MOD p DECOMPOSITIONS OF THE LOOP SPACES OF COMPACT SYMMETRIC SPACES

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Abstract. We give p-local homotopy decompositions of the loop spaces of compact, simply-connected symmetric spaces for quasi-regular primes. The factors are spheres, sphere bundles over spheres, and their loop spaces. As an application, upper bounds for the homotopy exponents are determined.

1. Introduction

If X is a topological space and there is a homotopy equivalence $X \simeq A \times B$ then there are induced isomorphisms of homotopy groups $\pi_m(X) \cong \pi_m(A) \oplus \pi_m(B)$ for every $m \geq 1$. So in order to determine the homotopy groups of a space it is useful to first try to decompose it as a product, up to homotopy equivalence. Ideally, the factors are simpler spaces which are easy to recognize, so that one can deduce homotopy group information about the original space X from known information about the factors. This approach has been very successful in obtaining important information about the homotopy groups of Lie groups [15, 31], Moore spaces [9], finite H-spaces [11], and certain manifolds [3, 2].

In practise, it helps if the initial space X is an H-space. Then the continuous multiplication can be used to multiply together maps from potential factors. For this reason, it is often the loop space of the original space that is decomposed up to homotopy, as looping introduces a multiplication and it simply shifts the homotopy groups of X down one dimension. It also helps to localize at a prime p, or rationally, in order to simplify the calculations while retaining p-primary features of the homotopy groups.

From now on, let p be an odd prime and assume that all spaces and maps have been localized at p. Harris [15] and Mimura, Nishida and Toda [31] gave p-local homotopy decompositions of torsion free simply-connected, simple compact Lie groups into products of irreducible factors. These were used, for example, in [31] to calculate the p-primary homotopy groups of the Lie group through a range, in [4] to calculate the v_1-periodic homotopy groups in certain cases, and in [13] to determine bounds on the homotopy exponents in certain cases. Here, the p-primary homotopy exponent of a space X is the least power of p that annihilates the p-torsion in $\pi_*(X)$.

It is natural to extend the decomposition approach to other spaces related to Lie groups. Some work has been done to determine homotopy decompositions of the loops on certain homogeneous spaces [1, 14] and analyze the exponent implications. In this paper we consider the loops on symmetric spaces with an eye towards deducing exponent information.

Compact, irreducible, simply-connected Riemannian symmetric spaces were classified by Cartan [6, 7] and an explicit list as homogeneous spaces was given in [22]. In an ad-hoc manner,

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the homotopy groups of symmetric spaces have been studied in several papers, for example [1, 5, 16, 17, 19, 29, 36, 37]. We give a more systematic approach.

A compact Lie group is *quasi-p-regular* if it is p-locally homotopy equivalent to a product of spheres and sphere bundles over spheres. Let G/H be a compact, irreducible, simply-connected Riemannian symmetric space where G is quasi-p-regular. Then for $p \geq 5$ we obtain p-local homotopy decompositions for $\Omega(G/H)$, which are stated explicitly in Theorems 5.4 and 5.8. It is notable that in all the decompositions, the factors are spheres, sphere bundles over spheres, and the loops on these spaces.

The key to our method is to replace the fibration

$$\Omega(G/H) \longrightarrow H \longrightarrow G \tag{1}$$

with a homotopy equivalent one

$$\prod(\mathrm{fib}(M(q_i))) \longrightarrow \prod M(A_i') \xrightarrow{\prod M(q_i)} \prod M(A_i) \tag{2}$$

using Cohen and Neisendorfer's construction of finite H-spaces [11] (see Theorem 2.2). Here, (2) is an H-fibration with a different H-structure from that in (1), but the maps $M(q_i)$ are simple enough to allow us to identify their homotopy fibres.

The paper is organized as follows. In Sections 2 through 4 we obtain the homotopy fibration (2) from (1), and prove properties about it. In Section 5 we identify the maps q_i in a case-by-case analysis, and thereby obtain a homotopy decomposition for $\Omega(G/H)$. In Section 6 we test the boundaries of our methods: examples are given to show that our methods can sometimes be extended to non-quasi-p-regular cases and sometimes not; we also give an example to show that our loop space decompositions sometimes cannot be delooped. In Section 7 we use the homotopy decompositions of $\Omega(G/H)$ to deduce homotopy exponent bounds for G/H.

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2. A decomposition method

Let G and H be Lie groups and let $\varphi\colon H \longrightarrow G$ be a group homomorphism. In this section we describe a method for producing a homotopy decomposition of the homotopy fibre of φ when both G and H are quasi-p-regular. In the case when φ is a group inclusion, this gives a homotopy decomposition of $\Omega(G/H)$. To do so, we first need some preliminary information.

The following is a consequence of the James construction [24]. For a path-connected, pointed space X, let $E\colon X \longrightarrow \Omega\Sigma X$ be the suspension map, which is adjoint to the identity map on ΣX.

Theorem 2.1. Let X be a path-connected space. Let Y be a homotopy associative H-space and suppose that there is a map $f\colon X \longrightarrow Y$. Then there is an extension

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "E"'] & Y \\ \Omega\Sigma X \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

where $\overline{f}$ is an H-map and it is the unique H-map (up to homotopy) with the property that $\overline{f} \circ E \simeq f$. □

Next, Cohen and Neisendorfer [11] gave a construction of finite p-local H-spaces satisfying many useful properties. The ones we need are listed below. For a $\mathbb{Z}/p\mathbb{Z}$-vector space V, let $\Lambda(V)$ be the exterior algebra on V. Take homology with mod-p coefficients.

Theorem 2.2. Fix a prime p. Let C_p be the collection of CW-complexes consisting of ℓ odd dimensional cells, where $\ell < p-1$. If $A \in C_p$ then there is a finite H-space $M(A)$ with the following properties:

(a) there is an isomorphism of Hopf algebras $H_*(M(A)) \cong \Lambda(\widetilde{H}_*(A))$;
(b) there are maps $M(A) \xrightarrow{s} \Omega\Sigma A \xrightarrow{\rho} M(A)$ such that $\rho \circ s$ is homotopic to the identity map on $M(A)$;
(c) the composite $A \xrightarrow{E} \Omega\Sigma A \xrightarrow{\rho} M(A)$ induces the inclusion of the generating set in homology.

Further, if $A, A', A'' \in C_p$ then:

(d) a map $f\colon A' \longrightarrow A$ induces a map $M(f)\colon M(A') \longrightarrow M(A)$;
(e) the maps ρ and s in part (b) are natural for maps $f\colon A' \longrightarrow A$;
(f) if there is a homotopy cofibration $A' \longrightarrow A \longrightarrow A''$ then there is a homotopy fibration $M(A') \longrightarrow M(A) \longrightarrow M(A'')$.

□

It will help to have some information about s_*. Let a be the composite

$$a\colon A \xrightarrow{E} \Omega\Sigma A \xrightarrow{\rho} M(A)$$

and let $\overline{E}$ be the composite

$$\overline{E}\colon A \xrightarrow{a} M(A) \xrightarrow{s} \Omega\Sigma A.$$

It may not be the case that $\overline{E}$ is homotopic to E. However, we will show that they induce the same map in homology modulo commutators. Recall by the Bott Samelson Theorem that $H_*(\Omega\Sigma A) \cong T(\widetilde{H}_*(A))$, where $T(\ )$ is the free tensor algebra functor. It is well known that for a $\mathbb{Z}/p\mathbb{Z}$-vector space V there is an algebra isomorphism $T(V) \cong UL\langle V\rangle$ where $L\langle V\rangle$ is the free Lie algebra generated by V and U is the universal enveloping algebra functor. Thus there is an algebra isomorphism $H_*(\Omega\Sigma A) \cong UL\langle\widetilde{H}_*(X)\rangle$.

Lemma 2.3. We have $(\overline{E})_* = E_*$ modulo commutators in $UL\langle\widetilde{H}_*(X)\rangle$.

Proof. Since s is a right homotopy inverse of ρ, we have $\rho\circ\overline{E} = \rho\circ s\circ a \simeq a$. By definition of a, we also have $\rho\circ E = a$. If $\ell < p-2$ then by [38], ρ is an H-map, so $\rho\circ(E-\overline{E}) \simeq \rho\circ E - \rho\circ\overline{E}$ is null homotopic. However, we would also like the statement of the lemma to hold for $\ell = p-1$ so we argue without knowing whether $\rho\circ(E-\overline{E})$ is null homotopic.

Define the space F and the map f by the homotopy fibration

$$F \xrightarrow{f} \Omega\Sigma A \xrightarrow{\rho} M(A).$$

By [11], this fibration is modelled in homology by the short exact sequence of algebras

$$0 \longrightarrow U[L,L] \xrightarrow{U(g)} UL \xrightarrow{U(\mathrm{ab})} UL_{ab} \longrightarrow 0$$

where L is the free Lie algebra generated by $\widetilde{H}_*(A)$, L_{ab} is the free abelian Lie algebra (that is, the bracket is identically zero) generated by $\widetilde{H}_*(A)$, $[L,L]$ is the Lie algebra kernel of the abelianization map $L \xrightarrow{\text{ab}} L_{ab}$, and g is the inclusion of $[L, L]$ into L. So $\rho_* \circ E_* = \rho_* \circ \overline{E}_*$ implies by exactness that $E_* - \overline{E}_*$ factors through $f_* = U(g)$. But as g is the map sending commutators of L into L, we obtain $E_* - \overline{E}_* = 0$ modulo commutators. $\square$

The following proposition is the key for decomposing $\Omega(G/H)$.

Theorem 2.4. Let $\varphi\colon H \longrightarrow G$ be a homomorphism of Lie groups. Suppose that there is a homotopy commutative diagram

$$\begin{array}{ccc} \bigvee_{i=1}^t A'_i & \xrightarrow{\bigvee_{i=1}^t q_i} & \bigvee_{i=1}^t A_i \\ \big\downarrow{\scriptstyle j'} & & \big\downarrow{\scriptstyle j} \\ H & \xrightarrow{\varphi} & G \end{array}$$

where $A'_i, A_i \in \mathcal{C}_p$ for $1 \le i \le t$, there are Hopf algebra isomorphisms $H_*(H) \cong \Lambda(\widetilde{H}_*(\bigvee_{i=1}^t A'_i))$ and $H_*(G) \cong \Lambda(\widetilde{H}_*(\bigvee_{i=1}^t A_i))$, and j', j induce the inclusions of the generating sets in homology. Then there is a homotopy commutative diagram

$$\begin{array}{ccc} \prod_{i=1}^t M(A'_i) & \xrightarrow{\prod_{i=1}^t M(q_i)} & \prod_{i=1}^t M(A_i) \\ \big\downarrow{\scriptstyle e'} & & \big\downarrow{\scriptstyle e} \\ H & \xrightarrow{\varphi} & G \end{array}$$

where e', e are homotopy equivalences.

Proof. First, since H and G are loop spaces, they are homotopy associative H-spaces, so Theorem 2.1 implies that the maps j' and j extend to H-maps $\bar{j}'\colon \Omega\Sigma(\bigvee_{i=1}^t A'_i) \longrightarrow H$ and $\bar{j}\colon \Omega\Sigma(\bigvee_{i=1}^t A_i) \longrightarrow G$. Since φ is an H-map, the uniqueness statement in Theorem 2.1 implies that there is a homotopy commutative diagram

$$\begin{array}{ccc} \Omega\Sigma(\bigvee_{i=1}^t A'_i) & \xrightarrow{\Omega\Sigma(\bigvee_{i=1}^t q_i)} & \Omega\Sigma(\bigvee_{i=1}^t A_i) \\ \big\downarrow{\scriptstyle \bar{j}'} & & \big\downarrow{\scriptstyle \bar{j}} \\ H & \xrightarrow{\varphi} & G. \end{array} \tag{3}$$

Second, the inclusion of a wedge summand $A_k \longrightarrow \bigvee_{i=1}^t A_i$ induces a map $\Omega\Sigma A_k \longrightarrow \Omega\Sigma(\bigvee_{i=1}^t A_i)$. The loop multiplication on $\Omega\Sigma(\bigvee_{i=1}^t A_i)$ lets us take the product of these maps for $1 \le k \le t$ to obtain a map $J\colon \prod_{i=1}^t \Omega\Sigma A_k \longrightarrow \Omega\Sigma(\bigvee_{i=1}^t A_i)$. This construction is natural for a map $\bigvee_{i=1}^t A'_i \xrightarrow{\bigvee_{i=1}^t q_i} \bigvee_{i=1}^t A_i$, so we obtain a homotopy commutative diagram

$$\begin{array}{ccc} \prod_{i=1}^t \Omega\Sigma A'_i & \xrightarrow{\prod_{i=1}^t \Omega\Sigma q_i} & \prod_{i=1}^t \Omega\Sigma A_i \\ \big\downarrow{\scriptstyle J'} & & \big\downarrow{\scriptstyle J} \\ \Omega\Sigma(\bigvee_{i=1}^t A'_i) & \xrightarrow{\Omega\Sigma(\bigvee_{i=1}^t q_i)} & \Omega\Sigma(\bigvee_{i=1}^t A_i). \end{array} \tag{4}$$

Third, since $A'_i, A_i \in \mathcal{C}_p$, by Theorem 2.2 (b) there are maps $s'_i\colon M(A'_i) \longrightarrow \Omega\Sigma A'_i$ and $s_i\colon M(A_i) \longrightarrow \Omega\Sigma A_i$ which have left homotopy inverses. The naturality property in Theorem 2.2 (e) then implies that there is a homotopy commutative diagram

$$\begin{tikzcd}
\prod_{i=1}^t M(A'_i) \arrow[r, "\prod_{i=1}^t M(q_i)"] \arrow[d, "\prod_{i=1}^t s'_i"] & \prod_{i=1}^t M(A_i) \arrow[d, "\prod_{i=1}^t s_i"] \\
\prod_{i=1}^t \Omega\Sigma A'_i \arrow[r, "\prod_{i=1}^t \Omega\Sigma q_i"] & \prod_{i=1}^t \Omega\Sigma A_i.
\end{tikzcd} \tag{5}$$

