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Reputation in Multi-unit Ascending Auction

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Reputation in Multi-unit Ascending Auction.

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Abstract

This paper considers a model of multi-unit ascending auction with two players and known values. This standard model is not robust to a small reputational perturbation. If reputation is one-sided, then the player without reputation lowers her demand in order to clear the market and stop the auction immediately at the reserve price. Hence, the player with reputation buys all the units she demands at the lowest possible price. If the reputation is on both sides, then the War of Attrition emerges. In any case, there is a unique equilibrium payoff profile. One feature of the equilibrium of the two-sided model is that market clearing is delayed and the expected realized price is higher than the reserve price.

Keywords: Mulit-unit auction, uniform price, ascending auction, reputation, complementarities, aggressive bidding.

JEL Classification: D44

1 Introduction

This paper considers a model of ascending multi-unit auction with two bidders and many units of a homogenous good for sale. In such an auction the bidding continues until the total demand of the bidders is equal to the total number of units available.

In the simplest model of this auction format, it is assumed that the bidders' marginal utility is common knowledge and constant. There is a great multiplicity of equilibria in this model – and consequently the model has no predictive power. This paper shows, that this multiplicity of equilibria is not robust. If we introduce an arbitrarily small but positive probability that bidders may be of a type that demands "large" bundles, even if prices are high, then the unique equilibrium payoff profile emerges. We will call such types "irregular".

If the reputation is one-sided, then in any equilibrium there is no delay. The player who has no reputation decreases his demand to the level that clears the market, in order to prevent the price from rising. If such a strong asymmetry occurs, the seller will not obtain any additional revenue beyond the reserve price. If the reputation is on both sides, then the equilibrium is mixed, and hence there is some delay. Each normal bidder decreases his demand to clear the market with some density over some price interval. However, some surplus stays with the bidders, since the expected price is less than the value of the object.

The introduction of irregular, aggressive types follows the approach of the reputation literature. In this approach the irregular types are assumed to be behavioral – they demand x units, as long as the price is lower than the value of the object, where exogenous number x is relatively "large" relative to the total supply. It is shown that there is only one equilibrium payoff profile in weakly undominated strategies in this version of the model.

It is argued later, however, that this kind of aggressive behavior of irregular types does not have to be assumed, but rather can be generated by a very plausible utility function; for that reason the irregular types do not have to be irrational per se. The utility function of the irregular type will have a very high marginal value for the x'th unit.

Why is this type plausible? Complementarities between objects may be an explanation for high marginal value for the x'th unit. The irregular type attaches some positive value to the objects as long as the entire bundle of size x is acquired; if he receives fewer units then they are all worthless, if he receives more than x, then all additional units are also worthless.

One might be tempted to say that irregular type is aggressive because he has valuation higher than the normal type. It is interesting to see that marginal values matter, rather than total valuation.

There are many real-life examples of multi-unit ascending auctions, where complementarities between objects are bound to play some role. The Federal Communications Commission (FCC) in the US was using multi-unit ascending auctions since 1994 in order to sell thousands of radio spectrum licenses for providing personal communications services such as mobile telephones, paging, or wireless computer networks. See McAfee and McMillan (1996) or Cramton (2001). Licenses varied in geographical coverage and amount of spectrum. It is recognized that complementarities among licenses are natural in these instances. For example, a license for Philadelphia region may be worth more to a company, if it wins the adjacent licenses in New York and Washington. Another example of multi-unit ascending auction was some of the UMTS auctions in Europe. German and Austrish governments auctioned off 12 blocks of spectrum, and each bidder could buy either two or three licenses.

2 Model

There is $L \geq 3$ indistinguishable units of a good for sale. The normal value of each unit is common knowledge and common, equal to $v > r$, where $r \geq 0$ is a reserve price. There are 2 bidders, each one can buy at most K units, where $\frac{L}{2} < K \leq L-1$. This upper bound is an exogenous parameter and could be set by the seller to prevent too much of a concentration of the objects in one hands.

The format of an auction is ascending. It will be modeled in a usual "clock" manner, where bidders announce their individual demands by means of usual "buttons". The price is continuously rising and each bidder announces how many units he wants to buy at any current price by keeping that number of buttons pushed. By releasing the buttons, the bidder can decrease her demand by one or more units. The buttons are not repushable, so the bidder cannot increase the demand.

One demand reduction may induce an immediate round of new reductions and all of them may happen at the same price. To keep track of what triggers what, assume that releasing a button automatically stops the clock for a short intermission, so that the opponent can accommodate this new situation, then the auction resumes.

The auction stops at the market clearing price, p^c, the first price where the excess demand is nonpositive. The distribution of objects to bidders is according to their demands at the market clearing price.

