

EXTERIOR POWER OPERATIONS ON HIGHER K-GROUPS VIA BINARY COMPLEXES

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ABSTRACT. We use Grayson’s binary multicomplex presentation of algebraic K -theory to give a new construction of exterior power operations on the higher K -groups of a (quasi-compact) scheme. We show that these operations satisfy the axioms of a λ -ring, including the product and composition laws. To prove the composition law we show that the Grothendieck group of the exact category of integral polynomial functors is the universal λ -ring on one generator.

INTRODUCTION

Exterior powers of vector bundles over a scheme X endow its Grothendieck group $K_0(X)$ with a family of operations $\lambda^r : K_0(X) \rightarrow K_0(X)$, $r = 0, 1, \dots$. These λ -operations allow us to define Adams operations and the γ -filtration on $K_0(X)$ and are, more generally, at the heart of Grothendieck’s Riemann-Roch theory (see [FL85]). This fundamental structure has been extended to the higher K -groups $K_n(X)$, $n \geq 0$, using a variety of sophisticated approaches and in various degrees of generality, by Kratzer [Kr80], Hiller [Hil81], Grayson [Gra89], Nenashev [Nen91] and Levine [Le97], and has been most profoundly studied and applied in Soulé’s seminal paper [So85]. Common to all these constructions is that they use homotopy theory.

In this paper we give a purely algebraic construction of the λ -operations on the higher K -groups of any quasi-compact scheme X . Our construction is explicit in the following sense: in his surprising paper [Gra12], Grayson has given explicit generators and relations for $K_n(X)$, and our construction describes explicit (albeit intricate) images of these generators under the λ -operations. Within the purely algebraic context of this paper, we prove moreover that our λ -operations satisfy the usual axioms, including the product and composition laws. In a forthcoming paper we address the problem of matching up our λ -operations with Hiller’s.

To describe our results in more precise terms, we now recall the definition of a λ -ring.

Definition. A *pre- λ -ring* is a commutative unital ring K together with maps $\lambda^r : K \rightarrow K$, $r \geq 1$, satisfying $\lambda^1(x) = x$ and the following axiom for all $x, y \in K$:

$$(1) \quad \lambda^r(x + y) = \lambda^r(x) + \sum_{i=1}^{r-1} \lambda^{r-i}(x) \lambda^i(y) + \lambda^r(y).$$

A *λ -ring* K is a pre- λ -ring satisfying the further axioms

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- (2) $\lambda^r(xy) = P_r(\lambda^1(x), \dots, \lambda^r(x), \lambda^1(y), \dots, \lambda^r(y))$
 (3) $\lambda^r(\lambda^s(x)) = P_{r,s}(\lambda^1(x), \dots, \lambda^{rs}(x)),$

where P_r and $P_{r,s}$ ($r, s > 0$) are certain universal integral polynomials (defined in such a way so the axioms (2) and (3) hold in every pre- λ -ring whose additive group is generated by elements l with $\lambda^r(l) = 0$ for all $r > 1$ and in which products of elements of this type are again of this type; for details see [FL85]).

Probably the most prominent example of a λ -ring is $K_0(X)$ (see [FL85]). The object of this paper is to make $K_*(X) = \bigoplus_{n \geq 0} K_n(X)$ into a λ -ring.

For each $n \geq 0$, Grayson [Gra12] associates to an exact category \mathcal{P} the exact category $(B_b^q)^n \mathcal{P}$ of so-called n -dimensional bounded acyclic binary complexes, and proves that $K_n(\mathcal{P})$ is isomorphic to a relatively simple-to-describe quotient of the Grothendieck group $K_0((B_b^q)^n \mathcal{P})$ (see section 1 for a detailed review of Grayson's construction). Using the Dold–Puppe construction [DP61], we inductively construct functors

$$\Lambda_n^r : (B_b^q)^n \mathcal{P}(X) \rightarrow (B_b^q)^n \mathcal{P}(X)$$

for all $r, n > 0$ from the usual exterior power functors $\Lambda^r : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, $r \geq 0$, on the category $\mathcal{P}(X)$ of vector bundles on X .

The following theorems are the main results of this paper.

Theorem A [Theorem 6.2]. *The functors Λ_n^r induce well-defined homomorphisms $\lambda^r : K_n(X) \rightarrow K_n(X)$ for $r, n > 0$.*

The tensor product induces the multiplication in the Grothendieck ring $K_0(X)$ and also an action of $K_0(X)$ on the higher K -groups $K_n(X)$. In particular, $K_*(X) = \bigoplus_{n \geq 0} K_n(X)$ carries the structure of a unital commutative ring in which the product of any two elements in $\bigoplus_{n \geq 1} K_n(X)$ is defined to be zero. Note that, if $n > 0$, axiom (1) for $x, y \in K_n(X)$ then follows from $\lambda^r : K_n(X) \rightarrow K_n(X)$ being a homomorphism (Theorem A). Furthermore, the formula in axiom (1) can be used to extend our operations $\lambda^r : K_n(X) \rightarrow K_n(X)$, $n \geq 0$, to a pre- λ -ring structure on $K_*(X)$.

Theorem B [Theorems 7.1, and 8.18]. *The pre- λ -ring $K_*(X)$ is a λ -ring.*

The first half of the paper is devoted to the construction of the exterior power functors Λ_n^r . Let $C_b \mathcal{P}(X)$ denote the category of bounded complexes in $\mathcal{P}(X)$. We use the Dold–Kan correspondence (reviewed along with the other necessary homological preliminaries in section 2) to obtain a chain-homotopy invariant functor $\Lambda_1^r : C_b \mathcal{P}(X) \rightarrow C_b \mathcal{P}(X)$ for each $r > 0$; if X is affine, then the bounded acyclic complexes in $\mathcal{P}(X)$ are precisely the contractible ones, so we obtain an endofunctor on the category of bounded acyclic chain complexes in $\mathcal{P}(X)$. By generalising and iterating this procedure over complexes of complexes we get the desired functors $\Lambda_n^r : (B_b^q)^n \mathcal{P}(X) \rightarrow (B_b^q)^n \mathcal{P}(X)$. This material is the subject of sections 3 and 4.

In the rather long section 5 we construct a ‘simplicial tensor product’ $\otimes_{\Delta, n}$ on $(B_b^q)^n \mathcal{P}(X)$. In defining exterior powers on $K_0(X)$ we obtain from a short exact sequence of vector bundles $0 \rightarrow V' \rightarrow V \rightarrow V'' \rightarrow 0$ a filtration of $\Lambda^r(V)$ whose successive quotients are $\Lambda^{r-i}(V') \otimes \Lambda^i(V'')$. We use our

simplicial tensor product of binary multicomplexes to obtain similar statements for short exact sequences in $(B_b^q)^n \mathcal{P}(X)$; our tensor product $\otimes_{\Delta, n}$ is to \otimes as the exterior powers Λ_n^r are to Λ^r . The main result of the section (Proposition 5.11) is that the product induced by $\otimes_{\Delta, n}$ on $K_n(X)$ vanishes.

In section 6 we pass our exterior powers from the affine case to general (quasi-compact) schemes and show that they induce well-defined operations $\lambda^r: K_n(X) \rightarrow K_n(X)$. As the product on $K_n(X)$ that is compatible with these operations is the zero product (by Proposition 5.11), it follows that the λ^r are group homomorphisms.

In section 7 we show that the resulting pre- λ -ring $K_*(X)$ satisfies the λ -ring axiom (2) concerning products.

The final λ -ring axiom (3) is proved in section 8. While the usual geometric splitting principle suffices to prove axiom (2) for $K_*(X)$ (see section 7) and both axioms (2) and (3) for $K_0(X)$, there seems to be no way of extending that approach to prove axiom (3) for $K_*(X)$. We will rather proceed as follows. As $K_0(X)$ is a λ -ring, there exist short exact sequences in $\mathcal{P}(X)$ that prove the relation $\lambda^r(\lambda^s(x)) = P_{r,s}(\lambda^1(x), \dots, \lambda^{rs}(x))$ in $K_0(X)$ when x is the class of a vector bundle V on X . We will see (see subsection 8.4) that if in fact these short exact sequences exist functorially in V , then we can inductively prove the existence of short exact sequences in $(B_b^q)^n \mathcal{P}(X)$ that prove the relation above when x is the class of an object in $(B_b^q)^n \mathcal{P}(X)$; in other words, we have then proved axiom (3) for $K_*(X)$.

We are therefore reduced to showing the existence of such short exact sequences of functors in V . This problem may be seen as a weak variant of the famous plethysm problem (see Remark 8.22). The crucial insight now is that it becomes attackable when we also require these functors to be polynomial (see Definition 8.1). On the one hand, this requirement guarantees the existence of appropriate base change functors and hence reduces the problem to $X = \text{Spec}(\mathbb{Z})$ (see subsections 8.1 and 8.4). On the other hand, it makes the computation of the corresponding Grothendieck group of functors feasible; this is the content of the following theorem, which we highlight as it may be of independent interest.

It is well-known that there exists a unique λ -ring structure on the ring $\mathbb{Z}[s_1, s_2, \dots]$ of integral polynomials in infinite variables such that $\lambda^r(s_1) = s_r$ for all r . Furthermore, let $\text{Pol}_{<\infty}(\mathbb{Z})$ denote the category of polynomial functors over \mathbb{Z} of bounded degree (whose Grothendieck group is easily seen to be a pre- λ -ring).

Theorem (Theorem 8.5). *The ring homomorphism*

$$\mathbb{Z}[s_1, s_2, \dots] \rightarrow K_0(\text{Pol}_{<\infty}(\mathbb{Z})), \quad s_i \mapsto [\Lambda^i]$$

is an isomorphism of pre- λ -rings.

This theorem obviously implies that the right-hand side is a λ -ring as well and hence that the short exact sequences of functors postulated above indeed exist. After interpreting polynomial functors as modules over certain Schur algebras following Krause [Kr13, Section 8.2] (see subsection 8.2), we will prove the theorem by following Serre's computation [Se68] of the

Grothendieck group of representations of the group scheme $\mathrm{GL}_{n,\mathbb{Z}}$ (see subsection 8.3). A crucial ingredient here is Green’s computation [Gre80] of the Grothendieck group of polynomial functors over a field.

The fundamental idea of proving λ -ring axioms for Grothendieck groups of complexes via the corresponding axioms for a Grothendieck group of appropriate functors is also sketched in an exchange of letters between Deligne and Grothendieck [DG67]. Both their correspondence and the introduction of Serre’s paper [Se68] already allude to a role of Serre’s result for λ -operations.

In a forthcoming paper we will complement the somewhat intricate constructions of this paper with simpler formulae that (help to) compute our λ -operations in certain cases. For instance, we will give formulae for our λ -operations when applied to K_1 -groups of rings or to external products $K_m(X) \times K_n(X) \rightarrow K_{m+n}(X)$.

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1. BINARY MULTICOMPLEXES AND ALGEBRAIC K -THEORY

In this section we review the description of algebraic K -groups in terms of binary complexes given in [Gra12]. We also prove a simple lemma about shifted binary complexes to justify a slight modification of Grayson’s description. The lemma is also useful for computations.

Recall that an *exact category* in the sense of Quillen [Qui73] is an additive category with a distinguished class of ‘short exact sequences’ that behave like the short exact sequences of an abelian category. A small exact category \mathcal{N} may also be thought of as a full subcategory of an ambient abelian category \mathcal{A} such that \mathcal{N} is closed under extensions in \mathcal{A} .¹ The category of chain complexes in an exact category is again an exact category, with short exact sequences defined to be those sequences of chain maps that are short exact in each degree. In this paper we consider only complexes that are concentrated in non-negative degrees, those with an underlying $\mathbb{Z}_{\geq 0}$ -graded object. We denote this category of chain complexes in \mathcal{N} by $C\mathcal{N}$. A chain complex is bounded if it has only finitely many non-zero objects. The exact subcategory of $C\mathcal{N}$ of bounded chain complexes is denoted $C_b\mathcal{N}$. An *acyclic* complex in an exact category is a chain complex N_\bullet in \mathcal{N} whose differentials $d_i: N_i \rightarrow N_{i-1}$ factor as $N_i \rightarrow Z_i \rightarrow N_{i-1}$ (with Z_i in \mathcal{N}), such that each $0 \rightarrow Z_{i+1} \rightarrow N_i \rightarrow Z_i \rightarrow 0$ is a short exact sequence in \mathcal{N} .² The full subcategories of

¹This is the *Gabriel–Quillen embedding theorem* [TT90], A.7.1, A.7.16.

²This is not in general the same thing as being a long exact sequence in the ambient abelian category \mathcal{A} . However in this paper we work only with *idempotent complete* exact categories, in which case the two notions coincide. See [Gra12] and [TT90] A.9.2.

acyclic complexes in $C\mathcal{N}$ and $C_b\mathcal{N}$ are also exact, and are denoted $C^q\mathcal{N}$ and $C_b^q\mathcal{N}$.

Since each of these categories of complexes is also an exact category, we can iterate their construction to define n -dimensional multicomplexes in \mathcal{N} . A 1-dimensional multicomplex in \mathcal{N} is simply a chain complex, an object of $C\mathcal{N}$. An $n + 1$ -dimensional multicomplex in \mathcal{N} is a chain complex in the exact category $C^n\mathcal{N}$ of n -dimensional multicomplexes in \mathcal{N} . We define categories of bounded and/or acyclic multicomplexes, $(C_b)^n\mathcal{N}$, $(C^q)^n\mathcal{N}$ and $(C_b^q)^n\mathcal{N}$ analogously. With these notions in place, we can define binary complexes and multicomplexes.

Definition 1.1.

- (1) A *binary complex* in an exact category \mathcal{N} is a triple $(N_\bullet, d, \tilde{d})$ consisting of: a $\mathbb{Z}_{\geq 0}$ -graded collection of objects of \mathcal{N} together with two differentials d and \tilde{d} such that (N_\bullet, d) and (N_\bullet, \tilde{d}) are chain complexes in \mathcal{N} . A binary complex can be regarded as pair of objects of $C\mathcal{N}$ that have the same underlying graded object. A morphism of binary complexes is a degree 0 map between these underlying objects that commutes with both differentials. The category of binary complexes in \mathcal{N} is denoted $B\mathcal{N}$. This is an exact category in the same way that $C\mathcal{N}$ is.
- (2) A *bounded acyclic* binary complex in \mathcal{N} is a binary complex such that the chain complexes (N_\bullet, d) and (N_\bullet, \tilde{d}) are bounded and acyclic. The category of bounded acyclic binary complexes in \mathcal{N} is denoted $B_b^q\mathcal{N}$. It is an exact subcategory of $B\mathcal{N}$.
- (3) An n -dimensional *binary multicomplex* is an object of the exact category $B^n\mathcal{N} = B \cdots B\mathcal{N}$ (defined in the same way as $C^n\mathcal{N}$). An n -dimensional bounded acyclic binary multicomplex is an object of $(B_b^q)^n\mathcal{N}$.

Remark 1.2. A (bounded acyclic) binary multicomplex N , of dimension n is equivalent to the following data: a (bounded) $\mathbb{Z}_{\geq 0}^n$ -graded collection of objects of \mathcal{N} equipped with *two* (acyclic) differentials, denoted d^i and \tilde{d}^i , in each direction $1 \leq i \leq n$ subject to the following commutativity requirements:

- (1) $d^i d^j = d^j d^i$
- (2) $d^i \tilde{d}^j = \tilde{d}^j d^i$
- (3) $\tilde{d}^i d^j = d^j \tilde{d}^i$
- (4) $\tilde{d}^i \tilde{d}^j = \tilde{d}^j \tilde{d}^i$

whenever $i \neq j$.

Another way to look at these commutativity restraints is that the various subsets of the differentials form (non-binary) multicomplexes: for each $i = 1, \dots, n$, choose d^i or \tilde{d}^i , and consider the object that has the same underlying \mathbb{Z}^n -graded object as N , but now has *one* acyclic differential in each direction i , given by d^i or \tilde{d}^i , depending on our choice. For each of the 2^n choices of differentials, the resulting object is a bounded acyclic multicomplex, i.e., an object of $(C_b^q)^n\mathcal{N}$; conversely, given a pair of differentials

d^i, \tilde{d}^i in each direction, if the 2^n choices all form objects of $(C_b^q)^n \mathcal{N}$, then the whole assembly is an object of $(B_b^q)^n \mathcal{N}$.

