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MINKOWSKI SPACE-TIME AND HYPERBOLIC GEOMETRY

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ABSTRACT

It has become generally recognized that hyperbolic (i.e. Lobachevskian) space can be represented upon one sheet of a two-sheeted cylindrical hyperboloid in Minkowski space-time. This paper aims to clarify the derivation of this result and to describe some further related ideas.

Firstly a simple justification is given of the stated property, which seems somewhat lacking in the literature. This is straightforward once it is shown that differential displacements on the hyperboloid surface are space-like elements in Minkowski space-time. This needs certain preliminary remarks on Minkowski space-time. Two other derivations are given which are valid in any pseudo-Euclidean space of the same type.

An alternative view comes from regarding Minkowski space-time projectively as a velocity space. This is possible with Minkowski's original representation but is best seen when Minkowski space-time is regarded differentially as a special case of the metric of General Relativity. Here the space may also be considered as differential space-time in the sense of Minkowski'. It may be considered as a projective space and in this case, as a velocity space which is a Lobachevsky space with hyperboloid representation. Projection of the hyperboloid to a disc or spherical ball gives an associated Beltrami-Klein representation of velocity space.

This geometrical representation has important application in physics being related to the hyperbolic theory of Special Relativity which was first proposed by Varićak in 1910 following Einstein's original 1905 paper. The Cayley metric for the velocity space representation leads to relativistic addition of two velocities.

The paper emphasizes the importance of Weierstrass coordinates as they are highly appropriate to the relativity application. They also show the close relation between the hyperboloid representation and the equivalent spherical one, the hyperbolic space being then regarded as a sphere of imaginary radius which has historically been a guiding idea and one closely related to Special Relativity.

1. Spherical and hyperbolic metrics

The relation between the metrics is here first illustrated in the familiar 3–dimension case.

The sphere:

$$x^2+y^2+z^2=R^2 \qquad (1.1)$$

If ρ is arc length from the pole and $\varphi = \rho/R$, the usual spherical parameterization is

$$x = R \sin\varphi \cos\theta = R \sin(\rho/R) \cos\theta$$
$$y = R \sin\varphi \sin\theta = R \sin(\rho/R) \sin\theta$$
$$z = R \cos\varphi \qquad = R \cos(\rho/R)$$
$$(1.2)$$

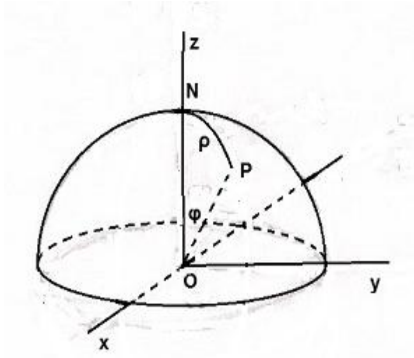

fig: A hemisphere in 3 dimensions

The spherical metric is then

$$ds^2 = = dx^2+dy^2+dz^2 = R^2 (d\varphi^2 + \sin^2\varphi\, d\theta^2) = d\rho^2 + R^2 \sin^2(\rho/R)\, d\theta^2 \qquad (1.3)$$

Cylindrical hyperboloid in two sheets:

$$-x^2-y^2+z^2 = R^2 \qquad (1.4)$$

Using hyperbolic functions the upper half–sheet it has parameterization similar to the spherical case:

$$x = R \sinh u \cos\theta = R \sinh(\rho/R) \cos\theta$$
$$y = R \sinh u \sin\theta = R \sinh(\rho/R) \sin\theta$$
$$z = R \cosh u \qquad = R \cosh(\rho/R)$$
$$(1.5)$$

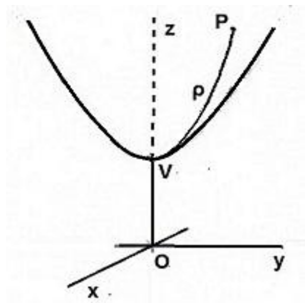

fig: Upper sheet of a 2-sheeted cylindrical hyperboloid

Here $u = \rho/R$ where ρ is arc length from the vertex V. A hyperbolic (i.e. Lobachevsky) metric arises if it is calculated according to

$$ds^2 = dx^2+dy^2- dz^2 = R^2 (du^2 + \sinh^2 u\, d\theta^2) = d\rho^2 + R^2 \sinh^2(\rho/R)\, d\theta^2 \qquad (1.6)$$

R is here the radius of negative curvature.

