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Reputation and Cooperation in the Repeated Second-price Auctions

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Abstract

This paper shows that there are strong reputational effects in a general class of second price auctions, including single-unit English and Vickrey auctions with interdependent values, multiunit ascending and uniform price auctions and a War of Attrition. It is based on recent results on reputation with symmetric discounting. If a reputation is one sided and bidders are patient, the bidder with reputation must obtain most of the surplus in the sequence of auctions, the other bidder and the seller get very little. If the reputation is two-sided then the bidders engage in a game akin to War of Attrition. The resulting payoff is very low for the bidders and very high for the seller. In any case, Folk Theorem fails: collusion in the second price auctions is impossible. The predictions of the model are that the path of prices is declining, in fact prices in the early auctions should reach levels that are higher than the value of the object and there should be a set of strong bidders emerging after a few auctions. A recent series of auctions of spectrum for UMTS services in Europe seems to fit both the assumptions and predictions of the model.

Keywords: Repeated Auctions, Ascending Auctions, Second-Price Auctions, Collusion, Reputation, Aggressive bidding

JEL Classification: D44, C72, L96,

1 Introduction

This paper shows that there are strong reputational effects in the repeated second-price auctions with entry fee and with two equally patient bidders. The reputation is for being very aggressive, that is, for bidding more than the object's true value to a normal opponent. It is shown that in the limit, as the discount factors of normal types converge to one, a unique payoff profile is supported as an equilibrium payoff profile.

If the reputation for being very aggressive is only one-sided, then the player without reputation will not challenge his opponent with reputation, but rather stay out and receive the outside option, in every auction except at most finitely many of them. Hence, the player with reputation will win almost all auctions and pay the reserve price, thus collecting almost the entire surplus. In this case, the auction sequence generates almost no revenue from bidding.

When the reputational profile is two-sided, then a kind of a War of Attrition between bidders emerges. The player who bids low loses his reputation and in turn receives very low payoff. On the other hand, waiting longer that the opponent promises a very high payoff. There is a unique equilibrium, mixed, in which each bidder bids low and loses his reputation with some likelihood in every point in time. When the model is entirely symmetric, then the equilibrium is symmetric. The War of Attrition is so severe in this case, that both bidders will receive payoffs equal to their outside option. The auction sequence generates very high prices in the initial auctions and low prices once the game reaches one-sided phase, but in total the revenue is the highest, in the limit equal to the entire surplus.

The most important predictions of the symmetric model can be summarized in the following way: the cooperation will not occur, the path of prices will be declining, the prices in early auctions will be higher than the true values of the object to a normal bidder, there will be a „strong" bidder emerging after a few auctions and finally, the expected discounted payoffs that normal types obtain before the repeated game are very low.

Literature on Reputation. Fudenberg and Levine (1989) in their seminal paper on reputation consider an environment in which one long-run player faces a sequence of short-run opponents (or one completely myopic opponent). They show that in the limit as the discount factor of the long-run player converges to one, this player in any Nash equilibrium obtains his Stackelberg payoff, as long as there is a nonzero probability that the long run player is of a crazy type that uses repeatedly a Stackelberg action.

Schmidt (1993) obtains a related result in the environment where the opponent of a long-run player is not completely myopic. He shows that when a player with reputation is sufficiently more patient than his somehow patient opponent, then in a certain class of games, called games with conflicting interests, the Nash equilibrium reputation effects exist. This result has a certain significance for the auction theory because all the second-price auctions belong to this family of games. As long as the conflict is strict (explained below), Schmidt's lower bound on the payoff of a more patient player with reputation is very high, otherwise this bound is lower, possibly close to this player's minmax payoff.

The most important single paper, that this paper relies on, is Cripps, Dekel and Pesendorfer (2005), henceforth CDP. They show that as long as the normal-form stage game belongs to a certain category, called games of strictly conflicting interests, then in any Nash equilibrium of the infinitely repeated version of that game with *equally patient players* and one-sided reputation, a player with reputation must get an expected payoff close to his maximal payoff, while his opponent must get a payoff close to his minmax. In particular, the perturbed version of Folk Theorem fails. It turns out that many versions of the second-price auctions are games of strictly conflicting interests, so the result of Cripps, Dekel and Pesendorfer can in principle be applied. The additional difficulty is that a stage game in CDP has a simple normal-form, while the auction is an incomplete-information game. But using the special properties of second-price auctions, their method is general enough to tackle this problem.

Strictly conflicting interests games contrast with other normal-form games, where the counterpart of the Folk Theorem holds. Cripps and Thomas (2000) and Chan (2000) show that if the unperturbed stage game is not a game of strictly conflicting interests and is not in dominant strategies, then there is a perfect Bayesian equilibrium in the repeated version of that stage game with reputation and symmetric discounting, in which players receive any feasible payoff greater than their minmax payoff. Since auctions other than second-price in general do not exhibit strictly conflicting interests, the conjecture is that the Folk Theorem for perturbed repeated games should hold for these auctions. This is only a conjecture because games with incomplete information, including auctions, were not studied by cited papers, nor they are studied here.

The part of this paper which deals with two-sided reputation draws from Abreu and Gul (2000), who consider two-sided reputation in the model of bargaining. The War of Attrition that emerges here is the direct counterpart of their result. They prove in their context that a set of equilibria of the discrete-time model converges to the unique equilibrium of the continuous time model when the time period becomes shorter.

Literature on Auctions. The results of this paper shed new light on the received auction literature.

The anti-Folk Theorem result of CDP applied to second-price auctions lie in a direct contradiction to the current standpoint of auction theory. Robinson (1985) and von Ungern-Sternberg (1988) observed that collusion can be easily sustained in the repeated (or even one shot) unperturbed English and Vickrey auctions. This is also true in other second-price auctions. For example, this property is discussed in McAdams (2000) or Ausubel and Schwartz (1999) in the context of multi-unit auctions. But, the analysis of this paper suggests that, loosely speaking, the cooperation (tacit collusion) among bidders is made more difficult by the choice of second-price auction rather than facilitated, so long as the environment allows for reputation gains and common discount factor is large. This anti-Folk Theorem result is especially clear in the symmetric reputation model discussed below, where both bidders in the limit must obtain payoffs essentially equal to their minmax value of the game, and the seller must get the entire ex ante surplus generated by the sequence of these auctions.

Another segment of the literature on auction theory relevant for this paper deals with small asymmetries in bidder's positions and the huge effect that these may have on the final outcome in the context of pure common-value second-price auction. The first paper of this kind is Bikhchandani (1988), which also happens to be the only paper about reputation in auctions. It shows that if a second-price auction with purely common values is perturbed a little – so that there is some nonzero probability that one of the bidders has a value slightly larger than regular players – then the very asymmetric equilibrium is selected as the unique equilibrium. If such game is repeated finitely many times, then this perturbation as well as this stage equilibrium are both very easy to sustain over time. Ultimately, the player with reputation wins all auctions and pays very little. Klemperer (1998) and (2000) also considers similar perturbation of a common-value auction.

The part of our paper that deals with one-sided reputation provides very similar conclusion as Bikhchandani (1988) and Klemperer (1998) and (2000), but in a different context and using completely different apparatus. Most importantly, the auction does not have to have purely common values. While there is a sense in which winner's curse is helpful in achieving this result, it is not necessary. Secondly, we deal with infinitely repeated games, where the possibility of collusion can, in principle, prevent the reputation to play a serious role. Bikhchandani (1988) considers only finitely repeated auctions, in which this problem can not arise. The picture that emerges from this research is that in the context of repeated auctions, it is the format of the second-price auction that is responsible for the result, and common values just help a little. As long as there is some cost of entry, whether it is „hard" monetary cost, or „soft" winner's curse, the repeated second-price auction with one-sided perturbation leads to a very asymmetric equilibrium.

One prediction of the model is that the prices should be declining. This "declining price anomaly" has been empirically confirmed in many cases (Ashenfelter 1989). There are several explanations of this phenomenon, but most of them assume that the auctions are sequential rather than repeated. The difference is that in the former, each bidder has a decreasing marginal valuation in number of units purchased. The simplest assumption of this type is that a bidder has a total demand of only one unit, so that once his demand is satisfied he vanishes from the game forever. These models are not appropriate to investigate cases where periods are similar in the sense that bidders do not care about the number of acquired items, as long as the average value-price margin is positive. Conversely, the model of reputational incentives developed below is clearly not valid to address questions of declining prices in auctions where bidders can or want to buy only one item.

There is, however, one paper in the literature of declining price anomaly relevant for our analysis. Von der Fehr (1994) considers two auctions conducted sequentially with a cost of participation. The potential losers in the second auction are revealed after they announced their bids in the first auction. Since the entry is costly, they choose not to participate, which decreases the competition in the second auction, eventually leading to a low price. This is similar to the outcome of the two-sided reputation model, where bidders engage in a War of Attrition, once it is revealed that one of them is "weak", the prices drop.

