Münster J. of Math. — (—), 999–999
DOI

Münster Journal of Mathematics

Functoriality and K-theory for $\mathrm{GL}_n(\mathbb{R})$

Sergio Mendes and Roger Plymen

Abstract. We investigate base change and automorphic induction $\mathbb{C}/\mathbb{R}$ at the level of K-theory for the general linear group $\mathrm{GL}_n(\mathbb{R})$. In the course of this study, we compute in detail the C^*-algebra K-theory of this disconnected group. This article is the archimedean companion of our previous article [12].

1. Introduction

In the general theory of automorphic forms, an important role is played by *base change* and *automorphic induction*, two examples of the principle of functoriality in the Langlands program [3]. Base change and automorphic induction have a global aspect and a local aspect [1]. In this article, we focus on the archimedean case of base change and automorphic induction for the general linear group $\mathrm{GL}_n(\mathbb{R})$, and we investigate these aspects of functoriality at the level of K-theory.

For $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ we have the Langlands classification and the associated L-parameters [10]. We recall that the domain of an L-parameter of $\mathrm{GL}_n(F)$ over an archimedean field F is the Weil group W_F. The Weil groups are given by

$$W_{\mathbb{C}} = \mathbb{C}^\times$$

and

$$W_{\mathbb{R}} = \langle j \rangle \mathbb{C}^\times$$

where $j^2 = -1 \in \mathbb{C}^\times$, $jc = \bar{c}j$ for all $c \in \mathbb{C}^\times$. Base change is defined by restriction of L-parameter from $W_{\mathbb{R}}$ to $W_{\mathbb{C}}$.

An L-parameter ϕ is *tempered* if $\phi(W_F)$ is bounded. Base change therefore determines a map of tempered duals.

Let X, Y be locally compact Hausdorff spaces, let X^+, Y^+ be their one-point compactifications. A map $f : X \to Y$ is *continuous at infinity* if it is the restriction of a continuous map from X^+ to Y^+. The K-theory groups K^0 and K^1 are contravariant functors from the category of locally compact Hausdorff spaces whose morphisms are maps continuous at infinity to the category of abelian groups, see [13, Prop. 2.6.10]. Now the tempered dual of $\mathrm{GL}_n(F)$ with $F = \mathbb{R}$ or $\mathbb{C}$ is a locally compact Hausdorff space. It seems natural to fuse together the Langlands functoriality which occurs in base change and automorphic induction with the K-theory functoriality. In this article, we accordingly study base change and automorphic induction at the level of K-theory.

We outline here the connection with the Baum-Connes correspondence. Let F denote $\mathbb{R}$ or $\mathbb{C}$ and let $G = G(F) = \mathrm{GL}_n(F)$. Let $C_r^*(G)$ denote the reduced C^*-algebra of G. The Baum-Connes correspondence is a canonical isomorphism [2][6][11]

$$\mu_F : K_*^{G(F)}(\underline{E}G(F)) \to K_* C_r^*(G(F))$$

where $\underline{E}G(F)$ is a universal example for proper actions of $G(F)$.

The noncommutative space $C_r^*(G(F))$ is strongly Morita equivalent to the commutative C^*-algebra $C_0(\mathcal{A}_n^t(F))$ where $\mathcal{A}_n^t(F)$ denotes the tempered dual of $G(F)$, see [15, §1.2][14]. As a consequence of this, we have

$$K_* C_r^*(G(F)) \cong K^* \mathcal{A}_n^t(F).$$

This leads to the following formulation of the Baum-Connes correspondence:

$$K_*^{G(F)}(\underline{E}G(F)) \cong K^* \mathcal{A}_n^t(F).$$

Base change and automorphic induction $\mathbb{C}/\mathbb{R}$ determine maps

$$\mathcal{BC}_{\mathbb{C}/\mathbb{R}} : \mathcal{A}_n^t(\mathbb{R}) \to \mathcal{A}_n^t(\mathbb{C})$$

and

$$\mathcal{AI}_{\mathbb{C}/\mathbb{R}} : \mathcal{A}_n^t(\mathbb{C}) \to \mathcal{A}_{2n}^t(\mathbb{R}).$$

This leads to the following diagrams

$$\begin{array}{ccc} K_*^{G(\mathbb{C})}(\underline{E}G(\mathbb{C})) & \xrightarrow{\mu_{\mathbb{C}}} & K^* \mathcal{A}_n^t(\mathbb{C})) \\ \downarrow & & \downarrow \mathcal{BC}^*_{\mathbb{C}/\mathbb{R}} \\ K_*^{G(\mathbb{R})}(\underline{E}G(\mathbb{R})) & \xrightarrow[\mu_{\mathbb{R}}]{} & K^* \mathcal{A}_n^t(\mathbb{R}). \end{array}$$

and

$$\begin{array}{ccc} K_*^{G(\mathbb{R})}(\underline{E}G(\mathbb{R})) & \xrightarrow{\mu_{\mathbb{R}}} & K^* \mathcal{A}_{2n}^t(\mathbb{R})) \\ \downarrow & & \downarrow \mathcal{AI}^*_{\mathbb{C}/\mathbb{R}} \\ K_*^{G(\mathbb{C})}(\underline{E}G(\mathbb{C})) & \xrightarrow[\mu_{\mathbb{C}}]{} & K^* \mathcal{A}_n^t(\mathbb{C}). \end{array}$$

where the left-hand vertical maps are the unique maps which make the diagrams commutative.

In section 2 we describe the tempered dual $\mathcal{A}_n^t(F)$ as a locally compact Hausdorff space.

In section 3 we compute the K-theory for the reduced C^*-algebra of $\mathrm{GL}_n(\mathbb{R})$. The real reductive Lie group $\mathrm{GL}_n(\mathbb{R})$ is not connected. If n is even our formulas show that we always have non-trivial K^0 and K^1. We also recall the K-theory for the reduced C^*-algebra of the complex reductive group $\mathrm{GL}_n(\mathbb{C})$, see [14]. In section 4 we recall the Langlands parameters for GL_n over archimedean local fields, see [10]. In section 5 we compute the base change map $\mathcal{BC} : \mathcal{A}_n^t(\mathbb{R}) \to \mathcal{A}_n^t(\mathbb{C})$ and prove that $\mathcal{BC}$ is a continuous proper map. At the level of K-theory, base change is the zero map for $n > 1$ (Theorem 5.5) and is nontrivial for $n = 1$ (Theorem 5.7). In section 6, we compute the automorphic induction map $\mathcal{AI} : \mathcal{A}_n^t(\mathbb{C}) \to \mathcal{A}_{2n}^t(\mathbb{R})$. Contrary to base change, at the level of K-theory, automorphic induction is nontrivial for every n (Theorem 6.3). In section 7, where we study the case $n = 1$, base change for K^1 creates a map

$$\mathcal{R}(\mathrm{U}(1)) \longrightarrow \mathcal{R}(\mathbb{Z}/2\mathbb{Z})$$

where $\mathcal{R}(\mathrm{U}(1))$ is the representation ring of the circle group $\mathrm{U}(1)$ and $\mathcal{R}(\mathbb{Z}/2\mathbb{Z})$ is the representation ring of the group $\mathbb{Z}/2\mathbb{Z}$. This map sends the trivial character of $\mathrm{U}(1)$ to $1 \oplus \varepsilon$, where ε is the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$, and sends all the other characters of $\mathrm{U}(1)$ to zero.

This map has an interpretation in terms of K-cycles. The K-cycle

$$(C_0(\mathbb{R}), L^2(\mathbb{R}), id/dx)$$

is equivariant with respect to $\mathbb{C}^\times$ and $\mathbb{R}^\times$, and therefore determines a class $\not{\partial}_{\mathbb{C}} \in K_1^{\mathbb{C}^\times}(\underline{E}\mathbb{C}^\times)$ and a class $\not{\partial}_{\mathbb{R}} \in K_1^{\mathbb{R}^\times}(\underline{E}\mathbb{R}^\times)$. On the left-hand-side of the Baum-Connes correspondence, base change in dimension 1 admits the following description in terms of Dirac operators:

$$\not{\partial}_{\mathbb{C}} \mapsto (\not{\partial}_{\mathbb{R}}, \not{\partial}_{\mathbb{R}})$$

This extends the results of [12] to archimedean fields.

We have, according to the Connes-Kasparov correspondence, the following isomorphism:

$$K_*\, C_r^*(\mathrm{GL}_n(\mathbb{R})) \simeq K^*_{\mathrm{O}(n)}(\mathbb{R}^n)$$

the equivariant K-theory of $\mathbb{R}^n$ with respect to the standard action of the orthogonal group $\mathrm{O}(n)$. This isomorphism opens the way to computing the K-theory of $C_r^*(\mathrm{GL}_n(\mathbb{R}))$ via equivariant K-theory: this program is carried out in the paper by Echterhoff and Pfante [8]. Our method of computing the K-theory of $C_r^*(\mathrm{GL}_n(\mathbb{R}))$ is quite different, as we have to keep track of the Langlands parameters.

After our article was posted on the arXiv, Chao and Wang sent us their article [7]. Their work and ours was done independently. There is some overlap, but we would like to describe the main differences. Their account of base change is different, as they place an emphasis on Galois-fixed points. In the context of the Connes-Kasparov isomorphism, they succeed in securing base change on maximal compact subgroups [7, §7.2]. On the other hand, their work does not include automorphic induction.