Let e' and e be the composites

$$e'\colon \prod_{i=1}^t M(A'_i) \xrightarrow{\prod_{i=1}^t s'_i} \prod_{i=1}^t \Omega\Sigma A'_i \xrightarrow{J'} \Omega\Sigma(\bigvee_{i=1}^t A'_i) \xrightarrow{\bar{j}'} H$$

$$e\colon \prod_{i=1}^t M(A_i) \xrightarrow{\prod_{i=1}^t s_i} \prod_{i=1}^t \Omega\Sigma A_i \xrightarrow{J} \Omega\Sigma(\bigvee_{i=1}^t A_i) \xrightarrow{\bar{j}} G.$$

Then juxtaposing (3), (4) and (5) we obtain a homotopy commutative diagram

$$\begin{tikzcd}
\prod_{i=1}^t M(A'_i) \arrow[r, "\prod_{i=1}^t M(q_i)"] \arrow[d, "e'"] & \prod_{i=1}^t M(A_i) \arrow[d, "e"] \\
H \arrow[r, "\varphi"] & G.
\end{tikzcd}$$

Finally, we show that e' and e are homotopy equivalences. By Whitehead's Theorem, it suffices to show that e' and e induce isomorphisms in homology or cohomology. Consider the restriction of e to $\bigvee_{i=1}^t A_i$, that is, consider the composite

$$\bigvee_{i=1}^t A_i \xrightarrow{\bigvee_{i=1}^t a_i} \prod_{i=1}^t M(A_i) \xrightarrow{\prod_{i=1}^t s_i} \prod_{i=1}^t \Omega\Sigma A_i \xrightarrow{J} \Omega\Sigma(\bigvee_{i=1}^t A_i) \xrightarrow{\bar{j}} G.$$

By the definition of a_i and Theorem 2.2 (c), $(a_i)_*$ is the inclusion of the generating set into $H_*(M(A_i))$. So if we can show that $(e \circ \bigvee_{i=1}^t a_i)_*$ is the inclusion of the generating set into $H_*(G)$, then e_* induces an isomorphism on generating sets. As $H_*(M(A))$ and $H_*(G)$ are primitively generated, dualizing to cohomology implies that e^* is an isomorphism on generating sets. Therefore, as e^* is an algebra map, it is an isomorphism in all degrees. The same argument holds for e'.

It remains to show that $(e \circ \bigvee_{i=1}^t a_i)_*$ is the inclusion of the generating set into $H_*(G)$. By definition of the map $\overline{E}$, we have $(\prod_{i=1}^t s_i) \circ \bigvee_{i=1}^t a_i = \bigvee_{i=1}^t \overline{E}_i$. So by Lemma 2.3, modulo commutators in $H_*(\prod_{i=1}^t \Omega\Sigma A_i)$, this map induces the same map in homology as $(\bigvee_{i=1}^t E_i)_*$. Observe that J is a product of H-maps and $\bar{j}$ is an H-map, so they induce algebra maps in homology. Therefore, as $H_*(G)$ is a commutative algebra, $(\bar{j} \circ J)_*$ sends all commutators in $H_*(\prod_{i=1}^t \Omega\Sigma A)$ to zero in $H_*(G)$. Thus $(\bar{j} \circ J \circ (\prod_{i=1}^t s_i) \circ \bigvee_{i=1}^t a_i)_* = (\bar{j} \circ J \circ (\bigvee_{i=1}^t E_i))_*$. The left map in this equality is $(e \circ \bigvee_{i=1}^t a_i)_*$. For the right map, by the definitions of J and $\bar{j}$, the composite $\bar{j} \circ J \circ (\bigvee_{i=1}^t E_i) \simeq j$. Thus $(e \circ \bigvee_{i=1}^t a_i)_* = j_*$. By hypothesis, j_* is the inclusion of the generating set into $H_*(G)$, and hence so is $(e \circ \bigvee_{i=1}^t a_i)_*$. $\square$

Let $\mathrm{fib}(M(q_i))$ be the homotopy fibre of the map $M(A'_i) \xrightarrow{M(q_i)} M(A_i)$. The homotopy commutative diagram in Theorem 2.4 implies that there is a homotopy fibration diagram

$$\begin{tikzcd} \prod_{i=1}^t \mathrm{fib}(M(q_i)) \arrow[r] \arrow[d, "\epsilon"] & \prod_{i=1}^t M(A'_i) \arrow[r, "\prod_{i=1}^t M(q_i)"] \arrow[d, "e'"] & \prod_{i=1}^t M(A_i) \arrow[d, "e"] \\ \Omega(G/H) \arrow[r] & H \arrow[r, "\varphi"] & G \end{tikzcd}$$

for some induced map ϵ of fibres. Since e', e are homotopy equivalences, the five-lemma implies that ϵ is as well. Thus we obtain the following.

Corollary 2.5. There is a homotopy equivalence

$$\Omega(G/H) \simeq_p \prod_{i=1}^{t} \mathrm{fib}(M(q_i)).$$

□

3. The quasi-p-regular case

In this section we aim towards Theorem 3.6, which shows that if H and G are both quasi-p-regular and satisfy mild restrictions on the factors, then the hypotheses of Theorem 2.4 are satisfied. To do this, we first need to study properties of the factors.

We begin by defining some spaces and maps following [31]. For $n \geq 2$, define the space $B(2n-1, 2n+2p-3)$ by the homotopy pullback

$$\begin{tikzcd} S^{2n-1} \arrow[r] \arrow[d, equal] & B(2n-1, 2n+2p-3) \arrow[r] \arrow[d] & S^{2n+2p-3} \arrow[d, "\frac{1}{2}\alpha_1(2n)"] \\ S^{2n-1} \arrow[r] & O(2n+1)/O(2n-1) \arrow[r] & S^{2n}. \end{tikzcd}$$

Notice that $H^*(B(2n-1, 2n+2p-3)) \cong \Lambda(x_{2n-1}, x_{2n+2p-3})$ and $\mathcal{P}^1(x_{2n-1}) = x_{2n+2p-3}$. In particular, $B(2n-1, 2n+2p-3)$ is a three-cell complex. Let $A(2n-1, 2n+2p-3)$ be the $(2n+2p-3)$-skeleton of $B(2n-1, 2n+2p-3)$ and let

$$i_{2n-1}\colon A(2n-1, 2n+2p-3) \longrightarrow B(2n-1, 2n+2p-3)$$

be the skeletal inclusion. Then $A(2n-1, 2n+2p-3)$ is a two-cell complex consisting of the bottom two cells in $B(2n-1, 2n+2p-3)$. Observe that $H^*(B(2n-1, 2n+2p-3)) \cong \Lambda(\widetilde{H}^*(A(2n-1, 2n+2p-3))$.

The space $B(2n-1, 2n+2p-3)$ is analogous to $M(A(2n-1, 2n+2p-3))$. It is introduced in addition to $M(A(2n-1, 2n+2p-3))$ because the standard homotopy decompositions of Lie groups due to Mimura, Nishida and Toda [31] are given in terms of the $B's$, and these will be used subsequently as a starting point for producing alternative decomposition in terms of $M(A)$'s via Theorem 2.4. For now, we note that the two are homotopy equivalent provided $p \geq 5$. (If $p = 3$ then as $A(2n-1, 2n+2p-3)$ has two cells, Theorem 2.2 does not apply - that is, the space $M(A(2n-1, 2n+2p-3))$ does not exist.)

Lemma 3.1. Let $p \geq 5$. If $n \geq 2$ then there is a homotopy equivalence

$$M(A(2n-1, 2n+2p-3)) \simeq_p B(2n-1, 2n+2p-3).$$

Proof. For simplicity, let $A = A(2n-1, 2n+2p-3)$, $B = B(2n-1, 2n+2p-3)$ and $i\colon A \longrightarrow B$ be i_{2n-1}. Since $p \geq 5$, by [28] the top cell splits off ΣB, that is, Σi has a left homotopy inverse $t\colon \Sigma B \longrightarrow \Sigma A$. Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "E"] \arrow[d, "i"] & \Omega\Sigma A \arrow[d, "\Omega\Sigma i"] \arrow[dr, equal] & & \\
B \arrow[r, "E"] & \Omega\Sigma B \arrow[r, "\Omega t"] & \Omega\Sigma A \arrow[r, "\rho"] & M(A)
\end{tikzcd}$$

where ρ is the map from Theorem 2.2. The left square homotopy commutes by the naturality of the suspension map E and the triangle homotopy commutes since t is a right homotopy inverse of Σi. Let $e = \rho \circ \Omega\Sigma t \circ E$ be the composition along the bottom row. By Theorem 2.2 (c), $\rho \circ E$ induces the inclusion of the generating set in homology, so the homotopy commutativity of the preceding diagram implies that $e \circ i$ does as well. But i is the inclusion of the $(2n+2p-3)$-skeleton, so it induces the inclusion of the generating set in homology. Thus e_* is a self-map of $\Lambda(x_{2n-1}, x_{2n+1p-3})$ which is an isomorphim on the generating set. As e is a map of spaces, e_* is a map of coalgebras, and any such map satisfies $\overline{\Delta} \circ e_* = (e_* \otimes e_*) \circ \overline{\Delta}$, where $\overline{\Delta}$ is the reduced diagonal. Applying the reduced diagonal to the product class $x_{2n-1} \otimes x_{2n+2p-3}$ we immediately see that e_* is also an isomorphism in degree $4n+2p-4$. Thus e_* is an isomorphism in all degrees and so e is a homotopy equivalence. $\square$

In what follows we need information about the homotopy sets $[A(2n-1, 2n+2p-3, S^{2m-1}]$ and $[A(2n-1, 2n+2p-3), B(2m-1, 2m+2p-3)]$ for various n and m. We do this now, starting by listing some known homotopy group calculations.

Lemma 3.2 (Toda [39])**.**

$$\pi_{2m-1+t}(S^{2m-1}) = \begin{cases} \mathbb{Z}/p\mathbb{Z} & t = 2i(p-1)-1, 1 \leq i \leq p-1 \\ \mathbb{Z}/p\mathbb{Z} & t = 2i(p-1)-2, m \leq i \leq p-1 \\ 0 & \text{otherwise for } 1 \leq t \leq 2p(p-1)-3 \end{cases}$$

$\square$

Lemma 3.3 (Mimura-Toda [32],Kishimoto [25])**.**

$$\pi_{3+t}(B(3, 2p+1)) = \begin{cases} \mathbb{Z}/p^2\mathbb{Z} & t = 2i(p-1)-1, 2 \leq i \leq p-1 \\ \mathbb{Z}_{(p)} & t = 2p-2 \\ 0 & \text{otherwise for } 1 \leq t \leq 2p(p-1)-3 \end{cases}$$

$$\pi_{2m-1+t}(B(2m-1, 2m+2p-3)) = \begin{cases} \mathbb{Z}/p^2\mathbb{Z} & t = 2i(p-1)-1, 2 \leq i \leq p-1 \\ \mathbb{Z}/p\mathbb{Z} & t = 2i(p-1)-2, m \leq i \leq p-1 \\ \mathbb{Z}_{(p)} & t = 2p-2 \\ 0 & \text{otherwise for } 1 \leq t \leq 2p(p-1)-3 \end{cases}$$

$\square$

Remark 3.4. Notice that if $0 < t \leq 4p-6$ and t is even then $\pi_{2m-1+t}(S^{2m-1}) \cong 0$, except in the one case when $m = 2$ and $\pi_{3+(4p-3)}(S^3) \cong \mathbb{Z}/p\mathbb{Z}$. Also, if $0 < t \leq 4p-6$ and t is even then $\pi_{3+t}(B(3, 2p+1)) \cong 0$ and $\pi_{2m-1+t}(B(2m-1, 2m+2p-3)) \cong 0$.

Lemma 3.5. Let $2 \leq m, n \leq p$. Select spaces A_m and B_n as follows:

$$A_m \in \{*, S^{2m-1}, A(2m-1, 2m+2p-3)\}$$

$$B_n \in \{*, S^{2n-1}, S^{2n+2p-3}, B(2n-1, 2n+2p-3), B(2n+2p-3, 2n+4p-5)\}.$$

Exclude the case when $A_m = A(2p-1, 4p-3)$ and $B_n = S^3$. If $m \neq n$ then $[A_m, B_n] \cong 0$.

Proof. If $A_m = *$ then we are done. Otherwise, the possible dimensions for the nontrivial cells of A_m are $2m-1$ and $2m+2p-3$. Observe that $\pi_{2m-1}(B_n) = \pi_{2n-1+t}(B_n)$ for $t = 2m-2n$, and $\pi_{2m+2p-3}(B_n) = \pi_{2n-1+t'}(B_n)$ for $t' = 2m+2p-2n-2$. In particular, both t and t' are even. Also, we may assume that $t, t' \geq 0$. As $m \neq n$ we obtain $t > 0$, and as $2 \leq m, n \leq p$, we also obtain $t' > 0$. Finally, since $2 \leq m, n \leq p$ we have $2m+2p \leq 4p$ and $2n \geq 4$. Thus $t < 4p-6$ and $t' \leq 4p-6$. So by Remark 3.4, $\pi_{2m-1}(B_n) \cong 0$ and, as the excluded case in the hypotheses rules out obtaining $\pi_{4p-3}(S^3)$, we also have $\pi_{2m+2p-3}(B_n) \cong 0$.