STRATEGIES: History at price p must describe how and when the bidders decreased their individual demands prior to price p. Thus, the history will be identified with a list of prices, when such changes occurred, and the resulting demands. Suppose that N of these changes occurred, let the history at price p be a list

$$h\left(p, z_1^N, z_2^N\right) = \left\{\left(p^0, z_1^0, z_2^0\right), \left(p^1, z_1^1, z_2^1\right), ..., \left(p^N, z_1^N, z_2^N\right)\right\} \text{ for } p \geq p^N,$$

where for every $n = 1, ..., N$ we have $p^n \geq p^{n-1}$ and $z_i^n \leq z_i^{n-1}$ for all $i = 1, 2$ and $z_i^n < z_i^{n-1}$ for some i. Each element (p^n, z_1^n, z_2^n) denotes an event that at least one of the bidders, say i, decreased his demand at price p^n to some new quantity $z_i^n < z_i^{n-1}$. If $p^n = p^{n-1}$ then after one of the bidders decreased his demand at price p^{n-1}, someone decreased the demand again immediately after the intermission. Initial element of this list denotes the starting point, where $p^0 = r$ is just the reserve price and $z_i^0 \leq K$ is an initial demand of player i. The final element of this list must have $z_1^N + z_2^N > L$ for the auction to still continue at and after p^N, otherwise the auction stops at p^N, and this is a market clearing price, $p^c = p^N$.

It is well known that a standard sealed-bid second-price auction with one object and two bidders is strategically equivalent to a clock version of an English auction. In the same way we will consider a sequential sealed-bid counterpart of the multi-unit clock auction described above. This auction will be regarded as a series of simultaneous-move games. The outcome of each such game determines the parameters of the game that will be played in the next stage. The final payoffs are determined in the final stage.

Specifically, fix an n'th intermission (stage) and a history $h(p^n, z_1^n, z_2^n)$. It means that the price and demands are (p^n, z_1^n, z_2^n). We will assume that bidders play a simultaneous-move game, in which bidder $i = 1, 2$ chooses two parameters: a "bid" $b_i \geq p^n$ and a demand $z_i \in \{z_i^n - 1, ..., 0\}$. If bidder 1 has a bid lower than bidder 2, then his bid becomes a new standing price during next intermission, $p^{n+1} = b_1 = \min\{b_1, b_2\}$, the new demand of bidder 1 is $z_1^{n+1} = z_1$, and the demand of bidder 2 remains unchanged at $z_2^{n+1} = z_2^n$. The new and commonly known history in the $(n+1)$'th intermission is

$$h'\left(p^{n+1}, z_1^{n+1}, z_2^{n+1}\right) = \left\{h\left(p^n, z_1^n, z_2^n\right), \left(b_1, z_1, z_2^n\right)\right\}$$

If there is a tie, $b_1 = b_2$, then the new demands for bidders 1 and 2 are equal to z_1 and z_2, respectively, and the new event added to the current history is $\left(p^{n+1}, z_1^{n+1}, z_2^{n+1}\right) = (b_1, z_1, z_2)$. The process continues until $z_1^{n+1} + z_2^{n+1} \leq L$.

A behavior of player i after history $h\left(p^N, z_1^N, z_2^N\right)$ is denoted by $F_i\left(z_i, b_i, h\left(p^N, z_1^N, z_2^N\right)\right)$, which is a probability that player i decreases his demand to $z_i \in \left\{z_1^N - 1, ..., 0\right\}$ at or before price $b_i \geq p^N$, after the history $h\left(p^N, z_1^N, z_2^N\right)$. If the history is null – before the game started – then the choice is an initial demand $z_i^0 \in \{0, 1, ..., K\}$.

Let $N-1$ be the last stage where there is an excess demand, that is $z_1^{N-1} + z_2^{N-1} > L$ and $z_1^N + z_2^N \leq L$. Then the market clearing price is $p^c = p^N$, and the distribution of objects among bidders is z_1^N and z_2^N.

Payoffs. Suppose that at the beginning of the auction nature chooses one of two types for each bidder. With prior probability $\mu_i \geq 0$ bidder i is irregular. With remaining probability, $1 - \mu_i$, bidder i is normal and he has gross payoff function linear in quantity, $V_{nor}(z) = vz$, where z is number of units acquired. The realized net payoff for normal bidder i is $\left(v - p^N\right) z_i^N$, where p^N is market clearing price.

We will focus on equilibria in which the weakly dominated strategies are not used. A behavioral strategy is weakly dominated if there is another behavioral strategy that brings payoff higher for all strategies of the opponent and strictly higher for some strategies of the opponent. For instance, bidding positive demands for prices higher than v is weakly dominated by a strategy that involves decreasing the demand to zero at price equal to v. Also a normal type who demands 1 unit has a weakly dominated strategy to decrease his demand to zero at a price less than v. This strategy is dominated by a strategy of bidding precisely one unit until the price reaches v.