We thank Paul Baum for a valuable exchange of emails. We also thank the referee for providing us with many detailed and constructive comments.

2. On the tempered dual of GL_n

Let $F = \mathbb{R}$. In order to compute the K-theory of the reduced C^*-algebra of $\mathrm{GL}_n(F)$ we need to parametrize the tempered dual $\mathcal{A}_n^t(F)$ of $\mathrm{GL}_n(F)$. Our key reference for the representation theory of $\mathrm{GL}_n(\mathbb{R})$ is Knapp [10].

Let M be a standard Levi subgroup of $\mathrm{GL}_n(F)$, i.e. a block-diagonal subgroup. Let ${}^0M$ be the subgroup of M such that the determinant of each block-diagonal is ± 1. Denote by $X(M) = \widehat{M/{}^0M}$ the group of *unramified characters* of M, consisting of those characters which are trivial on ${}^0M$.

Let $W(M) = N(M)/M$ denote the Weyl group of M. $W(M)$ acts on the discrete series $E_2({}^0M)$ of ${}^0M$ by permutations.

Now, choose one element $\sigma \in E_2({}^0M)$ for each $W(M)$-orbit. The *isotropy subgroup* of σ is defined to be

$$W_\sigma(M) = \{\omega \in W(M) : \omega.\sigma = \sigma\}.$$

Take one standard Levi subgroup M from each conjugacy class of Levi subgroups and one discrete series σ from each $W(M)$-orbit and form the disjoint union

$$\bigsqcup_{[M,\sigma]} X(M)/W_\sigma(M) = \bigsqcup_{[M]} \bigsqcup_{[\sigma]\in E_2({}^0M)} X(M)/W_\sigma(M). \tag{1}$$

The disjoint union has the structure of a locally compact, Hausdorff space and is called the *Harish-Chandra parameter space.*

Proposition 2.1. *There exists a bijection*

$$\begin{array}{ccc} \bigsqcup_{[M,\sigma]} X(M)/W_\sigma(M) & \longrightarrow & \mathcal{A}_n^t(\mathbb{R}) \\ \chi^\sigma & \mapsto & i_{\mathrm{GL}_n(\mathbb{R}),MN}(\chi^\sigma \otimes 1), \end{array}$$

where $\chi^\sigma(x) := \chi(x)\sigma(x)$ for all $x \in M$.

In view of the above bijection [15, 1.2], we will denote the Harish-Chandra parameter space by $\mathcal{A}_n^t(\mathbb{R})$.

We will see now the particular features of the archimedean case, starting with $\mathrm{GL}_n(\mathbb{R})$. Since the discrete series of $\mathrm{GL}_n(\mathbb{R})$ is empty for $n \geq 3$, we only need to consider partitions of n into 1's and 2's.

This allows us to to decompose n as $n = 2q + r$, where q is the number of 2's and r is the number of 1's in the partition. To this decomposition we associate the partition

$$n = (\underbrace{2, ..., 2}_{q}, \underbrace{1, ..., 1}_{r}),$$

which corresponds to the Levi subgroup

$$M \cong \underbrace{\mathrm{GL}_2(\mathbb{R}) \times ... \times \mathrm{GL}_2(\mathbb{R})}_{q} \times \underbrace{\mathrm{GL}_1(\mathbb{R}) \times ... \times \mathrm{GL}_1(\mathbb{R})}_{r}.$$

Varying q and r we determine a representative in each equivalence class of Levi subgroups. The subgroup ${}^0M$ of M is given by

$$ {}^0M \cong \underbrace{\mathrm{SL}_2^{\pm}(\mathbb{R}) \times ... \times \mathrm{SL}_2^{\pm}(\mathbb{R})}_{q} \times \underbrace{\mathrm{SL}_1^{\pm}(\mathbb{R}) \times ... \times \mathrm{SL}_1^{\pm}(\mathbb{R})}_{r}, $$

where

$$ \mathrm{SL}_m^{\pm}(\mathbb{R}) = \{g \in \mathrm{GL}_m(\mathbb{R}) : |det(g)| = 1\}. $$

In particular, $\mathrm{SL}_1^{\pm}(\mathbb{R}) = \{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$.

The representations in the discrete series of $SL_2^{\pm}(\mathbb{R})$, denoted $\mathcal{D}_\ell$ for $\ell \in \mathbb{N}$ ($\ell \geq 1$) are induced from $\mathrm{SL}_2(\mathbb{R})$ [10, p.399]:

$$ \mathcal{D}_\ell = i_{\mathrm{SL}_2^{\pm}(\mathbb{R}),\mathrm{SL}_2(\mathbb{R})}(\mathcal{D}_\ell^{+}), $$

where $\mathcal{D}_\ell^{+}$ acts in the space

$$ \{f : \mathcal{H} \to \mathbb{C} | f \text{ analytic }, \|f\|^2 = \int\int |f(z)|^2 y^{\ell-1} dxdy < \infty\}. $$

Here, $\mathcal{H}$ denotes the Poincaré upper half plane. The action of $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by

$$ \mathcal{D}_\ell^{+}(g)(f(z)) = (bz + d)^{-(\ell+1)} f(\frac{az + c}{bz + d}). $$

More generally, an element σ from the discrete series $E_2({}^0M)$ is given by

$$ \sigma = \mathcal{D}_{\ell_1} \otimes ... \otimes \mathcal{D}_{\ell_q} \otimes \tau_1 \otimes ... \otimes \tau_r, \tag{2} $$

where $\mathcal{D}_{\ell_i}$ ($\ell_i \geq 1$) are discrete series representations of $\mathrm{SL}_2^{\pm}(\mathbb{R})$ and τ_j is a representation of $\mathrm{SL}_1^{\pm}(\mathbb{R}) \cong \mathbb{Z}/2\mathbb{Z}$, i.e. $id = (x \mapsto x)$ or $sgn = (x \mapsto \frac{x}{|x|})$.

Finally we will compute the unramified characters $X(M)$, where M is the Levi subgroup associated to the partition $n = 2q + r$.

Let $x \in \mathrm{GL}_2(\mathbb{R})$. Any character of $\mathrm{GL}_2(\mathbb{R})$ is given by

$$ \chi(det(x)) = (sgn(det(x)))^{\varepsilon} |det(x)|^{it} $$

($\varepsilon = 0, 1, t \in \mathbb{R}$) and it is unramified provided that

$$ \chi(det(g)) = \chi(\pm 1) = (\pm 1)^{\varepsilon} = 1, \text{ for all } g \in \mathrm{SL}_2^{\pm}(\mathbb{R}). $$

This implies $\varepsilon = 0$ and any unramified character of $\mathrm{GL}_2(\mathbb{R})$ has the form

$$ \chi(x) = |det(x)|^{it}, \text{ for some } t \in \mathbb{R}. \tag{3} $$

Similarly, any unramified character of $\mathrm{GL}_1(\mathbb{R}) = \mathbb{R}^{\times}$ has the form

$$ \xi(x) = |x|^{it}, \text{ for some } t \in \mathbb{R}. \tag{4} $$

Given a block diagonal matrix $diag(g_1, ..., g_q, \omega_1, ..., \omega_r) \in M$, where $g_i \in \mathrm{GL}_2(\mathbb{R})$ and $\omega_j \in \mathrm{GL}_1(\mathbb{R})$, we conclude from (3) and (4) that any unramified character $\chi \in X(M)$ is given by

$$ \chi(diag(g_1, ..., g_q, \omega_1, ..., \omega_r)) = $$
$$ = |det(g_1)|^{it_1} \cdots |det(g_q)|^{it_q} \cdot |\omega_1|^{it_{q+1}} \cdots |\omega_r|^{it_{q+r}}, $$

for some $(t_1, ..., t_{q+r}) \in \mathbb{R}^{q+r}$. We can denote such element $\chi \in X(M)$ by $\chi_{(t_1,...,t_{q+r})}$. We have the following result.

Proposition 2.2. *Let M be a Levi subgroup of $\mathrm{GL}_n(\mathbb{R})$, associated to the partition $n = 2q + r$. Then, there is a bijection*

$$X(M) \to \mathbb{R}^{q+r} \ , \ \chi_{(t_1,...,t_{q+r})} \mapsto (t_1, ..., t_{q+r}).$$

Let us consider now $\mathrm{GL}_n(\mathbb{C})$. The tempered dual of $\mathrm{GL}_n(\mathbb{C})$ comprises the *unitary principal series* in accordance with Harish-Chandra [9]. The corresponding Levi subgroup is a maximal torus $T \cong (\mathbb{C}^\times)^n$. Denote by U the standard unipotent subgroup of $GL_n(\mathbb{C})$. The principal series representations are given by

$$\pi_{\ell,it} = i_{G,TU}(\sigma \otimes 1), \tag{5}$$

where σ=$\sigma_1 \otimes ... \otimes \sigma_n$ and $\sigma_j(z) = (\frac{z}{|z|})^{\ell_j} |z|_{\mathbb{C}}^{it_j}$ ($\ell_j \in \mathbb{Z}$ and $t_j \in \mathbb{R}$), with $|z|_{\mathbb{C}} = z\overline{z} = |z|^2$ [10, p.405].