Therefore, if $A_m = S^{2m-1}$ then $[A_m, B_n] \cong 0$. If $A_m = A(2m-1, 2m+2p-3)$ then the homotopy cofibration $S^{2m-1} \longrightarrow A_m \longrightarrow S^{2m+2p-3}$ implies that there is an exact sequence

$$\pi_{2m+2p-3}(B_n) \longrightarrow [A_m, B_n] \longrightarrow \pi_{2m-1}(B_n).$$

As the homotopy groups on the left and right are zero we obtain $[A_m, B_n] \cong 0$. $\square$

Return to Lie groups. Let G be a simply-connected, simple compact Lie group which is quasi-p-regular. Then by [31] there is a homotopy equivalence

$$G \simeq_p \prod_{m=2}^{p} B_m$$

where B_m is one of the following:

$$B_m \in \{*, S^{2m-1}, B(2m-1, 2m+2p-3), S^{2m+2p-3}, B(2m+2p-3, 2m+4p-5)\}.$$

Define A_m by the corresponding list:

$$A_m \in \{*, S^{2m-1}, A(2m-1, 2m+2p-3), S^{2m+2p-3}, A(2m+2p-3, 2m+4p-5)\}.$$

Notice that in each case, $H^*(B_m) \cong \Lambda(\widetilde{H}^*(A_m))$. Let j be the composite

$$j\colon \bigvee_{m=2}^{p} A_m \longrightarrow \prod_{m=2}^{p} B_m \xrightarrow{\simeq} G$$

where the left map is determined by the skeletal inclusion of A_m into B_m. Then there is an isomorphism $H^*(G) \cong \Lambda(\widetilde{H}^*(\bigvee_{m=1}^{p} A_m))$ for which j^* is the projection onto the generating set.

Now suppose that $H = H_1 \times H_2$ where H_1 and H_2 are simply-connected, simple compact Lie groups which are quasi-p-regular. (In theory, this could be generalized to a product of finitely many such Lie groups, but in practise two factors suffices. In fact, it will often be the case that H_1 is trivial.) By [31] there are homotopy equivalences

$$H_1 \simeq_p \prod_{m=2}^{p} B'_{m,1} \qquad H_2 \simeq_p \prod_{m=2}^{p} B'_{m,2}.$$

This time we impose a more stringent condition than in the case of G. We demand that

$$B'_{m,1}, B'_{m,2} \in \{*, S^{2m-1}, B(2m-1, 2m+2p-3)\}. \tag{6}$$

Let $A'_{m,1}$, $A'_{m,2}$ be the corresponding skeleta:

$$A'_{m,1}, A'_{m,2} \in \{*, S^{2m-1}, A(2m-1, 2m+2p-3)\}.$$

Let $B'_m = B'_{m,1} \times B'_{m,1}$ and $A'_m = A'_{m,1} \vee A'_{m,2}$. Then there is a homotopy equivalence

$$H \simeq_p \prod_{m=2}^{p} B'_m$$

and a map

$$j' \colon \bigvee_{m=2}^{p} A'_m \longrightarrow \prod_{m=2}^{p} B'_m \xrightarrow{\simeq} H$$

which induces the inclusion of the generating set in homology.

Theorem 3.6. Let G be a simply-connected, simple compact Lie group, let $H = H_1 \times H_2$ be a product of two such Lie groups, and let $\varphi\colon H \longrightarrow G$. be a homomorphism. Suppose that both G and H are quasi-p-regular, that the factors of H satisfy (6), and that if A'_m has $A(2p-1, 4p-3)$ as a wedge summand then $B_2 \neq S^3$. Then there is a homotopy commutative diagram

$$\begin{tikzcd} \bigvee_{m=2}^{p} A'_m \arrow[r, "\bigvee_{m=2}^{p} q_m"] \arrow[d, "j'"] & \bigvee_{m=2}^{p} A_m \arrow[d, "j"] \\ H \arrow[r, "\varphi"] & G \end{tikzcd}$$

where j' and j induce the inclusions of the generating sets in homology.

Proof. First, consider the composite

$$\theta_k \colon A'_k \hookrightarrow \bigvee_{m=2}^{p} A'_m \xrightarrow{j} H \xrightarrow{\varphi} G \xrightarrow{\simeq} \prod_{m=2}^{p} B_m.$$

By Lemma 3.5, $[A'_k, B_m] \cong 0$ unless $m = k$. Therefore θ_k factors as the composite

$$A'_k \xrightarrow{\lambda_k} B_k \xrightarrow{\text{incl}} \prod_{m=2}^{p} B_m \xrightarrow{\simeq} G$$

where λ_k is the projection of θ_k onto B_k.

Next, observe that if $B_k \in \{*, S^{2m-1}, S^{2m+2p-3}\}$ then $A_k = B_k$, so λ_k factors through the inclusion $A_k \longrightarrow B_k$ (which is the identity map). On the other hand, if $B_k = B(2m-1, 2m+2p-3)$ or $B_k = B(2m+2p-3, 2m+4p-5)$ then as the dimension of A'_k is at most $2m+2p-3$, we have λ_k factoring through the skeletal inclusion $A_k \longrightarrow B_k$. Thus, in any case, λ_k factors as a composite

$$A'_k \xrightarrow{q_k} A_k \hookrightarrow B_k$$

for some map q_k.

Putting this together, for each $2 \leq k \leq p$ we obtain a homotopy commutative diagram

$$\begin{tikzcd} A'_k \arrow[rr, "q_k"] \arrow[d, "\text{incl}"] & & A_k \arrow[r, "\text{incl}"] & B_k \arrow[d, "\text{incl}"] \\ \bigvee_{m=2}^{p} A'_m \arrow[r, "j'"] & H \arrow[r, "\varphi"] & G \arrow[r, "\simeq"] & \prod_{m=2}^{p} B_m. \end{tikzcd}$$

Taking the wedge sum of these diagrams for $2 \leq k \leq p$ and composing with the inverse equivalence $\prod_{m=2}^{p} B_m \xrightarrow{\simeq} G$ gives the diagram in the statement of the theorem. □

Remark 3.7. We will apply Theorem 3.6 in the case when G/H is a symmetric space. This requires that we also consider the possibility that $H = S^1 \times H_2$. Then $A = S^1 \vee A'_2$, and as G is simply-connected, the restriction of the composite $A \longrightarrow H \xrightarrow{\varphi} G$ to S^1 is null homotopic. We are left with the composite $A'_2 \longrightarrow H \xrightarrow{\varphi} G$, to which Theorem 3.6 applies. We obtain a homotopy commutative diagram

$$\begin{tikzcd} S^1 \vee (\bigvee_{m=2}^{p} A'_m) \arrow[r, "\text{pinch}"] \arrow[d, "j'"] & \bigvee_{m=2}^{p} A'_m \arrow[r, "\bigvee_{m=2}^{p} q_m"] & \bigvee_{m=2}^{p} A_m \arrow[d, "j"] \\ H \arrow[rr, "\varphi"] & & G \end{tikzcd}$$

4. Identifying the map q_m and the homotopy fibre of $M(q_m)$

The next step is to try to identify the maps q_m in Theorem 3.6 and the homotopy fibre of $M(q_m)$. Since j', j induce the inclusion of the generating set in homology, they induce the projection onto the generating set in cohomology. Thus $(q_m)^*$ is determined by the map of indecomposable modules induced by $H \xrightarrow{\varphi} G$:

$$Q\varphi^*\colon QH^*G \longrightarrow QH^*(H).$$

Based on the calculations to come in the subsequent sections, we will consider several possibilities for q_m with $(q_m)^* \neq 0$. In Proposition 4.2 we will show that this cohomology information is sufficient to determine the homotopy type of the fibre of $M(q_m)$.

At this point it is appropriate to notice that if $p = 3$ then Theorem 2.2 does not apply to the two cell complex $A(2n-1, 2n+2p-3)$. That is, the space $M(A(2n-1, 2n+2p-3))$ does not exist. To avoid this, from now on we will assume that all spaces and maps have been localized at a prime $p \geq 5$.

We begin by listing eight types of maps:

$v_1\colon A_m \longrightarrow A_m$
$v_2\colon S^{2m-1} \longrightarrow A(2m-1, 2m+2p-3)$
$v_3\colon A(2m-1, 2m+2p-3) \longrightarrow S^{2m+2p-3}$
$v_4\colon A(2m-1, 2m+2p-3) \longrightarrow A(2m+2p-3, 2m+4p-5)$
$v_5\colon S^{2m-1} \vee S^{2m-1} \longrightarrow S^{2m-1}$
$v_6\colon S^{2m-1} \vee S^{2m-1} \longrightarrow A(2m-1, 2m+2p-3)$
$v_7\colon S^{2m-1} \vee A(2m-1, 2m+2p-3) \longrightarrow A(2m-1, 2m+2p-3)$
$v_8\colon A(2m-1, 2m+2p-3) \vee A(2m-1, 2m+2p-3) \longrightarrow A(2m-1, 2m+2p-3)$.

Here, v_1 is a homotopy equivalence, v_2 is the inclusion of the bottom cell, $v_3$is the pinch map to the top cell, v_4 is the composite of the pinch map to the top cell and the inclusion of the bottom cell, v_5 is a homotopy equivalence when restricted to each wedge summand, v_6 is the inclusion of the bottom cell on each wedge summand, v_7 is the inclusion of the bottom cell when restricted to S^{2m-1} and is a homotopy equivalence when restricted to A_m, and v_8 is a homotopy equivalence when restricted to each copy of A_m.

Apply the functor M in Theorem 2.2 to the maps v_1 to v_8. Using the facts that $M(S^{2n-1}) \simeq_p S^{2n-1}$ and $M(X \vee Y) \simeq_p M(X) \times M(Y)$, we obtain maps:

$M(v_1)\colon M(A_m) \longrightarrow M(A_m)$
$M(v_2)\colon S^{2m-1} \longrightarrow M(A(2m-1, 2m+2p-3))$
$M(v_3)\colon M(A(2m-1, 2m+2p-3)) \longrightarrow S^{2m+2p-3}$
$M(v_4)\colon M(A(2m-1, 2m+2p-3)) \longrightarrow M(A(2m+2p-3, 2m+4p-5))$
$M(v_6)\colon S^{2m-1} \times S^{2m-1} \longrightarrow S^{2m-1}$
$M(v_6)\colon S^{2m-1} \times S^{2m-1} \longrightarrow M(A(2m-1, 2m+2p-3))$
$M(v_7)\colon S^{2m-1} \times M(A(2m-1, 2m+2p-3)) \longrightarrow M(A(2m-1, 2m+2p-3))$
$M(v_8)\colon M(A(2m-1, 2m+2p-3)) \times M(A(2m-1, 2m+2p-3)) \longrightarrow M(A(2m-1, 2m+2p-3))$.

Let $\mathrm{fib}(M(v_i))$ be the homotopy fibre of $M(v_i)$. In Lemma 4.2 we identify the homotopy type of $\mathrm{fib}(M(v_i)$ for $1 \leq i \leq 8$. First we need a preliminary lemma, which holds integrally or p-locally.

Lemma 4.1. Suppose that there are maps $X \xrightarrow{f} Y \xrightarrow{g} Z$ where Y and Z are H-spaces and g is an H-map. Let $h = g \circ f$. If m is the multiplication on Z, we obtain a composite

$$h \cdot g\colon X \times Y \xrightarrow{h \times g} Z \times Z \xrightarrow{m} Z.$$

Let F be the homotopy fibre of g. Then the homotopy fibre of $h \cdot g$ is homotopy equivalent to $X \times F$.

Proof. There is a homotopy equivalence $\theta\colon X \times Y \longrightarrow X \times Y$ given by sending (x, y) to $(x, \mu(x, y))$ where μ is the multiplication on Y. As g is an H-map, $h \cdot g$ is homotopic to the composite $\psi\colon X \times Y \xrightarrow{\theta} X \times Y \xrightarrow{\pi_2} Y \xrightarrow{g} Z$, where π_2 is the projection onto the second factor. The homotopy fibre of ψ is clearly $X \times F$, and so this is also the homotopy fibre of $h \cdot g$. □

Lemma 4.2. Let $p \geq 5$. The following hold:

(1) $\mathrm{fib}(M(v_1)) \simeq_p *$;
(2) $\mathrm{fib}(M(v_2)) \simeq_p \Omega S^{2m+2p-3}$;
(3) $\mathrm{fib}(M(v_3)) \simeq_p S^{2m-1}$;
(4) $\mathrm{fib}(M(v_4)) \simeq_p S^{2m-1} \times \Omega S^{2m+4p-5}$;
(5) $\mathrm{fib}(M(v_5)) \simeq_p S^{2m-1}$;
(6) $\mathrm{fib}(M(v_6)) \simeq_p S^{2m-1} \times \Omega S^{2m+2p-3}$;
(7) $\mathrm{fib}(M(v_7)) \simeq_p S^{2m-1}$;
(8) $\mathrm{fib}(M(v_8)) \simeq_p M(A(2m-1, 2m+2p-3)) \simeq B(2m-1, 2m+2p-3)$.

Proof. Since v_1 is a homotopy equivalence, it induces an isomorphism in homology, which implies by Theorem 2.2 (a) that $M(v_1)$ also induces an isomorphism in homology and so is a homotopy equivalence. It follows that $\mathrm{fib}(M(v_1)) \simeq_p *$, proving part (1).

By Theorem 2.2 (f), the homotopy cofibration $S^{2m-1} \longrightarrow A(2m-1, 2m+2p-3) \longrightarrow S^{2m+2p-3}$ induces a homotopy fibration $S^{2m-1} \longrightarrow M(A(2m-1, 2m+2q-3)) \longrightarrow S^{2m+2p-3}$. We immediately obtain $\mathrm{fib}(M(v_2)) \simeq_p \Omega S^{2m+2p-3}$ and $\mathrm{fib}(M(v_3)) \simeq_p S^{2m-1}$, proving parts (2) and (3).

