LS-CATEGORY OF MOMENT-ANGLE MANIFOLDS, MASSEY PRODUCTS, AND A GENERALISATION OF THE GOLOD PROPERTY

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Abstract. This paper is obtained as as synergy of homotopy theory, commutative algebra and combinatorics. We give various bounds for the Lusternik-Schnirelmann category of moment-angle complexes and show how this relates to vanishing of Massey products in $\mathrm{Tor}^{+}_{R[v_1,\ldots,v_n]}(R[K],R)$ for the Stanley-Reisner ring $R[K]$. In particular, we characterise the Lusternik-Schnirelmann category of moment-angle manifolds $\mathcal{Z}_K$ over triangulated d-spheres K for $d \leq 2$, as well as higher dimension spheres built up via connected sum, join, and vertex doubling operations. This characterisation is given in terms of the combinatorics of K, the cup product length of $H^*(\mathcal{Z}_K)$, as well as a certain generalisation of the Golod property. As an application, we describe conditions for vanishing of Massey products in the case of fullerenes and k-neighbourly complexes.

1. Introduction

A covering of a topological space X is said to be *categorical* if every set in the covering is open and contractible in X. That is, the inclusion map of each set into X is nullhomotopic. The *Lusternik-Schnirelmann category* (or simply *category*) $\mathrm{cat}(X)$ of X is the smallest integer k such that X admits a categorical covering by $k+1$ open sets $\{U_0,\ldots,U_k\}$.

Lusternik-Schnirelmann category and related invariants have been computed for polyhedral products of the form $(X,*)^K$ for certain nice spaces X in [18, 26], though the methods there do not extend to moment-angle complexes $\mathcal{Z}_K = (D^2,\partial D^2)^K$ for several reasons. For example, the contractibility of the disk D^2 does not give nice lower bounds in terms of the dimension of K. Indeed, there is a fairly large literature that is focused on determining those K for which the moment-angle complex $\mathcal{Z}_K$ has category 1 (c.f. [4, 5, 28, 30, 29, 34, 33, 6]).

Our work is motivated by a problem in commutative algebra and our results although stated in homotopy theory have direct applications in commutative algebra. Let R be a local ring. One of the fundamental aims of commutative algebra is to describe the homology ring of R, that is $\mathrm{Tor}_R(k,k)$, where k is a ground field. The first step in understanding $\mathrm{Tor}_R(k,k)$ is to obtain information about its Poincaré series $P(R)$, more specifically, whether $P(R)$ is a rational function. A far reaching contribution to this problem was made by Golod. A local ring R is *Golod* if all Massey products in $\mathrm{Tor}_{k[v_1,\ldots,v_n]}(R,k)$ vanish. Golod[25] proved that if a local ring is Golod, then its Poincaré series represents a rational function and it is determined by $P(\mathrm{Tor}_{k[v_1,\ldots,v_n]}(R,k))$. Although being Golod is an important property, not many Golod rings are known. Using our results on the homotopy type of moment angle complexes, we are able to use homotopy theory to gain some insight into these difficult homological-algebraic questions.

We will mostly be interested in moment-angle complexes $\mathcal{Z}_K$ over triangulated spheres K. These are known as *moment-angle manifolds* since it is here that $\mathcal{Z}_K$ takes the form of a topological manifold. Moment-angle complexes of this form have generated a lot of interest due to their connections to quasitoric manifolds in toric topology and intersection of quadrics in complex geometry, amongst other things. Their topology and cohomology is, however, very intricate, with many questions remaining open even for low dimensional K (see for example [12, 15, 24, 14, 40, 8, 38, 39]).

2010 *Mathematics Subject Classification.* Primary 55M30, 13H10, 05E40; Secondary 55U10.

Key words and phrases. polyhedral product, moment-angle complex, toric topology, Stanley-Reisner ring, Golod ring, Massey products, Lusternik-Schnirelmann category.

None-the-less, we will characterise those triangulations of d-spheres where $d \leq 2$ for which $\mathcal{Z}_K$ has a given category, as well as certain higher dimensional spheres built up via join, connected sum, and vertex doubling operations.

Our motivation for doing this is a combinatorial and algebraic characterisation of Golod complexes K and *co-H*-space (category ≤ 1) moment-angle complexes $\mathcal{Z}_K$ given in [28] in the case of flag complexes K. The authors there showed that both of these concepts are equivalent, and moreover, that they both coincide with chordality of the 1-skeleton of K and the triviality of the multiplication on $\mathrm{Tor}^+_{R[v_1,\dots,v_n]}(R[K],R)$ for $R = \mathbb{Z}$ or R any field, where $R[K]$ is the Stanley-Reisner ring of K over R. From the perspective of commutative algebra, an interesting algebraic consequence of this was that triviality of the multiplication on $\mathrm{Tor}^+_{R[v_1,\dots,v_n]}(R[K],R)$ implies that all higher Massey products are also trivial, at the very least when K is flag. [1] This depended on the general fact that the cohomology ring of a space of category ≤ 1 has trivial multiplication and Massey products vanish [23, 43]. It is natural to ask what the corresponding statement is for spaces with larger category, more so, if the characterisation for Golod flag complexes in [28] can be generalised in this sense. An answer to the first question was given by Rudyak in [42], which inspires the following definition.

Definition 1.1. A simplicial complex K on vertex set $[n]$ is *m-Golod* over R if

(1) $\mathrm{nill}(\mathrm{Tor}_{R[v_1,\dots,v_n]}(R[K],R)) \leq m+1$;
(2) Massey products $\langle v_1,\dots,v_k\rangle$ vanish in $\mathrm{Tor}^+_{R[v_1,\dots,v_n]}(R[K],R)$ whenever $v_i = a_1\cdots a_{m_i}$ and $v_j = b_1\cdots b_{m_j}$, and $m_i+m_j > m$ for some odd i and even j and $a_s, b_t \in \mathrm{Tor}^+_{R[v_1,\dots,v_n]}(R[K],R)$.

Proposition 1.2. *If $\mathrm{cat}(\mathcal{Z}_K) \leq m$, then K is m-Golod.* □

In this respect, the vanishing of certain Massey products can be established whenever the Lusternik-Schnirelmann category of a moment-angle complex is determined. Here the *nilpotency* $\mathrm{nill}\,A$ of a graded algebra A is the smallest integer k such that all length k products in the positive degree part A^+ vanish. Notice that K is $(m+1)$-Golod whenever it is m-Golod, and 1-Golodness of K coincides with the classical notion of Golodness [25], namely, that all products and (higher) Massey products are trivial in $\mathrm{Tor}^+_{R[v_1,\dots,v_n]}(R[K],R)$. All of this can be restated equivalently in terms of the cohomology of $\mathcal{Z}_K$ due to an isomorphism of graded commutative algebras $H^*(\mathcal{Z}_K;R) \cong \mathrm{Tor}_{R[v_1,\dots,v_n]}(R[K],R)$ when R is a field or $\mathbb{Z}$ [3]. We will consider (co)homology with integer coefficients.

Theorem 1.3. *If K is k-neighbourly, then $\mathrm{cat}(\mathcal{Z}_K) \leq \frac{1+\dim K}{k}$ and K is $\left(\frac{1+\dim K}{k}\right)$-Golod.* □

Theorem 1.4. *If K is any d-sphere for $d \leq 2$, or (under a few conditions) built up as a connected sum of joins of such spheres, then the following are equivalent: (a) $\mathrm{cat}(\mathcal{Z}_K) \leq k$; (b) K is k-Golod; (c) length $k+1$ cup products of positive degree elements in $H^*(\mathcal{Z}_K)$ vanish; (d) there does not exist a spherical filtration of full subcomplexes of K of length more than k. Moreover, $k \leq d+1$.* □

Applying *vertex doubling operations*, the range of spheres in Theorem 1.4 can be extended using the following.

Theorem 1.5. *If $K(J)$ is the simplicial wedge of K for some integer sequence $J = (j_1,\dots,j_n)$, then $\mathrm{cat}(\mathcal{Z}_{K(J)}) \leq \mathrm{cat}(\mathcal{Z}_K)$.* □

In the case when K is the boundary of the dual of a *fullerene* P we obtain the following result.

Theorem 1.6. *For a fullerene P, $\mathrm{cat}(\mathcal{Z}_{\partial P^*}) = 3$ and ∂P^* is 3-Golod.* □

We remark that the question of determining higher Massey products in $H^*(\mathcal{Z}_K)$ is an important one but notoriously difficult and equally interesting both for algebraist and topologists. Currently,

[1] A theorem in [7] claims this is true for all K, but a recent paper [37] provides a simple counter-example.

a systematic answer is known in the case of moment-angle complexes associated to one dimensional simplicial complexes, and only for triple Massey products of three dimensional cohomological classes (see [15]). The relation between Massey products to the Lusternik-Schnirelmann category of moment-angle complexes, one can obtain (for example)

Theorem 1.7. *For a fullerene P, all Massey products of decomposable elements in $H^+(\mathcal{Z}_{\partial P^*})$ vanish.* □

More general bounds for $\mathrm{cat}(\mathcal{Z}_K)$ will be given along the way, for instance, when K is formed by gluing simplicial complexes along common full subcomplexes. We note that many of the results in this paper extend to polyhedral products of the form $(Cone(X), X)^K$ in place of $\mathcal{Z}_K = (D^2, S^1)^K$.