An unramified character is given by

$$\chi \begin{pmatrix} z_1 & & \\ & \ddots & \\ & & z_n \end{pmatrix} = |z_1|_{\mathbb{C}}^{it_1} \times ... \times |z_n|_{\mathbb{C}}^{it_n}$$

and we can represent χ as $\chi_{(t_1,...,t_n)}$. Therefore, we have the following result.

Proposition 2.3. *Denote by T the standard maximal torus in $\mathrm{GL}_n(\mathbb{C})$. There is a bijection*

$$X(T) \to \mathbb{R}^n \ , \ \chi_{(t_1,...,t_n)} \mapsto (t_1, ..., t_n).$$

The Weyl group W is the symmetric group S_n and acts on $\mathbb{R}^n$ by permuting the components.

3. K-theory for GL_n

Using the Harish-Chandra parametrization of the tempered dual of $\mathrm{GL}_n(\mathbb{R})$ and $\mathrm{GL}_n(\mathbb{C})$ (recall that the Harish-Chandra parameter space is a locally compact, Hausdorff topological space) we can compute the K-theory of the reduced C^*-algebras $C_r^*\mathrm{GL}_n(\mathbb{R})$ and $C_r^*\mathrm{GL}_n(\mathbb{C})$.

3.1. **K-theory for $\mathrm{GL}_n(\mathbb{R})$.** We exploit the strong Morita equivalence described in [15, §1.2]. We note in passing that, in the proof of this strong Morita equivalence, the following ingredient is crucial: each tempered representation of $\mathrm{GL}_n(\mathbb{R})$, i.e. each unitary representation of $\mathrm{GL}_n(\mathbb{R})$ which is unitarily induced via parabolic induction from a discrete series representation of a Levi subgroup is irreducible, see [10, p.401]. We infer that

$$\begin{aligned} K_j(C_r^*\mathrm{GL}_n(\mathbb{R})) &= K^j(\textstyle\bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M)) \\ &= \textstyle\bigoplus_{(M,\sigma)} K^j(X(M)/W_\sigma(M)) \\ &= \textstyle\bigoplus_{(M,\sigma)} K^j(\mathbb{R}^{n_M}/W_\sigma(M)), \end{aligned} \tag{6}$$

where $n_M = q + r$ if M is a representative of the equivalence class of Levi subgroup associated to the partition $n = 2q + r$. Hence the K-theory depends on n and on each Levi subgroup.

For a given Levi subgroup M and a discrete series σ of ${}^0M$, the isotropy subgroup W_σ is a subgroup of the Weyl group $W(M)$, which is in turn a subgroup of the symmetric group S_n. The isotropy subgroup has the form $S_{n_1} \times ... \times S_{n_k}$ and acts on $\mathbb{R}^n$ by permuting the components. Write

$$\mathbb{R}^n \cong \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times ... \times \mathbb{R}^{n_k} \times \mathbb{R}^{n-n_1-...-n_k}.$$

If $n = n_1 + ... + n_k$ then we simply have $\mathbb{R}^n \cong \mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k}$.

The group $S_{n_1} \times ... \times S_{n_k}$ acts on $\mathbb{R}^n$ as follows.
S_{n_1} permutes the components of $\mathbb{R}^{n_1}$ leaving the remaining fixed;
S_{n_2} permutes the components of $\mathbb{R}^{n_2}$ leaving the remaining fixed;
and so on. If $n > n_1 + ... + n_k$ the components of $\mathbb{R}^{n-n_1-\cdots-n_k}$ remain fixed. This can be interpreted, of course, as the action of the trivial subgroup. As a consequence, one identifies the orbit spaces

$$\mathbb{R}^n/(S_{n_1} \times ... \times S_{n_k}) \cong \mathbb{R}^{n_1}/S_{n_1} \times ... \times \mathbb{R}^{n_k}/S_{n_k} \times \mathbb{R}^{n-n_1-...-n_k}$$

To compute the K-theory (6) we have to consider the following orbit spaces:

(i) $\mathbb{R}^n$, in which case $W_\sigma(M)$ is the trivial subgroup of the Weyl group $W(M)$;
(ii) $\mathbb{R}^n/(S_{n_1} \times ... \times S_{n_k})$, where $W_\sigma(M) = S_{n_1} \times ... \times S_{n_k} \subset W(M)$ (see the examples below).

The K-theory for $\mathbb{R}^n$ may be summarized as follows

$$K^j(\mathbb{R}^n) = \begin{cases} \mathbb{Z} \text{ if } n = j \mod 2 \\ 0 \text{ otherwise }. \end{cases}$$

Lemma 3.2. *For $n > 1$, $K^j(\mathbb{R}^n/S_n) = 0, j = 0, 1$.*

Proof. We consider the action of the symmetric group $W = S_n$ on $\mathbb{R}^n$. The subspace

$$\{t(1, \ldots, 1) : t \in \mathbb{R}\}$$

is fixed by W and the orthogonal subspace

$$\mathfrak{t} := \{(x_1, \ldots, x_n) : x_1 + \ldots + x_n = 0\}$$

is W-invariant. It follows that $\mathbb{R}^n/S_n \simeq \mathbb{R} \times \mathfrak{t}/W$. The action of W on $\mathfrak{t}$ is precisely the action of W on the Lie algebra $\mathfrak{t}$ of the standard maximal torus T of the Lie group $\mathrm{SL}_n(\mathbb{R})$. The closure $\overline{C}$ of a chamber $C \subset \mathfrak{t}$ is a fundamental domain for the action of W, see [5, Ch.5, §3]. The quotient $\mathfrak{t}/W$ is homeomeorphic to $\overline{C}$. Then we have

$$\mathbb{R}^n/W \simeq \mathbb{R} \times \overline{C}$$

Now $\overline{C}$ is a closed simplicial cone with vertex at the origin of $\mathbb{R}^n$. It has the topological type of a half-space in Euclidean space. Hence the K-theory of $\overline{C}$ is trivial. The Lemma follows immediately from the Künneth theorem applied to $\mathbb{R} \times \overline{C}$. □

Lemma 3.3. *$K^j(\mathbb{R}^n/(S_{n_1}\times ...\times S_{n_k}) = 0, j = 0,1$, where $S_{n_1}\times ...\times S_{n_k} \subset S_n$, unless $n_1 = \ldots = n_k = 1$.*

Proof. It suffices to prove for $\mathbb{R}^n/(S_{n_1}\times S_{n_2})$. The general case follows by induction on k.

Now, $\mathbb{R}^n/(S_{n_1}\times S_{n_2}) \cong \mathbb{R}^{n_1}/S_{n_1}\times \mathbb{R}^{n-n_1}/S_{n_2}$. Applying the Künneth formula and Lemma 3.2, the result follows. □

We give now some examples by computing $K_j C_r^* GL_n(\mathbb{R})$ for small n.

Example 3.4. We start with the case of $\mathrm{GL}_1(\mathbb{R})$. We have:

$$M = \mathbb{R}^\times \ , {}^0M = \mathbb{Z}/2\mathbb{Z} \ , W(M) = 1 \text{ and } X(M) = \mathbb{R}.$$

Hence,

$$\mathcal{A}_1^t(\mathbb{R}) \cong \bigsqcup_{\sigma\in\widehat{(\mathbb{Z}/2\mathbb{Z})}} \mathbb{R}/1 = \mathbb{R}\sqcup\mathbb{R}, \tag{7}$$

and the K-theory is given by

$$K_jC_r^*GL_1(\mathbb{R}) \cong K^j(\mathcal{A}_1^t(\mathbb{R})) = K^j(\mathbb{R}\sqcup\mathbb{R}) = K^j(\mathbb{R})\oplus K^j(\mathbb{R}) = \begin{cases} \mathbb{Z}\oplus\mathbb{Z} & ,j=1 \\ 0 & ,j=0. \end{cases}$$

Example 3.5. For $\mathrm{GL}_2(\mathbb{R})$ we have two partitions of $n = 2$ and the following data

Partition	M	${}^0M$	$W(M)$	$X(M)$	$\sigma \in E_2({}^0M)$
2+0	$\mathrm{GL}_2(\mathbb{R})$	$\mathrm{SL}_2^{\pm}(\mathbb{R})$	1	$\mathbb{R}$	$\sigma = \mathcal{D}_\ell^+, \ell\in\mathbb{N}$
1+1	$(\mathbb{R}^\times)^2$	$(\mathbb{Z}/2\mathbb{Z})^2$	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{R}^2$	$\sigma = \tau_1\otimes\tau_2$

with $\tau_i \in \widehat{\mathbb{Z}/2\mathbb{Z}} \simeq \{id, sgn\}$. Then the tempered dual is parametrized as follows

$$\mathcal{A}_2^t(\mathbb{R}) \cong \bigsqcup_{(M,\sigma)} X(M)/W_\sigma(M) = (\bigsqcup_{\ell\in\mathbb{N}}\mathbb{R})\sqcup(\mathbb{R}^2/S_2)\sqcup(\mathbb{R}^2/S_2)\sqcup\mathbb{R}^2,$$

and the K-theory groups are given by

$$K_jC_r^*GL_2(\mathbb{R}) \cong K^j(\mathcal{A}_2^t(\mathbb{R})) \cong (\bigoplus_{\ell\in\mathbb{N}} K^j(\mathbb{R}))\oplus K^j(\mathbb{R}^2) = \begin{cases} \bigoplus_{\ell\in\mathbb{N}}\mathbb{Z} & ,j=1 \\ \mathbb{Z} & ,j=0. \end{cases}$$

The general case of $\mathrm{GL}_n(\mathbb{R})$ will now be considered. It can be split in two cases: n even and n odd.