For part (4), since v_4 is the composite

$$A(2m-1, 2m+2p-3) \xrightarrow{v_3} S^{2m+2p-3} \xrightarrow{v_2} A(2m+2p-3, 2m+4p-5)$$

the naturality property in Theorem 2.2 implies that $M(v_4)$ is homotopic to the composite

$$M(2m-1, 2m+2p-3) \xrightarrow{M(A(v_3)} S^{2m+2p-3} \xrightarrow{M(v_2))} M(A(2m+2p-3, 2m+4p-5)).$$

Further, by [38], the maps $M(v_2)$ and $M(v_3)$ are H-maps so we obtain a homotopy pullback of H-spaces and H-maps

$$\begin{tikzcd} S^{2m-1} \arrow[r] \arrow[d, equal] & X \arrow[r] \arrow[d] & \Omega S^{2m+4p-5} \arrow[d, "\partial"] \\ S^{2m-1} \arrow[r] & M(A(2m-1, 2m+2p-3)) \arrow[r, "M(v_2)"] \arrow[d, "M(v_4)"] & S^{2m+2p-3} \arrow[d, "M(v_3)"] \\ & M(A(2m+2p-3, 2m+4p-5)) \arrow[r, equal] & M(A(2m+2p-3, 2m+4p-5)) \end{tikzcd} \tag{7}$$

which defines the H-space X and the H-map ∂. Note that $X \simeq_p \mathrm{fib}(M(v_4))$. In general, the attaching map for the $(2n+2p-3)$-cell in $M(A(2n-1, 2n+2p-3))$ is α_1, so the fibration connecting map $\partial\colon \Omega S^{2n+2p-3} \longrightarrow S^{2n-1}$ satisfies $\partial \circ E \simeq \alpha_1$. In our case, after looping (7), we obtain a composite of connecting maps $\Omega^2 S^{2m+4p-5} \xrightarrow{\Omega\partial} \Omega S^{2m+2p-3} \xrightarrow{\partial'} S^{2m-1}$ where the homotopy fibre of ∂' is $\Omega M(v_2)$. We have $\partial' \circ \Omega\partial \circ E^2 \simeq \alpha_1 \circ \alpha_1$, which is null homotopic by [39]. Thus $\partial' \circ \Omega\partial \circ E^2$ lifts through $\Omega M(v_2)$. Adjointing, this implies that $\partial \circ E$ lifts through $M(v_2)$ to a map $\lambda\colon S^{2m+4p-6} \longrightarrow M(A(2m-1, 2m+2p-3))$. By [38], $M(A(2m-1, 2m+2p-3))$ is homotopy associative, so by Theorem 2.1, λ extends to an H-map

$$\gamma\colon \Omega S^{2m+4p-5} \longrightarrow M(A(2m-1, 2m+2p-3)),$$

and as $M(v_2)$ is an H-map, the uniqueness property of Theorem 2.1 implies that $M(v_2) \circ \gamma \simeq \partial$. The pullback property of X therefore implies that γ pulls back to a map $\Omega S^{2m+4p-5} \longrightarrow X$ which is a right homotopy inverse for $X \longrightarrow \Omega S^{2m+4p-5}$. Since X is an H-space, this section implies that there is a homotopy equivalence $X \simeq_p S^{2m-1} \times \Omega S^{2m+4p-5}$.

Parts (5) through (8) are all special cases of Lemma 4.1. □

Next, we aim to show that if $(q_m)^* \neq 0$ in cohomology then q_m can be described in terms of the maps v_1 to v_8.

Lemma 4.3. Let $q_m\colon A'_m \longrightarrow A_m$ be a map as in Theorem 3.6 and suppose that, in cohomology, $(q_m)^* \neq 0$. Write u for an arbitrary unit in $\mathbb{Z}_{(p)}$. Then the following hold:

(1) if $A'_m = A_m$ then q_m is a homotopy equivalence;
(2) if $A'_m = S^{2m-1}$ and $A_m = A(2m-1, 2m+2p-3)$ then $q_m \simeq u \cdot v_2$;
(3) if $A'_m = A(2m-1, 2m+2p-3)$ and $A_m = S^{2m+2p-3}$ then $q_m \simeq u \cdot v_3$;
(4) if $A'_m = A(2m-1, 2m+2p-3)$ and $A_m = A(2m+2p-3, 2m+4p-5)$ then $q_m \simeq u \cdot v_4$.

Proof. For part (1), if $A'_m = A_m$ equals $*$ or S^{2m-1} then the assertion is clear. If they both equal $A(2m-1, 2m+2p-3)$ then recall that $H^*(A_m) = \mathbb{Z}/p\mathbb{Z}\{x_{2m-1}, x_{2m+2p-3}\}$ and $\mathcal{P}^1(x_{2m-1}) = x_{2m+2p-3}$. This Steenrod operation implies that if $(q_m)^*$ is nonzero on either generator then it is nonzero on both. Consequently, $(q_m)^*$ is an isomorphism and so q_m is a homotopy equivalence.

Part (2) is a consequence of the Hurewicz Theorem.

For parts (3) and (4), observe that there is a homotopy cofibration sequence

$$S^{2m-1} \xrightarrow{i} A(2m-1, 2m+2p-3) \xrightarrow{q} S^{2m+2p-3} \xrightarrow{\alpha_1(2m)} S^{2m}$$

where i is the inclusion of the bottom cell and q is the pinch map onto the top cell. For any space X, we obtain an induced exact sequence

$$\pi_{2m}(X) \longrightarrow \pi_{2m+2p-3}(X) \xrightarrow{q^*} [A(2m-1, 2m+2p-3), X] \xrightarrow{i_*} \pi_{2m-1}(X).$$

Taking $X = S^{2m+2p-3}$ or $X = A(2m+2p-3, 2m+4p-5)$, by connectivity $\pi_{2m}(X) \cong \pi_{2m-1}(X) \cong 0$, so q^* is an isomorphism. The Hurewicz Theorem implies in either case that $\pi_{2m+2p-3}(X)$ is isomorphic to $H^*(X)$. Therefore, in both cases, the homotopy class of q_m is determined by its image in cohomology, and the assertions follow. □

Arguing as for Lemma 4.3 we also obtain the following.

Lemma 4.4. Let $q_m\colon A'_{m,1} \vee A'_{m,2} \longrightarrow A_m$ be a map as in Theorem 3.6 and suppose that, in cohomology, $(q_m)^* \neq 0$ when projected to either $H^*(A'_{m,1})$ or $H^*(A'_{m,2})$. Write u, u' for arbitrary units in $\mathbb{Z}_{(p)}$. Then the following hold:

(5) if $A'_{m,1} = A'_{m,2} = S^{2m-1}$ and $A_m = S^{2m-1}$ then $q_m \simeq u \vee u'$ is a wedge sum of homotopy equivalences;
(6) if $A'_{m,1} = A'_{m,2} = S^{2m-1}$ and $A_m = A(2m-1, 2m+2p-3)$ then $q_m \simeq u \cdot v_2 \vee u' \cdot v_2$;
(7) if $A'_{m,1} = S^{2m-1}$, $A'_{m,2} = A(2m-1, 2mp+2p-3)$ and $A_m = A(2m-1, 2m+2p-3)$ then $q_m \simeq u \cdot v_2 \vee e'$ where e' is a homotopy equivalence;
(8) if $A'_{m,1} = A'_{m,2} = A(2m-1, 2mp+2p-3)$ and $A_m = A(2m-1, 2m+2p-3)$ then $q_m \simeq e \vee e'$ where e, e' are homotopy equivalences.

□

Lemmas 4.3 and 4.4 identify q_m in terms of the maps v_i, up to multiplication by units in $\mathbb{Z}_{(p)}$ or homotopy equivalences. Thus $M(q_m)$ can similarly be written in terms of the maps $M(v_i)$. As multiplication by a unit in $\mathbb{Z}_{(p)}$ or composition with a homotopy equivalence does not affect the homotopy type of the fibre, the homotopy fibre of $M(q_m)$ has the same homotopy type as the homotopy fibre of the corresponding $M(v_i)$'s. So Lemma 4.2 implies the following.

Proposition 4.5. Let $p \geq 5$ and let $q_m\colon A'_m \longrightarrow A_m$ be a map as in Theorem 3.6. If $(q_m)^* \neq 0$, then - listing cases as in Lemmas 4.3 and 4.4 - the homotopy fibre of $M(q_m)$ is as follows:

(1) $\mathrm{fib}(M(q_m)) \simeq_p *$;
(2) $\mathrm{fib}(M(q_m)) \simeq_p \Omega S^{2m+2p-3}$;
(3) $\mathrm{fib}(M(q_m)) \simeq_p S^{2m-1}$;
(4) $\mathrm{fib}(M(q_m)) \simeq_p S^{2m-1} \times \Omega S^{2m+4p-5}$;
(5) $\mathrm{fib}(M(q_m)) \simeq_p S^{2m-1}$;
(6) $\mathrm{fib}(M(q_m)) \simeq_p S^{2m-1} \times \Omega S^{2m+2p-3}$;
(7) $\mathrm{fib}(M(q_m)) \simeq_p S^{2m-1}$;
(8) $\mathrm{fib}(M(q_m)) \simeq_p M(A(2m-1, 2m+2p-3)) \simeq_p B(2m-1, 2m+2p-3)$.

□

5. Case by case analysis

In this section, we give homotopy decompositions of $\Omega(G/H)$ when G is quasi-p-regular using a case by case analysis. Note that when G is quasi-p-regular, H is automatically so by the classification of the symmetric space. The classical cases are considered first, followed by the exceptional cases.

5.1. **Classical cases.** The following homotopy decompositions for quasi-p-regular classical Lie groups are due to Mimura and Toda [32].

Theorem 5.1. For an odd prime p, there are homotopy equivalences:

G	p (odd)	
$SU(n)$	$p > n/2$	$\prod_{i=2}^{n-p+1} B(2i-1, 2i+2p-3) \times \prod_{j=\max(2,n-p+2)}^{\min(n,p)} S^{2j-1}$
$SO(2n+1)$	$p > n$	$\prod_{i=1}^{n-\frac{p-1}{2}} B(4i-1, 4i+2p-3) \times \prod_{j=n-\frac{p-3}{2}}^{\min(n,\frac{p-1}{2})} S^{4j-1}$
$Sp(n)$	$p > n$	$\prod_{i=1}^{n-\frac{p-1}{2}} B(4i-1, 4i+2p-3) \times \prod_{j=n-\frac{p-3}{2}}^{\min(n,\frac{p-1}{2})} S^{4j-1}$
$SO(2n)$	$p > n-1$	$\prod_{i=1}^{n-\frac{p+1}{2}} B(4i-1, 4i+2p-3) \times \prod_{j=n-\frac{p-1}{2}}^{\min(n-1,\frac{p-1}{2})} S^{4j-1} \times S^{2n-1}$

□

We will also use the following homotopy decompositions, due to Harris [15].

Theorem 5.2 ([15])**.** For an odd prime p, there are homotopy equivalences:

$$\begin{aligned}
SU(2n) &\simeq_p Sp(n) \times SU(2n)/Sp(n) \\
SU(2n+1) &\simeq_p Spin(2n+1) \times SU(2n+1)/Spin(2n+1) \\
SO(2n+1) &\simeq_p Spin(2n+1) \simeq_p Sp(n) \\
SO(2n) &\simeq_p Spin(2n) \simeq_p Spin(2n-1) \times S^{2n-1}
\end{aligned}$$

□

For expositional purposes, the $AIII$ case is examined first.

5.1.1. *Type AIII.* Assume that $2m \leq n$. Observe that $SU(n)/SU(n-m) = U(n)/U(n-m)$. Since the upper-left inclusion and the lower-right inclusions for $U(n)$ are conjugate and thus homotopic, the inclusion $U(m) \times U(n-m) \hookrightarrow U(n)$ is homotopic to

$$U(m) \times U(n-m) \overset{\iota_m \times \iota_{m-n}}{\hookrightarrow} U(n) \times U(n) \xrightarrow{\mu} U(n),$$

where $\iota_m : U(m) \hookrightarrow U(n)$ and $\iota_{n-m} : U(n-m) \hookrightarrow U(n)$ are the upper-left inclusions. By Lemma 4.1, for $m \leq n-m$ there is an integral homotopy equivalence

$$\Omega(U(n)/U(n-m) \times U(m)) \simeq U(m) \times \Omega(SU(n)/SU(n-m)).$$

By Theorem 5.1, there are homotopy equivalences

$$\begin{aligned}
SU(n-m) &= \prod_{i=2}^{n-m-p+1} B(2i-1, 2i+2p-3) \times \prod_{j=n-m-p+2}^{\min(p,n-m)} S^{2j-1} \\
SU(n) &= \prod_{i=2}^{n-p+1} B(2i-1, 2i+2p-3) \times \prod_{j=n-p+2}^{\min(p,n)} S^{2j-1}.
\end{aligned}$$

So if we define spaces A_i' and A_i for $i \le 2 \le p$ by

$$\bigvee_{i=2}^{p} A_i' = \bigvee_{i=2}^{n-m-p+1} A(2i-1, 2i+2p-3) \vee \bigvee_{j=n-m-p+2}^{\min(p,n-m)} S^{2j-1}$$
$$\bigvee_{i=2}^{p} A_i = \bigvee_{i=2}^{n-p+1} A(2i-1, 2i+2p-3) \vee \bigvee_{j=n-p+2}^{\min(p,n)} S^{2j-1}$$

then by Theorem 3.6 there is a homotopy commutative diagram

$$\begin{tikzcd} \bigvee_{i=2}^{p} A_i' \arrow[r, "\bigvee_{i=2}^{p} q_i"] \arrow[d] & \bigvee_{i=2}^{p} A_i \arrow[d] \\ SU(n-m) \arrow[r, "\varphi"] & SU(n). \end{tikzcd}$$

In each case, since φ^* is a projection, each $(q_i)^*$ is an epimorphism. So by Proposition 4.5 and Corollary 2.5 we have

$$\Omega(SU(n)/SU(n-m)) \simeq_p \prod_{i=2}^{p} \mathrm{fib}(M(q_i)) \simeq_p \prod_{j=n-m+1}^{n} \Omega S^{2j-1}.$$

Thus, for $p > n/2$, we obtain

$$\begin{aligned}\Omega(U(n)/U(m) \times U(n-m)) &\simeq_p U(m) \times \Omega(SU(n)/SU(n-m)) \\ &\simeq_p \prod_{j=1}^{m} S^{2j-1} \times \prod_{j=n-m+1}^{n} \Omega S^{2j-1}.\end{aligned}$$

Remark 5.3. Using a different approach, a homotopy decomposition for $\Omega(SU(n)/SU(n-m))$ is obtained in [1, 14] which holds for $n \le (p-1)(p-2)$. This range includes the quasi-p-regular cases and more. However, those methods do not extend to exceptional cases while ours do, so the argument above was given in detail for the sake of illustrating our approach.