Since this category of bounded acyclic binary complexes in \mathcal{N} is itself an exact category, we can form its Grothendieck group $K_0(B_b^q \mathcal{N})$. The main theorem of [Gra12] is a surprising connection between this group and the n^{th} higher K -group of \mathcal{N} . We call an n -dimensional binary multicomplex *diagonal* if the pair of differentials in some direction are equal, i.e, if $d^i = \tilde{d}^i$ for some $1 \leq i \leq n$. Grayson's theorem, which we shall hereafter use as our definition of the K -groups, says that $K_n(\mathcal{N})$ is isomorphic to the quotient of the Grothendieck group of $B_b^q \mathcal{N}$ by the subgroup generated by the classes of the diagonal bounded acyclic binary multicomplexes. More formally:

Theorem / Definition 1.3 ([Gra12], Corollary 7.4). *For \mathcal{N} an exact category and $n \geq 0$, the abelian group $K_n(\mathcal{N})$ is presented as follows. There is one generator for each bounded acyclic binary multicomplex of dimension n , and there are two families of relations:*

- (1) $[N'] + [N''] = [N]$ if there is a short exact sequence

$$0 \rightarrow N' \rightarrow N \rightarrow N'' \rightarrow 0$$

in $(B_b^q)^n \mathcal{N}$, and

- (2) $[D] = 0$ if D is a diagonal bounded acyclic binary multicomplex.

We remark that our statement of Theorem / Definition 1.3 is subtly different than the one originally given by Grayson. Our bounded acyclic binary multicomplexes are first-quadrant multicomplexes, those that are supported in $\mathbb{Z}_{\geq 0}^n$, whereas Grayson's do not have to satisfy this condition. The absolute lower bound for complexes is a technical constraint that we need in order to use the Dold–Kan correspondence. Our additional condition is harmless as the following proposition shows. For this, let $K_n^{\text{Gr}}(\mathcal{N})$ temporarily denote the n^{th} K -group of \mathcal{N} as defined in [Gra12].

Proposition 1.4. *For every exact category \mathcal{N} and every $n \geq 0$, the canonical homomorphism $K_n(\mathcal{N}) \rightarrow K_n^{\text{Gr}}(\mathcal{N})$ is bijective.*

Proof. For ease of presentation we shall prove this for $n = 1$ only: there is no additional difficulty for $n > 1$. Let $B_\infty^q \mathcal{N}$ denote the category of bounded acyclic binary complexes in \mathcal{N} that may be supported anywhere on \mathbb{Z} . For $i \geq 0$, let $B_{\geq -i}^q \mathcal{N}$ denote the full subcategory of $B_\infty^q \mathcal{N}$ consisting of complexes that are supported on $[-i, \infty]$. We then have $\cup_i B_{\geq -i}^q \mathcal{N} = B_\infty^q \mathcal{N}$ and hence $\lim_i K_0(B_{\geq -i}^q \mathcal{N}) = K_0(B_\infty^q \mathcal{N})$. Let T_i denote the subgroup of $K_0(B_{\geq -i}^q \mathcal{N})$ generated by diagonal complexes and let T denote the similarly defined subgroup of $K_0(B_\infty^q \mathcal{N})$. The resulting injective homomorphism $\lim_i T_i \rightarrow T$ is also surjective because all complexes are assumed to be bounded. We therefore obtain an isomorphism

$$\lim_i \left(K_0(B_{\geq -i}^q \mathcal{N}) / T_i \right) \cong \lim_i K_0(B_{\geq -i}^q \mathcal{N}) / \lim_i T_i \cong K_0(B_\infty^q \mathcal{N}) / T = K_1^{\text{Gr}}(\mathcal{N}).$$

The following lemma (after generalising it from $B_{\geq 0}^q \mathcal{N}$ to $B_{\geq -i}^q \mathcal{N}$) shows that, for every $i \geq 0$, ‘shifting’ induces a two-sided inverse to the negative

of the canonical homomorphism $K_0(B_{\geq -i}^q \mathcal{N})/T_i \rightarrow K_0(B_{\geq -i-1}^q \mathcal{N})/T_{i+1}$. Hence the canonical map

$$K_1(\mathcal{N}) = K_0(B_{\geq 0}^q \mathcal{N})/T_0 \rightarrow \lim_i \left(K_0(B_{\geq -i}^q \mathcal{N})/T_i \right) \cong K_1^{\text{Gr}}(\mathcal{N})$$

is an isomorphism, as was to be shown. \square

Definition 1.5. Let N_\bullet be an acyclic binary complex with differentials d and \tilde{d} . The k^{th} shift of N , denoted $N[k]$, is the acyclic binary complex that has the same collection of objects as N , but ‘shifted’ k places, i.e., $(N[k])_i = N_{i-k}$, and differentials given by $(-1)^k d$ and $(-1)^k \tilde{d}$.

Lemma 1.6. For any bounded acyclic binary complex N , and $k \in \mathbb{Z}_{\geq 0}$, we have $[N[k]] = (-1)^k [N]$ in $K_1 \mathcal{N}$.

Proof. It is enough to show that $[N[1]] = -[N]$. There is a short exact sequence

$$0 \rightarrow N_\bullet \rightarrow \text{cone}(N_\bullet) \rightarrow N_\bullet[1] \rightarrow 0,$$

where $\text{cone}(N_\bullet)$ denotes the mapping cone of the identity map $N_\bullet \xrightarrow{1} N_\bullet$ ($\text{cone}(N_\bullet)$ is a binary complex in the obvious way). So it suffices to show that $\text{cone}(N_\bullet)$ vanishes in $K_1 \mathcal{N}$. Let N_n be the left-most non-zero object of N_\bullet , and let $\text{trun}(N_\bullet)$ be the (not necessarily acyclic) binary complex formed by truncating N_\bullet to forget N_n : that is, $\text{trun}(N_\bullet)$ has a 0 in place of N_n . Then there is a short exact sequence

$$0 \rightarrow \text{cone}(\text{trun}(N_\bullet)) \rightarrow \text{cone}(N_\bullet) \rightarrow \Delta(N_n \xrightarrow{1} N_n) \rightarrow 0,$$

where $\Delta(N_n \xrightarrow{1} N_n)$ is the diagonal binary complex

$$0 \rightrightarrows N_n \xrightleftharpoons[1]{1} N_n \rightrightarrows 0,$$

which is supported in degrees $n+1$ and n . Mapping cones of identities are always acyclic, so $\text{cone}(\text{trun}(N_\bullet))$ is acyclic even when $\text{trun}(N_\bullet)$ is not. Since $\Delta(N_n \xrightarrow{1} N_n)$ is diagonal its class vanishes in $K_1 \mathcal{N}$, so the above short exact sequence yields the relation $[\text{cone}(N_\bullet)] = [\text{cone}(\text{trun}(N_\bullet))]$. We iterate this procedure by repeatedly truncating $\text{trun}(N_\bullet)$ to show that $[\text{cone}(N_\bullet)]$ is zero. \square

The same proof gives the analogous result for binary multicomplexes: for N in $(B_b^q)^n \mathcal{N}$ the class of N shifted one place in any of the n possible directions in $K_n(\mathcal{N})$ is $-[N]$. From this the actions of more general shifts (in multiple directions) follow immediately.

2. PRELIMINARIES FROM HOMOLOGICAL ALGEBRA

In this section we recall some preliminaries from the homological algebra of exact categories. We say what it means for an exact category to be idempotent complete or split, and show that the notions of acyclicity and contractibility of complexes coincide in exact categories that have both of these properties. We then review simplicial objects and the Dold–Kan correspondence. Finally we discuss functors of finite degree, a weakening of the concept of additive functors. These three topics may seem rather disjoint

here, but we bring them together in the next section to produce functors between categories of chain complexes that preserve boundedness and acyclicity, paving the way for a functor on binary multicomplexes that induces a map on K -theory.

Definition 2.1. An exact category \mathcal{N} is *idempotent complete* if every idempotent endomorphism in \mathcal{N} the category has a kernel.

This does not hold for example for the category of free modules over a ring when there exists a non-free projective module. All of the exact categories we use in this paper are idempotent complete. This is an assumption on the ‘base level’ exact categories we introduce, but will need to be proven for categories of multicomplexes (Lemma 3.4). Idempotent complete exact categories come with an embedding into an abelian category $\mathcal{N} \hookrightarrow \mathcal{A}$ that *supports long exact sequences*: a chain complex is acyclic in \mathcal{N} if and only if it is exact when considered as a chain complex of \mathcal{A} (see §1 of [Gra12]). Homological algebra is therefore quite straightforward in idempotent complete exact categories.

Contractible complexes in idempotent complete exact categories are always acyclic; indeed this is an equivalent characterisation of idempotent completeness ([Büh10], 10.9). Acyclic complexes in exact categories (even idempotent complete ones) are not usually contractible. There is a useful criterion for contractibility, however. Recall that a chain complex (C, d) is called *split* if there exist maps $s_n: C_{n-1} \rightarrow C_n$ such that $d_n s_n d_n = d_n$.

Lemma 2.2. *A chain complex in an idempotent complete exact category is contractible if and only if it is acyclic and split.*

Proof. It follows the definition of a chain homotopy that contractible complexes in idempotent complete exact categories are also split. Conversely, an elementary argument shows that if a complex in an exact category is acyclic and split, then the collection of splitting maps $\{s_n\}$ describes a homotopy from its identity map to its zero map. \square

If an acyclic complex is split, each of the constituent short exact sequences that it factors into is split: that is, isomorphic to a canonical direct sum sequence (the converse is obviously true as well). Recall that an exact category is called *split exact* if all of its declared short exact sequences are split. In such an exact category, all acyclic complexes are split. Hence the notions of contractibility and acyclicity coincide for complexes in a split exact category that is also idempotent complete. An example of such an exact category is the category $\mathcal{P}(R)$ of (finitely generated) projective modules over a ring R . That acyclic complexes are contractible in this category is key to the results of this paper.

We now turn to the Dold–Kan correspondence. To give its statement we need the language of simplicial objects. Recall that Δ denotes the *simplex category*: the category whose objects are the finite non-empty ordered sets $[n] = \{0 < 1 < \dots < n\}$ and whose morphisms are the order-preserving maps. A *simplicial object* in a category \mathcal{C} is a contravariant functor from Δ to \mathcal{C} , and the natural transformations between such functors make $\mathcal{C}^{\Delta^{\text{op}}}$ into a category. Equivalently, a simplicial object C in \mathcal{C} can be specified to be a

collection of objects $C_n, n \in \mathbb{N}$, of \mathcal{C} together with face maps $\delta_i: C_n \rightarrow C_{n-1}$ and degeneracy maps $\sigma_j: C_n \rightarrow C_{n+1}$, $i, j = 0, \dots, n$, satisfying various combinatorial identities. A morphism between simplicial objects C and D is a collection of morphisms $C_n \rightarrow D_n$ that commutes with the faces and degeneracies. A homotopy $h: f \simeq g$ between simplicial maps $f, g: C \rightarrow D$ is a simplicial morphism $h: C \times \Delta^1 \rightarrow D$ (where Δ^1 denotes the simplicial set corresponding to the ordered set $\{0 < 1\}$, as usual) such that $h|_{C \times \{0\}} = f$ and $h|_{C \times \{1\}} = g$; it can also be described as collection of morphisms $h_i: C_n \rightarrow D_{n+1}$, $i = 0, \dots, n$, which satisfy further combinatorial identities determined by compositions relating f, g , the h_i , and the faces and degeneracies of C and D . See, for example, Chapter 8 of [Wei94] for full definitions of simplicial objects and homotopies.

If $F: \mathcal{C} \rightarrow \mathcal{D}$ is a covariant functor, then post-composition with F induces a functor between categories of simplicial objects $\mathcal{C}^{\Delta^{\text{op}}} \rightarrow \mathcal{D}^{\Delta^{\text{op}}}$. Abusing notation, we shall also call this functor F . Importantly, if $h: f \simeq g$ is a simplicial homotopy between $f, g: C \rightarrow D$, then $F(h): F(f) \simeq F(g)$ is a simplicial homotopy between $F(f), F(g): F(C) \rightarrow F(D)$. The analogous statement for chain homotopies between chain maps is not true if F is not additive. The Dold–Kan correspondence shows that chain complexes and simplicial objects are equivalent in a non-obvious way, and allows us to induce homotopy-preserving functors between categories of chain complexes, even when the original functors are not additive.

Definition 2.3. Let \mathcal{P} be an additive category. Given a chain complex $C. \in \mathcal{CP}$, we define a simplicial object $\Gamma(C.) \in \mathcal{P}^{\Delta^{\text{op}}}$ as follows.

- (1) **Objects:** Given $p \leq n$, let η range over all surjections $[n] \twoheadrightarrow [p]$ in Δ , and let $C_p\langle\eta\rangle$ denote a copy of C_p that is labelled by η . For each n , set

$$\Gamma(C)_n := \bigoplus_{p \leq n} \bigoplus_{\eta} C_p\langle\eta\rangle.$$

- (2) **Maps:** If $\alpha: [m] \rightarrow [n]$ is a morphism in Δ , we describe $\Gamma(\alpha)$ by describing each $\Gamma(\alpha, \eta)$, the restriction of $\Gamma(\alpha)$ to the summand $C_p\langle\eta\rangle$ of $\Gamma(C)_n$. Let

$$[m] \xrightarrow{\eta'} [q] \xrightarrow{\varepsilon} [p]$$

be the unique epi-monic factorisation of $\eta\alpha$. Then

$$\Gamma(\alpha, \eta) := \begin{cases} 1: C_p\langle\eta\rangle \rightarrow C_p\langle\eta'\rangle & \text{if } q = p, \\ d_p: C_p\langle\eta\rangle \rightarrow C_{p-1}\langle\eta'\rangle & \text{if } q = p-1 \text{ and } \varepsilon = \varepsilon_p, \\ 0 & \text{otherwise.} \end{cases}$$

This construction extends to a functor³ $\Gamma: \mathcal{CP} \rightarrow \mathcal{P}^{\Delta^{\text{op}}}$.

Theorem (Dold–Kan correspondence). *If \mathcal{P} is idempotent complete, then the functor $\Gamma: \mathcal{CP} \rightarrow \mathcal{P}^{\Delta^{\text{op}}}$ is an equivalence of categories. Furthermore, Γ is exact and preserves homotopies.*

³Other authors (e.g., [Wei94]) use K in place of Γ ; we avoid this notation for obvious reasons.

Proof. Chapter 8 of [Wei94] proves this when \mathcal{P} is an abelian category. The general case is §1.2.3 of [Lur14]. \square

The inverse functor to Γ is most simply described for an abelian category.

Definition 2.4. Let A be a simplicial object in an abelian category \mathcal{A} .

- (1) The *associated chain complex* $C(A)$ has objects $C(A)_n = A_n$ and differential

$$d_n = \sum_{i=0}^n (-1)^i \delta_i: C(A)_n \rightarrow C(A)_{n-1}.$$

- (2) The subcomplex

$$D(A)_n = \sum_{i=0}^n \text{Im}(\sigma_i: A_{n-1} \rightarrow A_n)$$

is called the *degenerate subcomplex* of $C(A)$.

- (3) The *normalized Moore complex* $N(A)$ has objects

$$N_n(A) = A_n / D(A)_n$$

with the induced differential \bar{d}_n .

The associated chain complex splits globally as $C(A) = N(A) \oplus D(A)$.