For now, assume that irregular type of any bidder i is behavioral. That is, he demands $x_i \in \{1, ..., L-1\}$ for all prices, by assumption. Later on, a more complicated version of the model will be discussed, in which even irregular types are endowed with utility functions and are rational. It will be shown that under the assumption that bidders do not use weakly dominated strategies, the equilibrium already analyzed is still unique.

3 Literature versus predictions of the model.

It is known that there is a great multiplicity of equilibria in the unperturbed game, $\mu_i = 0$ for $i = 1, 2$. Very low prices, even equal to an exogenous reserve price, may be supported by an equilibrium.

Example 1 *Let $a \in \{1, ..., L-1\}$ be an arbitrary constant. Suppose that the bidders "split" the pie according to this constant – that is, they start bidding with $z_1^0 = a$ and $z_2^0 = L - a$ and then they never give up before the price is equal* *to value, that is* $F_i(z_i, b_i, \cdot) = 0$ *for all* $b_i \in (r, v)$ *and* $z_i \in \{0, ..., z_i^0 - 1\}$ *and* $F_i(0, b_i, \cdot) = 1$ *for* $b_i \geq v$.

Player 1 has no incentive to deviate because the market clearing price is r, and his ultimate payoff is $(v - r)a$. Any deviation to lower demand does not affect the price, but assigns less to him and hence the payoff is lower. Any deviation to a higher demand rises the price but does not affect the eventual assignment, hence the payoff is lower too.

Obviously, this collusive-seeming equilibrium is neither efficient nor revenue-maximizing. Moreover, there is many equilibria that differ in price and assignment of the units to the bidders.

Such a possibility of low revenues was first noticed by Wilson (1972), in the context of a divisible-object and sealed-bid version of the model. His point was elaborated by Back and Zender (1993) who showed that this type of auctions admits equilibria that lead to arbitrary low prices. The reason responsible for low prices is so-called demand reduction. Demand reduction is an incentive of each bidder to cut the demand to stop the auction even if the price is below the true value of the object. The cost of stopping the auction earlier is that the marginal unit is not acquired for the price that is lower than its value, and hence the potential surplus is lost to the opponent, but savings are that all inframarginal units are acquired for the price lower than otherwise. This incentive to lower the demand does not exists in single-unit ascending auction because there are no inframarginal units. For instance, in Example (1), inelastic demands are used to skyrocket the price, should the opponent try to acquire more units than this equilibrium postulates.

The equilibrium behavior in the model analyzed in this paper also recognizes this incentive for demand reduction, for the realized prices are always less than v. But since the equilibrium is unique, it is possible to address the question of comparative statics, and in particular the magnitude of the demand reduction. If the reputational profile is very asymmetric, say one of the bidders is known to be normal for sure, then the demand reduction leads the lowest price possible, r. The player reducing his demand is the one without reputation. On the other hand, if the model is symmetric, then the normal bidders engage in a struggle akin to War of Attrition, and the demand reduction is not that dramatic.

The literature also mentions a phenomenon called collusive-seeming equilibrium, which essentially is a different interpretation of the demand reduction, emphasizing rather the way how bidders can retain some surplus by coordinating their behavior. The bidding is seen as a negotiation or bargaining device to split the objects between the bidders. In Example (1), a constant a determines such a split. The reputational model of this paper may be seen as a formalization of this interpretation. The reputational profile may be interpreted as a profile of bargaining powers – that is, the larger the probability that a particular bidder is irregular and will not reduce the demand, the stronger this bidder appears to be, the sooner his opponent reduces his demand and eventually the higher payoff this bidder gets.

There were a few fairly promising attempts to give some recommendations to the seller who, for some reasons, must use uniform price mechanism. One argument, made by Back and Zender (1999) and McAdams (1999) gives the seller an opportunity to adjust the supply of shares after bidders announced their demands. Since bidders' collusive behavior is sensitive to this quantity, they are not able to coordinate on equilibrium so well. This again can easily be seen in terms of the Example (1); if the seller can decrease the number of units L after the demands are announced,[1] then the cooperation among bidders in a collusive-seeming way can effectively be punished by the seller.

Allowing for some alternation of total supply is closely related to the model of Klemperer and Meyer (1989). Their approach, in the context of an auction, is to assume that there are some noisy players who purchase some part of the total supply, leaving an uncertain residual for the strategic bidders. The probability of such a supply variation can be arbitrarily small, and still a unique equilibrium is selected (in the model with divisible objects), but this noise has to have a full support. The model in this paper introduces a perturbation for the same reason – to select a unique equilibrium – although the nature of this perturbation is different.