- $n = 2m$ even

Suppose n is even. For every partition $n = 2q + r$, either $W_\sigma(M) = 1$ or $W_\sigma(M) \neq 1$. If $W_\sigma(M) \neq 1$ then $\mathbb{R}^{n_M}/W_\sigma(M)$ is an orbit space for which the K-groups K^0 and K^1 both vanish. This happens precisely when $r > 2$ because there are exactly two distinct discrete series representations of $\mathbb{Z}/2\mathbb{Z}$ and therefore we have only two partitions, corresponding to the choices of $r = 0$ and $r = 2$, which contribute to the K-theory with non-zero K-groups

Partition	M	${}^0M$	$W(M)$
$2m$	$\mathrm{GL}_2(\mathbb{R})^m$	$\mathrm{SL}_2^{\pm}(\mathbb{R})^m$	S_m
$2(m-1)+2$	$\mathrm{GL}_2(\mathbb{R})^{m-1}\times(\mathbb{R}^{\times})^2$	$\mathrm{SL}_2^{\pm}(\mathbb{R})^{m-1}\times(\mathbb{Z}/2\mathbb{Z})^2$	$S_{m-1}\times(\mathbb{Z}/2\mathbb{Z})$

We also have $X(M)\cong\mathbb{R}^m$ for $n=2m$, and $X(M)\cong\mathbb{R}^{m+1}$, for $n=2(m-1)+2$.

For the partition $n=2m$ ($r=0$), an element $\sigma\in E_2({}^0M)$ is given by

$$\sigma=\mathcal{D}_{\ell_1}\otimes\ldots\otimes\mathcal{D}_{\ell_m}\ ,\ \ell_1>\ldots>\ell_m,\ \ell_i\in\mathbb{N}^m.$$

For the partition $n=2(m-1)+2$ ($r=2$), an element $\sigma\in E_2({}^0M)$ is given by

$$\sigma=\mathcal{D}_{\ell_1}\otimes\ldots\otimes\mathcal{D}_{\ell_{m-1}}\otimes id\otimes sgn\ ,\ \ell_1>\ldots>\ell_{m-1},\ \ell_i\in\mathbb{N}^{m-1}.$$

Therefore, the tempered dual has the following form

$$\mathcal{A}_n^t(\mathbb{R})=\mathcal{A}_{2m}^t(\mathbb{R})=(\bigsqcup_{\ell_1>\ldots>\ell_m}\mathbb{R}^m)\sqcup(\bigsqcup_{\ell'_1>\ldots>\ell'_{m-1}}\mathbb{R}^{m+1})\sqcup\mathcal{C},$$

with $\ell_i,\ell'_j\in\mathbb{N}$ and where $\mathcal{C}$ is a disjoint union of orbit spaces as in section 3. Note that the strictly decreasing condition is required in order to pick only one discrete series from each Weyl group orbit.

Theorem 3.6. *Suppose $n=2m$ is even. Then the K-groups are*

$$K_jC_r^*\mathrm{GL}_n(\mathbb{R})\cong\begin{cases}\bigoplus_{\ell_1>\ldots>\ell_m}\mathbb{Z} & ,j\equiv m(mod\,2)\\ \bigoplus_{\ell_1>\ldots>\ell_{m-1}}\mathbb{Z} & ,otherwise\end{cases}$$

with $\ell_i\in\mathbb{N}$. If $m=1$ then $K_jC_r^*\mathrm{GL}_2(\mathbb{R})\cong\mathbb{Z}$.

- $n=2q+1$ odd

If n is odd only one partition contributes to the K-theory of $\mathrm{GL}_n(\mathbb{R})$ with non-zero K-groups:

Partition	M	${}^0M$	$W(M)$	$X(M)$
$2q+1$	$\mathrm{GL}_2(\mathbb{R})^{q+1}\times\mathbb{R}^{\times}$	$\mathrm{SL}_2^{\pm}(\mathbb{R})^q\times(\mathbb{Z}/2\mathbb{Z})$	S_q	$\mathbb{R}^{q+1}$

An element $\sigma\in E_2({}^0M)$ is given by

$$\sigma=\mathcal{D}_{\ell_1}\otimes\ldots\otimes\mathcal{D}_{\ell_q}\otimes\tau\ ,\ \ell_1>\ldots>\ell_q,\ \ell_i\in\mathbb{N},\ \tau\in\widehat{\mathbb{Z}/2\mathbb{Z}}.$$

And the tempered dual is

$$\mathcal{A}_n^t(\mathbb{R})=\mathcal{A}_{2q+1}^t(\mathbb{R})=(\bigsqcup_{\ell_1>\ldots>\ell_q,\varepsilon}\mathbb{R}^{q+1})\sqcup\mathcal{C}$$

with $\ell_i\in\mathbb{N}$ and $\varepsilon\in\mathbb{Z}/2\mathbb{Z}$. The space $\mathcal{C}$ is a disjoint union of orbit spaces as in section 3.

Theorem 3.7. *Suppose $n = 2q + 1$ is odd. Then the K-groups are*

$$K_j C_r^* \mathrm{GL}_n(\mathbb{R}) \cong \begin{cases} \bigoplus_{\ell_1 > \ldots > \ell_q, \varepsilon} \mathbb{Z} & , j \equiv q+1 (mod 2) \\ 0 & , otherwise \end{cases}$$

with $\ell_i \in \mathbb{N}$ and $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Here, we use the following convention: if $q = 0$ then the direct sum is $\bigoplus_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z} \oplus \mathbb{Z}$.

We conclude that the K-theory of $C_r^* \mathrm{GL}_n(\mathbb{R})$ depends on essentially one parameter $q = \lfloor \frac{n}{2} \rfloor$ which gives the maximum number of $2's$ in the partitions of n into $1's$ and $2's$.

3.8. **K-theory for $\mathrm{GL}_n(\mathbb{C})$.** Let ${}^\circ T$ be the maximal compact subgroup of the maximal torus T of $\mathrm{GL}_n(\mathbb{C})$. Let σ be a unitary character of ${}^\circ T$. We note that $W = W(T),\ W_\sigma = W_\sigma(T)$. If $W_\sigma = 1$ then we say that the orbit $W \cdot \sigma$ is *generic.*

Theorem 3.9. *The K-theory of $C_r^* \mathrm{GL}_n(\mathbb{C})$ admits the following description. If $n = j \mod 2$ then K_j is free abelian on countably many generators, one for each generic W- orbit in the unitary dual of ${}^\circ T$, and $K_{j+1} = 0$.*

Proof. We exploit the strong Morita equivalence described in [14, Prop. 4.1]. We have a homeomorphism of locally compact Hausdorff spaces:

$$\mathcal{A}_n^t(\mathbb{C}) \cong \bigsqcup_{[\sigma]} X(T)/W_\sigma(T)$$

by the Harish-Chandra Plancherel Theorem for complex reductive groups [9], and the identification of the Jacobson topology on the left-hand-side with the natural topology on the right-hand-side, as in [14] . The result now follows from Lemma 3.2. □

Remark 3.10. Note that $[\sigma] = [T, \sigma]$ is labeled by $\ell_1 \geq ... \geq \ell_n$, with $\ell_i \in \mathbb{Z}$. Moreover, $W_\sigma(T)$ is trivial if and only if $\ell_1 > ... > \ell_n$.

4. Langlands parameters for GL_n

The Weil group of $\mathbb{C}$ is simply

$$W_\mathbb{C} \cong \mathbb{C}^\times,$$

and the Weil group of $\mathbb{R}$ can be written as disjoint union

$$W_\mathbb{R} \cong \mathbb{C}^\times \sqcup j\mathbb{C}^\times,$$

where $j^2 = -1$ and $jcj^{-1} = \bar{c}$ ($\bar{c}$ denotes complex conjugation). We shall use this disjoint union to describe the representation theory of $W_\mathbb{R}$.

An L-parameter is a continuous homomorphism

$$\phi : W_F \to \mathrm{GL}_n(\mathbb{C})$$

such that $\phi(w)$ is semisimple for all $w \in W_F$.

L-parameters are also called Langlands parameters. Two L-parameters are equivalent if they are conjugate under $\mathrm{GL}_n(\mathbb{C})$. The set of equivalence classes of

L-parameters is denoted by $\mathcal{G}_n$. The set of equivalence classes of L-parameters whose image is bounded is denoted by $\mathcal{G}_n^t$.