The normalized Moore complex defines a functor $N: \mathcal{A}^{\Delta^{\text{op}}} \rightarrow C\mathcal{A}$. It is exact and preserves homotopies, and is inverse to Γ (up to natural isomorphism). Now if \mathcal{P} is an idempotent complete exact category, then there is an embedding $\mathcal{P} \subseteq \mathcal{A}$ into an abelian category such that \mathcal{P} is closed under taking direct summands in \mathcal{A} . If P is an object of $\mathcal{P}^{\Delta^{\text{op}}} \subseteq \mathcal{A}^{\Delta^{\text{op}}}$, then the associated chain complex $C(P)$ is a chain complex \mathcal{A} with objects in \mathcal{P} . But $N(P)$ is a direct summand of $C(P)$, which has objects in \mathcal{P} , so $N(P)$ has objects in \mathcal{P} . Therefore N restricts to a functor $\mathcal{P}^{\Delta^{\text{op}}} \rightarrow C\mathcal{P}$. Furthermore the functor N is exact and preserves homotopies. See [Lur14] for further details.

We conclude our preliminaries by discussing functors of finite degree.

Definition 2.5. Let $F: \mathcal{C} \rightarrow \mathcal{D}$ be any functor between additive categories that satisfies $F(0) = 0$. Then there is a functorial decomposition

$$F(X \oplus Y) = F(X) \oplus \text{cr}_2(F)(X, Y) \oplus F(Y),$$

where $\text{cr}_2(F): \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{D}$ is the second *cross-effect functor* (see [EML54]), which is defined to be the kernel of the natural projection $F(X \oplus Y) \rightarrow F(X) \oplus F(Y)$. The functor F is said to have *degree* ≤ 1 if it is additive (i.e., if $\text{cr}_2(F)$ vanishes), and we say that F has *degree* $\leq d$ if $\text{cr}_2(F)(X, Y)$ is of degree $\leq d - 1$ in each argument. If F is of degree $\leq d$, then F is of degree $\leq d'$ for all $d' \geq d$. We say that F has *degree* d if it has degree $\leq d$ but does not have degree $\leq d - 1$.

Example 2.6. For R a non-zero commutative ring, the exterior power $\Lambda^r: \mathcal{P}(R) \rightarrow \mathcal{P}(R)$ has degree r for each $r > 0$. This follows from the canonical decomposition

$$\Lambda^r(X \oplus Y) \cong \Lambda^r(X) \oplus \left(\bigoplus_{i=1}^{r-1} \Lambda^{r-i}(X) \otimes \Lambda^i(Y) \right) \oplus \Lambda^r(Y).$$

If $F: \mathcal{P} \rightarrow \mathcal{Q}$ is an *additive* functor between exact categories, and if P is a bounded complex, then $NFT(P)$ is certainly bounded again. This also holds true for functors of finite degree, as the following lemma shows.

Lemma 2.7 ([SK10], Corollary 4.6). *Let P be a chain complex in $C\mathcal{P}$ of length ℓ , and let $F: \mathcal{P} \rightarrow \mathcal{Q}$ be a functor of degree d between exact categories. Then $NFT(P)$ has length less than or equal to $d\ell$. \square*

3. OPERATIONS ON ACYCLIC COMPLEXES

In this rather abstract section we describe how to use the Dold–Kan correspondence to extend a functor $F: \mathcal{P} \rightarrow \mathcal{P}$ on an idempotent complete exact category to a functor on each category of multicomplexes $F_n: C^n\mathcal{P} \rightarrow C^n\mathcal{P}$, $n \geq 1$. We show that if \mathcal{P} is split exact, then the extended functors F_n send acyclic multicomplexes to acyclic multicomplexes. We also show that if F is of finite degree, then each F_n preserves bounded multicomplexes and is also of finite degree.

Proposition 3.1. *Let $F: \mathcal{P} \rightarrow \mathcal{P}$ be a covariant functor on an idempotent complete exact category, with $F(0) = 0$. Let $F_1 := NFT: C\mathcal{P} \rightarrow C\mathcal{P}$ denote the induced functor. Then the following statements are true:*

- (1) $F_1(0) = 0$.
- (2) F_1 sends contractible complexes to contractible complexes.
- (3) If \mathcal{P} is split exact, then F_1 sends acyclic complexes to acyclic complexes.
- (4) If F is of degree at most d , then F_1 sends bounded complexes to bounded complexes and F_1 is again of degree at most d .

Proof. Part (1) is trivial.

For part (2), the functors $\Gamma: C\mathcal{P} \rightarrow \mathcal{P}^{\Delta^{\text{op}}}$ and $N: \mathcal{P}^{\Delta^{\text{op}}} \rightarrow C\mathcal{P}$ preserve homotopies and send 0 to 0, so they both send contractible objects to contractible objects. Furthermore, F sends homotopies in $\mathcal{P}^{\Delta^{\text{op}}}$ to homotopies in $\mathcal{P}^{\Delta^{\text{op}}}$ —if $h: f \sim g$ is a homotopy, then $F(h): F(f) \sim F(g)$ is a homotopy. Since F also has the property that $F(0) = 0$, we see that if $A \simeq 0$ in $\mathcal{P}^{\Delta^{\text{op}}}$, then $F(A) \simeq F(0) = 0$. Therefore $NFT(P)$ is contractible in $C\mathcal{P}$.

Following Lemma 2.2, the acyclic complexes in a split exact idempotent complete exact category coincide with the contractible ones, so part (3) follows from part (2).

Finally we consider part (4). The first part of this statement is Lemma 2.7. For the second part we note that, since N and Γ are additive, it is enough to show that $F: \mathcal{A}^{\Delta^{\text{op}}} \rightarrow \mathcal{B}^{\Delta^{\text{op}}}$ is of degree $\leq d$. This is proven by induction on d . \square

Part (3) of Proposition 3.1 may not hold in an exact category that is not split exact, as is shown in the following example.

Example 3.2. Let F be the degree 2 endofunctor $A \mapsto A^{\otimes 2}$ on the abelian category of abelian groups, and let C be the short exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{2} \mathbb{Z} \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

considered as an acyclic complex concentrated in degrees 0, 1 and 2. Then $NFT(C) = N \text{diag}(\Gamma(C) \otimes \Gamma(C))$ is homotopy equivalent to $\text{Tot}(C \otimes C)$

by the Eilenberg–Zilber theorem ([May92], §29)⁴. But the homology group $H_2(\text{Tot}(C. \otimes C.))$ is $\mathbb{Z}/2\mathbb{Z}$, so $NFT(C.)$ is not exact. Furthermore, the short exact sequence of functors

$$0 \longrightarrow N\Lambda^2\Gamma \longrightarrow NFT\Gamma \longrightarrow NSym^2\Gamma \longrightarrow 0$$

shows that at least one of $N\Lambda^2\Gamma(C.)$ or $NSym^2\Gamma(C.)$ is not exact either.

We now iterate the Dold–Kan correspondence to describe induced functors on categories of acyclic multicomplexes.

Definition 3.3. Let $F: \mathcal{P} \rightarrow \mathcal{P}$ be a covariant functor on an idempotent complete exact category. We define functors

$$F_n: C^n\mathcal{P} \rightarrow C^n\mathcal{P}$$

for all $n \geq 0$ recursively as follows:

- (1) $F_0 := F: \mathcal{P} \rightarrow \mathcal{P}$.
- (2) By regarding an object of $C^{n+1}\mathcal{P}$ as a chain complex in the exact category $C^n\mathcal{P}$, we define $F_{n+1} := NF_n\Gamma$.

To show that F_n sends acyclic multicomplexes to acyclic multicomplexes in a nice exact category, we need to know that $(C^q)^n\mathcal{P}$ satisfies the same hypotheses as \mathcal{P} . This is the content of the following technical lemma. The proof is not enlightening for the rest of the paper, so we relegate it to appendix A.

Lemma 3.4. *Let \mathcal{P} be an exact category. For all $n > 0$ we have the following:*

- (1) *If \mathcal{P} is idempotent complete, then $C^n\mathcal{P}$ and $(C^q)^n\mathcal{P}$ are also idempotent complete.*
- (2) *If \mathcal{P} is split exact, then $(C^q)^n\mathcal{P}$ is also split exact.*

The analogous results for the categories $C_b^n\mathcal{P}$ and $(C_b^q)^n\mathcal{P}$ of bounded multicomplexes also hold.

Corollary 3.5. *Let \mathcal{P} be a split exact idempotent complete exact category, and $F: \mathcal{P} \rightarrow \mathcal{P}$ a covariant functor such that $F(0) = 0$. Then for $n \geq 0$ the functors of Definition 3.3 restrict to functors*

$$F_n: (C^q)^n\mathcal{P} \rightarrow (C^q)^n\mathcal{P}.$$

Furthermore, if F is of finite degree, then F_n sends bounded multicomplexes to bounded multicomplexes. That is, each F_n restricts to a functor

$$F_n: (C_b^q)^n\mathcal{P} \rightarrow (C_b^q)^n\mathcal{P}.$$

Proof. We consider the unbounded case first. By Proposition 3.1 (1), we easily see that $F_n(0) = 0$ for all n . Assume that F_n restricts to a functor on the idempotent complete split exact category $(C^q)^n\mathcal{P}$. Regarding objects of $(C^q)^{n+1}\mathcal{P}$ as acyclic complexes in $(C^q)^n\mathcal{P}$, the functor $F_{n+1} = NF_n\Gamma$ restricts to a functor on $(C^q)^{n+1}\mathcal{P} = C^q((C^q)^n\mathcal{P})$, by Proposition 3.1 (3) and Lemma 3.4 (2). The first part of the result follows by induction. For the second part, if $F_0 = F$ is of finite degree, then the same induction over n shows that F_n is of finite degree for every n , by Proposition 3.1 (4).

⁴ See also Definition 5.4, Lemma 5.5.

In particular, for each $n \geq 1$, the functor $F_n = (F_{n-1})_1$ sends bounded complexes to bounded complexes, that is, it restricts to a functor

$$F_n: C_b^q((C^q)^{n-1}\mathcal{P}) \rightarrow C_b^q((C^q)^{n-1}\mathcal{P}).$$

But we can say more: considering P_\bullet in $(C_b^q)^n\mathcal{P}$ as a chain complex, each of its objects is in $(C_b^q)^{n-1}\mathcal{P}$, i.e., they are bounded. We claim that the objects of $F_n(P_\bullet) = NF_{n-1}\Gamma(P_\bullet)$ are also bounded. The objects of $\Gamma(P_\bullet)$ are finite direct sums of the objects of P_\bullet . Finite sums of bounded objects are bounded, so the objects of $\Gamma(P_\bullet)$ are bounded. Therefore, by the inductive hypothesis, the objects of $F_{n-1}\Gamma(P_\bullet)$ are also bounded. Finally, the objects of $NF_{n-1}\Gamma(P_\bullet)$ are direct summands of the objects of $F_{n-1}\Gamma(P_\bullet)$ (From Definition 2.4, after embedding into an abelian category), so they are bounded as well. Therefore F_n sends bounded chain complexes of bounded objects in $(C_b^q)^n\mathcal{P}$ to bounded chain complexes of bounded objects in $(C_b^q)^n\mathcal{P}$. This is exactly the statement that F_n restricts to a functor

$$F_n: (C_b^q)^n\mathcal{P} \rightarrow (C_b^q)^n\mathcal{P},$$

which was to be proved. \square

Remark 3.6. Throughout this chapter we work with the inductive definition of $(C_b^q)^n\mathcal{P}$, that is $(C_b^q)^n\mathcal{P} := C_b^q((C_b^q)^{n-1}\mathcal{P})$ for $n > 1$. As explained in Remark 1.2, one can instead think of objects in $(C_b^q)^n\mathcal{P}$ as $\mathbb{Z}_{\geq 0}^n$ -graded objects of \mathcal{N} (together with certain differentials) without specifying the order of directions in which the objects have been obtained in the inductive definition. The purpose of this remark is to convince the reader that our construction of the functors F_n given in this section (and hence our construction of exterior powers in the sequel) does not depend on the order of directions either. Rather than including a complete proof, we sketch the idea in the case $n = 2$. Let $F_0 = F$ be as before. The functor F_2 is defined as

$$N_h F_1 \Gamma_h = N_h N_v F_0 \Gamma_v \Gamma_h,$$

where the indices h and v indicated the horizontal and vertical directions respectively. It is quite straightforward to see that the composition $N_h N_v$ sends a bisimplicial object C to the double complex whose objects are obtained from the corresponding objects of C by dividing out the images of all of the horizontal and vertical degeneracy maps. This latter description of course does not depend on the order of N_h and N_v . One can show that the same holds for Γ_h and Γ_v by a similar argument, or just by recalling that Γ_h and Γ_v are adjoint to N_h and N_v respectively.

We can now describe the exterior power functors that we will use to induce operations on higher K -groups. The following example is the motivation for our work so far.

Main Example 3.7. Let $\mathcal{P}(R)$ be the category of finitely generated projective modules over a commutative ring R . This category is both idempotent complete and split exact. For each $r > 0$, the usual exterior power functor $\Lambda^r: \mathcal{P}(R) \rightarrow \mathcal{P}(R)$ satisfies the hypotheses of Corollary 3.5 (Λ^r has degree r). We therefore have induced functors

$$\Lambda_n^r: (C_b^q)^n\mathcal{P}(R) \rightarrow (C_b^q)^n\mathcal{P}(R)$$

for all $n \geq 0$.

In general, the complex $N\Lambda^r\Gamma(P_\bullet)$ is difficult to write down explicitly. The paper [SK10] gives an algorithm that addresses this problem. We conclude this section by computing $N\Lambda^r\Gamma(P_\bullet)$ for a very simple choice of P_\bullet .

Example 3.8. Let $\varphi: P \rightarrow Q$ be an isomorphism of invertible modules over some commutative ring R , considered as an acyclic complex concentrated in degrees 0 and 1:

$$\begin{array}{ccccccc} 0 & \longrightarrow & P & \xrightarrow{\varphi} & Q & \longrightarrow & 0, \\ 2 & & 1 & & 0 & & -1 \end{array}$$

or $P \xrightarrow{\varphi} Q$ for short. In Lemma 2.2 of [Köc01], the second author gives an explicit calculation of $N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)$ in terms of higher cross-effect functors (in fact, he does this for more general P, Q and φ). Specifically, in degree n we have:

$$N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)_n = \text{cr}_n(\Lambda^r)(P, \dots, P) \oplus \text{cr}_{n+1}(\Lambda^r)(Q, P, \dots, P).$$

We do not wish to expound on the theory of cross-effect functors here: the interested reader can see [EML54] or section 1 of [Köc01]. Instead we merely quote the properties of $\text{cr}_n(\Lambda^r)$ that we need. Firstly, $\text{cr}_n(\Lambda^r) = 0$ for $n > r$, as Λ^r is of degree r ; secondly, $\text{cr}_r(\Lambda^r)(P_1, \dots, P_r) = P_1 \otimes \dots \otimes P_r$; thirdly, if $n < r$ and if P_1, \dots, P_n are all invertible, then $\text{cr}_n(\Lambda^r)(P_1, \dots, P_n) = 0$. From these we see that:

$$N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)_n = \begin{cases} P^{\otimes r} & \text{if } n = r, \\ Q \otimes P^{\otimes(r-1)} & \text{if } n = r - 1, \\ 0 & \text{otherwise.} \end{cases}$$

We can also read off the differential $P^{\otimes r} \rightarrow Q \otimes P^{\otimes(r-1)}$ from Lemma 2.2 of *loc. cit.*: it is $\text{cr}_r(\Lambda^r)(\varphi, 1, \dots, 1) = \varphi \otimes 1 \otimes \dots \otimes 1$. So $N\Lambda^r\Gamma(P \xrightarrow{\varphi} Q)$ is the acyclic complex

$$\begin{array}{ccccccc} 0 & \longrightarrow & P \otimes P^{\otimes(r-1)} & \xrightarrow{\varphi \otimes 1} & Q \otimes P^{\otimes(r-1)} & \longrightarrow & 0. \\ r+1 & & r & & r-1 & & r-2 \end{array}$$

Of particular note is the special case in which P and Q are equal to R considered as a module over itself, and φ is given by multiplication by some $x \in R^\times$. Then $N\Lambda^r\Gamma(R \xrightarrow{x} R)$ is equal to the complex $(R \xrightarrow{x} R)$, shifted so that it is concentrated in degrees r and $r - 1$.

