A constant, radial, low-thrust problem including first order effects of J_2*

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I. Introduction

Tsien studied continuous low-thrust trajectories for Earth escape from circular orbit, and in particular, he provided analytic solutions to the constant radial thrust problem, thereafter also known as the *Tsien problem* [2]. A comprehensive analysis of the problem was presented by Battin [3], Boltz [4], and latter work by other authors [5, 6, 7, 8] has provided further insight to the problem. The Tsien problem is integrable and admits closed-form solutions in terms of standard elliptic integrals: the radial time evolution depends on the incomplete elliptic integrals of the first and second kinds, and the orbit evolution is known to depend on the incomplete elliptic integral of the third kind [6]. Many alternative analytical solutions have been proposed, both exact [9, 10, 11] and approximate [12], as well as many interesting and innovative applications that build on radial thrust configurations [13, 14].

However, the integrability of the problem does not hold when the Earth's oblateness perturbation is included in the dynamics. Since the dynamics of the J_2 problem are non-integrable (except for equatorial motion), the only possible closed-form solutions are obtained by approximating or averaging the full J_2 gravitational potential [15]. Among the many possibilities to approximate the J_2 problem, the concept of *intermediary orbits* is particularly useful. The aim of the intermediary is to capture in an integrable model the main effects of the dynamics of the original problem. If, besides, integrability is found after an infinitesimal contact transformation, the solution improves by incorporating the short-period dynamics into the intermediary. Among these intermediaries, Deprit's radial intermediary [16] was found particularly

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convenient because it provides a closed-form solution in terms of trigonometric functions, whereas other intermediaries rely on the evaluation of special functions.

We base on the radial intermediary, and obtain an integrable approximation to the constant, radial thrust problem which includes the main effects of the earth's oblateness. The analytical solution is valid as far as the thrust remains small and is not constrained to planar motion, thus providing notable improvements in the application of the Tsien problem to the maneuver design of Earth satellite orbits.

The paper starts with a brief review of the Tsien problem within Section 2, and in Section 3 the perturbation effects due to the Earth's oblateness are incorporated in the constant radial thrust problem. Finally, the accuracy of the formulation is assessed in Section 4, and the station-keeping of a GPS satellite is presented as an application example to support the suitability of the proposed method.

II. The Tsien Problem

The Tsien problem [2] is a planar motion with a constant, radial thrust acceleration. A spacecraft, initially in circular orbit of radius r_0 and velocity $r_0\,\omega_0$ (with $\omega_0 = \mu/r_0^3$ and μ is the gravitational parameter of the primary) is acted upon by a constant radial acceleration α. This acceleration is defined to be positive when pointing outwards, and negative when directed inwards. When the following non-dimensional parameter is defined

$$\varepsilon = \frac{8\,\alpha}{r_0\,\omega_0^2}$$

it is found that there is a limit value of α such that when $\varepsilon = 1$, the trajectory leads to an asymptotic circular orbit. Below that value the trajectory is bounded ($\varepsilon < 1$), whereas higher values ($\varepsilon > 1$) yield escape trajectories [11].

When the primary is assumed spherical and the radial thrust is the only perturbation acting upon the spacecraft, the Keplerian potential

$$V = -\frac{\mu}{r}$$

where r is the orbital radius, is modified by the disturbing potential

$$U = -\alpha\,r \tag{1}$$

Since the problem is conservative it can be approached from the point of view of the Hamiltonian formalism. Thus,

$$\mathcal{H} = T + V + U$$

where T is the kinetic energy, $T + V$ corresponds to the Kepler problem and U is the disturbing function of the Keplerian motion.