Remarks:
(1) Because of the negative sign before dz^2 the embedding space cannot be Euclidean; it must in fact be pseudo-Euclidean i.e. Euclidean form with a non-positive-definite metric. The most well-known example of such a space is *Minkowski space* (or *Minkowski space-time*) described below occurring in the Theory of Special Relativity. Positive-definiteness of the quadratic form for ds^2 can be shown in several ways, most simply by an algebraic proof of Carathéodory but also using the Cayley metric (see the appendix) or from the properties of Minkowski space (see below).
(2) The metrics for spherical and hyperbolic cases are interchangeable by substituting iR for R. The same is true for the equations for x and y though not for z. That also becomes interchangeable on rewriting the equation using a new variable t (= z/R) to give t = cosh (ρ/R). This is done with *Weierstrass coordinates* described below.

2. Minkowski space–time

Minkowski space-time (or just *Minkowski space*) is a 4 dimensional pseudo-Euclidean space of event-vectors (t, x, y, z) specifying events at time t and spatial position at x, y, z as seen by an observer assumed to be at (0, 0, 0, 0). The space has an indefinite metric form depending on the *velocity of light* c:

$$c^2 t^2 – x^2 – y^2 – z^2 \qquad (2.1)$$

This is invariant under a group of linear *Lorentz transformations* relating event-vectors (vectors for shortness) to those of other moving observers.

Vectors are classified in relation to the two sided *light cone*

$$c^2 t^2 = x^2 + y^2 + z^2 \qquad (2.2)$$

They are *time-like* if $c^2 t^2 > x^2+y^2+z^2$ or *space-like* if $c^2 t^2 < x^2+y^2+z^2$. Here they will be restricted to the *forward cone* $t > 0$ so that $ct > \sqrt{x^2 + y^2 + z^2}$ for time-like vectors implying that (t, x, y, z) is accessible by a light signal from (0, 0, 0, 0).

The *scalar product* of two vectors (t, x, y, z), (t', x', y', z') is defined as

$$ct\, ct' – x\, x' – y\, y' – z\, z' \qquad (2.3)$$

Lemma: The scalar product of two time-like vectors (t, x, y, z), (t', x', y', z') is positive:

$$ct\, ct' – xx' – yy' – zz' \; > \; 0 \qquad (2.4)$$

This follows from the Cauchy inequality (t, t' > 0 assumed):

$$xx' + yy' + zz' \; \le \sqrt{x^2 + y^2 + z^2}\sqrt{x'^2 + y'^2 + z'^2} \; < \; ct\, ct' \qquad (2.5)$$

Corollary: if two vectors (t, x, y, z), (t', x', y', z') satisfy

$$ct\, ct' – xx' – yy' – zz' \; = \; 0 \qquad (2.6)$$

at least one must be space-like.

Remark: it is possible to prove the stronger *reversed Cauchy inequality* for two time-like vectors (t, x, y, z), (t', x', y', z'):

$$ct\, ct' – xx' – yy' – zz' \; \ge \sqrt{c^2t^2 – x^2 – y^2 – z^2}\sqrt{c^2t'^2 – x'^2 – y'^2 – z'^2} \qquad (2.7)$$

with equality only if the vectors are proportional. See the appendix

3. The cylindrical hyperboloid in Minkowski space

From what has been said it can be deduced that in Minkowski space the upper sheet of a hyperboloid with equation

$$c^2 t^2 – (x^2+y^2+z^2) = \text{const.} > 0 \tag{3.1}$$

is a hyperbolic (Lobachevsky) space. This will follow once the method of calculating the metric is verified. All vectors lying on this hyperboloid are time-like and, by differentiation, the following condition holds

$$ct.cdt – x.dx – y.dy – z.dz = 0 \tag{3.2}$$

Since (t, x, y, z) is time-like by equation (3.1) it follows from the preceding corollary for (2.6) that (dt, dx, dy, dz) must be space-like satisfying

$$dx^2+dy^2+dz^2 – c^2 dt^2 > 0 \tag{3.3}$$

The same conclusion follows from the two alternative derivations in note (a) of the appendix.

