr} = \frac{2}{r^3}Uf$$

$$\phi_\theta = U\left(-r\sin\theta + \frac{1}{r}f'\right)$$

$$\phi_{\theta\theta} = U\left(-r\cos\theta + \frac{1}{r}f''\right)$$

$$\phi_{r\theta} = U\left(-\sin\theta - \frac{1}{r^2}f'\right)$$

Substituting into (E-10) for ϕ and T^2 we find

$$\left[\frac{1}{M_\infty^2} + \frac{\gamma-1}{2}\left(-\frac{2a_1 \cos 2\theta}{r^2}\right) - \frac{\gamma-1}{2}\left(1 + \frac{2a_1}{r^2}\cos 2\theta\right)\right.$$

$$\left.\left(\frac{f^2}{r^4} + \frac{f'^2}{r^4} - \frac{2}{r^2} f\cos\theta - \frac{2}{r^2} f'\sin\theta\right)\right]\left[\frac{2}{r^3} f + \frac{\cos\theta}{r}\right.$$

$$\left. - \frac{f}{r^3} - \frac{\cos\theta}{r} + \frac{f''}{r^3}\right] - \frac{1}{r}\left(1 + \frac{2a_1}{r^2}\cos 2\theta\right)$$

$$\left[\sin^2\theta - \frac{2f'}{r^2}\sin\theta + \frac{f'^2}{r^4}\right]\left(\frac{f''}{r^2} - \cos\theta\right)$$

$$+ \frac{2}{r}\left(1 + \frac{2a_1}{r^2}\cos 2\theta\right)\left(-\sin\theta\cos\theta + \frac{\cos\theta}{r^2} f + \frac{f\sin\theta}{r^2}\right.$$

$$\left. - \frac{ff'}{r^4}\right)\left(\sin\theta + \frac{f'}{r^2}\right) - \left(1 + \frac{2a_1}{r^2}\cos 2\theta\right)$$

$$\left(\cos^2\theta - \frac{2\cos\theta}{r^2} f + \frac{1}{r^4} f^2\right)\left(\frac{f}{r^3} + \cos\theta\right)$$

$$+ \left(1 - \frac{2\cos\theta}{r^2} f + \frac{f^2}{r^4} - \frac{2f'}{r^2}\sin\theta + \frac{f'^2}{r^4}\right)$$

$$\left\{\frac{2a_1\sin 2\theta}{r^3}\left(-\sin\theta + \frac{f'}{r^2}\right) + \frac{1}{r}\left(1 + \frac{4a_1}{r^2}\cos 2\theta\right)\right.$$

$$\left.\left(\cos\theta - \frac{1}{r^2} f\right)\right\} = 0$$

Retaining only the coefficients of r^{-3}, the highest order term remaining in the equation, we obtain

$$\frac{1}{M_\infty^2}\left(f + f''\right) + 2a_1\sin^2\theta\cos\theta\cos 2\theta$$

$$- f''\sin^2\theta - 4a_1\sin^2\theta\cos\theta\cos 2\theta$$

$$- 2\sin\theta\cos\theta f'$$

$$- 2\sin\theta\cos\theta f' + 2f'\sin\theta\cos\theta + 2f\sin^2\theta$$

$$- 2a_1\cos 2\theta\cos^3\theta - f\cos^2\theta + 2f\cos^2\theta$$

$$- 2a_1\sin 2\theta\sin\theta + 4a_1\cos 2\theta\cos\theta - f - 2f'\sin\theta\cos\theta$$

$$- 2f\cos^2\theta = 0$$

which simplifies to

$$f''(1 - M_\infty^2\sin^2\theta) - 4M_\infty^2\sin\theta\cos\theta f'$$
$$+\left[1 + (3\sin^2\theta - 2)M_\infty^2\right]f = -2a_1 M_\infty^2\cos 3\theta \qquad (E-12)$$

It can be verified by substitution that a solution of the left hand side put equal to zero is $f1(\theta) = \dfrac{\cos\theta}{1-M_\infty^2\sin^2\theta}$

so the general solution of (E-12) is of the form

$$f(\theta) = h(\theta) \quad f1 \quad (\theta)$$

Substituting into (E -10) gives

$$h\left\{(1-M_\infty^2\sin^2\theta)f_1'' - 4M_\infty^2\sin\theta\cos\theta f_1' + \left[1+(3\sin^2\theta-2)M_\infty^2\right]f_1\right\} + h'\left\{2(1-M_\infty^2\sin^2\theta)f_1' - 4M_\infty^2\sin\theta\cos\theta f_1\right\} + (1-M_\infty^2\sin^2\theta)h''f_1$$
$$= -2a_1 M_\infty^2\cos 3\theta$$

which simplifies to

$$h''\cos\theta - 2h'\sin\theta = -2a_1 M_\infty^2 \cos 3\theta$$

Integrating, we obtain

$$\cos^2\theta \; h' = B - a_1 M_\infty^2 \left(\tfrac{1}{4}\sin 4\theta + \tfrac{1}{2}\sin 2\theta\right)$$

where B is a constant.

$$h = B\sec^2\theta - 2a_1 M_\infty^2 \sin\theta\cos\theta$$

On integrating again we have

$$h(\theta) = A + B\tan\theta - a_1 M_\infty^2\sin^2\theta$$

where A is a constant.

Thus, the expression for $f(\theta)$ is

$$f(\theta) = \frac{A\cos\theta + B\sin\theta - a_1 M_\infty^2\sin^2\theta\cos\theta}{1 - M_\infty^2\sin^2 0}$$

Since the solution is symmetric about $\theta = 0$ and $\theta = \pi$ for a symmetric body we must have $B = 0$

Thus a two term series expansion for flow past a symmetric body is

$$\phi = U\cos\theta\left(r + \frac{(A - a_1 M_\infty^2\sin^2\theta)}{r(1-M_\infty^2\sin^2\theta)}\right) \qquad \text{(E-13)}$$

If we have values of χ, the perturbation potential for points well away from the body (e.g. for $r > 10$, $\frac{1}{r^2} < 0.01$ so the next term in the series is small) we can find the value of A by the method of least squares at the points at which $r = r_{n-1}$.

If there are ℓ points a distance r_{n-1} from the origin, we can minimize the difference between the values of the calculated $\chi_{i,n-1}$ and

$$\frac{(A - a_1 M_\infty^2 \sin^2\theta)\cos\theta_i}{r_{n-1}(1 - M_\infty^2 \sin^2\theta_i)} \quad \text{for } i = 1,2,\ldots,\ell$$

The square of the difference at any point is given by

$$\left(\frac{\cos\theta_i\,(A - a_1 M_\infty^2 \sin^2\theta_i)}{r_{n-1}(1 - M_\infty^2 \sin^2\theta_i)} - \chi_{i,n-1}\right)^2 \quad i = 1,2,\ldots,\ell$$

For the best fit we must minimize

$$D^2 = \sum_{i=1}^{\ell}\left(\frac{\cos\theta_i\,(A - a_1 M_\infty^2 \sin^2\theta_i)}{r_{n-1}(1 - M_\infty^2 \sin^2\theta_i)} - \chi_{i,n-1}\right)^2$$

For a minimum

$$\frac{dD^2}{dA} = \sum_{i=1}^{\ell} \frac{2\cos\theta_i}{r_{n-1}(1-M_\infty^2 \sin^2\theta_i)}\left(\left(\frac{A - a_1 M_\infty^2 \sin^2\theta_i}{r_{n-1}(1-M_\infty^2\sin^2\theta_i)}\right)\cos\theta_i - \chi_{i,n-1}\right) = 0$$

Thus

$$A = r_{n-1}\,\frac{\displaystyle\sum_{i=1}^{\ell}\left(\frac{\chi_{i,n-1}\cos\theta_i}{(1 - M_\infty^2\sin^2\theta_i)} + \frac{a_1 M_\infty^2\cos^2\theta_i \sin^2\theta_i}{r_{n-1}(1 - M_\infty^2\sin^2\theta_i)^2}\right)}{\displaystyle\sum_{i=1}^{\ell}\frac{\cos^2\theta_i}{(1-M_\infty^2\sin^2\theta_i)^2}} \qquad \text{(E-14)}$$

Also

$$\frac{d^2D^2}{dA^2} = \sum_{i=1}^{\ell}\frac{2\cos^2\theta_i}{r_{n-1}^2(1-M_\infty^2\sin^2\theta_i)^2} > 0$$

So the value of A gives a minimum sum of squares. Hence the values of χ can be found at a distance r_n from the body.

In the case of the plane flow past the unit circle no transformation is required, so

$$z = \sigma$$

and

$$a_1 = 0$$

Thus at large distances from the circle (E-13) becomes

$$\phi = U\left(r\cos\theta + \frac{A\cos\theta}{r\,(1-M_\infty^2\sin^2\theta)}\right) \qquad \text{(E-15)}$$

and the value of A is given by

$$A = \frac{r_{n-1}\displaystyle\sum_{i=1}^{\ell}\frac{\chi_{i,n-1}\cos\theta i}{(1-M_\infty^2\sin^2\theta i)}}{\dfrac{\cos^2\theta i}{(1-M_\infty^2\sin^2\theta i)^2}} \qquad \text{(E-16)}$$

For the ellipse however, from Appendix B we have that

$$a_1 = \lambda^2$$

so that at large distances from the ellipse (E-13) gives

$$\phi = U\left(r\cos\theta + \frac{\cos\theta}{r\,(1-M_\infty^2\sin^2\theta)}\,(A-\lambda^2M_\infty^2\sin^2\theta)\right) \qquad \text{(E-17)}$$

and (E-12) gives

$$A = \frac{r_{n-1}\displaystyle\sum_{i=1}^{\ell}\left(\frac{\chi_{i,n-1}\cos\theta i}{(1-M_\infty^2\sin^2\theta i)} + \frac{\lambda^2M_\infty^2\cos^2\theta i\sin^2\theta i}{r_{n-1}\,(1-M_\infty^2\sin^2\theta i)^2}\right)}{\displaystyle\sum_{i=1}^{\ell}\frac{\cos^2\theta i}{(1-M_\infty^2\sin^2\theta i)^2}} \qquad \text{(E-18)}$$

For the ellipse all the a i for $i > 1$ are zero, but this is not the case for the Karman-Trefftz profile, where from Appendix C (C-6) we have

$$a_0 = -b_k$$

and

$$a_1 = \frac{1}{3}k^2(m^2-1) \quad \text{(see appendix C)}$$

Thus for large $|\sigma|$ we can write

$$z = \sigma - b_k + \frac{\frac{1}{3} k^2 (m^2 - 1)}{}$$

and

$$T^2 = 1 - \frac{\frac{2}{3} k^2 (m^2 - 1)}{r^2} \cos 2\theta$$

(A-10) now becomes

$$\phi = U \left(r \cos\theta + \frac{\cos\theta}{r\,(1 - M_\infty^2 \sin^2\theta)} \left(A - \frac{1}{3} k^2 (m^2 - 1)\, M_\infty^2 \sin^2\theta \right) \right) \qquad \text{(E-19)}$$

at large distances from the profile where A is given by

$$A = \frac{\sum_{i=1}^{n} \frac{\chi_i \cos\theta_i}{(1 - M_\infty^2 \sin^2\theta_i)} + \frac{1}{3} \frac{M_\infty^2 k^2 (m-1) \cos^2\theta_i \sin^2\theta_i}{r_i^{\,n-1} (1 - M_\infty^2 \sin^2\theta_i)^2}}{\sum_{i=1} \cos^2\theta_i \,/\, (1 - M_\infty^2 \sin^2\theta_i)^2} \qquad \text{(E-20)}$$

APPENDIX F THE CONTRIBUTION TO THE VARIATIONAL INTEGRAL FROM EACH RECTANGLE IN PLANE FLOW

Consider rectangle 1 in figure 3. Define X_1 and X_2 by

$$X_1 = \chi_1 + \chi_2 - \chi_3 \; ; \qquad X_2 = \chi_2 + \chi_3 - \chi_1$$

where χ_1, χ_2, and χ_3 are the values of the potential at points 1, 2 and 3 respectively.