4. OPERATIONS ON BINARY MULTICOMPLEXES

The goal of this section is to extend the functors F_n between multicomplexes of the previous section to functors of *binary* multicomplexes. Together with the results of the previous section, this shows that if P_\bullet is a bounded acyclic binary multicomplex, then so is $\Lambda_n^r(P_\bullet)$.

Categories of binary complexes are not so well behaved as categories of complexes. In particular, the category of bounded acyclic binary complexes in a split exact category is not split exact.

Example 4.1. Let P be an object in a split exact category \mathcal{P} . The following diagram is an admissible epimorphism in the category of bounded acyclic binary complexes in \mathcal{P} :

$$\begin{array}{ccccc} P & \xrightarrow{i_1} & P \oplus P & \xrightarrow{p_1} & P \\ & \searrow i_2 & \downarrow \Sigma & \searrow p_2 & \downarrow \\ P & \xrightarrow{1} & P & \xrightarrow{1} & 0 \end{array}$$

(where i_1 and i_2 are the inclusions into the first and second summands, p_1 and p_2 are the corresponding projections, and $\Sigma = p_1 + p_2$). But there is no splitting $P \rightarrow P \oplus P$ that commutes with both the top and bottom differentials, so $B_b^q \mathcal{P}$ is not split exact.

This difficulty means that we cannot define exterior powers of binary multi-complexes recursively in exactly the way we have for multicomplexes. This problem is resolvable: we shall show that if P_\bullet is an object of $(C_b^q)^n \mathcal{P}(R)$, then the objects of $\Lambda_n^r(P_\bullet)$ are independent of the differentials of P_\bullet . Therefore it will make sense to define the exterior power of a binary complex by applying the exterior powers we developed above individually to the two differentials of the binary complex. The resulting pair of complexes will have the same objects, so we consider them as a binary complex.

Lemma 4.2. *Let $F: \mathcal{P} \rightarrow \mathcal{P}$ be a covariant functor on an idempotent complete exact category. If P_\bullet and Q_\bullet are chain complexes with the same underlying graded object, then $NFT(P_\bullet)$ and $NFT(Q_\bullet)$ have the same underlying graded object.*

Proof. Let $B \in \mathcal{P}^{\Delta^{\text{op}}}$ be a simplicial object. The objects of the complex $N(B)$ are given by

$$N(B)_n := B_n / \left(\sum_{i=0}^n \text{Im}(\sigma_i: B_{n-1} \rightarrow B_n) \right),$$

(after embedding \mathcal{P} in a suitable abelian category) where the σ_i are the degeneracies of B . It is enough therefore to show that the objects and degeneracy maps of $F\Gamma(P_\bullet)$ do not depend upon the differential of P_\bullet . The objects of $\Gamma(P_\bullet)$ are direct sums of the objects of P_\bullet , indexed by the surjections out of $[n]$ in Δ , and do not depend on the differential. The degeneracy operator $\sigma_i: \Gamma(P_\bullet)_{n-1} \rightarrow \Gamma(P_\bullet)_n$ is the image of the degeneracy map $\eta_i: [n] \rightarrow [n-1]$ in Δ . For any surjection $\eta: [n-1] \twoheadrightarrow [p]$, the composition $\eta\eta_i$ is also a surjection, so the monomorphism in the epi-monic factorisation of $\eta\eta_i$ is just the identity on $[p]$. Therefore, the degeneracy operator σ_i acts on $\Gamma(P_\bullet)_{n-1}$ by sending the summand corresponding to the surjection η by the identity to the summand of $\Gamma(P_\bullet)_n$ corresponding to the surjection $\eta_i\eta$. Thus σ_i does not depend on the differential of P_\bullet . Since the objects and degeneracies of $\Gamma(P_\bullet)$ only depend on the underlying graded object of P_\bullet , the same is true of $F\Gamma(P_\bullet)$. Therefore the objects of $NFT(P_\bullet)$ only depend on the underlying graded object as well. \square

Corollary 4.3. *Let $n \geq 1$, and let P_* , Q_* be objects of $(C_b^q)^n \mathcal{P}$. If P_* and Q_* have the same underlying \mathbb{Z}^n -graded object, then $F_n(P_*)$ and $F_n(Q_*)$ have the same underlying \mathbb{Z}^n -graded object.*

Proof. This is a straightforward induction on n . \square

We are now ready at last to define exterior powers of acyclic binary multicomplexes. Let P_* be an n -dimensional, bounded, acyclic binary multicomplex in \mathcal{P} , i.e., an object of $(B_b^q)^n \mathcal{P}$. We view the commutativity constraints on the differentials of P_* in the same way as described in Remark 1.2: as a collection of 2^n objects of $(C_b^q)^n \mathcal{P}$.

Definition 4.4. For a functor F that satisfies the hypotheses of Corollary 3.5, we define induced functors

$$F_n: (B_b^q)^n \mathcal{P} \longrightarrow (B_b^q)^n \mathcal{P}$$

by the following procedure. Let P_* be an object of $(B_b^q)^n \mathcal{P}$, viewed as a collection of 2^n (non-binary) multicomplexes in the manner described above. Since these multicomplexes all have the same underlying \mathbb{Z}^n -graded object, by Corollary 4.3 the same is true of the 2^n multicomplexes obtained by applying F_n (the functor defined on $(C_b^q)^n \mathcal{P}$ in Corollary 3.5) to the multicomplexes describing P_* . We define $F_n(P_*)$ to be the binary multicomplex described by the resulting collection of multicomplexes.

We now return to our main example of interest: the exterior power functors. Let R be a commutative ring. We have seen in 3.7 that the usual exterior power operations Λ^r satisfy the hypotheses of 3.5, so the exterior powers

$$\Lambda_n^r: (C_b^q)^n \mathcal{P}(R) \rightarrow (C_b^q)^n \mathcal{P}(R)$$

lift to exterior powers of binary multicomplexes

$$\Lambda_n^r: (B_b^q)^n \mathcal{P}(R) \rightarrow (B_b^q)^n \mathcal{P}(R)$$

for all $n \geq 0$ and $r \geq 1$.

5. SIMPLICIAL TENSOR PRODUCTS

In this section we develop a tensor product for multicomplexes that is compatible with the exterior powers we have defined in the previous sections. We show that the class of this product vanishes in the appropriate K -group, which will eventually be the key to showing that exterior power operations provide homomorphisms on higher K -groups.

5.1. Constructing simplicial tensor products. In this subsection, using the Dold–Kan correspondence again, we construct the so-called simplicial tensor product of multi-complexes and prove that it preserves acyclicity and boundedness of complexes.

Although we are ultimately interested in the products induced from the usual tensor products of modules (or sheaves), it is convenient in this section to work in the rather more abstract setting of a generic idempotent complete exact category with some form of well-behaved tensor product.

Definition 5.1. Let \mathcal{P} be an idempotent complete exact category. We say that a bi-additive bifunctor $\otimes: \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ is a *tensor product* if $P \otimes -$ and $- \otimes P$ are exact functors on \mathcal{P} for each object P of \mathcal{P} .

For the rest of this section, we fix such a category \mathcal{P} with a tensor product \otimes . The reader may wish to keep in mind the example $\mathcal{P} = \mathcal{P}(R)$, with the usual tensor product of R -modules.

Definition 5.2. Let P be an object of \mathcal{P} , and let $(Q_\bullet, d_Q), (R_\bullet, d_R)$ be chain complexes in \mathcal{P} .

- (1) By $P \otimes Q_\bullet$ we mean the chain complex whose i^{th} object is $P \otimes Q_i$, with differential $1 \otimes d_Q$. The complex $Q_\bullet \otimes P$ is defined analogously.
- (2) By $\text{Tot}(Q_\bullet \otimes R_\bullet)$ we mean the chain complex formed by taking the total complex of the bicomplex whose $(i, j)^{\text{th}}$ object is $Q_i \otimes R_j$, and whose differentials are $d^{\text{ver}} = d_Q \otimes (-1)^j$ and $d^{\text{hor}} = 1 \otimes d_R$. This bicomplex's i^{th} row is $Q_i \otimes R_\bullet$ and its j^{th} column is $Q_\bullet \otimes R_j$.

It is clear that if Q_\bullet and R_\bullet are bounded complexes, then the products $P \otimes Q_\bullet$ and $\text{Tot}(Q_\bullet \otimes R_\bullet)$ are bounded as well. We'll need a couple of properties of these products.

Lemma 5.3. *Let P_\bullet be a chain complex in \mathcal{P} .*

- (1) *The functor*

$$\begin{aligned} P_\bullet \otimes -: \mathcal{P} &\rightarrow C\mathcal{P} \\ Q &\mapsto P_\bullet \otimes Q \end{aligned}$$

is exact.

- (2) *If Q_\bullet is an acyclic complex in \mathcal{P} , then the complex $\text{Tot}(P_\bullet \otimes Q_\bullet)$ is acyclic.*

Proof. The first part is straightforward, as each $P_i \otimes -$ is an exact functor. For the second part, if Q_\bullet is acyclic, then, since acyclic complexes are spliced together from short exact sequences, each of the complexes $P_n \otimes Q_\bullet$ is acyclic. Therefore the rows of the bicomplex $P_\bullet \otimes Q_\bullet$ are acyclic. Our complexes are non-negative, so the total complex of this bicomplex is exact in an ambient abelian category by the acyclic assembly lemma ([Wei94], 2.7.3). Since \mathcal{P} is idempotent complete it supports long exact sequences, so $\text{Tot}(P_\bullet \otimes Q_\bullet)$ is acyclic in \mathcal{P} . \square

To define the simplicial tensor product of complexes we need to go beyond regular simplicial objects. A *bisimplicial object* B in \mathcal{P} is a functor $B: \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \mathcal{P}$. The *diagonal* of B is the simplicial object defined by pre-composition with the usual diagonal functor $\text{diag}: \Delta^{\text{op}} \rightarrow \Delta^{\text{op}} \times \Delta^{\text{op}}$:

$$\text{diag}(B) := B \circ \text{diag}: \Delta^{\text{op}} \rightarrow \Delta^{\text{op}} \times \Delta^{\text{op}} \rightarrow \mathcal{P}.$$

If C and D are simplicial objects in \mathcal{P} , then we define $C \otimes D$ to be the bisimplicial object given by $(C \otimes D)([m], [n]) = C_m \otimes D_n$ and $(C \otimes D)(\alpha, \beta) = C(\alpha) \otimes D(\beta)$ for $\alpha: [m] \rightarrow [m']$, $\beta: [n] \rightarrow [n']$. We can now push the tensor product around the Dold–Kan correspondence.

Definition 5.4. The *simplicial tensor product* of chain complexes P_\bullet and Q_\bullet in \mathcal{P} is defined to be

$$P_\bullet \otimes_\Delta Q_\bullet := N(\text{diag}(\Gamma(P_\bullet) \otimes \Gamma(Q_\bullet))).$$

A word of warning here: although the tensor product is an additive functor in each variable, the complex $P_\bullet \otimes_\Delta Q_\bullet$ is not equal to the product complex

$\text{Tot}(P. \otimes Q.)$ discussed above. They are related by the Eilenberg–Zilber theorem, which we shall use in the proof of the following lemma.

Lemma 5.5. *Let $P.$ and $Q.$ be chain complexes in \mathcal{P} , and suppose that at least one of them is acyclic. Then $P. \otimes_{\Delta} Q.$ is acyclic in \mathcal{P} .*

Proof. We suppose, without loss of generality, that $Q.$ is acyclic. By the Eilenberg–Zilber theorem ([May92] §29), the simplicial tensor product $P. \otimes_{\Delta} Q. = N \text{diag}(\Gamma(P.) \otimes \Gamma(Q.))$ is homotopy equivalent to $\text{Tot}(P. \otimes Q.)$, and is therefore acyclic by Lemma 5.3 (2) and Lemma 2.2. \square

The following is an analogue of Lemma 2.7 for the simplicial tensor product.

Lemma 5.6. *If $P.$ and $Q.$ are both bounded chain complexes in \mathcal{P} , of length k and l respectively. Then $P. \otimes_{\Delta} Q.$ is of length at most kl and so is bounded as well.*

Proof. Examining the Dold–Kan functors applied to a tensor product, one sees that the object $(P. \otimes_{\Delta} Q.)_n$ is equal to

$$N(\text{diag}(\Gamma(P.) \otimes \Gamma(Q.)))_n = \bigoplus_{\varphi} P_i \otimes Q_j,$$

where φ runs over all injections $[n] \hookrightarrow [i] \times [j]$ whose composition with the projections onto $[i]$ and $[j]$ gives surjections $[n] \twoheadrightarrow [i]$ and $[n] \twoheadrightarrow [j]$ (this is derived in [Law12]). The complexes $P.$ and $Q.$ are of length k and l , so $P_i = 0$ and $Q_j = 0$ for all $i > k$ and $j > l$. But for $n > kl$ there is no injection $[n] \hookrightarrow [i] \times [j]$, with $i \leq k$ and $j \leq l$, such that $[n] \twoheadrightarrow [i]$ and $[n] \twoheadrightarrow [j]$ are order-preserving surjections. So $(P. \otimes_{\Delta} Q.)_n = 0$ for $n > kl$. \square

We now verify that \otimes_{Δ} is a tensor product in the sense of Definition 5.1.

Proposition 5.7. *The simplicial tensor product \otimes_{Δ} is a tensor product on the idempotent complete exact category $C\mathcal{P}$ and restricts to a tensor product on the full subcategory $C_b^q\mathcal{P}$.*

Proof. If $P.$ and $Q.$ are in $C_b^q\mathcal{P}$, then so is $P. \otimes_{\Delta, n} Q.$, by Lemmas 5.5 and 5.6. So it remains to show that $- \otimes_{\Delta} -$ is bi-additive, and that the functors $P. \otimes_{\Delta} -$ and $- \otimes_{\Delta} P.$ are exact when $P.$ is in $C\mathcal{P}$.

The functors N and Γ are both additive and exact, so we only need to inspect $\text{diag}(- \otimes -)$. This is easily seen to be bi-additive, as $- \otimes -$ is bi-additive. Therefore $- \otimes_{\Delta} -$ is bi-additive as well.

Let B be a simplicial object in \mathcal{P} . For a short exact sequence of simplicial objects $0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0$, the sequence

$$0 \rightarrow \text{diag}(B \otimes A')_n \rightarrow \text{diag}(B \otimes A)_n \rightarrow \text{diag}(B \otimes A'')_n \rightarrow 0$$

is equal to

$$0 \rightarrow B_n \otimes A'_n \rightarrow B_n \otimes A_n \rightarrow B_n \otimes A''_n \rightarrow 0,$$

which is short exact since each $0 \rightarrow A'_n \rightarrow A_n \rightarrow A''_n \rightarrow 0$ is short exact and $B_n \otimes -$ is exact. So the sequence

$$0 \rightarrow \text{diag}(B \otimes A') \rightarrow \text{diag}(B \otimes A) \rightarrow \text{diag}(B \otimes A'') \rightarrow 0,$$

is short exact in every degree, for any simplicial object B in \mathcal{P} . Therefore the functor $\text{diag}(\Gamma(P.) \otimes -): \mathcal{P}^{\Delta^{\text{op}}} \rightarrow \mathcal{P}^{\Delta^{\text{op}}}$ is exact. The same is true for $\text{diag}(- \otimes \Gamma(P.))$. It follows that $P. \otimes_{\Delta} -$ and $- \otimes_{\Delta} P.$ are exact functors. \square

We are now ready to iteratively define simplicial tensor products on categories of multicomplexes.

Definition 5.8. We define *simplicial tensor products*

$$\otimes_{\Delta, n}: C^n \mathcal{P} \times C^n \mathcal{P} \rightarrow C^n \mathcal{P}$$

for all $n \geq 0$ recursively:

- (1) $\otimes_{\Delta, 0}: \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$ is the usual tensor product \otimes , and
- (2) by regarding objects P_\bullet and Q_\bullet of $C^{n+1} \mathcal{P}$ as chain complexes in the idempotent complete exact category $C^n \mathcal{P}$ with the tensor product $\otimes_{\Delta, n}$, we define $P_\bullet \otimes_{\Delta, n+1} Q_\bullet := N(\text{diag}(\Gamma(P_\bullet) \otimes_{\Delta, n} \Gamma(Q_\bullet)))$.