2 UMTS auctions in Europe

Years 2000 and 2001 witnessed an interesting natural experiment in repeated auctions. Many European governments started assigning licenses for Universal Mobile Telecommunications System (UMTS), also known as third-generation (3G) mobile telecommunication services. In many countries the mechanism was

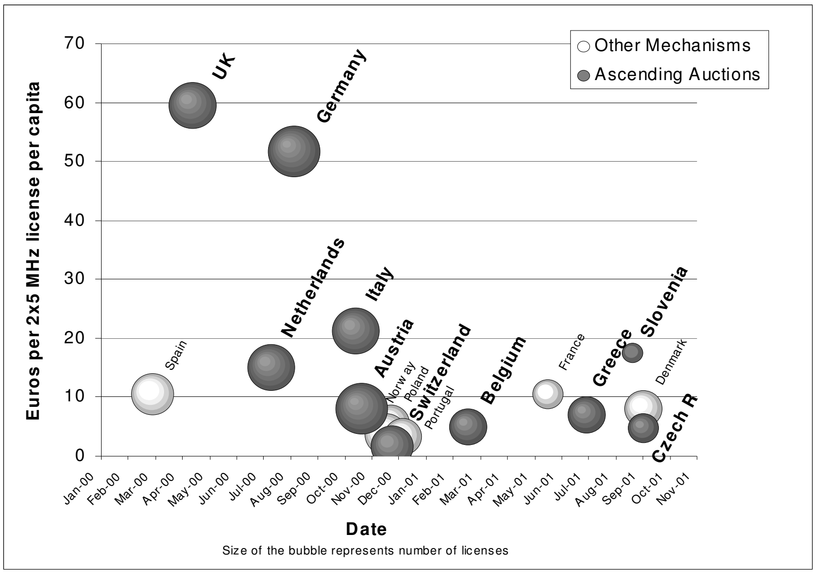

Figure 1: Cost of the European UMTS licenses.

some version of an ascending auction.[1] The first auction, conducted in the UK (see Figure 1)[2], brought staggering revenues, many times more than the initial media estimates. Also, in Germany, the third auction in this series, the revenues per license per capita were equally high. However, the prices in other auctions were significantly lower. The second and the fourth auction, in Netherlands and Italy, delivered prices that reached only between a fourth and a third of what the UK government collected. Austrian and Swiss governments, where the next two auctions had taken place, obtained only a tiny fraction of these initial revenues; in fact, the prices were almost equal to the low reserve prices. The price per license in later auctions, conducted in 2001, was equally low in Belgium, Greece and Czech Republic, and slightly higher in Slovenia.

The second interesting aspect is the trend in the number of bidders (see Figure 2). There were thirteen bidders in the UK, compared to five licenses. The excess demand of *eight* units was by far the highest among all the UMTS ascending auctions. In later auctions the excess demand went down, reaching zero in Swiss auction. In all later ascending actions, the number of bidders and ultimately the license holders was less than the number of licenses available; the excess demand was *negative*. This indicates that the governments relied on reserve prices rather than on bidding to establish prices.

The only exception was Denmark, where the government offered four licenses

[1] For an overview see Klemperer (2001), Jehiel and Moldovanu (2001), Cramton (2001a).
[2] Source: UMTS Forum (www.umts-forum.org)

UK:	8
Netherlands:	1
Germany:	1 - 3
Italy:	1
Austria:	0 - 2
Switzerland:	0
Later ascending auctions	< 0

Figure 2: Excess demand in European UMTS auctions

in a first-price, sealed-bid auction.[3] The number of incumbents was exactly four – equal to the number of licenses. Despite of that, this auction attracted a new entrant, who ultimately won one license. Given that in late 2001 the mood on financial markets and in the telecom industry was funereal, this auction was a remarkable success, both in terms of prices and entry.

The general public attributed price decline to a general slowdown of the global economy in the second half of 2000, and of the high-tech industries in particular. The story is that suddenly the prospects of this new service became grim, and hence telecommunication companies were not willing to pay as much as earlier for the right to provide the service. The popular view was, moreover, that companies were not rational in early auctions, that they were subject to some unexplainable bidding frenzy, in the context of wider "dot com" bubble and only later they came to their senses, realizing how devastating the bidding was. This paper argues that this view may be not that far from the truth, in the sense that bidders, being entirely rational, may have behaved *as if* they were irrational.

The literature about UMTS auctions in Europe tends to emphasize how the differences between the rules of the auction encourage or discourage entry of new bidders. Importance is attached to the issues such as the number of licenses compared to number of incumbents, reserve prices, activity rules, bid increments, etc. (see Klemperer (2001), Jehiel and Moldovanu (2001), Cramton (2001a)). These aspects certainly matter in inducing entry. But the sequential structure and the strategic interdependence between auctions, while acknowledged, is secondary in these papers. For example, Binmore and Klemperer (2001) (p. 16) write about the UK auction

> We thought subsequent 3G auctions might attract less entry since bidders would work out from the first auction who the likely winners were in future auctions. (...) Furthermore prices in the first auction might be driven higher if bidders thought that winning that auction

[3] It was a fourth-price sealed-bid auction with four licenses. But in the context of this paper this auction is a generalization of first-price auction with one object.

gave them a competitive advantage in the future auction.

Cramton (2001b) (p. 51) explains in a similar spirit:

> (...) the UK auction was the first in the sequence of European 3G auctions. The largest wireless operators believed that winning a license in the UK was an important first step in becoming or sustaining a major position in Europe. The UK was the foot in the door to Europe and potentially the world.

This present paper advocates and formalizes this dynamic point of view.

The model presented in this paper seems to fit roughly the case of UMTS auctions in Europe both in terms of assumptions as well as predictions. There was a number of international telecommunication companies that had a strategy for the whole of Europe, rather than for each country separately, so there was a strategic interdependence between subsequent auctions.

The assumption of two-sided reputation is well justified, not only because the auction market was new, with firms competing with each other in these circumstances for the first time, but also because the players faced significant uncertainty about the new technology and consumers' valuations. Firms could not exclude the possibility that they faced an aggressive type. Such an aggressive type may in fact be entirely rational, but may have higher estimate of the market's valuation or may face complemetarities among licenses in various countries.

The complementarities may explain declining prices in repeated auctions. The loser in the first auction does not have as much incentives for entry in the later auctions. Recognizing this strategic advantage from winning the initial auctions, bidders should bid more aggressively in a few initial actions than in later ones. This argument was also mentioned in the context of UMTS auctions in Europe. For example Klemperer (2001) writes about European UMTS auctions:

> Markets that were auctioned later were more valuable to those who had won earlier ones that fitted well with them in a network, and an early win also allowed a firm to influence suppliers about the development of the technology in ways that would help the firm in later markets. These "real" complementarities (...) further discouraged losers of early auctions form entering later auctions, especially ascending ones.

Cramton (2001a) writes in a related way (p. 25)

> To the extent that a UK license is complementary with UMTS licenses in other countries, then we should expect the UK licenses to sell for a premium, recognizing this complementary value.

We also refer to the complementarities among the items as one of the informal justification that exceptionally aggressive types may exist. As the above quotation indicates, such a justification may be very plausible. However, one of the points of this paper is that such complementarities do not have to be strong. It is merely enough that their existence is not ruled out by the bidders, and the reputational effects will do the rest.

While it is true that the predictions depend on the *relative* strength of both bidders – the probability of occurrence of the aggressive type of player 1 relative to that of player 2 – they do not depend on the *absolute* probability of that event. Namely, the probability of occurrence of aggressive types may be as close to zero as desired, and still the equilibrium payoff profile is unique in the limit.

3 Second-price sealed-bid auction with entry cost

This section introduces the repeated second-price auctions with entry cost. The most important result in the model where one player has reputation is that the bidder without reputation will stay out, while the bidder with reputation will enter and get the object for the lowest possible price, in all but finitely many periods. This result will be used to show that in a two-sided reputation version of the repeated model the incentive for maintaining reputation is so high that a very destructive War of Attrition emerges.

3.1 Stage Auction

Let the following stage auction be denoted Γ. Suppose that in each period there is one object for sale and there are 2 risk-neutral bidders. The seller and the mechanism will be exogenous to the model. The value of the object to each player in period t is determined by i's signal for that period, θ_i^t, an element of a discrete set of signals, $\Theta = \left\{\underline{\theta}, ..., \bar{\theta}\right\}$. The value of the object to the bidder i is $v_i\left(\theta_i^t\right) > 0$ for all $\theta_i^t \in \Theta$. Since the game is finite, there exists $\bar{v}$, such that for all signals and both players $v_i\left(\theta_i^t\right) < \bar{v}$. The reserve price is zero, but we assume that there is a nonzero but possible very small bidding cost, $F > 0$ and $v_i\left(\theta_i^t\right) > F$. We let $B = \left\{-1, 0, 1, 2, ..., \bar{B}\right\}$ denote the set of possible bids. A bid equal to -1 is interpreted as not enter. The signal and bid dependent payoff of player i in period t is denoted by $g_i\left(\theta_i^t\right)$, where the dependence on strategy profile is implicit:

$$g_i\left(\theta_i^t\right) = \begin{cases} v_i\left(\theta_i^t\right) - b_j - F & \text{if } 0 \leqslant b_j < b_i \\ -F & \text{if } 0 \leqslant b_i < b_j \\ \frac{1}{2}\left(v_i\left(\theta_i^t\right) - b_j\right) - F & \text{if } 0 \leqslant b_i = b_j \\ v_i\left(\theta_i^t\right) - F & \text{if } b_j < 0 \leqslant b_i \\ 0 & \text{if } b_i = -1 \end{cases}$$

The first three rows correspond to the situation in which both bidders enter. If i's bid is higher than j's then bidder i gets the object, pays the price equal to the bid of his opponent and pays an entry cost. If i bids less than his opponent, then he pays only an bidding cost and gets nothing else. If there is a tie, then we introduce a convention that both bidders get half of the surplus and pay the bidding cost. The fourth row corresponds to a situation when only bidder i enters. Then he wins an object automatically and pays only the bidding cost. The last row corresponds to a situation in which bidder i himself does not enter. Then he gets a payoff of zero.