To calculate the metric it is convenient to introduce the *proper time* from (0, 0, 0, 0) defined as

$$\tau = \surd\{t^2 – (x^2+y^2+z^2)/c^2\} \tag{3.4}$$

This is invariant under Lorentz transformation and so independent of observer. It is the same for all events represented by the hyperboloid (3.1) which gives the totality of events having the same proper time τ. Using τ, the hyperboloid can now be written

$$c^2 t^2 – (x^2+y^2+z^2) = c^2 \tau^2 \tag{3.5}$$

Parameterization using spherical coordinates with ρ as arc length from the vertex will be

$$\begin{aligned} t &= \tau \cosh (\rho/c\tau) \\ x &= c\tau \sinh (\rho/c\tau) \sin \varphi \cos \theta \\ y &= c\tau \sinh (\rho/c\tau) \sin \varphi \sin \theta \\ z &= c\tau \sinh (\rho/c\tau) \cos \varphi \end{aligned} \tag{3.6}$$

The metric is then found in Lobachevsky form as

$$ds^2 = dx^2+dy^2+dz^2 – c^2 dt^2 = d\rho^2 + (c\tau)^2 \sinh^2(\rho/c\tau) (d\varphi^2 + \sin^2\varphi\, d\theta^2) \tag{3.7}$$

4. Weierstrass coordinates

Weierstrass coordinates for spherical and hyperbolic surfaces resemble rectangular Cartesian coordinates but depend only on the surface and not on the embedding space. They are suitable for discussing the hyperboloid in pseudo-Euclidean space and its relation to the spherical analogue.

Weierstrass spherical coordinates: These are here illustrated for the 2-dimensional surface of a 3-dimensional sphere of radius R. The geodesics are great circle arcs. From an origin O on the sphere, two such arcs are constructed at right angles giving axes here called the ξ-axis and the η-axis. A point P on the sphere can be specified by the length ρ of the radial arc from P to the origin O together with the angle θ this arc makes with the ξ-axis. These give polar coordinates analogous to Cartesian polar coordinates. Alternatively, coordinates may be defined by constructing the arcs from P meeting the axes at right angles. These give *inner coordinates* (ξ, η) and *outer coordinates* (ξ', η') as in the diagram. Unlike Euclidean Cartesian coordinates, these are distinct.

The three *Weierstrass coordinates x, y, t* of the point P are most simply defined using the polar representation as

$$x = R \sin(\rho/R) \cos\theta$$
$$y = R \sin(\rho/R) \sin\theta$$
$$t = \cos(\rho/R) \qquad (4.1)$$

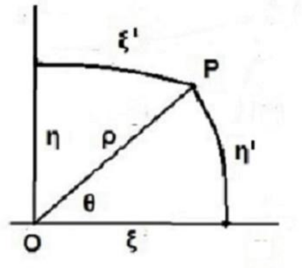

fig: The spherical case

Alternatively, by the spherical sine-rule, x, y are

$$x = R \sin(\xi'/R)$$
$$y = R \sin(\eta'/R) \qquad (4.2)$$

These Weierstrass coordinates satisfy

$$x^2 + y^2 + R^2 t^2 = R^2 \qquad (4.3)$$
$$dx^2 + dy^2 + R^2 dt^2 = d\rho^2 + R^2 \sin^2(\rho/R)\, d\theta^2 \qquad (4.4)$$