2. Preliminary

Recall the following concepts from [36, 13, 19, 22]. The *geometric category* $\mathrm{gcat}(X)$ of a space X is the smallest integer k such that X admits a categorical covering $\{U_0, \ldots, U_k\}$ of X with each U_i contractible (in itself), and the *category* $\mathrm{cat}(f)$ of a map $f\colon X \longrightarrow Y$ is the smallest k such that X admits an open covering $\{V_0, \ldots, V_k\}$ such that f restricts to a nullhomotopic map on each V_i. It is easy to see that $\mathrm{cat}(X) = \mathrm{cat}(\mathbb{1}\colon X \longrightarrow X)$, $\mathrm{cat}(f) \leq \min\{\mathrm{cat}(X), \mathrm{cat}(Y)\}$, $\mathrm{cat}(h \circ h') \leq \mathrm{cat}(h')$. For path-connected paracompact spaces,

$$\mathrm{cat}(f \times g) \leq \mathrm{cat}(f) + \mathrm{cat}(g),$$

which follows from the also well-known fact that $\mathrm{cat}(X \times Y) \leq \mathrm{cat}(X) + \mathrm{cat}(Y)$ together with the preceding inequalities. Unlike $\mathrm{cat}()$, $\mathrm{gcat}()$ is not a homotopy invariant, though one can obtain a homotopy invariant from $\mathrm{gcat}()$ by defining the *strong category*

$$\mathcal{C}\mathrm{at}(X) = \min\{\mathrm{gcat}(Y) \mid Y \simeq X\}.$$

In fact, the strong category satisfies $\mathcal{C}\mathrm{at}(X) - 1 \leq \mathrm{cat}(X) \leq \mathcal{C}\mathrm{at}(X) \leq \mathrm{gcat}(X)$. We shall let $\mathrm{cup}(X) = \mathrm{nill}\, H^*(X) - 1$ denote the length of the longest non-zero cup product of positive degree elements in $H^*(X)$. The main use of this is the classical lower bound

$$\mathrm{cup}(X) \leq \mathrm{cat}(X).$$

2.1. **Some General Bounds.** We begin by giving upper bounds for the Lusternik-Schnirelmann category over some general spaces.

Lemma 2.1. *Let A be a subcomplex of X and S an open subset of A. Then S is a deformation retract of an open subset U of X such that $U \cap A = S$.*

Proof. Let $\mathcal{I}_j$ be an index set for the j-cells e^j_α of $X - A$, $\Phi_\alpha\colon D^j \longrightarrow X$ its characteristic map, and $\phi_\alpha\colon \partial D^j \hookrightarrow D^j \xrightarrow{\Phi_\alpha} X$ its attaching map. Given a subset $B \subseteq X$, let $V_{\alpha,B}$ be the image of $\phi^{-1}_\alpha(B) \times [0, \frac{1}{2}) \subseteq D^j \cong (\partial D^j \times [0,1])/(\partial D^j \times \{1\})$ under Φ_α. Notice $V_{\alpha,B}$ deformation retracts onto a subspace of $\phi_\alpha(\partial D^j) \cap B$, and if $B \cap e^j_\alpha = \varnothing$, $B \cup V_{\alpha,B}$ deformation retracts onto B.

Construct $R_{i+1} \subseteq X$ such that $R_i \subseteq R_{i+1}$, R_i is a deformation retract of R_{i+1}, and $R_i \cap X^{\langle i \rangle}$ is open in the i-skeleton $X^{\langle i \rangle}$, by letting $R_0 = S$ and $R_{i+1} = R_i \cup \bigcup_{\alpha \in \mathcal{I}_{i+1}} V_{\alpha,S}$. Then $U = \bigcup_{i \geq 0} R_i$ is open in X, deformation retracts onto S, and $U \cap A = S$.

□

Lemma 2.2. *Given a filtration $X_0 \subseteq \cdots \subseteq X_m = X$ of subcomplexes of a CW-complex X, suppose $X_{i+1} - X_i$ is contractible in X for each i. Then $\mathrm{cat}(X) \leq \mathrm{cat}(X_0 \hookrightarrow X) + m \leq \mathrm{cat}(X_0) + m$.*

Proof. Let $k = \mathrm{cat}(X_0 \hookrightarrow X)$, and $\{U_0, \ldots, U_k\}$ be a categorical cover of the inclusion $X_0 \hookrightarrow X$. Note $V_i = X_{i+1} - X_i$ is open in X_{i+1} since the subcomplex X_i is closed in X_{i+1}. Then iterating Lemma 2.1, we have open subsets $\bar{U}_i$ and $\bar{V}_i$ that deformation retract onto each U_i and V_i respectively.

Since the U_i's and V_i's cover X and are contractible in X, so do the $\bar{U}_i$'s and $\bar{V}_i$'s, thus they form a categorical cover of X.

□

For any spaces X and Y, and a fixed basepoint $* \in X$, we let $X \rtimes Y = (X \times Y)/(* \times Y)$ denote the *right half-smash* of X and Y, and $Y \ltimes X = (Y \times X)/(Y \times *)$ the *left half-smash.*

Lemma 2.3. *If X and Y are CW-complexes and X is path-connected, then* $\mathrm{cat}(X \rtimes Y) = \mathrm{cat}(X)$.

Proof. Let $\tilde{X}$ be given by attaching the interval $[0,1]$ to X by identifying $0 \in [0,1]$ with the basepoint $* \in X$, and fix $1 \in \tilde{X}$ to be the basepoint. Given $k = \mathrm{cat}(\tilde{X})$ and $\{U_0, \dots, U_k\}$ a categorical cover of $\tilde{X}$, take the open cover $\{U_0 \rtimes Y, \dots, U_k \rtimes Y\}$ of $\tilde{X} \rtimes Y = (\tilde{X} \times Y)/(1 \times Y)$ (here $U_i \rtimes Y = U_i \times Y$ if $1 \notin U_i$). Notice that the contractions of each U_i in $\tilde{X}$ can be taken so that 1 remains fixed if $1 \in U_i$. If U_i contracts to a point b_i in $\tilde{X}$, $U_i \rtimes Y$ deforms onto $\{b_i\} \rtimes Y$ in $\tilde{X} \rtimes Y$, which in turn contracts to the basepoint in $\tilde{X} \rtimes Y$ by homotoping the coordinate b_i to 1. Therefore $\mathrm{cat}(\tilde{X} \rtimes Y) \leq k$, and we have $\mathrm{cat}(X \rtimes Y) \leq k$ since $X \simeq \tilde{X}$ and $X \rtimes Y \simeq \tilde{X} \rtimes Y$. Moreover, $\mathrm{cat}(X \rtimes Y) \geq k$ since X is a retract of $X \rtimes Y$.

□

Let $\mathcal{S}$ be m copies of the interval $[0,1]$ glued together at the endpoints 1 in some order. Given a collection of maps $X \xrightarrow{f_i} Y_i$ for $i = 1, \dots, m$, the *homotopy pushout* P of the maps f_i is the m-fold mapping cylinder

$$P = \big(Y_1 \coprod \cdots \coprod Y_m \coprod (X \times \mathcal{S})\big) / \sim$$

under the identification $(x,t) \sim f_i(x)$ whenever t is in the i^{th} copy of $[0,1]$ in $\mathcal{S}$ and $t = 0$.

Lemma 2.4. *Fix $m \geq 2$. For $i = 1, \dots, m$, let A_i and C_i be basepointed CW-complexes, $B_i = \prod_{j \neq i} A_j$, and E be a contractible space. Suppose $A_i \times E \xrightarrow{f_i} C_i$ are nullhomotopic maps, and P is the homotopy pushout of the maps $A_i \times E \times B_i \xrightarrow{f_i \times \mathbb{1}_{B_i}} C_i \times B_i$ for $i = 1, \dots, m$. Then*

$$\mathrm{cat}(P) \leq \max\{1, \mathrm{cat}(C_1), \dots, \mathrm{cat}(C_m)\}.$$

Proof. We proceed by induction on m. Start with $m = 2$. By Lemma 7.1 in [30], there is a splitting $P \simeq (\Sigma A_1 \wedge A_2) \vee (C_1 \rtimes A_2) \vee (C_2 \rtimes A_1)$. Thus using Lemma 2.3,

$$\mathrm{cat}(P) = \max\{\mathrm{cat}(\Sigma A_1 \wedge A_2), \mathrm{cat}(C_1 \rtimes A_2), \mathrm{cat}(C_2 \rtimes A_1)\} = \max\{1, \mathrm{cat}(C_1), \mathrm{cat}(C_2)\}.$$

The statement holds when $m = 2$.

Take $\mathcal{B}_0 = *$, $\mathcal{B}_\ell = \prod_{j \leq \ell} A_j$, $B_i' = \prod_{j \neq i, j < m} A_j$, and B_i as basepointed subspaces of $\mathcal{B} = \prod_j A_j$. Let P' be the homotopy pushout of $f_i \times \mathbb{1}_{B_i'}$ for $i = 1, \dots, m-1$ (these are all maps from $E \times B_m = E \times \mathcal{B}_{m-1}$). Suppose the lemma holds whenever $m < m'$ for some $m' > 2$. Let $m = m'$. Then $\mathrm{cat}(P') \leq \max\{1, \mathrm{cat}(C_1), \dots, \mathrm{cat}(C_{m-1})\}$. Notice that P is the homotopy pushout of $f_m \times \mathbb{1}_{B_m}$ and the inclusion $A_m \times E \times B_m \xrightarrow{\mathbb{1}_{A_m} \times g} A_m \times P'$, where g is the inclusion $W_{m-1} \subset P'$, and $W_\ell = E \times \mathcal{B}_\ell \times \{1\}$. We can deform W_ℓ into $W_{\ell-1}$ in P' as follows. First deform W_ℓ onto $f_\ell(A_\ell \times E) \times \mathcal{B}_{\ell-1}$ by moving it down the mapping cylinder $M = ((E \times B_m \times [0,1]) \coprod (C_\ell \times B_\ell))/\sim$ of P' and onto the base $C_\ell \times B_\ell$, then deform it onto $* \times \mathcal{B}_{\ell-1}$ in $C_\ell \times B_\ell$ using the nullhomotopy of f_ℓ. Finally, move $\mathcal{B}_{\ell-1}$ back up towards the top of the mapping cylinder M and into $W_{\ell-1}$. Composing these deformations for $\ell = m-1, \dots, 1$ gives a contraction in P' of W_{m-1} to a point. Thus, g is nullhomotopic, as is f_m. Since the lemma holds for the base case $m = 2$, $\mathrm{cat}(P) = \max\{1, \mathrm{cat}(P'), \mathrm{cat}(C_m)\} \leq \max\{1, \mathrm{cat}(C_1), \dots, \mathrm{cat}(C_m)\}$.