McAdams (1999) proposes a mechanism in which a seller gives a small monetary prize to reward a bidder who does not reduce the demand too early. The intuition is that in this case each bidder can win the prize by increasing the price by an arbitrarily small bit. Such a small change leads to a Bertrand-like bidding and eventually upsets the collusive-seeming equilibria. The realized price is equal to v. However, only equilibria in pure strategies are considered in this analysis. The model presented in this paper suggests however, that a mixed equilibrium with a random winner and random price is plausible. Such equilibrium is robust – small money prize perturbations will not be able to change the strategies of the bidders substantially, and the collusive-seeming equilibria apparently are not eliminated, the realized price would still occur below v.

From the point of view of the reputation literature the model of this paper is closely related to Abreu and Gul (2000), who analyze the bargaining problem with reputation on both sides. It was already mentioned that the multi-unit ascending auction can be seen as a mechanism in which bidders negotiate/bargain how to split the objects among themselves. The only substantial difference is how the cost of bargaining is specified – in the classical bargaining model this is discounting, in the multi-unit ascending auction this is increasing price. The equilibrium in the Example (1) corresponds to the well-known no delay outcome of the bargaining problem. See also Ausubel and Schwartz, (1999). The result of Abreu and Gul (2000) that there is some delay in reaching the agreement in bargaining corresponds to the similar delay result in the model below.

Abreu and Gul (2000) is itself related to the paper by Fudenberg and Tirole (1986), who introduce reputation into the standard model of a War of Attrition and receive the uniqueness of the equilibrium there. The uniqueness of the equilibrium payoff in the model of ascending multi-unit auction below is not

[1] They analyze a sealed-bid uniform price, rather than its ascending counterpart.

surprising then, given that such an auction is a form of War of Attrition.

4 Equilibrium with behavioral types

This section assumes that irregular type of bidder i demands x_i for all prices less than v. We will consider strategies in which normal bidder i starts demanding x_i at the reserve price, $z_i^0 = x_i$ for $i = 1, 2$. Let $X = x_1 + x_2$. Suppose that at the beginning of this ascending auction there is some excess demand, $X - L \geq 1$.

Let $y_i = L - x_j$ be a residual demand of bidder i, once the irregular demand of bidder j is fully satisfied. Note that $x_i > y_i$. Moreover, let $\omega_i = \frac{y_i}{x_i - y_i} > 0$.

Consider a family of strategies, in which if normal type of any bidder ever decreases his demand, then he decreases it to y_i. That is, for any $b_i \leq v$ we have $F_i\left(z_i, b_i, \{(r, x_1, x_2)\}\right) = 0$ if $z_i \neq y_i$. Define $\Pi_i\left(b_i\right) = \left(1 - \mu_i\right) F_i\left(y_i, b_i, \{(r, x_1, x_2)\}\right)$ to be the probability that (unconditional) player i decreases his demand to y_i before or at the time where the price reaches b_i. If these strategies are played then the game stops immediately after any of the bidders decides to reduce his demand – that is, there is only one stage, $p^c = p^1$. It is possible to pin down the entire strategy and ultimately the strategy profile. For instance, for any history involving demands $z_i^N < x_i$ and $z_j^N \leq x_j$ (after bidder i is revealed to be normal and bidder j is or is not revealed to be normal), bidder i gives up immediately, $F_i\left(L - z_j^N, \cdot, h\left(p^N, z_i^N, z_j^N\right)\right) = 1$, and his opponent gives up only at price equal to v, $F_j\left(\cdot, b_j, h\left(p^N, z_i^N, z_j^N\right)\right) = 0$ for all $b_j < v$.

If bidders use the above strategies, then $\left(\Pi_1(\cdot), \Pi_2(\cdot)\right)$ are enough to specify the payoff profile. The following proposition shows that if these functions have a particular form then they generate a unique equilibrium payoff profile.

Proposition 2 *If $0 \leq \left(\mu_i\right)^{\omega_i} \leq \left(\mu_j\right)^{\omega_j}$ and $0 < \mu_j$, then the unique Perfect Bayesian equilibrium payoff in weakly undominated strategies is generated by a strategy profile that involves*

$$\begin{cases} \Pi_j(b) = 1 - \left(\frac{v-b}{v-r}\right)^{\omega_i} \\ \Pi_i(b) = 1 - \mu_i\left(\mu_j\right)^{-\frac{\omega_j}{\omega_i}}\left(\frac{v-b}{v-r}\right)^{\omega_j} \end{cases} \tag{1}$$

for $b \in \left[r, v - (v-r)\left(\mu_j\right)^{\frac{1}{\omega_i}}\right]$.