In particular, the orbital motion is naturally described in the canonical set of polar-nodal variables (r, θ, ν, R, Θ, N), where r has been defined above, θ is the argument of latitude, ν is the right ascension of the ascending node, R is the radial velocity, Θ is the magnitude of the angular momentum vector, $N = \Theta\,\cos i$ is the projection of the angular momentum vector on the rotation axis of the primary, and i is the orbital inclination. Then, the Hamiltonian is written as

$$\mathcal{H} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} - \alpha\, r \tag{2}$$

Note that θ does not appear in the Hamiltonian so Θ is an integral of the problem. Moreover, ν and N do not appear in the Hamiltonian either. Thus, Eq. (2) is a radial Hamiltonian $\mathcal{H} \equiv \mathcal{H}(r, -, -, R, \Theta, -)$ which is also independent of the orbital plane since there is no information on the orbital inclination nor the line of nodes. Indeed, from Hamilton equations it is immediate that ν, Θ and N remain constant and the flow is separable. The radial motion is described by the differential equations

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial R} = R \tag{3a}$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial r} = \frac{\Theta^2}{r^3} - \frac{\mu}{r^2} + \alpha \tag{3b}$$

Combining Eq. (3a) with the Hamiltonian in Eq. (2), for each constant manifold $h = \mathcal{H}(r_0, -, -, R_0, \Theta_0, -)$ defined from the intitial conditions, the radial problem can be expressed as [9]

$$t - t_0 = \frac{1}{\sqrt{2}}\int_{r_0}^{r(t)} \frac{r\,\mathrm{d}r}{\sqrt{\alpha\, r^3 + h\, r^2 + \mu\, r - \frac{1}{2}\Theta^2}}$$

and the latter expression can be solved analytically as a function of elliptic integrals of the first and second kinds [3]. Once $r(t)$ is known, the argument of latitude is computed from the quadrature

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial\Theta} = \frac{\Theta}{r^2(t)} \tag{3c}$$

which can also be solved by introducing elliptic integrals of the third kind in the orbit solution [6]. Additional analytical solutions exist for the Tsien problem [9, 10, 11].

III. Including the Effects of the Oblateness

Let us reformulate the Tsien problem including not only a constant, radial acceleration but also the oblateness of the primary. Thus, the new disturbing potential becomes

$$U = -\alpha\, r + \frac{\mu}{r}\frac{R_\oplus^2}{r^2} J_2 P_2\left(\cos\phi\right) \tag{4}$$

where $R_\oplus$ is the equatorial radius of the primary, J_2 is the second degree zonal harmonic coefficient of the gravitational potential, P_2 is the Legendre polynomial of the second degree, and ϕ is the equatorial latitude. Using spherical trigonometry the equatorial latitude and the argument of latitude can be related by the expression

$$\cos\phi = \sin i\, \sin\theta$$

Note that the problem remains conservative.

When the orbital motion is described in polar-nodal variables, the Hamiltonian is written as

$$\mathcal{H} = \frac{1}{2}\left(R^2 + \frac{\Theta^2}{r^2}\right) - \frac{\mu}{r} - \alpha\, r - \frac{\mu}{r}\sigma\frac{p^2}{r^2}\left(1 - \frac{3}{2}\sin^2 i + \frac{3}{2}\sin^2 i\, \cos 2\theta\right) \tag{5}$$

where

$$p = \frac{\Theta^2}{\mu}$$

is the conic parameter, also called *semilatus rectum*, and we introduce the parameter

$$\sigma \equiv \sigma(\Theta) = \frac{1}{2} J_2 \frac{R_\oplus^2}{p^2} \tag{6}$$

The corresponding Hamilton equations are

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial R} = R \tag{7a}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial\Theta} = \frac{\Theta}{r^2}\Big[1 + 3\,\sigma\,(1+k)\,c^2\,(1-\cos 2\theta)\Big] \tag{7b}$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}t} = \frac{\partial\mathcal{H}}{\partial N} = \frac{\Theta}{r^2}\Big[-3\,\sigma\,(1+k)\,c\,(1-\cos 2\theta)\Big] \tag{7c}$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial r} = \frac{\Theta^2}{r^3}\left[\frac{k}{1+k} - 3\,\sigma\,(1+k)\left(1 - \frac{3}{2}\,s^2\,(1-\cos 2\theta)\right)\right] + \alpha \tag{7d}$$

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial\theta} = \frac{\Theta^2}{r^2}\Big[-3\,\sigma\,(1+k)\,s^2\,(1-\cos 2\theta)\Big] \tag{7e}$$

$$\frac{\mathrm{d}N}{\mathrm{d}t} = -\frac{\partial\mathcal{H}}{\partial\nu} = 0 \tag{7f}$$

where we abbreviated[a]

$$k \equiv k(r,\Theta) = \frac{p}{r} - 1, \qquad q \equiv q(R,\Theta) = \frac{p\,R}{\Theta}$$

and[b]

$$c \equiv c(\Theta,N) = \frac{N}{\Theta}, \qquad s \equiv s(\Theta,N) = \sqrt{1 - \frac{N^2}{\Theta^2}}$$