□

Lemma 2.5. *Fix $m \geq 2$, and for $i = 1, \dots, m$, let A_i, C_i, E be basepointed CW-complexes, E is path-connected, and let $B_i = \prod_{j \neq i} A_j$. Suppose $A_i \times E \xrightarrow{f_i} C_i$ are maps such that the restriction $(f_i)_{|A_i \times *}$*

of f_i to $A_i \times *$ is nullhomotopic, and P is the homotopy pushout of the maps $A_i \times E \times B_i \xrightarrow{f_i \times \mathbb{1}_{B_i}} C_i \times B_i$ for $i = 1, \dots, m$. Then

$$\mathrm{cat}(P) \le \max\{1, \mathrm{cat}(C_1), \dots, \mathrm{cat}(C_m)\} + \mathcal{C}\mathrm{at}(E).$$

Moreover, if each $C_i \times B_i$ is a subcomplex of some CW-complex X_i such that $X_i - C_i \times B_i$ is contractible in X_i, and P' is the homotopy pushout of the maps $A_i \times E \times B_i \xrightarrow{f_i \times \mathbb{1}_{B_i}} C_i \times B_i \hookrightarrow X_i$ for $i = 1, \dots, m$, then also

$$\mathrm{cat}(P') \le \max\{1, \mathrm{cat}(C_1), \dots, \mathrm{cat}(C_m)\} + \mathcal{C}\mathrm{at}(E).$$

Proof (part 1). Let $\mathcal{B} = A_1 \times \cdots \times A_m$, and $\mathcal{D} = \coprod_{i=1,\dots,m}(C_i \times B_i)$, and let $\mathcal{S}_t$ for $t < 1$ be m copies of the interval $[t, 1]$ glued together at the endpoints 1, and $\mathcal{S}'_t$ be its interior, namely, m copies of $(t, 1]$ glued at 1.

Let $k = \mathcal{C}\mathrm{at}(E)$ and take $E' \simeq E$ to be such that $k = \mathrm{gcat}(E')$. Then P is homotopy equivalent to the homotopy pushout Q of the maps $A_i \times E' \times B_i \xrightarrow{f'_i \times \mathbb{1}_{B_i}} C_i \times B_i$ for $i = 1, \dots, m$, where $A_i \times E' \xrightarrow{f'_i} C_i$ is the composite of f_i with the homotopy equivalence $A_i \times E' \xrightarrow{\mathbb{1}_{A_i} \times \simeq} A_i \times E$. Since E is path-connected and gcat() is unaffected by attaching an interval $[0, 1]$ to a space, we may assume that the homotopy equivalence $E' \xrightarrow{\simeq} E$ is basepointed for some $* \in E'$.

Let $U_0, \dots, U_k$ be an open cover of E' with each U_i a contractible subspace. Take Q_j to be the homotopy pushout of $A_i \times U_j \times B_i \xrightarrow{g_{i,j} \times \mathbb{1}_{B_i}} C_i \times B_i$ for $i = 1, \dots, m$, where $g_{i,j}$ is the restriction of f'_i to $A_i \times U_j$, and let $V_j = Q_j - \mathcal{D} \cong U_j \times \mathcal{B} \times \mathcal{S}'_0$. Since $g_{i,j} \times \mathbb{1}_{B_i}$ restricts $f'_i \times \mathbb{1}_{B_i}$, Q_j is a subspace of Q and V_j is open in Q. Moreover, we may contract V_j in Q to a point as follows. Let $\mathcal{B}_0 = *$ and $\mathcal{B}_\ell = \prod_{i \le \ell} A_i \subseteq \mathcal{B}$, and take the subspace $W_\ell = * \times \mathcal{B}_\ell \times \{1\}$ of $E' \times \mathcal{B} \times \mathcal{S}'_0 \subset Q$. We can deform W_ℓ into $W_{\ell-1}$ in Q, first by deforming W_ℓ onto $f'_\ell(A_\ell \times *) \times \mathcal{B}_{\ell-1}$ by moving it down the mapping cylinder $M = ((E' \times \mathcal{B} \times [0,1]) \coprod (C_\ell \times B_\ell))/\sim$ of Q and onto $C_\ell \times B_\ell$, then deforming it onto $* \times \mathcal{B}_{\ell-1}$ in $C_\ell \times B_\ell$ using the nullhomotopy of $(f'_\ell)_{|A_\ell \times *}$, and finally, moving $\mathcal{B}_{\ell-1}$ back up towards the top of the mapping cylinder M and into $E' \times \mathcal{B} \times \{1\}$. Composing these deformations of W_ℓ into $W_{\ell-1}$ in Q for $\ell = m, m-1, \dots, 1$, and deforming V_j onto W_m using contractibility of U_j and $\mathcal{S}'_0$ (onto 1), gives our contraction of V_j in Q to a point.

Assume $* \in U_0$. Since U_0 is contractible and $(g_{i,0})_{|A_i \times *} = (f'_i)_{|A_i \times *}$ is nullhomotopic, $g_{i,0}$ is also nullhomotopic. Lemma 2.4 then applies to Q_0, namely, we have

$$\mathrm{cat}(Q_0) \le \max\{1, \mathrm{cat}(C_1), \dots, \mathrm{cat}(C_m)\}.$$

Let $\mathcal{R} = \mathcal{S}'_0 - \mathcal{S}_{\frac{1}{2}} \cong \coprod_{i=1,\dots,m}(0, \frac{1}{2})$ and $\bar{\mathcal{R}} = \mathcal{S}_0 - \mathcal{S}_{\frac{1}{2}} \cong \coprod_{i=1,\dots,m}[0, \frac{1}{2})$, and consider the open subspace $Q'_0 = Q_0 \cup (E' \times \mathcal{B} \times \mathcal{R})$ of Q. Notice Q'_0 deformation retracts in the weak sense onto Q_0 by deformation retracting the subspace of Q_0

$$\big((E' \times \mathcal{B} \times \bar{\mathcal{R}}) \coprod \mathcal{D}\big)/\sim$$

onto $\mathcal{D}$, this being done by contracting each copy of $[0, \frac{1}{2})$ in the factor $\bar{\mathcal{R}}$ to 0, at the same time expanding $(U_0 \times \mathcal{B}) \times \mathcal{S}_{\frac{1}{2}}$ in Q'_0 by expanding each copy of $[\frac{1}{2}, 1]$ in the factor $\mathcal{S}_{\frac{1}{2}}$ outwards to $[0, 1]$. Then $\mathrm{cat}(Q'_0) = \mathrm{cat}(Q_0)$. So take $k' = \max\{1, \mathrm{cat}(C_1), \dots, \mathrm{cat}(C_m)\}$ and $\{U'_0, \dots, U'_{k'}\}$ to be a categorical cover for Q'_0. Notice that U'_i is open in Q since Q'_0 is, and $Q = \bigcup_{j=0}^n Q_j = Q'_0 \cup \bigcup_{j=1}^n V_j$. As each V_j is open and contractible in Q, then $\{U'_0, \dots, U'_{k'}, V_1, \dots, V_k\}$ is a categorical cover of Q. Therefore $\mathrm{cat}(P) = \mathrm{cat}(Q) \le k' + k$.

□

Proof (part 2). Since $C_i \times B_i$ is a subcomplex of X_i, P is a subspace of P' with

$$P' - P = \coprod_{i=1,\dots,m} (X_i - C_i \times B_i),$$

so $P' - P$ is open and contractible in P'. Notice each V_j is an open (and contractible) subset of P', while $S_j = U'_j \cap (\coprod_{i=1,\dots,m} X_i)$ is an open subset of $\mathcal{D}$. By Lemma 2.1, there exists an open subset R_j of $\coprod_{i=1,\dots,m} X_i$ that deformation retracts onto S_j such that $R_j \cap \mathcal{D} = S_j$. Then $R'_j = R_j \coprod (U'_j - S_j)$ is an open subset of P' that deformation retracts onto U'_j, thus is contractible in P'. Since $P' - P$ and V_j are both open in P', and $(P' - P) \cap V_j = \varnothing$, then the subspace $(P' - P) \coprod V_j$ is contractible in P'. We can therefore take $\{R'_0, \dots, R'_{k'}, (V_1 \coprod (P' - P)), V_2, \dots, V_k\}$ as a categorical cover for P', so $\mathrm{cat}(P') \le k' + k$.

□

3. Moment-Angle Complexes

Given a simplicial complex K on vertex set $[n]$ and a sequence of of pairs of spaces

$$\mathcal{S} = ((X_1, A_1), \dots, (X_n, A_n)),$$

$A_i \subseteq X_i$, the *polyhedral product* $\mathcal{S}^K$ is the subspace of $X^{\times n}$ defined by

$$\mathcal{S}^K = \bigcup_{\sigma \in K} Y_1^\sigma \times \cdots \times Y_n^\sigma,$$

where $Y_i^\sigma = X_i$ if $i \in \sigma$, or $Y_i^\sigma = A_i$ if $i \notin \sigma$. If the pairs (X_i, A_i) are all equal to the same pair (X, A), we usually write $\mathcal{S}^K$ as $(X, A)^K$. The *moment-angle complex* $\mathcal{Z}_K$ is defined as the polyhedral product $(D^2, \partial D^2)^K$, and the *real moment-angle complex* $\mathbb{R}\mathcal{Z}_K$ is the polyhedral product $(D^1, \partial D^1)^K$.

The *join* of two simplicial complexes K and L is the simplicial complex $K * L = \{\sigma \sqcup \tau \mid \sigma \in K, \tau \in L\}$, and one has $|K * L| \cong |K| * |L| \simeq \Sigma |K| \wedge |L|$ and $\mathcal{Z}_{K*L} \cong \mathcal{Z}_K \times \mathcal{Z}_L$. If $I \subseteq [n]$, $K_I = \{\sigma \in K \mid \sigma \subseteq I\}$ denotes the *full subcomplex* of K on vertex set I, in which case $\mathcal{Z}_{K_I}$ is a retract of $\mathcal{Z}_K$. Notice that if K_I and L_J are full subcomplexes of K and L, then $K_I * L_J$ is the full subcomplex $(K * L)_{I \sqcup J}$ of $K * L$. As a convention, we let $\mathcal{Z}_\varnothing = *$ when $\varnothing$ is on empty vertex set.

We let S^0 denote both the 0-sphere and the simplicial complex $\partial \Delta^1$ consisting of only two vertices. Generally, we assume our simplicial complexes (except $\varnothing$) are non-empty and have no ghost vertices, unless stated otherwise. Under this assumption, it is not difficult to see that the following holds.