Of course in the Euclidean representation the ratios ξ/R, η/R, ρ/R correspond to angles at the centre of the sphere and Rt to the z distance for axes having origin at the sphere centre

Weierstrass hyperbolic coordinates: These are similar but use the equations of hyperbolic trigonometry with R now the radius of negative curvature.

$$x = R \sinh(\rho/R) \cos\theta = R \sinh(\xi'/R)$$
$$y = R \sinh(\rho/R) \sin\theta = R \sinh(\eta'/R)$$
$$t = \cosh(\rho/R) \qquad (4.5)$$

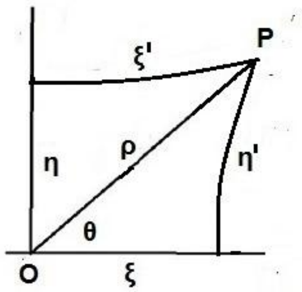

fig: The hyperbolic case

The hyperbolic sine-rule is used here. These coordinates satisfy

$$-x^2 - y^2 + R^2 t^2 = R^2 \qquad (4.6)$$
$$d x^2 + dy^2 - R^2 dt^2 = d\rho^2 + R^2 \sinh^2(\rho/R)\, d\theta^2 \qquad (4.7)$$

There is now a complete analogy between the spherical and hyperbolic cases and one transforms into the other by the substitution of iR for R. The limiting case as R tends to infinity is a Euclidean plane at t=1 with $d x^2 + dy^2 = d\rho^2 + \rho^2 d\theta^2$ and with (x, y) = (ξ, η) = (ξ', η')

5. Hyperbolic Weierstrass coordinates in 3 dimensions

The Weierstrass coordinates for the 2-dimensional surfaces given in the last section generalize to 3 dimensions by choosing an origin O at any point and then constructing geodesic arcs Oξ, Oη, Oζ mutually orthogonal at O as axes. The position of any point P may be specified by the length ρ of the arc OP together with the direction cosines l, m, n which OP makes at the origin with the three axes Oξ, Oη, Oζ.

In the hyperbolic case Weierstrass coordinates x, y, z, t can then be defined as

$$\begin{aligned} x &= R \sinh(\rho/R)\, l \\ y &= R \sinh(\rho/R)\, m \\ z &= R \sinh(\rho/R)\, n \\ t &= \cosh(\rho/R) \end{aligned} \qquad (5.1)$$

These equations may be written more shortly as

$$t = \cosh(\rho/R). \quad (x, y, z) = R \sinh(\rho/R)\, (l, m, n) \qquad (5.2)$$

The Weierstrass coordinates x, y, z, t give the hyperboloid representation

$$-x^2 - y^2 - z^2 + R^2 t^2 = R^2 \qquad (5.3)$$

From this, ρ can be identified as arc length from the vertex at (0, 0, 0, 1).

Using spherical coordinate representation, l, m, n are expressible as

$$(l, m, n) = (\sin\varphi \cos\theta, \sin\varphi \sin\theta, \cos\varphi) \qquad (5.4)$$

This gives the parameterization

$$\begin{aligned} x &= R \sinh(\rho/R) \sin\varphi \cos\theta \\ y &= R \sinh(\rho/R) \sin\varphi \sin\theta \\ z &= R \sinh(\rho/R) \cos\varphi \\ t &= \cosh(\rho/R) \end{aligned} \qquad (5.5)$$

The metric is then found as

$$dx^2 + dy^2 + dz^2 - R^2\, dt^2 = d\rho^2 + R^2 \sinh^2(\rho/R)\, (d\varphi^2 + \sin^2\varphi\, d\theta^2) \qquad (5.6)$$

These equations all transform to the corresponding spherical case on changing R to iR.

Remarks:
(1) The definition obviously extends to a space of any dimension.
(2) In the case of Minkowski space, R has the value c and it can be seen that the hyperboloid previously considered in section 3, equation (3.5) does not have the Weierstrass coordinate form unless new variables are taken as t/τ, x/τ, y/τ, z/τ. This is the velocity vector for Minkowski space as usually considered (i.e. suitable for motions with uniform velocity starting at the origin at t=0), It gives a special case of the velocity interpretation described in the next section. It was discussed in the conference paper [8].