Note that ν does not appear in the Hamiltonian and, therefore, N is an integral of the problem. However, as opposed to the Tsien problem, now θ and N (implicitly through the orbital inclination) do appear in the Hamiltonian. Also, in the following we restrict our analysis to the case of bounded motion around the primary and low values of the radial acceleration.

The flow derived from Eq. (5) is not expected to be integrable since general solutions to the J_2 problem are not known. However, an integrable approximation to the Hamiltonian (5) can be obtained by means of a transformation of variables. Particularly, Deprit's radial intermediary is designed to simplify the J_2 problem Hamiltonian by removing non-essential, short-periodic effects [16]. Let $\mathcal{T}$ be an infinitesimal contact transformation

$$\mathcal{T} : (r,\theta,\nu,R,\Theta,N) \to (r',\theta',\nu',R',\Theta',N')$$

defined by the corrections [15]

$$\Delta r = -\sigma\, p\left(1 - \frac{3}{2}s^2 - \frac{1}{2}s^2\,\cos 2\theta\right) \tag{8a}$$

$$\Delta\theta = -\sigma\left\{(1-6c^2)\,q + (1-2c^2)\,q\,\cos 2\theta - \left[\frac{1}{4} + k - \left(\frac{7}{4} + 3k\right)c^2\right]\sin 2\theta\right\} \tag{8b}$$

$$\Delta\nu = -\sigma\, c\left[(3+\cos 2\theta)\,q - \left(\frac{3}{2} + 2k\right)\sin 2\theta\right] \tag{8c}$$

$$\Delta R = -\sigma\frac{\Theta}{p}\,(1+k)^2\,s^2\,\sin 2\theta \tag{8d}$$

$$\Delta\Theta = \sigma\,\Theta\,s^2\left[\left(\frac{3}{2} + 2k\right)\cos 2\theta + q\,\sin 2\theta\right] \tag{8e}$$

$$\Delta N = 0 \tag{8f}$$

in which the right-hand side of the equations must be evaluated in original variables in the direct transformation

$$\xi' = \xi - \Delta\xi, \qquad \xi \in (r,\theta,\nu,R,\Theta,N)$$

and in prime variables for obtaining the inverse transformation

$$\xi = \xi' + \Delta\xi', \qquad \xi' \in (r',\theta',\nu',R',\Theta',N').$$

[a]Note that $k = e\,\cos f$ and $q = e\,\sin f$, where e is the orbital eccentricity and f the true anomaly.
[b]Note that $c = \cos i$ and $s = \sin i$, where i is the orbital inclination.

If the thrust remains small, say comparable to the oblateness perturbation

$$\alpha\, r \sim \frac{\mu}{r}\frac{1}{2}J_2\frac{R_\oplus^2}{r^2}$$

it can be checked that, up to second order effects of J_2, the infinitesimal contact transformation $\mathcal{T}$ converts the Hamiltonian in Eq. (5) into

$$\mathcal{H}' = \frac{1}{2}\left(R'^2 + \frac{\Theta'^2}{r'^2}\right) - \frac{\mu}{r'} - \alpha\, r' - \frac{\mu}{p'}\sigma'\frac{p'^2}{r'^2}\left(1 - \frac{3}{2}s'^2\right) \tag{9}$$

where the argument of latitude has been removed and hence Θ' is an integral of the problem. Then, Eq. (9) is a radial Hamiltonian $\mathcal{H}' \equiv \mathcal{H}'(r', -, -, R', \Theta', N')$ which is, therefore, integrable. For clarity in the notation primes will be dropped from here on.