Then at the centre of rectangle 1

$$\frac{\partial \chi}{\partial \theta} = \frac{X_1 - \chi_0}{2h_1} \quad \text{and} \quad \frac{\partial \chi}{\partial r} = \frac{X_2 - \chi_0}{2k_1}$$

At the centre of the rectangle r and θ are given by

$$\theta_1 = \theta + 0.5h_1 \qquad r_1 = r + 0.5k_1$$

Let T s be the value of the transform modulus at the centre of rectangle s (s = 1,2,3,4,) then the contribution to the integral from rectangle 1 is

$$J[\chi_0] = \left[(A_1\chi_0^2 + B_1\chi_0 + C_1)^{\alpha} + D_1\chi_0 + E_1\right] H_1$$

where

$$A_1 = -\frac{(\gamma-1)M_\infty^2}{8T_1^2}\left(\frac{1}{k_1^2} + \frac{1}{r_1^2 h_1^2}\right) \qquad \text{(F-1)}$$

$$B_1 = \frac{(\gamma-1)M_\infty^2}{2T_1^2}\left[\frac{\cos\theta_1}{k_1} - \frac{\sin\theta_1}{r_1 h_1} + \frac{1}{2}\left(\frac{X_2}{k_1^2} + \frac{X_1}{r_1^2 h_1^2}\right)\right] \qquad \text{(F-2)}$$

$$C_1 = 1 + \left(1 - \frac{1}{T_1^2}\right)\frac{(\gamma-1)M_\infty^2}{2} - \frac{(\gamma-1)M_\infty^2}{2T_1^2}\left[\frac{X_2\cos\theta_1}{k_1} - \frac{X_1\sin\theta_1}{r_1 h_1} + \frac{1}{4}\left(\frac{X_2^2}{k_1^2} + \frac{X_1^2}{r_1^2 h_1^2}\right)\right] \qquad \text{(F-3)}$$

$$D_1 = -\frac{\gamma M_\infty^2}{2T_1^2}\left(\frac{(r_1^2-1)\cos\theta_1}{r_1^2 k_1} - \frac{r_1^2+1}{h_1 r_1^3}\sin\theta_1\right) \qquad \text{(F-4)}$$

$$E_1 = -1 + \frac{\gamma M_\infty^2}{2T_1^2}\left(\frac{r_1^2-1}{r_1^2 k_1}X_2\cos\theta_1 + \frac{r_1^2+1}{h_1 r_1^3}X_1\sin\theta_1\right) \qquad \text{(F-5)}$$

and

$$H_1 = r_1 T_1^2 k_1 h_1 \qquad \text{(F-6)}$$

Similarly at the centre of rectangle 2

$$\frac{\partial \chi}{\partial \theta} = \frac{X3 + \chi_o}{2h_2} \quad ; \quad \frac{\partial \chi}{\partial r} = \frac{X4 - \chi_o}{2k1}$$

where

$$X3 = \chi_3 - \chi_4 - \chi_5 \quad \text{and } X4 = \chi_3 + \chi_4 - \chi_5$$

At the centre of rectangle 2, r and θ are given by

$\theta 2 = \theta - 0.5\, h_2$ and $r1 = r + 0.5k1$.

Thus by (3.4), for rectangle 2, we have

$$J\,[\chi_o] = \left[(A2\chi_o^2 + B_2\chi_o + C2)^{\alpha} + D_2\chi_o + E_2 \right] H\; 2 \tag{F-7}$$

where

$$A2 = -\frac{(\gamma - 1) M_\infty^2}{8T_2^2} \left(\frac{1}{k1^2} + \frac{1}{r1^2 h_2^2} \right) \tag{F-8}$$

$$B2 = \frac{(\gamma - 1) M_\infty^2}{2T_2^2} \left[\frac{\cos \theta 2}{k\,1} + \frac{\sin \theta 2}{r1\; h2} + \frac{1}{2}\left(\frac{X4}{k1^2} - \frac{X3}{r1h2^2} \right) \right] \tag{F-9}$$

$$C2 = 1 + \left(1 - \frac{1}{T2^2}\right) \frac{(\;-1)M\;\;^2}{2} - \frac{(\;-1)M_\infty^2}{2\,T2^2} \left[\frac{X4 \cos \theta 2}{k\,1} - \frac{X3 \sin \theta 2}{r1\; h2} + \frac{1}{4}\left(\frac{X4^2}{k1^2} + \frac{X3^2}{r1^2 h2^2} \right) \right] \tag{F-10}$$

$$D_2 = -\frac{\gamma M_\infty^2}{2T2^2} \left(\frac{r1^2 - 1 \cos \theta_2}{r1^2\; k1^2} + \frac{r1^2 + 1}{r1^3 h^2} \sin \theta\; 2 \right) \tag{F-11}$$

$$E2 = -1 + \frac{\gamma M_\infty^2}{2\; T2^2} \left(\frac{r1^2 - 1}{r1^2\; k1} X4 \cos\theta_1 - \frac{r1^2 + 1}{r1^3\; h^2} X3 \sin \theta\; 1 \right) \tag{F-12}$$

and

$$H2 = r1\; T2^2\; k1\; h2 \tag{F-13}$$

Similarly at the centre of rectangle 3

$$\frac{\partial}{\partial \theta} = \frac{X5 + \chi_o}{2\;h\;2} \qquad \frac{\partial \chi}{\partial r} = \frac{X6 + \chi_o}{2\;k\;2}$$

where

$$X5 = \chi_7 - \chi_5 - \chi_6, \quad X6 = \chi_5 - \chi_6 - \chi_7$$

$$\theta 2 = \theta - 0.5h2 \quad , \quad r2 = r - 0.5k2$$

Thus by (3.4), for rectangle 3, we have

$$J[\chi_o] = \left[(A3\chi_o^2 + B3\chi_o + C3)^{\alpha} + D3\chi_o + E3\right] H3 \quad (F\text{-}14)$$

where

$$A3 = -\frac{(\gamma-1)M_\infty^2}{8\,T3^2}\left(\frac{1}{k2^2} + \frac{1}{r2^2h2^2}\right) \quad (F\text{-}15)$$

$$B3 = -\frac{(\gamma-1)M_\infty^2}{2\,T\,3^2}\left[\frac{\cos\theta 2}{k_2} - \frac{\sin\theta 2}{r\,2\,h\,2} + \frac{1}{2}\left(\frac{X6}{k2^2} + \frac{X5}{r2^2h2^2}\right)\right] \quad (F\text{-}16)$$

$$C3 = 1 + \left(1 - \frac{1}{T\,3^2}\right)\frac{\gamma-1}{2}M_\infty^2 - \frac{(\gamma-1)M_\infty^2}{2\,T3^2}\left[\frac{X6\cos\theta 2}{k^2} - \frac{X5\sin\theta 2}{r\,2\,h\,2} + \frac{1}{4}\left(\frac{X6^2}{k2^2} + \frac{X5^2}{r2^2h2^2}\right)\right] \quad (F\text{-}17)$$

$$D3 = \frac{M_\infty^2}{2\,T3^2}\left[\frac{(r2^2-1)\cos\theta 2}{r2^2\,k2} - \frac{r2^2+1}{h2\,r2^3}\sin\theta\,2\right] \quad (F\text{-}18)$$

$$E3 = -1 + \frac{\gamma M_\infty^2}{2\,T3^2}\left(\frac{r2^2-1}{r2^2k2}X6\cos\theta\,2 - \frac{r2^2+1}{r2^3h2}X5\sin\theta\,2\right) \quad (F\text{-}19)$$

and

$$H3 = r\,2\,T\,3^2\,k\,2\,h\,2 \quad (F\text{-}20)$$

Similarly at the centre of rectangle 4

$$\frac{\partial\chi}{\partial\theta} = \frac{X7 - \chi_o}{2h1} \qquad \frac{\partial\chi}{\partial r} = \frac{X8 + \chi_o}{2k2}$$

where

$$X7 = \chi_1 + \chi_8 - \chi_7 \quad \text{and} \quad X8 = \chi_1 - \chi_7 - \chi_8$$

At the centre of the rectangle r and θ are given by

$$\theta 1 = \theta + 0.5h1 \qquad r2 = r - 0.5\,k2$$

Thus for rectangle 4 by (3.4)

$$J[\chi_o] = \left[(A4\,\chi_o^2 + B4\chi_o + C4)^{\alpha} + D4\,\chi_o + E4\right] H4 \quad (F\text{-}21)$$

where

$$A4 = -\frac{(\gamma-1)M_\infty^2}{8\,T4^2}\left(\frac{1}{k2^2} + \frac{1}{r2^2h1^2}\right) \quad (F\text{-}22)$$

$$B4 = -\frac{(\gamma-1)M_\infty^2}{2\,T4^2}\left[\frac{\cos\theta\,1}{k_2} + \frac{\sin\theta\,1}{r2\,h\,1} + \frac{1}{2}\left(\frac{X\,8}{k2^2} - \frac{X7}{r2^2h1^2}\right)\right] \quad (F\text{-}23)$$

$$C4 = 1 + \frac{(\gamma - 1)}{2} M_{\infty}^{2} \left(1 - \frac{1}{T4^{2}} \right) - \frac{(\gamma - 1) M_{\infty}^{2}}{2T4^{2}} \left[\frac{X8 \cos \theta 1}{K_2} \right.$$

$$\left. - X7 \frac{\sin \theta 1}{r_2 \, h \, 1} + \frac{1}{4} \left(\frac{X8^{2}}{k2^{2}} + \frac{X7^{2}}{r2^{2} h \, 1^{2}} \right) \right] \qquad (F\text{-}24)$$

$$D4 = \frac{\gamma M_{\infty}^{2}}{2 \, T4^{2}} \left(\frac{r_2^{2} - 1}{r2^{2} \, k^{2}} \cos \theta 1 + \frac{r2^{2} + 1}{r2^{3} \, h \, 1} \sin \theta 1 \right) \qquad (F\text{-}25)$$

$$E4 = -1 + \frac{\gamma M_{\infty}^{2}}{2 \, T4^{2}} \left(\frac{r2^{2} - 1}{r2^{2} k2} X8 \cos \theta 1 - \frac{r2^{2} + 1}{r2^{3} \, h \, 1} X7 \sin \theta 1 \right)$$

$$(F\text{-}26)$$

and

$$H4 \mathrel{\widehat{=}} r2 \, T4^{2} \, k2 \, h1 \qquad (F\text{-}27)$$

APPENDIX G THE FAR BOUNDARY CONDITION FOR AXISYMMETRIC FLOWS

The first step is to find the equation of motion for axisymmetric flow in terms of the velocity potential. The equations of motion are the continuity equation

$$\nabla . (\rho \underline{u}) = 0 \qquad \text{(A-1a)}$$

Bernoulli's equation

$$\frac{c^2}{\gamma - 1} + \frac{1}{2} q^2 = \text{constant} \qquad \text{(G-1)}$$

and, since the flow is irrotational

$$\nabla \times \underline{u} = 0 \qquad \text{(G-2)}$$

Since the flow is isentropic

$$C_\infty = \frac{U}{M_\infty} \quad \text{and}$$

$$\frac{c^2}{C_\infty^2} = \frac{p}{p_\infty} \cdot \frac{\rho_\infty}{\rho} = \left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1}$$

or

$$c^2 = \left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} \frac{U^2}{M_\infty^2}.$$

giving for (G-1)

$$\frac{U^2 \rho^{\gamma-1}}{\rho_\infty^{\gamma-1}(\gamma-1) M_\infty^2} + \frac{1}{2} |q|^2 = \frac{U^2}{(\gamma-1) M_\infty^2} + \frac{U^2}{2} \qquad \text{(G-3)}$$