3.1. **The Hochster Theorem.** When R is a field or $\mathbb{Z}$, it was shown in [11, 20, 3, 32] that there are isomorphisms of graded commutative algebras

$$H^*(\mathcal{Z}_K; R) \cong \mathrm{Tor}_{R[v_1,\dots,v_n]}(R[K], R) \cong \bigoplus_{I \subseteq [n]} \tilde{H}^*(\Sigma^{|I|+1} |K_I|; R). \tag{1}$$

The isomorphism on the left is induced by a quasi-isomorphism of DGAs between the Koszul complex of the Stanley-Reisner ring $R[K]$ and the cellular cochain complex of $\mathcal{Z}_K$ with coefficients in R. The multiplication on the right is given by maps $H^*(K_I) \otimes H^*(K_J) \longrightarrow H^{*+1}(K_{I \cup J})$ that are zero when $I \cap J \neq \varnothing$, otherwise they are induced by maps $\iota_{I,J} : |K_{I \cup J}| \longrightarrow |K_I * K_J| \cong |K_I| * |K_J| \simeq \Sigma |K_I| \wedge |K_J|$ geometrically realizing the canonical inclusions $K_{I \cup J} \hookrightarrow K_I * K_J$. One can iterate so that any length ℓ product $\bigotimes_{i=1}^{\ell} H^*(K_{I_i}) \longrightarrow H^{*+\ell-1}(K_{I_1 \cup \cdots \cup I_\ell})$ is induced by the inclusion

$$\iota_{I_1,\dots,I_\ell} : |K_{I_1 \cup \cdots \cup I_\ell}| \hookrightarrow |K_{I_1} * \cdots * K_{I_\ell}|$$

where the I_i's are mutually disjoint.

3.2. **A Necessary Condition.** Suppose $\mathrm{cat}(\mathcal{Z}_K) \le \ell - 1$, so cup products of length l vanish in $H^+(\mathcal{Z}_K)$. Then in light of the Hochster theorem, the inclusions $\iota_{I_1,\dots,I_\ell}$ must induce trivial maps on cohomology. In fact, their suspensions must be nullhomotopic by the following argument.

Let $\widehat{\mathcal{Z}}_K = \mathcal{Z}_K / \{(x_1, \dots, x_n) \in \mathcal{Z}_K \mid \text{at least one } x_i = *\}$. Fix $m = |I_1 \cup \cdots \cup I_\ell|$, $Y = \mathcal{Z}_{K_{I_1 \cup \cdots \cup I_\ell}}$, and $\hat{Y} = \widehat{\mathcal{Z}}_{K_{I_1 \cup \cdots \cup I_\ell}}$. Since Y is a retract of $\mathcal{Z}_K$, $\mathrm{cat}(Y) \le \ell - 1$. Recall from [36] that a path-connected basepointed CW-complex such as Y satisfies $\mathrm{cat}(Y) \le \ell - 1$ if and only if there is a map $Y \xrightarrow{\psi} FW_\ell(Y)$ such that the diagonal map $Y \xrightarrow{\triangle} Y^{\times \ell}$ factors up to homotopy as $Y \xrightarrow{\psi} FW_\ell(Y) \xrightarrow{include} Y^{\times \ell}$. Here $FW_\ell(Y) = \{(y_1, \dots, y_\ell) \in Y^{\times \ell} \mid \text{at least one } y_i = *\}$ is the fat wedge.

This implies the reduced diagonal map $\bar{\Delta}\colon Y \xrightarrow{\Delta} Y^{\times\ell} \longrightarrow Y^{\times\ell}/FW_\ell(Y) \xrightarrow{\cong} Y^{\wedge\ell}$ is nullhomotopic. Then so is $\zeta\colon Y \xrightarrow{\bar{\Delta}} Y^{\wedge\ell} \longrightarrow \bigwedge_j \mathcal{Z}_{K_{I_j}} \longrightarrow \bigwedge_j \widehat{\mathcal{Z}}_{K_{I_j}}$, where the second last map is the smash of the coordinate-wise projection maps onto each $\mathcal{Z}_{K_{I_j}}$, and the last map is the smash of quotient maps. This last nullhomotopic map ζ coincides with $Y \xrightarrow{q} \hat{Y} \xrightarrow{\hat{\iota}} \bigwedge_j \widehat{\mathcal{Z}}_{K_{I_j}}$, where q is the quotient map and $\hat{\iota}$ is the inclusion given simply by rearranging coordinates. Moreover, $\hat{\iota}$ is homeomorphic to $\Sigma^{m+1}\iota_{I_1,\dots,I_\ell}$ and Σq has a right homotopy inverse (c.f. [1], and also the proof of Proposition 2.5 and pg. 23 in [4]). It follows that $\Sigma^{m+1}\iota_{I_1,\dots,I_\ell}$ is nullhomotopic.

3.3. **Skeleta and Suspension on Coordinates.** Let $K^{(i)}$ denote the i-skeleton of K, and $K^{(-1)} = \varnothing$ on vertex set $[n]$. An inclusion of simplicial complexes $L \hookrightarrow K$ induces a canonical inclusion of CW-complexes $\mathcal{Z}_L \hookrightarrow \mathcal{Z}_K$. This gives $\mathcal{Z}_{K^{(i)}}$ and $\mathcal{Z}_{K^{(-1)}} = (\partial D^2)^{\times n} = (S^1)^{\times n}$ as subcomplexes of $\mathcal{Z}_K$.

Lemma 3.1 (Corollary 3.3 in [30]). *If K is on vertex set $[n]$ with no ghost vertices, then $\mathcal{Z}_{K^{(-1)}} = (\partial D^2)^{\times n}$ is contractible in $\mathcal{Z}_K$.* □

Lemma 3.2. *If $0 \le l \le \dim K$, then $\mathcal{Z}_{K^{(\ell)}} - \mathcal{Z}_{K^{(\ell-1)}}$ is contractible in $\mathcal{Z}_K$.*

Proof. We have a decomposition

$$\mathcal{Z}_{K^{(\ell)}} - \mathcal{Z}_{K^{(\ell-1)}} = \coprod_{\sigma\in K,\,|\sigma|=\ell+1} \tilde{Y}_1^\sigma \times \cdots \times \tilde{Y}_n^\sigma$$

where $\tilde{Y}_i^\sigma = D^2 - \partial D^2$ if $i \in \sigma$ and $\tilde{Y}_i^\sigma = \partial D^2$ if $i \notin \sigma$. This being a disjoint union of open subspaces of $\mathcal{Z}_{K^{(l)}}$, each of which can be deformed into $\mathcal{Z}_{K^{(-1)}}$ in $\mathcal{Z}_K$ (by contracting $\tilde{Y}_i^\sigma$ to a point in ∂D^2 whenever $i \in \sigma$). Thus $\mathcal{Z}_{K^{(\ell)}} - \mathcal{Z}_{K^{(\ell-1)}}$ can also be deformed into $\mathcal{Z}_{K^{(-1)}}$. Then $\mathcal{Z}_{K^{(\ell)}} - \mathcal{Z}_{K^{(\ell-1)}}$ is contractible in $\mathcal{Z}_K$ by Lemma 3.1.

□

Lemma 3.3. *If $-1 \le j \le \dim K$, then*

$$\operatorname{cat}(\mathcal{Z}_K) \le \operatorname{cat}(\mathcal{Z}_{K^{(j)}} \hookrightarrow \mathcal{Z}_K) + \dim K - j.$$

In particular, $\operatorname{cat}(\mathcal{Z}_K) \le \dim K + 1$.

Remark: Lemma 3.2 and 3.3 can be generalized to any filtration $L_j \subseteq \cdots \subseteq L_k = K$ satisfying $\partial\sigma \subseteq L_i$ whenever $\sigma \in L_{i+1}$ (in place of the skeletal filtration).

Proof. The skeletal filtration $K^{(j)} \subseteq \cdots \subseteq K^{(\dim K)} = K$ induces a filtration of subcomplexes $\mathcal{Z}_{K^{(j)}} \subseteq \cdots \subseteq \mathcal{Z}_K$, and for $0 \le j \le \dim K$, $\mathcal{Z}_{K^{(j)}} - \mathcal{Z}_{K^{(j-1)}}$ is contractible in $\mathcal{Z}_K$ by Lemma 3.2. The result then follows using Lemma 2.2. In particular, when $j = -1$, we get $\operatorname{cat}(\mathcal{Z}_K) \le \dim K + 1$ since $\operatorname{cat}(\mathcal{Z}_{K^{(-1)}} \hookrightarrow \mathcal{Z}_K) = 0$ by Lemma 3.1. □

Proposition 3.4. *Let $\mathcal{S} = ((X_1, A_1), \dots, (X_n, A_n))$ and $\mathcal{T} = ((\Sigma^{m_1}X_1, \Sigma^{m_1}A_1), \dots, (\Sigma^{m_n}X_n, \Sigma^{m_n}A_n))$ be sequences of pairs of spaces for some integers m_i and connected basepointed X_i. Then for any K (with no ghost vertices),*

$$\operatorname{cat}(\mathcal{T}^K) \le \operatorname{cat}(\mathcal{S}^K).$$

Proof. Let K be on vertex set $[n]$, $k = \operatorname{cat}(\mathcal{S}^K)$, and take a categorical cover $\{U_0, \dots, U_k\}$ of $\mathcal{S}^K$. For any open subset V of $\mathcal{S}^K$, define the following open subset V^1 of $\mathcal{T}^K$

$$V^1 = \left\{ ((t_1, x_1), \dots, (t_n, x_n)) \in \prod_{i=1}^n \Sigma^{m_i} X_i \;\middle|\; (x_1, \dots, x_n) \in V,\, t_i \in D^{m_i} \right\}.$$

In particular, $\mathcal{T}^K = (\mathcal{S}^K)^1$. Then $\{U_0^1, \dots, U_k^1\}$ is an open cover of $\mathcal{T}^K$. Since K has no ghost vertices, $A_i \subseteq X_i$, and each X_i is path-connected, then $\mathcal{S}^K$ is path-connected. Since $\Sigma^{m_i}X_i$ is the reduced suspension of the basepointed space X_i, we have identifications $(t, *) \sim * \in \Sigma^{m_i}X_i$. Then we can define a contraction of U_i^1 in $\mathcal{T}^K$ by contracting U_i in $\mathcal{S}^K$ to a point p and homotoping p to the basepoint $(*,\dots,*) \in \mathcal{S}^K$. Therefore, $\{U_0^1,\dots,U_k^1\}$ is a categorical cover of $\mathcal{Z}_K$.