The following iteration of Proposition 5.7 is now straightforward. The case $n = 0$ is an assumption of this section, and we iterate using $(C_b^q)^{n+1} \mathcal{P} = C_b^q((C_b^q)^n \mathcal{P})$.

Corollary 5.9. *For all $n \geq 0$, the simplicial tensor product $\otimes_{\Delta, n}$ is a tensor product in the sense of Definition 5.1 on $C^n \mathcal{P}$ and on $(C_b^q)^n \mathcal{P}$.* \square

In fact we can say a little more than this. The following lemma is crucial to the proof of the main result of this section.

Lemma 5.10. *Let P_\bullet be an object of $C_b((C_b^q)^n \mathcal{P})$, and let Q_\bullet be an object of $(C_b^q)^{n+1} \mathcal{P}$. Then $P_\bullet \otimes_{\Delta, n+1} Q_\bullet$ is an object of $(C_b^q)^{n+1} \mathcal{P}$.*

Proof. Noting that P_\bullet and Q_\bullet both have their objects in $(C_b^q)^n \mathcal{P}$, and that Q_\bullet is an acyclic complex of objects in that category, the lemma follows immediately from Lemmas 5.5 and 5.6 applied to the tensor product $\otimes_{\Delta, n}$ on the category $(C_b^q)^n \mathcal{P}$. \square

We can extend the simplicial tensor products to categories of binary complexes in the same way that we did for exterior powers in section 4. The simplicial tensor product of a pair of binary complexes $(P_\bullet, d_P, \tilde{d}_P)$ and $(Q_\bullet, d_Q, \tilde{d}_Q)$ is obtained by considering the pair of chain complexes $(P_\bullet, d_C) \otimes_{\Delta} (Q_\bullet, d_Q)$ and $(P_\bullet, \tilde{d}_P) \otimes_{\Delta} (Q_\bullet, \tilde{d}_Q)$ as a binary complex (it is straightforward to prove that they have the same underlying graded object, in the same manner as Lemma 4.2). The analogue of Corollary 4.3 then follows, and we define the simplicial tensor product of binary multicomplexes just as we did for a functor of one variable in Definition 4.4.

5.2. Vanishing of products. In this subsection we prove that the class of any simplicial tensor product vanishes in the corresponding K -group. Our proof resembles Grayson's procedure [Gra92, p. 103] of verifying that the second Euler characteristic of a doubly acyclic bicomplex vanishes.

Let $n > 0$, and let P_\bullet and Q_\bullet be n -dimensional bounded acyclic binary complexes of objects of \mathcal{P} . That is, P_\bullet and Q_\bullet are objects of $(B_b^q)^n \mathcal{P}$. Then the simplicial tensor product $P_\bullet \otimes_{\Delta, n} Q_\bullet$ is in $(B_b^q)^n \mathcal{P}$ as well by Corollary 5.9. Since the objects of $(B_b^q)^n \mathcal{P}$ are the generators of $K_n(\mathcal{P})$, one would like to use $\otimes_{\Delta, n}$ to induce a product $K_n(\mathcal{P}) \times K_n(\mathcal{P}) \rightarrow K_n(\mathcal{P})$. On first inspection this appears not to work, because the product $P_\bullet \otimes_{\Delta, n} Q_\bullet$ is not diagonal if only one of P_\bullet or Q_\bullet is diagonal. This is not a problem in the end though, since the whole product vanishes on $K_n(\mathcal{P})$.

Proposition 5.11. *Let $n > 0$. For any pair of n -dimensional bounded acyclic multicomplexes P_\bullet, Q_\bullet in $(B_b^q)^n \mathcal{P}$, the class $[P_\bullet \otimes_{\Delta, n} Q_\bullet]$ vanishes in $K_n(\mathcal{P})$.*

Proof. First we filter P_\bullet by degree. Regard P_\bullet as an acyclic binary complex of objects of $(B_b^q)^{n-1} \mathcal{P}$. For $i \geq 0$, let $P|_{[0, i]}$ be the binary complex obtained by ‘restricting’ P_\bullet to be supported on $[0, i]$. That is, $(P|_{[0, i]})_j$ is equal to P_j if $0 \leq j \leq i$, and $(P|_{[0, i]})_j = 0$ otherwise. The differentials on $P|_{[0, i]}$ are inherited from P_\bullet . We write $P_j[0]$ for P_j considered as a binary complex concentrated in degree 0. Then $P_j[j]$, which denotes P_j considered as a binary complex concentrated in degree j , is the quotient of the inclusion $P|_{[0, j-1]} \hookrightarrow P|_{[0, j]}$ (if $j \geq 1$). If P_\bullet is supported on $[0, n]$, so that $P_j = 0$ for $j > n$, we therefore have an n -stage filtration

$$P_0[0] = P|_{[0, 0]} \hookrightarrow P|_{[0, 1]} \hookrightarrow \cdots \hookrightarrow P|_{[0, n-1]} \hookrightarrow P|_{[0, n]} = P_\bullet$$

whose successive quotients determine short exact sequences

$$0 \rightarrow P|_{[0, j-1]} \rightarrow P|_{[0, j]} \rightarrow P_j[j] \rightarrow 0.$$

We take the simplicial tensor product with Q_\bullet of this whole filtration, obtaining sequences

$$(5.12) \quad 0 \rightarrow P|_{[0, j-1]} \otimes_{\Delta, n} Q_\bullet \rightarrow P|_{[0, j]} \otimes_{\Delta, n} Q_\bullet \rightarrow P_j[j] \otimes_{\Delta, n} Q_\bullet \rightarrow 0$$

for $j = 1, \dots, n$, which are short exact by Corollary 5.9.

By Lemma 5.10, all of the objects are in the right category, so each of the short exact sequences of (5.12) yields an equation

$$[P|_{[0, j]} \otimes_{\Delta, n} Q_\bullet] = [P|_{[0, j-1]} \otimes_{\Delta, n} Q_\bullet] + [P_j[j] \otimes_{\Delta, n} Q_\bullet]$$

in $K_n(\mathcal{P})$. Putting these together gives

$$[P_\bullet \otimes_{\Delta, n} Q_\bullet] = \sum_{j=0}^n [P_j[j] \otimes_{\Delta, n} Q_\bullet].$$

To proceed we need to assume a small lemma, for which the second type of relation in $K_n(\mathcal{P})$ (diagonal binary multicomplexes vanish) is crucial.

Lemma 5.13. *The following equality holds in $K_n(\mathcal{P})$:*

$$[P_j[j] \otimes_{\Delta, n} Q_\bullet] = (-1)^j [P_j[0] \otimes_{\Delta, n} Q_\bullet].$$

Continuing with the main proof, our equation now reads

$$[P_\bullet \otimes_{\Delta, n} Q_\bullet] = \sum_{j=0}^n (-1)^j [P_j[0] \otimes_{\Delta, n} Q_\bullet].$$

By inspection we see that $\Gamma(P_j[0])$ is the constant simplicial object which has P_j in each degree. The functor

$$\text{diag}(\Gamma(P_j[0]) \otimes_{\Delta, n-1} -): ((C_b)^{n-1} \mathcal{A})^{\Delta^{\text{op}}} \rightarrow ((C_b)^{n-1} \mathcal{A})^{\Delta^{\text{op}}}$$

is therefore isomorphic to the functor

$$P_j \otimes_{\Delta, n-1} -: ((C_b)^{n-1} \mathcal{A})^{\Delta^{\text{op}}} \rightarrow ((C_b)^{n-1} \mathcal{A})^{\Delta^{\text{op}}},$$

since they both have the same effect of ‘tensoring everywhere by P_j ’. This functor is additive, so we have an isomorphism of functors

$$N(P_j \otimes_{\Delta, n-1} \Gamma(-)) \cong P_j \otimes_{\Delta, n-1} -.$$

Hence

$$P_j[0] \otimes_{\Delta, n} Q_{\bullet} = N \operatorname{diag}(\Gamma(P_j[0]) \otimes_{\Delta, n-1} \Gamma(Q_{\bullet})) \cong P_j \otimes_{\Delta, n-1} Q_{\bullet},$$

so we have

$$[P_{\bullet} \otimes_{\Delta, n} Q_{\bullet}] = \sum_{j=0}^n (-1)^j [P_j \otimes_{\Delta, n-1} Q_{\bullet}].$$

There is an exact sequence

$$0 \rightarrow P_n \rightarrow P_{n-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow 0,$$

since P_{\bullet} is acyclic. The objects of Q_{\bullet} are in $(B_b^q)^{n-1} \mathcal{P}$, so $- \otimes_{\Delta, n-1} Q_{\bullet}$ is an exact functor by Lemma 5.3 (1), and so the following sequence is exact.

$$\begin{aligned} 0 \rightarrow P_n \otimes_{\Delta, n-1} Q_{\bullet} \rightarrow P_{n-1} \otimes_{\Delta, n-1} Q_{\bullet} \rightarrow \cdots \\ \cdots \rightarrow P_1 \otimes_{\Delta, n-1} Q_{\bullet} \rightarrow P_0 \otimes_{\Delta, n-1} Q_{\bullet} \rightarrow 0 \end{aligned}$$

Exact sequences translate into alternating sums in the Grothendieck group, so this exact sequence gives exactly the identity

$$\sum_{j=0}^n (-1)^j [P_j \otimes_{\Delta, n-1} Q_{\bullet}] = 0$$

in $K_0((B_b^q)^n \mathcal{P})$, and hence the same relation holds in $K_n(\mathcal{P})$. Therefore $[P_{\bullet} \otimes_{\Delta, n} Q_{\bullet}] = 0$, as required. \square

It remains to prove Lemma 5.13.

Proof of Lemma 5.13. Consider the following diagram as a short exact sequence of binary complexes concentrated in degrees j and $j-1$:

$$\begin{array}{ccc} 0 & \rightrightarrows & P_j \\ \downarrow & & \downarrow \\ P_j & \xrightleftharpoons[1]{1} & P_j \\ \downarrow & & \downarrow \\ P_j & \rightrightarrows & 0. \end{array}$$

We will use this diagram to show that $[P_j[j] \otimes_{\Delta, n} Q_{\bullet}] = -[P_j[j-1] \otimes_{\Delta, n} Q_{\bullet}]$. The argument can be iterated $j-1$ times to yield $[P_j[j] \otimes_{\Delta, n} Q_{\bullet}] = (-1)^j [P_j[0] \otimes_{\Delta, n} Q_{\bullet}]$ in $K_n(\mathcal{P})$, as required. For lack of a better notation, we will denote the middle row of the diagram by $(P_j = P_j)$. Then the diagram represents a short exact sequence of binary complexes

$$0 \rightarrow P_j[j-1] \rightarrow (P_j = P_j) \rightarrow P_j[j] \rightarrow 0,$$

which upon tensoring with Q_{\bullet} becomes the short exact sequence

$$0 \rightarrow P_j[j-1] \otimes_{\Delta, n} Q_{\bullet} \rightarrow (P_j = P_j) \otimes_{\Delta, n} Q_{\bullet} \rightarrow P_j[j] \otimes_{\Delta, n} Q_{\bullet} \rightarrow 0,$$

by Lemma 5.10. Since Q_\bullet is acyclic and has objects in $(B_b^q)^{n-1}\mathcal{P}$, each of the terms of this short exact sequence is an object of $(B_b^q)^n\mathcal{P}$ by Lemma 5.10, so we have a relation

$$[(P_j = P_j) \otimes_{\Delta, n} Q_\bullet] = [P_j[j-1] \otimes_{\Delta, n} Q_\bullet] + [P_j[j] \otimes_{\Delta, n} Q_\bullet],$$

in $K_0((B_b^q)^n\mathcal{P})$, and hence in $K_n(\mathcal{P})$. We claim that

$$[(P_j = P_j) \otimes_{\Delta, n} Q_\bullet] = 0$$

in $K_n(\mathcal{P})$, so that $[P_j[j] \otimes_{\Delta, n} Q_\bullet] = -[P_j[j-1] \otimes_{\Delta, n} Q_\bullet]$. We can filter Q_\bullet in the same manner that we have filtered P_\bullet in the main proof above,

$$Q_0[0] = Q|_{[0,0]} \hookrightarrow Q|_{[0,1]} \hookrightarrow \cdots \hookrightarrow Q|_{[0,n-1]} \hookrightarrow Q|_{[0,n]} = Q_\bullet$$

giving short exact sequences

$$0 \rightarrow Q|_{[0,i-1]} \rightarrow Q|_{[0,i]} \rightarrow Q_i[i] \rightarrow 0.$$

Upon tensoring with $(P_j = P_j)$, we have short exact sequences:

$$\begin{aligned} 0 \longrightarrow (P_j = P_j) \otimes_{\Delta, n} Q|_{[0,i-1]} &\longrightarrow (P_j = P_j) \otimes_{\Delta, n} Q|_{[0,i]} \\ &\longrightarrow (P_j = P_j) \otimes_{\Delta, n} Q_i[i] \longrightarrow 0, \end{aligned}$$

(by Lemma 5.10). Furthermore, since $(P_j = P_j)$ is an acyclic binary complex of objects of $(B_b^q)^{n-1}\mathcal{P}$, each of the terms of these short exact sequences is an object of $(B_b^q)^n\mathcal{P}$, by Lemma 5.10. We therefore have the equation

$$[(P_j = P_j) \otimes_{\Delta, n} Q_\bullet] = \sum_i [(P_j = P_j) \otimes_{\Delta, n} Q_i[i]]$$

in $K_0((B_b^q)^n\mathcal{P})$, and hence in $K_n(\mathcal{P})$. But $(P_j = P_j)$ is a diagonal binary complex, as is each $Q_i[i]$ (trivially). The simplicial tensor product of a pair of diagonal complexes is again diagonal, so each of the acyclic binary complexes $(P_j = P_j) \otimes_{\Delta, n} Q_i[i]$ is diagonal and hence vanishes in $K_n(\mathcal{P})$. Therefore $[(P_j = P_j) \otimes_{\Delta, n} Q_\bullet] = 0$, so the desired relation holds. \square

This finally completes the proof of Proposition 5.11. Having taken the trouble to set up an alternative product of bounded acyclic binary multicomplexes, one that is compatible with the exterior powers, we've now shown that (like the usual tensor product) it is always zero! It was not all for naught though: at least we know now that the induced operation $\otimes_{\Delta, n}: K_n(\mathcal{P}) \times K_n(\mathcal{P}) \rightarrow K_n(\mathcal{P})$ is well-defined. Furthermore, the vanishing of this product proves that the exterior power operations induce homomorphisms on $K_n(R)$ (and more generally, on the higher K -groups of schemes). This is shown in the next section.

6. EXTERIOR POWER OPERATIONS ON K -GROUPS OF SCHEMES

The goal of this section is to extend the endofunctor Λ_n^r defined in section 4 to bounded acyclic multicomplexes of locally free modules of finite rank on a scheme X , and to prove that it induces a well-defined operation λ^r on the higher K -group $K_n(X)$. We will see that, for $n > 0$, this operation λ^r is not just a map but in fact a homomorphism.

Let X be a quasi-compact scheme, and let $\mathcal{P}(X)$ be the category of locally free \mathcal{O}_X -modules of finite rank. Then $\mathcal{P}(X)$ is an exact category in the usual

sense. It is idempotent complete but not split exact in general. We write $K_n(X)$ for the K -group $K_n(\mathcal{P}(X))$.

As in section 3, we inductively define an endofunctor Λ_n^r on $C^n\mathcal{P}(X)$ for $r \geq 1$ and $n \geq 0$ as follows: the functor Λ_0^r is the usual r^{th} exterior power functor on $C^0\mathcal{P}(X) = \mathcal{P}(X)$, and Λ_n^r is defined as $N\Lambda_{n-1}^r\Gamma$, with N and Γ as introduced in section 2.