We can expand (A-1a) and (G-2) in cylindrical polar coordinates (x, R, ξ), and using the condition of axial symmetry we obtain

$$\frac{\partial}{\partial R}(R \rho u_R) + \frac{\partial}{\partial x}(R \rho u_x) = 0 \qquad \text{(G-4)}$$

$$\frac{\partial}{\partial x}(u_R) - \frac{\partial}{\partial R}(u_x) = 0 \qquad \text{(G-5)}$$

From (G-5) we can introduce a velocity potential ϕ so that

$$u_R = \frac{\partial \phi}{\partial R}$$

and

$$u_x = \frac{\partial \phi}{\partial x}$$

If we transform co-ordinates from (x, R) to (η_1, η_2) where $\sigma = \eta_1 + i\eta_2 = r\cos\theta + i r \sin\theta$ (G-4) becomes

$$\frac{\partial}{\partial \eta_2}(J_o R \rho u_R) + \frac{\partial}{\partial \eta_1}(J_o R \rho u_x) = 0 \qquad \text{(G-6)}$$

where Jo is the Jacobian of the transformation. If $T = \left|\frac{dz}{d\sigma}\right|$ is the transform modulus.

$$J_o = T^2$$

Since the coordinates (η_1, η_2) are orthogonal the element of length $ds = |dz|$ is given by

$$ds^2 = \rho_{11}^2 \, d\eta_1^2 + \rho_{22}^2 \, d\eta_2^2$$

since $\rho_{12} = \rho_{21} = 0$

But $ds^2 = |d\sigma|^2 = \left|\frac{dz}{d\sigma}\right|^2 |d\sigma|^2$

$$= T^2 (d\eta_1^2 + d\eta_2^2)$$

giving

$$\rho_{11} = T, \quad \rho_{22} = T$$

In the transformed plane the σ - plane the velocity components can be expressed as

$$u_{\eta_1} = h_{11} u_x + h_{12} u_R = T u_x$$

$$u_{\eta_2} = h_{21} u_x + h_{22} u_R = T u_R$$

Since $\eta_1 = r\cos\theta$ and $\eta_2 = r\sin\theta$

$$u_{\eta_1} = \cos\theta \, u_r - \frac{\sin\theta}{r} u_\theta$$

$$u_{\eta_2} = \sin\theta \, u_r + \frac{\cos\theta}{r} u_\theta$$

Thus (G-6) can be written

$$\frac{\partial}{\partial \eta_2}(T R \rho u_{\eta_2}) + \frac{\partial}{\partial \eta_1}(T R \rho u_{\eta_1}) = 0$$

Transforming again to r and θ we obtain

$$\left(\sin\theta \frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) T R \rho \left(\sin\theta \, u_r + \cos\theta \, u_\theta\right)$$
$$+\left(\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) T R \rho \left(\cos\theta \, u_r - \sin\theta \, u_\theta\right) = 0$$

which reduces to

$$\frac{\partial}{\partial r}(T R \rho u_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(T R \quad u_\theta) + \frac{1}{r} T R \rho u_r = 0$$

which is equivalent to

$$r\frac{\partial}{\partial r}(TR\rho u_r) + \frac{\partial}{\partial\theta}(TR\rho u_\theta) + TR\rho u_r = 0$$

giving finally

$$\frac{\partial}{\partial r}(rTR\rho u_r) + \frac{\partial}{\partial\Theta}(TR\rho u_\theta) = 0 \qquad \text{(G-7)}$$

Applying the same transformation to (G-5) gives

$$\frac{\partial}{\partial y_1}(Jo\, u_R) - \frac{\partial}{\partial y_2}(Jo\, u_\omega) = 0 \qquad \text{(G-8)}$$

In terms of r and θ, this is

$$\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right) T\left(\sin\theta\, u_r + \cos\theta\, u_\theta\right)$$

$$-\left(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\right) T\left(\cos\theta\, u_r - \frac{\sin\theta}{r}u_\theta\right) = 0$$

which reduces to

$$\frac{\partial}{\partial r}Tu_\theta - \frac{\partial}{\partial\theta}\left(Tu_r\right) + Tu_\theta = 0$$

giving finally

$$\frac{\partial}{\partial r}\left(rTu_\theta\right) - \frac{\partial}{\partial\theta}\left(Tu_r\right) = 0 \qquad \text{(G-9)}$$

Hence the velocity potential $\emptyset$ is now defined by

$$\left.\begin{aligned} \emptyset_r &= Tu_r \\ \emptyset_\theta &= rTu_\theta \end{aligned}\right\} \qquad \text{(G-10)}$$

Thus (G-3) and (G-7) can be written as

$$\frac{\partial}{\partial r}(rR\rho\emptyset_r) + \frac{\partial}{\partial\theta}\left(\frac{R\rho}{r}\emptyset_\theta\right) = 0 \qquad \text{(G-11)}$$

and

$$\frac{U^2\rho^{\gamma-1}}{\rho_\infty^{\gamma-1}(\gamma-1)M_\infty^2} + \frac{1}{2T^2}\left(\emptyset_r^2 + \frac{1}{r^2}\emptyset_\theta^2\right) = \frac{U^2}{(\gamma-1)M_\infty^2} + \frac{U^2}{2} \qquad \text{(G-12)}$$

Differentiating (G-12) with respect to r and θ, we obtain

$$\frac{U^2\rho^{\gamma-2}}{\rho_\infty^{\gamma-1}M_\infty^2}\frac{\partial\rho}{\partial r} - \frac{1}{T^3}\frac{\partial T}{\partial r}\left(\emptyset_r^2 + \frac{1}{r^2}\emptyset_\theta^2\right) +$$

$$\frac{1}{T^2}\left(\emptyset_r\emptyset_{rr} + \frac{1}{r^2}\emptyset_\theta\emptyset_{r\theta} - \frac{1}{r^3}\emptyset_\theta^2\right) = 0 \qquad \text{(G-13)}$$

$$\frac{U^2\rho^{\gamma-2}}{\rho_\infty^{\gamma-1}M_\infty^2}\frac{\partial\rho}{\partial\theta} - \frac{1}{T^3}\frac{\partial T}{\partial\theta}\left(\emptyset_r^2 + \frac{1}{r^2}\emptyset_\theta^2\right) + \frac{1}{T^2}\left(\emptyset_r\emptyset_{r\theta} + \frac{1}{r^2}\emptyset_\theta\emptyset_{\theta\theta}\right) = 0 \qquad \text{(G-14)}$$

Since $C^2 = \left(\frac{\rho}{\rho_\infty}\right)^{\gamma-1} \frac{U^2}{M_\infty^2}$ (G-13) and (G-14) are equivalent to

$$\frac{\partial \rho}{\partial r} = \frac{\rho}{C^2}\left[\frac{1}{T^3}\frac{\partial T}{\partial r}\left(\phi r^2 + \frac{1}{r^2}\phi\theta^2\right) - \frac{1}{T^2}\left(\phi r\,\phi rr + \frac{1}{r^2}\phi\theta\phi r\theta - \frac{1}{r^3}\phi\theta^2\right)\right] \qquad \text{(G-15)}$$

$$\frac{\partial \rho}{\partial \theta} = \frac{\rho}{C^2}\left[\frac{1}{T^3}\frac{\partial T}{\partial \theta}\left(\phi r^2 + \frac{1}{r^2}\phi\theta^2\right) - \frac{1}{T^2}\left(\phi r\phi r\theta + \frac{1}{r^2}\phi\theta\phi\theta\theta\right)\right] \qquad \text{(G-16)}$$

Expanding (G-11) we have

$$R\rho\phi r + r\frac{\partial R}{\partial r}\rho\phi r + rR\frac{\partial \rho}{\partial r}\phi r + rR\rho\phi rr + \rho\frac{\partial R}{\partial \theta}\phi\theta + \frac{R}{r}\frac{\partial \rho}{\partial \theta}\phi\theta + \frac{R\rho}{r}\phi\theta\theta = 0 \qquad \text{(G-17)}$$

Substituting for $\frac{\partial \rho}{\partial r}$ and $\frac{\partial \rho}{\partial \theta}$ from (G-15) and (G-16), (G-17) becomes

$$C^2R\phi r + C^2 r\frac{\partial R}{\partial r}\phi r + r^2 R\phi r\left[\frac{1}{T^3}\frac{\partial T}{\partial r}\left(\phi r^2 + \frac{1}{r^2}\phi\theta^2\right) - \frac{1}{T^2}\left(\phi r\phi rr + \frac{1}{r^2}\phi\theta\phi r\theta - \frac{1}{r^3}\phi\theta^3\right)\right] + C^2 rR\phi rr +$$
$$\frac{C^2}{r}\frac{\partial R}{\partial \theta}\phi\theta + \frac{R\phi\theta}{r}\left[\frac{1}{T^3}\frac{\partial T}{\partial \theta}\left(\phi r^2 + \frac{1}{r^2}\phi\theta^2\right) - \frac{1}{T^2}\left(\phi r\phi r\theta + \frac{1}{r^2}\phi\theta\phi\theta\theta\right)\right]$$
$$+ \frac{C^2R}{r}\phi\theta\theta = 0$$

or alternatively

$$C^2\left[\phi r\left(R + \frac{r\partial R}{\partial r}\right) + \frac{1}{r}\frac{\partial R}{\partial \theta}\phi\theta\right] + rR\phi rr\left(C^2 - \frac{1}{T^2}\phi r^2\right)$$
$$- \frac{2R}{rT^2}\phi r\phi\theta\phi r\theta + \frac{R}{r}\phi\theta\theta\left(C^2 - \frac{1}{r^2T^2}\phi_\theta^2\right) + \frac{R}{r^2T^2}\phi r\phi\theta^2$$
$$+ \frac{R}{T^3}\left(\phi r^2 + \frac{1}{r^2}\phi\theta^2\right)\left(r\frac{\partial T}{\partial r}\phi r + \frac{1}{r}\frac{\partial T}{\partial \theta}\phi\theta\right) = 0 \qquad \text{(G-18)}$$

For large $|\sigma|$ we can assume that the transformation from the symmetric cross-section in the z-plane to the unit circle in the σ-plane has the form

$$z = \sigma + a_0 + \frac{a_1}{\sigma} + \frac{a_2}{\sigma^2} + \ldots\ldots \qquad \text{(G-19)}$$

as in the two dimensional case.

