

# THE DUAL POLYHEDRAL PRODUCT, COCATEGORY AND NILPOTENCE

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ABSTRACT. The notion of a dual polyhedral product is introduced as a generalization of Hovey's definition of Lusternik-Schnirelmann cocategory. Properties established from homotopy decompositions that relate the based loops on a polyhedral product to the based loops on its dual are used to show that if  $X$  is a simply-connected space then the weak cocategory of  $X$  equals the homotopy nilpotency class of  $\Omega X$ . This answers a fifty year old problem posed by Ganea. The methods are applied to determine the homotopy nilpotency class of quasi- $p$ -regular exceptional Lie groups and sporadic  $p$ -compact groups.

## 1. INTRODUCTION

This paper establishes strong relationships between three different concepts in topology: polyhedral products, cocategory and homotopy nilpotency. Polyhedral products play a fundamental role in toric topology and have a growing number of applications to other areas of mathematics, such as group actions on graphs, intersections of quadrics, and coordinate subspace arrangements. Cocategory is dual to Lusternik-Schnirelmann category; the latter has been heavily studied since it was introduced in the 1930s, partly because of its connection to counting the critical points of functions between smooth manifolds. Homotopy nilpotency is the topological analogue of nilpotency in group theory, and it has powerful implications for homotopy theory, especially in the stable case.

Throughout the paper, assume that all spaces have the homotopy type of path-connected  $CW$ -complexes. It has long been thought that cocategory and homotopy nilpotency are very closely linked. One problem in establishing a good link is settling on the right definition of cocategory. Several different definitions exist, each trying to dualize some feature of Lusternik-Schnirelmann category. We use Hovey's definition, but others include those by Ganea [G1], Hopkins [Hop] and Murillo-Viruel [MV]. In all cases, a connection is made between the definition of cocategory in question and homotopy nilpotence. For example, Hovey and Murillo-Viruel show that, in their own notion of cocategory, if the cocategory of a simply-connected space  $X$  is  $m$  then iterated Samelson products of length  $\geq m + 1$  formed from the homotopy groups of  $\Omega X$  all vanish. Our main result is to show that if  $X$  is a simply-connected space then the weak cocategory of  $X$  (using Hovey's definition) is precisely equal to the homotopy nilpotency class of  $\Omega X$ . This has the added benefit of identifying Murillo-Viruel's and Hovey's notions of weak cocategory.

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To do this we first generalize the notion of cocategory to dual polyhedral products, and then use loop space decompositions to compare the based loops on a polyhedral product to the based loops on its dual. This results in a natural filtration of the based loops on a polyhedral product that should have many applications beyond those in this paper. A more detailed description is obtained in the special case of a thin product, which is relevant to cocategory.

The methods developed in the paper are sufficiently powerful to allow for explicit calculations of the homotopy nilpotency classes of quasi- $p$ -regular exceptional Lie groups and nonmodular  $p$ -compact groups. These cases would have previously been considered as completely unapproachable.

To describe our results more carefully, several definitions are required.

**The dual polyhedral product.** Let  $K$  be an abstract simplicial complex on  $m$  vertices, and assume the empty set belongs to  $K$ . For  $1 \leq i \leq m$ , let  $(X_i, A_i)$  be a pair of pointed  $CW$ -complexes, where  $A_i$  is a pointed subspace of  $X_i$ . Let  $(\underline{X}, \underline{A}) = \{(X_i, A_i)\}_{i=1}^m$  be the set of  $CW$ -pairs. For each simplex (face)  $\sigma \in K$ , let  $(\underline{X}, \underline{A})^\sigma$  be the subspace of  $\prod_{i=1}^m X_i$  defined by

$$(\underline{X}, \underline{A})^\sigma = \prod_{i=1}^m Y_i \quad \text{where} \quad Y_i = \begin{cases} X_i & \text{if } i \in \sigma \\ A_i & \text{if } i \notin \sigma. \end{cases}$$

The *polyhedral product* determined by  $(\underline{X}, \underline{A})$  and  $K$  is

$$(\underline{X}, \underline{A})^K = \bigcup_{\sigma \in K} (\underline{X}, \underline{A})^\sigma \subseteq \prod_{i=1}^m X_i.$$

For example, suppose each  $A_i$  is a point. If  $K$  is a disjoint union of  $m$  points then  $(\underline{X}, *)^K$  is the wedge  $X_1 \vee \cdots \vee X_m$ , and if  $K$  is the standard  $(m-1)$ -simplex then  $(\underline{X}, *)^K$  is the product  $X_1 \times \cdots \times X_m$ .

The polyhedral product is therefore a colimit over all the faces of  $K$ , ordered by inclusion. There is another way to describe the polyhedral product as a colimit. Let  $[m] = \{1, \dots, m\}$  and let  $I = \{i_1, \dots, i_k\}$  be a subset of  $[m]$ . Let  $K_I$  be the full subcomplex of  $K$  on the vertex set  $I$ , that is, the faces of  $K_I$  are those faces of  $K$  whose vertices are all in  $I$ . There is a map of simplicial complexes  $K_I \rightarrow K$  including  $K_I$  into  $K$ , and if the subsets of  $[m]$  are ordered by inclusion then there is a system of simplicial maps  $K_I \rightarrow K_J$  whenever  $I \subseteq J$ . These induce maps of polyhedral products  $\iota_{I,J}: (\underline{X}, \underline{A})^{K_I} \rightarrow (\underline{X}, \underline{A})^{K_J}$ . Notice that the only possible face of  $K$  which is not in some  $K_I$  for a proper subset  $I$  of  $[m]$  is  $\sigma = [m]$ . But in this case  $K$  equals the full simplex  $\Delta^{m-1}$ , and the polyhedral product  $(\underline{X}, \underline{A})^K$  equals  $X_1 \times \cdots \times X_m$ . Therefore, if  $K \neq \Delta^{m-1}$ , then

$$(\underline{X}, \underline{A})^K = \bigcup_{I \subsetneq [m]} (\underline{X}, \underline{A})^{K_I}.$$

The description of the polyhedral product as a colimit of its full subcomplexes lets us define a dual notion. Fix a simplicial complex  $K$  on the vertex set  $[m]$ . Since each  $(\underline{X}, \underline{A})^{K_I}$  is a pointwise inclusion into  $(\underline{X}, \underline{A})^K$ , it is a cofibration but not a fibration so the appropriate dual notion for

homotopy theory is not an inverse limit but a homotopy inverse limit. Order the subsets of  $[m]$  by reverse inclusion. If  $I \subseteq J$  then there does not exist a simplicial map  $K_J \rightarrow K_I$ . However, there is a map on the level of polyhedral products. If  $X^I$  is the product  $X_{i_1} \times \cdots \times X_{i_k}$  then, as in [DS, 2.2.3], the projection  $X^J \rightarrow X^I$  induces a map of polyhedral products  $\varphi_{J,I}: (\underline{X}, \underline{A})^{K_J} \rightarrow (\underline{X}, \underline{A})^{K_I}$  with the property that  $\varphi_{J,I} \circ \iota_{I,J}$  is the identity map on  $(\underline{X}, \underline{A})^{K_I}$ . Assemble  $\{(\underline{X}, \underline{A})^{K_I} \mid I \subseteq [m]\}$  and  $\{\varphi_{J,I} \mid I \subseteq J \subseteq [m]\}$  as the vertices and edges of an  $m$ -cube in order to take a homotopy inverse limit.

**Definition 1.1.** If  $K \neq \Delta^{m-1}$  then the *dual polyhedral product* is

$$(\underline{X}, \underline{A})_D^K = \varprojlim_{I \subsetneq [m]} (\underline{X}, \underline{A})^{K_I}.$$

**Cocategory.** A special case of the dual polyhedral product is the thin product. Let  $K$  be the simplicial complex on the vertex set  $[m]$  consisting of  $m$  disjoint points. In each pair of spaces  $(X_i, A_i)$ , take  $A_i = *$ . Then for  $I \subseteq [m]$  the full subcomplex  $K_I$  consists of  $|I|$  disjoint points, so

$$(\underline{X}, *)^{K_I} = \bigvee_{i \in I} X_i.$$

If  $J \subseteq I$ , the projection  $K_I \rightarrow K_J$  induces a map of polyhedral products  $\bigvee_{i \in I} X_i \rightarrow \bigvee_{j \in J} X_j$  which sends  $X_i$  to itself if  $i \in J$  or to the basepoint if  $i \notin J$ . Write  $\underline{X}$  for the set of spaces  $\{X_1, \dots, X_m\}$ .

**Definition 1.2.** The *thin product* of pointed spaces  $X_1, \dots, X_m$  is the space

$$P^m(\underline{X}) = (\underline{X}, *)_D^K$$

where  $K$  is the simplicial complex consisting of  $m$  disjoint points.

The thin product was defined by Hovey [Hov] as a dual to the fat wedge, and he used it to define a notion of cocategory that is dual to Lusternik-Schnirelmann category. Some of its properties have been determined by Hovey [Hov] and Anick [A, Lemma 6.24] in the case when  $m = 3$ .

In the case when each  $X_i$  equals a common space  $X$  for  $1 \leq i \leq m$ , write  $P^m(X)$  for  $P^m(\underline{X})$ . Let

$$\nabla: \bigvee_{i=1}^m X \rightarrow X$$

be the  $m$ -fold folding map.

**Definition 1.3.** Let  $X$  be a pointed space. The *cocategory* of  $X$  is the least  $m$  for which there exists an extension

$$\begin{array}{ccc} \bigvee_{i=1}^{m+1} X & \longrightarrow & P^{m+1}(X) \\ \downarrow \nabla & \nearrow & \\ X & & \end{array}$$

In this case, write  $\text{cocat}(X) = m$ .

For example, when  $m = 1$  then  $P^2(X)$  is the homotopy inverse limit of the system  $X \longrightarrow * \longleftarrow X$ . That is,  $P^2(X) \simeq X \times X$ . So  $\text{cocat}(X) = 1$  if and only if  $X$  is an  $H$ -space.

There is a weaker version of cocategory which will be important. Define the space  $F^{m+1}(X)$  and the map  $f^{m+1}(X)$  by the homotopy fibration

$$F^{m+1}(X) \xrightarrow{f^{m+1}(X)} \bigvee_{i=1}^{m+1} X \longrightarrow P^{m+1}(X).$$

**Definition 1.4.** Let  $X$  be a pointed space. The *weak cocategory* of  $X$  is the least  $m$  for which the composite

$$\begin{array}{ccc} F^{m+1}(X) & \xrightarrow{f^{m+1}(X)} & \bigvee_{i=1}^{m+1} X \\ & & \downarrow \nabla \\ & & X \end{array}$$

is null homotopic. In this case, write  $\text{wcocat}(X) = m$ .

Notice that the definitions immediately imply that  $\text{wcocat}(X) \leq \text{cocat}(X)$ .

**Homotopy nilpotency.** An  $H$ -group is a homotopy associative  $H$ -space with a homotopy inverse. Let  $G$  be an  $H$ -group. The commutator  $\bar{c}: G \times G \longrightarrow G$  is defined pointwise by  $\bar{c}(x, y) = xyx^{-1}y^{-1}$ . Observe that the restriction of  $\bar{c}$  to the wedge is null homotopic so  $\bar{c}$  extends to a map

$$c: G \wedge G \longrightarrow G.$$

Since  $\Sigma G \wedge G$  is a retract of  $\Sigma(G \times G)$ , the homotopy class of  $c$  is uniquely determined by that of  $\bar{c}$ . The map  $c$  is the Samelson product of the identity map on  $G$  with itself. For an integer  $m \geq 1$ , let  $G^{(m+1)}$  be the  $(m+1)$ -fold smash product of  $G$  with itself. Define the iterated Samelson product

$$c_m: G^{(m+1)} \longrightarrow G$$

by  $c_m = c \circ (1 \wedge c) \circ \cdots \circ (1 \wedge \cdots 1 \wedge c)$ . Notice that  $c_m$  has a universal property: any Samelson product of length  $m+1$  on  $G$  factors through  $c_m$ .

**Definition 1.5.** Let  $G$  be an  $H$ -group. The *homotopy nilpotency class* of  $G$  is the least  $m$  such that  $c_m$  is null homotopic but  $c_{m-1}$  is not. In this case, write  $\text{nil}(G) = m$ .

For example,  $\text{nil}(G) = 1$  if and only if  $G$  is homotopy commutative. Homotopy nilpotency in this formulation is due to Bernstein and Ganea [BGe], who related it to a notion of cocategory different from Hovey's. A different notion of homotopy nilpotency is due to Biedermann and Dwyer [BD] which occurs in the context of Goodwillie towers. A series of recent papers have explored the relationship between the two types of homotopy nilpotency and the various types of cocategory [BB, CS, CSV, E].

Our main theorem identifies (Hovey's) weak cocategory and (Bernstein-Ganea's) homotopy nilpotency.

**Theorem 1.6.** *Let  $X$  be a simply-connected space. Then  $\text{wccat}(X) = m$  if and only if  $\text{nil}(\Omega X) = m$ .*

One interesting consequence of Theorem 1.6 is the identification of Murillo-Viruel's and Hovey's notions of weak cocategory. The author would like to thank the referee for pointing this out. Write  $\text{MVccat}(X)$  and  $\text{MVwccat}(X)$  for the Murillo-Viruel definition of cocategory (see [MV, Definitions 3.4 and 3.9] for explicit definitions). In [MV, Remark 3.16] it is shown that  $\text{MVccat}(X) \leq \text{ccat}(X)$ , and the same argument shows that  $\text{MVwccat}(X) \leq \text{wccat}(X)$ . On the other hand, in [MV, Remark 4.13] it is shown that  $\text{nil}(\Omega X) \leq \text{MVwccat}(X)$ . Thus, using Theorem 1.6 we obtain a string of inequalities

$$\text{wccat}(X) \leq \text{nil}(\Omega X) \leq \text{MVwccat}(X) \leq \text{wccat}(X)$$

proving the following.

**Corollary 1.7.** *Let  $X$  be a simply-connected space. Then  $\text{MVwccat}(X) = \text{wccat}(X)$ .*  $\square$

The approach to proving Theorem 1.6 is to consider the homotopy fibration  $F^m(\underline{X}) \xrightarrow{f^m(\underline{X})} \bigvee_{i=1}^m X_i \longrightarrow P^m(\underline{X})$ . We identify the homotopy type of  $F^m(\underline{X})$  as a certain wedge of suspensions, and the homotopy class of  $f^m(\underline{X})$  as a wedge sum of Whitehead products. In the case when each  $X_i$  equals a common space  $X$ , this lets us play off the definition of weak cocategory, which involves the map  $f^m(X)$ , with the homotopy nilpotency of  $\Omega X$  by taking the adjoints of the Whitehead products to obtain Samelson products.

The identifications for  $F^m(\underline{X})$  and  $f^m(\underline{X})$  are obtained as special cases of much more general phenomena involving dual polyhedral products. The definition of  $(\underline{X}, \underline{A})_D^K$  as a homotopy inverse limit implies that there is a map  $(\underline{X}, \underline{A})^K \longrightarrow (\underline{X}, \underline{A})_D^K$ . Let  $F_{[m]}$  be its homotopy fibre. By comparing the homotopy types of  $\Omega(\underline{X}, \underline{A})^K$  and  $\Omega(\underline{X}, \underline{A})_D^K$ , we show that there is a homotopy equivalence

$$(1) \quad \Omega(\underline{X}, \underline{A})^K \simeq \Omega(\underline{X}, \underline{A})_D^K \times \Omega F_{[m]}.$$

In a way that can be made precise via certain idempotents,  $\Omega(\underline{X}, \underline{A})_D^K$  contains all the information about  $\Omega(\underline{X}, \underline{A})^K$  that involves only proper subsets of the ingredient pairs  $(X_i, A_i)$ , while  $\Omega F_{[m]}$  captures all of the information about  $\Omega(\underline{X}, \underline{A})^K$  that involves all  $m$  pairs  $(X_i, A_i)$  simultaneously. This leads to a filtration of the homotopy theory of  $\Omega(\underline{X}, \underline{A})^K$  obtained from its full subcomplexes.

**Theorem 1.8.** *For any polyhedral product  $(\underline{X}, \underline{A})^K$ , there is a homotopy equivalence*

$$\Omega(\underline{X}, \underline{A})^K \simeq \prod_{I \subseteq [m]} \Omega F_I$$

where  $F_I$  is the homotopy fibre of the map  $(\underline{X}, \underline{A})^{K_I} \longrightarrow (\underline{X}, \underline{A})_D^{K_I}$ .

In the special case when  $(\underline{X}, \underline{A})$  is of the form  $(\underline{CX}, \underline{X})$ , where  $CX$  is the reduced cone on  $X$ , more can be said. The polyhedral product  $(\underline{CX}, \underline{X})^K$  has been well studied. In [BBCG] it is shown that there is a homotopy equivalence

$$(2) \quad \Sigma(\underline{CX}, \underline{X})^K \simeq \bigvee_{I \notin K} \Sigma^2(|K_I| \wedge \widehat{X}^I)$$

where  $|K_I|$  is the geometric realization of  $K_I$ , and for  $I = \{i_1, \dots, i_k\}$ , we have  $\widehat{X}^I = X_{i_1} \wedge \dots \wedge X_{i_k}$ . In particular,  $\Sigma(\underline{CX}, \underline{X})^K$  is a wedge of suspensions of iterated smashes of the spaces  $X_i$ .

A great deal of work has been done in [BGr, GT2, GT3, IK1, IK2] to determine for which simplicial complexes the decomposition (2) desuspends. In [IK2] the notion of a totally homology fillable simplicial complex was introduced, which includes the more well known families of shifted, shellable and sequentially Cohen-Macaulay complexes. It was shown that if  $K$  is totally homology fillable then there is a homotopy equivalence

$$(3) \quad (\underline{CX}, \underline{X})^K \simeq \bigvee_{I \notin K} \Sigma(|K_I| \wedge \widehat{X}^I)$$

and  $\Sigma|K_I|$  is homotopy equivalent to a wedge of spheres. Consequently,  $(\underline{CX}, \underline{X})^K$  is homotopy equivalent to a wedge of suspensions of iterated smashes of the spaces  $X_i$ .

When  $K$  is totally homology fillable we show that the spaces  $F_I$  in Theorem 1.8 are also homotopy equivalent to wedges of suspensions of iterated smashes of the  $X_i$ 's. Further, in the special case of the thin product, the spaces  $F_{[m]}$  and  $F^m(\underline{X})$  are homotopy equivalent, and the map  $F^m(\underline{X}) \xrightarrow{f^m(\underline{X})} \bigvee_{i=1}^m X_i$  is a wedge sum of iterated Whitehead products.

This paper is organized as follows. In Part I we give homotopy decompositions of  $\Omega(\underline{X}, \underline{A})^K$  and  $\Omega(\underline{X}, \underline{A})_D^K$  via certain idempotents. Section 2 establishes the basic decomposition in terms of certain telescopes and Section 3 identifies the telescopes as certain loop spaces. Section 4 refines the decomposition in the case of  $\Omega(\underline{CX}, \underline{X})^K$  and  $\Omega(\underline{CX}, \underline{X})_D^K$  when  $K$  is totally homology fillable by showing that the factors are the based loops on certain wedges of suspensions. In Section 5 these results are then transferred to give analogous decompositions in the case of  $\Omega(\underline{X}, *)^K$  and  $\Omega(\underline{X}, *)_D^K$  for the same complexes  $K$ .

In Part II the role of Whitehead products is investigated. Section 6 recounts some of the homotopy theory surrounding a wedge of two spaces. In Section 7 a fundamental result is proved that does not seem to be in the literature: we show that Porter's decomposition of the homotopy fibre of the inclusion of a wedge into a product can be altered by a homotopy equivalence so that the maps from the fibre into the total space are described by Whitehead products. Along the way we also give a refined decomposition in the case when each space in the wedge is a suspension. Finally, in Section 8 all this is applied to give a homotopy decomposition of the homotopy fibre of the map from

the wedge into the thin product, and to identify the maps from the fibre to the wedge as certain Whitehead products.

In Part III the results from Parts I and II are used to prove Theorem 1.6 in Section 9. To explicitly calculate the homotopy nilpotency classes, a special class of  $H$ -spaces called retractile  $H$ -spaces is discussed in Section 10, and a criterion for applying Theorem 1.6 to retractile  $H$ -spaces is proved in Section 11. Examples are then given in Section 12.

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## Part 1. Homotopy decompositions of $\Omega(\underline{X}, \underline{A})^K$ and $\Omega(\underline{X}, \underline{A})_D^K$ .

### 2. DECOMPOSITIONS VIA IDEMPOTENTS

The focus for the most part is on  $\Omega(\underline{X}, \underline{A})^K$ , with the decomposition for  $\Omega(\underline{X}, \underline{A})_D^K$  being a consequence. The decomposition will be constructed by using a family of commuting idempotents. The idempotents will be defined on  $(\underline{X}, \underline{A})^K$  but in order to take a product of their telescopes a multiplication is needed, which is why loop spaces are taken. We begin with some general information about decompositions of  $H$ -groups using idempotents.

In general, let  $G$  be a path-connected  $H$ -group. Suppose that for  $1 \leq j \leq k$  there is a family of maps  $e_j: G \rightarrow G$  such that: (i) each  $e_j$  is an idempotent, (ii)  $e_i \circ e_j \simeq *$  if  $i \neq j$ , and (iii)  $1 \simeq e_1 + \cdots + e_k$ , where  $1$  is the identity map on  $G$  and the addition refers to the group structure on  $[G, G]$  induced by the multiplication on  $G$ . The family  $\{e_j\}_{j=1}^k$  is called a set of *mutually orthogonal idempotents*. Let  $T(e_j)$  be the telescope of  $e_j$ , that is,  $T(e_j) = \text{hocolim}_{e_j} G$ , and let  $G \rightarrow T(e_j)$  be the map to the telescope. Observe that, being a telescope, there is a map  $T(e_j) \rightarrow G$ , and as  $e_j$  is an idempotent the composite  $T(e_j) \rightarrow G \rightarrow T(e_j)$  is a homotopy equivalence. In particular,  $T(e_j)$  is a retract of an  $H$ -space and so is itself an  $H$ -space. Observe also that  $H_*(T(e_j)) \cong \text{Im}(e_j)_*$ . Since  $e_i \circ e_j \simeq *$  for  $i \neq j$  we may form the direct sum  $\oplus_{j=1}^k H_*(T(e_j))$ , and since  $1 \simeq e_1 + \cdots + e_k$  we obtain an isomorphism of modules

$$H_*(G) \cong \oplus_{j=1}^k H_*(T(e_j)).$$

The product of the telescope maps

$$G \longrightarrow \prod_{j=1}^k T(e_j)$$

therefore induces an isomorphism in homology. Since both  $G$  and  $\prod_{j=1}^k T(e_j)$  are  $H$ -spaces they are nilpotent. Therefore, by Dror's [Dr, Example 4.3] generalization of Whitehead's Theorem, the product of telescope maps is a homotopy equivalence.

**Lemma 2.1.** *Let  $G$  be a path-connected  $H$ -group and suppose that  $\{e_j\}_{j=1}^k$  is a family of mutually orthogonal idempotents on  $G$ . Then there is a homotopy equivalence  $G \rightarrow \prod_{j=1}^k T(e_j)$ .  $\square$*

Next, suppose that for  $1 \leq j \leq m$  there is a family of commuting idempotents  $e_j: G \rightarrow G$ . Observe that each pair  $(e_j, 1 - e_j)$  is mutually orthogonal, and the larger family of idempotents  $\{e_j, 1 - e_j\}_{j=1}^m$  all commute. Let  $\mathcal{J}$  be the collection of  $2^m$  sequences  $(a_1, \dots, a_m)$ , where each  $a_j \in \{0, 1\}$ . For  $(a_1, \dots, a_m) \in \mathcal{J}$ , define

$$f_{(a_1, \dots, a_m)}: G \rightarrow G$$

by the composite

$$f_{(a_1, \dots, a_m)} = f_{a_1} \circ \dots \circ f_{a_m} \quad \text{where} \quad f_{a_j} = \begin{cases} e_j & \text{if } a_j = 0 \\ 1 - e_j & \text{if } a_j = 1. \end{cases}$$

We record three properties of the maps  $f_{(a_1, \dots, a_m)}$ . First, since the idempotents  $\{e_j, 1 - e_j\}_{j=1}^m$  commute, each map  $f_{(a_1, \dots, a_m)}$  is also an idempotent. Second, if  $f_{(a'_1, \dots, a'_m)}$  is another such idempotent distinct from  $f_{(a_1, \dots, a_m)}$ , then at least one  $1 \leq j \leq m$  satisfies  $a'_j \neq a_j$ , for otherwise the two maps agree on every  $a_j$  and so are identical. Therefore the  $j^{\text{th}}$  term in the composite for  $f_{(a_1, \dots, a_m)}$  is  $e_j$  and that for  $f_{(a'_1, \dots, a'_m)}$  is  $1 - e_j$ , or vice-versa. But as  $e_j \circ (1 - e_j)$  is null homotopic and the idempotents commute, we obtain  $f_{(a_1, \dots, a_m)} \circ f_{(a'_1, \dots, a'_m)} \simeq *$ . Third, since  $1 = e_j + (1 - e_j)$  for each  $j$ , we obtain  $1 = (e_1 + (1 - e_1)) \circ \dots \circ (e_m + (1 - e_m))$ . Expanding gives  $1 = \sum_{(a_1, \dots, a_m) \in \mathcal{J}} f_{(a_1, \dots, a_m)}$ . The three properties together imply that the collection of idempotents  $\{f_{(a_1, \dots, a_m)}\}_{(a_1, \dots, a_m) \in \mathcal{J}}$  is mutually orthogonal. Therefore, if  $T(a_1, \dots, a_m)$  is the telescope of  $f_{(a_1, \dots, a_m)}$  then Lemma 2.1 implies the following.