6. Differential Minkowski space and velocity space

Minkowski space is frequently regarded as a special case occurring in General Relativity when the metric has constant coefficients. This view leads naturally to *differential Minkowski space* of 4-vectors (dt, dx, dy, dz) satisfying the usual rules of Minkowski space. This space of differential vectors is also a projective space of their ratios i.e. of velocities, having as absolute the cone

$$c^2dt^2 - (dx^2 + dy^2 + dz^2) = 0 \qquad (6.1)$$

This is a real conic so the resulting projective space is hyperbolic. Vectors lying within the cone are time-like, i.e.

$$dx^2 + dy^2 + dz^2 < c^2\, dt \qquad (dt > 0 \text{ assumed}) \qquad (6.2)$$

They represent physically feasible motions having velocity less than that of light:

$$(dx/dt)^2 + (dy/dt)^2 + (dz/dt) < c^2 \qquad (6.3)$$

It is consequently possible to define the *proper time differential*

$$d\tau = \sqrt{\{dt^2 - (dx^2 + dy^2 + dz^2)/c^2\}} = \sqrt{\{1 - v^2/c^2\}}\, dt \qquad (6.4)$$

where v is velocity. From this follows

$$c^2\, (dt/d\tau)^2 - (dx/d\tau)^2 - (dy/d\tau)^2 - (dz/d\tau)^2 = c^2 \qquad (6.5)$$

This is the Weierstrass form of a hyperbolic space having c as radius of negative curvature.

The Cayley-Klein metric: The projective distance u between two differential vectors is

$$u = \operatorname{arcosh} \frac{(c^2\, dtdt' - dxdx' - dydy' - dzdz')}{\sqrt{(c^2\, dt^2 - dx^2 - dy^2 - dz^2)}\sqrt{(c^2\, dt'^2 - dx'^2 - dy'^2 - dz'^2)}} \qquad (6.6)$$

$$= \operatorname{arcosh} \{c^2\, (dt/d\tau)\, (dt'/d\tau') - (dx/d\tau)\, (dx'/d\tau') - (dy/d\tau)\, (dy'/d\tau') - (dz/d\tau)\, (dz'/d\tau')\} \qquad (6.7)$$

With θ as angle between the two velocity vectors

$$\mathbf{v} = (dx/dt, dy/dt, dz/dt), \quad \mathbf{v'} = (dx'/dt', dy/dt, dy'/dt', dz'/dt') \qquad (6.8)$$

the distance u is seen to be

$$u = \operatorname{arcosh} \frac{c^2 - \mathbf{v.v'}}{\sqrt{(c^2 - \mathbf{v.v})}\sqrt{(c^2 - \mathbf{v'.v'})}} = \operatorname{arcosh} \frac{c^2 - v\, v' \cos\theta}{\sqrt{(c^2 - v^2)}\sqrt{(c^2 - v'^2)}} \qquad (6.9)$$

On changing from velocities to rapidities by v = c tanh w, v' = c tanh w' (6.9) gives

$$\cosh u = \cosh w \cosh w' - \cos\theta \sinh w \sinh w' \qquad (6.10)$$

This equation is the *hyperbolic cosine rule* for triangles in Lobachevsky space showing that u is a rapidity and is the sum of rapidities w, w' for a triangle having sides w, w' with included angle θ. It also has an interpretation of relative rapidity if the w, w' are vectors, one following the other.