Thus, the transform modulus T^2 and its derivatives still take the form of (E-8) and (E-9). Splitting (G-19) into real and imaginary parts we have

$$X + i R \simeq r e^{i\theta} + a_o + \frac{a_1}{r} e^{-i\theta}$$

$$= r \cos\theta + a_o + \frac{a_1}{r} \cos\theta + i \sin\theta \left(r - \frac{a_1}{r}\right)$$

giving

$$R \simeq \left(r - \frac{a_1}{r}\right) \sin\theta \qquad \text{(G-20)}$$

and
$$\frac{\partial R}{\partial r} \simeq \left(1 + \frac{a_1}{r^2}\right) \sin\theta$$

$$\frac{\partial R}{\partial \theta} \simeq \left(r - \frac{a_1}{r}\right) \cos\theta$$

Thus the equation of motion (G-18) becomes

$$U^2 \frac{1}{M_\infty^2} + \frac{(\gamma - 1)}{2} \left(1 - \frac{\phi_r^2 + \phi_\theta^2/r^2}{U^2 T^2}\right)\Bigg] \Bigg[2 r \sin\theta \, \phi_r$$

$$+ \left(1 - \frac{a_1}{r^2}\right) \cos\theta \, \phi_\theta + r^2 \left(1 - \frac{a_1}{r^2}\right) \sin\theta \, \phi_{rr}$$

$$+ \left(1 - \frac{a_1}{r^2}\right) \sin\theta \, \phi_{\theta\theta} + \frac{1}{T^2} \left(1 - \frac{a_1}{r^2}\right) \sin\theta$$

$$\Bigg[- r^2 \phi_{rr} \phi_r^2 - 2 \phi_r \phi_\theta \phi_{r\theta} - \frac{1}{r^2} \phi_\theta^2 \phi_{\theta\theta}$$

$$+ \frac{1}{r} \phi_r \phi_\theta^2 \Bigg] + \frac{1}{T^4} \left(1 - \frac{a_1}{r^2}\right) \sin\theta \left(\phi_r^2 + \frac{1}{r^2} \phi_\theta^2\right)$$

$$\Bigg[\frac{2 a_1}{r} \cos 2\theta \, \phi_r + \frac{2 a_1}{r^2} \sin 2\theta \, \phi_\theta \Bigg] = 0 \qquad \text{(G-21)}$$

We assume that for large $|\gamma|$ and hence large r that we can write ϕ in the form

$$\phi = U \left(r \cos\theta + \frac{f(\theta)}{r} \right) \qquad \text{(G-22)}$$

giving

$$\phi_r = U\left(\cos\theta - \frac{f}{r^2}\right)$$

$$\phi_{rr} = \frac{2Uf}{r^3}$$

$$\phi_\theta = U\left(-r\sin\theta + \frac{f'}{r}\right)$$

$$\phi_{\theta\theta} = U\left(-r\cos\theta + \frac{f''}{r}\right)$$

$$\phi_{r\theta} = U\left(-\sin\theta - \frac{f'}{r^2}\right)$$

Substituting into (G-21) gives

$$\frac{1}{M_\infty^2} + \frac{\gamma-1}{2}\left(-\frac{2al}{r^2}\cos 2\theta\right) - \frac{(\gamma-1)}{2}\left(1 + \frac{2al}{r^2}\cos 2\theta\right)\left\{-\frac{2\cos\theta}{r^2}f\right.$$

$$\left.+\frac{f^2}{r^4} - \frac{2\sin\theta}{r^2}f' + \frac{f'^2}{r^4}\right\}\Bigg]\Bigg[2r\sin\theta\left(\cos\theta - \frac{f}{r^2}\right)$$

$$+\left(1-\frac{al}{r^2}\right)\cos\theta\left(-r\sin\theta + \frac{f'}{r}\right) + \left(1-\frac{al}{r^2}\right)\frac{2f}{r}\sin\theta$$

$$+\left(1-\frac{al}{r^2}\right)\sin\theta\left(-r\cos\theta + \frac{f''}{r}\right)\Bigg] + \left(1+\frac{2al}{r^2}\cos 2\theta\right)\left(1-\frac{al}{r^2}\right)$$

$$\sin\theta\left[-\frac{2f}{r}\left(\cos^2\theta - \frac{2f\cos\theta}{r^2} + \frac{f^2}{r^4}\right) + 2\left(-r\sin\theta\cos\theta\right.\right.$$

$$\left.+\frac{f\sin\theta}{r} + \frac{f'\cos\theta}{r} - \frac{ff'}{r^3}\right)\left(\sin\theta + \frac{f'}{r^2}\right)$$

$$\left.-\left(\sin^2\theta - \frac{2f'}{r^2}\sin\theta + \frac{f'^2}{r^4}\right)\left(-2r\cos\theta + \frac{f''}{r} + \frac{f}{r}\right)\right]$$

$$+\left(1+\frac{4al}{r^2}\cos 2\theta\right)\left(1-\frac{al}{r^2}\right)\sin\theta\left(1 - \frac{2\cos\theta f}{r^2} + \frac{f^2}{r^4} - \frac{2\sin\theta f'}{r^2}\right.$$

$$\left.+\frac{f'^2}{r^4}\right)\left[\frac{2al}{r}\cos 2\theta\left(\cos\theta - \frac{f}{r^2}\right) + \frac{2al}{r}\sin 2\theta\left(-\sin\theta + \frac{f'}{r^2}\right)\right] = 0$$

Comparing coefficients of r^{-1} we obtain the following equation for $f(\theta)$

$$\frac{1}{M_\infty^2}\left[f''\sin\theta + f'\cos\theta + 2al\sin\theta\cos\theta\right]$$

$$-2f\sin\theta\cos^2\theta + f\sin^3\theta - f''\sin^3\theta - 4f'\sin^2\theta\cos\theta$$

$$+2al\sin\theta\cos 3\theta = 0$$

or on simplifying

$$f(1 - M_\infty^2\sin^2\theta) + f'\cos\theta\,(1 - 4M_\infty^2\,\mathrm{Sin}^2\theta$$
$$+ M_\infty^2 f\sin\theta\,(3\sin^2\theta - 2)$$
$$= -al\,(\sin 2\theta + 2M_\infty^2\,\mathrm{Cos}\,3\theta\sin\theta)$$

A solution of the corresponding homogeneous equation

$$\sin\theta\,(1-M_\infty^2\sin^2\theta)\,f''(\theta)+\cos\theta\,(1-4M_\infty^2\sin^2\theta)\,f1'(\theta)$$
$$+M_\infty^2\sin\theta\,(3\sin^2\theta-2)\,f(\theta)=0 \qquad \text{(G-24)}$$

is

$$f1(\theta)=\frac{1}{(1-M_\infty^2\sin^2\theta)^{\frac{1}{2}}}$$

which can be verified by substitution in (G-24). Thus the general solution of (G-23) is given by

$$f(\theta)=f1(\theta)\,h(\theta)$$

Substituting into (G-23) we have

$$\sin\theta\,(1-M_\infty^2\sin^2\theta)\,(f1''h+2f1'h'+f1\,h'')+$$
$$\cos\theta\,(1-4M_\infty^2\sin^2\theta)\,(f1'h+h'f1)+M_\infty^2\sin\theta\,(3\sin^2\theta-2)f1h$$
$$=-2a1\sin\theta\cos\theta\,(1+M_\infty^2-4M_\infty^2\sin^2\theta)$$

which simplifies to

$$\sin\theta\,(1-M_\infty^2\sin^2\theta)^{\frac{1}{2}}h''+\cos\theta\,(1-2M_\infty^2\sin^2\theta)(1-M_\infty^2\sin^2\theta)^{-\frac{1}{2}}h'$$
$$=2a1\sin\theta\cos\theta\,(1+M_\infty^2-4M_\infty^2\sin^2\theta)$$

Integrating we obtain

$$\sin\theta\,(1-M_\infty^2\sin^2\theta)^{\frac{1}{2}}h'(\theta)=B-a1(1+M_\infty^2)\sin^2\theta+2a1\,M_\infty^2\sin^4\theta$$

where B is a constant. Hence

$$h'(\theta)=\frac{B}{\sin\theta\sqrt{1-M_\infty^2\sin^2\theta}}-\frac{a1(1+M_\infty^2)\sin\theta}{(1-M_\infty^2\sin^2\theta)^{\frac{1}{2}}}+\frac{2a1\,M_\infty^2\sin^3\theta}{(1-M_\infty^2\sin^2\theta)^{\frac{1}{2}}}$$

Integrating again gives

$$h(\theta)=A'+\frac{B}{2}\log\left(\frac{(1-M_\infty^2\sin^2\theta)^{\frac{1}{2}}-\cos\theta}{(1-M_\infty^2\sin^2\theta)^{\frac{1}{2}}+\cos\theta}\right)+a1\cos\theta\,\left(1-M_\infty^2\sin^2\theta\right)^{\frac{1}{2}}$$
$$+\frac{a1(1+M_\infty^2)}{2M_\infty}\log\left(\frac{(1-M_\infty^2\sin^2\theta)^{\frac{1}{2}}-M_\infty\cos\theta}{(1-M_\infty^2\sin^2\theta)^{\frac{1}{2}}+M_\infty\cos\theta}\right)$$
$$+\frac{a1\,(1+M_\infty^2)}{M_\infty}\log\left(1-M_\infty^2\sin^2\theta+M_\infty\cos\theta\right)$$

where A' is a constant of integration.

This expression simplifies to

$$h(\theta) = A + \frac{B}{2} \log \left(\frac{(1-M_\infty^2 \sin^2 \theta)^{\frac{1}{2}} - \cos \theta}{(1-M_\infty^2 \sin^2 \theta)^{\frac{1}{2}} + \cos \theta} \right) + al \cos \theta \left(1-M_\infty^2 \sin^2 \theta\right)^{\frac{1}{2}}$$

where

$$A = A' + \frac{al}{2M_\infty} (1+M_\infty^2) \log (1-M_\infty^2)$$

Thus the general solution of (G-23) is given by

$$f(\theta) = \frac{A}{(1-M_\infty^2 \sin^2 \theta)^{\frac{1}{2}}} + \frac{B}{2\left(1-M_\infty^2 \sin^2 \theta\right)} \log\left(\frac{(1-M_\infty^2 \sin^2 \theta)^{\frac{1}{2}} - \cos \theta}{(1-M_\infty^2 \sin^2 \theta)^{\frac{1}{2}} + \cos \theta}\right)$$
$$+ \, al \cos \theta$$

For $f(\theta)$ to remain finite at $\theta = 0$ and $\theta = \pi$ we must have $B = 0$. Thus a two term series expansion for flow past a symmetric body is

$$\phi = U\left(r \cos \theta + \frac{1}{r}\left\{\frac{A}{(1 - M_\infty^2 \sin^2 \theta)^{\frac{1}{2}}} + al \cos \theta\right\}\right) \qquad \text{(G-2?)}$$

The leastsquares method is again used to find the value of A. At the last but one grid point the sum of squares of the difference between the series for χ and the perturbation potential $\chi_{i,n-1}$ is given by

$$D^2 = \sum_{i=1}^{l}\left(\frac{A}{rl\,(1-M^2 \sin^2 \theta)^{\frac{1}{2}}} + \frac{al \cos \theta i}{rl} - \chi_{i,n-1}\right)^2$$

To find the value of A which minimizes D^2 we require

$$\frac{\partial D^2}{\partial A} = \frac{1}{r_{n-1}} \sum_{i=1}^{l} \frac{1}{(1-M_\infty^2 \sin^2 \theta i)}\left(\frac{A}{rl\,(1-M^2 \sin \theta i)^{\frac{1}{2}}} + \frac{al \cos \theta i}{rl} - \chi_{i,n-1}\right) = 0$$

Hence

$$A = r_{n-1} \frac{\sum_{i=1}^{l} \left(al \cos \theta i / rl \,(1-M^2 \sin^2 \theta i) + \chi_{n-1} / (1-M^2 \sin^2 \theta i)\right)}{\sum_{i=1}^{l} 1 / \,(1-M^2 \sin^2 \theta i)}$$

Also

$$\frac{\partial^2 D^2}{\partial A^2} = \sum_{i=1}^{l} \frac{1}{r_{n-1}^2 (1-M_\infty^2 \sin^2 \theta i)} > 0$$

so the value of A obtained gives a minimum sum of squares. Hence the values of χ at a distance $r_{n-1} + h$ from the body can be found.

al takes the same values for the various bodies of revolution, as for the equivalent two-dimensional bodies.