Each player observes both bids after the auction; they constitute a public outcome, $y_t = \{b_i, b_{-i}\}$. This assumption is consistent with the sealed-bid format of the second-price auction, where the set of actual bids is announced publicly after the auction.

3.2 Repeated game.

Each bidder i observes an entire sequence of signals, $\theta_i = \left\{\theta_i^t\right\}_{t=0}^{\infty}$, before the repeated game starts. The alternative assumption would be that each player learns period t signal just before period t. However, the former assumption seems more adequate to model the European UMTS auctions. The time interval between auctions was so short that players were unlikely to update their assesment of the object's value between the auctions. Rather, they entered the entire sequence with some idea about the value of the sequence of objects, and they updated their information only via history of the play.

A repeated version of the second-price auction will be denoted by $\Gamma(\Delta, \rho)$, where Δ is a length of a time that elapses between auctions, and ρ is a discount rate of two bidders.

When the game is repeated, the stage payoff is interpreted as a flow rather than stock, so that the discounted payoff earned in t'th round is $\left[\int_{t\Delta}^{(t+1)\Delta} e^{-\rho z} dz\right] g_i\left(\theta_i^t\right)$. Note that the square bracket simplifies to $\frac{1}{\rho}\delta^t(1-\delta)$, where $\delta = e^{-\rho\Delta}$.

Let H denote the set of all paths of publicly observable outcomes and let path $h = \{y_0, y_1, ...\}$ be a typical element of this set. Let the $h^t = \{y_0, y_1, ..., y_t\}$ stand for the t-period public history.

The total payoff in the repeated game is a discounted sum of the sequence of stage payoffs, $\{g_i(\theta_i^v)\}_{v=0}^{\infty}$, normalized, as usually, by a constant to express per period and total payoffs in a comparable way:

$$
\begin{aligned}
u_i^t\left(\theta_i, h^{t-1}\right) &= E_{\theta_j | h^{t-1}, \theta_i}\left\{\rho \sum_{v=t}^{\infty}\left(\int_{(v-t)\Delta}^{(v-t+1)\Delta} e^{-\rho z} dz\right) g_i\left(\theta_i^v\right)\right\} \\
&= E_{\theta_j | h^{t-1}, \theta_i}\left\{(1-\delta)\sum_{v=t}^{\infty}\delta^{v-t} g_i\left(\theta_i^v\right)\right\}
\end{aligned}
$$

3.3 Perturbed game.

All models of reputation perturb slightly the baseline repeated game. They assume that there is a positive ex ante probability μ_i that player i is of a different type, called crazy, committed or behavioral, as opposed to normal or rational types in the baseline game. With such a change, each normal player may have an incentive to mimic crazy type for some time, in order to convince his opponent that he should behave in a particular way. For small μ_i this perturbed game is close to the baseline game, the question is whether the set of equilibria in the perturbed game is close to the set of equilibria in the original baseline game. The same question for auctions will be asked here.

We assume that there is a positive probability μ_i that player i is "crazy" at the beginning of a repeated auction. A crazy player always uses a prespecified pure bidding function, denoted $s_i^* : \Theta \to B$. With a remaining probability $1-\mu_i$ this player is normal, with a payoff function as above. Roughly speaking, the crazy type will bid aggressively, that is, he will be ready to offer very high bids, higher that the value of the object to his opponent.

This perturbed game will be called $\Gamma(\Delta, \rho, \mu)$, where $\mu = (\mu_1, \mu_2)$. Whenever $\mu_i > 0$ and $\mu_j = 0$ we say that the reputation is one-sided; if $\mu_i > 0$ and $\mu_j > 0$ then the reputation is two-sided; if, in addition, $\mu_i = \mu_j > 0$ then the reputation is symmetric.

By calling the player "crazy", this paper goes along the traditional terminology in the reputation literature. However, in second-price auctions aggressive bidding can be plausibly justified, though informally. For example, if the player has high private value for the object and low discount factor, then he may not be eager to join a collusive scheme. He may rather bid very aggressively every period, since his weakly dominant strategy is to bid his private value. Another example may involve complementarities among objects. The object is worth something only if it is bought together with a bunch of other objects. Therefore, the player may be determined to buy the missing piece of his most desired bundle. Whatever the case may be, the reputation argument is unique in that that it does not require that a particular player is eager to bid aggressively, it is enough to assume that his opponents believe with nonzero probability that such a case may indeed occur.

3.4 One-sided reputation.

This section presents a version of CDP result, slightly generalized to games of incomplete information of the format considered here. CDP consider a framework with complete-information, normal-form stage game that has strictly conflicting interests. They show that when such games are repeated, then a player who faces a possibly committed opponent will not play a best reply at most bounded number of periods, where the bound is independent of δ.

The following is one possible generalization of this concept to incomplete information games. Let $\bar{g}_i\left(\theta_i^t\right)$ be the maxmax payoff that player i can consume at t and define $\hat{g}_j\left(\theta_j^t\right)$ to be the minmax value of player j. In our case, $\bar{g}_i\left(\theta_i^t\right) = v_i\left(\theta_i^t\right) - F$ and $\hat{g}_j\left(\theta_j^t\right) = 0$. Let the expectations of these be denoted $\hat{g}_j = E\hat{g}_j\left(\theta_j^t\right)$ and $\bar{g}_i = E\bar{g}_i\left(\theta_i^t\right)$.

Definition 1 *Game Γ has **strictly conflicting interests** ex ante (in short SCI), if there exists a strategy $s_i^* : \Theta \to B$ such that if bidder i publicly commits* *to this strategy, and player j best replies to it in any way, then player j gets his ex ante expected minmax payoff $\hat{g}_j$ and player i gets his ex ante expected maxmax payoff $\bar{g}_j$. In addition, this profile of payoffs $(\bar{g}_i, \hat{g}_j)$ is the unique profile in which player i gets maxmax payoff.*

Let $R_j\left(s_i, \theta_j^t\right)$ to be the set of best replies of player j to strategy s_i. The finiteness of the SCI game implies the following fact, used in the proof of CDP. Suppose that player i commits to his best strategy s_i^* and suppose that player j does not play his stage best reply, $s_j \notin R_j\left(s_i^*, \theta_j^t\right)$. Then, there exists a number $l > 0$ such that player j's expected payoff is at most $\hat{g}_j - l$.

This establishes a certain minimal punishment relative to his minmax value that player j will suffer for sure in ex ante sense, if he does not play a best reply against his crazy opponent. The next definition specifies a stronger generalization of SCI property to games of incomplete information.

Definition 2 *Game Γ has* ***strictly conflicting interests ex post*** *(in short SCI ex post), if (a) it is SCI and (b) there exists $l > 0$ such that for every choice of nature, $\left(\theta_i^t, \theta_j^t\right) \in \Theta \times \Theta$, player j, by not playing a stage best reply, $s_j \notin R_j\left(s_i^*, \theta_j^t\right)$, obtains a stage payoff at most $\hat{g}_j\left(\theta_j^t\right) - l$.*

The reason why the case of stage game with SCI ex post is the simplest case is that if player j does not play a best reply then he expects to suffer a minimal loss, $l > 0$, not only in *expectation* but in *any possible realization* of nature's choice, $\left(\theta_i^t, \theta_j^t\right) \in \Theta \times \Theta$. It turns out that the method of CDP can still be used for this case almost without alteration. Naturally, there may be incomplete information games that have ex ante SCI but not SCI ex post. We will discuss an example of such a game later – second price auction without entry cost but interdependent values.

The example considered here – sealed-bid second-price auction with bidding cost and private values – is a SCI ex post game, and this fact is used in the proof of Proposition 2.

Proposition 1 *A sealed-bid second-price auction with entry cost defined above is a SCI ex post game.*

Proof. Suppose that bidder i commits to a strategy of bidding constant bid, $s_i^* \in B$, higher than the highest conceivable value of the object to any of the bidders, $s_i^* > \bar{v}$. Then the unique best response of bidder j is not to enter, because this guarantees a payoff of zero. Entering results in a loss relative to the minmax payoff of zero of at least $F > 0$, no matter what is the signal profile $\left(\theta_j^t, \theta_i^t\right)$.

For any $\left(\theta_j^t, \theta_i^t\right) \in \Theta \times \Theta$ the minmax payoff is $\hat{g}_j\left(\theta_j^t\right) = 0$ and the payoff of bidder i is $\bar{g}_i\left(\theta_i^t\right) = v_i\left(\theta_i^t\right) - F$. ■

The following proposition is the main result of this seciotn and is proved in the appendix.