Proposition 6.1. *For all $r, n > 0$, the functor Λ_n^r restricts to an endofunctor on the subcategory $(C_b^q)^n\mathcal{P}(X)$ of $C^n\mathcal{P}(X)$.*

Proof. Given any open affine subscheme $U = \text{Spec}(R)$ of X , a straightforward inductive argument shows that the following diagram commutes:

$$\begin{array}{ccc} C^n\mathcal{P}(X) & \xrightarrow{\Lambda_n^r} & C^n\mathcal{P}(X) \\ \downarrow & & \downarrow \\ C^n\mathcal{P}(U) & \xrightarrow{\Lambda_n^r} & C^n\mathcal{P}(U) \\ \parallel & & \parallel \\ C^n\mathcal{P}(R) & \xrightarrow{\Lambda_n^r} & C^n\mathcal{P}(R) \end{array}$$

The vertical arrows are induced by the restriction functor $\mathcal{P}(X) \rightarrow \mathcal{P}(U)$, $P \mapsto P|_U$, and the lower horizontal arrow is the functor Λ_n^r introduced in section 3. A complex in $C^n\mathcal{P}(X)$ is acyclic, or bounded, if and only if its restriction to every open affine subscheme has the respective property, so Proposition 6.1 follows from the results of section 3. \square

As in section 4, one easily deduces that, for any complex P_\bullet in $C^n\mathcal{P}(X)$, the objects in $\Lambda_n^r(P_\bullet)$ do not depend on the differentials in P_\bullet . We can therefore extend the endofunctor Λ_n^r to an endofunctor of $(B_b^q)^n\mathcal{P}(X)$, which we denote by Λ_n^r again. The goal of the rest of this section is to prove the following theorem.

Theorem 6.2. *Let $n > 0$ and $r > 0$. The endofunctor Λ_n^r of $(B_b^q)^n\mathcal{P}(X)$ induces a well-defined homomorphism $\lambda^r: K_n(X) \rightarrow K_n(X)$.*

Definition 6.3. The homomorphism λ^r in the previous theorem is called the r^{th} exterior power operation on $K_n(X)$.

Proof of Theorem 6.2. If P_\bullet is a diagonal multicomplex in $(B_b^q)^n\mathcal{P}(X)$, then the multicomplex $\Lambda_n^r(P_\bullet)$ is diagonal as well, by definition of Λ_n^r . It therefore suffices to show that the association $[P_\bullet] \mapsto [\Lambda_n^r(P_\bullet)]$ induces a well-defined homomorphism of groups

$$\lambda^r: K_0((B_b^q)^n\mathcal{P}(X)) \rightarrow K_n(X).$$

Thus we need to show that the equality

$$[\Lambda_n^r(P_\bullet)] = [\Lambda_n^r(P'_\bullet)] + [\Lambda_n^r(P''_\bullet)]$$

holds in $K_n(X)$ for every short exact sequence $0 \rightarrow P'_\bullet \rightarrow P_\bullet \rightarrow P''_\bullet \rightarrow 0$ in $(B_b^q)^n\mathcal{P}(X)$. The classes $[\Lambda_n^{r-i}(P'_\bullet) \otimes_{\Delta, n} \Lambda_n^i(P''_\bullet)]$, $i = 1, \dots, r-1$, vanish in $K_n(X)$ by Proposition 5.11 applied to the category $\mathcal{P} = \mathcal{P}(X)$, where the simplicial tensor product has been constructed inductively from the usual

tensor product of quasi-coherent \mathcal{O}_X -modules. So the desired equality is equivalent in $K_n(X)$ to the more familiar-looking identity

$$[\Lambda_n^r(P_\bullet)] = [\Lambda_n^r(P'_\bullet)] + \sum_{i=1}^{r-1} [\Lambda_n^{r-i}(P'_\bullet) \otimes_{\Delta, n} \Lambda_n^i(P''_\bullet)] + [\Lambda_n^r(P''_\bullet)].$$

In order to prove this latter formula, we cannot just apply the usual formula for the r^{th} exterior power of a direct sum because the given short exact sequence of binary complexes, $0 \rightarrow P'_\bullet \rightarrow P_\bullet \rightarrow P''_\bullet \rightarrow 0$, does not split in general, even if X is affine (see Example 4.1). Instead, by induction on n , we construct for every sequence $0 \rightarrow P'_\bullet \rightarrow P_\bullet \rightarrow P''_\bullet \rightarrow 0$ in $(B_b^q)^n \mathcal{P}(X)$ a natural induced filtration

$$\Lambda_n^r(P'_\bullet) \hookrightarrow \Lambda_n^{r-1}(P'_\bullet) \wedge_n \Lambda_n^1(P_\bullet) \hookrightarrow \cdots \hookrightarrow \Lambda_n^1(P'_\bullet) \wedge_n \Lambda_n^{r-1}(P_\bullet) \hookrightarrow \Lambda_n^r(P_\bullet)$$

of $\Lambda_n^r(P_\bullet)$ by certain sub-objects $\Lambda_n^{r-i}(P'_\bullet) \wedge_n \Lambda_n^i(P_\bullet)$, $i = 0, \dots, r$ of $\Lambda_n^r(P_\bullet)$, also belonging to $(B_b^q)^n \mathcal{P}(X)$, together with short exact sequences

$$(6.4) \quad 0 \longrightarrow \Lambda_n^{r-i+1}(P'_\bullet) \wedge_n \Lambda_n^{i-1}(P_\bullet) \longrightarrow \Lambda_n^{r-i}(P'_\bullet) \wedge_n \Lambda_n^i(P_\bullet) \longrightarrow \Lambda_n^{r-i}(P'_\bullet) \otimes_{\Delta, n} \Lambda_n^i(P''_\bullet) \longrightarrow 0,$$

$i = 1, \dots, n$.

For $n = 0$ and $i \in \{0, \dots, r\}$, the object $\Lambda_0^{r-i}(P') \wedge_0 \Lambda_0^i(P)$ is defined to be what is usually meant by $\Lambda^{r-i}(P') \wedge \Lambda^i(P)$: the image of the canonical homomorphism $\Lambda^{r-i}(P') \otimes \Lambda^i(P) \rightarrow \Lambda^r(P)$. It is well-known that these objects come with the required short exact sequences (6.4).

If $n > 0$ and if, for a moment, the sequence $0 \rightarrow P'_\bullet \rightarrow P_\bullet \rightarrow P''_\bullet \rightarrow 0$ is given in $C_b^q(C_b^q)^{n-1} \mathcal{P}(X)$ rather than in $(B_b^q)^n \mathcal{P}(X)$, we first note that applying the exact functor Γ to the sequence, we get the short exact sequence $0 \rightarrow \Gamma(P'_\bullet) \rightarrow \Gamma(P_\bullet) \rightarrow \Gamma(P''_\bullet) \rightarrow 0$ of simplicial objects in $(C_b^q)^{n-1} \mathcal{P}(X)$. By the inductive hypothesis, the complexes $\Lambda_{n-1}^{r-i}(\Gamma(P'_m)) \wedge_{n-1} \Lambda_{n-1}^i(\Gamma(P_m))$, $i = 0, \dots, n$, $m \geq 0$, belong to $(B_b^q)^{n-1} \mathcal{P}(X)$ and we have short exact sequences

$$\begin{aligned} 0 &\longrightarrow \Lambda_{n-1}^{r-i+1}(\Gamma(P'_m)) \wedge_{n-1} \Lambda_{n-1}^{i-1}(\Gamma(P_m)) \\ &\longrightarrow \Lambda_{n-1}^{r-i}(\Gamma(P'_m)) \wedge_{n-1} \Lambda_{n-1}^i(\Gamma(P_m)) \\ &\longrightarrow \Lambda_{n-1}^{r-i}(\Gamma(P'_m)) \otimes_{\Delta, n-1} \Lambda_{n-1}^i(\Gamma(P''_m)) \longrightarrow 0, \end{aligned}$$

$i = 1, \dots, r$, $m \geq 0$. These short exact sequences assemble to short exact sequences of simplicial objects in $(B_b^q)^{n-1} \mathcal{P}(X)$. By applying the exact functor N we finally obtain the required objects

$$\Lambda_n^{r-i}(P') \wedge_n \Lambda_n^i(P) := N(\Lambda_{n-1}^{r-i}(\Gamma(P')) \wedge_{n-1} \Lambda_{n-1}^i(\Gamma(P))),$$

$i = 0, \dots, r$, and the required short exact sequences (6.4). As the objects of the multicomplex $\Lambda_n^{r-i}(P') \wedge_n \Lambda_n^i(P)$ are independent of the differentials in the multicomplexes P'_\bullet and P_\bullet , this construction of \wedge_n passes to the category $(B_b^q)^n \mathcal{P}(X)$ as in section 4.

From Proposition 6.1 and section 5 we know that the complex $\Lambda_n^r(P_\bullet)$ and the complexes $\Lambda_n^{r-i}(P'_\bullet) \otimes_{\Delta, n} \Lambda_n^i(P''_\bullet)$, $i = 0, \dots, r$, belong to $(B_b^q)^n \mathcal{P}(X)$. Now a straightforward downwards induction on i based on the short exact

sequences (6.4) shows that the complexes $\Lambda_n^{r-i}(P'_\bullet) \wedge_n \Lambda_n^i(P_\bullet)$, $i = 0, \dots, r$ are bounded and acyclic, so they belong to $(B_b^q)^n \mathcal{P}(X)$ as was to be shown. \square

7. THE SECOND λ -RING AXIOM

Given a scheme X , there is a ‘trivial’ way to equip the graded abelian group $K_*(X) := \bigoplus_{n \geq 0} K_n(X)$ with a multiplication, and to extend the exterior power operations defined in the previous section to $K_*(X)$ so that they are compatible with addition in $K_*(X)$ in the usual sense. The main result of this section is that they are also compatible with multiplication in the expected way—that is to say, they satisfy the λ -ring axiom (2).

Let X be a quasi-compact scheme. We recall that $K_0(X)$ together with the usual exterior power operations $\lambda^r: K_0(X) \rightarrow K_0(X)$, $r \geq 0$, is a λ -ring as defined in the introduction (see chapter V of [FL85]). Furthermore, $K_n(X)$ is a $K_0(X)$ -module via $[P] \cdot [Q_\bullet] := [P \otimes Q_\bullet]$, for P in $\mathcal{P}(X)$ and Q_\bullet in $(B_b^q)^n \mathcal{P}(X)$, see also Definition 5.2 (1).

We define a multiplication on $K_*(X) := \bigoplus_{n \geq 0} K_n(X)$ by

$$(a_0, a_1, a_2, \dots) \cdot (b_0, b_1, b_2, \dots) = (a_0 b_0, a_0 b_1 + a_1 b_0, a_0 b_2 + a_1 b_1 + a_2 b_0, \dots);$$

in particular, the product of any two elements in $\bigoplus_{n \geq 1} K_n(X)$ is defined to be zero. With this multiplication, $K_*(X)$ is a commutative ring. Furthermore, we define exterior power operations $\lambda^r: K_*(X) \rightarrow K_*(X)$, $r \geq 0$, by the formula

$$\lambda^r((a_0, a_1, a_2, \dots)) = \left(\lambda^r(a_0), \sum_{i=0}^{r-1} \lambda^i(a_0) \lambda^{r-i}(a_1), \sum_{i=0}^{r-1} \lambda^i(a_0) \lambda^{r-i}(a_2), \dots \right).$$

By definition, we then have $\lambda^0(x) = 1$ and $\lambda^1(x) = x$ for all $x \in K_*(X)$. A straightforward calculation using Theorem 6.2 and the fact that $K_0(X)$ satisfies axiom (1) of a λ -ring shows that $K_*(X)$ also satisfies axiom (1). The next theorem addresses axiom (2).

Theorem 7.1. *The pre- λ -ring $K_*(X)$ defined above satisfies axiom (2) of a λ -ring.*

Proof. Axiom (2) holds for elements of the form $x = (a_0, 0, 0, \dots)$, $y = (b_0, 0, 0, \dots)$ in $K_*(X)$ because it holds for $K_0(X)$. It also holds for elements of the form $x = (0, a_1, a_2, \dots)$ and $y = (0, b_1, b_2, \dots)$ because $\lambda^r(0) = 0$ for all $r \geq 1$ and because every monomial in the ring $\mathbb{Z}[X_1, \dots, X_r, Y_1, \dots, Y_r]$ whose coefficient in $P_r(X_1, \dots, X_r, Y_1, \dots, Y_r)$ is non-zero is divisible by some product $X_i Y_j$. Furthermore, it suffices to check axiom (2) for x, y belonging to a set of additive generators of $K_*(X)$ because $K_*(X)$ satisfies axiom (1) and because axiom (2) is equivalent to the multiplicativity of the homomorphism

$$\begin{aligned} \lambda_t: K_*(X) &\rightarrow 1 + t K_*(X)[[t]] \\ x &\mapsto \sum_{r \geq 0} \lambda^r(x) t^r. \end{aligned}$$

We are therefore reduced to showing that the equality

$$(7.2) \quad \lambda^r(xy) = P_r(\lambda^1(x), \dots, \lambda^r(x), \lambda^1(y), \dots, \lambda^r(y))$$

holds in $K_n(X)$ for elements $y \in K_n(X)$ and $x \in K_0(X)$ of the form $x = [\mathcal{E}]$ for some locally free \mathcal{O}_X -module \mathcal{E} of finite rank.

We now invoke the projective bundle theorem ([Qui73], §8, Theorem 2.1). We remark that its proof in *loc. cit.* only relies on the additivity and resolution theorems, and not, for instance, on the dévissage theorem or localisation sequence. The additivity and resolution theorems have been proved in [Har15] within the context of Grayson's definition of higher K -groups, so the projective bundle theorem also has a proof within that context, without resorting to topological methods.

It is well-known that an iterated application of the projective bundle theorem yields the following splitting principle: there exists a projective morphism $f: Y \rightarrow X$ such that $f^*[\mathcal{E}]$ is the sum of invertible \mathcal{O}_Y -modules in $K_0(Y)$ and such that $f^*: K_*(X) \rightarrow K_*(Y)$ is injective. It is straightforward to check that $f^*: K_*(X) \rightarrow K_*(Y)$ is a homomorphism of (pre-) λ -rings. Using the above argument about additive generators again, we are therefore reduced to showing the equality (7.2) only when x is the class $[\mathcal{L}]$ of an invertible \mathcal{O}_X -module \mathcal{L} . In that case, (7.2) becomes the much simpler formula

$$\lambda^r([\mathcal{L}] \cdot y) = [\mathcal{L}^{\otimes r}] \cdot \lambda^r(y),$$

because $\lambda^2[\mathcal{L}] = \dots = \lambda^r[\mathcal{L}] = 0$, and because P_r satisfies the identity $P_r(1, 0, \dots, 0, Y_1, \dots, Y_r) = Y_r$ and has X -degree r (where X_i is defined to be of degree i for $i = 1, \dots, r$). Using the argument about additive generators again, it suffices to show that for any object P of $(B_b^q)^n \mathcal{P}(X)$, the object $\Lambda_n^r(\mathcal{L} \otimes P)$ is isomorphic to $\mathcal{L}^{\otimes r} \otimes \Lambda_n^r(P)$. This is well-known if $n = 0$, and follows by induction on n from the following chain of isomorphisms applied to each of the 2^n multicomplexes associated with the binary multicomplex P (which we again denote by P):

$$\begin{aligned} \Lambda_n^r(\mathcal{L} \otimes P) &= N\Lambda_{n-1}^r \Gamma(\mathcal{L} \otimes P) \\ &\cong N(\mathcal{L}^{\otimes r} \otimes \Lambda_{n-1}^r \Gamma(P)) \\ &\cong \mathcal{L}^{\otimes r} \otimes N\Lambda_{n-1}^r \Gamma(P) \\ &\cong \mathcal{L}^{\otimes r} \otimes \Lambda_n^r(P). \end{aligned} \quad \square$$

8. THE FINAL λ -RING AXIOM

The goal of this section is to prove that the pre- λ -ring $K_*(X)$ (introduced and proven to satisfy λ -ring axiom (2) in the previous section) also satisfies the final λ -ring axiom (3) and is therefore a λ -ring. The main ingredients are the language of polynomial functors, the identification of polynomial functors with modules over the Schur algebra, and Serre's method of computing the Grothendieck group of representations of the group scheme $\mathrm{GL}_{n, \mathbb{Z}}$.