When the flow past a sphere is being investigated no transformation is required and we have

$$x = r \cos \theta$$

$$R = r \sin \theta$$

In this case (G-18) is equivalent to

$$C^2 \left[2r \sin \theta \, \phi r + \cos \theta \, \phi \, \theta\right] + r^2 \sin \theta \, \phi rr \left(C^2 - \phi r^2\right) - 2 \sin \theta \, \phi r \, \phi \theta \, \phi r\theta + \sin \theta \, \phi \theta\theta \left(C^2 - \frac{1}{r^2} \phi \theta^2\right) + \frac{\sin \theta}{r} \, \phi \, r \, \phi \theta^2 = 0$$

or

$$C^2 \left[2 \, r \sin \theta \, \phi \, r + \cos \theta \, \phi \theta + r^2 \sin \theta \, \phi rr + \sin \theta \, \phi \theta\theta\right. - r^2 \sin \theta \, \phi r^2 \phi rr - 2 \sin \theta \, \phi r \phi \theta \phi r\theta - \frac{\sin \theta}{r} \phi \theta^2 \left(\frac{\phi \theta\theta}{r} - \phi r\right) = 0 \qquad \text{(G-26)}$$

Since the incompressible solution has a second term $O(r^{-2})$ we expect the far boundary solution to have a similar form. Thus we take

$$\phi = U \left(r \cos \theta + \frac{f(\theta)}{r2} \right)$$

so

$$\phi r = U \left(\cos \theta - \frac{2f}{r^3} \right)$$

$$\phi rr = \frac{6}{r^4} U f$$

$$\phi \theta = U \left(- r \sin \theta + \frac{f'}{r^2} \right)$$

$$\phi \theta\theta = U \left(- r \cos 0 + \frac{f''}{r^2} \right)$$

$$\phi r\theta = U \left(- \sin \theta - \frac{2f'}{r^3} \right)$$

Substituting these values into (G-26), we obtain, remembering that for the circle

$$C^2 = U^2 \left\{ \frac{1}{M_\infty^2} + \frac{\gamma - 1}{2} \left(\frac{1 - \phi r^2 + \phi \theta^2 / r^2}{U^2} \right) \right\}$$

the following differential equation for f (θ)

$$\left[\frac{1}{M_\infty^2} - \frac{\gamma-1}{2}\left(-\frac{4f\cos\theta}{r^3} + \frac{4f}{r^6} - \frac{2\sin\theta f'}{r^3} + \frac{f'^2}{r^6}\right)\right]$$

$$\times\left[\frac{2f\sin\theta}{r^2} + \frac{f'\cos\theta}{r^2} + \frac{f''}{r^2}\sin\theta\right] - \frac{6\sin\theta}{r^2}f\left(\cos^2\theta\right.$$

$$\left. - \frac{4f\cos\theta}{r^3} + \frac{4f^2}{r^6}\right) + 2\sin\theta\left(\sin\theta + \frac{2f'}{r^3}\right)\left(-r\sin\theta\cos\theta\right.$$

$$\left. + \frac{2f\sin\theta}{r^2} + \frac{f'\cos\theta}{r^2} - \frac{2ff'}{r^5}\right) - \frac{\sin\theta}{r}\left(r^2\sin^2\theta - \frac{2f'\sin\theta}{r}\right.$$

$$\left. + \frac{f'^2}{r^4}\right)\left[-2\cos\theta + \frac{f''}{r^3} + \frac{2f}{r^3}\right] = 0 \qquad \text{(G-28)}$$

Comparing coefficientsof r^{-2} we obtain

$$f''\sin\theta\,(1 - M_\infty^2\sin^2\theta) + f'\cos\theta\,(1 - 6M_\infty^2\sin^2\theta)$$
$$+ 2f\sin\theta\left[1 + M_\infty^2(4\sin^2\theta - 3)\right] = 0 \qquad \text{(G-29)}$$

after multiplying by M_∞^2

A solution of this equation is

$$f1(\theta) = \frac{\cos\theta}{(1 - M_\infty^2\sin^2\theta)^{3/2}}$$

which can be verified by substitution in (G-29). Thus the general solution of (G-29) is given by

$$f(\theta) = f1(\theta)\,h(\theta)$$

Substituting into (G-29) we have

$$g\left\{\sin\theta\,(1 - M_\infty^2\sin^2\theta)\,f1'' + f1'\cos\theta\,(1-6M_\infty^2\sin^2\theta) + \right.$$
$$\left. 2\,f1\sin\theta\left[1+M_\infty^2(4\sin^2\theta-3)\right]\right\} + g'\left\{2f1'\sin\theta\,(1-M_\infty^2\sin^2\theta)\right.$$
$$\left. + f1\cos\theta\,(1-6M_\infty^2\sin^2\theta)\right\} + h''\,f1\sin\theta\,(1 - M_\infty^2\sin^2\theta) = 0$$

which sim lifies to

$$\frac{h''\sin\theta\cos\theta}{(1-M_\infty^2\sin^2\theta)^{\frac{1}{2}}} - \frac{2h'\sin^2\theta}{(1-M_\infty^2\sin^2\theta)^{\frac{1}{2}}} = 0$$

or

$$h''(\theta)\cos^2\theta - 2\sin\theta\cos\theta\,h'(\theta) = 0$$

Thus, on integrating

$$\cos^2\theta\, h'(\theta) = B$$

where B is a constant.

Integrating again we have

$$h(\theta) = A + B\tan\theta$$

Thus the general solution of (G-29) is

$$f(\theta) = \frac{A\cos\theta + B\sin\theta}{(1-M_\infty^2\sin^2\theta)^{3/2}}$$

Since the solution for ϕ must be symmetric about $\theta = 0$ and $\theta = \pi$, we must have $B = 0$. Thus a two term series expansion for flow past a sphere is

$$\phi = U\left(r\cos\theta + \frac{A\cos\theta}{r^2(1-M_\infty^2\sin^2\theta)^{3/2}}\right)$$

As in the two-dimensional case and for a more general axisymmetric body, the least squares method is used to find the value of A. At the last but one grid point, the sum of squares of the difference between the series for χ and the perturbation potential $\chi_{i,n-1}$ is given by

$$D^2 = \sum_{i=1}^{\ell}\left(\frac{A\cos\theta_i}{r_{n-1}(1-M_\infty^2\sin^2\theta_i)^{3/2}} - \chi_{i,n-1}\right)$$

To find the value of A which makes this sum a minimum, we require

$$\frac{dD^2}{dA} = \sum_{i=1}^{\ell}\frac{2\cos\theta_i}{r_{n-1}^2(1-M_\infty^2\sin^2\theta_i)^{3/2}}\left(\frac{A\cos\theta_i}{r_{n-1}^2(1-M_\infty^2\sin^2\theta_i)^{3/2}} - \chi_{i,n-1}\right) = 0$$

Thus

$$A = r_{n-1}^2\,\frac{\displaystyle\sum_{i=1}^{\ell}\frac{\chi_{i,n-1}\cos\theta_i}{(1-M_\infty^2\sin^2\theta_i)^{3/2}}}{\displaystyle\sum_{i=1}^{\ell}\frac{\cos^2\theta_i}{(1-M_\infty^2\sin^2\theta_i)^3}}$$

Also

$$\frac{d^2D^2}{dA} = \sum_{i=1}^{\ell}\frac{2\cos^2\theta_i}{r_{n-1}^4(1-M_\infty^2\sin^2\theta_i)^3} > 0$$

for $M_\infty < 1$, so the value of A, obtained above, does give a minimum sum of squares. Hence the values of χ can be found at a distance r_n from the body.

APPENDIX H THE CONTRIBUTION TO THE VARIATIONAL INTEGRAL, FROM EACH RECTANGLE IN AXISYMMETRIC FLOW

Comparison of the expression (3.4) for plane flows with the expressions (6.4) and (6.5) for axisymmetric flows shows that the expressions for the Ai's, Bi's and Ci's are the same in both cases, but the Di's, Ei's and Hi's will be different.

If the form (6.4) is used by a similar method to Appendix F, we have that in rectangle 1 of figure 3.

$$D1 = \frac{\gamma M_\infty^2}{2T_1^2}\left(\chi_{or}\Big|_{\substack{r=r_1\\ \theta=\theta_1}} \frac{1}{k1} + \chi_{o\theta}\Big|_{\substack{r=r_1\\ \theta=\theta_1}} \frac{1}{r1^2 h1}\right) \qquad \text{(H-1)}$$

$$E1 = -1 + \frac{\gamma M_\infty^2}{2T1^2}\left(\chi_{or}\Big|_{\substack{r=r_1\\ \theta=\theta_1}} \frac{X2}{k1} + \chi_{o\theta}\Big|_{\substack{r=r_1\\ \theta=\theta_1}} \frac{X1}{r1^2 h1}\right) \qquad \text{(H-2)}$$

and

$$H1 = r1\ R11\ T1^2\ k_1\ h1 \qquad \text{(H-3)}$$

where the Xi have the same form as in Appendix F and R11 is the value of R at r = r1, θ = θ1.

In the rectangle 2, we obtain

$$D2 = -\frac{\gamma M_\infty^2}{2\,T_2^2}\left(\chi_{or}\Big|_{\substack{r=r_1\\ \theta=\theta_2}} \frac{1}{k1} - \chi_{o\theta}\Big|_{\substack{r=r_1\\ \theta=\theta_2}} \frac{1}{r_1^2\ h_2}\right) \qquad \text{(H-4)}$$

$$E2 = -1 + \frac{\gamma M_\infty^2}{2\,T_2^2}\left(\chi_{or}\Big|_{\substack{r=r_1\\ \theta=\theta_2}} \frac{X4}{k1} + \chi_{o\theta}\Big|_{\substack{r=r_1\\ \theta=\theta_2}} \frac{X5}{r_1^2 h_2}\right) \qquad \text{(H-5)}$$

and

$$H2 = r_1\ R_{12}\ T_2^2\ k_1\ h_2 \qquad \text{(H-6)}$$

where R_{12} is the value of R at $r = r_1$, $\theta = \theta_2$.

For rectangle 3, we have

$$D_3 = \frac{\gamma M_\infty^2}{2\,T_3^2}\left(\chi_{or}\Big|_{\substack{r=r_2\\ \theta=\theta_2}} \frac{1}{k2} + \chi_{o\theta}\Big|_{\substack{r=r_2\\ \theta=\theta_2}} \frac{1}{r_2^2\ h_2}\right) \qquad \text{(H-7)}$$

$$E_3 = -1 + \frac{\gamma M_\infty^2}{2T_3^2}\left(\chi_{or}\Big|_{\substack{r=r_2\\ \theta=\theta_2}} \frac{X6}{2} + \chi_{o\theta}\Big|_{\substack{r=r_2\\ \theta=\theta_2}} \frac{X7}{r_2^2\ h_2}\right) \qquad \text{(H-8)}$$

and

$$H_3 = r_2\ R_{22}\ T_3^2\ k_2\ h_2 \qquad \text{(H-9)}$$

where R_{22} is the value of R at $r = r_2$, $\theta = \theta_2$.

For rectangle 4, we have

$$D_4 = \frac{\gamma M_\infty^2}{2T_4}\left(\chi_{or}\Big|_{\substack{r=r_2\\ \theta=\theta_1}} \frac{1}{k_2} - \chi_{o\theta}\Big|_{\substack{r=r_2\\ \theta=\theta_1}} \frac{1}{r_2^2 h_1}\right) \qquad \text{(H-10)}$$

$$E_4 = -1 + \frac{\gamma M_\infty^2}{2T_4^2}\left(\chi_{or}\Big|_{\substack{r=r_2\\ \theta=\theta_1}} \frac{X8}{k_2} + \chi_{o\theta}\Big|_{\substack{r=r_2\\ \theta=\theta_1}} \frac{X7}{r_2^2 h_1}\right) \qquad \text{(H-11)}$$

and

$$H_4 = r_2 R_{21} T_4^2 k_2 h_1 \qquad \text{(H-12)}$$

where R_{21} is the value of R at $r = r_2$ $\theta = \theta_1$.

If the expression (6.5) is used rather than (6.4) the Hi are still the same but the expressions for Di and Ei need alteration.