7. Beltrami–Klein representation

This representation is of particular importance as it relates to a space where geodesic arcs are represented by straight lines. For illustration consider the previous hyperboloid representation

$$- x^2 – y^2 + R^2 t^2 = R^2 \qquad (7.1)$$

with parameterization

$$t = \cosh (\rho/R) \quad (x, y) = R \sinh (\rho/R) (l, m) \qquad (7.2)$$

By projection from the centre O of the hyperboloid the upper surface maps on to the disc inside a circle centred at the vertex V and conversely. Point (t, x, y) corresponds to a point with coordinates X, Y called *Beltrami coordinates* given by

$$(X, Y) = (x/t, y/t) = R \tanh (\rho/R) (l, m) \qquad (7.3)$$

Its radial distance from the centre of the disc is

$$r = \sqrt{(X^2+ Y^2)} = R \tanh (\rho/R) \qquad (7.4)$$

so

$$\rho = R \operatorname{artanh} (r/R) \qquad (7.5)$$

It may be shown that the disc is a Lobachevsky space where Euclidean distance is replaced by a distance defined projectively (Klein's definition).

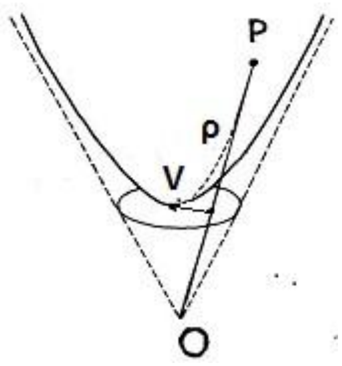

fig: Central projection to a disc

Relativity application: in its relativity form in differential Minkowski space, R becomes c and the hyperboloid takes the form (6,5) . The parameterization now becomes

$$V_0 = c \cosh (U/c), (V_1, V_2, V_3) = c \sinh (U/c) (l, m, n) \qquad (7.6)$$

In this equation U is the arc length of the geodesic from the vertex V. After projection into the sphere of radius c the velocity components (v_x, v_y, v_z) are found by division as

$$(v_x, v_y, v_z) = c \tanh (U/c) (l, m, n) \qquad (7.7)$$

with velocity of magnitude

$$v = \sqrt{(v_x^2 + v_y^2 + v_y^2)} = c \tanh U/c. \qquad (7.8)$$

so that

$$U = c \operatorname{artanh} (v/c) \qquad (7.9)$$

This identifies U as the *hyperbolic* or *relativistic velocity* (defined as scalar multiple cw of rapidity w which approximates v when v << c). Geometrically it is the length of the geodesic arc from the vertex, and it can be thought of as projecting into the normal velocity v represented by the length of the radial vector in a sphere of radius c.

APPENDIX & NOTES

1. Metrics for spherical and hyperbolic coordinates

Two methods are given here to justify the calculation of the metric element in the hyperbolic case
(a) The algebraic derivation of Carathéodory (*Variationsrechnung* ... 1935 [1])
(b) The Riemannian metric derived from the Cayley-Klein metric (as e.g. in Veblen & Young's: *Projective Geometry,* (Ginn 1909).)

(a) Carathéodory's derivation
Lemma: (Original notation) From (i), (ii) below follows the positive-definiteness of (iii)
(i) $y_0^2 - y_1^2 - y_2^2 - y_3^2 = 1$
(ii) $\eta_1 y_1 + \eta_2 y_2 + \eta_3 y_3 - \eta_0 y_0 = 0$
(iii) $Q = \eta_1^2 + \eta_2^2 + \eta_3^2 - \eta_0^2$

Proof: for any value of ρ

$$0 = \rho^2(y_1^2 + y_2^2 + y_3^2 - y_0^2 + 1) + 2\rho(\eta_1 y_1 + \eta_2 y_2 + \eta_3 y_3 - \eta_0 y_0) + (-Q + \eta_1^2 + \eta_2^2 + \eta_3^2 - \eta_0^2)$$
$$= -Q + (\rho y_1 + \eta_1)^2 + (\rho y_2 + \eta_2)^2 + (\rho y_3 + \eta_3)^2 - (\rho y_0 + \eta_0)^2 + \rho^2$$

On putting $\rho = -\eta_0/y_0$ Q is seen to be non–negative since it takes the form

$$Q = (1/y_0)^2\{(\eta_0 y_1 - \eta_1 y_0)^2 + (\eta_0 y_2 - \eta_2 y_0)^2 + (\eta_0 y_3 - \eta_3 y_0)^2 + \eta_0^2\}$$