Notably, Eq. (9) can be reorganized as follows

$$\mathcal{H} = \frac{1}{2}\left(R^2 + \frac{\tilde{\Theta}^2}{r^2}\right) - \frac{\mu}{r} - \alpha\, r \tag{10}$$

where the constant, *varied angular momentum* $\tilde{\Theta} \equiv \tilde{\Theta}(\Theta, N)$ is defined as

$$\tilde{\Theta} = \Theta\sqrt{1 - \sigma(2 - 3s^2)} \tag{11}$$

Hence, up to second order effects Eq. (10) is, in the new (prime) variables, formally equal to the constant, radial thrust problem of Eq. (2), but with the varied angular momentum $\tilde{\Theta}$ instead. Then, the radial motion for Eq. (10) is described by

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial R} = R \tag{12a}$$

$$\frac{\mathrm{d}R}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial r} = \frac{\tilde{\Theta}^2}{r^3} - \frac{\mu}{r^2} + \alpha \tag{12b}$$

which are analogous to Eqs. (3a-3b), and thus the same analytical solutions can be exploited to solve the radial motion of the problem, just by adjusting the value of the angular momentum.

However, the non-radial part of the Hamiltonian in Eq. (2) does not apply to the Hamiltonian (10) for two reasons. One is that Θ is the true canonical variable of the Hamiltonian (10), not $\tilde{\Theta}$. The other reason is that the latter Hamiltonian has a hidden dependency with N through the variable $\tilde{\Theta}(\Theta, N)$, whereas in the Tsien problem there is no such dependency. As a consequence, once $r(t)$ is known, the solution to the Hamiltonian in Eq. (10) must be completed by solving the following quadratures instead of the argument of latitude and the right ascension of the ascending node

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial \Theta} = \frac{\Theta}{r^2(t)}\left[1 - \sigma(1 - 6c^2)\right] \tag{12c}$$

$$\frac{\mathrm{d}\nu}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial N} = -3\,\frac{\Theta}{r^2(t)}\,\sigma\, c \tag{12d}$$

which can be solved analytically as well [9]. Since Θ and N are both first integrals of the Hamiltonian (10), the problem is closed.

Finally, the simple application of the inverse transformation $\mathcal{T}^{-1}$ recovers the solution in the original (non-prime) variables up to the truncation order.

IV. Accuracy and Applications to GPS Satellites

The accuracy of the proposed approximate solution degrades as the disturbing function U becomes more important. If $\alpha = 0$ and $\sigma = 0$ the problem becomes Keplerian, but as the oblateness increases, the Hamiltonian system deviates from integrable proportionally to σ. If the Hamiltonian in Eq. (5) is made dimensionless and the characteristic length and time are appropriately chosen, then the Hamiltonian can be conveniently written such that $\mu = 1$, the initial circular orbit has a radius $r_0 = 1$ and $\Theta = \Theta_0 = 1$. Note that in dimensionless form σ becomes, for a given primary, an explicit function of the equatorial radius $R_\oplus$, which is now scaled with the initial orbital radius:

$$\sigma \equiv \sigma(R_\oplus) = \frac{1}{2}\, J_2\, R_\oplus^2$$

As an immediate conclusion $R_\oplus \leqslant 1$ (see Table 1) and $\sigma(R_\oplus)$ has a maximum or *threshold* value $\sigma_{\mathrm{th}} = J_2/2$, which depends solely on the oblateness of the primary. High altitude orbits decrease the non-dimensional value of σ, which in turn improves the accuracy. However, an upper bound exists as well, where the third body perturbations become as relevant as the oblateness (see Figure 1). Therefore, the Medium Earth Orbit (MEO) region stands out as a promising target where the proposed model should work best in terms of accuracy. This region is mostly populated by global positioning satellite constellations, such as the GPS, Glonass or Galileo. In particular, the GPS orbit ($r_0 = 26\,578$ km, $i_0 = 55$ deg) will be taken as a benchmark.