**Lemma 2.2.** *Let  $G$  be a path-connected  $H$ -group and suppose that  $\{e_j\}_{j=1}^m$  is a family of commuting idempotents on  $G$ . Then there is a homotopy equivalence  $G \rightarrow \prod_{(a_1, \dots, a_m) \in \mathcal{J}} T(a_1, \dots, a_m)$ .  $\square$*

We wish to construct a family of commuting idempotents on  $(\underline{X}, \underline{A})^K$ . Recall from the Introduction that if  $I = (i_1, \dots, i_k) \subseteq [m]$  then  $K_I$  is a full subcomplex of  $K$  and there is a map of simplicial complexes  $K_I \rightarrow K$  but not a map  $K \rightarrow K_I$ . However, the situation improves on the level of polyhedral products. Let  $X^I = X_{i_1} \times \dots \times X_{i_k}$ , let  $X^I \rightarrow X^m$  be the map defined by sending the  $j^{\text{th}}$ -factor of  $X^I$  to the  $(i_j)^{\text{th}}$ -factor of  $X^m$ , and let  $X^m \rightarrow X^I$  be the projection. The inclusion of  $K_I$  into  $K$  induces a map  $\iota_I: (\underline{X}, \underline{A})^{K_I} \rightarrow (\underline{X}, \underline{A})^K$ . An immediate consequence of the definition of the polyhedral product as a union of coordinate subspaces of  $X^m = X_1 \times \dots \times X_m$  is the following lemma [DS, Lemma 2.2.3].

**Lemma 2.3.** *Let  $K_I$  be a full subcomplex of  $K$ . The following hold:*

- (a) *the inclusion  $X^I \rightarrow X^m$  induces a map of polyhedral products  $(\underline{X}, \underline{A})^{K_I} \rightarrow (\underline{X}, \underline{A})^K$  which equals  $\iota_I$ ;*



- (b) the projection  $X^m \rightarrow X^I$  induces a map of polyhedral products  $\varphi_I: (\underline{X}, \underline{A})^K \rightarrow (\underline{X}, \underline{A})^{K_I}$ ;
- (c) the composite  $(\underline{X}, \underline{A})^{K_I} \xrightarrow{\iota_I} (\underline{X}, \underline{A})^K \xrightarrow{\varphi_I} (\underline{X}, \underline{A})^{K_I}$  is the identity map.

□

For  $1 \leq j \leq m$ , let  $I_j = [m] \setminus \{j\}$ . By Lemma 2.3 there is a composite of polyhedral products

$$e_j: (\underline{X}, \underline{A})^K \xrightarrow{\varphi_{I_j}} (\underline{X}, \underline{A})^{K_{I_j}} \xrightarrow{\iota_{I_j}} (\underline{X}, \underline{A})^K.$$

**Lemma 2.4.** *The following hold:*

- (a) for  $1 \leq j \leq m$  the map  $e_j$  is an idempotent;
- (b) for any  $1 \leq j, k \leq m$  we have  $e_j \circ e_k = e_k \circ e_j$ .

*Proof.* Part (a) follows immediately from the definition of  $e_j$  and Lemma 2.3 (c). For part (b), if  $j = k$  then the statement is a tautology. Suppose that  $j \neq k$ . Let  $I_{j,k} = [m] \setminus \{j, k\}$ . Observe that the composites of projection and inclusion maps  $X^m \rightarrow X^{I_j} \rightarrow X^m \rightarrow X_{I_k} \rightarrow X^m$  and  $X^m \rightarrow X^{I_k} \rightarrow X^m \rightarrow X_{I_j} \rightarrow X^m$  both equal the composite  $X^m \rightarrow X^{I_{j,k}} \rightarrow X^m$ . Therefore, as in Lemma 2.3,  $e_j \circ e_k = e_k \circ e_j$ . □

By Lemma 2.4 the maps  $\{e_j\}_{j=1}^m$  are commuting idempotents on  $(\underline{X}, \underline{A})^K$ . However, this space is not an  $H$ -space in general so we must loop to obtain one. Since the loop map of an idempotent is an idempotent,  $\{\Omega e_j\}_{j=1}^m$  are commuting idempotents on  $\Omega(\underline{X}, \underline{A})^K$ . For  $(a_1, \dots, a_m) \in \mathcal{J}$ , define

$$f_{(a_1, \dots, a_m)}: \Omega(\underline{X}, \underline{A})^K \rightarrow \Omega(\underline{X}, \underline{A})^K$$

by the composite

$$f_{(a_1, \dots, a_m)} = f_{a_1} \circ \dots \circ f_{a_m} \quad \text{where} \quad f_{a_j} = \begin{cases} \Omega e_j & \text{if } a_j = 0 \\ 1 - \Omega e_j & \text{if } a_j = 1. \end{cases}$$

Then each  $f_{a_j}$  is an idempotent and as  $\{\Omega e_j\}_{j=1}^m$  commute, the composite  $f_{(a_1, \dots, a_m)}$  is also an idempotent. Let

$$T(a_1, \dots, a_m) = \varinjlim_{f_{(a_1, \dots, a_m)}} \Omega(\underline{X}, \underline{A})^K.$$

Lemma 2.2 implies the following.

**Proposition 2.5.** *Assume that  $(\underline{X}, \underline{A})^K$  is simply-connected. Then there is a homotopy equivalence*

$$\Omega(\underline{X}, \underline{A})^K \simeq \prod_{(a_1, \dots, a_m) \in \mathcal{J}} T(a_1, \dots, a_m).$$

□

We wish to relate the factors  $T(a_1, \dots, a_m)$  in the decomposition of  $\Omega(\underline{X}, \underline{A})^K$  in Proposition 2.5 to the factors that appear in the corresponding decomposition for  $\Omega(\underline{X}, \underline{A})^{K_I}$ .

**Lemma 2.6.** *If  $j \notin I$  then there is a commutative diagram*

$$\begin{array}{ccc} (\underline{X}, \underline{A})^K & \xrightarrow{e_j} & (\underline{X}, \underline{A})^K \\ \downarrow \varphi_I & & \downarrow \varphi_I \\ (\underline{X}, \underline{A})^{K_I} & \xlongequal{\quad} & (\underline{X}, \underline{A})^{K_I}. \end{array}$$

*Proof.* Recall that  $I_j = [m] \setminus \{j\}$ . Since  $j \notin I$ ,  $K_I$  is a full subcomplex of  $K_{I_j}$ . So the projection  $X^m \rightarrow X^I$  is the same as the composite  $X^m \rightarrow X^{I_j} \rightarrow X^m \rightarrow X^I$ . As in Lemma 2.3, the induced maps of polyhedral products  $(\underline{X}, \underline{A})^K \rightarrow (\underline{X}, \underline{A})^{K_I}$  and  $(\underline{X}, \underline{A})^K \rightarrow (\underline{X}, \underline{A})^{K_{I_j}} \rightarrow (\underline{X}, \underline{A})^K \rightarrow (\underline{X}, \underline{A})^{K_I}$  are the same. The lemma now follows.  $\square$

Let  $T(e_j)$  be the telescope of  $e_j$ . Taking telescopes of the horizontal maps in Lemma 2.6 immediately implies the following.

**Corollary 2.7.** *If  $j \notin I$  then there is a commutative diagram*

$$\begin{array}{ccc} (\underline{X}, \underline{A})^K & \xrightarrow{\quad} & T(e_j) \\ \downarrow \varphi_I & & \downarrow \\ (\underline{X}, \underline{A})^{K_I} & \xlongequal{\quad} & (\underline{X}, \underline{A})^{K_I}. \end{array}$$

$\square$

Now consider the map  $f_{(a_1, \dots, a_m)} = f_{a_1} \circ \dots \circ f_{a_m}$  where  $f_{a_j}$  is either  $\Omega e_j$  or  $1 - \Omega e_j$ . Suppose that  $j \notin I$ . If  $a_j = 0$  then  $f_{a_j} = \Omega e_j$ . By Lemma 2.6,  $\Omega \varphi_I \circ \Omega e_j = \Omega \varphi_I$ . So as the idempotents  $\{\Omega e_j \mid 1 \leq j \leq m\}$  commute, we obtain

$$\Omega \varphi_I \circ f_{a_1} \circ \dots \circ f_{a_{j-1}} \circ \Omega e_j \circ f_{a_{j+1}} \circ \dots \circ f_{a_m} = \Omega \varphi_I \circ f_{a_1} \circ \dots \circ f_{a_{j-1}} \circ f_{a_{j+1}} \circ \dots \circ f_{a_m}.$$

If  $a_j = 1$  then  $f_{a_j} = 1 - \Omega e_j$ . By Corollary 2.7,  $\Omega \varphi_I \circ (1 - \Omega e_j) \simeq *$ . So as the idempotents  $\{\Omega e_j \mid 1 \leq j \leq m\}$  commute, we obtain

$$\Omega \varphi_I \circ f_{a_1} \circ \dots \circ f_{a_{j-1}} \circ (1 - \Omega e_j) \circ f_{a_{j+1}} \circ \dots \circ f_{a_m} \simeq *.$$

Doing this for every  $j \notin I$  gives the following.

**Lemma 2.8.** *Let  $I = \{i_1, \dots, i_k\} \subseteq [m]$ . The following hold:*

- (a) *if  $a_j = 0$  for every  $j \notin I$  then  $\Omega \varphi_I \circ f_{a_1} \circ \dots \circ f_{a_m} \simeq \Omega \varphi_I \circ f_{a_{i_1}} \circ \dots \circ f_{a_{i_k}}$ ;*
- (b) *if  $a_j = 1$  for some  $j \notin I$  then  $\Omega \varphi_I \circ f_{a_1} \circ \dots \circ f_{a_m} \simeq *$ .*

$\square$

Let  $\mathcal{J}_I$  be the index set for the  $2^{|I|}$  idempotents  $f_{(a_{i_1}, \dots, a_{i_k})}$  used to decompose  $\Omega(\underline{X}, \underline{A})^{K_I}$  in Proposition 2.5. Notice that by Lemma 2.8 (a),  $f_{(a_{i_1}, \dots, a_{i_k})}$  corresponds precisely to the idempotent  $f_{(a_1, \dots, a_m)}$  on  $\Omega(\underline{X}, \underline{A})^K$  where every  $j \notin I$  has  $a_j = 0$ . Let  $T(a_{i_1}, \dots, a_{i_k})$  be the telescope of  $f_{(a_{i_1}, \dots, a_{i_k})}$ . Then Lemma 2.8 implies the following.

**Lemma 2.9.** *Let  $I = \{i_1, \dots, i_k\} \subseteq [m]$ . The following hold:*

(a) *if  $a_j = 0$  for every  $j \notin I$  then there is a commutative diagram*

$$\begin{array}{ccc} \Omega(\underline{X}, \underline{A})^K & \longrightarrow & T(a_1, \dots, a_m) \\ \downarrow \Omega\varphi_I & & \downarrow \simeq \\ \Omega(\underline{X}, \underline{A})^{K_I} & \longrightarrow & T(a_{i_1}, \dots, a_{i_k}); \end{array}$$

(b) *if  $a_j = 1$  for some  $j \notin I$  then the composite  $T(a_1, \dots, a_m) \hookrightarrow \Omega(\underline{X}, \underline{A})^K \xrightarrow{\Omega\varphi_I} \Omega(\underline{X}, \underline{A})^{K_I}$  is null homotopic.*

□

From Lemma 2.9 we obtain compatibility for the decompositions of  $\Omega(\underline{X}, \underline{A})^K$  and  $\Omega(\underline{X}, \underline{A})^{K_I}$ .

**Proposition 2.10.** *Let  $I \subseteq [m]$ . There is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega(\underline{X}, \underline{A})^K & \xrightarrow{\simeq} & \prod_{(a_1, \dots, a_m) \in \mathcal{J}} T(a_1, \dots, a_m) \\ \downarrow \Omega\varphi_I & & \downarrow \pi_I \\ \Omega(\underline{X}, \underline{A})^{K_I} & \xrightarrow{\simeq} & \prod_{(a_{i_1}, \dots, a_{i_k}) \in \mathcal{J}_I} T(a_{i_1}, \dots, a_{i_k}) \end{array}$$

where  $\pi_I$  projects away from factors with  $a_j = 1$  for some  $j \notin I$  and identifies  $T(a_1, \dots, a_m)$  with  $T(a_{i_1}, \dots, a_{i_k})$  for factors with  $a_j = 0$  for all  $j \notin I$ . □

Next, we bring in the dual polyhedral product. By definition,

$$(\underline{X}, \underline{A})_D^K = \varprojlim_{I \subsetneq [m]} (\underline{X}, \underline{A})^{K_I}$$

where the homotopy inverse limit is taken over the maps of polyhedral products  $(\underline{X}, \underline{A})^{K_J} \rightarrow (\underline{X}, \underline{A})^{K_I}$  induced by the projection  $X^J \rightarrow X^I$  when  $I \subseteq J$ . Looping, we obtain

$$\Omega(\underline{X}, \underline{A})_D^K = \varprojlim_{I \subsetneq [m]} \Omega(\underline{X}, \underline{A})^{K_I}.$$

On the other hand, by Proposition 2.10 the decompositions of the spaces  $\Omega(\underline{X}, \underline{A})^{K_I}$  in Proposition 2.5 are compatible with the maps  $\Omega(\underline{X}, \underline{A})^{K_I} \rightarrow \Omega(\underline{X}, \underline{A})^{K_J}$  and induce projections onto factors. Therefore  $\varprojlim_{I \subsetneq [m]} \Omega(\underline{X}, \underline{A})^{K_I}$  is precisely the product of all possible distinct factors that appear in the decompositions of  $\Omega(\underline{X}, \underline{A})^{K_I}$  for any  $I \subsetneq [m]$ . Put another way, the only factors of  $\Omega(\underline{X}, \underline{A})^K$  which are not also factors of  $\Omega(\underline{X}, \underline{A})_D^K$  are those that project trivially under every map  $\pi_I$  for all  $I \subsetneq [m]$ . There is only one such factor,  $T(1, \dots, 1)$ , so we obtain the following.

**Proposition 2.11.** *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega(\underline{X}, \underline{A})^K & \xrightarrow{\simeq} & \prod_{(a_1, \dots, a_m) \in \mathcal{J}} T(a_1, \dots, a_m) \\ \downarrow \Omega\varphi & & \downarrow \pi \\ \Omega(\underline{X}, \underline{A})_D^K & \xrightarrow{\simeq} & \prod_{(a_1, \dots, a_m) \in \mathcal{J} \setminus (1, \dots, 1)} T(a_1, \dots, a_m) \end{array}$$

where  $\pi$  is the projection. □

Since every factor of  $\Omega(\underline{X}, \underline{A})_D^K$  is also a factor of  $\Omega(\underline{X}, \underline{A})^K$  and the only additional factor of  $\Omega(\underline{X}, \underline{A})^K$  is  $T(1, \dots, 1)$ , Proposition 2.11 immediately implies the following.

**Theorem 2.12.** *Assume that  $(\underline{X}, \underline{A})^K$  is simply-connected. The map  $\Omega(\underline{X}, \underline{A})^K \xrightarrow{\Omega\varphi} \Omega(\underline{X}, \underline{A})_D^K$  has a right homotopy inverse and there is a homotopy equivalence*

$$\Omega(\underline{X}, \underline{A})^K \simeq \Omega(\underline{X}, \underline{A})_D^K \times T(1, \dots, 1).$$
□

### 3. FURTHER PROPERTIES OF THE DECOMPOSITIONS

Next, we show that each of the factors  $T(a_1, \dots, a_m)$  in the decompositions for  $\Omega(\underline{X}, \underline{A})^K$  and  $\Omega(\underline{X}, \underline{A})_D^K$  is a loop space, and prove Theorem 1.8. Define the space  $F_{[m]}$  be the homotopy fibration

$$F_{[m]} \longrightarrow (\underline{X}, \underline{A})^K \xrightarrow{\varphi} (\underline{X}, \underline{A})_D^K.$$

**Lemma 3.1.** *There is a homotopy equivalence  $T(1, \dots, 1) \simeq \Omega F_{[m]}$ .*

*Proof.* On the one hand, by Proposition 2.11, the homotopy fibre of  $\Omega\varphi$  is the same as that of  $\pi$ , which is  $T(1, \dots, 1)$ . On the other hand, by definition of  $F_{[m]}$ , the homotopy fibre of  $\Omega\varphi$  is  $\Omega F_{[m]}$ . Thus there is an induced map of fibres  $T(1, \dots, 1) \longrightarrow \Omega F_{[m]}$  which induces isomorphisms on homotopy groups by the Five Lemma. Since all spaces are  $CW$ -complexes, this implies the map of fibres is a homotopy equivalence. □

Fix a sequence  $(a_1, \dots, a_m) \in \mathcal{J}$ . Let  $\{a_{i_1}, \dots, a_{i_k}\}$  be the set consisting of all the elements in the sequence which are 1 and let  $\{a_{j_1}, \dots, a_{j_\ell}\}$  be the set consisting of all the elements in the sequence which are 0. Let  $I = \{i_1, \dots, i_k\}$  and  $J = \{j_1, \dots, j_\ell\}$ . Note that  $k + \ell = m$  and  $I \cap J = \emptyset$ .

By Lemma 2.9 (a) there is a homotopy equivalence  $T(a_1, \dots, a_m) \simeq T(a_{i_1}, \dots, a_{i_k})$ , where  $T(a_{i_1}, \dots, a_{i_k})$  is the telescope of the idempotent  $f_{a_{i_1}} \circ \dots \circ f_{a_{i_k}} = (1 - \Omega e_{i_1}) \circ \dots \circ (1 - \Omega e_{i_k})$  on  $\Omega(\underline{X}, \underline{A})^{K_I}$ . Note that each  $a_{i_t}$  equals 1 for  $1 \leq t \leq k$ , so applying Lemma 3.1 to the case of  $\Omega(\underline{X}, \underline{A})^{K_I}$  immediately implies the following.

**Lemma 3.2.** *There is a homotopy equivalence*

$$T(a_1, \dots, a_m) \simeq \Omega F_I$$

where  $I = \{i_1, \dots, i_k\}$  consists of those indices in  $[m]$  for which  $a_{i_t} = 1$  and  $F_I$  is the homotopy fibre of the map  $(\underline{X}, \underline{A})^{K_I} \rightarrow (\underline{X}, \underline{A})_D^{K_I}$ .  $\square$

Consequently, the decompositions of  $\Omega(\underline{X}, \underline{A})^K$  in Proposition 2.5 and of  $\Omega(\underline{X}, \underline{A})_D^K$  in Proposition 2.11 can be rewritten as follows.

**Theorem 3.3.** *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \Omega(\underline{X}, \underline{A})^K & \xrightarrow{\simeq} & \prod_{I \subseteq [m]} \Omega F_I \\ \downarrow \Omega\varphi & & \downarrow \pi \\ \Omega(\underline{X}, \underline{A})_D^K & \xrightarrow{\simeq} & \prod_{I \subsetneq [m]} \Omega F_I. \end{array}$$

where  $\pi$  is the projection.  $\square$

*Proof of Theorem 1.8.* This is simply the homotopy equivalence in the first row of Theorem 3.3.  $\square$

**Remark 3.4.** It is tempting to suspect that the homotopy decompositions in Theorem 3.3 deloop, that is, that there are homotopy equivalences  $(\underline{X}, \underline{A})^K \simeq \prod_{I \subseteq [m]} F_I$  and  $(\underline{X}, \underline{A})_D^K \simeq \prod_{I \subsetneq [m]} F_I$ . But  $(\underline{X}, \underline{A})^K$  and  $(\underline{X}, \underline{A})_D^K$  are not  $H$ -spaces so any hope of delooping the homotopy equivalences would come from the homotopy fibrations  $F_I \rightarrow (\underline{X}, \underline{A})^{K_I} \xrightarrow{\varphi} (\underline{X}, \underline{A})_D^{K_I}$  having a splitting of the form  $(\underline{X}, \underline{A})^{K_I} \rightarrow F_I$ . But this does not happen even in the simplest cases. For example, let  $K$  be two disjoint points. By the definition of the polyhedral product,  $(\underline{X}, \underline{A})^K = (X_1 \times A_2) \cup (A_1 \times X_2)$ . The proper full subcomplexes of  $K = \{1\} \amalg \{2\}$  are  $\{1\}$ ,  $\{2\}$  and  $\emptyset$ . The corresponding polyhedral products are  $X_1$ ,  $X_2$  and  $*$ . Therefore,  $(\underline{X}, \underline{A})_D^K$  is the homotopy inverse limit of the diagram  $X_1 \rightarrow * \leftarrow X_2$ , which is  $X_1 \times X_2$ . This implies that space  $F_{[2]}$  is the homotopy fibre of the inclusion  $(X_1 \times A_1) \cup (A_1 \times X_2) \rightarrow X_1 \times X_2$ . Specializing to  $A_1 = A_2 = *$ , this fibre is homotopy equivalent to that of the inclusion  $X_1 \vee X_2 \rightarrow X_1 \times X_2$ , which by the Hilton-Milnor Theorem, is  $\Omega X_1 * \Omega X_2$ . The only case when the map  $\Omega X_1 * \Omega X_2 \rightarrow X_1 \vee X_2$  has a left homotopy inverse is when at least one of  $X_1$  or  $X_2$  is trivial.

#### 4. FURTHER REFINEMENT IN THE CASE OF $(\underline{CX}, \underline{X})^K$

Recall from the Introduction that if the simplicial complex  $K$  is totally homology fillable (a property that includes shifted, shellable and sequentially Cohen-Macaulay complexes) then there is a homotopy equivalence

$$(\underline{CX}, \underline{X})^K \simeq \bigvee_{I \notin K} \Sigma |K_I| \wedge \widehat{X}^I$$

and  $\Sigma |K_I|$  is homotopy equivalent to a wedge of spheres. Thus  $(\underline{CX}, \underline{X})^K$  is homotopy equivalent to a wedge sum of spaces of the form  $\Sigma^t X_{i_1} \wedge \dots \wedge X_{i_k}$  for various  $t \geq 1$  and  $1 \leq i_1 < \dots < i_k \leq m$ . In this section we show that for this class of polyhedral products the spaces  $F_I$  that appear in the decompositions of  $\Omega(\underline{X}, \underline{A})^K$  and  $\Omega(\underline{X}, \underline{A})_D^K$  can be more explicitly identified.

To prepare, we require two general lemmas. For spaces  $B$  and  $C$  the *right half-smash* of  $B$  and  $C$  is the quotient space  $B \rtimes C = (B \times C) / \sim$  where  $(*, c) \sim *$ . It is well-known that if  $B$  is a co- $H$ -space then there is a homotopy equivalence  $B \rtimes C \simeq B \vee (B \wedge C)$ . The following lemma is well known and follows easily from the methods in [G2] (a more detailed statement and its proof appear later in Theorem 6.1).

**Lemma 4.1.** *Let  $B$  be a path-connected pointed space and  $C$  a simply-connected, pointed space. Let  $B \vee C \rightarrow C$  be the pinch map. Then there is a homotopy fibration*

$$B \rtimes \Omega C \xrightarrow{f} B \vee C \rightarrow C.$$

*This homotopy fibration is natural for maps  $B \rightarrow B'$  and  $C \rightarrow C'$*  □

One feature of the James construction [J] (see [Se, Proposition 7.9.1] for a more modern presentation) is that if  $Y$  is a pointed, path-connected space then there is a homotopy equivalence

$$\Sigma \Omega \Sigma Y \simeq \bigvee_{n=1}^{\infty} \Sigma Y^{(n)}$$

which is natural for maps  $Y \rightarrow Y'$ . An immediate consequence is the following.

**Lemma 4.2.** *Let  $X$  and  $Y$  be a pointed, path-connected spaces. Then there is a homotopy equivalence*

$$\Sigma X \wedge \Omega \Sigma Y \simeq \bigvee_{n=1}^{\infty} (\Sigma X \wedge Y^{(n)})$$

*which is natural for maps  $X \rightarrow X'$  and  $Y \rightarrow Y'$ .* □

We now give a construction that will identify the homotopy type of the space  $F_{[m]}$  in the case of a polyhedral product  $(\underline{CX}, \underline{X})^K$  where  $K$  is totally homology fillable. Recall that, for  $1 \leq j \leq m$ ,  $I_j = [m] \setminus \{j\}$  and  $e_j$  is the idempotent

$$e_j: (\underline{CX}, \underline{X})^K \rightarrow (\underline{CX}, \underline{X})^{K_{I_j}} \rightarrow (\underline{CX}, \underline{X})^K$$

induced by projecting  $(CX)^m$  to  $(CX)^{I_j}$  and then including back into  $(CX)^m$ . First consider  $e_1$ . Since  $(\underline{CX}, \underline{X})^K$  is homotopy equivalent to a wedge of spaces of the form  $\Sigma^t X_{i_1} \wedge \cdots \wedge X_{i_k}$  for various  $t \geq 1$  and  $1 \leq i_1 < \cdots < i_k \leq m$ , we can write

$$(\underline{CX}, \underline{X})^K \simeq \Sigma B_1 \vee \Sigma C_1$$

where each wedge summand of  $B_1$  has  $X_1$  as a smash factor and each wedge summand of  $C_1$  does not have  $X_1$  as a smash factor.