Lemma: (In the notation of the present paper) From (i), (ii) follows positive-definiteness of (iii):
(i) $c^2t^2 - x^2 - y^2 - z^2 = \text{const.}$
(ii) $x.dx + y.dy + z.dz - c^2 t.dt = 0$
(iii) $dx^2 + dy^2 + dz^2 - c^2 t^2$

(b) *Using Cayley-Klein metric*:
For a bilinear form B(x, x’) the projective distance ρ between coordinate vectors x, x' is

$$\cosh \rho = \frac{B(x, x')}{\sqrt{B(x, x)}\sqrt{B(x', x')}}$$

So

$$\sinh^2\rho = \frac{\{B(x, y)^2 - B(x, x).B(y, y)\}}{\{B(x, x)\, B(y, y)\}}$$

If y = x + dx, ρ is infinitesimal so that $\mathrm{sh}^2\rho = \rho^2 = ds^2$. Keeping only infinitesimals of second order on the right-hand side, there is found the relation of Riemannian to Cayley-Klein metrics

$$ds^2 = \frac{B(x, dx)^2}{B(x, x)^2} - \frac{B(dx, dx)}{B(x, x)}$$

Tangential displacements dx to a surface B(x, x) = C (const.>0) satisfy B(x, dx) = 0 so have

$$ds^2 = -B(dx, dx)/C$$

In particular, on setting $B(x, x) = x_0^2 - x_1^2 - x_2^2 - x_3^2$

$$ds^2 = (dx_1^2 + dx_2^2 + dx_3^2 - dx_0^2)/C$$

2. Minkowski space

(a) Minkowski described this space in his 1908 Cologne lecture "Space and Time" [2] published posthumously in 1909. Minkowski's notation is used in the present paper as it relates easily to Weierstrass coordinates which is not so with other notations e.g. (x_1, x_2, x_3, x_4), (x_0, x_1, x_2, x_3).

(b) *The inverse Cauchy inequality*: for two time-like vectors (t, x, y, z), (t', x', y', z'):

$$ct\,ct' - xx' - yy' - zz' \;\geq\; \sqrt{\{(ct)^2 - x^2 - y^2 - z^2\}}\;\sqrt{\{(ct')^2 - x'^2 - y'^2 - z'^2\}}$$

Equality holds only if the vectors are proportional.

Proof: The following simple proof is due to Aczél 1956 [3]. Let

$$\begin{aligned} f(\lambda) &= \lambda^2\{(ct)^2 - x^2 - y^2 - z^2\} - 2\lambda\{ct\,ct' - xx' - yy' - zz'\} + \{(ct')^2 - x'^2 - y'^2 - z'^2\} \\ &= (\lambda\, ct - ct')^2 - (\lambda x - x')^2 - (\lambda y - y')^2 - (\lambda z - z')^2 \end{aligned}$$

When $\lambda \to \pm\infty$, $f(\lambda) > 0$ while when $\lambda = t'/t$, $f(\lambda) \leq 0$ being zero only for proportional vectors. So $f(\lambda) = 0$ has both roots real implying the discriminant of the quadratic $f(\lambda)$ is non-negative i.e.

$$(ct\,ct' - xx' - yy' - zz')^2 \geq \{(ct)^2 - x^2 - y^2 - z^2\}\{(ct')^2 - x'^2 - y'^2 - z'^2\}$$

The result follows since the quantity inside the left hand brackets has been proved to be positive.

3. The hyperboloid in Minkowski space

(a) Minkowski commented on the relevance of non-Euclidean geometry to the geometry of space-time in his 1907 paper (reproduced with discussion in the paper of Scott-Walter 1999 [4])
Note that Minkowski's 4-velocity differs there from the current definition. This appears to have been a mistake. Minkowski changed it in his 1908 'Die Grundgleichungen ….' paper equation (19).
(b) The representation of a Lobachevsky space on a hyperboloid has recently become known as the *hyperboloid model*. The paper of Reynolds 1993 [5] gives details of the geometry (though in a different presentation to that here) together with useful historical comments.
(c) The observation that events having the same proper time lie on a Lobachevsky space was made from a different point of view in the paper of Törnebohm 1964 [6].