5.1.2. *Type CII.* Assume $2m \le n$. Similar to the type $AIII$ case, for $n < p$ we have

$$\begin{aligned}\Omega(Sp(n)/(Sp(m) \times Sp(n-m))) &\simeq_p Sp(m) \times \Omega(Sp(n)/Sp(n-m)) \\ &\simeq_p \prod_{j=1}^{m} S^{4j-1} \times \prod_{j=n-m+1}^{n} \Omega S^{4j-1}.\end{aligned}$$

5.1.3. *Type BDI.* Similar to the type $AIII$ case, we have

$$\Omega(SO(n)/(SO(m) \times SO(n-m))) \simeq_p SO(m) \times \Omega(SO(n)/SO(n-m)),$$

where $2m \le n$. By Theorem 5.2, for p odd there are homotopy equivalences $SO(2k+1) \simeq_p Sp(k)$ and $SO(2k+2) \simeq_p Sp(k) \times S^{2k+1}$. Therefore, we obtain homotopy equivalences:

$$\begin{aligned}
&\Omega(SO(2n+1)/SO(2(n-m)+1)) \simeq_p \Omega(Sp(n)/Sp(n-m))\\
&\Omega(SO(2n+1)/SO(2(n-m)+2)) \simeq_p S^{2(n-m)+1} \times \Omega(Sp(n)/Sp(n-m))\\
&\Omega(SO(2n+2)/SO(2(n-m)+1)) \simeq_p \Omega S^{2n+1} \times \Omega(Sp(n)/Sp(n-m))\\
&\Omega(SO(2n+2)/SO(2(n-m)+2)) \simeq_p S^{2(n-m)+1} \times \Omega S^{2n+1} \times \Omega(Sp(n)/Sp(n-m)).
\end{aligned}$$

Complete decompositions are now obtained from the CII case.

5.1.4. *Types AI, AII.* Homotopy decompositions of $SU(2n)/Sp(n)$ and $SU(2n+1)/SO(2n+1)$ are given in [31, Thm 4.1] as sub-decompositions of $SU(n)$.

$$SU(2n)/Sp(n) \simeq_p \prod_{i=1}^{n-\frac{p+1}{2}} B(4i+1, 4i+2p-1) \times \prod_{j=\min(1,n-\frac{p-1}{2})}^{\min(n-1,\frac{p-1}{2})} S^{4j+1} \quad (p > n)$$

$$SU(2n+1)/SO(2n+1) \simeq_p \prod_{i=1}^{n-\frac{p-1}{2}} B(4i+1, 4i+2p-1) \times \prod_{j=\min(1,n-\frac{p-3}{2})}^{\min(n,\frac{p-1}{2})} S^{4j+1} \quad (p > n).$$

For $SU(2n)/SO(2n)$, by [33, Theorem 6.7], $Q\varphi^*\colon QH^t(SU(2m)) \longrightarrow QH^t(SO(2m))$ is nontrivial for $t \in \{3, 7, \ldots, 4m-5\}$. So arguing as in the $AIII$ case, we obtain a homotopy equivalence

$$\Omega SU(2n)/SO(2n) \simeq_p \Omega S^{2n} \times \prod_{i=1}^{n-\frac{p+1}{2}} \Omega B(4i+1, 4i+2p-1) \times \prod_{j=\min(1,n-\frac{p-1}{2})}^{\min(n-1,\frac{p-1}{2})} \Omega S^{4j+1} \quad (p > n).$$

5.1.5. *Types $CI, DIII$.* For the type CI case of $Sp(n)/U(n)$, $Sp(n)$ is quasi regular when $p > n$ and then $U(n) \simeq_p \prod_{i=1}^{n} S^{2i-1}$. By [33, Theorem 5.8], $Q\varphi^*\colon QH^t(Sp(n)) \longrightarrow QH^t(U(n))$ is nontrivial for $t \in \{3, 7, \ldots, 4[n/2]\}$. So arguing as in the $AIII$ case we obtain a homotopy equivalence

$$\Omega(Sp(n)/U(n)) \simeq_p \prod_{j=0}^{[\frac{n-1}{2}]} S^{4j+1} \times \prod_{j=[\frac{n+2}{2}]}^{n} \Omega S^{4j-1}, \quad (p > n).$$

For the type $DIII$ case of $SO(2n)/U(n)$, we can reduce it to a type CII case by

$$SO(2n)/U(n) = SO(2n-1)/U(n-1) \simeq_p Sp(n-1)/U(n-1).$$

Summarising the results for classical cases, we have the following.

Theorem 5.4. For $p \geq 5$, there are homotopy equivalences:

Type	G/H	$p \geq 5$	Homotopy type of $\Omega(G/H)$
AI	$SU(2n+1)/SO(2n+1)$	$p > n$	$\prod_{i=1}^{n-\frac{p-1}{2}} \Omega B(4i+1, 4i+2p-1) \times \prod_{j=\min(1, n-\frac{p-3}{2})}^{\min(n, \frac{p-1}{2})} \Omega S^{4j+1}$
	$SU(4n+2)/SO(4n+2)$	$p = 2n+1$	$\prod_{i=1}^{n-1} \Omega B(4i+1, 4i+2p-1) \times \Omega S^{8n+1} \times \Omega S^{8n+3}$
	$SU(2n)/SO(2n)$	$p > 2n$	$\Omega S^{2n} \times \prod_{i=1}^{n-\frac{p+1}{2}} \Omega B(4i+1, 4i+2p-1) \times \prod_{j=\min(1, n-\frac{p-1}{2})}^{\min(n-1, \frac{p-1}{2})} \Omega S^{4j+1}$
AII	$SU(2n)/Sp(n)$	$p > n$	$\prod_{i=1}^{n-\frac{p+1}{2}} \Omega B(4i+1, 4i+2p-1) \times \prod_{j=\max(1, n-\frac{p-1}{2})}^{\min(n-1, \frac{p-1}{2})} \Omega S^{4j+1}$
$AIII$	$\frac{U(n)}{U(m) \times U(n-m)}$ †	$p > n/2$	$\prod_{j=1}^{m} S^{2j-1} \times \prod_{j=n-m+1}^{n} \Omega S^{2j-1}$
BDI	$\frac{SO(2n+1)}{SO(2m) \times SO(2(n-m)+1)}$ †	$p > n$	$\prod_{j=1}^{m-1} S^{4j-1} \times S^{2m-1} \times \prod_{j=n-m+1}^{n} \Omega S^{4j-1}$
	$\frac{SO(2n+1)}{SO(2m-1) \times SO(2(n-m)+2)}$ ‡	$p > n$	$\prod_{j=1}^{m-1} S^{4j-1} \times S^{2(n-m)+1} \times \prod_{j=n-m+1}^{n} \Omega S^{4j-1}$
	$\frac{SO(2n+2)}{SO(2m+1) \times SO(2(n-m)+1)}$ †	$p > n$	$\prod_{j=1}^{m} S^{4j-1} \times \Omega S^{2n+1} \times \prod_{j=n-m+1}^{n} \Omega S^{4j-1}$
	$\frac{SO(2n+2)}{SO(2m) \times SO(2(n-m)+2)}$ ‡	$p > n-1$	$\prod_{j=1}^{m-1} S^{4j-1} \times S^{2m-1} \times S^{2(n-m)+1} \times \Omega S^{2n+1} \times \prod_{j=n-m+1}^{n} \Omega S^{4j-1}$
CI	$Sp(n)/U(n)$	$p > n$	$\prod_{j=0}^{[\frac{n-1}{2}]} S^{4j+1} \times \prod_{j=[\frac{n+2}{2}]}^{n} \Omega S^{4j-1}$
CII	$\frac{Sp(n)}{Sp(m) \times Sp(n-m)}$ †	$p > n$	$\prod_{j=1}^{m} S^{4j-1} \times \prod_{j=n-m+1}^{n} \Omega S^{4j-1}$
$DIII$	$SO(2n)/U(n)$	$p > n-1$	$\prod_{j=0}^{[\frac{n-2}{2}]} S^{4j+1} \times \prod_{j=[\frac{n+1}{2}]}^{n-1} \Omega S^{4j-1}$

- for †, we assume $2m \leq n$
- for ‡, we assume $2m \leq n+1$

□

Remark 5.5. Terzić's computation of the rational homotopy groups of classical symmetric spaces in [37] can be reproduced from the decompositions above. Our list corrects a typo in her description of the rational homotopy type of $SO(2n)/U(n)$. See also Remark 5.9 for the exceptional cases.

Remark 5.6. Mimura [30] showed that the homotopy decompositions for types AI and AII deloop. He also showed that these cases hold for $p = 3$ as well, and the AII case can be strengthened to hold for $p \geq n$.

5.2. **Exceptional cases.** The following homotopy decompositions for quasi-p-regular exceptional Lie groups are due to Mimura and Toda [32].

Theorem 5.7. For an odd prime p, there are homotopy equivalences:

G	p	
G_2	5	$B(3,11)$
	≥ 7	$S^3 \times S^{11}$
F_4	5	$B(3,11) \times B(15,23)$
	7	$B(3,15) \times B(11,23)$
	11	$B(3,23) \times S^{11} \times S^{15}$
	≥ 13	$S^3 \times S^{11} \times S^{15} \times S^{23}$
E_6	5	$B(3,11) \times B(9,17) \times B(15,23)$
	7	$B(3,15) \times B(11,23) \times S^9 \times S^{17}$
	11	$B(3,23) \times S^9 \times S^{11} \times S^{15} \times S^{17}$
	≥ 13	$S^3 \times S^9 \times S^{11} \times S^{15} \times S^{17} \times S^{23}$
E_7	11	$B(3,23) \times B(15,35) \times S^{11} \times S^{19} \times S^{27}$
	13	$B(3,27) \times B(11,35) \times S^{15} \times S^{19} \times S^{23}$
	17	$B(3,35) \times S^{11} \times S^{15} \times S^{19} \times S^{23} \times S^{27}$
	≥ 19	$S^3 \times S^{11} \times S^{15} \times S^{19} \times S^{23} \times S^{27} \times S^{35}$
E_8	11	$B(3,23) \times B(15,35) \times B(27,47) \times B(39,59)$
	13	$B(3,27) \times B(15,39) \times B(23,47) \times B(35,59)$
	17	$B(3,35) \times B(15,47) \times B(27,59) \times S^{23} \times S^{39}$
	19	$B(3,39) \times B(23,59) \times S^{15} \times S^{27} \times S^{35} \times S^{47}$
	23	$B(3,47) \times B(15,59) \times S^{23} \times S^{27} \times S^{35} \times S^{39}$
	29	$B(3,59) \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S^{47}$
	≥ 31	$S^3 \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S^{47} \times S^{59}$

$\square$

In analyzing the loop space of an exceptional symmetric space corresponding to a map $\varphi\colon H \longrightarrow G$ between quasi-p-regular Lie groups, we will use the following strategy.

Strategy:

(1) Use Theorem 2.4 to replace $H \xrightarrow{\varphi} G$ by $\prod_{m=2}^{p} M(A'_m) \xrightarrow{\prod_{m=2}^{p} M(q_m)} \prod_{m=2}^{p} M(A_m)$.
(2) Determine those q_m which are nontrivial in cohomology via the induced map of indecomposable modules, $Q\varphi^*\colon QH^*(G) \longrightarrow QH^*(H)$.
(3) Observe that the remaining maps $q_m\colon A'_m \longrightarrow A_m$ are trivial because either A'_m or A_m is trivial.
(4) Deduce the homotopy fibre of $M(q_m)$ from Proposition 4.5 or from the fact that $M(q_m)$ is trivial.
(5) Use Corollary 2.5 to obtain $\Omega(G/H) \simeq \prod_{m=2}^{p} \mathrm{fib}(M(q_m))$.