**Lemma 4.3.** *The following hold:*

- (a) *the restriction of  $e_1$  to  $\Sigma B_1$  is null homotopic;*
- (b) *the restriction of  $e_1$  to  $\Sigma C_1$  is the inclusion of  $\Sigma C_1$  into  $(\underline{CX}, \underline{X})^K$ ;*

(c) part (a) implies that there is a homotopy commutative diagram

$$\begin{array}{ccc} \Sigma B_1 \vee \Sigma C_1 & \xrightarrow{q_1} & \Sigma C_1 \\ \downarrow \simeq & & \downarrow \\ (\underline{CX}, \underline{X})^K & \longrightarrow & (\underline{CX}, \underline{X})^{K_{I_1}} \end{array}$$

where  $q_1$  is the pinch map.

*Proof.* The wedge decomposition of  $(\underline{CX}, \underline{X})^K$  is natural with respect to maps of simplicial complexes. Applying this to the composite  $e_1: (\underline{CX}, \underline{X})^K \rightarrow (\underline{CX}, \underline{X})^{K_{I_1}} \rightarrow (\underline{CX}, \underline{X})^K$ , the fact that  $I_1 = \{2, \dots, m\}$  implies that any wedge summand of  $(\underline{CX}, \underline{X})^K$  involving  $X_1$  is mapped trivially to  $(\underline{CX}, \underline{X})^{K_1}$  while any wedge summand not involving  $X_1$  is mapped identically to itself by  $e_1$ . This proves parts (a) and (b). Part (c) follows immediately from part (a).  $\square$

Define the space  $G_1$  and the map  $g_1$  by the homotopy fibration

$$G_1 \xrightarrow{g_1} \Sigma B_1 \vee \Sigma C_1 \xrightarrow{q_1} \Sigma C_1.$$

By Lemmas 4.1 and 4.2 there are natural homotopy equivalences

$$(4) \quad G_1 \simeq \Sigma B_1 \rtimes \Omega \Sigma C_1 \simeq \Sigma B_1 \vee (\Sigma B_1 \wedge \Omega \Sigma C_1) \simeq \Sigma B_1 \vee \left( \bigvee_{n=1}^{\infty} (\Sigma B_1 \wedge (C_1)^{(n)}) \right).$$

Observe that, by definition, each wedge summand in  $B_1$  is a smash product with  $X_1$  as a factor, so every wedge summand of  $B_1 \wedge (C_1)^{(n)}$  is also a smash product with  $X_1$  as a factor. Therefore, every wedge summand of  $G_1$  is the suspension of a smash product that has  $X_1$  as a factor. We now separate out those wedge summands that also have  $X_2$  as a factor. As the wedge summands are all suspensions, we can write

$$G_1 = \Sigma B_2 \vee \Sigma C_2$$

where each wedge summand of  $B_2$  is a smash product with  $X_1$  and  $X_2$  as factors and each wedge summand of  $C_2$  is a smash product with  $X_1$  as a factor but not  $X_2$ . Let  $\psi_1$  be the composite

$$\psi_1: G_1 \xrightarrow{g_1} \Sigma B_1 \vee \Sigma C_1 \xrightarrow{\simeq} (\underline{CX}, \underline{X})^K.$$

**Lemma 4.4.** *For the composite  $G_1 = \Sigma B_2 \vee \Sigma C_2 \xrightarrow{\psi_1} (\underline{CX}, \underline{X})^K \xrightarrow{e_2} (\underline{CX}, \underline{X})^K$  the following hold:*

- (a) *the restriction of  $e_2 \circ \psi_1$  to  $\Sigma B_2$  is null homotopic;*
- (b) *the restriction of  $e_2 \circ \psi_1$  to  $\Sigma C_2$  is homotopic to the restriction of  $\psi_1$  to  $\Sigma C_1$ ;*

(c) part (a) implies that there is a homotopy commutative diagram

$$\begin{array}{ccc}
 \Sigma B_2 \vee \Sigma C_2 & \xrightarrow{q_2} & \Sigma C_2 \\
 \downarrow \simeq & & \downarrow \\
 G_1 & & \\
 \downarrow \psi_1 & & \downarrow \\
 (\underline{CX}, \underline{X})^K & \longrightarrow & (\underline{CX}, \underline{X})^{K_{I_2}}
 \end{array}$$

where  $q_2$  is the pinch map.

*Proof.* The proofs of parts (a) and (b) start by reorganizing the data in order to apply the naturality of Lemma 4.1. First consider  $(\underline{CX}, \underline{X})^K \simeq \Sigma B_1 \vee \Sigma C_1$ . As all the wedge summands of  $\Sigma B_1$  and  $\Sigma C_1$  are suspensions, we may write  $\Sigma B_1 = \Sigma B_{1,1} \vee \Sigma B_{1,2}$  and  $\Sigma C_1 = \Sigma C_{1,1} \vee \Sigma C_{1,2}$  where  $\Sigma B_{1,1}$  ( $\Sigma C_{1,1}$  respectively) consists of those wedge summands of  $\Sigma B_1$  ( $\Sigma C_1$ ) having  $X_1$  as a smash factor but not  $X_2$ , and  $\Sigma B_{1,2}$  ( $\Sigma C_{1,2}$ ) consists of those wedge summands of  $\Sigma B_1$  ( $\Sigma C_1$ ) having both  $X_1$  and  $X_2$  as smash factors. The pinch map  $\Sigma B_1 \vee \Sigma C_1 \xrightarrow{q_1} \Sigma C_1$  can then be rewritten as a pinch map  $\Sigma B_{1,1} \vee \Sigma B_{1,2} \vee \Sigma C_{1,1} \vee \Sigma C_{1,2} \longrightarrow \Sigma C_{1,1} \vee \Sigma C_{1,2}$ .

Let  $\Sigma B'_2 = \Sigma B_{1,2} \vee \Sigma C_{1,2}$  and let  $\Sigma C'_2 = \Sigma B_{1,1} \vee \Sigma C_{1,1}$ . Notice that  $(\underline{CX}, \underline{X})^K \simeq \Sigma B'_2 \vee \Sigma C'_2$  where  $\Sigma B'_2$  consists of all wedge summands in  $(\underline{CX}, \underline{X})^K$  which are smash products with  $X_2$  as a factor and  $\Sigma C'_2$  consists of all wedge summands in  $(\underline{CX}, \underline{X})^K$  which are smash products not having  $X_2$  as a factor. As in Lemma 4.3, the restriction of  $e_2$  to  $\Sigma B'_2$  is null homotopic and the restriction to  $\Sigma C'_2$  is the inclusion of  $\Sigma C'_2$  into  $(\underline{CX}, \underline{X})^K$ . Therefore the composite  $\Sigma B'_2 \vee \Sigma C'_2 \xrightarrow{\simeq} (\underline{CX}, \underline{X})^K \longrightarrow (\underline{CX}, \underline{X})^{K_{I_2}}$  factors through the pinch map  $q'_2: \Sigma B'_2 \vee \Sigma C'_2 \longrightarrow \Sigma C'_2$ . Reordering the wedge summands,  $q'_2$  can be regarded as the wedge sum  $\Sigma B_{1,1} \vee \Sigma B_{1,2} \vee \Sigma C_{1,1} \vee \Sigma C_{1,2} \xrightarrow{q_B \vee q_C} \Sigma B_{1,1} \vee \Sigma C_{1,1}$  of the pinch maps  $q_B: \Sigma B_1 = \Sigma B_{1,1} \vee \Sigma B_{1,2} \longrightarrow \Sigma B_{1,1}$  and  $q_C: \Sigma C_1 = \Sigma C_{1,1} \vee \Sigma C_{1,2} \longrightarrow \Sigma C_{1,1}$ .

Putting  $q_1$  and  $q'_2$  together, by the naturality of Lemma 4.1 there is a homotopy fibration diagram

$$\begin{array}{ccccc}
 \Sigma B_1 \rtimes \Omega \Sigma C_1 & \longrightarrow & \Sigma B_{1,1} \vee \Sigma B_{1,2} \vee \Sigma C_{1,1} \vee \Sigma C_{1,2} & \xrightarrow{q_1} & \Sigma C_{1,1} \vee \Sigma C_{1,2} \\
 \downarrow q_B \rtimes \Omega q_C & & \downarrow q_B \vee q_C & & \downarrow q_C \\
 \Sigma B_{1,1} \rtimes \Omega \Sigma C_{1,1} & \longrightarrow & \Sigma B_{1,1} \vee \Sigma C_{1,1} & \xrightarrow{q} & \Sigma C_{1,1}
 \end{array}$$

where  $q$  is the pinch map. Putting the left square together with the factorization of  $e_2$  through  $q'_2$  gives a homotopy commutative diagram

$$\begin{array}{ccccccc}
 \Sigma B_1 \rtimes \Omega \Sigma C_1 & \longrightarrow & \Sigma B_{1,1} \vee \Sigma B_{1,2} \vee \Sigma C_{1,1} \vee \Sigma C_{1,2} & = & \Sigma B'_2 \vee \Sigma C'_2 & \xrightarrow{\simeq} & (\underline{CX}, \underline{X})^K \\
 \downarrow q_B \rtimes \Omega q_C & & \downarrow q_B \vee q_C & & \downarrow q'_2 & & \downarrow \\
 \Sigma B_{1,1} \rtimes \Omega \Sigma C_{1,1} & \longrightarrow & \Sigma B_{1,1} \vee \Sigma C_{1,1} & = & \Sigma C'_2 & \longrightarrow & (\underline{CX}, \underline{X})^{K_{I_2}}.
 \end{array}$$



Notice that  $G_1 \simeq \Sigma B_1 \rtimes \Omega \Sigma C_1$  and the top row is homotopic to  $\psi_1$ .

The maps  $q_B$  and  $q_C$  pinch out any wedge summands of  $\Sigma B_1$  and  $\Sigma C_1$  respectively that have  $X_2$  as a smash factor. As  $q_B$  and  $q_C$  are suspensions, the naturality of the wedge decomposition (4) of  $G_1 \simeq \Sigma B_1 \rtimes \Omega \Sigma C_1$  implies that  $q_B \rtimes \Omega q_C$  pinches out any wedge summand of  $G_1$  having both  $X_1$  and  $X_2$  as factors, and sends any wedge summand having  $X_1$  as a smash factor but not  $X_2$  identically to itself in  $\Sigma B_{1,1} \rtimes \Omega \Sigma C_{1,1}$ . That is,  $q_B \rtimes \Omega q_C$  is the same as the map  $G_1 = \Sigma B_2 \vee \Sigma C_2 \xrightarrow{q_2} \Sigma C_2$ . Therefore, the homotopy commutativity of the previous diagram implies that the restriction of  $e_2 \circ \psi_1$  to  $\Sigma B_2$  is null homotopic and the restriction to  $\Sigma C_2$  is the restriction of  $\psi_1$ . This proves parts (a) and (b). Part (c) follows immediately from part (a).  $\square$

Now proceed as before by taking the homotopy fibre of the pinch map  $G_1 = \Sigma B_2 \vee \Sigma C_2 \xrightarrow{q_2} \Sigma C_2$ . Iterating, for  $1 \leq j < m$  we obtain homotopy fibrations

$$G_{j+1} \xrightarrow{q_{j+1}} G_j \simeq \Sigma B_j \vee \Sigma C_j \xrightarrow{q_j} \Sigma C_j$$

where  $q_j$  is the pinch map; every wedge summand of  $B_j$  is a smash product with  $X_1, \dots, X_j$  as factors; every wedge summand of  $C_j$  is a smash product with  $X_1, \dots, X_{j-1}$  as factors but not  $X_j$ ; and there are natural homotopy equivalences

$$G_{j+1} \simeq \Sigma B_j \rtimes \Omega \Sigma C_j \simeq \Sigma B_j \vee (\Sigma B_j \wedge \Omega \Sigma C_j) \simeq \Sigma B_j \vee \left( \bigvee_{n=1}^{\infty} (\Sigma B_j \wedge (C_j)^{(n)}) \right).$$

Further, if  $\psi_j$  is the composite

$$\psi_j: G_j \xrightarrow{g_j} G_{j-1} \xrightarrow{\psi_{j-1}} (\underline{CX}, \underline{X})^K$$

then arguing as in Lemma 4.4 the following hold.

**Lemma 4.5.** *For the composite  $G_j = \Sigma B_j \vee \Sigma C_j \xrightarrow{\psi_j} (\underline{CX}, \underline{X})^K \xrightarrow{e_j} (\underline{CX}, \underline{X})^K$  the following hold:*

- (a) *the restriction of  $e_j \circ \psi_j$  to  $\Sigma B_j$  is null homotopic;*
- (b) *the restriction of  $e_j \circ \psi_j$  to  $\Sigma C_j$  is homotopic to the restriction of  $\psi_j$  to  $\Sigma C_j$ ;*
- (c) *part (a) implies that there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Sigma B_j \vee \Sigma C_j & \xrightarrow{q_j} & \Sigma C_j \\ \downarrow \simeq & & \downarrow \\ G_j & & \\ \downarrow \psi_j & & \downarrow \\ (\underline{CX}, \underline{X})^K & \longrightarrow & (\underline{CX}, \underline{X})^{K_{I_j}} \end{array}$$

where  $q_j$  is the pinch map.

$\square$

For  $1 \leq j \leq m$ , let  $\theta_j$  be the composite

$$\theta_j: \Sigma C_j \hookrightarrow \Sigma B_j \vee \Sigma C_j = G_j \xrightarrow{\psi_j} (\underline{CX}, \underline{X})^K$$

and let  $\theta$  be the product of the maps  $\Omega\theta_j$ :

$$\theta: \prod_{j=1}^m \Omega \Sigma C_j \longrightarrow \Omega(\underline{CX}, \underline{X})^K.$$

Let  $\theta \cdot \psi_m$  be the composite

$$\theta \cdot \psi_m: \left( \prod_{j=1}^m \Omega \Sigma C_j \right) \times \Omega G_m \xrightarrow{\theta \times \psi_m} \Omega(\underline{CX}, \underline{X})^K \times \Omega(\underline{CX}, \underline{X})^K \xrightarrow{\mu} \Omega(\underline{CX}, \underline{X})^K$$

where  $\mu$  is the loop multiplication.

**Lemma 4.6.** *Suppose that  $K$  is totally homology fillable. Then the map  $(\prod_{j=1}^m \Omega \Sigma C_j) \times \Omega G_m \xrightarrow{\theta \cdot \psi_m} \Omega(\underline{CX}, \underline{X})^K$  is a homotopy equivalence.*

*Proof.* In general, the homotopy fibration  $B \rtimes \Omega C \longrightarrow B \vee C \longrightarrow C$  in Lemma 4.1 has a section  $C \longrightarrow B \vee C$  given by the inclusion of the wedge summand. Therefore, after looping, there is a homotopy equivalence  $\Omega C \times (\Omega B \rtimes \Omega C) \xrightarrow{\simeq} \Omega(B \vee C)$ . In our case, for  $1 \leq j < m$ , from the homotopy fibration  $G_{j+1} \xrightarrow{g_{j+1}} G_j = \Sigma B_j \vee \Sigma C_j \xrightarrow{q_j} \Sigma C_j$  we obtain a homotopy equivalence  $\Omega \Sigma C_j \times \Omega G_{j+1} \xrightarrow{\simeq} \Omega G_j$ . Iteratively substituting the homotopy equivalence for  $\Omega G_{j+1}$  into that for  $\Omega G_j$ , we obtain a homotopy equivalence  $(\prod_{j=1}^m \Omega \Sigma C_j) \times \Omega G_m \xrightarrow{\simeq} \Omega(\underline{CX}, \underline{X})^K$ . Notice that the restriction of this homotopy equivalence to  $\Omega \Sigma C_j$  is the definition of  $\Omega\theta_j$  and the restriction to  $\Omega G_m$  is  $\Omega\psi_m$ . Thus the equivalence is  $\theta \cdot \Omega\psi_m$ .  $\square$

Theorem 3.3 and Lemma 4.6 give different homotopy equivalences for  $\Omega(\underline{CX}, \underline{X})^K$ . We wish to compare them.

**Lemma 4.7.** *The composite  $G_m \xrightarrow{\psi_m} (\underline{CX}, \underline{X})^K \xrightarrow{\varphi} (\underline{CX}, \underline{X})_D^K$  is null homotopic.*

*Proof.* Fix an integer  $j$  for  $1 \leq j < m$ . Consider the homotopy fibration  $G_{j+1} \xrightarrow{g_{j+1}} G_j \xrightarrow{q_j} \Sigma C_j$ . By Lemma 4.5 (c), the composite  $\gamma_j: G_j \xrightarrow{\psi_j} (\underline{CX}, \underline{X})^K \longrightarrow (\underline{CX}, \underline{X})^{K_{I_j}}$  factors through  $q_j$ . Therefore  $\gamma_j \circ g_{j+1}$  is null homotopic. By definition,  $\psi_{j+1} = \psi_j \circ g_{j+1}$ , so the composite  $G_{j+1} \xrightarrow{\psi_{j+1}} (\underline{CX}, \underline{X})^K \longrightarrow (\underline{CX}, \underline{X})^{K_{I_j}}$  is null homotopic. The recursive definition of  $\psi_m$  implies that it factors through  $\psi_j$  so the composite  $G_m \xrightarrow{\psi_m} (\underline{CX}, \underline{X})^K \longrightarrow (\underline{CX}, \underline{X})^{K_{I_j}}$  is null homotopic. This holds for all  $1 \leq j \leq m$ , so  $\psi_m$  composes trivially into  $(\underline{CX}, \underline{X})^{K_{I_j}}$  for all  $j$ . But this implies that  $\psi_m$  composes trivially to  $(\underline{CX}, \underline{X})^{K_I}$  for every proper subset  $I$  of  $[m]$  because the projection  $(CX)^m \longrightarrow (CX)^I$  has to factor through some  $(CX)^{I_t}$ . Looping,  $\Omega\psi_m$  composes trivially to  $\Omega(\underline{CX}, \underline{X})^{K_I}$  for any  $I \subsetneq [m]$ . But by Proposition 2.11,  $\Omega(\underline{CX}, \underline{X})_D^K$  decomposes as a product, each factor of which is a factor of  $\Omega(\underline{CX}, \underline{X})^{K_I}$  for some  $I$ , and this decomposition is compatible with  $\Omega\varphi$ . Thus the composite  $\Omega G_m \xrightarrow{\Omega\psi_m} \Omega(\underline{CX}, \underline{X})^K \xrightarrow{\Omega\varphi} \Omega(\underline{CX}, \underline{X})_D^K$  is null homotopic.

To deloop this, observe that as  $G_m$  is a suspension there is a map  $G_m \rightarrow \Sigma\Omega G_m$  which is a right homotopy inverse of the evaluation map  $ev: \Sigma\Omega G_m \rightarrow G_m$ . The naturality of the evaluation map implies that there is a homotopy commutative diagram

$$\begin{array}{ccccccc} G_m & \longrightarrow & \Sigma\Omega G_m & \xrightarrow{\Sigma\Omega\psi_m} & \Sigma\Omega(\underline{CX}, \underline{X})^K & \xrightarrow{\Sigma\Omega\varphi} & \Sigma\Omega(\underline{CX}, \underline{X})_D^K \\ & \searrow & \downarrow ev & & \downarrow ev & & \downarrow ev \\ & & G_m & \xrightarrow{\psi_m} & (\underline{CX}, \underline{X})^K & \xrightarrow{\varphi} & (\underline{CX}, \underline{X})_D^K \end{array}$$

The null homotopy for  $\Omega\varphi \circ \Omega\psi_m$  therefore implies that  $\varphi \circ \psi_m$  is also null homotopic.  $\square$

Recall that there is a homotopy fibration  $F_{[m]} \rightarrow (\underline{CX}, \underline{X})^K \xrightarrow{\varphi} (\underline{CX}, \underline{X})_D^K$ .

**Corollary 4.8.** *There is a lift*

$$\begin{array}{ccc} & G_m & \\ \lambda \swarrow & \downarrow \psi_m & \\ F_{[m]} & \longrightarrow & (\underline{CX}, \underline{X})^K \xrightarrow{\varphi} (\underline{CX}, \underline{X})_D^K \end{array}$$

for some map  $\lambda$ . Further,  $\Omega\lambda$  has a left homotopy inverse.

*Proof.* The existence of the lift follows immediately from Lemma 4.7. By Lemma 4.6,  $\Omega\psi_m$  has a left homotopy inverse. The fact that  $\lambda$  factors through  $\psi_m$  then implies that  $\Omega\lambda$  also has a left homotopy inverse.  $\square$

**Lemma 4.9.**  $\prod_{j=1}^m \Omega\Sigma C_j \xrightarrow{\theta} \Omega(\underline{CX}, \underline{X})^K \xrightarrow{\Omega\varphi} \Omega(\underline{CX}, \underline{X})_D^K$  has a left homotopy inverse.

*Proof.* Fix an integer  $j$  for  $1 \leq j < m$ . By Lemma 4.6, the map  $\Omega\Sigma C_j \xrightarrow{\Omega\theta_j} \Omega(\underline{CX}, \underline{X})^K$  has a left homotopy inverse. On the other hand, by Lemma 4.5 (b),  $e_j \circ \theta_j \simeq \theta_j$ . This implies that the composite  $\Omega\Sigma C_j \xrightarrow{\Omega\theta_j} \Omega(\underline{CX}, \underline{X})^K \rightarrow \Omega(\underline{CX}, \underline{X})^{K_j} \rightarrow \Omega(\underline{CX}, \underline{X})^K$  has a left homotopy inverse. Therefore the composite  $\Omega\Sigma C_j \xrightarrow{\Omega\theta_j} \Omega(\underline{CX}, \underline{X})^K \rightarrow \Omega(\underline{CX}, \underline{X})^{K_j}$  has a left homotopy inverse. By definition of  $(\underline{CX}, \underline{X})_D^K$  as a homotopy limit, the map  $(\underline{CX}, \underline{X})^K \rightarrow (\underline{CX}, \underline{X})^{K_j}$  factors through  $(\underline{CX}, \underline{X})_D^K$ . Thus the composite  $\Omega\Sigma C_j \xrightarrow{\Omega\theta_j} \Omega(\underline{CX}, \underline{X})^K \xrightarrow{\Omega\varphi} \Omega(\underline{CX}, \underline{X})_D^K$  has a left homotopy inverse. This is true for all  $1 \leq j \leq m$ . Collectively, no overlap in the factors in the different retractions occurs because the product map  $\prod_{j=1}^m \Omega\Sigma C_j \xrightarrow{\theta} \Omega(\underline{CX}, \underline{X})^K$  has a left homotopy inverse. Hence the composite  $\prod_{j=1}^m \Omega\Sigma C_j \xrightarrow{\theta} \Omega(\underline{CX}, \underline{X})^K \xrightarrow{\Omega\varphi} \Omega(\underline{CX}, \underline{X})_D^K$  also has a left homotopy inverse.  $\square$

**Proposition 4.10.** *Let  $K$  be a totally homology fillable simplicial complex. Then the map  $G_m \xrightarrow{\lambda} F_{[m]}$  in Corollary 4.8 is a homotopy equivalence.*

**Remark 4.11.** It is worth repeating at this point that  $G_m$  is homotopy equivalent to a wedge of spaces of the form  $\Sigma^t X_{i_1} \wedge \cdots \wedge X_{i_k}$  where  $t \geq 1$  and  $\{1, \dots, m\} \subseteq \{i_1, \dots, i_k\}$ . That is, each  $X_i$  for  $1 \leq i \leq m$  appears as a smash factor in every wedge summands of  $G_m$ .

*Proof.* On the one hand, by Theorem 3.3, the homotopy fibration  $F_{[m]} \rightarrow \Omega(\underline{CX}, \underline{X})^K \xrightarrow{\Omega\varphi} \Omega(\underline{CX}, \underline{X})_D^K$  splits to give a homotopy equivalence  $\Omega(\underline{CX}, \underline{X})^K \simeq \Omega(\underline{CX}, \underline{X})_D^K \times \Omega F_{[m]}$ . On the other hand, by Lemma 4.6, the composite  $(\prod_{j=1}^m \Omega \Sigma C_j) \times \Omega G_m \xrightarrow{\theta \cdot \psi_m} \Omega(\underline{CX}, \underline{X})^K$  is a homotopy equivalence. We compare the two decompositions.