Proposition 2 *Let $\Gamma(\Delta, \rho, (\mu_i, 0))$ be the repeated and perturbed version of stage game with SCI ex post, Γ. Let $\mu_i > 0$ be probability that player i is crazy, using each period the constant "commitment" bid $s_i^* > \bar{v}$ independent of the signal sequence θ_i.*

Then the number of periods in which player j does not play a best response to s_i^* along the history in which player i keeps playing s_i^* is bounded by a number that does not depend on δ.

Corollary 1 *Let $\Gamma(\Delta, \rho, (\mu_i, 0))$ be the repeated and perturbed version of stage game with SCI ex post, Γ. Let $\mu_i > 0$ be probability that player i is crazy, using each period the constant "commitment" bid $s_i^* > \bar{v}$ independent of the signal sequence θ_i. Moreover, suppose that θ_i^t is independent over time, so that $\bar{g}_i = E\bar{g}_i\left(\theta_i^t\right)$ is constant. Then, as $\delta \to 1$, the payoff of i converges in distribution to a degenerate distribution at $\bar{g}_i$.*

3.5 Two-sided reputation

In many environments it seems plausible to allow both bidders to have positive prior probability of being crazy, $\mu_1 > 0$ and $\mu_2 > 0$.[4] The stage game is the same as studied above, the second-price auction with entry cost, where the crazy type of any player bids the same constant bid, $s^* > \bar{v}$.

This section draws from Abreu and Gul (2000), who study two-sided reputation in a bargaining game. Their result that the equilibria of a discrete-time version of the model converge to the unique equilibrium of the continuous time model is also our main motivation for studying the continuous time model. Thus suppose in this section that time is continuous. This game will be denoted $\Gamma(0, \rho, \mu)$, where $\Delta = 0$ stands for the length of a time period. For simplicity, assume also that signals are independent across time and let $\bar{g}_i = E\bar{g}_i\left(\theta_i^t\right)$.

The game is formulated in such a way that once a bidder loses his reputation, he cannot get it back. Therefore, there can be at most three phases: phase of two-sided reputation, phase of one-sided reputation and phase of no reputation at all. We deal with the bidders' incentives to reveal their normality in the first phase, thus ending it. So for simplicity, we truncate the game once it reaches the second phase, and endow the players with payoffs that they can expect in that second phase. In particular, if player j reveals his normality before player i, then the continuation payoff profile is assumed to be $(\bar{g}_i, 0)$. Note that in the assymetric reputation game, as $\Delta \to 0$, the distribution of equilibrium payoffs of the player with reputiation converges to a degenerate distribution at $\bar{g}_i$, and this limit payoff will be assumed as a continution payoff for the player with reputation.

Similarly, once the game reaches the third phase immediately after the first one, we truncate the game and provide the players with any individually rational

[4] This seems to be a more realistic assumption for the case of European UMTS auctions. In newly formed auction markets bidders meet for the first time, like in this example, and they should be treated by the model in an equal a priori way. This would allow for the asymmetry to emerge endogenously.

and feasible payoff profile. Let G be the set of such payoffs. This profile is assumed to be arbitrary within G, since a priori we cannot say anything specific about the continuation payoff when both players are normal. Moreover, we do know that the Folk Theorem with public monitoring holds in this case, so we can expect that any such payoff profile can indeed be freely assumed in the limit $\Delta \to 0$.

The Folk Theorem holding in the third phase is essentially the only difference between our framework and the bargaining context of Abreu and Gul (2000). Bargaining model has a unique continuation payoff, while in repeated games this payoff profile can be any individually rational and feasible profile.

Let $\Pi_i(t)$ be the probability that bidder i plays a bid lower than s^* by the time t, thus revealing that he is normal. Let $\tilde{g}$ be an instantaneous payoff of any bidder, if both continue to use action s^*. Since the stage game is SCI ex post and s^* is not a stage best reply to s^*, we have $\tilde{g} < 0$. Consider a mixed equilibrium in which both players use a strictly increasing and atomless $\Pi_i(\cdot)$ on a certain time interval $(0, T)$ and constant $\Pi_i(\cdot)$ after date T. Let $u_j(t)$ be the total discounted payoff of the player j when he exits at time $t > 0$ and his opponent uses a strategy $\Pi_i(t)$.

$$u_j(t) = \Pi_i(0)\,\bar{g}_j + \int_0^t \left[\tilde{g}\left(1 - e^{-\rho v}\right) + e^{-\rho v}\bar{g}_j\right]\pi_i(v)\,dv + (1 - \Pi_i(t))\,\tilde{g}\left(1 - e^{-\rho t}\right)$$

Since the equilibrium is mixed, it must be that $u_j'(t) = 0$ for all $t \in (0, T)$. This condition implies that

$$\frac{\pi_i(t)}{1 - \Pi_i(t)} = \rho\frac{-\tilde{g}}{\bar{g}_j} =: \lambda_i > 0$$

The explicit solution for strategy of bidder i is

$$\Pi_i(t) = 1 - c_i e^{-\lambda_i t} \text{ for } i = 1, 2$$

where c_i is yet unknown. In order to pin down (c_1, c_2) we will use the final and initial conditions. If the game is still in phase 1 at time T, then both players are known to be crazy for sure. That is $\Pi_i(T) = 1 - \mu_i$, implying that $T = \frac{1}{\lambda_i}\ln\frac{1}{\mu_i}$. On the other hand, only one player can exit at $t = 0$ with a positive mass, let i be this player, $\Pi_i(0) \geqslant \Pi_j(0) = 0$. This last condition implies that $c_j = 1 \geq c_i$. The latter inequality imposes condition $\left(\mu_j\right)^{\frac{1}{\lambda_j}} \geq \left(\mu_i\right)^{\frac{1}{\lambda_i}}$. Provided that this condition is satisfied, the following strategy profile forms an equilibrium

$$\begin{cases} \Pi_i(t) = 1 - \left(\frac{1}{\mu_j}\right)^{\frac{\lambda_i}{\lambda_j}} \mu_i e^{-\lambda_i t} \\ \Pi_j(t) = 1 - e^{-\lambda_j t} \end{cases} \tag{1}$$

Proposition 3 *Let $\left(\mu_j\right)^{\frac{1}{\lambda_j}} \geq \left(\mu_i\right)^{\frac{1}{\lambda_i}}$. The strategy profile (1) is a unique equilibrium.*

Proof. This is an equilibrium by construction. Uniqueness is shown in the following series of lemmas. Let (Π_i, Π_j) be any equilibrium. Let

$$T_i = \inf \left\{ t : \Pi_i(t) = \lim_{t' \to \infty} \Pi_i(t') \right\}$$

be the time at which it is known that player i is never going to concede in the future. The following series of lemmas establishes uniqueness:

1. $T_1 = T_2$.

   Suppose not, $T_1 < T_2$. Then player 1 exits with probability zero after T_1. A rational player 2 will never wait with exiting after that date. Let $T = T_1 = T_2$.

2. If Π_j is discontinuous at any given point t, then there exists $\varepsilon > 0$ such that Π_i is constant on $(t - \varepsilon, t)$ and Π_i is not discontinuous at t.

   Suppose that Π_j is discontinuous at t. Then player i who is supposed to exit before t but sufficiently close to t, is better off by waiting a tiny instant to some date just after t. The gain is discrete since a positive mass of players j exit at t, while the loss due to waiting is arbitrary small. For the second part suppose that Π_i is also discontinuous at t. Then there is positive probability that both bidders exit at t. The continuation payoff in the case of a tie is arbitrary within G. But since the game is SCI, there exists $\tau > 0$ such that any continuation payoff profile $(g_i, g_j) \in G$ must satisfy $g_j \leq \tau(\bar{g}_i - g_i)$, hence there is at least one bidder (say i) who gets a continuation payoff less than $\frac{\tau \bar{g}_i}{\tau+1} < \bar{g}_i$. Such bidder will have an incentive to wait a little. The gain is discrete, $\frac{\bar{g}_i}{\tau+1}$, and the loss is arbitrary small.

3. If Π_i is continuous at t then u_j is continuous at t.

   See the definition of u_j, above.

4. There is no interval $(t_1, t_2) \subseteq (0, T)$ such that both Π_i and Π_j are constant on (t_1, t_2). Both Π_i and Π_j are strictly increasing on $(0, T)$.

   For the first claim suppose not, and let t_* be a supremum of upper bounds of all such intervals, so that at least one player exits with positive probability at every date just after t_*. Let i be a player, whose u_i is continuous at t_* (there is at least one). Note that u_i and u_j are both strictly decreasing on the interval (t_1, t_*), so for a fixed $t \in (t_1, t_*)$ there exists a positive constant ε such that for every $s \in (t_* - \varepsilon, t_* + \varepsilon)$ we have $u_i(t) > u_i(s)$. In particular, this means that player i cannot exit with positive probability at dates $s \in (t_*, t_* + \varepsilon)$ and Π_i is constant for all dates in $(t_1, t_* + \varepsilon)$. But then u_j is strictly decreasing on the interval $(t_1, t_* + \varepsilon)$, so player j cannot exit at these dates and Π_j is constant there too. We reach a contradiction, because t_* was assumed to be the supremum of all such intervals. The second part is a by-product of the above.