Let F be either $\mathbb{R}$ or $\mathbb{C}$. Let $\mathcal{A}_n(F)$ (resp. $\mathcal{A}_n^t(F)$) denote the smooth dual (resp. tempered dual) of $\mathrm{GL}_n(F)$. The local Langlands correspondence is a bijection

$$\mathcal{G}_n(F) \to \mathcal{A}_n(F).$$

When we restrict to bounded parameters, we obtain a bijection which we will denote ${}_F\mathcal{L}_n$:

$$ {}_F\mathcal{L}_n : \mathcal{G}_n^t(F) \to \mathcal{A}_n^t(F) \tag{8}$$

L-parameters for $W_{\mathbb{C}}$. A 1-dimensional L-parameter for $W_{\mathbb{C}}$ is a character of $\mathbb{C}^\times$:

$$\chi_{\ell,t}(z) := (\frac{z}{|z|})^\ell \otimes |z|_{\mathbb{C}}^t$$

where $|z|^2 = |z|_{\mathbb{C}} = z\overline{z}$, $\ell \in \mathbb{Z}$ and $t \in \mathbb{C}$. The unitary characters are therefore given by

$$\chi_{\ell,it}(re^{i\theta}) = r^{2it}e^{i\ell\theta}$$

with $t \in \mathbb{R}$ and $\ell \in \mathbb{Z}$.

L-parameters for $W_{\mathbb{R}}$. The 1-dimensional L-parameters for $W_{\mathbb{R}}$ are as follows

$$(+,t)(z) = |z|_{\mathbb{R}}^t \quad \text{and} \quad (+,t)(j) = 1$$
$$(-,t)(z) = |z|_{\mathbb{R}}^t \quad \text{and} \quad (+,t)(j) = -1$$

We may now describe the local Langlands correspondence for $\mathrm{GL}(1,\mathbb{R})$:

$$(+,t) \mapsto 1 \otimes |.|_{\mathbb{R}}^t$$
$$(-,t) \mapsto sgn \otimes |.|_{\mathbb{R}}^t$$

The Weil group $W_{\mathbb{R}}$ admits 2-dimensional irreducible representations, denoted $\varphi_{\ell,t}$ with $\ell \in \mathbb{Z}, \ell \neq 0$, and $t \in \mathbb{R}$. They are defined in [10, (3.3)]:

$$\varphi_{\ell,t}(z) = \begin{pmatrix} \chi_{\ell,t}(z) & 0 \\ 0 & \chi_{\ell,t}(\overline{z}) \end{pmatrix}, \ \varphi_{\ell,t}(j) = \begin{pmatrix} 0 & (-1)^\ell \\ 1 & 0 \end{pmatrix}.$$

We will need one crucial property, namely

$$\varphi_{\ell,t}|_{W_{\mathbb{C}}} = \chi_{\ell,t} \oplus \chi_{-\ell,t} \tag{9}$$

and the single equivalence

$$\varphi_{\ell,t} \simeq \varphi_{-\ell,t}$$

The L-parameter $\varphi_{\ell,it}$ corresponds, via the Langlands correspondence, to the discrete series:

$$\varphi_{\ell,it} \mapsto D_\ell \otimes |\det(.)|_{\mathbb{R}}^{it}, \ \text{ with } \ \ell \in \mathbb{N}, t \in \mathbb{R}.$$

according to (3.4) in [10].

Lemma 4.1. [10] *Every finite-dimensional semi-simple representation ϕ of $W_{\mathbb{R}}$ is fully reducible, and each irreducible representation has dimension one or two.*

5. Base change

We may state the base change problem for archimedean fields in the following way. Consider the archimedean base change $\mathbb{C}/\mathbb{R}$. We have $W_{\mathbb{C}} \subset W_{\mathbb{R}}$ and there is a natural map

$$Res_{W_{\mathbb{C}}}^{W_{\mathbb{R}}} : \mathcal{G}_n(\mathbb{R}) \longrightarrow \mathcal{G}_n(\mathbb{C}) \tag{10}$$

called *restriction*. By the local Langlands correspondence for archimedean fields [3, Theorem 3.1, p.236][10], there is a base change map

$$\mathcal{BC} : \mathcal{A}_n(\mathbb{R}) \longrightarrow \mathcal{A}_n(\mathbb{C}) \tag{11}$$

such that the following diagram commutes

$$\begin{tikzcd} \mathcal{A}_n(\mathbb{R}) \arrow[r, "\mathcal{BC}"] & \mathcal{A}_n(\mathbb{C}) \\ \mathcal{G}_n(\mathbb{R}) \arrow[u, "{}_{\mathbb{R}}\mathcal{L}_n"] \arrow[r, "Res_{W_{\mathbb{C}}}^{W_{\mathbb{R}}}"'] & \mathcal{G}_n(\mathbb{C}) \arrow[u, "{}_{\mathbb{C}}\mathcal{L}_n"'] \end{tikzcd}$$

Arthur and Clozel's book [1] gives a full treatment of base change for GL_n. The case of archimedean base change can be captured in an elegant formula [1, p. 71]. We briefly review these results.

Given a partition $n = 2q + r$ let χ_i $(i = 1, ..., q)$ be a ramified character of $\mathbb{C}^\times$ and let ξ_j $(j = 1, ..., r)$ be a ramified character of $\mathbb{R}^\times$. Since the χ_i's are ramified, $\chi_i(z) \neq \chi_i^\tau(z) = \chi_i(\overline{z})$, where τ is a generator of $Gal(\mathbb{C}/\mathbb{R})$. By Langlands classification [10], each χ_i defines a discrete series representation $\pi(\chi_i)$ of $\mathrm{GL}_2(\mathbb{R})$, with $\pi(\chi_i) = \pi(\chi_i^\tau)$. Denote by $\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r)$ the *generalized principal series representation* of $\mathrm{GL}_n(\mathbb{R})$

$$\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r) = i_{\mathrm{GL}_n(\mathbb{R}),MN}(\pi(\chi_1) \otimes ... \otimes \pi(\chi_q) \otimes \xi_1 \otimes ... \otimes \xi_r \otimes 1). \tag{12}$$

The base change map for the generalized principal series representation is given by induction from the Borel subgroup $B(\mathbb{C})$ [1, p. 71]:

$$\mathcal{BC}(\pi) = i_{\mathrm{GL}_n(\mathbb{C}),B(\mathbb{C})}(\chi_1, \chi_1^\tau, ..., \chi_q, \chi_q^\tau, \xi_1 \circ N, ..., \xi_r \circ N), \tag{13}$$

where $N = N_{\mathbb{C}/\mathbb{R}} : \mathbb{C}^\times \longrightarrow \mathbb{R}^\times$ is the norm map defined by $z \mapsto z\overline{z}$.

We illustrate the base change map with two simple examples.

Example 5.1. For $n = 1$, base change is simply composition with the norm map

$$\mathcal{BC} : \mathcal{A}_1^t(\mathbb{R}) \to \mathcal{A}_1^t(\mathbb{C}) \ , \ \mathcal{BC}(\chi) = \chi \circ N.$$

Example 5.2. For $n = 2$, there are two different kinds of representations, one for each partition of 2. According to (12), $\pi(\chi)$ corresponds to the partition

$2 = 2+0$ and $\pi(\xi_1, \xi_2)$ corresponds to the partition $2 = 1+1$. Then the base change map is given, respectively, by

$$\mathcal{BC}(\pi(\chi)) = i_{\mathrm{GL}_2(\mathbb{C}),B(\mathbb{C})}(\chi, \chi^\tau),$$

and

$$\mathcal{BC}(\pi(\xi_1, \xi_2)) = i_{\mathrm{GL}_2(\mathbb{C}),B(\mathbb{C})}(\xi_1 \circ N, \xi_2 \circ N).$$

5.3. **The base change map.** Now, we define base change as a map of topological spaces and study the induced K-theory map. A continuous map $f : X \to Y$ between topological spaces is *proper* if $f^{-1}(K)$ is a compact subset of X for every compact subset K of Y. If f is a proper map between locally compact Hausdorff spaces then f is continuous at infinity, see [13, Prop. 2.6.4]. So proper maps are morphisms in the category of locally compact Hausdorff spaces, see [13, Prop. 2.6.6].

Proposition 5.4. *The base change map $\mathcal{BC} : \mathcal{A}^t_n(\mathbb{R}) \to \mathcal{A}^t_n(\mathbb{C})$ is a continuous proper map.*

Proof. First, we consider the case $n = 1$. As we have seen in Example 5.1, base change for GL(1) is the map given by $\mathcal{BC}(\chi) = \chi \circ N$, for all characters $\chi \in \mathcal{A}^t_1(\mathbb{R})$, where $N : \mathbb{C}^\times \to \mathbb{R}^\times$ is the norm map.

Let $z \in \mathbb{C}^\times$. We have

$$\mathcal{BC}(\chi)(z) = \chi(|z|^2) = |z|^{2it}. \tag{14}$$

A generic element from $\mathcal{A}^t_1(\mathbb{C})$ has the form

$$\mu(z) = (\frac{z}{|z|})^\ell |z|^{it}, \tag{15}$$

where $\ell \in \mathbb{Z}$ and $t \in \mathbb{R}$, as stated before. Viewing the Pontryagin duals $\mathcal{A}^t_1(\mathbb{R})$ and $\mathcal{A}^t_1(\mathbb{C})$ as topological spaces by forgetting the group structure, and comparing (14) and (15), the base change map can be defined as the following continuous map

$$\begin{array}{rcc} \varphi : \mathcal{A}^t_1(\mathbb{R}) \cong \mathbb{R} \times (\mathbb{Z}/2\mathbb{Z}) & \longrightarrow & \mathcal{A}^t_1(\mathbb{C}) \cong \mathbb{R} \times \mathbb{Z} \\ \chi = (t, \varepsilon) & \mapsto & (2t, 0) \end{array}$$

A compact subset of $\mathbb{R} \times \mathbb{Z}$ in the connected component $\{\ell\}$ of $\mathbb{Z}$ has the form $K \times \{\ell\} \subset \mathbb{R} \times \mathbb{Z}$, where $K \subset \mathbb{R}$ is compact. We have

$$\varphi^{-1}(K \times \{\ell\}) = \left\{ \begin{array}{ll} \varnothing & ,\text{if } \ell \neq 0 \\ \frac{1}{2}K \times \{\varepsilon\} & ,\text{if } \ell = 0, \end{array} \right.$$

where $\varepsilon \in \mathbb{Z}/2\mathbb{Z}$. Therefore $\varphi^{-1}(K \times \{\ell\})$ is compact and φ is proper.