8.1. Polynomial functors. In this subsection we introduce the notion of polynomial functors and state that the Grothendieck group of the category of polynomial functors over \mathbb{Z} is isomorphic to the universal λ -ring in one variable, see Theorem 8.5 below. This theorem will allow us to prove the final λ -ring axiom for $K_*(X)$ in subsection 8.4. The proof of Theorem 8.5 will occupy the next two subsections.

We recall $\mathcal{P}(S)$ denotes the category of \mathcal{O}_S -modules that are locally free of finite rank on a scheme S . We define a category $\underline{\mathcal{P}}(S)$ 'enriched in schemes over S ' as follows. The objects are the same as the objects of $\mathcal{P}(S)$, and for

every $V, W \in \mathcal{P}(S)$ we have an S -scheme

$$\underline{\mathrm{Hom}}(V, W) := \underline{\mathrm{Spec}}_S \mathrm{Sym}^\bullet(\mathcal{H}\mathrm{om}(V, W)^\vee).$$

This is the ‘physical vector bundle’ corresponding to the locally free \mathcal{O}_S -module $\mathcal{H}\mathrm{om}(V, W)$ and we have

$$\underline{\mathrm{Hom}}(V, W)(T) = \mathrm{Hom}_{\mathcal{O}_T}(V_T, W_T)$$

for every S -scheme T . In fact, by Yoneda’s Lemma, we may think of $\underline{\mathrm{Hom}}(V, W)$ as the functor which associates $\mathrm{Hom}_{\mathcal{O}_T}(V_T, W_T)$ with every S -scheme T . The latter viewpoint is used in a lot of literature about polynomial functors. Composition in $\underline{\mathcal{P}}(S)$ is given by the natural maps

$$\underline{\mathrm{Hom}}(U, V) \times_S \underline{\mathrm{Hom}}(V, W) \rightarrow \underline{\mathrm{Hom}}(U, W)$$

of schemes over S , and the identities are given by the obvious sections $\mathrm{id}_V \in \underline{\mathrm{Hom}}(V, V)(S)$.

Definition 8.1. A *polynomial functor* F over S is an enriched functor $F: \underline{\mathcal{P}}(S) \rightarrow \underline{\mathcal{P}}(S)$. A *morphism of polynomial functors* is a natural transformation. We denote the category of polynomial functors over S by $\mathrm{Pol}(S)$.

In other words, a polynomial functor consists of objects $FV \in \mathcal{P}(S)$, $V \in \mathcal{P}(S)$, and of morphisms of S -schemes

$$F: \underline{\mathrm{Hom}}(V, W) \rightarrow \underline{\mathrm{Hom}}(FV, FW), \quad V, W \in \mathcal{P}(S)$$

which satisfy the usual functor axioms. In less precise terms, Ff being a morphism of S -schemes means that, if for instance $S = \mathrm{Spec}(k)$, k a field, the map $Ff: \mathrm{Hom}(V, W) \rightarrow \mathrm{Hom}(FV, FW)$ is given by polynomials in coordinates of V and W . Note that we do not ask Ff to be additive. Every polynomial functor F induces an ‘ordinary’ endofunctor of $\mathcal{P}(S)$, denoted by F again. A morphism $\eta: F \rightarrow G$ consists of a morphism of \mathcal{O}_S -modules

$$\eta_V: FV \rightarrow GV$$

for every $V \in \mathcal{P}(S)$, satisfying the usual conditions for a natural transformation.

Example 8.2 (Exterior powers). Functoriality of Λ^d implies that for all $V, W \in \mathcal{P}(S)$ we have a map

$$\mathrm{Hom}(V, W) \longrightarrow \mathrm{Hom}(\Lambda^d V, \Lambda^d W).$$

This is *a priori* a map of sets, but its formation commutes with base change $T \rightarrow S$, and hence by Yoneda it defines a map of S -schemes

$$\underline{\mathrm{Hom}}(V, W) \longrightarrow \underline{\mathrm{Hom}}(\Lambda^d V, \Lambda^d W).$$

We obtain a polynomial functor $\Lambda^d: \underline{\mathcal{P}}(S) \rightarrow \underline{\mathcal{P}}(S)$.

The category $\mathrm{Pol}(S)$ is a $\Gamma(S, \mathcal{O}_S)$ -linear category. We declare a sequence $0 \rightarrow F \rightarrow G \rightarrow H \rightarrow 0$ in $\mathrm{Pol}(S)$ to be exact if the sequence

$$0 \rightarrow FV \rightarrow GV \rightarrow HV \rightarrow 0$$

is exact for every V ; this way $\mathrm{Pol}(S)$ becomes an exact category [To13, Section 2.1.1]. It carries a tensor product

$$\otimes: \mathrm{Pol}(S) \times \mathrm{Pol}(S) \rightarrow \mathrm{Pol}(S)$$

as well as exterior power operators

$$\Lambda^n: \text{Pol}(S) \rightarrow \text{Pol}(S), \quad F \mapsto \Lambda^n F := \Lambda^n \circ F.$$

These data turn $K_0(\text{Pol}(S))$ into a pre- λ -ring. To prove this, one proceeds as in the proof of Theorem 6.2. As there, the category $\text{Pol}(S)$ is in general not split exact, but for every short exact sequence as above, one can construct a natural filtration

$$0 \subset \Lambda^n F \subset F \wedge \dots \wedge F \wedge G \subset \dots \subset F \wedge G \wedge \dots \wedge G \subset \Lambda^n G$$

of $\Lambda^n G$ whose successive quotients are isomorphic to $\Lambda^{n-k} F \otimes \Lambda^k G$, $k = 0, \dots, n$.

Less evident is that for every morphism $f: T \rightarrow S$ there is a natural *base change functor* $f^*: \text{Pol}(S) \rightarrow \text{Pol}(T)$. This can be constructed as follows. Let $F: \mathcal{P}(S) \rightarrow \mathcal{P}(S)$ be a polynomial functor. Given $V \in \mathcal{P}(T)$ one chooses an open cover (U_i) of T , vector bundles $V_i \in \mathcal{P}(S)$ and isomorphisms $\alpha_i: (f^* V_i)|_{U_i} \rightarrow V|_{U_i}$. These define glueing data $\alpha_{ij} := \alpha_i^{-1} \alpha_j$ and one constructs the desired $(f^* F)V$ by glueing the bundles $f^*(FV_i)$ over the U_{ij} using the maps $F(\alpha_{ij})$. Note that the expression $F(\alpha_{ij})$ makes sense as F is a polynomial functor. For an alternative description of f^* , see Remark 8.9. Thus, every polynomial functor $F \in \text{Pol}(S)$ induces a family of functors

$$F_T: \mathcal{P}(T) \rightarrow \mathcal{P}(T),$$

indexed by $T \rightarrow S$, and that the F_T commute with base change.

The functor f^* is exact, and commutes with the operations \otimes and Λ^n , so that f^* induces a morphism

$$f^*: K_0(\text{Pol}(S)) \rightarrow K_0(\text{Pol}(T))$$

of pre- λ -rings.

Definition 8.3. A polynomial functor $F \in \text{Pol}(S)$ is said to be *homogeneous of degree d* if, for every $V \in \mathcal{P}(S)$, the diagram

$$\begin{array}{ccc} \mathbb{G}_{m,S} & \xrightarrow{x \mapsto x^d} & \mathbb{G}_{m,S} \\ \downarrow & & \downarrow \\ \underline{\text{Hom}}(V, V) & \xrightarrow{F} & \underline{\text{Hom}}(FV, FV) \end{array}$$

commutes; here, the vertical morphisms are given by scalar multiplication. We denote by $\text{Pol}_d(S)$ the category of polynomial functors homogenous of degree d , and by $\text{Pol}_{<\infty}(S)$ the category of polynomial functors that are finite direct sums of homogeneous polynomial functors.

Example 8.4. The polynomial functor Λ^d is homogenous of degree d . The infinite direct sum $\bigoplus_{d \geq 0} \Lambda^d$ is well-defined as it becomes finite when applied to any V ; it is a polynomial functor, but not in $\text{Pol}_{<\infty}(S)$.

Let $\mathbb{Z}[s_1, s_2, \dots]$ denote the ring of symmetric functions, with s_i the i^{th} elementary symmetric function. This is a λ -ring, with $\lambda^i(s_1) = s_i$, also called the universal λ -ring in one variable, see [Yau10, §1.3]. It is also a graded ring with $\deg s_d = d$.

Theorem 8.5. *The ring homomorphism*

$$\mathbb{Z}[s_1, s_2, \dots] \rightarrow K_0(\text{Pol}_{<\infty}(\mathbb{Z})), \quad s_i \mapsto [\Lambda^i]$$

is an isomorphism of pre- λ -rings.

The proof of this theorem will be given at the end of subsection 8.3.

Corollary 8.6. *$K_0(\text{Pol}_{<\infty}(\mathbb{Z}))$ is a λ -ring.* \square

8.2. The Schur algebra. The object of this subsection is to relate polynomial functors to the Schur algebra, see [Ro63, Ch. I & Ch. IV] and [Kr13, §2] for details.

Throughout this subsection, R is a commutative ring. If M is a locally free R -module and d a non-negative integer, then the R -module of degree d divided powers is the module of symmetric degree d tensors:

$$\Gamma^d M = \Gamma_R^d M = (M^{\otimes d})^{S_d}.$$

If A is an associative and locally free R -algebra and M is moreover an A -module, then $\Gamma_R^d A$ is a sub- R -algebra of $A^{\otimes d}$ and the obvious multiplication of $\Gamma_R^d A$ on $\Gamma_R^d M$ turns $\Gamma_R^d M$ into a $\Gamma_R^d A$ -module.

Let n be a positive integer. Consider the *Schur algebra* $\Gamma^d \text{Mat}(n, R)$ of R associated with n and d . It is free as an R -module. For every R -module V , the module $V^n = \text{Hom}(R^n, V)$ is a right $\text{Mat}(n, R)$ -module, hence $\Gamma^d(V^n)$ is a right $\Gamma^d \text{Mat}(n, R)$ -module.

Lemma 8.7. *If V is a projective R -module, then $\Gamma^d(V^n)$ is a projective right $\Gamma^d \text{Mat}(n, R)$ -module.*

Proof. If V is a direct summand of W , then $\Gamma^d(V^n)$ is a direct summand of $\Gamma^d(W^n)$, so without loss of generality we may assume that V is a free R -module. Then $\Gamma^d(V^n)$ is a direct sum of $\Gamma^d \text{Mat}(n, R)$ -modules of the form

$$\Gamma^{d_1}(R^n) \otimes_R \cdots \otimes_R \Gamma^{d_i}(R^n)$$

with $\sum d_i = d$. By [ABW88, Prop. 2.1] these are projective over the Schur algebra $\Gamma^d \text{Mat}(n, R)$, and the lemma follows. \square

We denote by $\mathcal{M}(R, n, d)$ the category of finitely generated left modules over the Schur algebra $\Gamma^d \text{Mat}(n, R)$, and by $\mathcal{M}_p(R, n, d)$ the full sub-category consisting of those modules whose underlying R -module is projective.

We have a ‘truncation’ functor,

$$\text{Pol}_d(R) \rightarrow \mathcal{M}_p(R, n, d), \quad F \mapsto F(R^n),$$

where the structure of left $\Gamma^d \text{Mat}(n, R)$ -module on $F(R^n)$ is defined as follows. We have a map

$$\underline{\text{End}}(R^n) \xrightarrow{F} \underline{\text{End}}(F(R^n))$$

which is homogeneous of degree d . By the universal property of divided powers (see [Ro63, Proposition IV.1] and [Fe98, Proposition 2.5.1]), this map is induced by an R -module homomorphism

$$\Gamma^d \text{End}(R^n) \rightarrow \text{End}(F(R^n)),$$

which is moreover multiplicative, hence giving $F(R^n)$ the structure of a $\Gamma^d \text{End}(R^n)$ -module.

Theorem 8.8. *If $n \geq d$, the functor $\text{Pol}_d(R) \rightarrow \mathcal{M}_p(R, n, d)$ is an equivalence of categories.*

Proof. See [Kr13], where the same result is shown for polynomial functors taking values in arbitrary R -modules, and arbitrary $\Gamma^d \text{Mat}(n, R)$ -modules. The same argument works in our context, we only need to check that the inverse functor maps $\mathcal{M}_p(R, n, d)$ to $\text{Pol}_d(R)$ (that is, that the inverse functor preserves ‘finite type and projective’).

The inverse functor is defined as follows. Let M be a $\Gamma^d \text{Mat}(n, R)$ -module. Then we define a functor

$$F_M: \text{Mod}(R) \rightarrow \text{Mod}(R), V \mapsto \Gamma^d(V^n) \otimes_{\Gamma^d \text{Mat}(n, R)} M,$$

where the right $\Gamma^d \text{Mat}(n, R)$ -module structure on $\Gamma^d(V^n)$ is inherited from the structure of right $\text{Mat}(n, R)$ -module on $V^n = \text{Hom}(R^n, V)$. Formation of F_M commutes with base change.

Now assume that both M and V are finitely generated and projective R -modules. Then the module $F_M(V)$ is also finitely generated. We claim that $F_M(V)$ is also projective. By Lemma 8.7 the module $\Gamma^d(V^n)$ is projective, hence a direct summand of a free $\Gamma^d \text{Mat}(n, R)$ -module $\oplus_I \Gamma^d \text{Mat}(n, R)$, and hence $F_M(V)$ is a direct summand of a projective R -module $\oplus_I M$. \square

Remark 8.9. Theorem 8.8 gives an alternative way for producing the base change of a polynomial functor. If $R \rightarrow S$ is a map of commutative rings, and if M is a $\Gamma^d \text{Mat}(n, R)$ -module, then the base change $M \otimes_R S$ is a $\Gamma^d \text{Mat}(n, S)$ -module, since formation of $\Gamma^d \text{Mat}(n, -)$ commutes with base change.

8.3. The Grothendieck group of polynomial functors over \mathbb{Z} . We fix n and d satisfying $n \geq d$. For brevity we write $\mathcal{M}(R) := \mathcal{M}(R, n, d)$ and $\mathcal{M}_p(R) := \mathcal{M}_p(R, n, d)$. Furthermore we write $\mathbb{Z}[s_1, s_2, \dots]_d$ for the weighted degree d part of the polynomial ring $\mathbb{Z}[s_1, s_2, \dots]$. It is equal to $\mathbb{Z}[s_1, \dots, s_n]_d$.

In this subsection, following [Se68], we compute the Grothendieck group $K_0(\mathcal{M}_p(\mathbb{Z}))$. Together with Theorem 8.8 this then implies Theorem 8.5.

If R is an integral domain, there is a natural homomorphism

$$K_0(\mathcal{M}_p(R)) \xrightarrow{\text{Thm. 8.8}} K_0(\text{Pol}_d(R)) \rightarrow \mathbb{Z}[s_1, s_2, \dots]_d$$

given by sending a polynomial functor F to the weights of the action of \mathbb{G}_m^n on $F(R^n)$.

Theorem 8.10. *For every field K the map $K_0(\mathcal{M}(K)) \rightarrow \mathbb{Z}[s_1, s_2, \dots]_d$ is an isomorphism.*

Proof. See [Gre80, 2.2 & 3.5, esp. Remark (ii)]. The book [Gre80] assumes the field K to be infinite, but this assumption is only used in relating modules over $\Gamma^d \text{Mat}(n, K)$ to representations of the monoid $\text{Mat}(n, K)$, as opposed to representations of the *monoid scheme* $\text{Mat}_{n, K}$, which would also work over a finite field K . See also [Ja03, II.A]. \square

We will deduce from the cases $K = \mathbb{Q}$ and $K = \mathbb{F}_p$ in this theorem that the map

$$K_0(\text{Pol}_d(\mathbb{Z})) \rightarrow \mathbb{Z}[s_1, s_2, \dots]_d$$

is an isomorphism. The proof is essentially identical to Serre's proof that

$$K_0(\mathrm{GL}_{n,\mathbb{Z}}) \rightarrow K_0(\mathrm{GL}_{n,\mathbb{Q}})$$

is an isomorphism, see [Se68].