By Lemma 4.9, the composite  $h: \prod_{j=1}^m \Omega \Sigma C_j \xrightarrow{\theta} \Omega(\underline{CX}, \underline{X})^K \xrightarrow{\Omega\varphi} \Omega(\underline{CX}, \underline{X})_D^K$  has a left homotopy inverse. By Corollary 4.8, the map  $\Omega G_m \xrightarrow{\Omega\psi_m} \Omega(\underline{CX}, \underline{X})^K$  lifts to a map  $\Omega G_m \xrightarrow{\Omega\lambda} \Omega F_{[m]}$  and  $\Omega\lambda$  has a left homotopy inverse. Thus the product map

$$\Gamma: \Omega(\underline{CX}, \underline{X})^K \xrightarrow{\simeq} \left( \prod_{j=1}^m \Omega \Sigma C_j \right) \times \Omega G_m \xrightarrow{h \times \Omega\lambda} \Omega(\underline{CX}, \underline{X})_D^K \times \Omega F_{[m]} \xrightarrow{\simeq} \Omega(\underline{CX}, \underline{X})^K$$

has the property that  $h \times \Omega\lambda$  has a left homotopy inverse. In particular,  $h \times \Omega\lambda$  induces an injection in homology and therefore  $\Gamma_*$  is an injection in homology with any coefficients. Taking  $\mathbb{Z}/p\mathbb{Z}$  or  $\mathbb{Q}$  coefficients,  $\Gamma_*$  is a self-map of a finite type module which is an injection, and so it is an isomorphism. Thus  $\Gamma$  induces an isomorphism in mod- $p$  and rational homology and so induces an isomorphism in integral homology. Therefore  $\Gamma$  is a homotopy equivalence. This implies that  $h \times \Omega\lambda$  is a homotopy equivalence. As  $h \times \Omega\lambda$  is a product map, each of  $h$  and  $\Omega\lambda$  must therefore be a homotopy equivalence.

Finally, since  $\Omega\lambda$  is a homotopy equivalence, it induces an isomorphism on homotopy groups, and therefore so does  $\lambda$ . Hence  $\lambda$  is also a homotopy equivalence, as asserted.  $\square$

By Theorem 3.3, each factor in the homotopy decompositions of  $\Omega(\underline{CX}, \underline{X})^K$  and  $\Omega(\underline{CX}, \underline{X})_D^K$  is of the form  $\Omega F_I$  where  $F_I$  is the homotopy fibre of the map  $(\underline{CX}, \underline{X})^{K_I} \rightarrow (\underline{CX}, \underline{X})_D^{K_I}$ . Further, by [IK2] any subcomplex of a totally homology fillable simplicial complex is itself totally homology fillable. Therefore Proposition 4.10 applies to each  $F_I$  to obtain the following.

**Theorem 4.12.** *Let  $K$  be a totally homology fillable simplicial complex. Then the homotopy decompositions*

$$\Omega(\underline{CX}, \underline{X})^K \simeq \prod_{I \subset [m]} \Omega F_I \quad \Omega(\underline{CX}, \underline{X})_D^K \simeq \prod_{I \subsetneq [m]} \Omega F_I$$

*in Theorem 3.3 have the property that each space  $F_I$  is homotopy equivalent to a wedge of summands, where each wedge summand is the suspension of a smash product having  $X_i$  as a factor for all  $i \in I$ .  $\square$*

## 5. A GENERALIZATION TO $(\underline{X}, *)^K$

In this section we show that Theorem 4.12 gives useful information for a wider range of polyhedral products. Let  $\{X_i\}_{i=1}^m$  be a collection of pointed  $CW$ -complexes and let  $K$  be a simplicial complex on the vertex set  $[m]$ . Bahri, Bendersky, Cohen and Gitler [BBCG, Corollary 2.32], relying heavily on a result of Denham and Suciu [DS], show that there is a homotopy fibration

$$(5) \quad (\underline{CX}, \underline{X})^K \rightarrow (\underline{X}, *)^K \rightarrow \prod_{i=1}^m X_i.$$

The spaces  $(\underline{X}, *)^K$  include many familiar spaces: if  $K$  is  $m$  disjoint points then  $(\underline{X}, *)^K$  is homotopy equivalent to the wedge  $X_1 \vee \cdots \vee X_m$ ; if  $K$  is the boundary of the standard  $(m-1)$ -simplex then  $(\underline{X}, *)^K$  is the fat wedge in  $\prod_{i=1}^m X_i$ ; and crucially to toric topology, if each  $X_i = \mathbb{C}P^\infty$  then  $(\underline{X}, *)^K$  is the Davis-Januszkiewicz space  $DJ_K$ .

Each vertex  $i$  of  $K$  is the full subcomplex  $K_{\{i\}}$ , and the polyhedral product  $(\underline{X}, *)^{K_{\{i\}}}$  is simply  $X_i$ . So including  $i$  into  $K$  we obtain a map of polyhedral products  $X_i \longrightarrow (\underline{X}, *)^K$  which, when composed to  $\prod_{i=1}^m X_i$ , is the inclusion of the  $i^{\text{th}}$  factor. After looping we can take the product of all such maps for  $1 \leq i \leq m$  to obtain a section for the map  $\Omega(\underline{X}, *)^K \longrightarrow \prod_{i=1}^m \Omega X_i$ , implying the following.

**Lemma 5.1.** *The homotopy fibration (5) splits after looping, resulting in a homotopy equivalence*

$$\Omega(\underline{X}, *)^K \simeq \left( \prod_{i=1}^m \Omega X_i \right) \times \Omega(\underline{C\Omega X}, \underline{\Omega X})^K.$$

□

We wish to show that the homotopy equivalence in Lemma 5.1 is compatible with that in Theorem 4.12. The following lemma does this. Write  $F_I(\underline{X}, *)$  and  $F_I(\underline{C\Omega X}, \underline{\Omega X})$  for the spaces  $F_I$  that appear in the respective decompositions of  $\Omega(\underline{X}, *)^K$  and  $\Omega(\underline{C\Omega X}, \underline{\Omega X})^K$  in Theorem 3.3.

**Lemma 5.2.** *Let  $1 \leq i \leq m$ . The following hold:*

- (a)  $F_{\{i\}}(\underline{X}, *) \simeq X_i$ ;
- (b)  $F_{\{i\}}(\underline{C\Omega X}, \underline{\Omega X}) \simeq *$ ;
- (c) *for  $I \subseteq [m]$  and  $I \neq \{i\}$  for any  $1 \leq i \leq m$ , the map  $(\underline{C\Omega X}, \underline{\Omega X})^K \longrightarrow (\underline{X}, *)^K$  induces a homotopy equivalence  $F_I(\underline{C\Omega X}, \underline{\Omega X}) \simeq F_I(\underline{X}, *)$ .*

*Proof.* In general, for  $(\underline{X}, \underline{A})^K$ , the definition of the polyhedral product implies that  $(\underline{X}, \underline{A})^{K_{\{i\}}} = X_i$ . By the definition of the dual polyhedral product as a homotopy colimit over the proper subcomplexes of  $K$ , we obtain  $(\underline{X}, \underline{A})_D^{K_{\{i\}}} = *$ , and by definition,  $F_{\{i\}}$  is the homotopy fibre of the map  $(\underline{X}, \underline{A})^{K_{\{i\}}} \longrightarrow (\underline{X}, \underline{A})_D^{K_{\{i\}}}$ . Thus  $F_{\{i\}} \simeq X_i$ . In particular, we obtain  $F_{\{i\}}(\underline{X}, *) \simeq X_i$  and  $F_{\{i\}}(\underline{C\Omega X}, \underline{\Omega X}) \simeq C\Omega X_i \simeq *$ . Part (c) now follows from parts (a) and (b) and Lemma 5.1. □

Consequently, by applying Theorem 4.12 to  $\Omega(\underline{C\Omega X}, \underline{\Omega X})^K$  we obtain the following.

**Theorem 5.3.** *Let  $K$  be a totally homology fillable simplicial complex. Then there are homotopy decompositions*

$$\Omega(\underline{X}, *)^K \simeq \left( \prod_{i=1}^m \Omega X_i \right) \times \prod_{\substack{I \subseteq [m] \\ I \neq \{i\}}} \Omega F_I \quad \Omega(\underline{X}, *)_D^K \simeq \left( \prod_{i=1}^m \Omega X_i \right) \times \prod_{\substack{I \subseteq [m] \\ I \neq \{i\}}} \Omega F_I$$

where each  $F_I$  is homotopy equivalent to a wedge of summands, and each wedge summand is the suspension of a smash product having  $\Omega X_i$  as a factor for all  $i \in I$ . □

In particular, if each  $X_i$  is  $\mathbb{C}P^\infty$  then  $(\underline{X}, *)^K = DJ(K)$  and  $\Omega X_i \simeq S^1$ . So Theorem 5.3 gives a homotopy decomposition for  $\Omega DJ(K)$  in which every  $F_I$  is homotopy equivalent to a wedge of spheres.

## Part 2. Whitehead products.

### 6. WHITEHEAD PRODUCTS AND GANEA'S THEOREM

In studying thin products it is important to have a good grip on the homotopy theory of the wedge  $\bigvee_{i=1}^m X_i$ . In this section we examine the special case when  $m = 2$ , in Section 7 we examine the general case when  $m \geq 2$ , and in Section 8 we relate this to thin products.

Consider the following two maps: the inclusion  $i: X \vee Y \longrightarrow X \times Y$  of the wedge into the product and the pinch map  $q: X \vee Y \longrightarrow Y$  onto the right wedge summand. Observe that  $q$  is also the composite  $X \vee Y \xrightarrow{i} X \times Y \xrightarrow{\pi_2} Y$ , where  $\pi_2$  is the projection onto the second factor. Define spaces  $F$  and  $G$ , and maps  $f$  and  $g$ , by the homotopy pullback diagram

$$\begin{array}{ccccc} F & \longrightarrow & G & \longrightarrow & X \\ \parallel & & \downarrow g & & \downarrow i_1 \\ F & \xrightarrow{f} & X \vee Y & \xrightarrow{i} & X \times Y \\ & & \downarrow q & & \downarrow \pi_2 \\ & & Y & \xlongequal{\quad} & Y \end{array}$$

where  $i_1$  is the inclusion of the first factor. Ganea [G2] identified the homotopy type of  $F$  as  $\Sigma\Omega X \wedge \Omega Y$  and the homotopy classes of  $f$  as a Whitehead product. The homotopy type of  $G$  is well known to be  $X \rtimes \Omega Y$  but the homotopy class of  $g$  is not readily identifiable in terms of otherwise known maps. When  $X$  is a suspension then  $X \rtimes \Omega Y \simeq X \vee (X \wedge \Omega Y)$ , in which case the homotopy class of  $g$  should be identifiable. As the author can find no reference for this, we give a proof. There is no claim of anything new here, as the methods and results go back to Ganea [G2] and he surely knew everything stated in this section.

Let  $i_X: X \longrightarrow X \vee Y$  and  $i_Y: Y \longrightarrow X \vee Y$  be the inclusions of the respective wedge summands. Define  $ev_X$  and  $ev_Y$  by the composites  $ev_X: \Sigma\Omega X \xrightarrow{ev} X \xrightarrow{i_X} X \vee Y$  and  $ev_Y: \Sigma\Omega Y \xrightarrow{ev} Y \xrightarrow{i_Y} X \vee Y$ , where  $ev$  is the canonical evaluation map.

**Theorem 6.1.** *The following hold:*

- (a) *There is a homotopy fibration*

$$\Sigma\Omega X \wedge \Omega Y \xrightarrow{[ev_X, ev_Y]} X \vee Y \xrightarrow{i} X \times Y;$$

- (b) *there is a homotopy fibration*

$$X \rtimes \Omega Y \xrightarrow{g} X \vee Y \xrightarrow{q} Y$$

where the restriction of  $g$  to  $X$  is  $i_X$ ;

- (c) if  $X = \Sigma X'$ , then there is a choice of a homotopy equivalence  $\Sigma X' \rtimes \Omega Y \simeq \Sigma X' \vee (\Sigma X' \wedge \Omega Y)$  such that the restriction of  $g$  to  $\Sigma X' \wedge \Omega Y$  is  $[i_X, ev_Y]$ .

*Proof.* The goal is really part (c) since Ganea [G2, Equation (9) and Lemma 5.1] proved part (a) and part (b) is well known (see, for example, [Se, Theorem 7.7.7]). However, to set up for part (c) we re-prove the identifications of the homotopy types of the fibres of  $i$  and  $q$  in parts (a) and (b).

In general, for a space  $Z$ , let  $PZ$  be the path space of  $Z$ , where paths start at the basepoint of  $Z$  at time  $t = 0$ . Let  $ev_1: PZ \rightarrow Z$  be the map which evaluates a path at time  $t = 1$ . The homotopy fibre of a map  $h: A \rightarrow Z$  is homotopy equivalent to the topological pullback of  $h$  and  $ev_1$ .

In our case, let  $Q$  be the topological pullback of the maps  $X \vee Y \xrightarrow{i} X \times Y$  and  $PX \times PY \xrightarrow{ev_1 \times ev_1} X \times Y$ . Notice that the part of  $Q$  sitting over  $* \vee Y$  is  $\Omega X \times PY$ , the part sitting over  $X \vee *$  is  $PX \times \Omega Y$ , and the part sitting over  $* \vee *$  is  $\Omega X \times \Omega Y$ . Thus  $Q = \Omega X \times PY \cup_{\Omega X \times \Omega Y} PX \times \Omega Y$ . Let  $C\Omega X$  be the reduced cone on  $\Omega X$ , where the cone point is at  $t = 1$ . It is well known that there is a homotopy equivalence  $\psi_X: C\Omega X \rightarrow PX$  given by  $\psi_X(t, \omega) = \omega_t$ , where  $\omega_t$  is the path defined by  $\omega_t(s) = \omega((1-t)s)$ . Note that  $\omega_0 = \omega$  and  $\omega_1$  is the constant path to the basepoint of  $X$ . Thus  $Q \simeq \Omega X \times C\Omega Y \cup_{\Omega X \times \Omega Y} C\Omega X \times \Omega Y$ , and the right side is the definition of the join  $\Omega X * \Omega Y$ . Ganea shows that the composite  $\Sigma \Omega X \wedge \Omega Y \simeq \Omega X * \Omega Y \xrightarrow{\simeq} Q \rightarrow X \vee Y$  is the Whitehead product  $[ev_X, ev_Y]$ , proving part (a).

Next, let  $P$  be the topological pullback of the maps  $X \vee Y \xrightarrow{q} Y$  and  $PY \xrightarrow{ev_1} Y$ . Notice that the part of  $P$  sitting over  $* \vee Y$  is  $* \times PY$ , the part sitting over  $X \vee *$  is  $X \times \Omega Y$ , and the part sitting over  $* \vee *$  is  $* \times \Omega Y$ . Thus  $P = * \times PY \cup_{* \times \Omega Y} X \times \Omega Y$ . The composite  $X \rightarrow P \rightarrow X \vee Y$  is exactly  $i_X$  and contracting  $PY$  we obtain  $P \simeq X \rtimes \Omega Y$ . This proves part (b).

For part (c), observe that the projection  $X \times Y \xrightarrow{\pi_2} Y$  induces a projection  $PX \times PY \xrightarrow{\pi_2} PY$ , so there is an induced map of topological pullbacks  $Q \rightarrow P$ . In terms of the identifications for  $Q$  and  $P$  above, this map of pullbacks is

$$\Omega X \times PY \cup_{\Omega X \times \Omega Y} PX \times \Omega Y \rightarrow * \times PY \cup_{* \times \Omega Y} X \times \Omega Y$$

where  $PX \times PY \xrightarrow{ev_1 \times 1} X \times PY$  has been restricted to the subspaces  $\Omega X \times PY$ ,  $PX \times \Omega Y$  and  $\Omega X \times \Omega Y$ . Notice that the composite  $C\Omega X \xrightarrow{\psi_X} PX \xrightarrow{ev_1} X$  sends  $(t, \omega)$  to  $\omega_t(1) = \omega(1-t)$ . That is,  $ev_1 \circ \psi_X = -ev$ , where  $\Sigma \Omega X \xrightarrow{ev} X$  is the canonical evaluation map. Thus the composite  $\Sigma \Omega X \wedge \Omega Y \simeq \Omega X * \Omega Y \xrightarrow{\simeq} Q \rightarrow P \simeq X \rtimes \Omega Y \rightarrow X \wedge \Omega Y$  is  $-ev \wedge 1$ . Consequently, if  $X = \Sigma X'$  and  $s$  is the composite  $s: \Sigma X' \wedge \Omega Y \xrightarrow{\Sigma E \wedge 1} \Sigma \Omega \Sigma X' \wedge \Omega Y \xrightarrow{\simeq} Q \rightarrow P \simeq X \rtimes \Omega Y$ , then  $\Sigma X' \wedge \Omega Y \xrightarrow{\gamma} \Sigma X' \rtimes \Omega Y \rightarrow \Sigma X' \wedge \Omega Y$  is homotopic to  $-id$ , where  $id$  is the identity map. Thus  $s$  is a section for the right map in the cofibration  $\Sigma X' \xrightarrow{j} \Sigma X' \rtimes \Omega Y \rightarrow \Sigma X' \wedge \Omega Y$ , where  $j$  is the inclusion. Therefore  $\Sigma X' \vee (\Sigma X' \wedge \Omega Y) \xrightarrow{j+s} X \rtimes \Omega Y$  is a homotopy equivalence.

Finally, from the pullback map  $Q \longrightarrow P$  and the definition of  $s$  there is a homotopy commutative diagram

$$\begin{array}{ccccc} \Sigma X' \wedge \Omega Y & \xrightarrow{\Sigma E \wedge 1} & \Sigma \Omega \Sigma X' \wedge \Omega Y & \xrightarrow{\simeq} & Q \longrightarrow X \vee Y \\ & \searrow s & & \downarrow & \parallel \\ & & & P & \longrightarrow X \vee Y. \end{array}$$

By part (a) the map  $\Sigma \Omega \Sigma X' \wedge \Omega Y \simeq Q \longrightarrow X \vee Y$  is homotopic to  $[ev_X, ev_Y]$ . Therefore the top row is homotopic to  $[i_X, ev_Y]$ . Hence, choosing the homotopy equivalence for  $P \simeq \Sigma X' \rtimes \Omega Y$  determined by  $j + s$ , the restriction of  $\Sigma X' \rtimes \Omega Y \longrightarrow X \vee Y$  to  $\Sigma X' \wedge \Omega Y$  is homotopic to  $[i_X, ev_Y]$ .  $\square$

## 7. WHITEHEAD PRODUCTS AND PORTER'S THEOREM

Define the space  $\Gamma(\underline{X})$  and the map  $\gamma(\underline{X})$  by the homotopy fibration

$$\Gamma(\underline{X}) \xrightarrow{\gamma(\underline{X})} \bigvee_{i=1}^m X_i \longrightarrow \prod_{i=1}^m X_i.$$

When  $m = 2$  Theorem 6.1 (a) identifies  $\Gamma(\underline{X})$  as  $\Sigma \Omega X_1 \wedge \Omega X_2$  and the homotopy class of  $\gamma(\underline{X})$  as a Whitehead product  $[ev_X, ev_Y]$ . Porter [P] partially generalized this by identifying the homotopy type of  $\Gamma(\underline{X})$  for any  $m \geq 2$ .

**Theorem 7.1.** *Let  $X_1, \dots, X_m$  be simply-connected, pointed CW-complexes. Then there is a homotopy equivalence*

$$\Gamma(\underline{X}) \simeq \bigvee_{k=2}^m \left( \bigvee_{1 \leq i_1 < \dots < i_k \leq m} (k-1)(\Sigma \Omega X_{i_1} \wedge \dots \wedge \Omega X_{i_k}) \right)$$

which is natural for maps  $X_i \longrightarrow Y_i$ .  $\square$

However, Porter did not identify the homotopy class of the map  $\gamma(\underline{X})$ . It is folklore that it too is determined by Whitehead products that depend on evaluation maps, but there seems to be no proof of this in the literature. As this is important in what follows, in this section we carry out the identification.

For  $1 \leq j \leq m$ , let

$$s_j : X_j \longrightarrow \bigvee_{i=1}^m X_i$$

be the inclusion of the  $j^{\text{th}}$ -wedge summand. Let  $t_j$  be the composite

$$t_j : \Sigma \Omega X_j \xrightarrow{ev_j} X_j \xrightarrow{s_j} \bigvee_{i=1}^m X_i$$

where  $ev_j$  is the canonical evaluation map. We say that an iterated Whitehead product has *length*  $k$  if it is formed from  $k$  ingredient maps. For example,  $[t_{i_1}, t_{i_2}]$  has length 2.



**Theorem 7.2.** *Let  $X_1, \dots, X_m$  be simply-connected, pointed CW-complexes. Then there is a natural homotopy equivalence*

$$\Gamma(\underline{X}) \simeq \bigvee_{k=2}^m \left( \bigvee_{1 \leq i_1 < \dots < i_k \leq m} (k-1)(\Sigma \Omega X_{i_1} \wedge \dots \wedge \Omega X_{i_k}) \right)$$

*which may be chosen so that the restriction of  $\gamma(\underline{X})$  to  $\Sigma \Omega X_{i_1} \wedge \dots \wedge \Omega X_{i_k}$  is an iterated Whitehead product of length  $k$  formed from the maps  $t_{i_1}, \dots, t_{i_k}$ .*

Theorem 7.2 will be proved in two stages. First we consider the special case when each  $X_i = \Sigma Y_i$  and in Theorem 7.4 identify the homotopy fibre of the inclusion  $\bigvee_{i=1}^m \Sigma Y_i \rightarrow \prod_{i=1}^m \Sigma Y_i$ , but in a formulation different from that of Theorem 7.2. This formulation will also be useful in subsequent sections. Second, we consider the wedge sum of evaluation maps  $\bigvee_{i=1}^m \Sigma \Omega X_i \rightarrow \bigvee_{i=1}^m X_i$  and use the formulation for the first stage in the case of  $\Sigma Y_i = \Sigma \Omega X_i$  in order to prove Theorem 7.2.

**7.1. Identifying the homotopy fibre of the inclusion  $\bigvee_{i=1}^m \Sigma Y_i \rightarrow \prod_{i=1}^m \Sigma Y_i$ .** This starts with a refinement of Theorem 6.1 (c) in the case of suspensions. It is notationally convenient in what follows to pinch to the first wedge summand rather than the second. It was shown that there is a homotopy fibration  $\Omega \Sigma X \ltimes \Sigma Y \xrightarrow{g} \Sigma X \vee \Sigma Y \xrightarrow{q} \Sigma X$  where  $q$  is the pinch map,  $\Omega \Sigma X \ltimes \Sigma Y \simeq (\Omega \Sigma X \wedge \Sigma Y) \vee \Sigma Y$ , the restriction of  $g$  to  $\Sigma Y$  is the inclusion  $i_{\Sigma Y}$ , and the restriction of  $g$  to  $\Omega \Sigma X \wedge \Sigma Y$  is the Whitehead product  $[ev_{\Sigma X}, i_{\Sigma Y}]$ . By Lemma 4.2 there is a homotopy equivalence  $\Omega \Sigma X \wedge \Sigma Y \simeq \bigvee_{n=1}^{\infty} \Sigma X^{(n)} \wedge Y$ .

For  $n \geq 1$ , define

$$ad^n(i_{\Sigma X})(i_{\Sigma Y}): \Sigma X^{(n)} \wedge Y \rightarrow \Sigma X \vee \Sigma Y$$

recursively by  $ad^1(i_{\Sigma X})(i_{\Sigma Y}) = [i_{\Sigma X}, i_{\Sigma Y}]$  and, for  $n \geq 2$ ,  $ad^n(i_{\Sigma X})(i_{\Sigma Y}) = [i_{\Sigma X}, ad^{n-1}(i_{\Sigma X})(i_{\Sigma Y})]$ . The following was essentially proved in [N, Theorem 9.3.1].

**Theorem 7.3.** *Let  $X$  and  $Y$  be pointed, path-connected CW-complexes. There is a homotopy fibration*

$$\left( \bigvee_{n=1}^{\infty} \Sigma X^{(n)} \wedge Y \right) \vee \Sigma Y \xrightarrow{\Psi} \Sigma X \vee \Sigma Y \xrightarrow{q} \Sigma X$$

*where the restriction of  $\Psi$  to  $\Sigma Y$  is  $i_{\Sigma Y}$  and the restriction of  $\Psi$  to  $\Sigma X^{(n)} \wedge Y$  is the iterated Whitehead product  $ad^n(i_{\Sigma X})(i_{\Sigma Y})$ . All of this is natural for maps  $X \rightarrow X'$  and  $Y \rightarrow Y'$ .*

*Proof.* The precise statement of [N, Theorem 9.3.1] is that the composite

$$(6) \quad \Omega \left( \left( \bigvee_{n=1}^{\infty} \Sigma X^{(n)} \wedge Y \right) \vee \Sigma Y \right) \times \Omega \Sigma X \xrightarrow{\Omega \psi \times \Omega i_{\Sigma X}} \Omega(\Sigma X \vee \Sigma Y) \times \Omega(\Sigma X \vee \Sigma Y) \xrightarrow{\mu} \Omega(\Sigma X \vee \Sigma Y)$$

is a homotopy equivalence, where  $\mu$  is the loop multiplication. We will show that this implies the existence of the homotopy fibration in the statement of the theorem.