The Case $n > 1$. Base change determines a map $\mathcal{BC} : \mathcal{A}^t_n(\mathbb{R}) \to \mathcal{A}^t_n(\mathbb{C})$ of topological spaces. Let $X = X(M)/W_\sigma(M)$ be a connected component of $\mathcal{A}^t_n(\mathbb{R})$. Then, X is mapped under $\mathcal{BC}$ into a connected component $Y = Y(T)/W_{\sigma'}(T)$ of $\mathcal{A}^t_n(\mathbb{C})$. Given a generalized principal series representation

$$\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r)$$

where the χ_i's are ramified characters of $\mathbb{C}^\times$ and the ξ's are ramified characters of $\mathbb{R}^\times$, then

$$\mathcal{BC}(\pi) = i_{G,B}(\chi_1, \chi_1^\tau, ..., \chi_q, \chi_q^\tau, \xi_1 \circ N, ..., \xi_r \circ N).$$

Here, $N = N_{\mathbb{C}/\mathbb{R}}$ is the norm map and τ is the generator of $Gal(\mathbb{C}/\mathbb{R})$.

We associate to π the usual parameters uniquely defined for each character χ and ξ. For simplicity, we write the set of parameters as a $(q+r)$-uple:

$$(t, t') = (t_1, ..., t_q, t'_1, ..., t'_r) \in \mathbb{R}^{q+r} \cong X(M).$$

Now, if $\pi(\chi_1, ..., \chi_q, \xi_1, ..., \xi_r)$ lies in the connected component defined by the fixed parameters $(\ell, \varepsilon) \in \mathbb{Z}^q \times (\mathbb{Z}/2\mathbb{Z})^r$, then

$$(t, t') \in X(M) \mapsto (t, t, 2t') \in Y(T)$$

is a continuous proper map.

It follows that

$$\mathcal{BC} : X(M)/W_\sigma(M) \to Y(T)/W_{\sigma'}(T)$$

is continuous and proper since the orbit spaces are endowed with the quotient topology. □

Theorem 5.5. *The functorial map induced by base change*

$$K_j(C_r^*\mathrm{GL}_n(\mathbb{C})) \stackrel{K_j(\mathcal{BC})}{\longrightarrow} K_j(C_r^*\mathrm{GL}_n(\mathbb{R}))$$

is zero for $n > 1$.

Proof. We start with the case $n > 2$. Let $n = 2q + r$ be a partition and M the associated Levi subgroup of $\mathrm{GL}_n(\mathbb{R})$. Denote by $X_\mathbb{R}(M)$ the unramified characters of M. As we have seen, $X_\mathbb{R}(M)$ is parametrized by $\mathbb{R}^{q+r}$. On the other hand, the only Levi subgroup of $GL_n(\mathbb{C})$ for $n = 2q + r$ is the diagonal subgroup $X_\mathbb{C}(M) = (\mathbb{C}^\times)^{2q+r}$.

If $q = 0$ then $r = n$ and both $X_\mathbb{R}(M)$ and $X_\mathbb{C}(M)$ are parametrized by $\mathbb{R}^n$. But then in the real case an element $\sigma \in E_2({}^0M)$ is given by

$$\sigma = i_{\mathrm{GL}_n(\mathbb{R}),P}(\chi_1 \otimes ... \otimes \chi_n),$$

with $\chi_i \in \widehat{\mathbb{Z}/2\mathbb{Z}}$. Since $n \geq 3$ there is always repetition of the χ_i's. It follows that the isotropy subgroups $W_\sigma(M)$ are all nontrivial and the spaces $\mathbb{R}^n/W_\sigma$ are orbit spaces for which the K-theory groups vanish, see Lemma 3.3.

If $q \neq 0$, then $X_\mathbb{R}(M)$ is parametrized by $\mathbb{R}^{q+r}$ and $X_\mathbb{C}(M)$ is parametrized by $\mathbb{R}^{2q+r}$ (see Propositions 2.2 and 2.3).

Base change creates a map

$$\mathbb{R}^{q+r} \longrightarrow \mathbb{R}^{2q+r}.$$

Composing with the stereographic projections we obtain a map

$$S^{q+r} \longrightarrow S^{2q+r}$$

between spheres. Any such map is nullhomotopic [4, Proposition 17.9]. Therefore, the induced K-theory map

$$K^j(S^{2q+r}) \longrightarrow K^j(S^{q+r})$$

is the zero map.

The Case $n=2$. For $n=2$ there are two Levi subgroups of $\mathrm{GL}_2(\mathbb{R})$, $M_1 \cong \mathrm{GL}_2(\mathbb{R})$ and the diagonal subgroup $M_2 \cong (\mathbb{R}^\times)^2$. By Proposition 2.2 $X(M_1)$ is parametrized by $\mathbb{R}$ and $X(M_2)$ is parametrized by $\mathbb{R}^2$. The maximal torus T of $\mathrm{GL}_2(\mathbb{C})$ is the diagonal subgroup $(\mathbb{C}^\times)^2$. From Proposition 2.3 we have $X(T) \cong \mathbb{R}^2$.

Since $K^1(\mathcal{A}_2^t(\mathbb{C})) = 0$ by Theorem 5.1, we only have to consider the K^0 functor. The only contribution to $K^0(\mathcal{A}_2^t(\mathbb{R}))$ comes from $M_2 \cong (\mathbb{R}^\times)^2$ and we have (see Example 3.5)

$$K^0(\mathcal{A}_2^t(\mathbb{R})) \cong \mathbb{Z}.$$

For the Levi subgroup $M_2 \cong (\mathbb{R}^\times)^2$, base change is

$$\begin{array}{ccc} \mathcal{BC} : \mathcal{A}_2^t(\mathbb{R}) & \longrightarrow & \mathcal{A}_2^t(\mathbb{C}) \\ \pi(\xi_1,\xi_2) & \mapsto & i_{\mathrm{GL}_2(\mathbb{C}),B(\mathbb{C})}(\xi_1 \circ N, \xi_2 \circ N), \end{array}$$

Therefore, it maps a class $[t_1,t_2]$, which lies in the connected component $(\varepsilon_1,\varepsilon_2)$, into the class $[2t_1,2t_2]$, which lies in the connect component $(0,0)$. In other words, base change maps a generalized principal series $\pi(\xi_1,\xi_2)$ into a component of $\mathcal{A}_2^t(\mathbb{C})$ whose discrete factor is a nongeneric orbit. It follows from Theorem 3.9 that

$$K^0(\mathcal{BC}) : K^0(\mathcal{A}_2^t(\mathbb{R})) \to K^0(\mathcal{A}_2^t(\mathbb{C}))$$

is the zero map. □

5.6. **Base change in one dimension.** In this section we consider base change for GL_1.

Theorem 5.7. *The functorial map induced by base change*

$$K_1(C_r^*\mathrm{GL}_1(\mathbb{C})) \overset{K_1(\mathcal{BC})}{\longrightarrow} K_1(C_r^*\mathrm{GL}_1(\mathbb{R}))$$

is given by $K_1(\mathcal{BC}) = \Delta \circ Pr$, where Pr is the projection of the zero component of $K^1(\mathcal{A}_1^t(\mathbb{C}))$ into $\mathbb{Z}$ and Δ is the diagonal $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$.

Proof. For GL_1, base change

$$\chi \in \mathcal{A}_1^t(\mathbb{R}) \mapsto \chi \circ N_{\mathbb{C}/\mathbb{R}} \in \mathcal{A}_1^t(\mathbb{C})$$

induces a map

$$K^1(\mathcal{BC}) : K^1(\mathcal{A}_1^t(\mathbb{C})) \to K^1(\mathcal{A}_1^t(\mathbb{R})).$$

Any character $\chi \in \mathcal{A}_1^t(\mathbb{R})$ is uniquely determined by a pair of parameters $(t,\varepsilon) \in \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$. Similarly, any character $\mu \in \mathcal{A}_1^t(\mathbb{C})$ is uniquely determined by a pair of parameters $(t,\ell) \in \mathbb{R} \times \mathbb{Z}$. The discrete parameter ε (resp., ℓ) labels each connected component of $\mathcal{A}_1^t(\mathbb{R}) = \mathbb{R} \sqcup \mathbb{R}$ (resp., $\mathcal{A}_1^t(\mathbb{C}) = \bigsqcup_{\mathbb{Z}} \mathbb{R}$).