5. Both Π_i and Π_j are continuous on $(0, T)$.

   Suppose Π_j has a jump at t, then Π_i is constant just before t. This contradicts that both are strictly increasing.

6. Both u_i and u_j are differentiable on $(0, T)$.

   Utilities u_i and u_j are continuous on $(0, T)$ because Π_i and Π_j are. Since Π_i and Π_j are strictly increasing u_i must be constant, hence differentiable.

It follows that $u_i'(t) = 0$ for all $t \in (0, T)$. Hence, an equilibrium must have the above form. ■

As a corollary, we obtain a strong conclusion for the symmetric reputation model of repeated second-price auctions with entry cost.

Corollary 2 *Let Γ be a second-price sealed-bid auction with entry cost, and let $\Gamma(0, \rho, \mu)$ be its symmetric continuous-time version, $\mu_1 = \mu_2$. Then the equilibrium payoff for both players is equal to minmax of Γ.*

Proof. Plug in the equilibrium strategies into the payoff $u_i(t)$ and evaluate it at the exit time $t = 0$. ■

The intuition behind these results is clear. When the game is still in a two-sided reputation phase, both players engage in the form of War of Attrition. Each player wants to win the status of being the only player with reputation, which allows him to consume the surplus associated with such status in the one-sided reputation phase. The competition is so dramatic that in effect both players get the lowest conceivable payoff.

Note that μ may be arbitrarily close to zero and the result is still valid, hence the reputational effects do not vanish as both μ_1 and μ_2 converge to zero.

Now suppose that prior probability belief is not entirely symmetric, $(\mu_j)^{\frac{1}{\lambda_j}} > (\mu_i)^{\frac{1}{\lambda_i}} > 0$. Then there is a mass, $\Pi_i(0) = 1 - \left(\frac{1}{\mu_j}\right)^{\frac{\lambda_i}{\lambda_j}} \mu_i$, of normal types of player i who exit at $t = 0$. Thanks to that, player j obtains a payoff higher than his minmax:

$$u_j(0) = \Pi_i(0)\, \bar{g}_j > 0$$

Moreover, if we fix μ_j and let $\mu_i \to 0$ then the "disadvantaged" player i exits at $t = 0$ in the limit, that is $\Pi_i(0) \to 1$, and the payoff of the "advantaged" player j converges to his maxmax $\bar{g}_j$, which is also his payoff in the one-sided reputation model.

This conclusion does not depend on the magnitude of μ_j, only on the relative difference between μ_i and μ_j. Both of these parameters can be arbitrarily close to zero, yet one of the players may get almost the maxmax payoff and the other may get minmax in the unique equilibrium - it is enough that one of these parameters is sufficiently smaller than the other.

4 Discussion

It was mentioned in the introduction that the current literature recognizes second-price auctions as especially susceptible to collusion between bidders (Robinson (1985) and von Ungern-Sternberg (1988)). Various collusive equilibria are possible even if players are not very patient.

To illustrate this phenomenon, let us consider a very simple example of a complete information second-price sealed-bid auction. Let the value of the object be 100. The following strategy profile forms a perfect equilibrium of $\Gamma(\Delta, \rho)$ for any $\rho > 0$ and $\Delta > 0$: "Players use a symmetric coin to determine who is the designated winner. If tail is observed then player 1 bids 101 and player 2 bids 0, if heads is observed then vice versa, as long as there were no deviations observed. After any deviation players bid as in static symmetric equilibrium, that is they bid 100." Even if players are completely myopic this is still an equilibrium, although in that case such an equilibrium is in weakly dominated strategies. The same kind of collusive equilibrium can be found in the single-unit War of Attrition. For the case of multi-unit second-price auctions, similar equilibria exit (McAdams (2000) and Ausubel and Schwartz (1999)).

On the other hand, collusion in the first price auction is not that simple. Each player has a positive current gain from cheating his partners (which is not true in the second-price auction). Therefore, only a sufficiently high discount factor together with a threat of future punishment can discourage him from deviation.

The intuition built by the above examples is reversed, however, if we consider slightly perturbed auctions. This paper shows that by allowing for arbitrarily small reputation for toughness, the essentially unique and extreme equilibrium is selected as $\Delta \to 0$. In this sense the cooperation among the players is impossible.

Do such reputation effects exist in other auctions – say in the first price auction? The conjectural answer is negative. Auctions other than second-price are not games with strictly conflicting interests. Therefore, the one-sided reputation result of Proposition 2 does not hold. Even more, there is a literature (Cripps and Thomas (2000) and Chan (2000)) showing that if the underlying stage game is any game which does not belong to a family of games with strictly conflicting interests nor it belongs to a family of games with dominant strategies, then the reputation results do not exist. The first-price auction does not belong to either of these groups, so ultimately we obtain the completely opposite conclusion than in the current literature. Roughly, it can be stated as

Conclusion 1 *The cooperation among bidders is more difficult to sustain in the second-price auction than in the first price auction in the model with positive reputation and patient players.*

Symmetric two-sided reputation model applied to second-price auctions produces a few testable hypothesis, some of them distinguish this model from others.

First prediction of the model is that

Conclusion 2 *prices should fall over time in the model of symmetric reputation.*

The game starts with the phase in which players are very aggressive, announcing very high bids, leading to very high realized prices. At certain date, one contestant effectively gives up. From that point on, there is only finitely many periods in which the price is higher than the reserve. Altogether, the incentive to maintain reputation developed in this paper provides yet another explanation of so-called declining price anomaly (Ashenfelter (1989)). It is also consistent with the price trend in the European auctions.

Secondly, the model predicts that the prices in early auctions are not only higher than in later auctions, but also

Conclusion 3 *prices in early auctions are higher than the bidder's estimate of the value of the object in the model of symmetric reputation.*

If observed bids are higher than these estimates, then it does not have to be an account of a "frenzy" bidding or "irrationality" of the players, as has been expressed on such occasions. This paper provides an explanation why rational bidders may choose to bid and pay more than any reasonable estimate of the value of the object. In the case of European UMTS auctions it is obviously very difficult to establish what was the telecoms' estimate of value of the license. However, a few years after the first auction, the claim that telecoms overpaid for the most expensive licenses does not seem to be unreasonable.

Next prediction is that

Conclusion 4 *a bidder who wins early auctions should win almost all later auctions in the model of symmetric reputation.*

In other words, there is a strong bidder emerging after a certain number of auctions. This is particularly sharp in the model above, since a player who wins at price less than s^* for the first time, wins in all but finitely many auction after that. This is an attractive property because it endogenizes the asymmetries between bidders. It has to be mentioned that in European auctions there were dozens of new entrants initially, as well as a great number of incumbents. After the series of auctions only a limited number of telecoms will remain as continent-wide players.

One of the most important products of the analysis above is the following:

Conclusion 5 *the expected payoff of a normal bidder is equal to his minmax in the model of symmetric reputation.*

Thus the fact that virtually all the bidders that participated in pan-European auctions faced serious financial instability afterwards is not a coincidence nor it is a proof of their irrationality.

Finally, low prices alone – below the predicted prices in the one shot auction – do not prove the existence of the bidding ring. The analysis above implies that

Conclusion 6 *(a) low prices may be an effect of very asymmetric reputation profile, rather than cooperation among bidders and (b) small asymmetries may have huge effect on payoff profile, even if the auction is not in almost common values.*

If one of the bidders has μ_i much lower than his opponent, then there is high probability that he will bid low, stopping the auction early. In order to understand the nature of low prices in the repeated second-price auctions it may be useful to check who tends to be the winner. Winner rotation is not consistent with the reputational model developed above, so the explicit cooperation between bidders could be a valid explanation. On the other hand, if the set of winners is constant over auctions, then the reputation is a possible explanation. This last conclusion is similar as in Bikhchandani (1988) and Klemperer (1998) and (2000). They show than the one-shot second-price auction with almost common values has equilibria that change much with tiny variation of the exogenous parameters. Small asymmetries between bidders are magnified, so that the player with a small edge can obtain the object for the price equal to the reserve price. We show in this paper that this feature of second-price auctions is more general, as long as the game is repeated with patient players. In particular, the second-price auction does not have to be in almost common values.

A natural question is what the seller thinks about the bidder's reputation.[5] Is he happy because the cooperation between bidders is impossible?

It is immediate to realize that the seller's equilibrium payoff depends on whether the reputation is symmetric or asymmetric. Suppose the game starts with two-sided and symmetric reputation: $\mu_1 = \mu_2 > 0$. Then the expected payoff profile of both bidders is $(\hat{g}_1, \hat{g}_2)$. It implies that

Conclusion 7 *in symmetric reputation model, almost the entire ex ante total surplus goes to the seller.*

Now, consider an asymmetric reputation case, $\mu_i > 0$ and $\mu_j = 0$. Then bidder j almost always bids reserve price and loses, hence the seller obtains only reserve price as a revenue in almost every period. His discounted payoff is really low, in fact

Conclusion 8 *in the one-sided reputation model, the seller's total discounted revenue is almost zero entirely determined by the reserve price.*

In the intermediate cases, when reputation is two-sided but with various degree of asymmetry, the conclusion about the seller's revenue is equally easy to obtain – the more asymmetric the reputation, the smaller the seller's payoff.