4. Weierstrass coordinates

These were introduced by Killing who attended Weierstrass' 1872 lectures (now lost). The subject is not widely treated in current literature. An older book giving clear treatment is that of Sommerville 1914 reprinted in 2005 [7]. The geometrical figures occurring here, usually called Lambert quadri-laterals, were described by the 13th century Arab scholar Ibn al-Haytham (Alhazen).

5. Hyperbolic Weierstrass coordinates in 3 dimensions

In the velocity interpretation only the ratios of the coordinates t, x, y, z are important which corresponds to a projective geometry as described in the next section. The interpretation is valid inside the cone of vectors satisfying the condition for time-like vectors in Minkowski space. However it applies only to uniform motion starting at time t = 0 from the origin. It was described in the writer's conference paper of 1996 [8].

6. Differential Minkowski space and velocity space

(a) An early account of velocity representation on a hyperboloid in Minkowski space was given by Wick [9] for application in physics.
(b) The idea of treating the Lobachevsky interpretation by differential space representation was proposed briefly many years ago by Pauli in connection with Varićak's reinterpretation of Special Relativity in Lobachevsky space (see the footnote to p.74 in the English version of Pauli's 'Theory of Relativity' [10]). However details were not given and the idea was not followed up. Differential Minkowski space was discussed in relation to Carathéodory's approach to the Special Relativity in the writer's paper [11] and in the 2006 monograph revised as [12]
(c) The idea of rapidity space goes back to Varićak (see below}. More recently it was discussed by Rhodes & Simon [13], Giulini [14] and Ungar [15] as well as the writer [12].
(d) The equation (6.5) may equally be expressed in terms of the Minkowski velocity 4-vector:

$$(V_0, V_1, V_2, V_3) = (c\, dt/d\tau,\ dx/d\tau,\ dy/d\tau,\ dz/d\tau)$$

as

$$V_0^2 - V_1^2 - V_2^2 - V_3^2 = c^2$$

and the Cayley-Klein metric as

$$\operatorname{arcosh} \{V_0 V'_0 - V_1 V'_1 - V_2 V'_2 - V_3 V'_3\}$$

But this representation is here less appropriate than that with Weierstrass coordinates.
(d) The hyperbolic theory of special relativity largely developed from a generalization of the classical addition of velocities that v, v' inclined to each other at an angle θ have resultant

$$v = \sqrt{(v^2 + 2\, v\, v' \cos\theta + v'^2)}$$

the direction being found geometrically by simple vector addition. In the 1905 Special Theory of Relativity, Einstein showed that, relativistically, this formula should be replaced by

$$v = \sqrt{\left\{\frac{(v^2 + 2\, v\, v' \cos\theta + v'^2) - (v\, v' \sin\theta)^2/c^2}{1 + v\, v' \cos\theta/c^2}\right\}}$$

But the geometrical interpretation was left unclear. In 1909 Sommerfeld [16] showed how, by using Minkowski's complex space-time representation, velocities are added vectorially on the surface of a sphere of imaginary radius. Varićak 1910 [18], (1912) [19], then gave a corresponding interpretation in Lobachevsky space. As shown above, this addition formula follows directly from the geometrical view of the present paper.

7. The Beltrami-Klein representation

Beltrami published in 1868 soon after posthumous publication of Riemann's famous dissertation and he was the first to use Riemannian geometry for spaces of negative curvature. Klein in 1872 used projective geometry for Beltrami's representation. Beltrami also described two other representations of hyperbolic geometry usually ascribed to Poincaré who discussed them many years later. The papers of Beltrami and Klein are available in the book of Stillwell 1996 [19] in English translation with a commentary.

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