Base change maps each component ε of $\mathcal{A}_1^t(\mathbb{R})$ into the component 0 of $\mathcal{A}_1^t(\mathbb{C})$, sending $t \in \mathbb{R}$ to $2t \in \mathbb{R}$. The map $t \mapsto 2t$ is homotopic to the identity. At the level of K^1, the base change map is given by $K_1(\mathcal{BC}) = \Delta \circ Pr$, where Pr is the projection of the zero component of $K^1(\mathcal{A}_1^t(\mathbb{C}))$ into $\mathbb{Z}$ and Δ is the diagonal $\mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$.

□

6. AUTOMORPHIC INDUCTION

We begin this section by describing the action of $Gal(\mathbb{C}/\mathbb{R})$ on $\widehat{W_{\mathbb{C}}} = \widehat{\mathbb{C}^\times}$. Take $\chi = \chi_{\ell,t} \in \widehat{\mathbb{C}^\times}$ and let τ denote the nontrivial element of $Gal(\mathbb{C}/\mathbb{R})$. Then, $Gal(\mathbb{C}/\mathbb{R})$ acts on $\widehat{\mathbb{C}^\times}$ as follows:

$$\chi^\tau(z) = \chi(\overline{z}).$$

Hence,

$$\chi^\tau_{\ell,t}(z) = (\frac{\overline{z}}{|z|})^\ell |z|_{\mathbb{C}}^{it} = (\frac{z}{|z|})^{-\ell} |z|_{\mathbb{C}}^{it}$$

and we conclude that

$$\chi^\tau_{\ell,t}(z) = \chi_{-\ell,t}(z).$$

In particular,

$$\chi^\tau = \chi \Leftrightarrow \ell = 0 \Leftrightarrow \chi = |.|_{\mathbb{C}}^{it}$$

i.e, χ is unramified.

Note that $W_{\mathbb{C}} \subset W_{\mathbb{R}}$, with index $[W_{\mathbb{R}} : W_{\mathbb{C}}] = 2$. Therefore, there is a natural induction map

$$Ind_{\mathbb{C}/\mathbb{R}} : \mathcal{G}_1^t(\mathbb{C}) \to \mathcal{G}_2^t(\mathbb{R}).$$

By the local Langlands correspondence for archimedean fields [3, 10], there exists an automorphic induction map $\mathcal{AI}_{\mathbb{C}/\mathbb{R}}$ such that the following diagram commutes

$$\begin{tikzcd}
\mathcal{A}_1^t(\mathbb{C}) \arrow[r, "\mathcal{AI}_{\mathbb{C}/\mathbb{R}}"] & \mathcal{A}_2^t(\mathbb{R}) \\
\mathcal{G}_1^t(\mathbb{C}) \arrow[u, "{}_{\mathbb{C}}\mathcal{L}_1"] \arrow[r, "Ind_{\mathbb{C}/\mathbb{R}}"'] & \mathcal{G}_2^t(\mathbb{R}) \arrow[u, "{}_{\mathbb{R}}\mathcal{L}_2"']
\end{tikzcd}$$

Proposition 6.1. *If $\ell \neq 0$ then we have*

$$\mathrm{Ind}_{\mathbb{C}/\mathbb{R}}(\chi_{\ell,t}) \simeq \mathrm{Ind}_{\mathbb{C}/\mathbb{R}}(\chi_{-\ell,t}) \simeq \varphi_{\ell,t}$$

If $\ell = 0$ then we have

$$\mathrm{Ind}_{\mathbb{C}/\mathbb{R}}(\chi_{0,t}) = (+, 2t) \oplus (-, 2t)$$

Proof. It is enough to use Frobenius reciprocity. We start with $\ell \neq 0$, and apply (9):

$$\begin{aligned} < Ind_{\mathbb{C}/\mathbb{R}}(\chi_{\ell,t}), \varphi_{\ell,t} > &=< \chi_{\ell,t}, Res_{\mathbb{C}/\mathbb{R}}(\varphi_{\ell,t}) > \\ &=< \chi_{\ell,t}, \chi_{\ell,t} \oplus \chi_{-\ell,t} > \\ &= 1 \end{aligned}$$

$$\begin{aligned} < Ind_{\mathbb{C}/\mathbb{R}}(\chi_{-\ell,t}), \varphi_{\ell,t} > &=< \chi_{-\ell,t}, Res_{\mathbb{C}/\mathbb{R}}(\varphi_{\ell,t}) > \\ &=< \chi_{-\ell,t}, \chi_{\ell,t} \oplus \chi_{-\ell,t} > \\ &= 1 \end{aligned}$$

$$\begin{aligned} < Ind_{\mathbb{C}/\mathbb{R}}(\chi_{0,t}), (+, 2t) > &=< \chi_{0,t}, Res_{\mathbb{C}/\mathbb{R}}(+, 2t) > \\ &=< \chi_{0,t}, \chi_{0,t} > \\ &= 1 \end{aligned}$$

$$\begin{aligned} < Ind_{\mathbb{C}/\mathbb{R}}(\chi_{0,t}), (-, 2t) > &=< \chi_{0,t}, Res_{\mathbb{C}/\mathbb{R}}(-, 2t) > \\ &=< \chi_{0,t}, \chi_{0,t} > \\ &= 1 \end{aligned}$$

□

6.2. **The automorphic induction map.** In the case of $\mathrm{GL}_{2n}(\mathbb{R})$ we will have to consider the discrete series representations

$$D_{|\ell_1|} \otimes |\det(.)|^{it_1} \otimes \cdots \otimes D_{|\ell_n|} \otimes |\det(.)|^{it_n}$$

on the Levi subgroup $M = \mathrm{GL}_2(\mathbb{R}) \times \cdots \times \mathrm{GL}_2(\mathbb{R}) \subset \mathrm{GL}_{2n}(\mathbb{R})$. Let $P = MN$ be the corresponding parabolic subgroup, and, using a classical notation, denote by

$$D_{|\ell_1|} \otimes |\det(.)|^{it_1} \times \cdots \times D_{|\ell_n|} \otimes |\det(.)|^{it_n}$$

the corresponding irreducible tempered representations of $\mathrm{GL}_{2n}(\mathbb{R})$ obtained via parabolic induction.

In the same notation, denote by

$$\chi_{\ell_1,it_1} \times \cdots \times \chi_{\ell_n,it_n}$$

the irreducible tempered representation of $\mathrm{GL}_n(\mathbb{C})$ coming via parabolic induction from the the unitary character $\chi_{\ell_1,it_1} \otimes \cdots \otimes \chi_{\ell_n,it_n}$ on the standard maximal torus of $\mathrm{GL}_n(\mathbb{C})$.

Define $\pi(|\ell_j|, t_j)$ as follows:

$$\pi(|\ell_j|, t_j) = D_{|\ell_j|} \otimes |\det|^{it_j} \quad \text{if} \quad \ell_j \neq 0 \tag{16}$$

$$= 1 \otimes |\det|^{2it} \times sgn \otimes |\det|^{2it} \quad \text{if} \quad \ell_j = 0 \tag{17}$$

Consider now the locally compact Hausdorff space

$$\mathfrak{E}(|\ell_1|, \ldots, |\ell_n|) := \{\pi(|\ell_1|, t_1) \times \cdots \times \pi(|\ell_n|, t_n) : t_1, \ldots, t_n \in \mathbb{R}\}$$

which is a subspace of the tempered dual of $\mathrm{GL}_{2n}(\mathbb{R})$, and the locally compact Hausdorff space

$$\mathfrak{F}(\ell_1, \ldots, \ell_n) := \{\chi_{\ell_1, it_1} \times \cdots \times \chi_{\ell_n, it_n} : t_1, \ldots, t_n \in \mathbb{R}\}$$

which is a subspace of the tempered dual of $\mathrm{GL}_n(\mathbb{C})$.

Then the automorphic induction map $\mathcal{AI}_n$ maps the space $\mathfrak{F}(\ell_1, \ldots, \ell_n)$ bijectively onto the space $\mathfrak{E}(|\ell_1|, \ldots, |\ell_n|)$ via the natural identification of the coordinates $t_1, \ldots, t_n$:

$$\mathcal{AI}_n : \mathfrak{F}(\ell_1, \ldots, \ell_n) \simeq \mathfrak{E}(|\ell_1|, \ldots, |\ell_n|)$$

We have the functorial K-theory map

$$K_j(\mathcal{AI}_n) : K^j(\mathfrak{E}(|\ell_1|, \ldots, |\ell_n|)) \simeq K^j(\mathfrak{F}(i_1, \ldots, i_n)) \tag{18}$$

whenever $i_1 = \pm\ell_1, \ldots, i_n = \pm\ell_n$.

Suppose first that the ℓ_j are all distinct, with none of them 0. Then $\mathfrak{E}(|\ell_1|, \ldots, |\ell_n|)$ and $\mathfrak{F}(\ell_1, \ldots, \ell_n)$ are n-dimensional Euclidean spaces. In the isomorphism (18), a generator for the left-hand-side, denoted $\delta(\ell_1, \ldots, \ell_n)$, will correspond to a generator for the right-hand-side, denoted $\varepsilon(i_1, \ldots, i_n)$.