If initially the reputation is symmetric, the seller expects prices to be very high in early auctions and then they should converge to the reserve price in later auctions. This means that the seller may have an incentive to change the format

[5] This is an interesting question despite the fact that we do not treat the seller as a player.

of the auction later in the game after the prices drop, or change the reserve price. He may expect that higher prices will occur after such changes if bidders are not expected to collude perfectly, because in other auctions one-sided reputation effects may not be so strong. This trend also could be observed in European auctions, where in later auctions many governments relied on reserve prices rather than on bidding between telecom companies, and Denmark decided to run an auction corresponding to the first-price sealed-bid auction. Obviously, if the seller cannot commit to the auction format before the whole repeated game is played, then the result brakes down, because the bidders should expect that the seller will change the rules and that they will not be able to consume their reputation.

Up to now we were investigating the incentives of a seller or sellers who offer the objects in a prearranged sequence of auctions. Now imagine that there is a number of distinct sellers that sell in the sequence, but sellers have certain freedom in choosing the timing of his auction. The commitment assumption is much less justified, nevertheless still possible, as many governments were preparing the rules of the UMTS auctions simultaneously, before they observed collapse on the market. Since the declining prices are expected in a symmetric reputation case, sellers want to conduct their auction as early as possible. In a way, seller wants to preempt other sellers. This incentive is admitted directly by Binmore and Klemperer (2001), who were involved in the process of designing the UMTS auction in the U.K. They write (p. 17)

> "Beginning the planning so far in advance of the auction proved shrewd move by the U.K. government. It allowed us plenty of time to develop and test our ideas and, just as importantly, it allowed for a sustained marketing campaign without Britain being overtaken in the race to be first on the European scene (indeed worldwide) with a 3G auction".

5 Extensions

This section reconsiders the model with one-sided reputation. Its main objective is to observe that the result of Proposition 2 holds in many different versions of second price or ascending auctions.

5.1 English auction with entry cost

Consider an ascending auction rather than sealed-bid second-price (English rather than Vickrey). In the context of this model the only difference is that the public outcome does not include the bid/action of the winner. In other words, $y_t = \{\min(b_i, b_j)\}$. It turns out that the proof of Proposition 2 is not affected. In particular, the only thing that changes is the definition of the public path h^{t-1}, but all arguments where this public path is used, remain unchanged. As a consequence, the Proposition 2 extends to the repeated English auction with entry cost.

5.2 Second-price all-pay auctions and Wars of Attrition

Consider an auction where the winner pays the loser's bid and the loser pays his own bid. As previously, values are private and the entire sequence of values is known to the bidder. The payoff is

$$g_i\left(\theta_i^t\right)=\begin{cases} v_i\left(\theta_i^t\right)-b_j & \text{if } 0\leqslant b_j<b_i \\ -b_i & \text{if } 0\leqslant b_i<b_j \\ \frac{1}{2}v_i\left(\theta_i^t\right)-b_i & \text{if } 0\leqslant b_i=b_j \\ v_i\left(\theta_i^t\right) & \text{if } b_j<0\leqslant b_i \\ 0 & \text{if } b_i=-1 \end{cases}$$

In sealed-bid version of the auction bids are submitted sealed but are revealled publicly after the auction. The ascending version of the model is called War of Attrition and only losing bid is revealled publicly.

Such second price all-pay auction is also a game with SCI ex post and the Proposition (2) applies.

5.3 Multiunit uniform price and ascending auctions

Consider a simple version of a multiunit uniform price auction. The seller has $K\geqslant 3$ indivisible objects for sale. There are two risk neutral bidders. Suppose that the underlying value of $k\geq 1$ units to bidder i is $v_i\left(k,\theta_i^t\right)>0$, where θ_i^t is an actual signal realization, and the function v_i is increasing in both arguments and that $v_i\left(0,\theta_i^t\right)=0$. If initial excess demand is one unit, $k_1+k_2=K+1$ then bidders announce only one number, a bid $b_i\in B$ (as a function of signal sequence and history), and the the stage game's payoffs can be expressed as

$$g_i\left(\theta_i^t\right)=\begin{cases} v_i\left(k_i,\theta_i^t\right)-k_ib_j & \text{if } b_j<b_i \\ v_i\left(k_i-1,\theta_i^t\right)-\left(k_i-1\right)b & \text{if } 0\leqslant b_i<b_j \\ \frac{v_i\left(k_i,\theta_i^t\right)+v_i\left(k_i-1,\theta_i^t\right)}{2}-\left(k_i-\frac{1}{2}\right)b_i & \text{if } 0\leqslant b_i=b_j \\ 0 & \text{if } b_i=-1 \end{cases}$$

This is essentially a form of all-pay auction, where it is striclty better to lose at lower prices than at higher prices. Such auctions are also games with SCI ex post and the Proposition (2) applies again. This would apply almost immediately to a multiunit auction in which the excess demand is more than one unit, although the notation would be more complex.

5.4 Second-price auction with interdependent values and no entry cost

This section considers a standard second-price auction *without* entry cost. We do not provide the one-sided reputational result, but we conjecture one. The idea is to use winner's curse in the second-price auction with interdependent values as a substitute for entry cost. We show below that such a game, when endowed with the reserve price high enough, is in fact a game with SCI, although not with SCI ex post.

All the above results in the generic second-price auctions rely on the fact that in the stage game, *conditional on losing*, a bidder facing a committed opponent *strictly* prefers to stop bidding earlier, rather than later. In a standard single-unit second-price auction without the cost of entry, the loser always receives a payoff of zero, no matter what is the profile of bids. In principle, the loser can "punish" aggressive winner by bidding more than the reserve price r, and still lose with probability one. For exactly this reason English and Vickrey auctions with private values and no bidding cost do not have to be games with SCI ex post. Now, we are going to argue that for the case of interdependent values (and the reserve price sufficiently high) these auctions are in fact games with SCI ex ante.

Assume that signal and bid dependent payoffs are the same as in the our baseline model, except that entry cost is zero and reserve price, r, may be positive. The space of bids is now $B = \{r-1, r, r+1, ..., \bar{B}\}$, where bid $r-1$ is interpreted as "not enter". Value of the object to player i will depend on his signal, but also on signal of his opponent. For simplicity assume that the value functions are symmetric, $v\left(\theta_i^t, \theta_j^t\right)$ for $i, j = 1, 2$. In particular, the payoff for bidder i is

$$\begin{cases} v\left(\theta_i^t, \theta_j^t\right) - b_j & \text{if } r \leqslant b_j < b_i \\ 0 & \text{if } r \leqslant b_i < b_j \\ \frac{1}{2}\left[v\left(\theta_i^t, \theta_j^t\right) - b_j\right] & \text{if } r \leqslant b_i = b_j \\ v\left(\theta_i^t, \theta_j^t\right) - r & \text{if } b_j < r \leqslant b_i \\ 0 & \text{if } r - 1 = b_i \end{cases}$$

Let $r_* = v\left(\bar{\theta}, \underline{\theta}\right) - 1$. Consider a bidding function $s_i^* : \Theta \to B$ defined by

$$s_i^*\left(\theta_i^t\right) = \begin{cases} \bar{v} & \text{if } \theta_i^t \neq \underline{\theta} \\ r+1 & \text{if } \theta_i^t = \underline{\theta} \end{cases}$$

for $r > r_*$. We have the following proposition

Proposition 4 *Let Γ be a single-unit English auction with interdependent values, $v : \Theta \times \Theta \to \mathcal{R}$ for $i = 1, 2$ with a reserve price r. Then Γ is a game with SCI ex ante if and only if $r > r_*$.*

Proof. (If) Suppose $r > r_*$. Let player i be committed to the bidding function, $s_i^* : \Theta \to B$, defined above. Consider his opponent, player j. If j wins the auction, then he obtains a payoff equal to

$$v\left(\theta_j^t, \theta_i^t\right) - s_i^*\left(\theta_i^t\right) = \begin{cases} v\left(\theta_j^t, \theta_i^t\right) - \bar{v} & \text{if } \theta_i^t \neq \underline{\theta} \\ v\left(\theta_j^t, \theta_i^t\right) - (r+1) & \text{if } \theta_i^t = \underline{\theta} \end{cases}$$

If $\theta_i^t \neq \underline{\theta}$, then this payoff is negative. If $\theta_i^t = \underline{\theta}$ then this payoff is strictly less than

$$v\left(\theta_j^t, \underline{\theta}\right) - (r_* + 1) \leq 0$$

In any case, winning guarantees strictly negative payoff. Since losing guarantees payoff of zero, player j must maximize the probability of losing the auction. Bidder j loses the auction for sure, if he bids less than the lowest bid of bidder i, which is $r+1$. In effect, player j is forced to stay out or enter and bid the reserve price r in order to lose for sure. In any case, player i grabs the whole object for the lowest possible price r.