Note that there may be a multiplicity of equilibria, but there is a unique equilibrium payoff. The unique equilibrium payoff profile can be obtained by evaluating the utility functions, defined below, at price r. The payoff ultimately is

$$\begin{cases} u_i(r) = y_i(v-r) \\ u_j(r) = \left[x_j\left(1 - \mu_i\left(\mu_j\right)^{-\frac{\omega_j}{\omega_i}}\right) + y_j\mu_i\left(\mu_j\right)^{-\frac{\omega_j}{\omega_i}}\right](v-r) \end{cases}$$

When the reputation is one-sided, $\mu_i = 0$ and $\mu_j > 0$, then player i decreases his demand to y_i immediately at price r, $\Pi_i(r) = 1$, and the payoff profile is

$$\begin{aligned} u_i(r) &= y_i(v - r) \\ u_j(r) &= x_j(v - r) \end{aligned}$$

4.1 Proof of the Proposition (2).

Let $Z^N = z_1^N + z_2^N$ be the total demand at stage $N = 0, 1, ...$ Let also μ_i^N be a posterior probability belief in the N'th intermission that player i is irregular, where $\mu_i^0 = \mu_i$ is a prior probability belief.

The idea of the proof is to find an equilibrium by backwards induction. This reduces the entire game to a series of games. Moreover, each such game turns out to be a simple timing game, namely a War of Attrition.

Lemma (3) below deals with the last stage of the auction, where the excess demand is equal to one unit. It shows that if reputation is one-sided, then player who is known to be normal decreases his demand immediately. Lemma (4) provides an inductive step, which is to show that if in one-sided model a player without reputation reduces his demand immediately when the excess demand is up to some $D - 1$, then in one-sided model a player without reputation reduces his demand immediately when the excess demand is D. Lemma (5) combines these two results to conclude that in one-sided model, the player without reputation reduces immediately his demand to the level just enough to stop the auction. Thanks to this, the auction at any stage is equivalent to a simple War of Attrition and representable by the above $(\Pi_1(\cdot), \Pi_2(\cdot))$.

Lemma 3 *Suppose that in stage N the excess demand is one, $Z^N - L = 1$, and suppose that $0 \leq \left(\mu_i^N\right)^{\omega_i} \leq \left(\mu_j^N\right)^{\omega_j}$ and $0 < \mu_j^N$ (or $z_j^N = x_j$, $z_i^N \leq x_i$). Let $y_i^N = L - z_j^N$ and $\omega_i^N = \frac{y_i^N}{x_i^N - y_i^N}$. Then the unique equilibrium payoff is generated by the strategies*

$$\begin{cases} \Pi_j^N(b) = 1 - \left(\frac{v-b}{v-p^N}\right)^{\omega_i^N} \\ \Pi_i^N(b) = 1 - \mu_i^N \left(\mu_j^N\right)^{\left(-\omega_j^N/\omega_i^N\right)} \left(\frac{v-b}{v-p^N}\right)^{\omega_j^N} \end{cases}$$

Proof. Any decrease of the demand by any of the bidders stops the auction. Obviously a normal type of bidder i will decrease his demand by one unit, to $L - z_j^N$. Let $\Pi_i(b_i) = \left(1 - \mu_i^N\right) F_i\left(y_i^N, b_i, h\left(p^N, z_1^N, z_2^N\right)\right)$ denote a probability that player i decreases his demand to $L - z_j^N$ by the time the price reaches b_i, stopping the auction. For simplicity of notation, let us ignore the superscript N in this proof, and write μ_i, z_i, y_i, p and ω_i instead of μ_i^N, z_i^N, y_i^N, p^N and ω_i^N.

If a normal bidder i is the first to decrease his demand at price b, then his realized payoff is $y_i(v - b)$. On the other hand, if i's opponent, bidder j, decides to decrease his demand first, at some price b_j, then i receives the quantity that he demanded, z_i. The resulting payoff for normal player i in this case is $z_i(v - b_j)$.

Given the strategy of player j summarized in $\Pi_j(\cdot)$, the expected payoff of player i from exiting at price b is

$$u_i(b) = z_i(v-p)\Pi_j(p) + \int_p^b z_i(v-b_j)\,d\Pi_j(b_j) + y_i(v-b)(1-\Pi_j(b))$$

Let

$$R_i = \inf\left\{b : \Pi_i(b) = \lim_{b' \to v} \Pi_i(b')\right\}$$

be the time at which it is known that player i is never going to decrease his demand in the future.

Suppose that the situation involves a reputation is on both sides, $0 < (\mu_i)^{\omega_i} \leq (\mu_j)^{\omega_j} < 1$. Let us establish a few properties of equilibria.