The image of the generator $\delta(\ell_1, \ldots, \ell_n)$ under $K_j(\mathcal{AI}_n)$ has 2^n components, which lie in the K-theory groups $K^j(\mathfrak{F}(i_1, \ldots, i_n))$ with $i_1 = \pm\ell_1, \ldots, i_n = \pm\ell_n$. The component in $K^j(\mathfrak{F}(i_1, \ldots, i_n))$ is $\varepsilon(i_1, \ldots, i_n)$. This is automorphic induction at the level of K-theory.

Now we re-consider the space $\mathfrak{F}(\ell_1, \ldots, \ell_n)$. If two or more of the ℓ_j are equal, then $\mathfrak{F}(\ell_1, \ldots, \ell_n)$ is the Cartesian product of locally compact Hausdorff spaces, each of which is either a symmetric product of real lines, or a Euclidean space. Then we have $K^j(\mathfrak{F}(\ell_1, \ldots, \ell_n) = 0$ for $j = 0, 1$ by Lemma (3.2) and the Künneth theorem [13, 3.3.15]. So the map in (18) is the zero map.

This leaves one case to be considered, namely when some of the ℓ_j are equal to 0. We start with the case when one of the ℓ_j is 0, say $\ell_1 = 0$. Define

$$\begin{aligned} &\mathfrak{X}(0, \ldots, |\ell_n|) : \\ &= \{1 \otimes |\det|^{2is_1} \times sgn \otimes |\det|^{2it_1} \times \cdots \times \pi(|\ell_n|, t_n) : s_1, t_1, \ldots, t_n \in \mathbb{R}\} \end{aligned}$$

We then have an injective map

$$\mathcal{AI}_n : \mathfrak{F}(0, \ldots, \ell_n) \to \mathfrak{X}(0, \ldots, |\ell_n|)$$

The dimensions of these two Euclidean spaces are n and $n + 1$. The parity difference implies that the induced K-theory map is the zero map.

If several of the ℓ_j are equal to 0, say $\ell_j = 0$ for $1 \le j \le k$, then we will correspondingly have an injective map

$$\mathcal{AI}_n : \mathfrak{F}(0, \ldots, 0, \ldots, \ell_n) \to \mathfrak{X}(0, \ldots, 0, \ldots, |\ell_n|)$$

where $\mathfrak{X}(0, \ldots, 0, \ldots |\ell_n|)$ denotes a space modelled on $\mathfrak{X}(0, \ldots, |\ell_n|)$ but including the term

$$1 \otimes |\det|^{2is_1} \times sgn \otimes |\det|^{2it_1} \times \cdots \times 1 \otimes |\det|^{2is_k} \times sgn \otimes |\det|^{2it_k}$$

The space $\mathfrak{X}(0,\dots,0,\dots,|\ell_n|)$ will be a Cartesian product of locally compact Hausdorff spaces, each of which is either a symmetric product of real lines, or a Euclidean space. Such spaces are trivial in K-theory.

This leads to our final result. Let $K_j(\mathcal{AI}_n)$ denote the functorial K-theory map induced by automorphic induction.

Theorem 6.3. *Consider the functorial map induced by automorphic induction*

$$K_j(C_r^*\mathrm{GL}_{2n}(\mathbb{R})) \stackrel{K_j(\mathcal{AI}_n)}{\longrightarrow} K_j(C_r^*\mathrm{GL}_n(\mathbb{C})).$$

Suppose that $n \equiv j \mod 2$*, and let* $0 < \ell_1 < \dots < \ell_n$*. The* K^j*-generator* $\delta(\ell_1,\dots,\ell_n)$ *is determined by the discrete series representations* $D_{\ell_1},\dots,D_{\ell_n}$*. The image of this generator under* $K_j(\mathcal{AI}_n)$ *has* 2^n *components, which lie in the* K*-theory groups* $K^j(\mathfrak{F}(i_1,\dots,i_n))$ *with* $i_1 = \pm\ell_1,\dots,i_n = \pm\ell_n$*. The component in* $K^j(\mathfrak{F}(i_1,\dots,i_n))$ *is* $\varepsilon(i_1,\dots,i_n)$*.*

7. K-CYCLE

The standard maximal compact subgroup of $\mathrm{GL}_1(\mathbb{C})$ is the circle group $U(1)$, and the maximal compact subgroup of $\mathrm{GL}_1(\mathbb{R})$ is $\mathbb{Z}/2\mathbb{Z}$. Base change for K^1 creates a map

$$\mathcal{R}(\mathrm{U}(1)) \longrightarrow \mathcal{R}(\mathbb{Z}/2\mathbb{Z})$$

where $\mathcal{R}(\mathrm{U}(1))$ is the representation ring of the circle group $\mathrm{U}(1)$ and $\mathcal{R}(\mathbb{Z}/2\mathbb{Z})$ is the representation ring of the group $\mathbb{Z}/2\mathbb{Z}$. This map sends the trivial character of $\mathrm{U}(1)$ to $1 \oplus \varepsilon$, where ε is the nontrivial character of $\mathbb{Z}/2\mathbb{Z}$, and sends all the other characters of $\mathrm{U}(1)$ to zero.

This map has an interpretation in terms of K-cycles. The real line $\mathbb{R}$ is a universal example for the action of $\mathbb{C}^\times$ and $\mathbb{R}^\times$. The K-cycle

$$(C_0(\mathbb{R}), L^2(\mathbb{R}), id/dx) \tag{19}$$

is equivariant with respect to $\mathbb{C}^\times$ and $\mathbb{R}^\times$. The actions are

$$\mathbb{C}^\times \times \mathbb{R} \to \mathbb{R}, \quad (z,y) \mapsto \log|z| + y$$

$$\mathbb{R}^\times \times \mathbb{R} \to \mathbb{R}, \quad (x,y) \mapsto \log|x| + y$$

The K-cycle (19) therefore determines a class $\partial\!\!\!/_{\mathbb{C}} \in K_1^{\mathbb{C}^\times}(\underline{E}\mathbb{C}^\times)$ and a class $\partial\!\!\!/_{\mathbb{R}} \in K_1^{\mathbb{R}^\times}(\underline{E}\mathbb{R}^\times)$. On the left-hand-side of the Baum-Connes correspondence, base change in dimension 1 admits the following description in terms of Dirac operators:

$$\partial\!\!\!/_{\mathbb{C}} \mapsto (\partial\!\!\!/_{\mathbb{R}}, \partial\!\!\!/_{\mathbb{R}})$$

It would be interesting to interpret the automorphic induction map at the level of equivariant K-theory:

$$\mathcal{AI}^* : K^*_{\mathrm{O}(n)}(\mathbb{R}^n) \longrightarrow K^*_{\mathrm{U}(n)}(\mathbb{C}^n).$$

REFERENCES

[1] J. Arthur, L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Math. Studies 120, Princeton University Press, Princeton, 1989.

[2] P. Baum, A. Connes, N. Higson, Classifying space for proper actions and K-theory for group C^*-algebras, Contemporary Math., **167** (1994) 241–291.

[3] J. Bernstein, S. Gelbart et al, *An introduction to the Langlands program*, Birkhauser 2003.

[4] R. Bott, L.W. Tu, *Differential Forms in Algebraic Topology*, Springer-Verlag, New York 1982.

[5] N. Bourbaki, *Lie groups and Lie algebras*, Chapters 4-6, Springer 2002.

[6] J. Chabert, S. Echterhoff, R. Nest, The Connes-Kasparov conjecture for almost connected groups and for linear p-adic groups, Publications Math. I.H.E.S. **97** (2003) 239–278.

[7] K. F. Chao, H. Wang, Langlands functoriality in K-theory for C^*-algebras. I. Base change, preprint 2015.

[8] S. Echterhoff, O. Pfante, Equivariant K-theory of finite-dimensional real vector spaces, Munster J. of Math. **2** (2009) 65–94.

[9] Harish-Chandra, The Plancherel theorem for complex semi-simple Lie groups, Trans. Amer. Math. Soc. **76** (1954) 485–528.

[10] A. W. Knapp, Local Langlands correspondence: the archimedean case, Proc. Symp. Pure. Math. **55**, 1994, 393–410.

[11] V. Lafforgue, K-théorie bivariante pour les algèbres de Banach et conjecture de Baum-Connes, Invent. Math. **149** (2002) 1–95.

[12] S. Mendes, R.J. Plymen, Base change and K-theory for $\mathrm{GL}(n)$, J. Noncommut. Geom., **1** (2007), 311–331.

[13] E. Park, *Complex topological K-theory*, Cambridge studies in advanced mathematics **111** (2008).

[14] M. G. Penington, R. J. Plymen, The Dirac operator and the principal series for complex semisimple Lie groups, J. Functional Analysis, **53** (1983) 269–286.

[15] R. J. Plymen, The reduced C*-algebra of the p-adic group $\mathrm{GL}(n)$, J. Functional Analysis **72** (1987) 1–12.

S. Mendes
ISCTE, Av. das Forças Armadas
1649-026, Lisbon, Portugal
E-mail: `sergio.mendes@iscte.pt`

R.J. Plymen
Southampton University
Southampton SO17 1BJ, England
and
School of Mathematics
Manchester University
Manchester M13 9PL, England
E-mail: `r.j.plymen@soton.ac.uk`
`plymen@manchester.ac.uk`