(Only if) Suppose now that $r_* \geq r$ and suppose that there is i's bidding function $\tilde{s} : \Theta \to B$, such that j's best reply is to bid r or $r-1$, giving player i the highest payoff of $v\left(\theta_i^t, \theta_j^t\right) - r$. It is impossible that $\min_\theta \tilde{s}(\theta) \geq r+2$, because j's could also best reply with $r+1$ and still lose the auction with probability one. On the other hand, if $\min_\theta \tilde{s}(\theta) \leq r+1$ then bidder j, who observed signal $\bar{\theta}$, would bid $s_j\left(\bar{\theta}\right) = r+1$, rather than r or $r-1$. To see this, note that if j observes $\bar{\theta}$, then his value is $v\left(\bar{\theta}, \theta_i^t\right)$, so the payoff of j from winning is at least

$$\begin{aligned} v\left(\bar{\theta}, \theta_i^t\right) - (r+1) &\geq v\left(\bar{\theta}, \underline{\theta}\right) - (r+1) \\ &= r_* + 1 - r - 1 \geq 0 \end{aligned}$$

This contradicts initial assumption that only r and $r-1$ are j's uniform best replies. Hence the game does not have SCI ex ante. ■

The interesting feature of the auction without entry cost but with interdependent values is the nature of strictness of conflicting interests. It is a little different than in auctions, where the explicit cost of bidding or entering prevents the loser from bidding high bids. In the auctions of this section this force is more subtle – the responsible factor is the winner's curse. The player will not want to challenge the crazy type because winning against such a type is terrible news about the value of the object. Bidding only slightly above the reserve price induces a positive probability of occurrence of such an event.

The requirement that $r > r_*$ is key, otherwise the designated loser would be able to costlessly use bids higher than r to give the winner a payoff that is lower than maxmax.

It is interesting to see that the more the values are private, the higher the reserve price is required for the above proposition. The realized prices must be above r_* and below the highest possible value of the object. In other words, they must occur in the range between $v\left(\bar{\theta}, \underline{\theta}\right)$ and $v\left(\bar{\theta}, \bar{\theta}\right)$. The "size" of this interval depends on how strongly the signal of the other player affects this bidder's value. The closer to private values, the narrower this range is. In the limit, when the values are private, this range becomes just one point, since one's value does not depend on the second argument at all. As a consequence, the reserve would have to be at least as high as the highest value of the object. Obviously, much before that limit is reached, r would become higher than the value of the object for player i, thus making the proposition pointless.

This game, even if it is a game with SCI, is not with SCI ex post. This is one of the reasons why the proof of Proposition 2 cannot be directly applied here. To see that this game is not with SCI ex post, note that in such games player j must face a loss in *every realization* of the nature's move, if he does not play a stage best reply. In the example of this section, however, player j may bid say $r+1$, which is not a stage best reply against a committed opponent, and still there may be a realization of i's signal such that player i bids strictly more than $r+1$. In this case bidder j costlessly challenges bidder i.

6 Proof of Proposition (2)

Consider the following incomplete information stage second-price auction Γ with SCI ex post. Each player observes his sequence of signals at the beginning of the whole game, $\theta_1 = \left\{\theta_1^t\right\}_{t=0}^{\infty}$ and $\theta_2 = \left\{\theta_2^t\right\}_{t=0}^{\infty}$. The values are private. The minmax value of the auction in period t is $\hat{g}_j\left(\theta_j^t\right)$, and the maxmax is $\bar{g}_i\left(\theta_i^t\right)$. For the sealed-bid second-price auction with an entry cost these are $\hat{g}_j\left(\theta_j^t\right) = 0$ and $\bar{g}_i\left(\theta_i^t\right) = v_i\left(\theta_i^t\right) - F$. For War of Attrition they are $\hat{g}_j\left(\theta_j^t\right) = 0$ and $\bar{g}_i\left(\theta_i^t\right) = v_i\left(\theta_i^t\right)$. In the multi unit uniform price auction they are $\hat{g}_j\left(\theta_j^t\right) = v_i\left(k_i - 1, \theta_i^t\right) - \left(k_i - 1\right) r$ and $\bar{g}_i\left(\theta_i^t\right) = v_i\left(k_i, \theta_i^t\right) - k_i r$.

Proposition 5 *Let $\Gamma\left(\Delta, \rho, \left(\mu_i, 0\right)\right)$ be the repeated and perturbed version of stage game with SCI ex post, Γ. Let $\mu_i > 0$ be probability that player i is crazy, using each period the constant "commitment" bid $s_i^* > \bar{v}$ independent of the signal sequence θ_i.*

Then the number of periods in which player j does not play a best response to s_i^* along the history in which player i keeps playing s_i^* is bounded by a number that does not depend on δ.

Proof. Step 1. Consider any sequence of signals, $\theta_j = \left\{\theta_j^t\right\}_{t=0}^{\infty}$ and $\theta_i = \left\{\theta_i^t\right\}_{t=0}^{\infty}$, and suppose that bidder i continues with bids $s_i^* > \bar{v}$. Bidder j may use a mixed strategy – a distribution over bids as a function of a history and signal sequence, θ_j. Consider a pure strategy s_j of bidder j, which lies in the support of his mixed strategy. Any path generated by this pure strategy occurs in equilibrium. For any path consistent with $\left(\theta_j, s_j\right)$ and s^*, we can define T to be the first period, such that player i always plays a s^* at and after this period, and hence player j receives a continuation payoff equal to minmax zero. Let ι_t be an indicator function that takes value of one if j does not play a stage best reply at t, and zero otherwise. Let $n\left(t\right) = \sum_{z=0}^{t-1} \iota_z$ be the number of periods in which player j does not play a stage best reply along any such path up to period t. Therefore, $n\left(T\right)$ is the total number of periods when player i does not play a stage best reply along this path.

Fix an equilibrium. For every θ_i, pick j's signal and pure strategy $\left(\tilde{\theta}_j, \tilde{s}_j\right)$, so that, in pariod $t=0$, the following payoff of player i from playing s_i^* forever is the lowest among all equilibrium signal-strategy pairs $\left(\theta_j, s_j\right)$

$$\left(1-\delta\right) \sum_{z=t}^{\infty} \delta^{z-t} \left[\left(1 - \tilde{\iota}_z\right) \bar{g}_i\left(\theta_i^z\right) + \tilde{\iota}_z\left(-M\right)\right], \tag{2}$$

where $\tilde{\iota}_t$ is the indicator function generated by this signal-strategy of j. That is, no matter what is the signal-strategy of j, player i can guarantee himself at least this payoff, simply by repeating s_i^* forever. That signal and pure strategy generates a unique history, given that s_i^* keeps occurring in every period. Let this public path of play be called h, and the history at t be denoted h^{t-1}.

Note that for any $t>0$, after the history h^{t-1} occurred, the lower bound on payoff of normal i is still given by (2). If there was another equilibrium strategy of player j or his signal profile, that would give i lower payoff at $t>0$ than this bound, then our original signal-strategy would not minimize (2) over all (θ_j, s_j).

Step 2. Along this path, let the stage payoff at t of player j be denoted by $g_{jt}^e\left(\theta_i^t, \theta_j^t\right)$. Observe that since bidder i submits bid s_i^*, this stage payoff of j is no more than j's minmax. That is, bidder j can get the minmax by playing a stage best reply; if he does not play a best reply, then by SCI ex post his payoff is strictly less than the minmax. In other words, there exists $l>0$ such that for any (θ_i, θ_j),

$$g_{jt}^e\left(\theta_i^t, \theta_j^t\right) \leq \hat{g}_j\left(\theta_j^t\right) - \tilde{\iota}_t l \tag{3}$$

This provides a θ_i-independent upper bound on the stage payoffs of player j, as long as s_i^* is played.

Step 3. If in period t player i plays an action that is different than s_i^*, then let $c_{it}\left(\theta_i, \theta_j\right)$ be the continuation payoff that player i will receive at t. That is

$$c_{it}\left(\theta_i, \theta_j\right) = (1-\delta)\sum_{z=t}^{\infty}\delta^{z-t} g_{izt}\left(\theta_i^z, \theta_j^z\right)$$

where $g_{izt}\left(\theta_i^z, \theta_j^z\right)$ is a payoff in period z that i will receive in this equilibrium, when i plays not s_i^* for the first time at t. The stage game is SCI ex post for every state and period z and t, so there exists positive $\tau<\infty$ such that for all (θ_1, θ_2)

$$g_{jzt}\left(\theta_i^z, \theta_j^z\right) - \hat{g}_j\left(\theta_j^z\right) \leq \tau\left(\bar{g}_i\left(\theta_i^z\right) - g_{izt}\left(\theta_i^z, \theta_j^z\right)\right)$$

Taking the discounted sum of LHS and RHS for all periods $z \geq t$, we obtain this relationship also for every discounted payoff profile $c_{it}\left(\theta_i, \theta_j\right)$, $c_{jt}\left(\theta_i, \theta_j\right)$ and every profile of signal sequences (θ_i, θ_j):

$$c_{jt}\left(\theta_i, \theta_j\right) - (1-\delta)\sum_{z=t}^{\infty}\delta^{z-t}\hat{g}_j\left(\theta_j^z\right) \leq \tau\left[(1-\delta)\sum_{z=t}^{\infty}\delta^{z-t}\bar{g}_i\left(\theta_i^z\right) - c_{it}\left(\theta_i, \theta_j\right)\right] \tag{4}$$

Player i's continuation payoff in period t has to be at least as high as the payoff form playing s_i^* forever, because this is always an option of player i. By continuing to play s_i^* forever player i will obtain a payoff at least

$$(1-\delta)\sum_{z=t}^{\infty}\delta^{z-t}\left[\left(1-\tilde{\iota}_z\right)\bar{g}_i\left(\theta_i^z\right) + \tilde{\iota}_z(-M)\right] \leq c_{it}\left(\theta_i, \theta_j\right) \tag{5}$$

This assumes that the worst $\left(\tilde{\theta}_j, \tilde{s}_j\right)$ from the perspective of player i occurred, minimizing the bound (2). If some other signal was observed by j, or if some other pure strategy is used by j, then player i will receive payoff $c_{it}(\theta_i, \theta_j)$ that is definitely better than the bound (5).