Observe that the naturality of the Whitehead product, the fact that  $q \circ i_{\Sigma X}$  is the identity map on  $\Sigma X$ , and the fact that  $q \circ i_{\Sigma Y}$  is null homotopic together imply that

$$q \circ ad^1(i_{\Sigma X})(i_{\Sigma Y}) \simeq ad^1(q \circ i_{\Sigma X})(q \circ i_{\Sigma Y}) \simeq ad^1(1_{\Sigma X})(*) = [i_{\Sigma X}, *] \simeq *.$$

Inducting on the definition of  $ad^n$  then implies that  $q \circ ad^n(i_{\Sigma X})(i_{\Sigma Y}) \simeq *$  for each  $n \geq 1$ . Thus  $q \circ \Psi$  is null homotopic. Recall from the discussion preceding the theorem that the homotopy fibre of  $q$  is  $\Omega \Sigma X \ltimes \Sigma Y$ . So we obtain a lift

$$(7) \quad \begin{array}{ccc} & (\bigvee_{n=1}^{\infty} \Sigma X^{(n)} \wedge Y) \vee \Sigma Y & \\ \ell \swarrow & \downarrow \Psi & \\ \Omega \Sigma X \ltimes \Sigma Y & \xrightarrow{g} & \Sigma X \vee \Sigma Y \end{array}$$

for some map  $\ell$ . If  $\ell$  is a homotopy equivalence, then  $\Psi$  may be substituted for the homotopy fibre of  $q$ , giving the homotopy fibration asserted by the theorem.

It remains to show that  $\ell$  is a homotopy equivalence. The homotopy equivalence (6) implies that the map  $\Omega \Psi$  induces an injection in homology. Looping the homotopy commutative triangle in (7) then implies that  $\Omega \ell$  induces an injection in homology. In the discussion preceding the theorem it was observed that  $(\bigvee_{n=1}^{\infty} \Sigma X^{(n)} \wedge Y) \vee \Sigma Y$  and  $\Omega \Sigma X \ltimes \Sigma Y$  have the same homotopy type. As both spaces are simply-connected, their loop spaces then also have the same homotopy type. Thus  $(\Omega \ell)_*$  being an injection implies that it is an isomorphism. Now Dror's [Dr] generalization of Whitehead's Theorem implies that  $\Omega \ell$  is a homotopy equivalence. Hence  $\ell$  induces an isomorphism on homotopy groups, so as all spaces have the homotopy type of  $CW$ -complexes,  $\ell$  is a homotopy equivalence.  $\square$

We will use Theorem 7.3 iteratively. Let  $Y_1, \dots, Y_m$  be pointed, path-connected  $CW$ -complexes. Let  $A_0 = \bigvee_{i=1}^m \Sigma Y_i$  and for  $1 \leq j \leq m$  let

$$s_j: \Sigma Y_j \longrightarrow \bigvee_{i=1}^m \Sigma Y_i$$

be the inclusion of the  $j^{th}$ -wedge summand. Let  $B_1 = \bigvee_{i=2}^m Y_i$  so that  $A_0 = \Sigma Y_1 \vee \Sigma B_1$ . By Theorem 7.3 there is a homotopy fibration

$$A_1 \xrightarrow{\Psi_1} \Sigma Y_1 \vee \Sigma B_1 \xrightarrow{q_1} \Sigma Y_1$$

where  $q_1$  is the pinch map,  $A_1 = (\bigvee_{n=1}^{\infty} \Sigma Y_1^{(n)} \wedge B_1) \vee \Sigma B_1$ , the restriction of  $\Psi_1$  to  $\Sigma B_1$  is the inclusion  $i_{\Sigma B_1}$ , and the restriction of  $\Psi_1$  to  $\Sigma Y_1^{(n)} \wedge B_1$  is the iterated Whitehead product  $ad^n(i_{\Sigma Y_1})(i_{\Sigma B_1})$ . Since  $B_1$  is the wedge  $\bigvee_{i=2}^m Y_i$  and the inclusion  $i_{\Sigma B_1}$  is the wedge sum of the inclusions  $s_j$  for  $2 \leq j \leq m$ , and since  $i_{\Sigma Y_1} = s_1$ , the Whitehead product  $ad^n(i_{\Sigma Y_1})(i_{\Sigma B_1})$  is the wedge sum of the Whitehead products  $ad^n(s_1)(s_j)$  for  $2 \leq j \leq m$ .

Observe that  $A_1$  is a suspension and has  $\Sigma Y_2$  as a wedge summand (via it being a wedge summand of  $\Sigma B_1$ ). Write

$$A_1 = \Sigma Y_2 \vee \Sigma B_2$$

where  $B_2 = (\bigvee_{i=3}^m Y_i) \vee (\bigvee_{n=1}^\infty Y_1^{(n)} \wedge B_1)$ . Now iterate this process by considering the pinch map  $\Sigma Y_2 \vee \Sigma B_2 \xrightarrow{q_2} \Sigma Y_2$ . We obtain a sequence of homotopy fibrations

$$A_k \xrightarrow{\Psi_k} \Sigma Y_k \vee \Sigma B_k \xrightarrow{q_k} \Sigma Y_k$$

for  $1 \leq k \leq m$ , where  $A_{k-1} = \Sigma Y_k \vee \Sigma B_k$ ,  $q_k$  is the pinch map,  $A_k = (\bigvee_{n=1}^\infty \Sigma Y_k^{(n)} \wedge B_k) \vee \Sigma B_k$ , the restriction of  $\Psi_k$  to  $\Sigma B_k$  is the inclusion  $i_{\Sigma B_k}$  into  $\Sigma Y_k \vee \Sigma B_k$ , and the restriction of  $\Psi_k$  to  $\Sigma Y_k^{(n)} \wedge B_k$  is the iterated Whitehead product  $ad^n(i_{\Sigma Y_k})(i_{\Sigma B_k})$ . Consider the maps  $i_{\Sigma Y_k}$  and  $i_{\Sigma B_k}$  composing into  $\bigvee_{i=1}^m \Sigma Y_i$  by the composite

$$\Phi_{k-1}: A_{k-1} = \Sigma Y_k \vee \Sigma B_k \xrightarrow{\Psi_{k-1}} A_{k-2} \longrightarrow \cdots \longrightarrow A_1 \xrightarrow{\Psi_1} A_0 = \bigvee_{i=1}^m \Sigma Y_i.$$

In the iteration we have  $B_k = (\bigvee_{i=k+1}^m Y_i) \vee (\bigvee_{n=1}^\infty Y_{k-1}^{(n)} \wedge B_{k-1})$ , the restriction of  $\Phi_{k-1}$  to  $\bigvee_{i=k+1}^m \Sigma B_i$  is the inclusion, the restriction of  $\Phi_{k-1}$  to  $\bigvee_{n=1}^\infty \Sigma Y_{k-1}^{(n)} \wedge B_{k-1}$  is a wedge sum of iterated Whitehead products formed from the maps  $s_j$  for  $1 \leq j \leq m$ , and the restriction of  $\Phi_{k-1}$  to  $\Sigma Y_k$  is the inclusion. Thus  $\Phi_{k-1} \circ i_{\Sigma B_k}$  is a wedge sum of the inclusions  $s_j$  for  $k+1 \leq j \leq m$  and iterated Whitehead products of the  $s_j$ 's for  $1 \leq j \leq m$ , and  $\Phi_{k-1} \circ i_{\Sigma Y_k}$  is the inclusion  $s_j$ . Hence  $\Phi_k = \Phi_{k-1} \circ \Psi_k$  is the wedge sum of the inclusions  $s_j$  for  $k+1 \leq j \leq m$  and iterated Whitehead products of the  $s_j$ 's for  $1 \leq j \leq m$ .

Consequently, at the end of the iteration, we have  $A_m$  a wedge of suspensions and the composite

$$\Phi_m: A_m \xrightarrow{\Psi_m} A_{m-1} \longrightarrow \cdots \longrightarrow A_1 \xrightarrow{\Psi_1} \bigvee_{i=1}^m \Sigma Y_i$$

is a wedge sum of iterated Whitehead products formed from the maps  $s_j$  for  $1 \leq j \leq m$ .

**Theorem 7.4.** *There is a homotopy fibration*

$$A_m \xrightarrow{\Phi_m} \bigvee_{i=1}^m \Sigma Y_i \longrightarrow \prod_{i=1}^m \Sigma Y_i$$

which is natural for maps  $Y_i \longrightarrow Y'_i$ .

*Proof.* Noting that  $\Phi_1 = \Psi_1$ , we start with the homotopy fibration  $A_1 \xrightarrow{\Phi_1} \bigvee_{i=1}^m \Sigma Y_i \xrightarrow{q_1} \Sigma Y_1$ . Assume inductively that for some  $2 \leq k \leq m$  that there is a homotopy fibration  $A_{k-1} \xrightarrow{\Phi_{k-1}} \bigvee_{i=1}^m \Sigma Y_i \longrightarrow \prod_{i=1}^{k-1} \Sigma Y_i$ . Observe that the map  $A_{k-1} \xrightarrow{q_k} \Sigma Y_k$  factors as the composite  $A_{k-1} \xrightarrow{\Phi_{k-1}} \bigvee_{i=1}^m \Sigma Y_i \longrightarrow \Sigma Y_k$ . Therefore there is a homotopy fibration diagram

$$\begin{array}{ccccc} A_{k-1} & \xrightarrow{\Phi_{k-1}} & \bigvee_{i=1}^m \Sigma Y_i & \longrightarrow & \prod_{i=k}^m \Sigma Y_i \\ \downarrow q_k & & \downarrow & & \parallel \\ \Sigma Y_{k-1} & \longrightarrow & \prod_{i=k-1}^m \Sigma Y_i & \xrightarrow{\pi_{k-1}} & \prod_{i=1}^{k-1} \Sigma Y_i \end{array}$$

where  $\pi_{k-1}$  is the projection. Since the left square is a homotopy pullback, from the homotopy fibration  $A_k \xrightarrow{\Psi_k} A_{k-1} \xrightarrow{q_{k-1}} \Sigma Y_k$  and the fact that  $\Phi_k = \Phi_{k-1} \circ \Psi_k$ , we obtain a homotopy fibration  $A_k \xrightarrow{\Phi_k} \bigvee_{i=1}^m \Sigma Y_i \longrightarrow \prod_{i=k-1}^m \Sigma Y_i$ . By induction, such a homotopy fibration exists for all  $2 \leq k \leq m$ , so when  $k = m$  we obtain a homotopy fibration  $A_m \xrightarrow{\Phi_m} \bigvee_{i=1}^m \Sigma Y_i \longrightarrow \prod_{i=1}^m \Sigma Y_i$  as asserted.

Finally, since  $\Phi_m$  consists of iterated Whitehead products formed from the inclusions  $s_j$  for  $1 \leq j \leq m$ , the naturality of the Whitehead product and the inclusions implies that this homotopy fibration is natural for maps  $Y_i \longrightarrow Y'_i$ .  $\square$

**7.2. Identifying the homotopy fibre of the inclusion  $\bigvee_{i=1}^m X_i \longrightarrow \prod_{i=1}^m X_i$ .** We turn to Theorem 7.2. The idea is to use the information in Theorem 7.4 in the case of  $\bigvee_{i=1}^m \Sigma \Omega X_i$  and compose it with evaluation maps to  $\bigvee_{i=1}^m X_i$ . We first consider the basic case of Theorem 7.3 applied to  $\Sigma \Omega X \vee \Sigma \Omega Y$  and compose with evaluation map to  $X \vee Y$ . Note here that we assume  $X$  and  $Y$  are simply-connected so that  $\Omega X$  and  $\Omega Y$  are path-connected, as needed in Theorem 7.3.

**Lemma 7.5.** *Let  $X$  and  $Y$  be simply-connected, pointed CW-complexes. There is a lift*

$$\begin{array}{ccc} \bigvee_{n=1}^{\infty} \Sigma \Omega X^{(n)} \wedge \Omega Y & \xrightarrow{\bigvee_{n=1}^{\infty} ad^n(i_{\Sigma \Omega X})(i_{\Sigma \Omega Y})} & \Sigma \Omega X \vee \Sigma \Omega Y \\ \downarrow \lambda & & \downarrow ev \vee ev \\ \Sigma \Omega X \wedge \Omega Y & \xrightarrow{[ev_X, ev_Y]} & X \vee Y \end{array}$$

for some map  $\lambda$ .

*Proof.* By Theorem 6.1 (a), there is a homotopy fibration  $\Sigma \Omega X \wedge \Omega Y \xrightarrow{[ev_X, ev_Y]} X \vee Y \xrightarrow{i} X \times Y$ . The naturality of the Whitehead product implies that the composition  $(ev_X \vee ev_Y) \circ ad^n(i_{\Sigma \Omega X})(i_{\Sigma \Omega Y})$  is the Whitehead product  $ad^n(ev_X)(ev_Y)$ . Every Whitehead product into  $X \vee Y$  composes trivially with  $i$  and so lifts through  $[ev_X, ev_Y]$ . The lemma now follows.  $\square$

**Remark 7.6.** In analogy with the definition of  $t_j$ , let  $t_X = ev \circ i_{\Sigma \Omega X}$  and  $t_Y = ev \circ i_{\Sigma \Omega Y}$ . Then Lemma 7.5 equivalently says that  $ad^n(t_X)(t_Y)$  factors through  $[ev_X, ev_Y] = [t_X, t_Y]$ .

Recall that the map  $t_j$  is defined as the composite  $\Sigma \Omega X_j \xrightarrow{ev_j} X_j \xrightarrow{s_j} \bigvee_{i=1}^m X_i$ . The naturality of the evaluation map implies that  $t_j$  can also be regarded as the composite

$$t_j: \Sigma \Omega X_i \xrightarrow{s_j} \bigvee_{i=1}^m \Sigma \Omega X_i \xrightarrow{\bigvee_{i=1}^m ev_i} \bigvee_{i=1}^m X_i.$$

Now run through the construction of the spaces  $A_k$  with the input being  $Y_i = \Omega X_i$  for  $1 \leq i \leq m$ . Starting with  $A_0 = \bigvee_{i=1}^m \Sigma \Omega X_i$ , the first homotopy fibration  $A_1 \xrightarrow{\Psi_1} A_0 \xrightarrow{q_0} \Sigma \Omega X_1$  has

$$A_1 = \bigvee_{n=1}^{\infty} \left( \Omega X_1^{(n)} \wedge \left( \bigvee_{i=2}^m \Sigma \Omega X_i \right) \right) \vee \left( \bigvee_{i=2}^m \Sigma \Omega X_i \right),$$

the restriction of  $\Psi_1$  to  $\bigvee_{i=1}^m \Sigma \Omega X_i$  is a wedge of the inclusions  $s_j$  for  $2 \leq j \leq m$ , and the restriction of  $\Psi_1$  to  $\Omega X_1^{(n)} \wedge (\bigvee_{i=2}^m \Sigma \Omega X_i)$  is a wedge of iterated Whitehead products  $ad^n(s_1)(s_j)$  for  $2 \leq j \leq m$ . Composing  $\Phi_1 = \Psi_1$  with the wedge of evaluation maps

$$e: \bigvee_{i=1}^m \Sigma \Omega X_i \xrightarrow{\bigvee_{i=1}^m ev_i} \bigvee_{i=1}^m X_i$$

the maps  $s_j$  are sent to  $t_j$  and, by Lemma 7.5, the iterated Whitehead products  $ad^n(s_1)(s_j)$  factor through  $[t_1, t_j]$ . At the next stage, writing  $A_1 = \Sigma B_2 \vee \Sigma \Omega X_2$ , we have

$$A_2 = \bigvee_{n=1}^{\infty} \left( \Omega X_2^{(n)} \wedge \Sigma B_2 \right) \vee \Sigma B_2.$$

First,  $\Sigma B_2$  consists of all wedge summands in  $A_1$  except  $\Sigma \Omega X_2$ , and  $e \circ \Phi_2$  maps these into  $\bigvee_{i=1}^m X_i$  exactly as does  $e \circ \Phi_1$ , that is, by a  $t_j$  if  $3 \leq j \leq m$  (excluding  $t_2$  since  $\Sigma \Omega X_2$  is not in  $B_2$ ) or by factoring through a  $[t_1, t_j]$  for  $2 \leq j \leq m$ . Second,  $\Phi_2$  maps the summands  $\bigvee_{n=1}^{\infty} \left( \Omega X_2^{(n)} \wedge \Sigma B_2 \right)$  into  $\bigvee_{i=1}^m \Sigma \Omega X_i$  by iterated Whitehead products  $ad^n(s_2)(f)$  where  $f$  comes from  $\Phi_1$ . Composing with the wedge of evaluations  $e$ , the naturality of the Whitehead product implies that  $e \circ f$  factors through a  $t_j$  for  $3 \leq j \leq m$  or a  $[t_1, t_j]$  for  $2 \leq j \leq m$ , and Lemma 7.5 then implies that  $e \circ ad^n(s_2)(f) = ad^n(e \circ s_2)(e \circ f)$  factors through  $[t_2, t_j]$  for  $3 \leq j \leq m$  or  $[t_2, [t_1, t_j]]$  for  $2 \leq j \leq m$ . Refining a bit, observe that if  $j = 2$  then  $[t_2, [t_1, t_2]] \simeq [t_1, [t_2, t_1]] \circ (\Sigma 1 \wedge T)$  where  $1$  is the identity map on  $\Omega X_2$  and  $T: \Omega X_1 \wedge \Omega X_2 \longrightarrow \Omega X_2 \wedge \Omega X_1$  interchanges wedge summands. But  $[t_2, [t_2, t_1]] = ad^2(t_2)(t_1)$  and, by Remark 7.6,  $ad^2(t_2)(t_1)$  factors through  $[t_2, t_1]$ . Thus, in fact,  $e \circ ad^n(s_2)(f) = ad^n(e \circ s_2)(e \circ f)$  factors through  $[t_2, t_j]$  for  $3 \leq j \leq m$  or  $[t_2, [t_1, t_j]]$  for  $3 \leq j \leq m$ .

Continuing in this manner we see that  $e \circ \Phi_m$  factors through a wedge sum of the Whitehead products  $[t_{r_s}, [t_{r_{s-1}}, [\dots, [t_{r_1}, t_j] \dots]]$  where  $j > r_1$ ,  $1 \leq r_1 < \dots < r_s \leq m$ , and  $j \notin \{r_1, \dots, r_s\}$ .

**Remark 7.7.** We need to re-index the enumeration of these Whitehead products. Let  $C$  be the collection of iterated Whitehead products of the form  $[t_{r_s}, [t_{r_{s-1}}, [\dots, [t_{r_1}, t_j] \dots]]$  where  $j > r_1$  and  $1 \leq r_1 < \dots < r_s \leq m$ . Fixing a sequence  $(i_1, \dots, i_k)$  with  $1 \leq i_1 < \dots < i_k \leq m$ , we wish to determine how many brackets in  $C$  are formed from this sequence. Note that  $r_1$  must be the smallest index, so we must take  $r_1 = i_1$ . Note that  $j$  only needs to be bigger than  $r_1$ , so there are  $k-1$  possible choices for  $j$ ; we can take  $j = i_s$  for  $2 \leq s \leq k$ . Suppose that  $j = i_{s_0}$ . Then the remaining terms are  $i_2, \dots, \hat{i}_{s_0}, \dots, i_k$  where  $\hat{i}_{s_0}$  is removed from the list. As this list is ordered, and we are substituting them in for the ordered list  $r_2, \dots, r_{k-1}$ , we are forced to take  $(r_2, \dots, r_k) = (i_2, \dots, \hat{i}_{s_0}, \dots, i_k)$ . Thus in total there are  $k-1$  possible brackets in  $C$  we can form from  $(i_1, \dots, i_k)$ .

Conversely, any element in  $C$  corresponds to a sequence  $(i_1, \dots, i_{s+1})$  by simply taking the sequence  $(r_1, \dots, r_s)$  and inserting  $j$  in order. Thus there is a one-to-one correspondence between  $C$  and the collection of  $k-1$  brackets for each sequence  $(i_1, \dots, i_k)$  with  $1 \leq i_1 < \dots < i_k \leq m$ .

To compress notation, let

$$(8) \quad W(\underline{X}) = \bigvee_{k=2}^m \left( \bigvee_{1 \leq i_1 < \dots < i_k \leq m} (k-1)(\Sigma \Omega X_{i_1} \wedge \dots \wedge \Omega X_{i_k}) \right).$$

Using Remark 7.7 to correspond indexing sets, let

$$\Theta(\underline{X}): W(\underline{X}) \longrightarrow \bigvee_{i=1}^m X_i$$

be the wedge sum of the Whitehead products in  $C$ . We have proved the following.

**Lemma 7.8.** *Let  $X_1, \dots, X_m$  be simply-connected, pointed CW-complexes. Then there is a factorization*

$$\begin{array}{ccc} A_m & \xrightarrow{\Phi_m} & \bigvee_{i=1}^m \Sigma \Omega X_i \\ \downarrow \epsilon & & \downarrow \bigvee_{i=1}^m ev \\ W(\underline{X}) & \xrightarrow{\Theta(\underline{X})} & \bigvee_{i=1}^m X_i \end{array}$$

for some map  $\epsilon$ . □

Lemma 7.8 will play a key role in the proof of Theorem 7.2. Before starting the proof, some additional preliminary information is needed.

**Lemma 7.9.** *The homotopy fibration  $\Gamma(\underline{X}) \xrightarrow{\gamma(\underline{X})} \bigvee_{i=1}^m X_i \longrightarrow \prod_{i=1}^m X_i$  has the property that  $\Omega\gamma(\underline{X})$  has a left homotopy inverse.*

*Proof.* For  $1 \leq j \leq m$ , the composite  $X_j \xrightarrow{s_j} \bigvee_{i=1}^m X_i \longrightarrow \prod_{i=1}^m X_i$  is the inclusion of the  $j^{th}$  factor. After looping the maps  $\Omega s_j$  can be multiplied together to obtain a right homotopy inverse  $s$  for the map  $\Omega(\bigvee_{i=1}^m X_i) \longrightarrow \prod_{i=1}^m \Omega X_i$ . Consequently there is a homotopy equivalence

$$\prod_{i=1}^m \Omega X_i \times \Omega \Gamma(\underline{X}) \xrightarrow{s \times \Omega \gamma(\underline{X})} \Omega \left( \bigvee_{i=1}^m X_i \right) \times \Omega \left( \bigvee_{i=1}^m X_i \right) \xrightarrow{\mu} \Omega \left( \bigvee_{i=1}^m X_i \right)$$

where  $\mu$  is the loop multiplication. Therefore  $\Omega\gamma(\underline{X})$  has a left homotopy inverse. □

For  $1 \leq i \leq m$ , we have the evaluation map  $ev_i: \Sigma \Omega X_i \longrightarrow X_i$ . Let

$$E_i: \Omega X_i \longrightarrow \Omega \Sigma \Omega X_i$$

be the suspension map.. Since  $ev_i$  is the left adjoint of the identity map on  $\Omega X_i$  and  $E_i$  is the right adjoint of the identity map on  $\Sigma \Omega X_i$ , the composite  $\Omega ev_i \circ E_i$  is homotopic to the identity map on  $\Omega X_i$ . Building on this, for any sequence  $1 \leq i_1 < \dots < i_k \leq m$ , there is a composite

$$\Sigma \Omega X_{i_1} \wedge \dots \wedge \Omega X_{i_k} \longrightarrow \Sigma \Omega \Sigma \Omega X_{i_1} \wedge \dots \wedge \Omega \Sigma \Omega X_{i_k} \longrightarrow \Sigma \Omega X_{i_1} \wedge \dots \wedge \Omega X_{i_k}$$

where the left map is  $\Sigma E_{i_1} \wedge \dots \wedge E_{i_k}$ , the right map is  $\Sigma \Omega ev_{i_1} \wedge \dots \wedge \Omega ev_{i_k}$ , and the composite is homotopic to the identity map. Doing this for each of the wedge summands in (8) we obtain maps

$$(9) \quad W(\underline{X}) \xrightarrow{\xi} W(\Sigma \Omega \underline{X}) \xrightarrow{\zeta} W(\underline{X})$$

whose composite is a homotopy equivalence.

*Proof of Theorem 7.2.* Consider the diagram

$$(10) \quad \begin{array}{ccccc} & & & \Omega W(\underline{X}) & \\ & & & \downarrow \Omega \xi & \\ & \Omega A_m & \xleftarrow{\Omega \theta} & \Omega W(\underline{\Sigma \Omega X}) & \\ & \downarrow \Omega \epsilon & \searrow \Omega \Phi_m & \swarrow \Omega \gamma(\underline{\Sigma \Omega X}) & \downarrow \Omega \zeta \\ & \Omega W(\underline{X}) & & \Omega(\bigvee_{i=1}^m \Sigma \Omega X_i) & \Omega W(\underline{X}) \\ & \searrow \Omega \Theta(\underline{X}) & & \downarrow \Omega(\bigvee_{i=1}^m ev_i) & \swarrow \Omega \gamma(\underline{X}) \\ & & \Omega(\bigvee_{i=1}^m X_i) & & \\ & & \downarrow Q(\underline{X}) & & \\ & & \Omega W(\underline{X}). & & \end{array}$$

Here,  $Q(\underline{X})$  is a left homotopy inverse of  $\Omega \gamma(\underline{X})$  that exists by Lemma 7.9. The left quadrilateral homotopy commutes by Lemma 7.8, and the right quadrilateral homotopy commutes by the naturality of Theorem 7.1. For the triangle, by Lemma 7.8 and Theorem 7.1 respectively, both  $A_m \xrightarrow{\Phi_m} \bigvee_{i=1}^m \Sigma \Omega X_i \longrightarrow \prod_{i=1}^m \Sigma \Omega X_i$  and  $W(\underline{\Sigma \Omega X}) \xrightarrow{\gamma(\underline{\Sigma \Omega X})} \bigvee_{i=1}^m \Sigma \Omega X_i \longrightarrow \prod_{i=1}^m \Sigma \Omega X_i$  are homotopy fibrations, so there is a homotopy equivalence  $\theta: W(\underline{\Sigma \Omega X}) \longrightarrow A_m$  such that  $\gamma(\underline{\Sigma \Omega X}) \simeq \Phi_m \circ \theta$ . Looping this give the homotopy commutativity of the upper triangle above. Therefore the entire diagram homotopy commutes.