□

Notice that the $(i+1)$-skeleton $(\mathbb{R}\mathcal{Z}_K)^{(i+1)}$ of $\mathbb{R}\mathcal{Z}_K$ is equal to $\mathbb{R}\mathcal{Z}_{K^{(i)}}$ (this is not true for the complex moment-angle complex $\mathcal{Z}_K$).

Corollary 3.5. *For any K (with no ghost vertices),*

$$\operatorname{cat}(\mathcal{Z}_K) \le \operatorname{cat}(\mathbb{R}\mathcal{Z}_K) \tag{2}$$

and if $\mathbb{R}\mathcal{Z}_K$ is not contractible and $i \ge 0$, then

$$\operatorname{cat}(\mathcal{Z}_{K^{(i)}}) \le \operatorname{cat}(\mathbb{R}\mathcal{Z}_{K^{(i)}}) \le \operatorname{cat}(\mathbb{R}\mathcal{Z}_K). \tag{3}$$

Proof. Inequality (2) and the first inequality in (3) follow from Proposition 3.4. By the main corollary of Theorem 1 in [17], the i-skeleton $X^{(i)}$ of any connected non-contractible CW-complex X satisfies that $\operatorname{cat}(X^{(i)}) \le \operatorname{cat}(X)$. Since $(\mathbb{R}\mathcal{Z}_K)^{(i+1)} = \mathbb{R}\mathcal{Z}_{K^{(i)}}$ holds for real moment-angle complexes, the last inequality follows.

□

It is plausible that the second bound can be strengthened to $\operatorname{cat}(\mathcal{Z}_{K^{(i)}}) \le \operatorname{cat}(\mathcal{Z}_K)$. In any case, even if it is true, we will sometimes need a sharper bound.

Let X and Y be path-connected paracompact spaces, and $\mathcal{U} = \{U_0,\dots,U_k\}$ and $\mathcal{V} = \{V_0,\dots,V_\ell\}$ be categorical covers of X and Y, respectively. We recall James' construction of a categorical cover $\mathcal{W} = \{W_0,\dots,W_{k+\ell}\}$ of $X \times Y$ from the covers $\mathcal{U}$ and $\mathcal{V}$ (see [36], page 333).

Let $\{\pi_j\}_{j\in\{0,\dots,k\}}$ be a partition of unity subordinate to the cover $\mathcal{U}$. For any subset $S \subseteq \{0,\dots,k\}$, define

$$W_{\mathcal{U}}(S) = \{x \in X \mid \pi_j(x) > \pi_i(x) \text{ for any } j \in S \text{ and } i \notin S\},$$

and for any point $p \in X$, let

$$S_{\mathcal{U}}(p) = \{j \in \{0,\dots,k\} \mid \pi_j(p) > 0\}$$

(since the context is clear, let $W(S) = W_{\mathcal{U}}(S)$ and $S(p) = S_{\mathcal{U}}(p)$). Then $W(S)$ is an open subset of X and $X = \bigcup_{S\subseteq\{0,\dots,k\}} W(S)$ (given $x \in X$, $x \in W(S)$ where $S = \{i \mid \pi_i(x) = \max\{\pi_1(x),\dots,\pi_k(x)\}\}$). Moreover, $W(S') \cap W(S) = \emptyset$ when $S \notin S'$ and $S' \notin S$ (in particular, when $|S| = |S'|$ and $S \ne S'$), and $W(S) \subseteq U_j$ whenever $j \in S$. Therefore $W(S)$ is contractible in X. Then so is the disjoint union of open sets

$$U_i' = \coprod_{\substack{S=S(p) \text{ for some } p\in X\\ |S|=i+1}} W(S). \tag{4}$$

Since $W(S) = \emptyset$ when $S \ne S(p)$ for every $p \in X$, the set $\{U_0',\dots,U_k'\}$ forms a categorical cover of X. We obtain a categorical cover $\{V_0',\dots,V_\ell'\}$ of Y from $\mathcal{V}$ by an analogous construction.

Now let $\bar{U}_i = U_{k-i}' \cup \cdots \cup U_k'$ and $\bar{V}_j = V_{\ell-j}' \cup \cdots \cup V_\ell'$, and for $-1 \le s \le k+\ell$, let $C_{-1} = \emptyset$ and

$$C_s = \bigcup_{\substack{i+j=s\\ i\le k,\, j\le \ell}} \bar{U}_i \times \bar{V}_j$$

Take $W_s = C_s - C_{s-1}$. Notice that

$$W_s = \coprod_{\substack{i+j=s\\ i\le k,\, j\le\ell}} U_i' \times V_j'. \tag{5}$$

This defines a categorical cover $\mathcal{W}$ of $X \times Y$.

Given subcomplexes $B \subseteq Y$ and $A \subseteq X$, consider the polyhedral product

$$\mathcal{X}^{S^0} = X \times B \cup_{A\times B} A \times Y$$

over the sequence $\mathcal{X} = ((X,A),(Y,B))$.

Lemma 3.6. *If $X - A$ is contractible in X and $Y - B$ is contractible in Y, then*

$$\mathrm{cat}(\mathcal{X}^{S^0}) \le \mathrm{cat}(A) + \mathrm{cat}(B) + 1.$$

Proof. Suppose we have categorical covers $\{R_1,\dots,R_k\}$ and $\{S_1,\dots,S_\ell\}$ of A, and B. By Lemma 2.1, we have open subsets $U_i \subseteq X$ and $V_i \subseteq Y$ such that U_i and V_i deformation retract onto R_i and S_i respectively, and $U_i \cap A = R_i$ and $V_i \cap B = S_i$ for $i \ge 1$. Then taking $U_0 = X - A$ and $V_0 = Y - B$, $\mathcal{U} = \{U_0,\dots,U_k\}$ and $\mathcal{V} = \{V_0,\dots,V_\ell\}$ are categorical covers of X and Y.

Notice that

$$X \times Y - U_0 \times V_0 = \mathcal{X}^{S^0},$$

and since $R_i = U_i \cap A = U_i - U_0$ and $S_j = V_j \cap B = V_j - V_0$ for $i,j \ge 1$,

$$D_{i,j} := U_i \times V_j - U_0 \times V_0 = (R_i \times V_j) \cup_{R_i \times S_j} (U_i \times S_j). \tag{6}$$

Notice that $D_{i,j}$ is contractible in $\mathcal{X}^{S^0}$ by deformation retracting the factor U_i onto R_i and V_j onto S_j, then contracting $R_i \times S_j$ in $A \times B$.

Take the categorical cover $\mathcal{W} = \{W_0,\dots,W_{k+\ell}\}$ of $X \times Y$ constructed from $\mathcal{U}$ and $\mathcal{V}$ as above. By (5),

$$W_s - U_0 \times V_0 = \coprod_{\substack{i+j=s\\ i\le k,\, j\le \ell}} (U_i' \times V_j' - U_0 \times V_0),$$

and by (4),

$$U_i' \times V_j' - U_0 \times V_0 = \coprod_{\substack{S=S_{\mathcal{U}}(p) \text{ for some } p\in X\\ T=S_{\mathcal{V}}(q) \text{ for some } q\in Y\\ |S|=i+1,\,|T|=j+1}} (W_{\mathcal{U}}(S) \times W_{\mathcal{V}}(T) - U_0 \times V_0).$$

These are disjoint unions of open subsets of $\mathcal{X}^{S^0}$. Since $W_{\mathcal{U}}(S)$ is contained in some $U_{i'}$ and $W_{\mathcal{V}}(T)$ is contained in some $V_{j'}$, it follows that $(W_{\mathcal{U}}(S) \times W_{\mathcal{V}}(T) - U_0 \times V_0)$ is contained in $D_{i',j'}$, so it is contractible in $\mathcal{X}^{S^0}$. Therefore, so are the disjoint unions $U_i' \times V_j' - U_0 \times V_0$ and $W_s - U_0 \times V_0$. Moreover, since $W_{k+\ell} = U_k' \times V_\ell'$, and $U_k' = W_{\mathcal{U}}(\{0,\dots,k\})$ and $V_\ell' = W_{\mathcal{V}}(\{0,\dots,\ell\})$ are contained in $U_{i'}$ and $V_{j'}$ respectively for each $i' \in \{0,\dots,k\}$ and $j' \in \{0,\dots,\ell\}$, $W_{k+\ell} - U_0 \times V_0 = \varnothing$. Then

$$\{(W_0 - U_0 \times V_0),\dots,(W_{k+\ell-1} - U_0 \times V_0)\}$$

is a categorical cover of $\mathcal{X}^{S^0}$.

□

Corollary 3.7. *Let K and L be simplicial complexes with $d = \dim K$ and $d' = \dim L$ (so $\dim K * L = d + d' + 1$). Then*

$$\mathrm{cat}(\mathcal{Z}_{(K*L)^{(d+d')}}) \le \mathrm{cat}(\mathcal{Z}_{K^{(d-1)}}) + \mathrm{cat}(\mathcal{Z}_{L^{(d'-1)}}) + 1.$$

Proof. Notice $(K*L)^{(d+d')} = (K * L^{(d'-1)}) \cup_{(K^{(d-1)} * L^{(d'-1)})} (K^{(d-1)} * L)$, $\mathcal{Z}_{K*L} = \mathcal{Z}_K \times \mathcal{Z}_L$, so

$$\begin{aligned}\mathcal{Z}_{(K*L)^{(d+d')}} &= (\mathcal{Z}_{K*L^{(d'-1)}}) \cup_{\mathcal{Z}_{(K^{(d-1)}*L^{(d'-1)})}} (\mathcal{Z}_{K^{(d-1)}*L})\\ &= (\mathcal{Z}_K \times \mathcal{Z}_{L^{(d'-1)}}) \cup_{\mathcal{Z}_{K^{(d-1)}} \times \mathcal{Z}_{L^{(d'-1)}}} (\mathcal{Z}_{K^{(d-1)}} \times \mathcal{Z}_L),\end{aligned}$$

and $\mathcal{Z}_K - \mathcal{Z}_{K^{(d-1)}}$ and $\mathcal{Z}_L - \mathcal{Z}_{L^{(d'-1)}}$ are contractible in $\mathcal{Z}_K$ and $\mathcal{Z}_L$ by Lemma 3.2. The result follows by Lemma 3.6.