1. $R_1 = R_2$. If $R < v$, then for all $b \in (R, v)$, $\Pi_1(b) = 1 - \mu_1$ and $\Pi_2(b) = 1 - \mu_2$, where $R = R_1 = R_2$.

   Suppose not, $R_1 < R_2$. Then player 1 exits with probability zero in the interval (R_1, v). A normal type of player 2 will never wait with exiting after R_1.

2. $R > p$

   If $R = p$, then no player should decrease his demand at p but rather should wait a little, proving that he is irregular.

3. If Π_j is discontinuous at any given point $b \in (p, v)$, then there exists $\varepsilon > 0$ such that Π_i is constant on $(b - \varepsilon, b)$. If Π_j is discontinuous at any given point $b \in [p, v)$ then Π_i is not discontinuous at b.

   Suppose that Π_j is discontinuous at b. Then normal player i who is supposed to exit before b but sufficiently close to b, is better off by waiting a tiny instant to some date just after b. The gain is discrete since a positive mass of players j exit at b and $z_i > y_i$, while the loss due to waiting is arbitrary small. For the second part suppose that Π_i is also discontinuous at b. Then there is positive probability that both bidders exit at b. On the other hand, if i waits a bit, this gives a discrete increase of a payoff, while a loss due to waiting is arbitrary small.

4. If Π_i is continuous at b then u_j is continuous at b.

   See the definition of u_j, above.

5. There is no interval $(b_1, b_2) \subseteq (p, R)$ such that both Π_i and Π_j are constant on (b_1, b_2). Both Π_i and Π_j are strictly increasing on (p, R).

   For the first claim suppose not, and let b_* be a supremum of upper bounds of all such intervals, so that at least one bidder exits with positive probability at every price just after b_*. Let i be a bidder, whose u_i is continuous at b_* (there is at least one). Note that u_i and u_j are both strictly decreasing on the interval (b_1, b_*), so for a fixed $b \in (b_1, b_*)$ there exists a positive constant ε such that for every $s \in (b_* - \varepsilon, b_* + \varepsilon)$ we have $u_i(b) > u_i(s)$. In particular, this means that bidder i cannot exit with positive probability at dates $s \in (b_*, b_* + \varepsilon)$ and Π_i is constant for all dates in $(b_1, b_* + \varepsilon)$. But then u_j is strictly decreasing on the interval $(b_1, b_* + \varepsilon)$, so bidder j cannot exit at these dates and Π_j is constant there too. We reach a contradiction, because b_* was assumed to be the supremum of all such intervals. The second part is a by-product of the above.

6. Both Π_i and Π_j are continuous on (p, R).

   Suppose Π_j has a jump at b, then Π_i is constant just before b. This contradicts that both are strictly increasing.

7. Both u_i and u_j are differentiable on (p, R).

   Utilities u_i and u_j are continuous on (p, R) because Π_i and Π_j are. Since Π_i and Π_j are strictly increasing u_i must be constant, hence differentiable.

8. It follows that $u_i'(b) = 0$ for all $b \in (p, R)$

The condition, $u_i'(b) = 0$ for $b \in (p, R)$, implies that

$$\frac{1}{\omega_i}(v - b) = \frac{1 - \Pi_j(b)}{\pi_j(b)}$$

The general solution of the above condition is explicit and has the following form

$$\Pi_j(b) = 1 - (1 - \Pi_j(R)) \left(\frac{v - b}{v - R} \right)^{\omega_i}$$

where the point $(R, \Pi_j(R))$ pins down the particular solution - the point yet unknown. Since $\omega_i > 0$ this function is increasing.

One boundary condition is $\Pi_j(R) = 1 - \mu_j$ and $\Pi_i(R) = 1 - \mu_i$, which implies

$$\begin{cases} \Pi_j(b) = 1 - \mu_j \left(\frac{v-b}{v-R} \right)^{\omega_i} \\ \Pi_i(b) = 1 - \mu_i \left(\frac{v-b}{v-R} \right)^{\omega_j} \end{cases} \tag{2}$$

The additional boundary conditions are $\Pi_i(p) \geq \Pi_j(p) = 0$, which imply

$$\begin{aligned} R &= v - (v - p)(\mu_j)^{\frac{1}{\omega_i}} \\ (\mu_i)^{\omega_i} &\leq (\mu_j)^{\omega_j} \end{aligned}$$

Eliminating R from the equations (2) gives the result.