Since the intermediary misses second order terms of J_2, the more important of which are proportional to $1 - 21\cos^4 i$ (cf. Eq. (5) of Ref. 17), an immediate insight is that the accuracy of the intermediary solution depends on inclination, with better performance for high inclination orbits —say between 55 and 125 degrees. The degradation of the accuracy can be studied by integrating the exact Hamiltonian (5) from Equations (7), integrating the approximate Hamiltonian (10) from Equations (12) and comparing both solutions in non-

Table 1. **Values for the dimensionless parameter $\sigma(R_\oplus)$ in Earth orbit**

Orbital Altitude	Dimensionless $R_\oplus$	Dimensionless $\sigma(R_\oplus)$
0 km	1.0000	$5.4131 \cdot 10^{-4}$
400 km	0.9410	$4.7931 \cdot 10^{-4}$
800 km	0.8886	$4.2738 \cdot 10^{-4}$
2 000 km	0.7613	$3.1372 \cdot 10^{-4}$
5 000 km	0.5606	$1.7010 \cdot 10^{-4}$
10 000 km	0.3894	$8.2093 \cdot 10^{-5}$
20 000 km	0.2418	$3.1648 \cdot 10^{-5}$
35 786 km	0.1513	$1.2387 \cdot 10^{-5}$

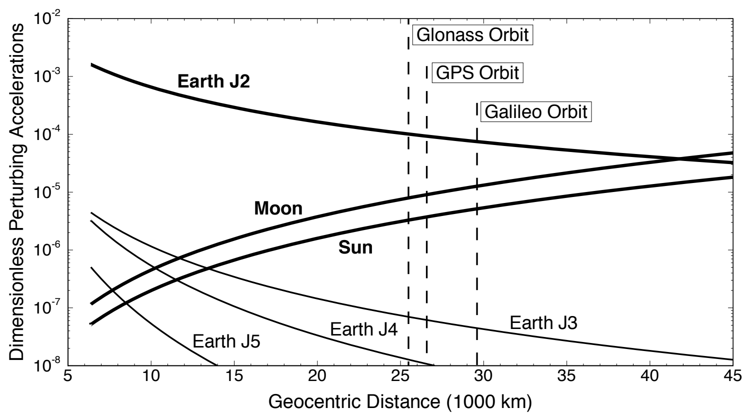


Figure 1. **Dimensionless perturbing acceleration in Earth orbit.**

prime variables [1]. A slight variability of the errors must be accepted which depends predominantly on the initial inclination i_0 (and by extension N) as well as the argument of latitude θ_0. This dependency is not surprising as Eqs. (8) of the contact transformation $\mathcal{T}$ depend on these variables. Figure 2 displays the position error after 10 orbital revolutions assuming initially circular orbits in the MEO region, in particular at the GPS orbital distance. The root mean square (RMS) of the error stays below 22 m, whereas the maximum peak value is 44 m. When the particular orbit of the GPS constellation ($i_0 = 55$ deg) is mapped to Figure 2, position errors restrain to an RMS of 3.7 m with peak values of 6.4 m. To put these numbers in context, the errors one would obtain by neglecting the J_2 effects for a GPS satellite would be as high as 97 km with a peak value of 173 km. This evidences that the effects of the oblateness upon MEO trajectories are substantial and omitting this source of orbital perturbation might have important consequences. Thus, the proposed formulation enables a meaningful increase in accuracy while preserving a fully analytical formulation.

To study the accuracy of the approximation when a non-null radial thrust is included, it is useful to introduce the parameter Λ, defined as the ratio between the two disturbing accelerations:

$$\Lambda(R_\oplus) = \frac{\alpha}{\sigma(R_\oplus)}$$

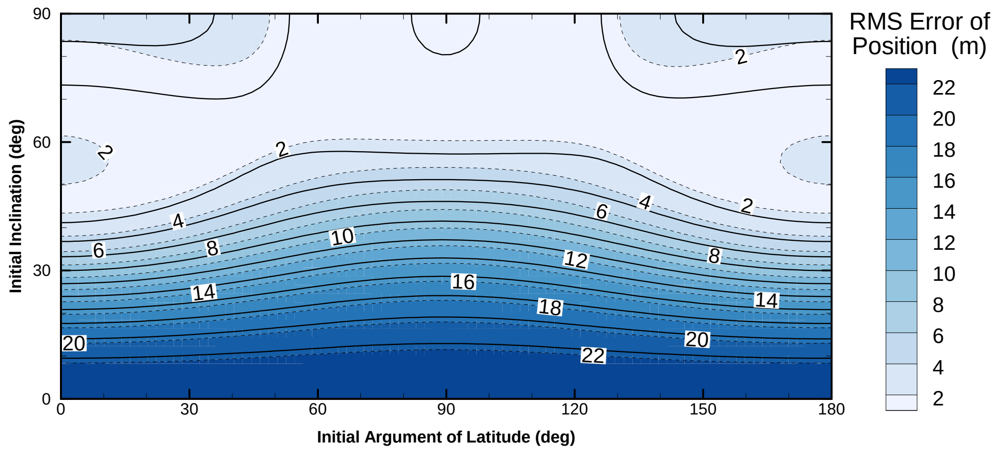
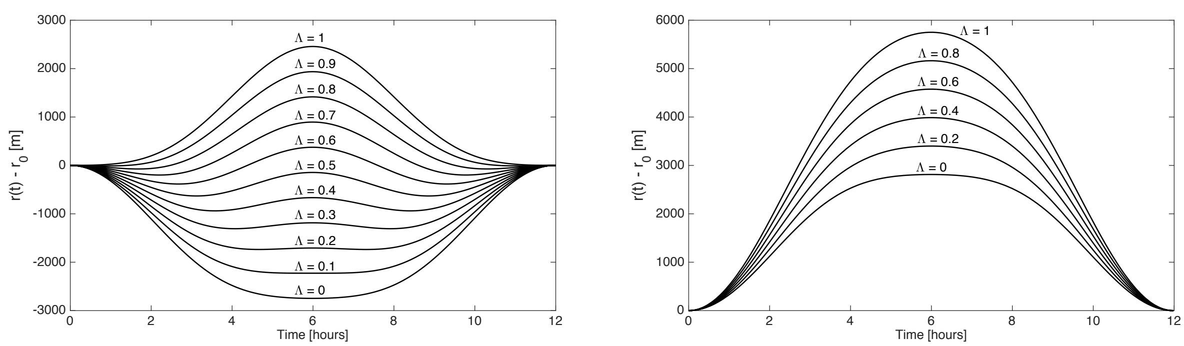


Figure 2. RMS of position error after 10 orbits in MEO with Λ=0 (dashed line) and Λ=1/2 (solid line).

Thus, the initial assumption that the thrust and the oblateness perturbation remain comparable implies that $\Lambda \sim 1$. Figure 3 quantifies the radial displacements of a GPS spacecraft throughout a single orbit for different thrust values. Note that, due to the earth oblateness, spacecraft make radial excursions up to several kilometers even with a null thrust ($\Lambda = 0$), and these values may reduce or accentuate depending on the thrust level and the initial value of θ_0.

The magnitude of the thrust grows linearly with Λ, such that for GPS satellites (mass of 1630 kg), $\Lambda = 1$ corresponds to 82 mN, which is a reasonable value for onboard electric propulsion systems. This limitation suggests the low-thrust station-keeping of GPS satellites as a reasonable application example where the present formulation shows major benefits.



a) $\theta_0 = 0$ deg *b)* $\theta_0 = 90$ deg

Figure 3. Differences in the orbital radius of GPS satellites with J_2 and radial thrust.

The station-keeping of a constellation often requires to maintain an equally spaced distribution of satellites within the same orbital plane. Therefore, the relative longitude of the satellites within the orbit may need to be accordingly adjusted, i.e. spacecraft need to be relocated or repositioned in the same orbit but at an angle $\Delta\theta$ ahead or behind their actual position.