□

3.4. **Missing Face Complexes.** Take K on vertex set $[n]$. We fix the basepoint in the unreduced suspension $\Sigma|K| = (|K| \times [0,1])/\sim$ to be the tip of the double cone corresponding to 1 under the indentifications $(x,0) \sim 0$ and $(x,1) \sim 1$. Let $MF(K) = \{\sigma \subseteq [n] \mid \sigma \notin K, \partial\sigma \subseteq K\}$ be the collection of (minimal) missing faces of K. We will need a somewhat more flexible alternative to the *directed missing face complexes* defined in [27].

Definition 3.8. K on vertex set $[n]$ is called a *homology missing face complex* (or *HMF-complex*) if for each non-empty $I \subseteq [n]$, K_I is a simplex or there exists a subcollection $\mathcal{C}_I \subseteq MF(K_I)$ such that the wedge sum of suspended inclusions

$$\gamma_I\colon \bigvee_{\sigma\in\mathcal{C}_I} \Sigma|\partial\sigma| \longrightarrow \Sigma|K_I|$$

induces an isomorphism on homology (therefore it is a homotopy equivalence since it is a map between suspensions).

Remark 3.9. Given $H_*(K_I)$ is torsion-free, since each $\Sigma|\partial\sigma|$ is a sphere, one needs only to find γ_I that induces surjection on homology in order for K to be an HMF-complex.

Proposition 3.10. *If K is an HMF-complex, then $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres or is contractible. Therefore* $\mathrm{cat}(\mathcal{Z}_K) \leq 1$ *and* $\mathcal{C}\mathrm{at}(\mathcal{Z}_K) \leq 1$.

Proof. For each $I \subseteq [n]$, either K_I is a simplex, boundary of a simplex, or else for each $\sigma \in \mathcal{C}_I$, we can pick an $i_\sigma \in I$ such that $\partial\sigma \subseteq K_{I-\{i_\sigma\}}$, so each inclusion $|\partial\sigma| \longrightarrow |K_I|$ factors through inclusions $|\partial\sigma| \longrightarrow |K_{I-\{i_\sigma\}}| \longrightarrow |K_I|$. Take the composite

$$f\colon \Sigma|K_I| \xrightarrow{\gamma_I^{-1}} \bigvee_{\sigma\in\mathcal{C}_I} \Sigma|\partial\sigma| \longrightarrow \bigvee_{i\in I} \Sigma|K_{I-\{i_\sigma\}}| \longrightarrow \Sigma|K_I|$$

where γ_I^{-1} is a homotopy inverse of γ_I, the second last map includes the summand $\Sigma|\partial\sigma|$ into the summand $\Sigma|K_{I-\{i_\sigma\}}|$, and the last map is the standard inclusion on each summand. Since the composite of the last two maps is γ_I, f is a homotopy equivalence. Then K is an *extractible complex* as defined in [33]. Therefore $\mathcal{Z}_K$ is homotopy equivalent to a wedge of spheres or contractible by Corollary 3.3 therein. □

3.5. **Gluing and Connected Sum.** If L and K are simplicial complexes and C is a full subcomplex common to both L and K, then we obtain a new simplicial complex $L \cup_C K$ by gluing L and K along C. One can always glue along simplices since they are always full subcomplexes. When $C = \varnothing$, $L \cup_C K$ is just the disjoint union $L \sqcup K$.

Given $\sigma \in K$, define the *deletion* of the face σ from K to be the simplicial complex given by

$$K\backslash\sigma = \{\tau \in K \mid \sigma \nsubseteq \tau\}.$$

If σ is a common face of L and K, define the *connected sum* $L\#_\sigma K$ to be the simplicial complex $(L\backslash\sigma) \cup_{\partial\sigma} (K\backslash\sigma)$. In other words, $L\#_\sigma K$ is obtained by deleting σ from L and K and gluing along the boundary $\partial\sigma$. As a convention, we let $\mathcal{Z}_\varnothing = *$ when $\varnothing$ is on empty vertex set.

Proposition 3.11. *If C is a (possibly empty) full subcomplex common to $K_1,\dots,K_m$, then*

$$\mathrm{cat}(\mathcal{Z}_{K_1\cup_C\cdots\cup_C K_m}) \leq \max\{1, \mathrm{cat}(\mathcal{Z}_{K_1}),\dots,\mathrm{cat}(\mathcal{Z}_{K_m})\} + \mathcal{C}\mathrm{at}(\mathcal{Z}_C).$$

Moreover, if each K_i is the $d_i - 1$ skeleton of some d_i dimensional simplicial complex $\bar{K}_i$, and C is also a full subcomplex of each $\bar{K}_i$, then

$$\mathrm{cat}(\mathcal{Z}_{\bar{K}_1\cup_C\cdots\cup_C \bar{K}_m}) \leq \max\{1, \mathrm{cat}(\mathcal{Z}_{K_1}),\dots,\mathrm{cat}(\mathcal{Z}_{K_m})\} + \mathcal{C}\mathrm{at}(\mathcal{Z}_C).$$

Proof. Let K_i be on vertex set $[n_i]$, and C has ℓ vertices. If C is on vertex set $[n_i]$, possibly with ghost vertices, the inclusion $C \hookrightarrow K_i$ induces a coordinate-wise inclusion $(\partial D^2)^{\times n_i - \ell} \times \mathcal{Z}_C \xhookrightarrow{f_i} \mathcal{Z}_{K_i}$. By Lemma 3.1, f_i is nullhomotopic when restricted to $(\partial D^2)^{\times n_i - \ell} \times *$. Let $N_i = \Sigma_{j \neq i} n_j$. Note $\mathcal{Z}_{K_1 \cup_C \cdots \cup_C K_m}$ is the pushout of $(\partial D^2)^{\times n_i - \ell} \times \mathcal{Z}_C \times (\partial D^2)^{\times N_i - \ell} \xhookrightarrow{f_i \times \mathbb{1}} \mathcal{Z}_{K_i} \times (\partial D^2)^{\times N_i - \ell}$ for $i = 1, \ldots, m$. Since each of these maps are inclusions of subcomplexes, $\mathcal{Z}_{K_1 \cup_C \cdots \cup_C K_m}$ is homotopy equivalent to the homotopy pushout P of these maps. The first inequality therefore follows from the first part of Lemma 2.5.

By Lemma 3.2, $\mathcal{Z}_{\bar{K}_i} - \mathcal{Z}_{K_i}$ is contractible in $\mathcal{Z}_{\bar{K}_i}$, so the second equality follows from the second part of Lemma 2.5.

□

Example 3.12. In particular, when C is a simplex $\operatorname{cat}(\mathcal{Z}_{L \cup_C K}) \leq \max\{1, \operatorname{cat}(\mathcal{Z}_L), \operatorname{cat}(\mathcal{Z}_K)\}$ and $\operatorname{cat}(\mathcal{Z}_{\bar{L} \cup_C \bar{K}}) \leq \max\{1, \operatorname{cat}(\mathcal{Z}_L), \operatorname{cat}(\mathcal{Z}_K)\}$ since $\mathcal{Z}_C$ is contractible. These also hold when C is the empty simplex and $\mathcal{Z}_C = *$ (in which case $\bar{L} \cup_C \bar{K} = \bar{L} \sqcup \bar{K}$ and $L \cup_C K = L \sqcup K$). When C is the boundary of a simplex, $\operatorname{cat}(\mathcal{Z}_{L \cup_C K}) \leq \max\{1, \operatorname{cat}(\mathcal{Z}_L), \operatorname{cat}(\mathcal{Z}_K)\} + 1$ since $\mathcal{Z}_C$ here is a sphere.

The bound in Proposition 3.11 is not always optimal, sometimes far from it. If K and Δ^{n-1} are on vertex set $[n]$ and L is formed by gluing Δ^{n-1} and $\{n+1\} * K$ along K, then $\mathcal{Z}_L$ is a *co-H*-space by [33] so $\operatorname{cat}(\mathcal{Z}_L) = 1$ (in fact, it is not difficult to directly show that $\mathcal{Z}_L \simeq \Sigma^2 \mathcal{Z}_K$). On the other hand, Proposition 3.11 gives $\operatorname{cat}(\mathcal{Z}_L) \leq \max\{1, \operatorname{cat}(\mathcal{Z}_K)\} + \mathcal{C}\operatorname{at}(\mathcal{Z}_K)$ since $\mathcal{Z}_{\{n+1\} * K} \cong D^2 \times \mathcal{Z}_K \simeq \mathcal{Z}_K$ and $\mathcal{Z}_{\Delta^n}$ is contractible.

Corollary 3.13. *Suppose*

$$K = L_1 \#_{\sigma_1} L_2 \#_{\sigma_2} \cdots \#_{\sigma_{k-1}} L_k$$

where $\dim L_i = d$, σ_i *is a d-face common to* L_i *and* L_{i+1}, *and* $\sigma_i \cap \sigma_j = \varnothing$ *when* $i \neq j$. *Then*

$$\operatorname{cat}(\mathcal{Z}_K) \leq \max\{1, \operatorname{cat}(\mathcal{Z}_{{L_1}^{(d-1)}}), \ldots, \operatorname{cat}(\mathcal{Z}_{{L_k}^{(d-1)}})\} + 1.$$

Proof. Take the disjoint unions

$$C = \partial\sigma_1 \sqcup \cdots \sqcup \partial\sigma_{k-1},$$

$$K_1 = \bigsqcup_{1 \leq 2i+1 \leq k-1} L_{2i+1}$$

$$K_2 = \bigsqcup_{2 \leq 2i \leq k-1} L_{2i},$$

and take the iterated face deletions $K_1' = K_1 \backslash (\sigma_1 \sqcup \cdots \sqcup \sigma_{k-1})$ and $K_2' = K_2 \backslash (\sigma_1 \sqcup \cdots \sqcup \sigma_{k-1})$. Then C is a full subcomplex common to both K_1' and K_2', and to both ${K_1'}^{(d-1)}$ and ${K_2'}^{(d-1)}$. Moreover, ${K_1'}^{(d-1)} = {K_1}^{(d-1)}$ and ${K_2'}^{(d-1)} = {K_2}^{(d-1)}$, and $K = K_1' \cup_C K_2'$, so by the second part of Proposition 3.11,

$$\operatorname{cat}(\mathcal{Z}_K) \leq \max\{1, \operatorname{cat}(\mathcal{Z}_{{K_1}^{(d-1)}}), \operatorname{cat}(\mathcal{Z}_{{K_2}^{(d-1)}})\} + \mathcal{C}\operatorname{at}(\mathcal{Z}_C)$$