Lemma 8.11 (Projective resolutions). *The canonical map*

$$K_0(\mathcal{M}_p(\mathbb{Z})) \rightarrow K_0(\mathcal{M}(\mathbb{Z}))$$

is an isomorphism.

Proof. (Compare [Se68, §2.2 & §2.3].) Let M be a finitely generated module over $\Gamma^d \mathrm{Mat}(n, \mathbb{Z})$. If M can be generated by m elements, we obtain a presentation

$$0 \rightarrow P_0 \rightarrow P_1 \rightarrow M \rightarrow 0$$

with $P_1 = (\Gamma^d \mathrm{Mat}(n, \mathbb{Z}))^m$. Both P_1 and P_0 are torsion-free, hence projective as \mathbb{Z} -modules. The same argument as [Se68, Prop. 4] shows that $[P_0] - [P_1] \in K_0(\mathcal{M}_p(\mathbb{Z}))$ is independent of the choice of presentation, and that $M \mapsto [P_0] - [P_1]$ defines a two-sided inverse to the map of the proposition. \square

Lemma 8.12 (Localization sequence). *The obvious sequence*

$$\bigoplus_{\ell \text{ prime}} K_0(\mathcal{M}(\mathbb{F}_\ell)) \rightarrow K_0(\mathcal{M}(\mathbb{Z})) \rightarrow K_0(\mathcal{M}(\mathbb{Q})) \rightarrow 0$$

is exact.

Proof. The argument is identical to [Se68, Théorème 1]. The main point is to verify that every $\Gamma^d \mathrm{Mat}(n, \mathbb{Q})$ -module V of finite \mathbb{Q} -dimension contains a $\Gamma^d \mathrm{Mat}(n, \mathbb{Z})$ -submodule Λ with $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda = V$. To construct such Λ , take an arbitrary sub- \mathbb{Z} -module Λ_0 with $\mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_0 = V$, and take $\Lambda := \Gamma^d \mathrm{Mat}(n, \mathbb{Z}) \Lambda_0$. \square

Lemma 8.13 (Decomposition maps). *For every prime ℓ there is a unique homomorphism d_ℓ making the triangle*

$$\begin{array}{ccc} K_0(\mathcal{M}_p(\mathbb{Z})) & \longrightarrow & K_0(\mathcal{M}(\mathbb{Q})) \\ \downarrow & \swarrow d_\ell & \\ K_0(\mathcal{M}(\mathbb{F}_\ell)) & & \end{array}$$

commute.

Proof. The argument is identical to [Se68, Thm 2]. \square

Lemma 8.14. *The composition*

$$K_0(\mathcal{M}(\mathbb{Q})) \xrightarrow{d_\ell} K_0(\mathcal{M}(\mathbb{F}_\ell)) \longrightarrow K_0(\mathcal{M}(\mathbb{Z}))$$

is the zero map.

Proof. See [Se68, Lemme 4]. Similarly to there, one uses that for every \mathbb{Z} -torsion-free $\Gamma^d \mathrm{Mat}(n, \mathbb{Z})$ -module Λ the map $\Lambda \rightarrow \ell\Lambda$, $x \mapsto \ell x$, is an isomorphism of $\Gamma^d \mathrm{Mat}(n, \mathbb{Z})$ -modules. \square

Proposition 8.15. *For every prime ℓ , the map d_ℓ is an isomorphism.*

Proof. By Theorem 8.10 the functors

$$\Lambda^{d_1} \otimes \Lambda^{d_2} \otimes \dots \otimes \Lambda^{d_m}$$

with $\sum d_i = d$ define a basis of $K_0(\mathcal{M}(\mathbb{Q}))$ and of $K_0(\mathcal{M}(\mathbb{F}_\ell))$. Since the map d_ℓ preserves this basis, it is an isomorphism. \square

Corollary 8.16. *The canonical map*

$$K_0(\mathcal{M}(\mathbb{Z})) \rightarrow K_0(\mathcal{M}(\mathbb{Q}))$$

is an isomorphism.

Proof. By Proposition 8.15 and Lemma 8.14 the maps

$$K_0(\mathcal{M}(\mathbb{F}_\ell)) \rightarrow K_0(\mathcal{M}(\mathbb{Z}))$$

are the zero maps. But then the localization sequence of Lemma 8.12 shows that the map $K_0(\mathcal{M}(\mathbb{Z})) \rightarrow K_0(\mathcal{M}(\mathbb{Q}))$ is an isomorphism. \square

Proof of Theorem 8.5. The degree d part of the homomorphism

$$\mathbb{Z}[s_1, s_2, \dots] \rightarrow K_0(\text{Pol}_{<\infty}(\mathbb{Z})), \quad s_i \mapsto [\Lambda^i],$$

is obviously inverse to the composition of the isomorphisms

$$K_0(\text{Pol}_d(\mathbb{Z})) \cong K_0(\mathcal{M}_p(\mathbb{Z})) \cong K_0(\mathcal{M}(\mathbb{Z})) \cong K_0(\mathcal{M}(\mathbb{Q})) \cong \mathbb{Z}[s_1, s_2, \dots]_d$$

given by Theorem 8.8, Lemma 8.11, Corollary 8.16 and Theorem 8.10, respectively, and is hence bijective and compatible with exterior power operations. \square

Remark 8.17. The category $\text{Pol}_d(R)$ is equivalent with the category of weight d representations of the monoid $\text{Mat}_{n,R}$, which forms a full subcategory of the category of representations of $\text{GL}_{n,R}$. Rather than translating Serre's argument from the GL_n to the Mat_n context, one could also deduce our result from Serre's. However, some care has to be taken because the right adjoint to the inclusion, mapping a $\text{GL}_{n,\mathbb{Z}}$ -representation V to the largest sub-representation that extends to $\text{Mat}_{n,\mathbb{Z}}$, is not exact, see [Ja03, II.A].

8.4. Proof of the final λ -ring axiom. In this subsection, we derive from Corollary 8.6 that, for every quasi-compact scheme X , the pre- λ -ring $K_*(X)$ satisfies the final λ -ring axiom. Together with Theorem 7.1, this implies that $K_*(X)$ is a λ -ring and finishes the proof of this paper's main result.

Theorem 8.18. *The ring $K_*(X)$ equipped with the exterior power operations defined in section 7 satisfies axiom (3) of a λ -ring.*

Proof. Let $r, s \geq 1$ and $n \geq 0$. For every $x \in K_n(X)$ we want to show that the identity

$$(8.19) \quad \lambda^r(\lambda^s(x)) = P_{r,s}(\lambda^1(x), \dots, \lambda^{rs}(x))$$

holds in $K_n(X)$. We recall that, if $n \geq 1$, all products occurring on the right-hand side of (8.19) are trivial (and hence that the right-hand side of (8.19) happens to be just a multiple of $\lambda^{rs}(x)$). We will show the stronger statement that the identity (8.19) in fact holds in $K_0((B_b^q)^n \mathcal{P}(X))$ for all $x \in K_0((B_b^q)^n \mathcal{P}(X))$. Now the products occurring on the right-hand side

of (8.19) are induced by the simplicial tensor product introduced in section 5; these products become trivial in $K_n(X)$ by Proposition 5.11. By a standard argument (see the proof of Theorem 7.1) we may assume that x is the class of an object P of $(B_b^q)^n \mathcal{P}(X)$. One easily checks, for instance using the Gabriel–Quillen embedding theorem [TT90, A.7.1, A.7.16], that for every exact category \mathcal{P} and any skeletally small category \mathcal{I} , the category of functors from \mathcal{I} to \mathcal{P} is again an exact category in the obvious way. In particular, the category $\text{End}((B_b^q)^n \mathcal{P}(X))$ of endo-functors of $(B_b^q)^n \mathcal{P}(X)$ is an exact category. Furthermore it carries a tensor product and exterior power operations (given by $F \mapsto \Lambda_n^d \circ F$). Via the homomorphism $K_0(\text{End}((B_b^q)^n \mathcal{P}(X))) \rightarrow K_0((B_b^q)^n \mathcal{P}(X))$ given by $F \mapsto F(P)$, the desired identity now follows from the even stronger identity

$$(8.20) \quad [\Lambda_n^r \circ \Lambda_n^s] = P_{r,s}([\Lambda_n^1], \dots, [\Lambda_n^{rs}])$$

in $K_0(\text{End}((B_b^q)^n \mathcal{P}(X)))$, which we now prove. We remember that the identity (8.20) (with the subscripts n omitted) holds in the Grothendieck group $K_0(\text{Pol}_{<\infty}(\mathbb{Z}))$ by Corollary 8.6. Then it also holds in $K_0(\text{Pol}_{<\infty}^0(\mathbb{Z}))$ where $\text{Pol}_{<\infty}^0(\mathbb{Z})$ denotes the full subcategory of $\text{Pol}_{<\infty}(\mathbb{Z})$ consisting of functors F satisfying $F(0) = 0$; this follows from the fact that the canonical inclusion $\text{Pol}_{<\infty}^0(\mathbb{Z}) \hookrightarrow \text{Pol}_{<\infty}(\mathbb{Z})$ is split by $F \mapsto (V \mapsto \ker(F(V) \rightarrow F(0)))$. The identity (8.20) therefore follows from Corollary 8.6 once we have shown that we have a pre- λ -ring homomorphism

$$(8.21) \quad K_0(\text{Pol}_{<\infty}^0(\mathbb{Z})) \rightarrow K_0(\text{End}((B_b^q)^n \mathcal{P}(X)))$$

that sends the class of the identity functor to the class of the identity functor. By base change (see subsection 8.1), every functor in $\text{Pol}(\mathbb{Z})$ induces a functor in $\text{Pol}(U)$ for every open subset U of X and this construction is compatible with restriction with respect to any inclusion of open subsets of X . The constructions of sections 3, 4 and 6 therefore inductively induce a functor

$$\text{Pol}_{<\infty}^0(\mathbb{Z}) \rightarrow \text{End}((B_b^q)^n \mathcal{P}(X));$$

this functor is exact and compatible with tensor products and exterior power operations as one easily verifies by induction on n . Thus it induces the desired homomorphism (8.21) and the proof of Theorem 8.18 is complete. \square

Remark 8.22. We have seen in the previous proof that the λ -ring axiom (3) already holds in $K_0((B_b^q)^n \mathcal{P}(X))$, i.e., before dividing out the subgroup generated by classes of diagonal multicomplexes. The same holds true for the λ -ring axiom (2). This can be shown similarly by using Corollary 8.6 or by using the characteristic-free Cauchy decomposition as constructed in [ABW82]. Whereas Corollary 8.6 only proves the existence of short exact sequences, Akin, Buchsbaum and Weyman *explicitly* construct short exact sequences in [ABW82] that prove axiom (2) of a λ -ring. The problem of explicitly describing short exact sequences of polynomial functors that prove axiom (3) seems however to be even harder than the famous and related plethysm problem in representation theory. Such explicit short exact sequences for the plethysm $\Lambda^2 \circ \Lambda^2$ can be found on page 175 of the paper [AB85] by Akin and Buchsbaum. Although there also exist solutions of the

classical plethysm problem for $\Lambda^r \circ \Lambda^2$ and $\Lambda^2 \circ \Lambda^s$, we are not aware of any corresponding characteristic-free short exact sequences.

APPENDIX A. PROOF OF LEMMA 3.4

In this appendix we prove Lemma 3.4, which states that:

- (1) If \mathcal{P} is an idempotent complete exact category, then so are $C^n\mathcal{P}$ and $(C^q)^n\mathcal{P}$.
- (2) If \mathcal{P} is a split exact category, then so is $(C^q)^n\mathcal{P}$.

Note that to prove each of these statements it is enough to prove the case $n = 1$.

Proof of Lemma 3.4 (1). Let $e: P_\bullet \rightarrow P_\bullet$ be an idempotent map of chain complexes. Then each map $e_n: P_n \rightarrow P_n$ is an idempotent of \mathcal{P} and so has a kernel $\ker(e_n)$ which is an object of \mathcal{P} . By the universal property of kernels, the chain map on P_\bullet induces a map $\ker(e_n) \rightarrow \ker(e_{n-1})$ for each n , and these assemble to form a chain complex of kernels. Thus every idempotent in $C\mathcal{P}$ has a kernel in $C\mathcal{P}$, so $C\mathcal{P}$ is idempotent complete. To show that $C^q\mathcal{P}$ is idempotent complete as well, we must show that this kernel chain complex is acyclic in \mathcal{P} if P_\bullet is. To do this, it suffices to consider the case when the complex is a short exact sequence; the general case then follows because \mathcal{P} supports long exact sequences.

If $P_\bullet = (0 \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow 0)$ and if $e: P_\bullet \rightarrow P_\bullet$ is an idempotent chain map, then P_\bullet is isomorphic to a sequence of the form

$$0 \longrightarrow \ker(e_2) \oplus \operatorname{im}(e_2) \longrightarrow \ker(e_1) \oplus \operatorname{im}(e_1) \longrightarrow \ker(e_0) \oplus \operatorname{im}(e_0) \longrightarrow 0.$$

Furthermore, as the morphisms in this short exact sequence commute with the idempotents $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, they split as direct sums of induced morphisms. Hence the sequence

$$0 \longrightarrow \ker(e_2) \longrightarrow \ker(e_1) \longrightarrow \ker(e_0) \longrightarrow 0$$

is exact as well. □

Proof of Lemma 3.4 (2). We wish to show that every admissible monomorphism $i: P_\bullet \rightarrowtail Q_\bullet$ in $C^q\mathcal{P}$ is split; that is, that there exists a chain map $s: Q_\bullet \rightarrow P_\bullet$ such that each $s_n i_n: P_n \rightarrow Q_n$ is the identity. Let us restrict to the case in which P_\bullet and Q_\bullet are short exact sequences of \mathcal{P} . Consider the following diagram in a split exact category,

$$\begin{array}{ccccc} P' & \xrightarrow{j_P} & P & \xrightarrow{q_P} \twoheadrightarrow & P'' \\ \downarrow i' & & \downarrow i & & \downarrow i'' \\ Q' & \xrightarrow{j_Q} & Q & \xrightarrow{q_Q} \twoheadrightarrow & Q'' \end{array}$$

and fix a splitting s'' for i'' . We claim that there exist splittings s' and s of i' and i such that the resulting $s_\bullet: Q_\bullet \rightarrow P_\bullet$ is a chain map (and hence a splitting of i_\bullet). The general case follows from this claim. Indeed, since acyclic complexes are spliced together from short exact sequences, we construct a splitting for a monomorphism of acyclic complexes $i: P_\bullet \rightarrowtail Q_\bullet$ by splitting each monomorphism of short exact sequences separately. The part of the claim concerning a fixed splitting s'' allows us to choose these splittings of

short exact sequences in a compatible manner (beginning in degree 0). So it is enough to prove the claim.

We choose compatible splittings h_P and t_P of j_P and q_P respectively, i.e., $j_P h_P + t_P q_P = 1$. We also choose a splitting s_0 for i and now set $s = j_P h_{Ps_0} + t_P s'' q_Q$. Then we compute:

- (1) $si = j_P h_{Ps_0} i + t_P s'' q_Q i = j_P h_P + t_P s'' i'' q_P = j_P h_P + t_P q_P = 1$,
- (2) $q_P s = q_P j_P h_{Ps_0} + q_P t_P s'' q_Q = s'' q_Q$,

so s is a splitting for i and s, s'' commute with q_P, q_Q . We therefore get an induced map of kernels $s': Q' \rightarrow P'$ satisfying $j_P s' = s j_Q$. Moreover $j_P s' i' = s j_Q i' = s i j_P = j_P$, and j_P is monic, so $s' i' = 1$. \square

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