By (9), the composite  $\Omega \zeta \circ \Omega \xi$  is a homotopy equivalence and, by definition,  $Q(\underline{X})$  is a left homotopy inverse for  $\Omega \gamma(\underline{X})$ . Therefore, on the right side of the diagram, the composite  $Q(\underline{X}) \circ \Omega \gamma(\underline{X}) \circ \Omega \zeta \circ \Omega \xi$  is a homotopy equivalence. The homotopy commutativity of the diagram therefore implies that  $Q(\underline{X}) \circ \Omega \Theta(\underline{X}) \circ \Omega \epsilon \circ \Omega \theta \circ \Omega \xi$  is a homotopy equivalence. Let  $a = \Omega \epsilon \circ \Omega \theta \circ \Omega \xi$ , let  $b = Q(\underline{X}) \circ \Omega \Theta(\underline{X})$  and consider the composite of self-maps

$$\Omega W(\underline{X}) \xrightarrow{a} \Omega W(\underline{X}) \xrightarrow{b} \Omega W(\underline{X}).$$

We have just seen that  $b \circ a$  is a homotopy equivalence. This implies that, in integral homology,  $a_*$  is an injection. Since  $H_*(\Omega W(\underline{X}))$  is a finite type module, a self-map which is an injection must be an isomorphism, and hence  $a_*$  is an isomorphism. Noting that  $W(\underline{X})$  is a suspension so that  $\Omega W(\underline{X})$  is nilpotent, Dror's [Dr] generalization of Whitehead's Theorem implies that  $a$  is a homotopy equivalence. (While we do not strictly need this, notice that  $a$  being a homotopy equivalence implies that  $b$  is also a homotopy equivalence, since  $b = (b \circ a) \circ a^{-1}$  and both  $b \circ a$  and  $a^{-1}$  are homotopy equivalences.)

Let  $c = \Omega\zeta \circ \Omega\xi \circ a^{-1}$ . As both  $\Omega\zeta \circ \Omega\xi$  and  $a^{-1}$  are homotopy equivalences, so is  $c$ . Further, the definition of  $a$  and the homotopy commutativity of (10) implies that

$$\Omega\Theta(\underline{X}) \simeq \Omega\Theta(\underline{X}) \circ a \circ a^{-1} = \Omega\Theta(\underline{X}) \circ \Omega\epsilon \circ \Omega\theta \circ \Omega\xi \circ a^{-1} \simeq \Omega\gamma(\underline{X}) \circ \Omega\zeta \circ \Omega\xi \circ a^{-1} = \Omega\gamma(\underline{X}) \circ c.$$

Thus there is a homotopy commutative diagram

$$(11) \quad \begin{array}{ccc} & \Omega W(\underline{X}) & \\ \swarrow c^{-1} & \downarrow \Omega\Theta(\underline{X}) & \\ \Omega W(\underline{X}) & \xrightarrow{\Omega\gamma(\underline{X})} & \Omega(\bigvee_{i=1}^m X_i). \end{array}$$

We claim that (11) deloops. In general, if  $g: \Sigma A \rightarrow \Sigma B$  is a map with the property that  $\Omega g$  is a homotopy equivalence, then  $g$  is a homotopy equivalence. For  $\Omega g$  induces an isomorphism on homotopy groups and therefore, as both  $\Sigma A$  and  $\Sigma B$  are simply-connected, so does  $g$ . As we are assuming spaces are  $CW$ -complexes, this implies that  $g$  is a homotopy equivalence. In our case, by definition,  $a = \Omega\bar{a}$  for  $\bar{a} = \epsilon \circ \theta \circ \xi$ . As  $\Omega\bar{a}$  is a homotopy equivalence, so is  $\bar{a}$ . Therefore  $a^{-1} \simeq \Omega(\bar{a}^{-1})$ , implying that  $c = \Omega\bar{c}$  for  $\bar{c} = \zeta \circ \xi \circ \bar{a}^{-1}$ . Again, as  $c$  is a homotopy equivalence, so is  $\bar{c}$ . Since  $W(\underline{X})$  is a suspension it retracts off  $\Sigma\Omega W(\underline{X})$ , implying that for any space  $B$  the map  $[W(\underline{X}), B] \rightarrow [\Omega W(\underline{X}), \Omega B]$  sending  $g$  to  $\Omega g$  is an injection. Hence the homotopy commutativity of (11), with  $c^{-1}$  written as  $\Omega\bar{c}^{-1}$ , deloops to give a homotopy commutative diagram

$$(12) \quad \begin{array}{ccc} & W(\underline{X}) & \\ \swarrow \bar{c}^{-1} & \downarrow \Theta(\underline{X}) & \\ W(\underline{X}) & \xrightarrow{\gamma(\underline{X})} & \bigvee_{i=1}^m X_i \end{array}$$

where  $\bar{c}^{-1}$  is a homotopy equivalence.

By Theorem 7.1 there is a homotopy fibration  $W(\underline{X}) \xrightarrow{\gamma(\underline{X})} \bigvee_{i=1}^m X_i \rightarrow \prod_{i=1}^m X_i$ . Thus (12) implies that there is a homotopy fibration  $W(\underline{X}) \xrightarrow{\Theta(\underline{X})} \bigvee_{i=1}^m X_i \rightarrow \prod_{i=1}^m X_i$ . This proves the first assertion of the theorem. The naturality assertion follows from the fact that  $\Theta(\underline{X})$  is natural because the Whitehead product is natural.  $\square$

## 8. THIN PRODUCTS AND WHITEHEAD PRODUCTS

Recall from the Introduction that if  $X_1, \dots, X_m$  are pointed spaces then the thin product is defined by  $P^m(\underline{X}) = (\underline{X}, *)_D^K$  where  $K$  is the simplicial complex consisting of  $m$  disjoint points. Noting that  $(\underline{X}, *)^K \simeq \bigvee_{i=1}^m X_i$ , from the map  $(\underline{X}, *)^K \xrightarrow{\varphi} (\underline{X}, *)_D^K$  we obtain a homotopy fibration

$$F^m(\underline{X}) \xrightarrow{f^m(\underline{X})} \bigvee_{i=1}^m X_i \xrightarrow{\varphi} P^m(\underline{X}).$$

For  $1 \leq j \leq m$ , recall again that

$$s_j: X_j \rightarrow \bigvee_{i=1}^m X_i$$



is the inclusion of the  $j^{\text{th}}$ -wedge summand. Suppose that for  $1 \leq k \leq t$  there are maps  $a_k: \Sigma Y_k \rightarrow X_{j_k}$  where  $1 \leq j_k \leq m$ . Let  $b_k$  be the composite

$$b_k: \Sigma Y_k \xrightarrow{a_k} X_{j_k} \xrightarrow{s_{j_k}} \bigvee_{i=1}^m X_i.$$

Let

$$w: \Sigma Y_1 \wedge \cdots \wedge Y_t \rightarrow \bigvee_{i=1}^m X_i$$

be an iterated Whitehead product formed from the maps  $b_k$ . We say that  $w$  has length  $t$  and involves the maps  $\{s_{j_1}, \dots, s_{j_t}\}$ .

**Theorem 8.1.** *Let  $w$  be a Whitehead product on  $\bigvee_{i=1}^m X_i$  formed from the maps  $b_k$ . Suppose that  $w$  has length  $t \geq m$  and involves all the maps  $s_j$  for  $1 \leq j \leq m$ . Then  $w$  lifts through  $F^m(\underline{X}) \xrightarrow{f^m(\underline{X})} \bigvee_{i=1}^m X_i$ .*

*Proof.* Recall that  $K$  is  $m$  disjoint points. For any proper subset  $I \subset [m]$ , by Lemma 2.3 the projection  $X^m \rightarrow X^I$  induces a map of polyhedral products  $(\underline{X}, *)^K \rightarrow (\underline{X}, *)^{K_I}$  which, in this case, is the equivalent to the pinch map  $p_I: \bigvee_{i=1}^m X_i \rightarrow \bigvee_{i \in I} X_i$ . Observe that if  $j \notin I$  then the composite  $\Sigma X_j \xrightarrow{s_j} \bigvee_{i=1}^m X_i \xrightarrow{p_I} \bigvee_{i \in I} X_i$  is trivial, since  $s_j$  is the inclusion of  $X_j$  into the wedge. Since  $I$  is a proper subset of  $[m]$ , we can always find a  $j$  such that  $p_I \circ s_j$  is null homotopic. As  $w$  is a Whitehead product involving all the maps  $s_j$  for  $1 \leq j \leq m$ , the naturality of the Whitehead product implies that  $p_I \circ w$  is null homotopic. This holds for any proper subset  $I$  of  $[m]$ , so  $p_I \circ w$  is null homotopic for all  $I \subsetneq [m]$ .

Let  $\tilde{w}: Y_1 \wedge \cdots \wedge Y_t \rightarrow \Omega(\bigvee_{i=1}^m X_i)$  be the adjoint of  $w$  and consider the composite

$$Y_1 \wedge \cdots \wedge Y_t \xrightarrow{\tilde{w}} \Omega\left(\bigvee_{i=1}^m X_i\right) \xrightarrow{\Omega\varphi} \Omega P^m(\underline{X}).$$

By Theorem 5.3, every factor in the homotopy decomposition of  $\Omega P^m(\underline{X}) = \Omega(\underline{X}, *)_D^K$  is also a factor of  $\Omega(\underline{X}, *)^{K_I}$  for some  $I \subsetneq [m]$ , and the decomposition is compatible with the maps  $\Omega p_I$ . Therefore as  $p_I \circ w$  is null homotopic for all  $I \subsetneq [m]$ , so is  $\Omega p_I \circ \tilde{w}$ . Thus  $\Omega\varphi \circ \tilde{w}$  is null homotopic, implying that its adjoint  $\varphi \circ w$  is null homotopic. Hence  $w$  lifts through  $f^m(\underline{X})$ .  $\square$

A special case is when each of the spaces  $X_i$  equals a common space  $X$ . Then we write  $P^m(X)$  for the thin product, and have a homotopy fibration  $F^m(X) \xrightarrow{f^m(X)} \bigvee_{i=1}^m X \rightarrow P^m(X)$ . In this case Theorem 8.1 refines. Consider the composite  $F^m(X) \xrightarrow{f^m(X)} \bigvee_{i=1}^m X \xrightarrow{\nabla} X$  which is used to define the weak cocategory of  $X$ .

**Lemma 8.2.** *Let  $w$  be a Whitehead product on  $X$  of length  $t \geq m$ . Then  $w$  lifts through  $\nabla$  to a Whitehead product on  $\bigvee_{i=1}^m X$  of length  $t$  involving all the inclusions  $s_j$  for  $1 \leq j \leq m$ .*

*Proof.* Suppose that  $w$  is a Whitehead product on  $X$  of length  $t \geq m$ , where  $w$  is formed from maps  $a_k: \Sigma Y_k \rightarrow X$  for  $1 \leq k \leq t$ . For  $1 \leq k \leq m$ , let  $b_j$  be the composite  $\Sigma Y_j \xrightarrow{a_j} X \xrightarrow{s_j} \bigvee_{i=1}^m X$ . If  $t > m$ , then for  $m < k \leq t$ , let  $b_k$  be the composite  $\Sigma Y_k \xrightarrow{a_k} X \xrightarrow{s_m} \bigvee_{i=1}^m X$ . Let  $\bar{w}$  be the Whitehead product of the maps  $b_k$  for  $1 \leq k \leq t$ , where the bracketing order is the same as for  $w$ . Then the naturality of the Whitehead product implies that  $w \simeq \nabla \circ \bar{w}$ . Thus  $\bar{w}$  lifts  $w$  through  $\nabla$ , it has the same length as  $w$ , and involves all  $m$  inclusions  $s_j$ .  $\square$

Combining the lifts in Lemma 8.2 and Theorem 8.1 we immediately obtain the following.

**Theorem 8.3.** *Any Whitehead product on  $X$  of length  $t \geq m$  lifts through the composite  $F^m(X) \xrightarrow{f^m(X)} \bigvee_{i=1}^m X \xrightarrow{\nabla} X$ .*  $\square$

**Remark 8.4.** It is interesting to compare Theorems 8.3 with [Hov, Theorem 2]. Both prove the same statement but in different language, we use polyhedral products in the proof of Theorem 8.1 while Hovey uses “flasque diagrams”. However, the key point in both cases is the same: that the Whitehead products considered on  $\bigvee_{i=1}^m X$  vanish when composed to any proper subwedge.

Next, we aim towards Theorem 8.6 which is a sort of converse to Theorem 8.1. While Theorem 8.1 says that any Whitehead product of length  $t \geq m$  involving all  $m$  maps  $s_j$  lifts through  $f^m(\underline{X})$ , Theorem 8.6 says that the homotopy class of  $f^m(\underline{X})$  is completely determined by length  $t \geq m$  Whitehead products involving all  $m$  maps  $s_j$ . To see this we modify the proof of Proposition 4.10 that identified the homotopy type of  $F^m(\underline{X})$  in order to take into account the Whitehead product information in Theorems 7.2 and 6.1.

First, we require a general lemma.

**Lemma 8.5.** *Suppose that there are maps*

$$f = \bigvee_{i=1}^s f_i: \bigvee_{i=1}^s \Sigma A_i \rightarrow Z \quad g = \bigvee_{j=1}^t g_j: \bigvee_{j=1}^t \Sigma B_j \rightarrow Z.$$

*Then the Whitehead product  $[f, g]$  is homotopic to the wedge sum of Whitehead products  $\bigvee_{i=1}^s \bigvee_{j=1}^t [f_i, g_j]$ .*

*Proof.* Denoting the adjoint of a map  $u$  by  $\tilde{u}$ , it is equivalent to show that the Samelson product of  $\langle \tilde{f}, \tilde{g} \rangle$  is homotopy equivalent to the wedge of Samelson products  $\bigvee_{i=1}^s \bigvee_{j=1}^t \langle \tilde{f}_i, \tilde{g}_j \rangle$ . But this is clear from the pointwise definition of the Samelson product of two maps  $u$  and  $v$  as  $\langle u, v \rangle(x, y) = u(x)v(y)u(x)^{-1}v(y)^{-1}$ .  $\square$

**Theorem 8.6.** *Let  $X_1, \dots, X_m$  be a simply-connected, pointed CW-complexes. Then the homotopy fibration*

$$F^m(\underline{X}) \xrightarrow{f^m(\underline{X})} \bigvee_{i=1}^m X_i \longrightarrow P^m(\underline{X})$$

*satisfies the following:*

- (a)  $F^m(\underline{X}) \simeq \bigvee_{\alpha \in \mathcal{I}} \Sigma(\Omega X_1)^{(\alpha_1)} \wedge \cdots \wedge (\Omega X_m)^{(\alpha_m)}$  where  $\alpha_i \geq 1$  for each  $1 \leq i \leq m$ ;
- (b) the restriction of  $f^m(\underline{X})$  to  $\Sigma(\Omega X_1)^{(\alpha_1)} \wedge \cdots \wedge (\Omega X_m)^{(\alpha_m)}$  is an iterated Whitehead product of length  $t \geq m$  formed from the maps  $t_j: \Sigma \Omega X_j \xrightarrow{ev} X_j \xrightarrow{s_j} \bigvee_{i=1}^m X_i$ , where each  $t_j$  for  $1 \leq j \leq m$  appears at least once;
- (c) parts (a) and (b) are natural for maps of spaces  $X_i \rightarrow Y_i$ .

*Proof.* The construction in Section 4 worked for any polyhedral product  $(\underline{CX}, \underline{X})^K$  where  $K$  is a totally homology fillable simplicial complex. Specialize to the case of  $(\underline{C\Omega X}, \underline{\Omega X})^K$  when  $K$  is  $m$  disjoint points. Then the homotopy fibration  $(\underline{C\Omega X}, \underline{\Omega X})^K \rightarrow (\underline{X}, *)^K \rightarrow \prod_{i=1}^m X_i$  is a model for the homotopy fibration  $\Gamma(\underline{X}) \xrightarrow{\gamma(\underline{X})} \bigvee_{i=1}^m X_i \rightarrow \prod_{i=1}^m X_i$ . Therefore, by Theorem 7.2 there is a homotopy decomposition of  $(\underline{C\Omega X}, \underline{\Omega X})^K$  as a wedge sum of spaces  $\Sigma \Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k}$  which can be chosen so that the restriction of the map  $(\underline{C\Omega X}, \underline{\Omega X})^K \xrightarrow{\gamma(\underline{X})} (\underline{X}, *)^K \simeq \bigvee_{i=1}^m X_i$  to a wedge summand  $\Sigma \Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_t}$  is an iterated Whitehead product of the maps  $t_j$ .

With  $(\underline{C\Omega X}, \underline{\Omega X})^K \rightarrow \bigvee_{i=1}^m X_i$  as the starting point in this case, the first step in the construction in Section 4 was to divide the wedge summands of  $(\underline{C\Omega X}, \underline{\Omega X})^K$  so that  $(\underline{C\Omega X}, \underline{\Omega X})^K \simeq \Sigma B_1 \vee \Sigma C_1$ , where  $\Sigma B_1$  consists of those wedge summands having  $\Omega X_1$  as a smash factor and  $\Sigma C_1$  consists of those wedge summands not having  $\Omega X_1$  as a smash factor. Pinching to  $\Sigma C_1$ , we obtain a homotopy fibration  $\Sigma B_1 \rtimes \Omega \Sigma C_1 \xrightarrow{g_1} \Sigma B_1 \vee \Sigma C_1 \rightarrow \Sigma C_1$ . By Theorem 7.3, there is a homotopy equivalence

$$\Sigma B_1 \rtimes \Omega \Sigma C_1 \simeq \Sigma B_1 \vee \bigvee_{n=1}^{\infty} (\Sigma B_1 \wedge (C_1)^{(n)})$$

which may be chosen so that the restriction of  $g_1$  to  $\Sigma B_1$  is  $i_L$  and the restriction to  $\Sigma B_1 \wedge (C_1)^{(n)}$  is an iterated Whitehead product of the maps  $i_L$  and  $i_R$  where  $i_L$  appears once and  $i_R$  appears  $n$  times. Regarding  $\Sigma B_1$  and  $\Sigma C_1$  as a wedge of spaces of the form  $\Sigma \Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k}$ , by Lemma 8.5 each iterated Whitehead product of the maps  $i_L$  and  $i_R$  is homotopic to a wedge sum of iterated Whitehead products of the inclusion maps of the summands  $\Sigma \Omega X_{i_1} \wedge \cdots \wedge \Omega X_{i_k}$  into  $(\underline{C\Omega X}, \underline{\Omega X})^K$ . Therefore, as  $(\underline{C\Omega X}, \underline{\Omega X})^K \xrightarrow{\gamma(\underline{X})} \bigvee_{i=1}^m X_i$  is a wedge sum of iterated Whitehead products of the maps  $t_j$ , the naturality of the Whitehead product implies that the composite  $\Sigma B_1 \rtimes \Omega \Sigma C_1 \xrightarrow{g_1} \Sigma B_1 \vee \Sigma C_1 \simeq (\underline{C\Omega X}, \underline{\Omega X})^K \xrightarrow{\gamma(\underline{X})} \bigvee_{i=1}^m X_i$  is homotopic to a wedge sum of iterated Whitehead products (of iterated Whitehead products) of the maps  $t_j$ . Further, as each wedge summand of  $\Sigma B_1 \rtimes \Omega \Sigma C_1$  has  $\Omega X_1$  as a smash factor, each of the Whitehead products has  $t_1$  appearing at least once.

The next step in the construction identifying the homotopy type of  $F^m(\underline{X})$  was to divide  $G_1 \simeq \Sigma B_1 \rtimes \Omega \Sigma C_1$  into a wedge  $\Sigma B_2 \vee \Sigma C_2$  where each wedge summand of  $\Sigma B_2$  has both  $\Omega X_1$  and  $\Omega X_2$  as smash factors while the wedge summands of  $\Sigma C_2$  have  $\Omega X_1$  as a smash factor but not  $\Omega C_2$ . Then one pinches  $G_1$  to  $\Sigma C_2$  and calls the homotopy fibre  $G_2$ . This process is iterated until  $G_m$  is identified as being homotopy equivalent to  $F^m(\underline{X})$ . At each step in the iteration we may argue as in the previous paragraph to identify the composite  $G_i \rightarrow G_{i-1} \rightarrow \cdots \rightarrow G_1 \rightarrow (\underline{C\Omega X}, \underline{\Omega X})^K \rightarrow \bigvee_{i=1}^m X_i$  as

homotopic to a wedge sum of iterated Whitehead products of the maps  $t_j$ . As each of  $\Omega X_1, \dots, \Omega X_m$  appears as a smash factor in each wedge summand of  $G_m$ , each of the corresponding Whitehead products has  $t_1, \dots, t_m$  appearing at least once. This proves parts (a) and (b).

The naturality statement in part (c) follows from the naturality of Theorems 7.2 and 6.1.  $\square$

When each  $X_i$  equals a common space  $X$  Theorem 8.6 implies the following.

**Corollary 8.7.** *Let  $X$  be a simply-connected, pointed CW-complex. Then the homotopy fibration*

$$F^m(X) \xrightarrow{f^m(X)} \bigvee_{i=1}^m X \longrightarrow P^m(X)$$

satisfies the following:

- (a)  $F^m(X) \simeq \bigvee_{\alpha \in \mathcal{I}} \Sigma(\Omega X)^{(t_\alpha)}$ ;
- (b) the restriction of  $f^m(X)$  to  $\Sigma(\Omega X)^{(t_\alpha)}$  is an iterated Whitehead product of length  $t \geq m$  formed from the maps  $t_j: \Sigma \Omega X \xrightarrow{ev} X \xrightarrow{s_j} \bigvee_{i=1}^m X$  where each  $t_j$  for  $1 \leq j \leq m$  appears at least once;
- (c) parts (a) and (b) are natural for a map of spaces  $X \rightarrow Y$ .

$\square$

We also need a refined version of Theorem 8.6 in the case when each  $X_i$  is a suspension.

**Theorem 8.8.** *Let  $Y_1, \dots, Y_m$  be simply-connected, pointed CW-complexes. Then the homotopy fibration*

$$F^m(\underline{\Sigma Y}) \xrightarrow{f^m(\underline{\Sigma Y})} \bigvee_{i=1}^m \Sigma Y_i \longrightarrow P^m(\underline{\Sigma Y})$$

satisfies the following:

- (a)  $F^m(\underline{\Sigma Y}) \simeq \bigvee_{\beta \in \mathcal{J}} \Sigma(Y_1)^{(\beta_1)} \wedge \dots \wedge (Y_m)^{(\beta_m)}$  where  $\beta_i \geq 1$  for each  $1 \leq i \leq m$ ;
- (b) the restriction of  $f^m(\underline{\Sigma Y})$  to  $\Sigma(Y_1)^{(\beta_1)} \wedge \dots \wedge (Y_m)^{(\beta_m)}$  is an iterated Whitehead product of length  $t \geq m$  formed from the maps  $\Sigma Y_j \xrightarrow{s_j} \bigvee_{i=1}^m \Sigma Y_i$ , where each  $s_j$  for  $1 \leq j \leq m$  appears at least once;
- (c) parts (a) and (b) are natural for maps of spaces  $Y_i \rightarrow Z_i$ .

*Proof.* Argue exactly as for Theorem 8.6 using Theorem 7.4 as the starting point rather than Theorem 7.2, and organizing the  $B_i, C_i$  wedge summands so that  $B_i$  contains all the smash products having  $Y_1, \dots, Y_i$  as factors (rather than  $\Omega \Sigma Y_1, \dots, \Omega \Sigma Y_i$ ) and  $C_i$  contains all the smash products having  $Y_1, \dots, Y_{i-1}$  as factors but not  $Y_i$ .  $\square$

### Part 3. Cocategory and Nilpotence.

#### 9. COCATEGORY AND NILPOTENCE I

The purpose of this section is to prove main result of the paper, Theorem 1.6: if  $X$  is simply-connected then  $\text{wcocat}(X) = m$  if and only if  $\text{nil}(\Omega X) = m$ .

*Proof of Theorem 1.6.* We will use the homotopy fibration

$$(13) \quad F^{m+1}(X) \xrightarrow{f^{m+1}(X)} \bigvee_{i=1}^{m+1} X \longrightarrow P^{m+1}(X)$$

from Corollary 8.7.

Suppose that  $\text{wcocat}(X) = m$ . By the definition of weak cocategory, this implies that the composition  $F^{m+1}(X) \xrightarrow{f^{m+1}(X)} \bigvee_{i=1}^{m+1} X \xrightarrow{\nabla} X$  is null homotopic. Consider the Samelson product  $(\Omega X)^{(m+1)} \xrightarrow{c_m} \Omega X$ , where  $c_m$  is the  $(m+1)$ -fold Samelson product of the identity map on  $\Omega X$  with itself. Let  $w_m: \Sigma(\Omega X)^{(m+1)} \rightarrow X$  be the adjoint of  $c_m$ . Then  $w_m$  is the  $(m+1)$ -fold Whitehead product of the evaluation map  $\Sigma \Omega X \xrightarrow{\text{ev}} X$  with itself. By Theorem 8.3,  $w_m$  factors through  $\nabla \circ f^{m+1}(X)$ . But by hypothesis,  $\nabla \circ f^{m+1}(X)$  is null homotopic. Therefore  $w_m$  is null homotopic and hence its adjoint  $c_m$  is null homotopic. This implies that  $\text{nil}(\Omega X) \leq m$ ; that is,  $\text{nil}(\Omega X) \leq \text{wcocat}(X)$ .