It is clear that C is an HMF-complex, so $\mathcal{Z}_C$ is homotopy equivalent to a wedge of spheres and $\mathcal{C}\operatorname{at}(\mathcal{Z}_C) = 1$ (alternatively, this follows from Theorem 10.1 in [29]). Moreover, we can think of $\mathcal{Z}_{{K_1}^{(d-1)}}$ as being built up iteratively by gluing $\bigsqcup_{1 \leq 1 \leq j} \mathcal{Z}_{{L_{2i+1}}^{(d-1)}}$ and $\mathcal{Z}_{{L_{2j+3}}^{(d-1)}}$ along the empty simplex, so iterating the first inequality in Example 3.12,

$$\operatorname{cat}(\mathcal{Z}_{{K_1}^{(d-1)}}) \leq \max\{\, 1,\ \operatorname{cat}(\mathcal{Z}_{{L_1}^{(d-1)}}),\ \operatorname{cat}(\mathcal{Z}_{{L_3}^{(d-1)}}),\ \ldots\}.$$

Likewise, $\operatorname{cat}(\mathcal{Z}_{{K_2}^{(d-1)}}) \leq \max\{\, 1,\ \operatorname{cat}(\mathcal{Z}_{{L_2}^{(d-1)}}),\ \operatorname{cat}(\mathcal{Z}_{{L_4}^{(d-1)}}),\ \ldots\}$. The inequality in the lemma follows. □

4. Triangulated Spheres

Let $\mathcal{C}_0 = \{S^0\}$, $\mathcal{C}_1$, and $\mathcal{C}_2$ consist of all triangulated 0,1, and 2-spheres, and for $d \geq 3$, let $\mathcal{C}_d$ be the class of triangulated d-spheres defined by $K \in \mathcal{C}_d$ if

(1) $K = L_1 * \cdots * L_k$ for some $L_i \in \mathcal{C}_{d_i}$, $d_i \leq 2$, and $d_1 + \cdots + d_k = d - k + 1$;
(2) $K = K_1 \#_{\sigma_1} \cdots \#_{\sigma_{\ell-1}} K_\ell$ where σ_i is a d-face common to K_i and K_{i+1} with $\sigma_i \cap \sigma_j = \varnothing$ when $i \neq j$, and each $K_i = L_{1,i} * \cdots * L_{k_i,i}$ is of the form (1) such that each $L_{j,i}$ is not the boundary of a simplex.

The join $L * L'$ is the simplicial complex $\{\sigma \sqcup \sigma' \mid \sigma \in L, \sigma' \in L'\}$, and the connected sum $K \#_\sigma K'$ is given topologically by gluing triangulations K and K' of S^d along a common d-face σ, and deleting its interior.

Remark 4.1. If L and L' are boundaries ∂P^* and $\partial P'^*$ of the duals of simple polytopes P and P', then $L * L'$ is the boundary of $(P \times P')^*$, while $L \#_\sigma L'$ is the boundary of dual Q^*, where Q is obtained by taking the vertex cut at the vertices of P and P' that are dual to σ, gluing along the new hyperplane and removing it after gluing.

Our goal in this section will be to show the following.

Theorem 4.2. *If K on vertex set $[n] = \{1, \ldots, n\}$ is any triangulated d-sphere for $d = 0, 1, 2$, or $K \in \mathcal{C}_d$ for $d \geq 3$, then the following are equivalent.*

(1) *K is m-Golod over $\mathbb{Z}$;*
(2) $\mathrm{nill}(\mathrm{Tor}_{\mathbb{Z}[v_1,\ldots,v_n]}(\mathbb{Z}[K], \mathbb{Z})) \leq m + 1$ *(equivalently* $\mathrm{cup}(\mathcal{Z}_K) \leq m$*);*
(3) *for any filtration of full subcomplexes*
$$\partial \Delta^{d+2-\ell} = K_{I_\ell} \subsetneq K_{I_{\ell-1}} \subsetneq \cdots \subsetneq K_{I_1} = K$$
such that $|K_{I_i}| \cong S^{d+1-i}$, we have $\ell \leq m$;
(4) $\mathrm{cat}(\mathcal{Z}_K) \leq m$.

Moreover, $1 \leq m \leq d + 1$; that is, K satisfies any of the above for some m which cannot be greater than $d + 1$. □

4.1. **Well-Behaved Triangulations.**

Definition 4.3. We say a triangulation K of a d-sphere S^d on vertex set $[n]$ is *well-behaved* if for all $I \subseteq [n]$ and any $k < d$:

(1) if $|K_I| \cong S^k$ and $K_I \subseteq K_J$ for some $J \subseteq [n]$ such that $|K_J| \cong S^{k+1}$, then $|K_{J-I}| \simeq S^0$;
(2) if $H_*(K_I) \cong H_*(S^k)$, then there is $I' \subseteq [n]$ such that $K_{I'} \subseteq K_I$ and $|K_{I'}| \cong S^k$;
(3) if I is a face in K, then $|K_{[n]-I}| \simeq *$;
(4) $H_*(K_I)$ is torsion-free.

As we will see, conditions (2)-(4) are there to keep the well-behaved condition invariant under join and connected sum operations.

Lemma 4.4. *Triangulations K of 0, 1, and 2-spheres are well-behaved.*

Proof. This is trivial for $d = 0, 1$. When $d = 2$, Conditions (1) and (3) are also clear. Since $|K|$ is a sphere and $|K_I|$ has the homotopy type of a dimension ≤ 1 CW-complex when $I \subsetneq [n]$, $H_*(K_I)$ is torsion-free. To see that Condition (2) holds, notice that if $H_*(K_I) \cong H_*(S^0)$ then we can pick $K_{I'} = S^0 \subseteq K_I$ ($|I'| = 2$). If $H_*(K_I) \cong H_*(S^1)$, we obtain a 1-dimensional full subcomplex $K_{I''} \subseteq K_I$ such that $H_*(K_{I''}) \cong H_*(S^1)$ via a sequence of elementary collapses of 2-faces in K_I, and then we can choose $K_{I'} \subseteq K_{I''}$ such that $|K_{I'}| \cong S^1$. □

If K and L are triangulations of the d-sphere S^d, then so is $K \#_\sigma L$. If K and L are triangulations of S^d and $S^{d'}$, then $K * L$ is a triangulation of $S^{d+d'+1}$ since $|K * L| \cong |K| * |L|$.

Lemma 4.5. *Suppose K and L are both triangulations of S^d, and σ is a d-face common to K and L. If K and L are well-behaved, then so is $K\#_\sigma L$.*

Lemma 4.6. *If K and L are well-behaved triangulations of S^d and $S^{d'}$, then $K * L$ is a well-behaved triangulation of $S^{d+d'+1}$.*

We will need three lemmas before we can prove these.

Lemma 4.7. *Suppose K and L are well-behaved triangulations on vertex sets $\mathcal{V}$ and $\mathcal{V}'$, and let N be a full subcomplex of $K * L$.*

If $H_*(N) \cong H_*(S^k)$ for some k, then for some $I \subseteq \mathcal{V}$ and $J \subseteq \mathcal{V}'$, we have $N = K_I * L_J$ such that $H_*(K_I) \cong H_*(S^i)$ and $H_*(L_J) \cong H_*(S^j)$. Moreover, if $|N| \cong S^k$, then $|K_I| \cong S^i$ and $|L_J| \cong S^j$.

Proof. Since N is a full subcomplex of $K * L$, $N = K_I * L_J$ where $I \sqcup J$ is the vertex set of N. Since K and L are well-behaved, $H_*(K_I)$ and $H_*(L_J)$ are torsion-free, and since

$$\tilde{H}_{*+1}(N) \cong \tilde{H}_{*+1}(|K_I| * |L_J|) \cong \tilde{H}_*(|K_I| \wedge |L_J|) \cong \tilde{H}_*(K_I) \otimes \tilde{H}_*(L_J), \tag{7}$$

then $H_*(N)$ is torsion-free.

If $I = \varnothing$ then $N = K_\varnothing * L_J = \varnothing * L_J = L_J$, and the lemma follows. Similarly when $J = \varnothing$. Assume $I \neq \varnothing$ and $J \neq \varnothing$. Suppose $H_*(N) \cong H_*(S^k)$. Then by (7), we must have $H_*(K_I) \cong H_*(S^i)$ and $H_*(L_J) \cong H_*(S^j)$ for some i, j such that $i + j + 1 = k$.

Now suppose, moreover, that we have $|N| \cong S^k$. Suppose $|K_I| \ncong S^i$. Since K is well-behaved and $H_*(K_I) \cong H_*(S^i)$, there exists a full subcomplex $K_{I'} \subseteq K_I$ such that $|K_{I'}| \cong S^i$, and $K_{I'} \neq K_I$ since $|K_I| \ncong S^i$. Then $H_k(K_{I'} * L_J) \cong H_k(K_I * L_J) = H_k(N) \cong \mathbb{Z}$, so under the inlusion $K_{I'} * L_J \subseteq K_I * L_J = N$, there exists a cycle ω in the simplicial chain subgroup $C_k(K_{I'} * L_J) \subseteq C_k(N)$ that represents a generator $\mathbb{Z} \subseteq H_k(K_{I'} * L_J)$. But since $\dim(K_{I'} * L_J) = \dim N = k$, $K_{I'} * L_J \neq K_I * L_J$, and the k-faces of N generate N (since $|N|$ is a sphere), $K_{I'} * L_J$ does not contain all k-faces of N, so ω must be a cycle in $C_k(N)$ that does not contain all k-faces of N. But this is impossible since $|N| \cong S^k$, so any k-dimensional cycle must contain all the k-faces, so we must have $|K_I| \cong S^i$. By a similar argument, $|L_J| \cong S^j$.