For rectangle 1 we now have

$$D_1 = \frac{-\gamma M_\infty^2}{2}\left(\frac{\partial r}{\partial x}\Big|_{\substack{r=r_1\\ \theta=\theta_1}} \frac{1}{k_1} + \frac{\partial \theta}{\partial x}\Big|_{\substack{r=r_1\\ \theta=\theta_1}} \frac{1}{r_1^2 h_1}\right) \qquad \text{(H-13)}$$

and

$$E_1 = -1 + \frac{\gamma M_\infty^2}{2}\left(\frac{\partial r}{\partial x}\Big|_{\substack{r=r_1\\ \theta=\theta_1}} \frac{X2}{k_1} + \frac{\partial \theta}{\partial x}\Big|_{\substack{r=r_1\\ \theta=\theta_1}} \frac{X1}{r_1^2 h_1}\right) \qquad \text{(H-14)}$$

Similarly for rectangle 2, we obtain

$$D_2 = -\frac{\gamma M_\infty^2}{2}\left(\frac{\partial r}{\partial x}\Big|_{\substack{r=r_1\\ \theta=\theta_2}} \frac{1}{k_1} - \frac{\partial \theta}{\partial x}\Big|_{\substack{r=r_1\\ \theta=\theta_2}} \frac{1}{r_1^2 h_2}\right) \qquad \text{(H-15)}$$

and

$$E_2 = -1 + \frac{\gamma M_\infty^2}{2}\left(\frac{\partial r}{\partial x}\Big|_{\substack{r=r_1\\ \theta=\theta_2}} \frac{X4}{k_1} + \frac{\partial \theta}{\partial x}\Big|_{\substack{r=r_1\\ \theta=\theta_2}} \frac{X3}{r_1^2 h_2}\right) \qquad \text{(H-16)}$$

In rectangle 3, the expressions are

$$D_3 = \frac{\gamma M_\infty^2}{2}\left(\frac{\partial r}{\partial x}\Big|_{\substack{r=r_2\\ \theta=\theta_2}} \frac{1}{k_2} + \frac{\partial \theta}{\partial x}\Big|_{\substack{r=r_2\\ \theta=\theta_2}} \frac{1}{r_2^2 h_2}\right) \qquad \text{(H-17)}$$

and

$$E_3 = -1 + \frac{\gamma M_\infty^2}{2}\left(\frac{\partial r}{\partial x}\Big|_{\substack{r=r_2\\ \theta=\theta_2}} \frac{X6}{k_2} + \frac{\partial \theta}{\partial x}\Big|_{\substack{r=r_2\\ \theta=\theta_2}} \frac{X5}{r_2^2 h_2}\right) \qquad \text{(H-18)}$$

Finally for rectangle 4, we have

$$D_4 = \frac{\gamma M_\infty^2}{2}\left(\left.\frac{\partial r}{\partial x}\right|_{\substack{r=r_2\\ \theta=\theta_1}} \frac{1}{k_2} - \left.\frac{\partial \theta}{\partial x}\right|_{\substack{r=r_2\\ \theta=\theta_1}} \frac{1}{r_2^{\,2} h_1}\right) \qquad \text{(H-19)}$$

and

$$E_4 = -1 + \frac{\gamma M_\infty^2}{2}\left(\left.\frac{\partial r}{\partial x}\right|_{\substack{r=r_2\\ \theta=\theta_1}} \frac{X8}{k_2} + \left.\frac{\partial \theta}{\partial x}\right|_{\substack{r=r_2\\ \theta=\theta_2}} \frac{X_7}{r_2^{\,2} h1}\right) \qquad \text{(H-20)}$$

APPENDIX I SERIES SOLUTION AT THE TRAILING-EDGE OF A KARMAN-TREFFTZ PROFILE

It was stated in Section 6 that the series solution for the velocity potential of the flow past a cone had been found by Mangler (1948), in the form

$$\phi = -K \rho_c^{\nu} F (\Psi) \qquad (6.6)$$

to within a constant (see fig. 5)

Consider a Karman-Trefftz profile with the part near the trailing edge made up of a cone. The x - coordinate of the trailing edge is mk, so using polar coordinates s_c and ψ we have

$$x + iR = s_c e^{i\psi_c} = mk + \rho_c e^{i\Theta}$$
$$= mk - \rho_c e^{i\Psi}$$

giving

$$\rho_c e^{i\Psi} = -(x - mk) - iR$$
$$= -\frac{mk(\rho_1^{2m} - \rho_2^{2m})}{(\rho_1^{2m} + \rho_2^{2m} - 2\rho_1^m \rho_2^m \cos m(\phi_2 - \phi_1))} + mk$$
$$- \frac{2i\rho_1^m \rho_2^m \, mk \sin m(\phi_2 - \phi_1)}{(\rho_1^{2m} + \rho_2^{2m} - 2\rho_1^m \rho_2^m \cos m(\phi_2 - \phi_1))}$$

from Appendix C.

$$= \frac{-2mk\rho_2^m}{(\rho_1^{2m} + \rho_2^{2m} - 2\rho_1^m\rho_2^m \cos m(\phi_2 - \phi_1))}\Big[(\rho_1^m \cos m(\phi_2 - \phi_1) - \rho_2^m) + i\rho_1^m \sin m(\phi_2 - \phi_1)\Big]$$

giving

$$\rho_c = \frac{-2mk\rho_2^m}{(\rho_1^{2m} + \rho_2^{2m} - 2\rho_1^m\rho_2^m \cos m(\phi_2 - \phi_1))^{\frac{1}{2}}}$$

We require the form of ρ near the trailing edge, so we take $r = 1 + \epsilon$ and θ small.

Then

$$\rho_1^2 = r^2 - 2(b_k - k) r \cos\theta + (b_k - k)^2$$
$$\simeq 1 + 2\epsilon + \epsilon^2 - 2(b_k - k)(1 + \epsilon)(1 - \frac{\theta^2}{2}) + (b_k - k)^2$$
$$= 4k^2 + 4k\epsilon + \frac{\epsilon^2}{4k^2} + (b_k - k)\theta^2$$

so

$$\rho_1 \simeq 2k\left(1 + \frac{\epsilon}{k} + \frac{\epsilon^2}{4k^2} + \frac{(b_k - k)}{4k^2}\theta^2\right)^{\frac{1}{2}}$$

$$\simeq 2k\left(1 + \frac{\epsilon}{2k} + \frac{(b_k - k)\,\theta^2}{4k^2}\right)$$

Also

$$\begin{aligned}\rho_2^2 &= r^2 - 2r\cos\theta + 1 \\ & \quad 1 + 2\epsilon + \epsilon^2 - 2(1+\epsilon)\left(1 - \frac{\theta^2}{2}\right) + 1 \\ &= \epsilon^2 + \theta^2\end{aligned}$$

Thus

$$\rho_2 \simeq (\epsilon^2 + \theta^2)^{\frac{1}{2}}$$

Therefore, substituting into the expression for ρ_c, we have

$$\rho_c \simeq 2mk\,(\epsilon^2+\theta^2)^{\frac{m}{2}}\left[(2k)^{2m}\left(1 + \frac{m\epsilon}{k} + \frac{m\,(b_k - k)\,\theta^2}{2k^2} + \frac{m(2m-1)}{4k^2}\epsilon^2\right) + (\epsilon^2+\theta^2)^m - 2\,(2k)^m\left(1 + \frac{m\epsilon}{2k}\right)(\epsilon^2+\theta^2)^{\frac{m}{2}}\cos m(\phi_2 - \phi_1)\right]^{-\frac{1}{2}}$$

$$\simeq \frac{2\,mk\,(\epsilon^2+\theta^2)^{\frac{m}{2}}}{(2k)^m}\left[1 + \frac{m\epsilon}{k}\right]^{-\frac{1}{2}}$$

$$\simeq \frac{m\,(\epsilon^2+\theta^2)^{\frac{m}{2}}}{(2k)^{m-1}} \qquad \text{(I - 1)}$$

Now

$$\cos\Psi = \frac{-(x - mk)}{\rho_c}$$

$$= -\left(\rho_1^m\cos m(\phi_2-\phi_1) - \rho_2^m\right)\left[\rho_1^{2m} + \rho_2^{2m} - 2\rho_1^m\rho_2^m\cos m(\phi_2-\phi_1)\right]^{-\frac{1}{2}}$$

$$\simeq -\left((2k)^m\left(1 + \frac{m\epsilon}{2k} + \frac{m\,(b_k-k)\,\theta^2}{4k^2} + \frac{m\,(m-1)\,\epsilon^2}{8k^2}\right)\cos m(\phi_2-\phi_1) - (\epsilon^2+\theta^2)^{\frac{m}{2}}\right)\left[(2k)^{2m}\left(1 + \frac{m\epsilon}{k} + m\,\frac{(b_k-k)}{2k^2}\theta^2 + \frac{m\,(2m-1)}{4k^2}\epsilon^2\right) + (\epsilon^2+\theta^2)^m - 2\,(2k)^m\left(1 + \frac{m\epsilon}{2k}\right)(\epsilon^2+\theta^2)^{\frac{m}{2}}\cos m\,(\phi_2-\phi_1)\right]^{-\frac{1}{2}}$$

$$\simeq \left(\left(1 + \frac{m\epsilon}{2k} + m\,\frac{(b_k-k)\,\theta^2}{4k^2} + \frac{m\,(m-1)\,\epsilon^2}{8k^2}\cos m\,(\phi_2-\phi_1)\right.\right.$$

$$- \frac{(\epsilon^2 + \theta^2)^{\frac{m}{2}}}{(2k)^m} \left[1 - \frac{m\epsilon}{2k} - \frac{m(b_k - k)\theta^2}{4k^2} - \frac{m(2m-1)}{8k^2}\epsilon^2 \right.$$

$$\left. + \frac{3m^2\epsilon^2}{8k^2} + \frac{1}{(2k)^m} (\epsilon^2 + \theta^2)^{\frac{m}{2}} \cos m(\phi 2 - \phi 1) \right]$$

$$\simeq - \cos m(\phi_2 - \phi_1) + \frac{(\epsilon^2 + \theta^2)^{\frac{m}{2}}}{(2k)^m} \sin^2 m(\phi_2 - \phi 1)$$

Mangler and Leuteritz, in work mentioned by Mangler (1948), wrote (6.6) in the form

$$\phi + Cc = - K \rho_c^{\nu} \eta_{\nu}(\Omega) \qquad \text{(I-2)}$$

where Cc is a constant

$$\eta_{\nu}(\Omega) = F(\Psi)$$

and

$$\Omega = \cos \Psi$$

Now, from the work referred to in Mangler (1948)

$$\eta_{\nu}(\Omega) = \sum_{n=0}^{\infty} \frac{b_n (\Omega + 1)^n}{n!}$$

where the constants b_n are given by the relation

$$\frac{b_{n+1}}{b_n} = \frac{n(n+1) - \nu(\nu+1)}{2(n+1)}$$

which has the solution

$$b_n = \prod_{k=0}^{n-1} \frac{k(k+1) - \nu(\nu+1)}{2(k+1)} b_0$$

and b_0 can be taken as 1. Thus

$$b_n = (-\tfrac{1}{2})^n \prod_{k=0}^{n-1} \frac{\nu(\nu+1) - k(k+1)}{(k+1)} \qquad n=1,2 \ldots.$$

and

$$\eta_{\nu}(\Omega) = 1 + \sum_{n=1}^{\infty} \left(-\frac{(\Omega+1)}{2} \right)^n \prod_{k=0}^{n-1} \frac{\nu(\nu+1) - k(k+1)}{(k+1)^2}$$

Hence

$$\phi + Cc \simeq K \left[\frac{m(\epsilon^2 + \theta^2)^{\frac{m}{2}}}{(2k)^{m-1}} \left[1 + \sum_{n=1}^{\infty} \left(-\frac{(\Omega+1)^n}{2} \right) \prod_{k=0}^{n-1} \frac{\nu(\nu+1) - k(k+1)}{(k+1)^2} \right. \right.$$

Since

$$\phi = U(r\cos\theta + \chi)$$

$$\chi + C_c = -r\cos\theta - \frac{K m^{\nu}(\epsilon^2 + \theta^2)^{\frac{m\nu}{2}}}{(2k)^{\nu(m-1)}}\eta_\nu(\Omega) \qquad \text{(I-3)}$$