Now suppose that reputation is only on one side, $\mu_j > 0$ and $\mu_i = 0$. Suppose that $R > p$ is the last price at which normal bidder i exits. Points 3-8 above are unaffected by the assumption of one-sided reputation. The construction of the equilibrium leads to equations (2) and a conclusion that $\Pi_i(b) = 1$ for all $b > p$. This contradicts the assumption that $R > p$. Hence, $p = R$. There may be a lot of equilibria, but in each equilibrium we have $\Pi_i(b) = 1$, for all $b > p$, which is consistent with the claim. ∎

Lemma 4 *Fix $D = 2, 3, ..., X - L$. Suppose that $Z^{N+1} - L \in \{1, ..., D-1\}$ and $\mu_j^{N+1} > 0$, $\mu_i^{N+1} = 0$ (or $z_j^{N+1} = x_j$ and $z_i^{N+1} < x_i$) implies that player i collects only $L - x_j$ units at market clearing price p^{N+1}. Then $Z^N - L = D$ and $\mu_j^N > 0$, $\mu_i^N = 0$ implies that player i reduces his demand with probability one at price p^N.*

Proof. Consider any intermission N in which the excess demand is D and player i has already revealed his normality and player j is still possibly irregular, $\mu_j^N > 0$ and $\mu_i^N = 0$, that is $z_j^N = x_j$ and $z_i^N < x_i$.

Suppose that bid of player i is less than the bid of j, that is $b_i < b_j$. Then in the next stage the demand will be at most $D-1$ and $\mu_j^{N+1} > 0$, $\mu_i^{N+1} = 0$. Also $p^{N+1} = b_i$, $z_j^N = z_j^{N+1} = x_j$. Therefore, by the hypothesis of the lemma, player i will receive a payoff $y_i (v - b_i)$, where $y_i = L - x_j$, and player j will receive a payoff $x_j (v - b_i)$. Let $\Pi_i (b_i)$ denote a probability that player i decreases his demand by the time the price reaches b_i.

Suppose that bid of i is not less than the bid of j, that is $b_i \geq b_j = p^{N+1}$. Then in the next stage both players are revealed to be normal and there is a multiplicity of equilibrium continuation payoffs. Consider the most difficult case for the claim in the lemma, that is the case in which player i gets the largest possible incentive to bid high b_i, and player j gets the largest possible incentive to bid small b_j, so that $b_i \geq b_j$ is the most likely. Namely, suppose that if $b_i \geq b_j$, then player j gets $(x_j - 1)(v - b_j)$ and player i gets $z_i^N (v - b_j)$ which are the highest conceivable payoffs that both bidders can obtain in this situation. If in this case bidder i reduces his demand immediately and with probability one, as the lemma claims, then he will reduce his demand immediately and with probability one in all other cases. Let $\Pi_j (b_j)$ denote a probability that player j decreases his demand by the time the price reaches b_j.

Hence, if bidder i and j decides to reduce their demands at b then the expected payoffs are, respectively,

$$\begin{aligned} u_i (b) &= z_i^N \left(v - p^N\right) \Pi_j \left(p^N\right) + \int_{p^N}^{b} z_i^N (v - b_j)\, d\Pi_j (b_j) + y_i (v - b) (1 - \Pi_j (b)) \\ u_j (b) &= x_j \left(v - p^N\right) \Pi_i \left(p^N\right) + \int_{p^N}^{b} x_j (v - b_i)\, d\Pi_i (b_i) + (x_j - 1)(v - b)(1 - \Pi_i (b)) \end{aligned}$$

This problem is completely analogous to the one in Lemma (3). The only difference is that x_i is replaced by z_i^N and y_j is replaced by $(x_j - 1)$. Still, however, this is a War of Attrition, since $z_i^N > y_i$ and $x_j > (x_j - 1)$. Hence, in any equilibrium of one-sided model, bidder without reputation reduces his demand with probability one immediately at price p^N. ■

Lemma 5 *Consider any stage N, in which $\mu_j^N > 0$, $\mu_i^N = 0$ (or $z_j^N = x_j$, $z_i^N < x_i$). Then with probability one player i reduces his demand at p^N (in one or more intermissions) and receives a payoff $y_i \left(v - p^N\right)$.*