5.2.1. *Type G.* Recall that $SO(4) \simeq_p S^3 \times S^3$ for $p \geq 5$. For $p = 5$, by Theorem 3.6 there is a homotopy commutative diagram

$$\begin{tikzcd}
S^3 \vee S^3 \arrow[r] \arrow[d] & A(3,11) \arrow[d] \\
SO(4) \arrow[r, "\varphi"] & G_2.
\end{tikzcd}$$

Since

$$\varphi^* : QH^3(G_2; \mathbb{F}_p) \to QH^3(SO(4); \mathbb{F}_p) \cong QH^3(S^3 \times S^3; \mathbb{F}_p)$$

is non-trivial, Proposition 4.5 implies that there is a homotopy equivalence

$$\Omega(G_2/SO(4)) \simeq_p S^3 \times \Omega S^{11} \quad (p = 5).$$

For $p > 5$, the space $A(3, 11)$ is replaced by $S^3 \vee S^{11}$ and arguing as in the $p = 5$ case we obtain

$$\Omega(G_2/SO(4)) \simeq_p S^3 \times \Omega S^{11} \quad (p > 5).$$

5.2.2. *Type FI.* By Theorem 5.1 there are homotopy equivalences

$$SU(2) \cdot Sp(3) \simeq_p \begin{cases} S^3 \times B(3, 11) \times S^7 & (p = 5) \\ S^3 \times S^3 \times S^7 \times S^{11} & (p > 5). \end{cases}$$

It is shown in [22] that

$$H^*(FI; \mathbb{F}_p) = \mathbb{F}_p[f_4, f_8]/(r_{16}, r_{24}), \quad (p \geq 5)$$

for some relations r_{16}, r_{24} in degrees 16 and 24 respectively. Thus

$$Q\varphi^* : QH^m(F_4; \mathbb{F}_p) \to QH^m(SU(2) \cdot Sp(3); \mathbb{F}_p)$$

is non-trivial for $m \in \{3, 11\}$ and $p \geq 5$. When $p = 5$, by Theorem 3.6 there is a homotopy commutative diagram

$$\begin{tikzcd} S^3 \vee A(3, 11) \vee S^7 \arrow[r] \arrow[d] & A(3, 11) \vee A(15, 23) \arrow[d] \\ SU(2) \cdot Sp(3) \arrow[r, "\varphi"] & F_4. \end{tikzcd}$$

Proposition 4.5 therefore implies that there is a homotopy equivalence

$$\Omega FI \simeq_5 S^3 \times S^7 \times \Omega B(15, 23).$$

For $p > 5$, arguing similarly we obtain

$$\Omega FI \simeq_p S^3 \times S^7 \times \Omega S^{15} \times \Omega S^{23}.$$

5.2.3. *Type FII.* By Theorem 5.1 there are homotopy equivalences

$$Spin(9) \simeq_p \begin{cases} B(3, 11) \times B(7, 15) & (p = 5) \\ B(3, 15) \times S^7 \times S^{11} & (p = 7) \\ S^3 \times S^7 \times S^{11} \times S^{15} & (p > 7). \end{cases}$$

Since

$$H^*(FII; \mathbb{Z}) = \mathbb{Z}[x_8]/(x_8^3),$$

we have

$$Q\varphi^* : QH^m(BF_4; \mathbb{F}_p) \to QH^m(BSpin(9); \mathbb{F}_p)$$

non-trivial for $m \in \{3, 11, 15\}$ and $p \geq 5$. Therefore, arguing as in the FI case, we obtain homotopy equivalences

$$\Omega(F_4/Spin(9)) \simeq_p S^7 \times \Omega S^{23} \quad (p \geq 5).$$

5.2.4. *Type EIV.* It will be convenient to describe the EIV case before that of EI. We contribute nothing new to this case. By [16], for odd primes p there is a homotopy equivalence

$$E_6 \simeq_p E_6/F_4 \times F_4.$$

So from the decompositions of E_6 and F_4 in Theorem 5.7 one obtains homotopy equivalences

$$E_6/F_4 \simeq \begin{cases} B(9,17) & (p=5) \\ S^9 \times S^{17} & (p \geq 7). \end{cases}$$

5.2.5. *Type EI.* By [20], for odd primes p there is an isomorphism

$$H^*(EI;\mathbb{F}_p) = \mathbb{F}_p[e_8]/(e_8^3) \otimes E(e_9, e_{17}).$$

Notice that the right side is abstractly isomorphic to $H^*(F_4/Spin(9);\mathbb{F}_p) \otimes H^*(E_6/F_4;\mathbb{F}_4)$. Observe that at odd primes, $PSp(4) \simeq_p Spin(9)$ so $EI = E_6/PSp(4) \simeq_p E_6/Spin(9)$. Let

$$\phi : E_6/F_4 \to E_6$$

be the inclusion from the homotopy equivalence $E_6 \simeq_p F_4 \times E_6/F_4$ and let

$$\psi : F_4/Spin(9) \to E_6/Spin(9)$$

be the map of quotient spaces induced from the factorization of the group homomorphism $Spin(9) \longrightarrow E_6$ through F_4. From the homotopy fibration sequence $E_6 \xrightarrow{\partial} E_6/Spin(9) \longrightarrow BSpin(9) \longrightarrow BE_6$ there is a homotopy action

$$\theta\colon E_6 \times E_6/Spin(9) \longrightarrow E_6/Spin(9)$$

which extends $\partial \vee id$. The composition

$$\theta \circ (\phi \times \psi) : E_6/F_4 \times F_4/Spin(9) \to E_6/Spin(9),$$

therefore induces an isomorphism in mod-p cohomology and so is a homotopy equivalence. Combined with the identification of EIV and FII cases, we obtain homotopy equivalences

$$\Omega E_6/PSp(4) \simeq \begin{cases} \Omega B(9,17) \times S^7 \times \Omega S^{23} & (p=5) \\ \Omega S^9 \times \Omega S^{17} \times S^7 \times \Omega S^{23} & (p \geq 7). \end{cases}$$

5.2.6. *Type EII.* By Theorem 5.1 there are homotopy equivalences

$$SU(2)\cdot SU(6) \simeq_p \begin{cases} S^3 \times B(3,11) \times S^5 \times S^7 \times S^9 & (p=5) \\ S^3 \times S^3 \times S^5 \times S^7 \times S^9 \times S^{11} & (p>5). \end{cases}$$

By [21], for $p \geq 5$

$$H^*(E_6/SU(2)\cdot SU(6);\mathbb{F}_p) = \mathbb{F}_p[x_4, x_6, x_8]/(r_{16}, r_{18}, r_{24})$$

for some relations r_{16}, r_{18}, r_{24} in degrees $15, 18, 24$ respectively. Thus for $p \geq 5$

$$Q\varphi^* : QH^m(E_6;\mathbb{F}_p) \to QH^m(SU(2)\cdot SU(6);\mathbb{F}_p),$$

is non-trivial for $m \in \{3, 9, 11\}$. For $p = 5$, by Theorem 3.6 there is a homotopy commutative diagram

$$\begin{array}{ccc} S^3 \vee A(3,11) \vee S^5 \vee S^7 \vee S^9 & \longrightarrow & A(3,11) \vee A(9,17) \vee A(15,23) \\ \downarrow & & \downarrow \\ SU(2)\cdot SU(6) & \xrightarrow{\varphi} & E_6. \end{array}$$

Proposition 4.5 therefore implies that there is a homotopy equivalence

$$\Omega(E_6/SU(2)\cdot SU(6)) \simeq_5 S^3\times S^5\times S^7\times \Omega S^{17}\times \Omega B(15,23).$$

For $p>5$, arguing similarly we obtain

$$\Omega(E_6/SU(2)\cdot SU(6)) \simeq_p S^3\times S^5\times S^7\times \Omega S^{15}\times \Omega S^{17}\times \Omega S^{23}\quad (p>7).$$

5.2.7. *Type EIII.* By Theorem 5.1 there are homotopy equivalences

$$Spin(10)\simeq_p Spin(9)\times S^9\simeq \begin{cases} S^9\times B(3,11)\times B(7,15) & (p=5)\\ S^9\times B(3,15)\times S^7\times S^{11} & (p=7).\end{cases}$$

It is shown in [23, 40] that for $p\geq 5$

$$H^*(E_6/T^1\cdot Spin(10);\mathbb{F}_p)=\mathbb{F}_p[x_2,x_8]/(r_{18},r_{24})$$

for some relations r_{18}, r_{24} in degrees $18, 24$. Thus

$$Q\varphi^*: QH^m(E_6;\mathbb{F}_p)\to QH^m(T^1\cdot Spin(10);\mathbb{F}_p)$$

is non-trivial for $m\in\{3,9,11,15\}$ for $p\geq 5$. Therefore, arguing as in the *EII* case (but modifying slightly to account for the S^1 term by using Remark 3.7) we obtain homotopy equivalences

$$\Omega(E_6/T^1\cdot Spin(10))\simeq_p S^1\times\Omega S^{17}\times S^7\times \Omega S^{23}\quad (p\geq 5).$$

5.2.8. *Type EV.* By Theorem 5.1, for $p\geq 11$ there are homotopy equivalences

$$SU(8)/\{\pm I\}\simeq_p SU(8)\simeq_p S^3\times S^5\times S^7\times S^9\times S^{11}\times S^{13}\times S^{15}.$$

By the Appendix,

$$Q\varphi^*: QH^m(E_7;\mathbb{F}_p)\to QH^m(SU(8)/\{\pm I\};\mathbb{F}_p)$$

is non-trivial for $m\in\{3,11,15\}$ when $p\geq 11$. For $p=11$, by Theorem 3.6 there is a homotopy commutative diagram

$$\begin{tikzcd} S^3\vee S^5\vee S^7\vee S^9\vee S^{11}\vee S^{13}\vee S^{15} \arrow[r] \arrow[d] & A(3,23)\vee A(15,35)\vee S^{11}\vee S^{19}\vee S^{27} \arrow[d] \\ SU(8)/\{\pm I\} \arrow[r, "\varphi"] & E_7. \end{tikzcd}$$

Proposition 4.5 therefore implies that there is a homotopy equivalence

$$\Omega E_7/(SU(8)/\{\pm I\})\simeq_5 S^5\times S^7\times S^9\times S^{13}\times\Omega S^{19}\times\Omega S^{23}\times\Omega S^{27}\times\Omega S^{35}.$$

For $p>11$, arguing similarly we obtain

$$\Omega E_7/(SU(8)/\{\pm I\})\simeq_p S^5\times S^7\times S^9\times S^{13}\times\Omega S^{19}\times\Omega S^{23}\times\Omega S^{27}\times\Omega S^{35}\quad (p\geq 11).$$

5.2.9. *Type EVI.* By Theorem 5.1, there are homotopy equivalences

$$Spin(12) \simeq_p S^3 \times S^7 \times S^{11} \times S^{11} \times S^{15} \times S^{19} \quad (p \geq 11).$$

By [35], for $p \geq 5$

$$H^*(E_7/T^1 \cdot Spin(12); \mathbb{F}_p) = \mathbb{F}_p[x_2, x_8, x_{12}]/(r_{24}, r_{28}, r_{36})$$

for some relations r_{24}, r_{28}, r_{36} in degrees $24, 28, 36$ respectively. From the fibre sequence

$$S^2 \hookrightarrow E_7/T^1 \cdot Spin(12) \to E_7/SU(2) \cdot Spin(12)$$

we therefore obtain

$$H^*(E_7/SU(2) \cdot Spin(12); \mathbb{F}_p) = \mathbb{F}_p[x_4, x_8, x_{12}]/I$$

for some ideal I consisting of elements of degrees ≥ 24. Hence

$$Q\varphi^* : QH^m(E_7; \mathbb{F}_p) \to QH^m(SU(2) \cdot Spin(12); \mathbb{F}_p)$$

is non-trivial for $m \in \{3, 11, 15, 19\}$ when $p \geq 11$. Therefore, arguing as in the EV case we obtain homotopy equivalences

$$\Omega E_7/SU(2) \cdot Spin(12) \simeq_p S^3 \times S^7 \times S^{11} \times \Omega S^{23} \times \Omega S^{27} \times \Omega S^{35} \quad (p \geq 11).$$

5.2.10. *Type EVII.* By Theorem 5.7 there are homotopy equivalences

$$E_6 \simeq_p \begin{cases} B(3, 23) \times S^9 \times S^{11} \times S^{15} \times S^{17} & (p = 11) \\ S^3 \times S^9 \times S^{11} \times S^{15} \times S^{17} \times S^{23} & (p > 11). \end{cases}$$

By [8, 41], for $p \geq 11$

$$H^*(E_7/T^1 \cdot E_6; \mathbb{F}_p) = \mathbb{F}_p[x_2, x_{10}, x_{18}]/(r_{20}, r_{28}, r_{36})$$

for some relations r_{20}, r_{28}, r_{36} in degrees $20, 28, 36$ respectively. Thus

$$Q\varphi^* : QH^m(E_7; \mathbb{F}_p) \to QH^m(T^1 \cdot E_6; \mathbb{F}_p)$$

is non-trivial for $m \in \{3, 11, 15, 23\}$ when $p \geq 11$. Therefore, arguing as in the EV case (modifying slightly to account for the S^1 term by using Remark 3.7) we obtain homotopy equivalences

$$\Omega E_7/T^1 \cdot E_6 \simeq_p S^1 \times S^9 \times S^{17} \times \Omega S^{19} \times \Omega S^{27} \times \Omega S^{35} \quad (p \geq 11).$$

5.2.11. *Type EVIII.* Using Theorem 5.1, there are homotopy equivalences

$$Ss(16) \simeq_p S^{15} \times Sp(7) \simeq_p \begin{cases} B(3, 23) \times B(7, 27) \times S^{11} \times S^{15} \times S^{15} \times S^{19} & (p = 11) \\ B(3, 27) \times S^7 \times S^{11} \times S^{15} \times S^{15} \times S^{19} \times S^{23} & (p = 13) \\ S^3 \times S^7 \times S^{11} \times S^{15} \times S^{15} \times S^{19} \times S^{23} \times S^{27} & (p \geq 17). \end{cases}$$

By [18] and [26],

$$Q\varphi^* : QH^*(E_8; \mathbb{F}_p) \to QH^*(Ss(16); \mathbb{F}_p)$$

is non-trivial for $m \in \{3, 15, 23, 27\}$ when $p > 5$. For $p = 11$, by Theorem 3.6 there is a homotopy commutative diagram

$$\begin{tikzcd} A(3,23) \vee A(7,27) \vee S^{11} \vee S^{15} \vee S^{15} \vee S^{19} \arrow[r] \arrow[d] & A(3,23) \vee A(15,35) \vee A(27,47) \vee A(39,59) \arrow[d] \\ Ss(16) \arrow[r, "\varphi"] & E_8. \end{tikzcd}$$