Combining (4) and (5) we have

$$\begin{aligned} c_{jt}(\theta_i, \theta_j) - (1-\delta)\sum_{z=t}^{\infty}\delta^{z-t}\hat{g}_j\left(\theta_j^z\right) &\leq \tau\left[(1-\delta)\sum_{z=t}^{\infty}\delta^{z-t}\left[\bar{g}_i\left(\theta_i^z\right) - (1-\tilde{\iota}_z)\,\bar{g}_i\left(\theta_i^z\right) - \tilde{\iota}_z(-M)\right]\right] \\ &\leq \tau 2M(1-\delta)\sum_{z=t}^{\infty}\delta^{z-t}\tilde{\iota}_z \end{aligned}$$

In period t and after h^{t-1} was observed, there will be at most $n(T) - n(t)$ periods where player j does not play a best reply. The sum on the RHS is the largest if $n(T) - n(t)$ periods in which not a best reply is played occur at the beginning. Hence

$$c_{jt}(\theta_i, \theta_j) - (1-\delta)\sum_{z=t}^{\infty}\delta^{z-t}\hat{g}_j\left(\theta_j^z\right) \leq \tau 2M\left(1-\delta^{n(T)-n(t)}\right) \tag{6}$$

This provides a θ_i-independent upper bound on the continuation payoff of player j after the event that i plays an action different than s_i^*.

Step 4. Let $\pi_v^t(\theta_i)$ be the probability belief of j at period t and after observing h^{t-1}, that normal player i who observed θ_i stops playing crazy bid at period $v \geq t$. Let $\pi_v^t = E_{\theta_i|(\theta_j, h^{t-1})}\pi_v^t(\theta_i)$ be the expected probability belief of j conditional on $\left(\theta_j, h^{t-1}\right)$ that normal player i plays a bid different than s_i^* for the first time at some period $v \geq t$. To save notation let $\pi_v = \pi_v^0$. Note that since i is crazy with probability μ_i we have

$$\sum_{z=0}^{T-1}\pi_z \leq 1 - \mu_i \tag{7}$$

The Bayesian updating is particularly simple here, namely if $\left(\theta_j, h^{t-1}\right)$ was observed, then j's posterior at t is

$$\pi_v^t = \frac{\pi_v}{1 - \sum_{z=0}^{t-1}\pi_z} \tag{8}$$

After history h^{t-1}, player j has the expected payoff at t, which must be at least as high as the minmax. Hence, for every $t = 0, 1, ..., T-1$

$$\begin{aligned} (1-\delta)\sum_{r=t}^{\infty}\delta^{r-t}\hat{g}_j\left(\theta_j^r\right) \leq\; & E_{\theta_i|(\theta_j, h^{t-1})}\sum_{v=t}^{T-1}\pi_v^t(\theta_i)\left\{\begin{array}{c} (1-\delta)\sum_{r=t}^{v-1}\delta^{r-t}g_{jr}^e\left(\theta_i^r, \theta_j^r\right) + \\ +\delta^{v-t}c_{jv}(\theta_i, \theta_j) \end{array}\right\} + \\ & + E_{\theta_i|(\theta_j, h^{t-1})}\left(1 - \sum_{v=t}^{T-1}\pi_v^t(\theta_i)\right)\left\{\begin{array}{c} (1-\delta)\sum_{r=t}^{T-1}\delta^{r-t}g_{jr}^e\left(\theta_i^r, \theta_j^r\right) + \\ +(1-\delta)\sum_{r=T}^{\infty}\delta^{r-t}\hat{g}_j\left(\theta_j^r\right) \end{array}\right\} \end{aligned}$$

Subtracting the discounted minmax from both sides we get

$$
\begin{aligned}
0 \quad \leq \quad & E_{\theta_i|(\theta_j,h^{t-1})} \sum_{v=t}^{T-1} \pi_v^t(\theta_i) \left\{ \begin{array}{c} (1-\delta)\sum_{r=t}^{v-1} \delta^{r-t}\left[g_{jr}^e\left(\theta_i^r,\theta_j^r\right)-\hat{g}_j\left(\theta_j^r\right)\right]+ \\ +\delta^{v-t}\left[c_{jv}\left(\theta_i,\theta_j\right)-(1-\delta)\sum_{r=v}^{\infty}\delta^{r-v}\hat{g}_j\left(\theta_j^r\right)\right] \end{array} \right\} + \\
& +E_{\theta_i|(\theta_j,h^{t-1})}\left(1-\sum_{v=t}^{T-1}\pi_v^t(\theta_i)\right)\left\{(1-\delta)\sum_{r=t}^{T-1}\delta^{r-t}\left[g_{jr}^e\left(\theta_i^r,\theta_j^r\right)-\hat{g}_j\left(\theta_j^r\right)\right]\right\}
\end{aligned}
$$

However, we obtained the upper bounds (3) and (6) on j's payoffs. The expected "upper bound of payoffs" must be no less than the expected "payoffs", hence it must be that for every $t = 0, 1, ..., T-1$

$$
\begin{aligned}
0 \quad \leq \quad & E_{\theta_i|(\theta_j,h^{t-1})} \sum_{v=t}^{T-1} \pi_v^t(\theta_i) \left\{(1-\delta)\sum_{r=t}^{v-1}\delta^{r-t}\left(-\tilde{\imath}_r l\right)+\delta^{v-t}\tau 2M\left(1-\delta^{n(T)-n(v)}\right)\right\} \\
& +E_{\theta_i|(\theta_j,h^{t-1})}\left(1-\sum_{v=t}^{T-1}\pi_v^t(\theta_i)\right)\left\{(1-\delta)\sum_{r=t}^{T-1}\delta^{r-t}\left(-\tilde{\imath}_r l\right)\right\}
\end{aligned}
$$

Since these upper bounds are θ_i-independent, we can write that for every $t = 0, 1, ..., T-1$

$$
\begin{aligned}
0 \quad \leq \quad & \sum_{v=t}^{T-1} \pi_v^t \left\{\sum_{r=t}^{v-1}\delta^{r-t}\left(-\tilde{\imath}_r l\right)+\delta^{v-t}\tau 2M\frac{1-\delta^{n(T)-n(v)}}{1-\delta}\right\} \\
& +\left(1-\sum_{v=t}^{T-1}\pi_v^t\right)\left\{\sum_{r=t}^{T-1}\delta^{r-t}\left(-\tilde{\imath}_r l\right)\right\}
\end{aligned}
$$

Use the Bayesian updating formula (8) to eliminate π_v^t,

$$
\begin{aligned}
0 \quad \leq \quad & \sum_{v=t}^{T-1} \frac{\pi_v}{1-\sum_{z=0}^{t-1}\pi_z} \left\{\sum_{r=t}^{v-1}\delta^{r-t}\left(-\tilde{\imath}_r l\right)+\delta^{v-t}\tau 2M\frac{1-\delta^{n(T)-n(v)}}{1-\delta}\right\} \\
& +\left(1-\sum_{v=t}^{T-1}\frac{\pi_v}{1-\sum_{z=0}^{t-1}\pi_z}\right)\left\{\sum_{r=t}^{T-1}\delta^{r-t}\left(-\tilde{\imath}_r l\right)\right\}
\end{aligned}
$$

Now, multiply by the denominator and use (7), to get $t = 0, 1, ..., T-1$

$$
0 \leq \sum_{v=t}^{T-1} \pi_v \left\{\sum_{r=t}^{v-1}\delta^{r-t}\left(-\tilde{\imath}_r\right)+\delta^{v-t}\frac{\tau 2M}{l}\frac{1-\delta^{n(T)-n(v)}}{1-\delta}\right\}+\mu_i\left\{\sum_{r=t}^{T-1}\delta^{r-t}\left(-\tilde{\imath}_r\right)\right\} \tag{9}
$$

This system of inequalities is a simpler version of inequalities in CDP. They show that there is an independent of δ upper bound on $n(T)$ (see CDP). Hence the conclusion.

Observe that in any equilibrium player i gets the payoff that is higher than (2), and in the same time the number of periods when $\tilde{\imath}_t = 1$ is bounded. ■

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