The constants K and C_c are found by comparing the value of the potential obtained by this series with that obtained by the variational method at the third radial grid points out from the body at $\theta = 0$ and $\theta = h_{\ell-1}$

$$\chi_{\ell-1,3} = -C_c - (1 + k_1 + k_2)\cos h_{\ell-1} - \frac{K m^{\nu}(h_{\ell-1}^2 + (k_1+k_2)^2)^{\frac{m\nu}{2}}}{(2k)^{\nu(m-1)}} \times \eta_\nu(\xi_{\ell-1,3})$$

and

$$\chi_{\ell,3} = -C_c - (1 + k_1 + k_2) - \frac{K m^{\nu}(k_1 + k_2)^{m\nu}}{(2k)^{\nu(m-1)}}\eta_\nu(\Omega_{\ell,3})$$

Subtracting these two we obtain

$$\chi_{\ell-1,3} - \chi_{\ell,3} = (1+k_1+k_2)(1-\cos h_{\ell-1}) + \frac{K m^{\nu}}{(2k)^{\nu(m-1)}}\Big\{(k_1+k_2)^{m\nu}\eta_\nu(\Omega_{\ell,3}) - (h_{\ell-1}^2 + (k_1+k_2)^2)^{\frac{m\nu}{2}}\eta_\nu(\Omega_{\ell-1,3})\Big\}$$

Hence

$$K = \frac{(2k)^{\nu(m-1)}\left[\chi_{\ell-1,3} - \chi_{\ell,3} - (1+k_1+k_2)(1-\cos h_{\ell-1})\right]}{m^{\nu}\left[(k_1+k_2)^{m\nu}\eta_\nu(\Omega_{\ell,3}) - (h_{\ell-1}^2 + (k_1+k_2)^2)^{\frac{m\nu}{2}}\eta_\nu(\Omega_{\ell-1,3})\right]}$$

$$C_c = \frac{\left[\chi_{\ell,3}(h_{\ell-1}^2 + (k_1+k_2)^2\right]^{\frac{m\nu}{2}} - \chi_{\ell-1,3}(k_1+k_2)^{m\nu}\eta_\nu(\Omega_{\ell-1,3})}{\left[(k_1+k_2)^{m\nu}\eta_\nu(\Omega_{\ell,3})\right.}$$

$$\frac{+(1+k_1+k_2)\left\{(h_{\ell-1}^2+(k_1+k_2)^2)^{\frac{m\nu}{2}}\eta_\nu(\Omega_{\ell-1,3}) - (k_1+k_2)^{\frac{m\nu}{2}}\eta_\nu(\Omega_{\ell,3})\cos h_{\ell-1}\right\}}{\left.-(h_{\ell-1}^2 + (k_1+k_2)^2)^{\frac{m\nu}{2}}\eta_\nu(\Omega_{\ell-1,3})\right]}$$

The values of $\chi_{\ell-1,1}$, $\chi_{\ell-1,2}$, $\chi_{\ell,1}$ and $\chi_{\ell-1,2}$ are then found by substituting these values of K and C_c in (I-3) with the requisite values of r and θ.

SYMBOLS

A	constant in r^{-1} term in series expansion of the potential for large r.
As s=1,2,3,4	coefficient of χ_{ij}^2 term in quadratic expression raised to the power α for the contribution to the integral for rectangle s.
A1	$\sqrt{\frac{\partial x}{\partial \xi_1}} = \sqrt{\rho_{11}}$
A2	$\sqrt{\frac{\partial y}{\partial \xi_2}} = \sqrt{\rho_{22}}$
a	major axis of an ellipse
aj j=1,2,..	coefficients of powers of σ^{-j} in series expansion of z .
B	boundary of body C.
Bs s=1,2,3,4	coefficient of χ_{ij} term in quadratic expression raised to the power α for the contribution to the integral for rectangle s.
b	minor axis of an ellipse
b_k	1 - k
b_n n=0,...	$(-\frac{1}{2})^n \prod_{k=0}^{n-1} \frac{\nu(\nu+1) - k(k+1)}{(k+1)}$ coefficients in series solution for the potential at the vertex of a cone.
C	the profile being investigated.
CL	lift
C_{R1}	large circle radius R_1
Cs s=1,2,3,4	terms independent of χ_{ij} in the quadratic expression raised to the power α for the contribution to the integral for rectangle s.

C	local speed of sound
C_c	constant term in velocity potential near the vertex of a cone
C_o	stagnation speed of sound
C_1	average of major and minor axes of an ellipse
C_∞	speed of sound at infinity
C'	aerofoil chord for the map to the unit circle
D	constant of integration
D_s $s=1,2,3,4$	coefficient of $\chi_{i\,j}$ in contribution to the integral for the rectangle s.
E	constant proportional to the circulation.
E_s $s=1,2,3,4$	terms independent of $\chi_{i\,j}$ in the linear part of the contribution to the integral for rectangle s.
F	non-dimensional difference between free stream and incompressible velocity potentials.
f	function of θ in r^{-1} term in the series expansion of ϕ at the far boundary.
$g(\chi_{i\,j})$	$\sum_{s=1}^{4} \left[\alpha(A_s \chi_{i\,j}^2 + B_s \chi_{i\,j} + C_s)^{\alpha-1}(2A_s \chi_{i\,j}+B_s)+D_s \right] H_s$
H_s	factor multiplying all other terms in the contribution to the integral for rectangle s
h	constant mass flux across B
h_i	$i = 1,2 \ldots \ell-1$ mesh steps in θ direction
h_1	$\sqrt{\dfrac{\partial x}{\partial r}}$
h_2	$\sqrt{\dfrac{\partial y}{\partial \theta}}$
$I[u]$	Bateman-Kelvin integral
J	Jacobian of the transformation $= h_1^2\, h_2^2$
$J[u]$	Bateman-Dirichlet integral
$J_{i\,j}$	$i=1,2,\ldots,\ell$ $j=1,2,\ldots,n-1$ contribution to the integral from the rectangle with sides length h_{i-1} and k_j

$\bar{J}$	$\sum_{i=1}^{\ell} \sum_{j=1}^{n-1} J_{ij}$
K	constant relating pressure to ρ^{γ}
k	The poles of the Karman-Trefftz profile are at $\pm$ k
k_j	$j = 1,2,.. n - 1$ mesh steps in r direction
l	number of grid points in θ direction
M	local Mach number
M_{∞}	free stream Mach number
m	$2 - \frac{\tau}{\pi}$
n	number of grid points in r direction
$\hat{n}$	unit normal to the body
p	pressure
pL	non-dimensional pressure
p_o	stagnation pressure
p_{∞}	pressure at infinity
q	speed
qm	initial velocity
R	distance from axis of symmetry in axisymmetric flows
R_f	value of r at far boundary
R_1	large value of r
$R_{k l}$	$k=1,2;.. \ell =1,2$ value of R at $r = r_k$, $\theta = \theta_l$
r	distance from the origin in the transformed plane
r_j	$j = 1,..., n$ value of r at grid points.
r_1	value of r in rectangles 1 and 2
r_2	value of r in rectangles 3 and 4
s	arc length of contour B
s_c	distance from the origin in the physical plane.
T	transform modulus
T_s	value of T in rectangle s

U free stream velocity

$\underline{u}$ velocity vector

u_R velocity component in R direction

u_r velocity component in r direction

u_x velocity component in x direction (axisymmetric flow)

u_1 velocity component in x direction (plane flow)

u_2 velocity component in y direction

u_θ angular velocity component

$u_{\eta 1}$ velocity component in $\eta 1$direction

$u_{\eta 2}$ velocity components in $\eta 2$ direction

V flow region

W $\int_c \left(\rho \phi \frac{\partial \phi}{\partial n} - U \rho_\infty \chi \frac{\partial x}{\partial n} \right) ds$

w relaxation parameter

X_s s=1,2, ...8 expression for sum or difference of three of the χ_s

x coordinate along the body axis

y coordinate perpendicular to the body axis in two-dimensional flows

$\underline{Z}$ arbitrary vector

z complex variable in the physical plane.

α the ratio ...... $\gamma / \gamma - 1$

α_I angle of incidence

β $1 / \gamma - 1$

β_M $\sqrt{1 - M_\infty^2}$

Γ circulation

γ ratio of the specific heats 1.405

$\delta_{n\ell}$	Kronecker delta = 0 $n \neq \ell$, 1 $n = \ell$
ϵ	small parameter
ζ	intermediate plane in which the transformation of a Karman-Trefftz profile is a circle with its centre at a place other than the origin.
η_1, η_2	Cartesian coordinates in the σ - plane for axisymmetric flows
θ	the angle, the line joining the cone vertex to any point makes with the positive x - axis.
θ	arg (σ) in the transformed plane.
θ_i	$i = 1,2, \ldots, \ell$ value of θ at grid points.
θ_1	value of θ in rectangles 1 and 4
θ_2	value of θ in rectangles 2 and 3
K	constant in the expression for the velocity potential near the cone vertex
Λ	$\underline{\Sigma} . \underline{\Sigma}$.
λ	$\frac{1}{2} (a^2 - b^2)^{\frac{1}{2}}$
μ	thickness ratio of ellipse
ν	constant related to the cone angle lying between 1 and 2
ξ	angle in cylindrical polar coordinates
ξ_1, ξ_2	cartesian coordinates in the σ - plane for plane flow
ρ	density
ρ_c	distance from the vertex of the cone
$\rho_{n\ell}$	$\delta_{n\ell} \; A_\ell \; A_\ell$
ρ_o	stagnation density
ρ_1	$(r^2 - 2 \, (b_k - k) \, r \cos \theta + (b_k - k)^2)^{\frac{1}{2}}$.
ρ_2	$(r^2 - 2 \, r \cos \theta + 1)^{\frac{1}{2}}$

ρ_∞	density at infinity
σ	complex variable in the circle plane
σ'	inverse plane of σ-plane i.e. complex variable on the interior of the unit circle.
τ	trailing edge angle of a Karman-Trefftz profile.
$\underline{I}$	momentum $\rho \underline{u}$
ϕ	velocity potential
ϕ extr	extremal value of ϕ
ϕ_o	velocity potential for incompressible flow.
ϕ_1	$\tan^{-1}$ $(r \sin \theta / (r \cos \theta - b_k + k))$
ϕ_2	$\tan^{-1}$ $(r \sin \theta / (r \cos \theta - 1))$
ϕ_∞	velocity potential for free stream flow
χ	non-dimensional perturbation potential
χ_A	difference between velocity potential for compressible and incompressible flow.
$\chi_{i\,j}^{(n)}$	nth. approximation to the solution $\chi_{i\,j}$.
$\chi_{i\,j}$	$i = 1,2,\ldots.\ell$, $j = 1,\ldots,$ n value of χ at the grid point i, j
χ_s	$s = 1,2,\ldots,9$ value of χ at point s in the nine-point scheme.
χ_o	non-dimensional incompressible potential ϕ_o / U
χ'	perturbation potential $U\chi$
$\overline{\chi}_{i\,j}^{(n)}$	value of $\chi_{i\,j}^{(n)}$ before relaxation
Φ	$\pi - \theta$
ψ	stream function
ψ_c	arg (z) in the physical plane
Ω	$\cos \Phi$