Proposition 4.5 therefore implies that there is a homotopy equivalence

$$\Omega E_8/Ss(16) \simeq_{11} S^7 \times S^{11} \times S^{15} \times S^{19} \times \Omega S^{35} \times \Omega S^{47} \times \Omega B(39,59).$$

For $p > 11$, arguing similarly we obtain

$$\Omega E_8/Ss(16) \simeq_p \begin{cases} S^7 \times S^{11} \times S^{15} \times S^{19} \times \Omega S^{39} \times \Omega S^{47} \times \Omega B(35,59) & (p = 13) \\ S^7 \times S^{11} \times S^{15} \times S^{19} \times \Omega S^{35} \times \Omega S^{39} \times \Omega S^{47} \times \Omega S^{59} & (p \geq 17). \end{cases}$$

5.2.12. *Type EIX.* Recall the four cases for the homotopy decomposition of E_7 in Theorem 5.7 when $p \geq 11$. By [34],

$$H^*(E_8/T^1 \cdot E_7; \mathbb{F}_p) = \mathbb{F}_p[x_2, x_{12}, x_{20}]/(r_{40}, r_{48}, r_{60}),$$

for some relations r_{40}, r_{48}, r_{60} in degrees $40, 48, 60$ respectively. From the fibre sequence

$$S^2 \hookrightarrow E_8/T^1 \cdot E_7 \to E_8/SU(2) \cdot E_7$$

we obtain

$$H^*(E_8/SU(2) \cdot E_7; \mathbb{F}_p) = \mathbb{F}_p[x_4, x_{12}, x_{20}]/I$$

where I is some ideal consising of elements in degrees ≥ 40. Thus

$$Q\varphi^m : QH^*(E_8; \mathbb{F}_p) \to QH^m(SU(2) \cdot E_7; \mathbb{F}_p)$$

is non-trivial for $m \in \{3, 15, 23, 27, 35\}$ when $p \geq 11$. Arguing similarly to the *EVIII* case we obtain homotopy equivalences

$$\Omega E_8/SU(2) \cdot E_7 \simeq_p \begin{cases} S^3 \times S^{11} \times S^{19} \times \Omega S^{47} \times \Omega B(39,59) & (p = 11) \\ S^3 \times S^{11} \times S^{19} \times \Omega S^{39} \times \Omega S^{47} \times \Omega S^{59} & (p \geq 13). \end{cases}$$

Summarising the results for the exceptional cases, we have the following (together with exponent information which will be proved later in Section 7).

Theorem 5.8. For p an odd prime, there are homotopy equivalences:

Type	G/H	Homotopy type of $\Omega(G/H)$	Exponent
G	$G_2/SO(4)$	$S^3 \times \Omega S^{11}$ $p \geq 5$	$= p^5$
FI	$F_4/SU(2) \cdot Sp(3)$	$S^3 \times S^7 \times \Omega B(15,23)$ $p = 5$ $S^3 \times S^7 \times \Omega S^{15} \times \Omega S^{23}$ $p \geq 7$	$\leq 5^{12}$ $= p^{11}$
FII	$F_4/Spin(9)$	$S^7 \times \Omega S^{23}$ $p \geq 5$	$= p^{11}$
EI	$E_6/PSp(4)$	$S^7 \times \Omega B(9,17) \times \Omega S^{23}$ $p = 5$ $S^7 \times \Omega S^9 \times \Omega S^{17} \times \Omega S^{23}$ $p \geq 7$	$= 5^{11}$ $= p^{11}$
EII	$E_6/SU(2) \cdot SU(6)$	$S^3 \times S^5 \times S^7 \times \Omega S^{17} \times \Omega B(15,23)$ $p = 5$ $S^3 \times S^5 \times S^7 \times \Omega^{15} \times \Omega S^{17} \times \Omega S^{23}$ $p \geq 7$	$\leq 5^{12}$ $= p^{11}$
$EIII$	$E_6/T^1 \cdot Spin(10)$	$S^1 \times S^7 \times \Omega S^{17} \times \Omega S^{23}$ $p \geq 5$	$= p^{11}$
EIV	E_6/F_4	$\Omega B(9,17)$ $p = 5$ $\Omega S^9 \times \Omega S^{17}$ $p \geq 7$	$\leq 5^9$ $= p^8$
EV	$E_7/(SU(8)/\{\pm I\})$	$S^5 \times S^7 \times S^9 \times S^{13} \times \Omega S^{19} \times \Omega S^{23} \times \Omega S^{27} \times \Omega S^{35}$ $p \geq 11$	$= p^{17}$
EVI	$E_7/SU(2) \cdot Spin(12)$	$S^3 \times S^7 \times S^{11} \times \Omega S^{23} \times \Omega S^{27} \times \Omega S^{35}$ $p \geq 11$	$= p^{17}$
$EVII$	$E_7/T^1 \cdot E_6$	$S^1 \times S^9 \times S^{17} \times \Omega S^{19} \times \Omega S^{27} \times \Omega S^{35}$ $p \geq 11$	$= p^{17}$
$EVIII$	$E_8/Ss(16)$	$S^7 \times S^{11} \times S^{15} \times S^{19} \times \Omega S^{35} \times \Omega B(39,59) \times \Omega S^{47}$ $p = 11$ $S^7 \times S^{11} \times S^{15} \times S^{19} \times \Omega B(35,59) \times \Omega S^{39} \times \Omega S^{47}$ $p = 13$ $S^7 \times S^{11} \times S^{15} \times S^{19} \times \Omega S^{35} \times \Omega S^{39} \times \Omega S^{47} \times \Omega S^{59}$ $p \geq 17$	$\leq 11^{30}$ $\leq 13^{30}$ $= p^{29}$
EIX	$E_8/SU(2) \cdot E_7$	$S^3 \times S^{11} \times S^{19} \times \Omega B(39,59) \times \Omega S^{47}$ $p = 11$ $S^3 \times S^{11} \times S^{19} \times \Omega S^{39} \times \Omega S^{47} \times \Omega S^{59}$ $p \geq 13$	$\leq 11^{30}$ $= p^{29}$

□

Remark 5.9. Two of the decompositions in the previous table deloop. Harris [16] showed that $E_6/F_4 \simeq_5 B(9,17)$ and $E_6/F_4 \simeq_p S^9 \times S^{17}$ for $p \geq 7$, and in this paper we show that $E_6/PSp(4) \simeq_p E_6/F_4 \times F_4/Spin(9)$ for $p \geq 3$.

Remark 5.10. Terzic's computation of the rational homotopy groups ([37]) can be easily reproduced from these decompositions. We found minor mistakes in her calculations for $G_2/SO(4)$ and $E_6/SU(2)\cdot SU(6)$. See also Remark 5.10 for classical cases.

6. Limitations and extensions of the methods

In this section we examine the boundaries of our methods and results. It is natural to ask whether the loop space decompositions of symmetric spaces deloop, and whether the methods can be extended to apply in cases that are not quasi-p-regular.

6.1. **Impossibility of delooping.** We gave decompositions for the loop spaces of symmetric spaces. It is reasonable to ask whether they actually come from decompositions of symmetric spaces themselves. Kumpel [27] and Mimura [30] showed that if the homotopy fibration $H \longrightarrow G \longrightarrow G/H$ is totally non-cohomologous to zero then the symmetric space will decompose, delooping our results. This holds for $SU(2n+1)/SO(2n+1)$, $SU(2n)/Sp(n)$, $Spin(2n)/Spin(2n-1)$ and E_6/F_4. However, in general a delooping does not exist, as we now see with the particular example of $FI = F_4/SU(2)\cdot Sp(3)$.

We have shown that

$$\Omega FI \simeq_5 S^3 \times S^7 \times \Omega B(15,23).$$

However, this decomposition does not deloop, as we now show. The following calculation will be needed.

Theorem 6.1 ([22])**.**

$$H^*(FI;\mathbb{F}_p) = \frac{\mathbb{F}_p[f_4,f_8]}{(f_4^3-12f_4f_8+8f_{12},\, f_4f_{12}-3f_8^2,\, f_8^3-f_{12}^2)}.$$

□

In particular,

$$H^*(FI;\mathbb{F}_5) = \frac{\mathbb{F}_5[f_4,f_8,f_{12}]}{(f_4^3-2f_4f_8-2f_{12},\, f_4f_{12}-3f_8^2,\, f_8^3-f_{12}^2)}.$$

We will show that this ring cannot be a non-trivial tensor product of two rings. From the relations we obtain

$$\begin{aligned} 3(f_4^3-2f_4f_8-2f_{12}) &\Rightarrow f_{12} = 3f_4^3 - f_4f_8 \\ f_4f_{12}-3f_8^2 &\Rightarrow f_4^4 - 2f_4^2f_8 - f_8^2 \\ f_8^3 - f_{12}^2&. \end{aligned}$$

If a splitting exists, there should be a substitution

$$f_4 \mapsto f_4, f_8 \mapsto af_8' + bf_4^2, \quad a \in \mathbb{F}_5^\times, b \in \mathbb{F}_5$$

such that the relation $f_4^4 - 2f_4^2f_8 - f_8^2$ lies in $\mathbb{F}_5[f_4] \cup \mathbb{F}_5[f_8']$. However, this is impossible. Therefore there is no non-trivial product decomposition for FI localised at $p = 5$.

6.2. **Non-quasi-p-regular cases.** We study examples of Lie group homomorphisms $H \xrightarrow{\varphi} G$ when H and/or G are not quasi-p-regular. In the first three examples, the methods from Sections 2 to 4 hold and a homotopy decomposition of $\Omega(G/H)$ is obtained, while in the final two examples potential obstructions appear.

All the examples occur at the prime $p = 7$, and relate to the homotopy equivalences

$$E_7 \simeq_7 B(3,15,27) \times B(11,23,35) \times S^{19}$$
$$E_8 \simeq_7 B(3,15,27,39) \times B(23,35,47,59)$$

established in [31].

1. $\mathbf{EV} = \mathbf{E_7/(SU(8)/\{\pm I\})}$. Here, $SU(8)/\{\pm I\} \simeq_7 SU(8) \simeq_7 B(3,15) \times S^5 \times S^7 \times S^9 \times S^{11} \times S^{13}$. We hope to apply Theorem 2.4. Consider the composite

$$\phi\colon A(3,15) \vee S^5 \vee S^7 \vee S^9 \vee S^{11} \vee S^{13} \longrightarrow SU(8) \xrightarrow{\varphi} E_7 \xrightarrow{\simeq} B(3,15,27) \times B(11,23,35) \times S^{19}.$$

By [31], the homotopy groups of $B(3,15,27) \times B(11,23,35) \times S^{19}$ are zero in dimensions $\{5,7,9,13\}$, so ϕ factors through a map $\phi'\colon A(3,15) \vee S^{11} \longrightarrow B(3,15,27) \times B(11,23,35) \times S^{19}$. As well, by [31] $\pi_t(B(11,23,35)) = 0$ for $t \in \{3,15\}$, $\pi_{11}(B(3,15,27)) = 0$ and $\pi_t(S^{19}) = 0$ for $t \in \{3,11,15\}$, so the map ϕ' is determined by the maps $\phi'_1\colon A(3,15) \longrightarrow B(3,15,27)$ and $\phi'_2\colon S^{11} \longrightarrow B(11,23,35)$. The 15-skeleton of $B(3,15,27)$ is $A(3,15)$ so ϕ'_1 factors as a composite $A(3,15) \xrightarrow{g_1} A(3,15,27) \longrightarrow B(3,15,27)$ for some map g_1. Similarly, ϕ'_2 factors as a composite $S^{11} \xrightarrow{g_2} A(11,23,35) \longrightarrow B(11,23,35)$ for some map g_2. Hence there is a homotopy commutative diagram

$$\begin{tikzcd}
A(3,15) \vee S^5 \vee S^7 \vee S^9 \vee S^{11} \vee S^{13} \arrow[r, "Q"] \arrow[d] & A(3,15) \vee S^{11} \arrow[r, "g_1 \vee g_2"] & A(3,15,27) \vee A(11,23,35) \vee S^{19} \arrow[d] \\
SU(8) \arrow[rr, "\varphi"] & & E_7
\end{tikzcd}$$

where Q is the pinch map. Therefore, noting that $M(S^{2n+1}) \simeq S^{2n+1}$, by Theorem 2.4 and Corollary 2.5, the homotopy fibre of the map $SU(8) \xrightarrow{\varphi} E_7$ is homotopy equivalent to the homotopy fibre of the composite

$$M(A(3,15)) \times S^5 \times S^7 \times S^9 \times S^{11} \times S^{13} \xrightarrow{\pi} M(A(3,15)) \times S^{11}$$
$$\xrightarrow{M(g_1)\times M(g_2)} M(A(3,15,27)) \times M(A(11,23,35)) \times S^{19}$$

where π is the projection.