Conversely, suppose that  $\text{nil}(\Omega X) = m$ . By Corollary 8.7,  $F^{m+1}(X) \simeq \bigvee_{\alpha \in \mathcal{I}} \Sigma(\Omega X)^{(t_\alpha)}$  and the restriction of  $f^{m+1}(X)$  to each wedge summand is an iterated Whitehead product of length  $\geq m+1$ . Therefore  $\nabla \circ f^{m+1}(X)$  maps each wedge summand  $\Sigma(\Omega X)^{(t_\alpha)}$  to  $X$  by an iterated Whitehead product of length  $\geq m+1$ . The adjoint  $(\Omega X)^{(t_\alpha)} \rightarrow \Omega X$  is a Samelson product of length  $\geq m+1$ , and so factors through  $c_m$ . By hypothesis,  $c_m$  is null homotopic. Therefore, so is  $(\Omega X)^{(t_\alpha)} \rightarrow \Omega X$  and its adjoint  $\Sigma(\Omega X)^{(t_\alpha)} \rightarrow \Sigma \Omega X$ . This is true for every  $\alpha \in \mathcal{I}$ , so  $\nabla \circ f^{m+1}(X)$  is null homotopic, implying that  $\text{wcocat}(X) \leq m$ ; that is,  $\text{wcocat}(X) \leq \text{nil}(\Omega X)$ .

Combining the previous two paragraphs we obtain  $\text{wcocat}(X) = m$  if and only if  $\text{nil}(\Omega X) = m$ .  $\square$

#### 10. RETRACTILE $H$ -SPACES

We now aim towards explicit calculations of the homotopy nilpotency class of quasi- $p$ -regular exceptional Lie groups and nonmodular  $p$ -compact groups in Section 12. This requires introducing a special family of finite  $H$ -spaces. Throughout this section we will assume that all spaces and maps have been localized (or completed) at a prime  $p$ , and homology is taken with mod- $p$  coefficients. For a  $\mathbb{Z}/p\mathbb{Z}$ -vector space  $V$ , let  $\Lambda(V)$  be the exterior algebra generated by  $V$ .

**Definition 10.1.** Let  $B$  be an  $H$ -space. Suppose that there is a space  $A$  and a map  $i: A \rightarrow B$  satisfying the following:

- (i)  $H_*(B) \cong \Lambda(\tilde{H}_*(A))$ ;
- (ii)  $i_*$  induces the inclusion of the generating set;

(iii)  $\Sigma i$  has a left homotopy inverse.

Then the triple  $(A, i, B)$  is a *retractile*  $H$ -space.

If  $(A, i, B)$  is a retractile  $H$ -space then many properties of  $B$  are determined by the restriction to  $A$  [GT1, Th1, Th2]. We will show this is also the case for  $\text{nil}(G)$  when  $G$  is a finite loop space and  $(A, i, G)$  is retractile. A large family of retractile  $H$ -spaces was constructed in different ways by Cooke, Harper and Zabrodsky [CHZ] and Cohen and Neisendorfer [CN]. They show that if  $p$  is a prime and  $A$  is a finite  $CW$ -complex consisting of  $\ell$  odd dimensional cells, where  $\ell < p - 1$ , then there exists a  $p$ -local  $H$ -space  $B$  and a map  $i: A \rightarrow B$  which makes  $(A, i, B)$  retractile. Further, in the boundary case when  $\ell = p - 1$ , they show that if there happens to be a finite  $H$ -space  $B$  and a map  $A \xrightarrow{i} B$  inducing an isomorphism  $H_*(B) \cong \Lambda(\tilde{H}_*(A))$ , then  $\Sigma i$  has a left homotopy inverse so  $(A, i, B)$  is retractile. Notice that products of retractile  $H$ -spaces are retractile. For if each of  $\{(A_j, i_j, B_j)\}_{j=1}^m$  is retractile then so is  $(\bigvee_{j=1}^m A_j, \bigvee_{j=1}^m i_j, \prod_{j=1}^m B_j)$ .

Specific examples of retractile  $H$ -spaces are given by certain simply-connected simple compact Lie groups and  $p$ -compact groups. Fix a prime  $p$ . By [MNT], when localized at  $p$  every simply-connected, simple compact Lie group  $G$  without  $p$ -torsion in cohomology decomposes as a product of  $k$  spaces for some  $1 \leq k \leq p - 1$  spaces. If the rank is low with respect to  $p$  then by [Th2] the factors are all retractile, and so their product - the original group  $G$  - is also retractile. The precise ranks involved are as follows:

$$\begin{array}{ll}
 SU(n) & \text{if } n \leq (p-1)^2 + 1 \\
 Sp(n) & \text{if } 2n \leq (p-1)^2 \\
 Spin(2n+1) & \text{if } 2n \leq (p-1)^2 \\
 Spin(2n) & \text{if } 2(n-1) \leq (p-1)^2 \\
 G_2, F_4, E_6 & \text{if } p \geq 5 \\
 E_7, E_8 & \text{if } p \geq 7.
 \end{array}
 \tag{14}$$

Recall that  $X$  is a  $p$ -compact group if  $X \simeq \Omega BX$  where  $BX$  is a  $p$ -complete space and the mod- $p$  homology of  $X$  is finite. When  $p$  is odd there is a classification of simply-connected simple  $p$ -compact groups without  $p$ -torsion in cohomology. We give the description in [Da, Introduction], which distills the classification from deep results in [AGMV]. There are four families: (i) simply-connected, simple compact Lie groups; (ii) an infinite family of spaces that decompose as a product of spheres, sphere bundles over spheres, and factors of  $SU(n)$ ; (iii) 30 special nonmodular cases; and (iv) 4 exotic modular cases. The retractile cases in (i) are covered by (14). The retractile cases in (ii) depend on  $SU(n)$  being retractile, and from [Da] it follows that every case in (iii) and (iv) is retractile. So

to (14) we add a second list:

- (15) (a) an infinite family of  $p$ -compact groups that decompose as a product of spheres, sphere bundles over spheres, and factors of  $SU(n)$  for  $n \leq (p-1)^2 + 1$ ;  
 (b) the 30 special nonmodular  $p$ -compact groups (see [Da, Table 3.2]);  
 (c) the 4 exotic modular  $p$ -compact groups.

Since a  $p$ -compact group  $G$  is a loop space, it has a classifying space  $BG$  and there is a homotopy equivalence  $e: G \rightarrow \Omega BG$ . Let  $\overline{ev}$  be the composite

$$\overline{ev}: \Sigma G \xrightarrow{\Sigma e} \Sigma \Omega BG \xrightarrow{ev} BG$$

where  $ev$  is the canonical evaluation map. Let  $j$  be the composite

$$j: \Sigma A \xrightarrow{\Sigma i} \Sigma G \xrightarrow{\overline{ev}} BG.$$

A key property is the following.

**Lemma 10.2.** *Let  $G$  be a  $p$ -compact group from list (14) or (15). Then there is a homotopy commutative diagram*

$$\begin{array}{ccc} \Sigma G & \xrightarrow{\overline{ev}} & BG \\ \downarrow r & & \parallel \\ \Sigma A & \xrightarrow{j} & BG \end{array}$$

where  $r$  is a left homotopy inverse of  $\Sigma i$ .

*Proof.* By hypothesis, the map  $\Sigma A \xrightarrow{\Sigma i} \Sigma G$  has a right homotopy inverse, so  $\Sigma G \simeq \Sigma A \vee C$  for some space  $C$ . In [GT1] the retractile properties were used to precisely describe the complementary factor  $C$  in the case when  $G$  is from the list (14), but the proof adapts immediately to list (15) as well. It is well known that there is a homotopy fibration  $\Sigma G \wedge G \xrightarrow{\mu^*} \Sigma G \xrightarrow{\overline{ev}} BG$  where  $\mu^*$  is the canonical Hopf construction. In [GT1] it is shown there is a homotopy equivalence

$$(16) \quad \Sigma A \vee C \xrightarrow{\Sigma i + s} \Sigma G$$

where  $C$  is a retract of  $\Sigma G \wedge G$  and  $s$  factors through the Hopf construction  $\mu^*$ . Consequently,  $s$  composes trivially with  $\overline{ev}$ , so from the decomposition in (16) we obtain a homotopy commutative diagram

$$\begin{array}{ccc} \Sigma G & \xrightarrow{\overline{ev}} & BG \\ \downarrow r & & \parallel \\ \Sigma A & \xrightarrow{j} & BG \end{array}$$

where  $r$  is a left homotopy inverse of  $\Sigma i$ . □

For a space  $Y$ , let  $E: Y \rightarrow \Omega\Sigma Y$  be the suspension map, which is defined as the (right) adjoint of the identity map on  $\Sigma Y$ . Notice that the evaluation map  $\Sigma\Omega Y \xrightarrow{ev} Y$  is the (left) adjoint of the identity map on  $\Omega Y$ . So the composite  $\Omega Y \xrightarrow{E} \Omega\Sigma\Omega Y \xrightarrow{\Omega ev} \Omega Y$  is homotopic to the identity map. This leads to the following corollary of Lemma 10.2.

**Corollary 10.3.** *The composite  $G \xrightarrow{E} \Omega\Sigma G \xrightarrow{\Omega r} \Omega\Sigma A \xrightarrow{\Omega j} \Omega BG$  is homotopic to the homotopy equivalence  $e$ .*

*Proof.* By Lemma 10.2,  $j \circ r \simeq \bar{ev} = ev \circ \Sigma e$ . Thus  $\Omega j \circ \Omega r \circ E \simeq \Omega ev \circ \Omega \Sigma e \circ E$ . The naturality of  $E$  implies that the latter composite is homotopic to  $\Omega ev \circ E \circ e$ . But  $\Omega ev \circ E$  is homotopic to the identity map on  $\Omega BG$ , so we obtain  $\Omega j \circ \Omega r \circ E \simeq e$ , as asserted.  $\square$

## 11. COCATEGORY AND NILPOTENCE II

In this section a restricted version of Theorem 1.6 is proved.

**Definition 11.1.** Let  $G$  be an  $H$ -group. Suppose that  $Y$  is a pointed, path-connected space and there is a map  $f: Y \rightarrow G$ . We say that  $G$  has *homotopy nilpotency class  $m$  with respect to  $f$*  if the composite  $Y^{(m+1)} \xrightarrow{f^{(m+1)}} G^{(m+1)} \xrightarrow{c_m} G$  is null homotopic but  $c_{m-1} \circ f^{(m)}$  is nontrivial. In this case we write  $\text{nil}(Y, f, G) = m$ .

**Definition 11.2.** Let  $X$  and  $Y$  be pointed, path-connected spaces, and suppose that there is a map  $g: Y \rightarrow X$ . We say that  $X$  has *weak cocategory  $m$  with respect to  $g$*  if the composite  $F_{m+1}(Y) \xrightarrow{f_{m+1}(Y)} \bigvee_{i=1}^{m+1} Y \xrightarrow{\nabla} Y \xrightarrow{g} X$  is null homotopic but  $g \circ \nabla \circ f_m(Y)$  is nontrivial. In this case we write  $\text{wccat}(Y, g, X) = m$ .

**Remark 11.3.** Definition 11.2 is dual to the weak category of a map (see [CLOT, Exercise 2.8]).

**Theorem 11.4.** *Let  $G$  be a  $p$ -compact group from list (14) or (15). Then  $\text{wccat}(\Sigma A, j, BG) = m$  if and only if  $\text{nil}(A, i, G) = m$ .*

*Proof.* The naturality of the thin product and the naturality of the decomposition of  $F^{m+1}(X)$  in Corollary 8.7 implies that there is a homotopy fibration diagram

$$\begin{array}{ccccc} F^{m+1}(\Sigma A) & \xrightarrow{f^{m+1}(\Sigma A)} & \bigvee_{i=1}^{m+1} \Sigma A & \longrightarrow & P^{m+1}(\Sigma A) \\ \downarrow g^{m+1}(j) & & \downarrow \bigvee_{i=1}^{m+1} j & & \downarrow P^{m+1}(j) \\ F_{m+1}(BG) & \xrightarrow{f^{m+1}(BG)} & \bigvee_{i=1}^{m+1} BG & \longrightarrow & P^{m+1}(BG) \end{array}$$

where  $F^{m+1}(\Sigma A) \simeq \bigvee_{\alpha \in \mathcal{I}} \Sigma(\Omega\Sigma A)^{(t_\alpha)}$ ,  $F^{m+1}(BG) \simeq \bigvee_{\alpha \in \mathcal{I}} \Sigma(\Omega BG)^{(t_\alpha)}$ , and  $g^{m+1}(j) \simeq \bigvee_{\alpha \in \mathcal{I}} \Sigma(\Omega j)^{(t_\alpha)}$ .

Suppose that  $\text{wccat}(\Sigma A, j, BG) = m$ . Then the composite  $F^{m+1}(\Sigma A) \xrightarrow{f^{m+1}(\Sigma A)} \bigvee_{i=1}^{m+1} \Sigma A \xrightarrow{\nabla} \Sigma A \xrightarrow{j} BG$  is null homotopic. Consider the Samelson product  $A^{(m+1)} \xrightarrow{i^{(m+1)}} G^{(m+1)} \xrightarrow{c_m} G$ .



Its adjoint is the composite  $\Sigma A^{(m+1)} \xrightarrow{\Sigma i^{(m+1)}} \Sigma G^{(m+1)} \xrightarrow{w_m} BG$ , where  $w_m$  is the  $(m+1)$ -fold Whitehead product of  $\bar{e}\bar{v}$  with itself, which is adjoint to  $c_m$ .

We claim that there is a lift

$$(17) \quad \begin{array}{ccc} & & F^{m+1}(\Sigma A) \\ & \nearrow & \downarrow f^{m+1}(\Sigma A) \\ & & \bigvee_{i=1}^{m+1} \Sigma A \\ & \nearrow & \downarrow j \circ \nabla \\ \Sigma A^{(m+1)} & \xrightarrow{\Sigma i^{(m+1)}} \Sigma G^{(m+1)} \xrightarrow{w_m} BG. \end{array}$$

The lift is constructed in two stages. First, we give a specific lift of  $w_m \circ \Sigma i^{(m+1)}$  through  $j \circ \nabla$ . Let  $s'_j: \Sigma A \rightarrow \bigvee_{i=1}^{m+1} \Sigma A$  and  $s_j: BG \rightarrow \bigvee_{i=1}^{m+1} BG$  be the inclusions of the  $j^{\text{th}}$ -wedge summands, and let  $t'_j$  and  $t_j$  be the composites  $t'_j: \Sigma \Omega \Sigma A \xrightarrow{ev} \Sigma A \xrightarrow{s'_j} \bigvee_{i=1}^{m+1} \Sigma A$  and  $t_j: \Sigma G \xrightarrow{\bar{e}\bar{v}} BG \xrightarrow{s_j} \bigvee_{i=1}^{m+1} BG$ . Let  $\bar{w}'_m = [t'_1, [t'_2, \dots, [t'_m, t'_{m+1}] \dots]]$  and  $\bar{w}_m = [t_1, [t_2, \dots, [t_m, t_{m+1}] \dots]]$ . Since  $G \simeq \Omega BG$ , for convenience we write  $\Omega \Sigma A \xrightarrow{\Omega j} \Omega BG$  as  $\Omega \Sigma A \xrightarrow{\Omega j} G$ . Consider the diagram

$$(18) \quad \begin{array}{ccccccc} \Sigma A^{(m+1)} & \xrightarrow{\Sigma E^{(m+1)}} & \Sigma \Omega \Sigma A^{(m+1)} & \xrightarrow{\bar{w}'_m} & \bigvee_{i=1}^{m+1} \Sigma A & \xrightarrow{\nabla} & \Sigma A \\ & \searrow \Sigma i^{(m+1)} & \downarrow \Sigma(\Omega j)^{(m+1)} & & \downarrow \bigvee_{i=1}^{m+1} j & & \downarrow j \\ & & \Sigma G^{(m+1)} & \xrightarrow{\bar{w}_m} & \bigvee_{i=1}^{m+1} BG & \xrightarrow{\nabla} & BG. \end{array}$$

The left triangle homotopy commutes since  $i \simeq j \circ E$ . The middle square homotopy commutes by the naturality of the Whitehead product. The right square homotopy commutes by the naturality of the fold map. Thus the entire diagram homotopy commutes. Observe that since  $\Sigma A \xrightarrow{\Sigma E} \Sigma \Omega \Sigma A \xrightarrow{ev} \Sigma A$  is homotopic to the identity map, we have  $t'_j \circ E \simeq s'_j$ . Thus the composite  $\bar{w}'_m \circ \Sigma E^{(m+1)}$  along the top row in (18) homotopic to  $[s'_1, [s'_2, \dots, [s'_m, s'_{m+1}] \dots]]$ . On the other hand, since  $w_m \simeq \nabla \circ \bar{w}_m$ , the bottom direction around (18) is homotopic to  $w_m \circ \Sigma i^{(m+1)}$ . Thus the homotopy commutativity of (18) implies that  $[s'_1, [s'_2, \dots, [s'_m, s'_{m+1}] \dots]]$  lifts  $w_m \circ \Sigma i^{(m+1)}$  through  $j \circ \nabla$ . Further, as  $\bar{w}'_m$  is an  $(m+1)$ -fold Whitehead product involving all  $m+1$  spaces in the wedge  $\bigvee_{i=1}^{m+1} \Sigma A$ , Theorem 8.1 implies that  $\bar{w}'_m$  lifts through  $f^{m+1}(\Sigma A)$  to  $F^{m+1}(\Sigma A)$ .

By hypothesis,  $\text{wocat}(\Sigma A, j, BG) = m$ , so the composite  $F^{m+1}(\Sigma A) \xrightarrow{f^{m+1}(\Sigma A)} \bigvee_{i=1}^{m+1} \Sigma A \xrightarrow{\nabla} \Sigma A \xrightarrow{j} BG$  is null homotopic. Therefore the lift of  $w_m \circ \Sigma i^{(m+1)}$  through  $\nabla \circ j \circ f^{m+1}(\Sigma A)$  in (17) implies that  $w_m \circ \Sigma i^{(m+1)}$  is null homotopic. Taking adjoints,  $w_m \circ i^{(m+1)}$  is null homotopic, and so  $\text{nil}(A, i, G) \leq m$ . That is,  $\text{nil}(A, i, G) \leq \text{wocat}(\Sigma A, j, BG)$ .

Conversely, suppose that  $\text{nil}(A, i, G) \leq m$ . Then the composite  $A^{(m+1)} \xrightarrow{i^{(m+1)}} G^{(m+1)} \xrightarrow{c_m} G$  is null homotopic. Taking adjoints, the composite  $\Sigma A^{(m+1)} \xrightarrow{\Sigma i^{(m+1)}} \Sigma G^{(m+1)} \xrightarrow{w_m} BG$  is null homotopic. Consider the composite  $F^{m+1}(\Sigma A) \xrightarrow{f^{m+1}(\Sigma A)} \bigvee_{i=1}^{m+1} \Sigma A \xrightarrow{\nabla} \Sigma A \xrightarrow{j} BG$ . By Corollary 8.7 we have  $F^{m+1}(\Sigma A) \simeq \bigvee_{\gamma \in \mathcal{J}_{m+1}} \Sigma A^{(t_\gamma)}$  and  $f^{m+1}(\Sigma A)$  restricted to  $\Sigma A^{(t_\gamma)}$  is an iterated Whitehead

product  $w_{t_\gamma}$  of length  $\geq m+1$  formed from the inclusions  $\Sigma A \hookrightarrow \bigvee_{i=1}^{m+1} \Sigma A$  of the wedge summands. Thus each  $w_{t_\gamma}$  factors through  $w_m \circ \Sigma i^{(m+1)}$ , which by hypothesis is null homotopic. Therefore  $w_{t_\gamma}$  is null homotopic. As this is true for every wedge summand of  $F^{m+1}(\Sigma A)$ , the map  $f^{m+1}(\Sigma A)$  is null homotopic. Thus  $j \circ \nabla \circ f^{m+1}(\Sigma A)$  is null homotopic, implying that  $\text{wccat}(\Sigma A, j, BG) \leq m$ . That is,  $\text{wccat}(\Sigma A, j, BG) \leq \text{nil}(A, i, G)$ .

Hence  $\text{wccat}(\Sigma A, j, BG) = m$  if and only if  $\text{nil}(A, i, G) = m$ .  $\square$

The next theorem links the equivalences in Theorems 1.6 and 11.4.

**Theorem 11.5.** *Let  $G$  be a  $p$ -compact group from list (14) or (15). Then  $\text{wccat}(BG) = m$  if and only if  $\text{wccat}(\Sigma A, j, BG) = m$ .*

*Proof.* In general, by Corollary 8.7, for a space  $X$  there is a homotopy fibration

$$(19) \quad F^{m+1}(X) \xrightarrow{f^{m+1}(X)} \bigvee_{i=1}^{m+1} X \longrightarrow P^m(X)$$

where the right map is the inclusion of the wedge into the thin product;  $F^{m+1}(X) \simeq \bigvee_{\alpha \in \mathcal{I}} \Sigma(\Omega X)^{(t_\alpha)}$ ; and  $f^{m+1}(X)$  is a wedge sum of iterated Whitehead products of maps of the form  $\Sigma \Omega X \xrightarrow{ev} X \xrightarrow{s_j} \bigvee_{i=1}^{m+1} X$ . Moreover, all of this is natural for maps of spaces  $X \rightarrow Y$ . Recall that  $(A, i, G)$  being retractile means that there is a map  $\Sigma G \xrightarrow{r} \Sigma A$  which is a left homotopy inverse of  $\Sigma i$ . Starting from the composite  $\Sigma G \xrightarrow{r} \Sigma A \xrightarrow{j} BG$ , consider the diagram

$$(20) \quad \begin{array}{ccccc} \bigvee_{\alpha \in \mathcal{I}} \Sigma G^{(t_\alpha)} & \xrightarrow{\quad \simeq \quad} & \bigvee_{\alpha \in \mathcal{I}} \Sigma(\Omega BG)^{(t_\alpha)} & & \\ \downarrow \bigvee_{\alpha \in \mathcal{I}} \Sigma E^{(t_\alpha)} & & \parallel & & \\ \bigvee_{\alpha \in \mathcal{I}} \Sigma(\Omega \Sigma G)^{(t_\alpha)} & \xrightarrow{\bigvee_{\alpha \in \mathcal{I}} \Sigma(\Omega r)^{(t_\alpha)}} & \bigvee_{\alpha \in \mathcal{I}} \Sigma(\Omega \Sigma A)^{(t_\alpha)} & \xrightarrow{\bigvee_{\alpha \in \mathcal{I}} \Sigma(\Omega j)^{(t_\alpha)}} & \bigvee_{\alpha \in \mathcal{I}} \Sigma(\Omega BG)^{(t_\alpha)} \\ \downarrow \simeq & & \downarrow \simeq & & \downarrow \simeq \\ F^{m+1}(\Sigma G) & \longrightarrow & F^{m+1}(\Sigma A) & \longrightarrow & F^{m+1}(BG) \\ \downarrow f^{m+1}(\Sigma G) & & \downarrow f^{m+1}(\Sigma A) & & \downarrow f^{m+1}(BG) \\ \bigvee_{i=1}^{m+1} \Sigma G & \xrightarrow{\bigvee_{i=1}^{m+1} r} & \bigvee_{i=1}^{m+1} \Sigma A & \xrightarrow{\bigvee_{i=1}^{m+1} j} & \bigvee_{i=1}^{m+1} BG \\ \downarrow \nabla & & \downarrow \nabla & & \downarrow \nabla \\ \Sigma G & \xrightarrow{\quad r \quad} & \Sigma A & \xrightarrow{\quad j \quad} & BG. \end{array}$$

The bottom squares homotopy commute by the naturality of the fold map, the middle and top squares homotopy commute by the naturality of (19) and the naturality of the decomposition of  $F^{m+1}(X)$ . By Corollary 10.3,  $\Omega j \circ \Omega r \circ E$  is a homotopy equivalence, so the top rectangle also homotopy commutes.

Suppose that  $\text{wccat}(BG) = m$ . Then the composite  $F^{m+1}(BG) \xrightarrow{f^{m+1}(BG)} \bigvee_{i=1}^{m+1} BG \xrightarrow{\nabla} BG$  is null homotopic. The homotopy commutativity of (20) therefore implies that the composite  $F^{m+1}(\Sigma A) \xrightarrow{f^{m+1}(\Sigma A)} \bigvee_{i=1}^{m+1} \Sigma A \xrightarrow{\nabla} \Sigma A \xrightarrow{j} BG$  is null homotopic. Thus  $\text{wccat}(\Sigma A, j, BG) \leq m$ .