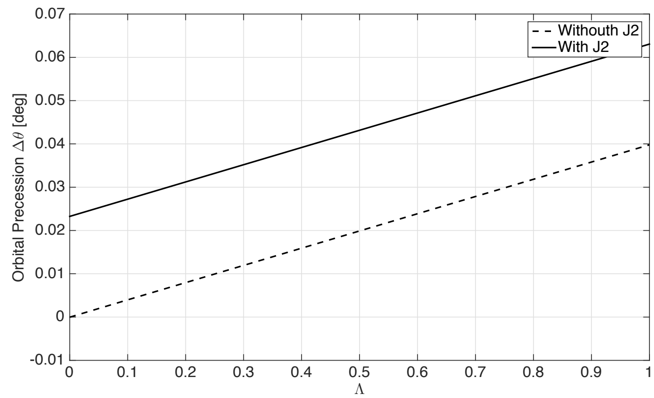
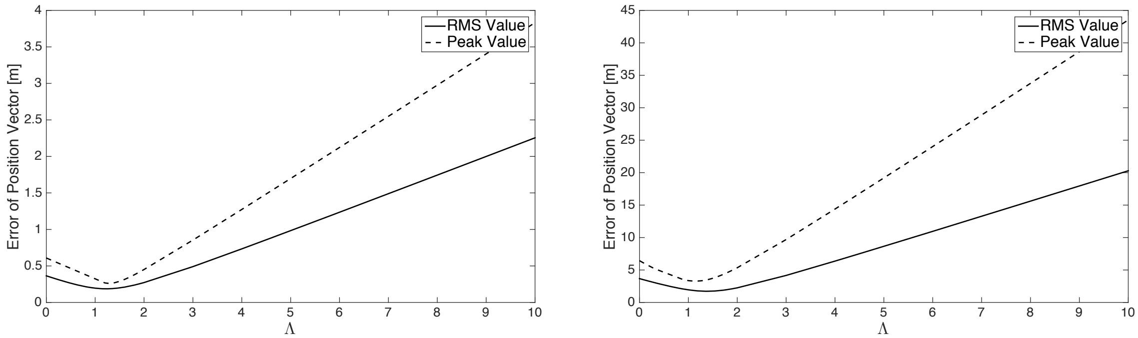


Figure 4. Orbital precession per orbit as a function of the radial thrust.

For radial thrust problems in the bounded regime ($\varepsilon < 1$) the resulting non-Keplerian orbits exhibit a precession of the periapse along with a stretching of the orbital period, which can be computed analytically [11]. The combination of these features suggests that the radial thrust could be employed as a technique for satellite repositioning within an orbit [12]. Figure 4 shows the orbital precession (per orbit) that can be achieved for GPS satellites and how it is affected by the oblateness and the radial thrust. Even though the precession rate is low for the considered thrust levels, when multiplied by the orbital radius it could yield azimuthal errors on the order of 10 km if the oblateness is not accounted for. As opposed to the latter example, it can be observed from Figure 5 that using the proposed formulation, the positional error after one single orbit is at sub-meter level for reasonable values of the radial thrust, and even throughout 10 orbital revolutions the error (either RMS or peak values) does not exceed a 10 m threshold. As a matter of fact, numerical simulations show that even if $\Lambda \gg 1$ the solution still remains acceptably accurate for many engineering applications, and undoubtedly far more accurate than the solution one would obtain by excluding the oblateness effect from the model. Indeed, thrust levels of $\Lambda \sim 10$ (i.e. a thrust magnitude close to 1 N) would still be reasonably valid for station-keeping or repositioning purposes.

V. Conclusions

A formulation is presented within the Hamiltonian formalism, which describes the constant, radial, low-thrust problem and incorporates first order effects of the J_2 zonal harmonic. The formulation exploits the radial intermediary of Deprit to yield an approximate problem which remains integrable, yet captures the essential effects of the oblateness perturbation. The proposed formulation leads to akin expressions as for the radial thrust problem around spherical bodies, also known as the Tsien problem, so the spectrum of available analytical solutions to solve the latter problem can be conveniently exploited to solve the approximate



a) Errors throughout 1 orbital revolution *b)* Errors throughout 10 orbital revolutions

Figure 5. Errors per orbit in the position of GPS satellites as a function of Λ.

dynamics.

The quality of the approximation decreases linearly with the value of J_2 and increases proportionally to the inverse square of the orbital distance from the primary. Also, even though the thrust level is assumed to be comparable to the oblateness perturbation, in practice larger radial accelerations pose only a small penalty to the accuracy. Therefore, this formulation is found to be particularly suitable to study low-thrust orbital maneuvers and the station-keeping of satellites in the MEO orbital region. The accuracy of the approximation is also found to depend on the initial conditions of the problem (namely on the initial orbital inclination and argument of latitude) as well as on the thrust level.

The station-keeping of GPS satellites is proposed as an application example, revealing that the proposed formulation is accurate up to sub-meter level throughout a single orbit, whereas studying the problem using analytical solutions to the Tsien problem (which does not incorporate the J_2 effects) would have led to errors on the order of 10 km.

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