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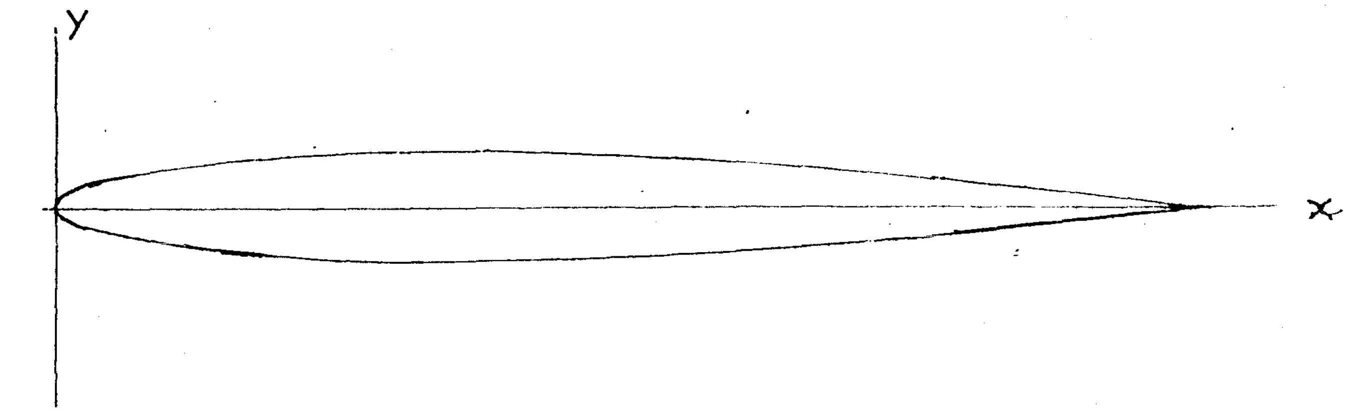
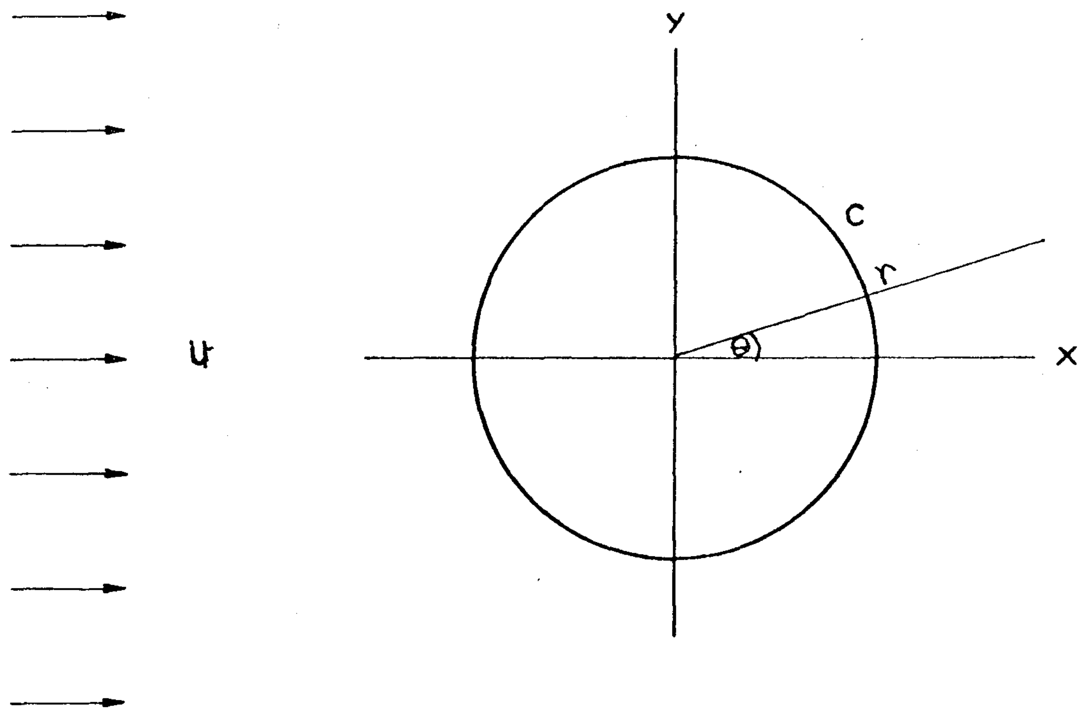


(a) Physical (z) plane



(b) Circle (σ) plane

Fig.1 Sketch of flow field

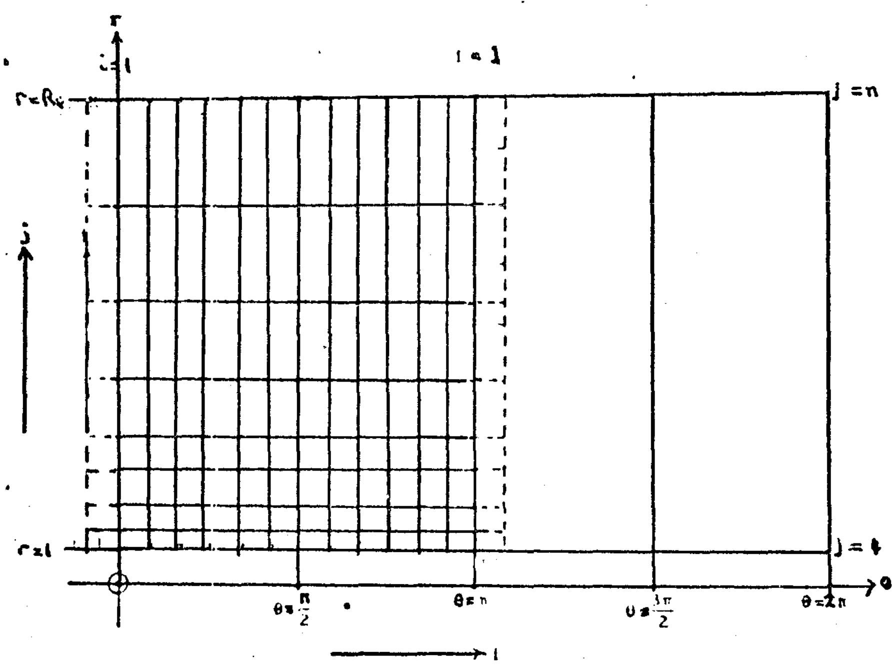


Fig. 2 Subdivision of the modified domain in the σ plane

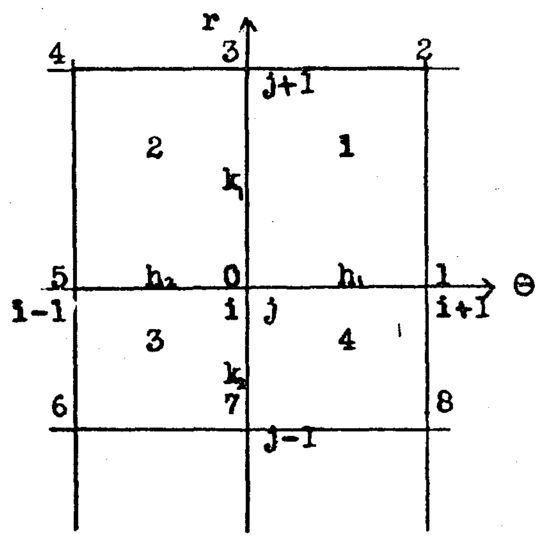
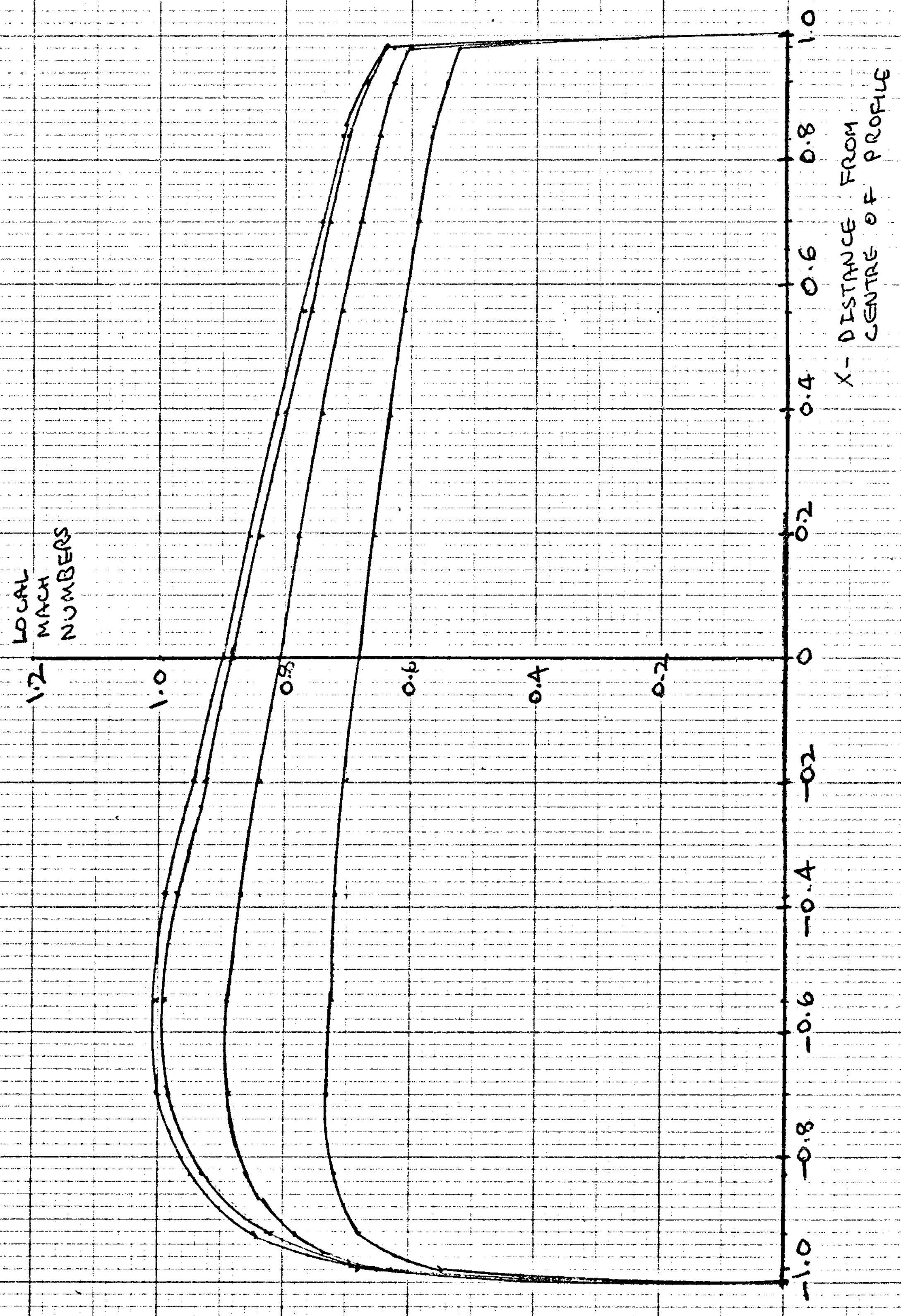


Fig. 3 Nine point scheme for finding the potential at the point i,j

Fig. 4 Local Mach Numbers on the surface of a 10% thick Karman-Trefftz profile.



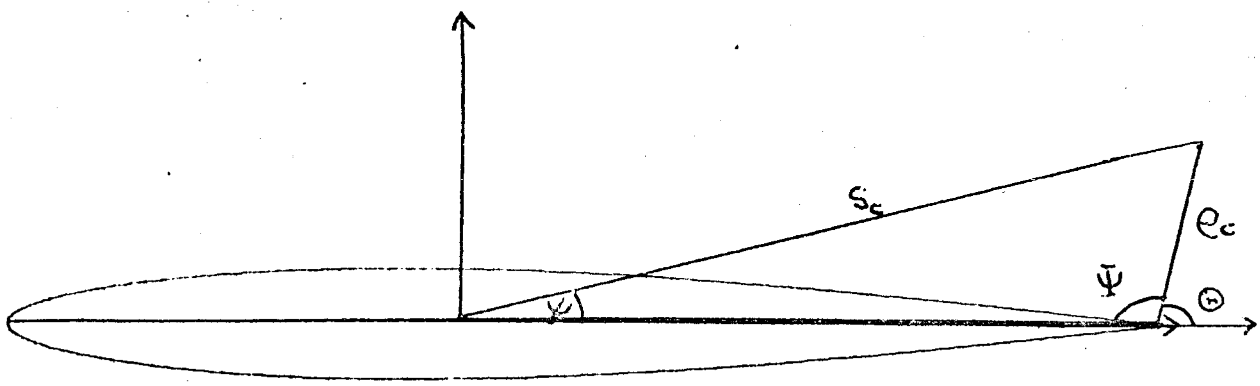


Fig. 5 Terminology used in finding the potential near the trailing edge of a body of revolution with a Karman-Trefftz profile as cross-section.