Suppose that  $\text{wocat}(\Sigma A, j, BG) = m$ . Then the composite  $F^{m+1}(\Sigma A) \xrightarrow{f^{m+1}(\Sigma A)} \bigvee_{i=1}^{m+1} \Sigma A \xrightarrow{\nabla} \Sigma A \xrightarrow{j} BG$  is null homotopic. The homotopy commutativity of (20) therefore implies that the upper direction around the diagram is null homotopic. But as the top row is a homotopy equivalence, this implies that the composite  $F^{m+1}(BG) \xrightarrow{f^{m+1}(BG)} \bigvee_{i=1}^{m+1} BG \xrightarrow{\nabla} BG$  is null homotopic. That is,  $\text{wocat}(BG) \leq m$ .

Hence  $\text{wocat}(BG) = m$  if and only if  $\text{wocat}(\Sigma A, j, BG) = m$ .  $\square$

Combining the equivalences in Theorems 1.6, 11.4 and 11.5 we obtain the following.

**Theorem 11.6.** *Let  $G$  be a  $p$ -compact group from (14) or (15). Then  $\text{nil}(G) = m$  if and only if  $\text{nil}(A, i, G) = m$ .*  $\square$

This reduction from checking whether  $G^{(m+1)} \xrightarrow{c_m} G$  is null homotopic to checking whether  $A^{(m+1)} \xrightarrow{i^{(m+1)}} G^{(m+1)} \xrightarrow{c_m} G$  is null homotopic gives a very practical tool for determining the homotopy nilpotency class of certain finite loop spaces. This will be used in the next section to give explicit calculations.

## 12. EXAMPLES

Fix an odd prime  $p$ . Let  $G$  be a  $p$ -localized simply-connected simple compact Lie group, or in the  $p$ -complete setting, let  $G_p^\wedge$  be a  $p$ -compact group. In the  $p$ -local case,  $G$  is rationally homotopy equivalent to a product of odd dimensional spheres  $\prod_{i=1}^\ell S^{2n_i-1}$ , and in the  $p$ -complete case,  $G_p^\wedge$  is rationally homotopy equivalent to the rationalization of a product of odd dimensional  $p$ -completed spheres  $\prod_{i=1}^\ell (S^{2n_i-1})_p^\wedge$ . Reordering the indices if need be, we may assume that  $n_1 \leq \dots \leq n_\ell$ . The *type* of  $G$  is  $\{n_1, \dots, n_\ell\}$ . From now on we drop the usual  $(\ )_p^\wedge$  notation for  $p$ -completion for convenience, the context making it clear when it should be used. In the  $p$ -local ( $p$ -complete) case we say that  $G$  is  *$p$ -regular* if there is a  $p$ -local (or  $p$ -complete) homotopy equivalence  $G \simeq \prod_{i=1}^\ell S^{2n_i-1}$ . It is *quasi- $p$ -regular* if there is a  $p$ -local (or  $p$ -complete) homotopy equivalence  $G \simeq \prod_{i=1}^\ell B_i$ , where each  $B_i$  is either a sphere or a sphere bundle over a sphere.

Kaji and Kishimoto [KK] have determined the homotopy nilpotency classes of all  $p$ -regular  $p$ -compact groups. Kishimoto [K] determined the homotopy nilpotency class of all quasi- $p$ -regular cases of  $SU(n)$ . In both cases the calculations were intense and involved detailed information about nontrivial Samelson products on the group in question. In contrast, very little is known about nontrivial Samelson products in exceptional Lie groups, so one would ordinarily not hope to be able to determine their homotopy nilpotency classes. Nevertheless, in Theorem 12.1 we will calculate the homotopy nilpotency class in most quasi- $p$ -regular cases.

The sphere bundles over spheres that appear as factors have a particular form, which we describe first. Let  $\pi_m^S(S^n)$  be the  $m^{\text{th}}$ -stable homotopy group of  $S^n$ . The least nonvanishing  $p$ -torsion homotopy group of  $S^3$  is  $\pi_{2p}(S^3) \cong \mathbb{Z}/p\mathbb{Z}$ . Let  $\alpha: S^{2p} \rightarrow S^3$  represent a generator. This map

is stable and it represents the generator in  $\pi_{n+2p-3}^S(S^n) \cong \mathbb{Z}/p\mathbb{Z}$ . For  $n \geq 1$ , define the space  $B(2n+1, 2n+2p-1)$  by the homotopy pullback

$$\begin{array}{ccccc} S^{2n+1} & \longrightarrow & B(2n+1, 2n+2p-1) & \longrightarrow & S^{2n+2p-1} \\ \parallel & & \downarrow & & \downarrow \alpha \\ S^{2n+1} & \longrightarrow & S^{4n+3} & \xrightarrow{w} & S^{2n+2} \end{array}$$

where  $w$  is the Whitehead product of the identity map with itself.

Restrict to  $p \geq 7$  and consider the exceptional simple compact Lie groups which are quasi- $p$ -regular but not  $p$ -regular. By [MNT] a complete list is as follows:

$$(21) \quad \begin{array}{ll} F_4 & p=7 \quad B(3, 15) \times B(11, 23) \\ & p=11 \quad B(3, 23) \times S^{11} \times S^{15} \\ E_6 & p=7 \quad F_4 \times S^9 \times S^{17} \\ & p=11 \quad F_4 \times S^9 \times S^{17} \\ E_7 & p=11 \quad B(3, 23) \times B(15, 35) \times S^{11} \times S^{19} \times S^{27} \\ & p=13 \quad B(3, 27) \times B(11, 35) \times S^{15} \times S^{19} \times S^{23} \\ & p=17 \quad B(3, 35) \times S^{11} \times S^{15} \times S^{19} \times S^{23} \times S^{27} \\ E_8 & p=11 \quad B(3, 23) \times B(15, 35) \times B(27, 47) \times B(39, 59) \\ & p=13 \quad B(3, 27) \times B(15, 39) \times B(23, 47) \times B(35, 59) \\ & p=17 \quad B(3, 35) \times B(15, 47) \times B(27, 59) \times S^{23} \times S^{39} \\ & p=19 \quad B(3, 39) \times B(23, 59) \times S^{15} \times S^{27} \times S^{35} \times S^{47} \\ & p=23 \quad B(3, 47) \times B(15, 59) \times S^{23} \times S^{27} \times S^{35} \times S^{39} \\ & p=29 \quad B(3, 59) \times S^{15} \times S^{23} \times S^{27} \times S^{35} \times S^{39} \times S^{47}. \end{array}$$

McGibbon [M] showed that none of the Lie groups listed in (21) is homotopy commutative. Thus, in every case,  $\text{nil}(G) \geq 2$ .

Complete at  $p \geq 7$  and consider the nonmodular  $p$ -compact groups in cases 4 through 34 of the Shepard-Todd numbering which are quasi- $p$ -regular but not  $p$ -regular. We exclude case 28 which

is  $F_4$ . As in [Da], a complete list is given by:

	<i>Case</i>	<i>Prime</i>	<i>Space</i>
	5	7	$B(11, 23)$
	9	17	$B(15, 47)$
	10	13	$B(23, 47)$
	14	19	$B(11, 47)$
	16	11	$B(39, 59)$
	17	41	$B(39, 119)$
	18	31	$B(59, 119)$
	20	19	$B(23, 59)$
	24	11	$B(7, 27) \times S^{11}$
	25	7	$B(11, 23) \times S^{17}$
	26	13	$B(11, 35) \times S^{23}$
(22)	27	19	$B(23, 59) \times S^{13}$
	29	13	$B(15, 39) \times S^7 \times S^{23}$
	29	17	$B(7, 39) \times S^{15} \times S^{23}$
	30	11	$B(3, 23) \times B(39, 59)$
	30	19	$B(3, 39) \times S^{27} \times S^{59}$
	30	29	$B(3, 59) \times S^{23} \times S^{39}$
	31	13	$B(15, 39) \times B(23, 47)$
	31	17	$B(15, 47) \times S^{23} \times S^{39}$
	32	13	$B(23, 47) \times B(35, 59)$
	32	19	$B(23, 59) \times S^{35} \times S^{47}$
	33	13	$B(11, 35) \times S^7 \times S^{19} \times S^{23}$
	34	31	$B(23, 83) \times S^{11} \times S^{35} \times S^{47} \times S^{59}$
	34	37	$B(11, 83) \times S^{23} \times S^{35} \times S^{47} \times S^{59}$

Saumell [Sa] showed that the only groups in (22) which are homotopy commutative are:  $B(15, 47)$  at  $p = 17$ ,  $B(11, 47)$  at  $p = 19$ ,  $B(39, 119)$  at  $p = 41$ ,  $B(23, 59)$  at  $p = 19$ , and  $B(7, 27) \times S^{11}$  at  $p = 11$ . Otherwise, the groups are not homotopy commutative, in which case  $\text{nil}(G) \geq 2$ .

We now state the main result of this section, to be proved following several preparatory lemmas.

**Theorem 12.1.** *Let  $p \geq 7$  and let  $G$  be a quasi- $p$ -regular  $p$ -compact group from (21) or (22). If  $G$  is not homotopy commutative then  $\text{nil}(G) = 2$ .*

We begin with some properties of the space  $B(2n+1, 2n+2p-1)$ , which has been well studied. It is a three-cell complex whose mod- $p$  homology is  $H_*(B(2n+1, 2n+2p-1)) \cong \Lambda(x_{2n+1}, x_{2n+2p-1})$ ,

where  $|x_t| = t$ . Let  $A(2n+1, 2n+2p-1)$  be the  $(2n+2p-1)$ -skeleton of  $B$ . Then

$$H_*(B(2n+1, 2n+2p-1)) \cong \Lambda(\tilde{H}_*(A(2n+1, 2np+1)))$$

and the skeletal inclusion  $i: A(2n+1, 2n+2p-1) \rightarrow B(2n+1, 2n+2p-1)$  induces the inclusion of the generating set in homology. By [CHZ, CN],  $\Sigma i$  has a right homotopy inverse. Thus we obtain the following.

**Lemma 12.2.** *For  $n \geq 1$ ,  $(A(2n+1, 2n+2p-1), i, B(2n+1, 2n+2p-1))$  is a retractile triple.  $\square$*

Let  $G$  be a quasi- $p$ -regular  $p$ -compact group such that  $G \simeq \prod_{i=1}^{\ell} B_i$ , where each  $B_i$  is a sphere or space of the form  $B(2n_i+1, 2n_i+2p-1)$ . If  $B_i = S^{2n_i-1}$  let  $A_i = S^{2n_i-1}$  and let  $I_i: A_i \rightarrow B_i$  be the identity map. If  $B_i = B(2n_i+1, 2n_i+2p-1)$  let  $A_i = A(2n_i+1, 2n_i+2p-1)$ . Let  $A = \bigvee_{i=1}^{\ell} A_i$  and let  $I: A \rightarrow B$  be the wedge sum of the maps  $I_i$  for  $1 \leq i \leq \ell$ . The retractile property in Lemma 12.2 implies that  $(A, I, G)$  is also retractile. Referring to  $G$  itself as being retractile, this is stated as follows.

**Lemma 12.3.** *Let  $G$  be a quasi- $p$ -regular  $p$ -compact group which is a product of spheres and spaces of the form  $B(2n+1, 2n+2p-1)$ . Then  $G$  is retractile.  $\square$*

Some elementary information about the homotopy groups of  $G$  is also required. This requires a few preliminary lemmas.

**Lemma 12.4.** *Let  $X$  be a finite CW-complex with cells in dimensions  $\{m_1, \dots, m_s\}$ . Let  $Y$  be a space with the property that  $\pi_{m_i}(Y) = 0$  for  $1 \leq i \leq s$ . Then any map  $f: X \rightarrow Y$  is null homotopic.*

*Proof.* For a positive integer  $t$ , let  $X_t$  be the  $t$ -skeleton of  $X$ . Since  $X$  has cells in dimensions  $\{m_1, \dots, m_s\}$ , for  $1 \leq i \leq s$  there are cofibration sequences

$$\bigvee S^{m_i-1} \xrightarrow{f_i} X_{m_{i-1}} \rightarrow X_{m_i} \xrightarrow{q_i} \bigvee S^{m_i}$$

for attaching maps  $f_i$  and pinch maps  $q_i$ , where  $X_{m_0}$  is the basepoint.

Clearly the restriction of  $X \xrightarrow{f} Y$  to  $X_{m_0}$  is null homotopic. Suppose inductively that the restriction of  $f$  to  $X_{m_{i-1}}$  is null homotopic. Then from the cofibration  $X_{m_{i-1}} \rightarrow X_{m_i} \xrightarrow{q_i} \bigvee S^{m_i}$  we see that the restriction of  $f$  to  $X_{m_i}$  factors through  $q_i$  to a map  $e_i: \bigvee S^{m_i} \rightarrow Y$ . But as  $\pi_{m_i}(Y) = 0$ , the map  $e_i$  is null homotopic. Therefore the restriction of  $f$  to  $X_{m_i}$  is null homotopic. By induction, the restriction of  $f$  to  $X_{m_s} = X$  - that is,  $f$  itself - is null homotopic.  $\square$

A space  $B$  is spherically resolved by spheres  $S^{n_1}, \dots, S^{n_t}$  if for  $1 \leq j < t$  there are homotopy fibrations

$$S^{n_j} \rightarrow B_j \rightarrow B_{j+1}$$

where  $B_1 = B$  and  $B_t = S^{n_t}$ . For example,  $SU(n)$  is spherically resolved by  $S^3, S^5, \dots, S^{2n-1}$ . From Lemma 12.4 we obtain the following refinement in the case of spherically resolved spaces.

**Lemma 12.5.** *Let  $X$  be a finite CW-complex with cells in dimensions  $\{m_1, \dots, m_s\}$ . Let  $B$  be spherically resolved by spheres  $S^{n_1}, \dots, S^{n_t}$ . If  $\pi_{m_i}(S^{n_j}) = 0$  for all  $1 \leq i \leq s$  and  $1 \leq j \leq t$ , then any map  $f: X \rightarrow B$  is null homotopic.*

*Proof.* Proceed by downward induction on  $t$ . When  $t = n$ , as  $B_t = S^{n_t}$ , the hypotheses immediately imply that  $\pi_{m_i}(B_t) = 0$  for  $1 \leq i \leq s$ . If  $j < t$ , suppose inductively that  $\pi_{m_i}(B_{j+1}) = 0$  for  $1 \leq i \leq s$ . Then as  $\pi_{m_i}(S^{n_j}) = 0$  for  $1 \leq i \leq s$ , the long exact sequence of homotopy groups induced by the homotopy fibration  $S^{n_j} \rightarrow B_j \rightarrow B_{j+1}$  implies that  $\pi_{m_i}(B_j) = 0$  for  $1 \leq i \leq s$ . As  $B = B_1$ , by induction we have  $\pi_{m_i}(B) = 0$  for  $1 \leq i \leq s$ . Lemma 12.4 then implies that any map  $X \xrightarrow{f} B$  is null homotopic.  $\square$

A special case is  $B(3, 2p+1)$ . This can be regarded as the homotopy fibre of the map  $\bar{\alpha}: S^{2p+1} \rightarrow BS^3$  which is adjoint to  $S^{2p} \xrightarrow{\alpha} S^3$ . Since  $\alpha$  has order  $p$ , so does its adjoint, and therefore  $\bar{\alpha} \circ p$  is null homotopic, implying that the degree  $p$  map on  $S^{2p+1}$  lifts to the homotopy fibre of  $\bar{\alpha}$ . That is, there is a map  $c: S^{2p+1} \rightarrow B(3, 2p+1)$  with the property that the composite  $S^{2p+1} \xrightarrow{c} B(3, 2p+1) \rightarrow S^{2p+1}$  is of degree  $p$ . For a space  $X$ , let  $X\langle 3 \rangle$  be the three-connected cover of  $X$ . Notice that the map  $c$  lifts to a map  $\bar{c}: S^{2p+1} \rightarrow B(3, 2p+1)\langle 3 \rangle$ . The following property was proved by Toda [To1].

**Lemma 12.6.** *The map  $S^{2p+1} \xrightarrow{\bar{c}} B(3, 2p+1)\langle 3 \rangle$  is  $(2p^2 - 2)$ -connected. In particular,  $\bar{c}$  induces an isomorphism on the homotopy groups  $\pi_m$  for  $m \leq 2p^2 - 2$ .*  $\square$

**Proposition 12.7.** *Let  $p$  be an odd prime. Let  $G$  be a quasi- $p$ -regular  $p$ -compact group which is a product of spheres and spaces of the form  $B(2n+1, 2n+2p-1)$ . Suppose that either:*

- (i)  $G$  has type  $\{n_1, \dots, n_\ell\}$  where  $3n_\ell < \min\{n_1p, n_1 + p^2 - p\}$ ; or
- (ii)  $G \simeq B(3, 2p+1) \times Y$  where  $Y$  is as in part (i) and its type also satisfies  $2 < n_1 \leq p$  and  $p < n_\ell$ .

*Then  $\text{nil}(G) \leq 2$ .*

*Proof. Part (i):* By Lemma 12.3, the hypotheses on  $G$  imply that there is a retractile triple  $(A, i, G)$ . By Theorem 11.6, to show that  $\text{nil}(G) \leq 2$  it suffices to show that the composite  $A \wedge A \wedge A \xrightarrow{i \wedge i \wedge i} G \wedge G \wedge G \xrightarrow{c_2} G$  is null homotopic. We will show more, that any map  $f: A \wedge A \wedge A \rightarrow G$  is null homotopic.

Since  $G$  is quasi- $p$ -regular of type  $\{n_1, \dots, n_\ell\}$ , it is spherically resolved by the odd dimensional spheres  $S^{2n_1-1}, \dots, S^{2n_\ell-1}$ . The type of  $G$  also implies that  $A$  has a CW-structure with  $\ell$  cells, in dimensions  $\{2n_1 - 1, \dots, 2n_\ell - 1\}$ . So  $A \wedge A \wedge A$  has a CW-structure consisting of cells in dimensions  $\{m_1, \dots, m_s\}$  where each  $m_i$  is odd. Therefore, by Lemma 12.5, if  $\pi_m(S^{2n_i-1}) = 0$  for each  $m \in \{m_1, \dots, m_s\}$  and each  $1 \leq i \leq \ell$ , then  $f$  is null homotopic.

The dimension of  $A \wedge A \wedge A$  is  $6n_\ell - 3$ , so we consider  $\pi_m(S^{2n_i-1})$  for  $m$  an odd number with  $m \leq 6n_\ell - 3$ . Observe that the stable range for  $S^{2n_i-1}$  is  $2n_i p - 4$ . That is,  $\pi_m(S^{2n_i-1}) \cong \pi_m^S(S^{2n_i-1})$  if  $m < 2n_i p - 3$ . Since  $n_1 \leq \dots \leq n_\ell$ ,  $S^{2n_i-1}$  has the stable range of least dimension. So if  $6n_\ell - 3 < 2n_1 p - 3$ , then  $\pi_m(S^{2n_i-1})$  is stable for every  $m \in \{m_1, \dots, m_s\}$  and every  $1 \leq i \leq \ell$ . This inequality is equivalent to  $3n_\ell < n_1 p$ , which is one of the hypotheses. Therefore, we need only consider the stable homotopy groups  $\pi_m^S(S^{2n_i-1})$ .

Since  $m$  is odd, the stable homotopy group  $\pi_m^S(S^{2n_i-1})$  is in an even stem. By [To2], the even stable stem of least dimension is  $2p(p-1)-2$  (the stem of the stable generator  $\beta_1$ ). Thus  $\pi_m^S(S^{2n_i-1}) = 0$  for odd  $m$  whenever  $m < (2n_i - 1) + 2p(p-1) - 2$ . As  $n_1 \leq \dots \leq n_\ell$  and  $m \leq 6n_\ell - 3$ , we have  $\pi_m^S(S^{2n_i-1}) = 0$  whenever  $6n_\ell - 3 < (2n_1 - 1) + 2p(p-1) - 2$ . This inequality is equivalent to  $3n_\ell < n_1 + p^2 - p$ , which is one of the hypotheses.

Hence  $\pi_m(S^{2n_i-1}) = 0$  for each  $m \in \{m_1, \dots, m_s\}$  and for each  $1 \leq i \leq n$ , and the proof of part (i) is complete.

*Part (ii):* Since  $G \simeq B(3, 2p+1) \times Y$  with  $Y$  as in part (i), by Lemma 12.3 there is a retractile triple  $(A, i, G)$  where  $A = A_1 \vee A_2$  for retractile triples  $(A_1, i_1, B(3, 2p+1))$  and  $(A_2, i_2, Y)$ . Notice that  $A_1$  is the  $(2p+1)$ -skeleton of  $B(3, 2p+1)$ .

Consider the composite  $\theta: A \wedge A \wedge A \xrightarrow{i \wedge i \wedge i} G \wedge G \wedge G \xrightarrow{c_2} G$ . Observe that  $A \wedge A \wedge A$  is 8-connected, so  $\theta$  factors through the three-connected cover  $G\langle 3 \rangle$ . Since  $G \simeq B(3, 2p+1) \times Y$  and  $n_1 > 2$  implies that  $Y$  is at least 4-connected, we have  $G\langle 3 \rangle \simeq B(3, 2p+1)\langle 3 \rangle \times Y$ . By Lemma 12.6,  $\pi_k(B(3, 2p+1)\langle 3 \rangle) \cong \pi_k(S^{2p+1})$  for  $k < 2p^2 - 1$ . Thus, in this dimensional range, it is as if  $G$  is spherically resolved by  $S^{2n_1-1}, \dots, S^{2n_\ell-1}$  and  $S^{2p+1}$ . The hypothesis that  $p < n_\ell$  implies that the dimension of  $A \wedge A \wedge A$  is that of  $A_2 \wedge A_2 \wedge A_2$ , which is  $6n_\ell - 3$ . The hypothesis that  $n_1 \leq p$  implies that the arguments for  $\pi_m(S^{2n_i-1}) = 0$  in part (1) also imply that  $\pi_m(S^{2p+1}) = 0$ . Thus if  $6n_\ell - 3 < 2p^2 - 1$ , then the argument in part (i) goes through without change to show that  $\theta$  is null homotopic and  $\text{nil}(G) \leq 2$ .

It remains to show that  $6n_\ell - 3 < 2p^2 - 1$ . By hypothesis,  $3n_\ell < \min\{n_1 p, n_1 + p^2 - p\}$  and  $n_1 \leq p$ . If  $\min\{n_1 p, n_1 + p^2 - p\} = n_1 p$  then  $3n_\ell < n_1 p$  and  $n_1 \leq p$  implies that  $6n_\ell - 3 < 2p^2 - 3 < 2p^2 - 1$ . If  $\min\{n_1 p, n_1 + p^2 - p\} = n_1 + p^2 - p$  then  $3n_\ell < n_1 + p^2 - p$  and  $n_1 \leq p$  implies that  $3n_\ell < p^2$  and so  $6n_\ell - 3 < 2p^2 - 3 < 2p^2 - 1$ . In either case,  $6n_\ell - 3 < 2p^2 - 1$ , as required.  $\square$

*Proof of Theorem 12.1.* It has already been mentioned that every group  $G$  in (21) and every non-homotopy commutative group  $G$  in (22) has  $\text{nil}(G) \geq 2$ . In each such case, except  $E_8$  at  $p = 11$ , the hypotheses of Proposition 12.7 hold, so  $\text{nil}(G) \leq 2$ . Hence  $\text{nil}(G) = 2$ .

The one outstanding case is  $E_8$  at  $p = 11$ . As in part (ii) of Proposition 12.7, write  $E_8 \simeq B(3, 23) \times Y$  where  $Y$  has type  $\{8, 14, 18, 20, 24, 30\}$ . Now  $n_1 = 8$  and  $n_\ell = 30$ , but  $3n_\ell$  is not less than  $\min\{n_1 p, n_1 + p^2 - p\} = \min\{88, 118\}$ . The one obstruction that appears in the argument



proving Proposition 12.7 corresponds to  $6n_\ell - 3 = 177$  (the dimension of  $A \wedge A \wedge A$ ) being less than  $2n_1p - 3 = 173$  (the stable range of  $S^{15}$ ). But from the homotopy fibration  $W_8 \longrightarrow S^{15} \xrightarrow{E^2} \Omega^2 S^{17}$  induced by the double suspension  $E^2$ , since  $W_8$  is homotopy equivalent to the Moore space  $P^{175}(11)$  in dimensions  $\leq 180$  and  $\pi_{177}(P^{175}(11)) = 0$ , we see that  $\pi_{177}(S^{15})$  is stable, and so the argument in Proposition 12.7 goes through. Therefore  $\text{nil}(E_8) = 2$  in this case as well.  $\square$

**Remark 12.8.** There are other quasi- $p$ -regular exceptional Lie groups:  $G_2$ ,  $F_4$  and  $E_6$  at  $p = 3$  and  $p = 5$ . The cases  $F_4$  and  $E_6$  at 3 have torsion in mod-3 homology; therefore a characterization of homotopy nilpotent Lie groups by Rao [R] implies that, localized at 3, we have  $\text{nil}(F_4) = \infty$  and  $\text{nil}(E_6) = \infty$ . McGibbon [M] showed that  $G_2$  at  $p = 5$  is homotopy commutative; therefore localized at 5 we have  $\text{nil}(G_2) = 1$ . The remaining cases all have torsion free homology and are not homotopy commutative. Moreover, they do not satisfy the hypotheses of Proposition 12.7 and fail to do so in a way that does not allow for a sidestepping argument as for  $E_8$  at  $p = 11$  in the proof of Theorem 12.1. So the precise value of  $\text{nil}(G)$  in these cases remains undetermined.

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