In the Appendix it is shown that

$$Q\varphi^*\colon QH^m(E_7) \longrightarrow QH^m(SU(8)/\{\pm I\})$$

is nontrivial for $m \in \{3,11,15\}$. Thus g_1^* and g_2^* are onto in mod-7 cohomology, implying that $M(g_1)^*$ and $M(g_2)^*$ are onto in mod-7 cohomology. Therefore, arguing as in Proposition 4.5, there is a homotopy equivalence

$$\Omega(E_7/(SU(8)/\{\pm I\}) \simeq_7 S^5 \times S^7 \times S^9 \times S^{13} \times \Omega S^{27} \times \Omega B(23,35) \times \Omega S^{19}.$$

2. $\mathbf{EVI} = \mathbf{E_7/SU(2)\cdot Spin(12)}$. Here, $SU(2)\cdot Spin(12) \simeq_7 SU(2) \times Spin(12) \simeq_7 S^3 \times B(3,15) \times B(7,19) \times S^{11} \times S^{11}$. Arguing as in the previous case, we obtain maps $g_1\colon S^3 \vee A(3,15) \longrightarrow$

$A(3,15,27)$, $g_2\colon S^{11} \vee S^{11} \longrightarrow A(11,23,35)$ and $g_3\colon A(7,19) \longrightarrow S^{19}$ and a homotopy commutative diagram

$$\begin{tikzcd}
(S^3 \vee A(3,15)) \vee (S^{11} \vee S^{11}) \vee A(7,19) \arrow[r, "g_1\vee g_2\vee g_3"] \arrow[d] & A(3,15,27) \vee A(11,23,35) \vee S^{19} \arrow[d] \\
S^3 \times Spin(12) \arrow[r, "\varphi"] & E_7
\end{tikzcd}$$

As in Section 5.2.8, $Q\varphi^*$ is nonzero in degrees $\{3, 11, 15, 19\}$, so arguing as in the previous case we obtain a homotopy equivalence

$$\Omega(E_7/SU(2)\cdot Spin(12)) \simeq_7 S^3 \times \Omega S^{27} \times S^{11} \times \Omega B(23,35) \times S^7.$$

3. **EVII** = $\mathbf{E_7/T^1\cdot E_6}$. Here, $T^1 \cdot E_6 \simeq_7 T^1 \times E_6 \simeq_7 S^1 \times B(3,15) \times B(11,23) \times S^9 \times S^{17}$. Arguing as in the first case, we obtain maps $g_1\colon A(3,15) \longrightarrow A(3,15,27)$ and $g_2\colon A(11,23) \longrightarrow A(11,23,35)$, and a homotopy commutative diagram

$$\begin{tikzcd}
S^1 \vee A(3,15) \vee A(11,23) \vee S^9 \vee S^{17} \arrow[r, "Q"] \arrow[d] & A(3,15) \vee A(11,23) \arrow[r, "g_1\vee g_2"] & A(3,15,27) \vee A(11,23,35) \vee S^{19} \arrow[d] \\
S^1 \times E_6 \arrow[rr, "\varphi"] & & E_7
\end{tikzcd}$$

where Q is the pinch map. As in Section 5.2.9, $Q\varphi^*$ is nonzero in degrees $\{3, 11, 15, 23\}$, so arguing as in the first case we obtain a homotopy equivalence

$$\Omega(E_7/T^1\cdot E_6) \simeq_7 S^1 \times \Omega S^{27} \times \Omega S^{35} \times S^9 \times S^{17} \times \Omega S^{19}.$$

4. **EVIII** = $\mathbf{E_8/Ss(16)}$. Here, $Ss(16) \simeq_7\simeq Spin(16) \simeq_7 B(3,15,27) \times B(7,19) \times B(11,23) \times S^{15}$. We hope to apply Theorem 2.4. Consider the composite

$$\phi\colon A(3,15,27) \vee A(7,19) \vee A(11,23) \vee S^{15} \longrightarrow Spin(16)$$
$$\xrightarrow{\varphi} E_8 \xrightarrow{\simeq} B(3,15,27,39) \times B(23,35,47,59).$$

By [31], the homotopy groups of $B(3,15,27,39) \times B(23,35,47,59)$ are zero in dimensions $\{7,19\}$ so ϕ factors through a map $\phi'\colon A(3,15,27) \vee A(11,23) \vee S^{15}) \longrightarrow B(3,15,27,39) \times B(23,35,47,59)$. By [31], $\pi_t(B(23,35,47,59) = 0$ for $t \in \{3,15,27\}$ and $\pi_t(B(3,15,27,35) = 0$ for $t \in \{11,23\}$, so the map ϕ' is determined by maps $\phi'_1\colon A(3,15,27) \vee S^{15} \longrightarrow B(3,15,27,39)$ and $\phi'_2\colon A(11,23) \longrightarrow B(23,35,47,59)$. Notice that the 27-skeleton of $B(3,15,27,39)$ is $A(3,15,27) \cup e^{18}$, and $\pi_{27}(S^{18}) \cong \mathbb{Z}/7\mathbb{Z}$. Thus there is a potential obstruction to lifting ϕ'_1 to a map $A(3,15,27) \vee S^{15} \longrightarrow A(3,15,27,39)$. It is unclear whether the obstruction vanishes. If not, then Theorem 2.4 cannot be applied and the homotopy type of $\Omega(E_8/Ss(16))$ at $p = 7$ would remain undetermined.

5. **EVIX** = $\mathbf{E_8/SU(2)\cdot E_7}$. As in the previous example, we obtain an obstruction to lifting $\phi'_1\colon S^3 \vee A(3,15,27) \longrightarrow B(3,15,27,39)$ to $A(3,15,27,39)$, which leaves unresolved the homotopy type of $\Omega(E_8/SU(2)\cdot E_7)$ at $p = 7$.

Remark 6.2. An important difference between the three E_7 examples that worked and the two E_8 examples that did not is that the domain in the three E_7 examples were all quasi-p-regular while this was not the case in the E_8 examples.

7. Exponents

Recall that, for a prime p, the *p-primary homotopy exponent* of a space X is the least power of p that annihilates the p-torsion in $\pi_*(X)$. If the p-primary exponent is p^r, write $\exp_p(X) = p^r$. The homotopy decompositions of $\Omega(G/H)$ allow us to find precise exponents or upper and lower bounds on the exponent of G/H.

Observe that in every homotopy decomposition of $\Omega(G/H)$ in Theorems 5.4 and 5.8, the factors are either spheres, sphere bundles over spheres, or the loops on either of these two. Exponent information about these spaces is known. A precise exponent for spheres was determined in [10], and exponent bounds for spaces of the form $B(2m-1, 2m+2p-3)$ was determined in [13].

Theorem 7.1 ([10])**.** Let $p \geq 5$. Then $\exp_p(S^{2n+1}) = p^n$. □

Theorem 7.2 ([13])**.** Let $p \geq 5$. Then $\exp_p(B(3, 2p+1)) = p^{p+1}$ and for $m > 2$,

$$p^{m+p-2} \leq \exp_p(B(2m-1, 2m+2p-3)) \leq p^{m+p-1}.$$

□

Suppose that X is a product of spheres and spaces $B(2i-1, 2i+2p-3)$ for various i. Rationally, X is homotopy equivalent to a product of odd dimensional spheres, say $X \simeq_{\mathbb{Q}} \prod_{i=1}^{\ell} S^{2m_i+1}$. The *type* of X is the list $\{m_1, \dots, m_\ell\}$ where - relabelling if necessary - we may assume that $m_1 \leq \cdots \leq m_\ell$. Theorems 7.1 and 7.2 immediately imply that the exponent of X depends only on the exponent of the factors of X containing a generator in cohomology of degree $2m_\ell + 1$. Explicitly, $\exp_p(X) = p^{m_\ell}$ if each factor of X containing a generator in cohomology of degree $2m_\ell + 1$ is a sphere, and $p^{m_\ell} \leq \exp_p(X) \leq p^{m_\ell+1}$ if at least one factor of X containing a generator in cohomology of degree $2m_\ell+1$ is $B(2m_\ell-2p+3, 2m_\ell+1)$. In our case, observe that the homotopy decompostions for $\Omega(G/H)$ in the classical cases listed in Theorem 5.4 imply that the factor containing a generator in cohomology of maximal degree is of the form $B(2i-1, 2i+2p-3)$ only for $SU(2n+1)/SO(2n+1)$, $SU(2n)/SO(2n)$ and $SU(2n)/Sp(n)$ when $n = p-1$. Thus we have the following.

Theorem 7.3. For $p \geq 5$, there are exponent bounds:

Type	G/H	$p \geq 5$	Exponent
AI	$SU(2n+1)/SO(2n+1)$	$p > n$	$\begin{cases} \leq p^{4n+2} & \text{if } p-1 = n \\ = p^{4n+1} & \text{if } p-1 > n \end{cases}$
	$SU(2n)/SO(2n)$	$p > n$	$\begin{cases} \leq p^{4n} & \text{if } p-1 = n \\ = p^{4n-1} & \text{if } p-1 > n \end{cases}$
AII	$SU(2n)/Sp(n)$	$p > n$	$\begin{cases} \leq p^{4n} & \text{if } p-1 = n \\ = p^{4n-1} & \text{if } p-1 > n \end{cases}$
$AIII$	$\dfrac{U(n)}{U(m) \times U(n-m)}$ †	$p > n/2$	$= p^{2n-1}$
BDI	$\dfrac{SO(2n+1)}{SO(2m) \times SO(2(n-m)+1)}$ †	$p > n$	$= p^{4n-1}$
	$\dfrac{SO(2n+1)}{SO(2m-1) \times SO(2(n-m)+2)}$ ‡	$p > n$	$= p^{4n-1}$
	$\dfrac{SO(2n+2)}{SO(2m+1) \times SO(2(n-m)+1)}$ †	$p > n$	$= p^{4n-1}$
	$\dfrac{SO(2n+2)}{SO(2m) \times SO(2(n-m)+2)}$ ‡	$p > n-1$	$= p^{4n-1}$
CI	$Sp(n)/U(n)$	$p > n$	$= p^{4n-1}$
CII	$\dfrac{Sp(n)}{Sp(m) \times Sp(n-m)}$ †	$p > n$	$= p^{4n-1}$
$DIII$	$SO(2n)/U(n)$	$p > n-1$	$= p^{4n-3}$

- for †, we assume $2m \leq n$
- for ‡, we assume $2m \leq n+1$

□

Theorems 7.1 and 7.2 also imply the exponent bounds listed in Theorem 5.8.

Appendix

For $p > 5$, we show that

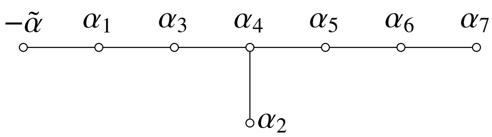
$$Qi^* : QH^m(E_7; \mathbb{F}_p) \to QH^m(SU(8)/C; \mathbb{F}_p)$$

is non-trivial for $m \in \{3, 11, 15\}$, where $C = \{\pm I\}$. To see this we show that

$$Qi^* : QH^m(BE_7; \mathbb{F}_p) \to QH^m(B(SU(8)/C; \mathbb{F}_p))$$

is non-trivial for $m \in \{4, 12, 16\}$ via the Weyl group invariant subrings.

The extended Dynkin-Coxeter diagram for E_7 is as follows:



We adopt a basis t_i's satisfying

$$\tilde{\alpha} = t_1 - t_2,\ \alpha_1 = t_3 - t_2,\ \alpha_2 = \frac{(t_1 + \cdots + t_4) - (t_5 + \cdots + t_8)}{2},\ \alpha_i = t_{i+1} - t_i\ (3 \leq i \leq 7).$$

The Weyl group $W(A_7)$ for $SU(8)/C$ is generated by the reflection corresponding to α_i ($i \neq 2$) and $\tilde{\alpha}$, and

$$H^*(B(SU(8)/C); \mathbb{F}_p) = H^*(BT; \mathbb{F}_p)^{W(A_7)} = \mathbb{F}_p[c_2, \dots, c_8],$$

where c_i is the i-th elementary symmetric polynomial in t_j's. Let ρ be the reflection corresponding to α_2. We check that algebra generators in degrees 4, 12, and 16 contain c_2, c_6 and c_8, respectively, in

$$H^*(BE_7; \mathbb{F}_p) = H^*(BT; \mathbb{F}_p)^{W(E_7)} = \mathbb{F}_p[c_2, \dots, c_8]^\rho.$$

Let a_i and b_i be the i-th elementary symmetric polynomials in $t_1, \dots, t_4$ and $t_5, \dots, t_8$, respectively. Notice that $\alpha_2 = \frac{t_1+t_2+t_3+t_4-t_5-t_6-t_7-t_8}{2} = a_1$ and $c_i = \sum_{j+k=i} a_j b_k$.

Denote $\frac{\alpha_2}{2}$ by τ, for short. Since $\rho(t_i) = t_i - \tau$ for $i \leq 4$ and $\rho(t_i) = t_i + \tau$ for $i \geq 4$, we can compute $\rho(a_i)$ and $\rho(b_i)$ easily, and this yields the following:

$$\begin{aligned}
&\rho(c_2) = c_2,\\
&\rho(c_3) = c_3 + 2(a_2 - b_2)\tau,\\
&\rho(c_4) \equiv c_4 + 3(a_3 - b_3)\tau - 3(a_2 + b_2)\tau^2 \mod (\tau^4),\\
&\rho(c_5) \equiv c_5 + 4(a_4 - b_4)\tau - 2(a_3 + b_3)\tau^2 \mod (\tau^4),\\
&\rho(c_6) \equiv c_6 + (a_3 b_2 - a_2 b_3)\tau - 2a_2 b_2 \tau^2 - 2(a_3 - b_3)\tau^3 \mod (\tau^4),\\
&\rho(c_8) \equiv c_8 + (a_4 b_3 - a_3 b_4)\tau + (a_4 b_2 + a_2 b_4 - a_3 b_3)\tau^2 - (a_3 b_2 - a_2 b_3)\tau^3 \mod (\tau^4).
\end{aligned}$$

We then conclude a generator x_i in degree i satisfies the following by computing modulo (τ^2):

$$\begin{aligned}
&x_4 = c_2,\\
&x_{12} \equiv c_6 - \tfrac{1}{6}c_2 c_4 + \tfrac{1}{8}c_3^2 \mod (a_1),\\
&x_{16} \equiv c_8 - \tfrac{1}{4}c_2 c_6 - \tfrac{1}{8}c_3 c_5 + \tfrac{1}{12}c_4^2 \mod (a_1).
\end{aligned}$$

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