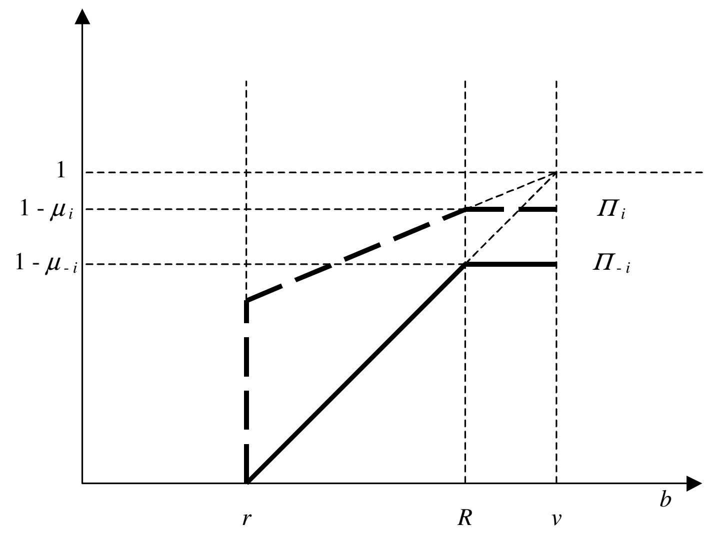

Figure 1: Linear example of an equilibrium in a multi-unit ascending auction

Proof. This claim is true if the excess demand of one unit, $Z^N - L = 1$, by lemma (3). Hence the hypothesis of lemma (4) is satisfied for $D = 2$. Applying this lemma repeatedly to any $D = 2, 3, ..., X - L$ proves the result. ■

The lemma (5) states the Proposition (2) in the case of one-sided reputation, $\mu_i = 0$ and $\mu_j > 0$. To prove the Proposition (2) in the case of two-sided reputation, $0 < (\mu_i)^{\omega_i} \leq (\mu_j)^{\omega_j}$, consider a history $h(r, x_1, x_2)$. If bids are $b_i \leq b_j$ then in the next stage bidder i is known to be normal, hence by lemma (5) his payoff is $y_i(v - b_i)$, otherwise it is $x_i(v - b_j)$. This problem is again analogous to the one in Lemma (3), except that $N = 1$, $z_1^N = x_1$, $z_2^N = x_2$ and $p^N = r$, hence the result.

4.2 Example

Let $L = 3$ let $x_i = x_j = 2$. This means that $\omega_i = \omega_j = 1$ and if $\mu_i \leq \mu_j$ then the equilibrium strategies are

$$\begin{cases} \Pi_j(b) = 1 - \frac{v-b}{v-r} \\ \Pi_i(b) = 1 - \frac{\mu_i}{\mu_j}\frac{v-b}{v-r} \end{cases}$$

See Figure 1. To be more general, the functions $\Pi_j(b)$ and $\Pi_i(b)$ are both linear if the irregular type demands two thirds of the total quantity.

5 Rationalizing irregular types

This section shows in a simplified case that the irregular type does not have to behave in a way that is assumed exogenously.

Assumption. Suppose that $K = x_i = x_j$ and that $X - L = 1$.

Under this assumption we obtain just a simple timing game. The auction stops when any of the bidders decreases his demand. As long as the auction is not finished, there is just one simple history possible, namely the history in which both bidders demanded precisely K units. Our linear example above satisfies this assumption.

The normal type has a value function as assumed above, $V_{nor}(z) = vz$. Now we will assume that irregular type has a function

$$V_{irr}(z) = \begin{cases} 0 & \text{if } z < x_i \\ vx_i & \text{if } z = x_i \end{cases}$$

Note that the irregular type has weakly lower valuation than the normal type, $V_{irr}(z) \leq V_{nor}(z)$, but his *marginal* valuation for the last unit is much higher than the one of the normal bidder.

This valuation function of irregular type formalizes a notion that units or objects in this auction may be complementary to each other. Irregular type values a bundle of x_i units only if the whole bundle is acquired. Anything less is worthless. But whether any given player is normal or irregular is not known to the public, and prior probability belief is given by the distribution μ_i and $1 - \mu_i$.

The rational type defined above does not have to announce demand equal to x_i, as the behavioral type in the previous section is assumed. It turns out that there are multiple equilibria in this slightly augmented model. For instance, consider a strategy profile in which player 1 yields immediately, regardless of type. That is the normal type decreases his demand to $x_1 - 1$, and the irregular type decreases his demand to 0. On the other hand, player 2 demands x_2 units until the price reaches or exceeds v. In this equilibrium the market is cleared at the reserve price and no player of no type has incentives to deviate.

This equilibrium, however, is implausible, because it relies on weakly dominated strategies. Namely, irregular type of player 1 could use a strategy in which he decreases his demand at v, rather than at the reserve price. The former strategy weakly dominates the latter, because both guarantee the payoff of at least zero, but for some strategies of the opponent (not used in this equilibrium) the former strategy gives positive profit.

We obtain

Proposition 6 *There exists a unique equilibrium payoff profile in weakly undominated strategies.*

Proof. Just note that irregular player has a unique weakly dominant strategy, to decrease the demand from x to zero, when the price reaches v. Given this behavior of irregular type, we obtain precisely the behavioral model in which there is a unique equilibrium payoff profile by Proposition (2). ■

Is the assumption at the beginning of this section restrictive? Potentially yes, because in a more general case, when the excess demand is more than one, we obtain a model in which more complicated histories can occur. Decreasing demand at certain prices may serve as a signaling device. Moreover, it was shown that irregular types will not demand less than their x, but there may be a reason for them to demand more than x.

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