□

Lemma 4.8. *Let K be a triangulation of a d-sphere S^d on vertex set $[n]$. If $H_*(K_I)$ is torsion-free for some $I \subseteq [n]$, $I \neq [n]$, then so is $H_*(K_I \backslash \sigma)$ for any d-face $\sigma \in K_I$, and the inclusion $|\partial\sigma| \hookrightarrow |K_I \backslash \sigma|$ induces an injection on homology groups with torsion-free cokernel.*

Proof. Since $|K_I| \ncong |K| \cong S^d$, any cycle in the simplicial chain group $C_d(K)$ consists of all the d-faces of K, and there are no cycles in $C_d(K_I)$ and $C_d(K_I \backslash \sigma)$, so $H_i(K_I) \cong H_i(K_I \backslash \sigma) = 0$ for $i \geq d$. By the homology long exact sequence for the cofibration sequence $|K_I \backslash \sigma| \hookrightarrow |K_I| \longrightarrow |K_I|/|K_I \backslash \sigma| \cong S^d$, we have $H_i(K_I \backslash \sigma) \cong H_i(K_I)$ for $i < d-1$, and a short exact sequence

$$0 \longrightarrow H_d(S^d) \longrightarrow H_{d-1}(K_I \backslash \sigma) \longrightarrow H_{d-1}(K_I) \longrightarrow 0.$$

Therefore $H_{d-1}(K_I \backslash \sigma)$ is torsion-free since $H_{d-1}(K_I)$ and $H_d(S^d) \cong \mathbb{Z}$ are, and the second map in the sequence has torsion-free cokernel. Since $|K_I|$ is the homotopy cofiber of $S^{d-1} \cong |\partial\sigma| \hookrightarrow |K_I \backslash \sigma|$, the second map is induced by this inclusion in degree $d-1$, so it induces an injection with torsion-free cokernel on each homology group. □

Lemma 4.9. *Suppose K and L are triangulations of S^d on vertex sets $\mathcal{V}$ and $\mathcal{V}'$, σ is a d-face common to K and L, and take $K\#_\sigma L$ on vertex set $\mathcal{W}$.*

If $H_*(K_I)$ and $H_*(L_J)$ are torsion-free for each $I \subseteq \mathcal{V}$ and $J \subseteq \mathcal{V}'$, then $H_*((K\#_\sigma L)_\mathcal{I})$ is torsion-free for each subset $\mathcal{I} \subseteq \mathcal{W}$.

Proof. Let $Z = K\#_\sigma L$. Given $\mathcal{I} \subseteq \mathcal{W}$, and $I = \mathcal{V} \cap \mathcal{I}$ and $J = \mathcal{V}' \cap \mathcal{I}$, we have $|Z_\mathcal{I}|$ is homotopy equivalent to: (a) $|K_I| \vee |L_J|$ when $\mathcal{I} \cap \sigma \neq \varnothing$ and $\sigma \notin \mathcal{I}$; (b) $|K_I| \coprod |L_J|$ when $\mathcal{I} \cap \sigma = \varnothing$; (c) otherwise it is equal to $|K_I|\#_\sigma|L_J|$ when $\sigma \subseteq \mathcal{I}$. Therefore $H_*(Z_\mathcal{I})$ is torsion free in the first two cases, or the last case when $\mathcal{I} = \mathcal{W}$. To see that it is torsion-free in the last case when $\mathcal{I} \neq \mathcal{W}$, notice that one of $I \neq \mathcal{V}$ or $J \neq \mathcal{V}'$. If $I = \mathcal{V}$ or $J = \mathcal{V}'$, then $|Z_\mathcal{I}| \cong |L_J|$ or $|Z_\mathcal{I}| \cong |K_I|$ respectively, and we are done. Assume $I \neq \mathcal{V}$ and $J \neq \mathcal{V}'$. Notice that $|K_I\backslash\sigma| \cup |L_J\backslash\sigma| = |Z_\mathcal{I}|$, $|K_I\backslash\sigma| \cap |L_J\backslash\sigma| = |\partial\sigma|$, and by Lemma 4.8, $H_*(K_I\backslash\sigma)$ and $H_*(L_J\backslash\sigma)$ are torsion-free, with the inclusions $|\partial\sigma| \hookrightarrow |K_I\backslash\sigma|$ and $|\partial\sigma| \hookrightarrow |L_J\backslash\sigma|$ inducing injections on homology groups with torsion-free cokernel. Then it follows by the Mayer-Vietoris sequence for $(|Z_\mathcal{I}|, |K_I\backslash\sigma|, |L_J\backslash\sigma|)$ that $H_*(Z_\mathcal{I})$ is torsion free. □

Proof of Lemma 4.5. Let $Z = K\#_\sigma L$ on vertex set $\mathcal{W}$, and K and L be on vertex sets $\mathcal{V}$ and $\mathcal{V}'$. By Lemma 4.9, any full subcomplex $Z_\mathcal{I}$ has torsion-free homology, so condition (4) in Definition 4.3 holds for $Z_\mathcal{I}$.

Since Z is $K\backslash\sigma$ and $L\backslash\sigma$ glued along the boundary of the d-simplex $\partial\sigma$, $\mathcal{I}$ is a face in K or L whenever it is a face in Z. Then K and L being well-behaved, we have $|Z_{\mathcal{W}-\mathcal{I}}| \simeq |K_{\mathcal{V}-\mathcal{I}}| \vee |L_{\mathcal{V}'-\mathcal{I}}| \simeq *$ when $\mathcal{I} \cap \sigma \neq \varnothing$. Since $|L\backslash\sigma| \cong |K\backslash\sigma| \cong D^d$, then $|Z_{\mathcal{W}-\mathcal{I}}| \cong |K_{\mathcal{V}-\mathcal{I}}| \simeq *$ or $|Z_{\mathcal{W}-\mathcal{I}}| \cong |L_{\mathcal{V}'-\mathcal{I}}| \simeq *$ when $\mathcal{I} \cap \sigma = \varnothing$. Condition (3) holds as well.

If $|Z_\mathcal{I}| \cong S^k$ for some $k < d$, then $Z_\mathcal{I}$ is either a full subcomplex of one of K or L, or else $Z_\mathcal{I} = \partial\sigma$ (otherwise $Z_\mathcal{I}$ would have a $(k-1)$-face contained in three k-faces) so the only case to check is the last one $Z_\mathcal{I} = \partial\sigma$. Here, condition (2) in Definition 4.3 clearly holds. Since K and L are well-behaved, $|K_{\mathcal{V}-\sigma}| \simeq |L_{\mathcal{V}'-\sigma}| \simeq *$, so it follows that $|Z_{\mathcal{W}-\sigma}| \simeq S^0$, and condition (1) holds as well.

□

Proof of Lemma 4.6. Let K and L be on vertex sets $\mathcal{V}$ and $\mathcal{V}'$, $Z = K * L$ be on vertex set $\mathcal{W} = \mathcal{V} \sqcup \mathcal{V}'$, and let $\mathcal{I} \subseteq \mathcal{W}$. Then $Z_\mathcal{I} = K_I * L_J$ where $I \sqcup J = \mathcal{I}$, and since $H_*(K_I)$ and $H_*(L_J)$ are torsion-free, $\tilde{H}_{*+1}(Z_\mathcal{I}) \cong \tilde{H}_*(K_I) \otimes \tilde{H}_*(L_J)$ is torsion-free.

If $\mathcal{I}$ is a face in Z, then I and J are faces in K and L, so we have $|K_{\mathcal{V}-I}| \simeq *$ and $|L_{\mathcal{V}'-J}| \simeq *$ since K and L are well-behaved. Therefore $|Z_{\mathcal{W}-\mathcal{I}}| = |K_{\mathcal{V}-I} * L_{\mathcal{V}'-J}| \simeq *$, and condition (3) of Definition 4.3 holds.

Suppose $H_*(Z_\mathcal{I}) \cong H_*(S^k)$. Suppose $I \neq \varnothing$ and $J \neq \varnothing$. By Lemma 4.7, $H_*(K_I) \cong H_*(S^i)$ and $H_*(L_J) \cong H_*(S^j)$ for some i,j such that $i + j + 1 = k$. Since K and L and well-behaved, this implies there are $K_{I'} \subseteq K_I$ and $L_{J'} \subseteq L_J$ such that $|K_I| \cong S^i$ are $|L_J| \cong S^j$. Then $Z_{I' \sqcup J'} = K_{I'} * L_{J'} \subseteq K_I * K_J = Z_\mathcal{I}$ and $|Z_{I' \sqcup J'}| \cong S^{i+j+1}$. Thus condition (2) holds when $I \neq \varnothing$ and $J \neq \varnothing$.

Now consider that case where $I = \varnothing$ or $J = \varnothing$. Without loss of generality, suppose $I = \varnothing$. We have $Z_\mathcal{I} = K_\varnothing * L_J = \varnothing * L_J = L_J$, so if $H_*(Z_\mathcal{I}) \cong H_*(S^k)$, then $H_*(L_J) \cong S^k$, and we have $L_{J'} \subseteq L_J = Z_\mathcal{I}$ such that $|L_{J'}| \cong S^k$ since L is well-behaved. Thus condition (2) holds when $I = \varnothing$ or $J = \varnothing$.

Let $S^{-1} = \varnothing$. Suppose $|Z_\mathcal{I}| \cong S^k$ and $Z_\mathcal{I} \subseteq Z_\mathcal{J}$ such that $|Z_\mathcal{J}| \cong S^{k+1}$. Then $Z_\mathcal{J} = K_{I'} * L_{J'}$ where $I' \sqcup J' = \mathcal{J} \subsetneq \mathcal{I} = I \sqcup J$, so $K_I \subseteq K_{I'}$ and $L_J \subseteq L_{J'}$, and by Lemma 4.7, we must have $|K_I| \cong S^i$, $|L_J| \cong S^j$, $|K_{I'}| \cong S^{i'}$, $|L_{J'}| \cong S^{j'}$, where $i + j + 1 = k$ and $i' + j' + 1 = k + 1$. Since $K_I \subseteq K_{I'}$ and $L_J \subseteq L_{J'}$, then $i \leq i'$ and $j \leq j'$, so either (a) $i' = i$ and $j' = j + 1$; or (b) $i' = i + 1$ and $j' = j$. Without any loss of generality, assume (a) holds. Then $K_I = K_{I'}$ (i.e. $I = I'$), $K_{I-I'} = \varnothing$, and

$$|Z_{\mathcal{J}-\mathcal{I}}| = |K_{I'-I} * L_{J'-J}| = |L_{J'-J}| \simeq S^0,$$

with the last homotopy equivalence since L is well-behaved. It follows that condition (1) of Definition 4.3 holds.

□

4.2. **Spherical Filtrations and Filtration Length.**

Definition 4.10. Given a triangulation of a d-sphere K on vertex set $[n]$, suppose we have a filtration of full subcomplexes

$$\partial\Delta^{d+2-\ell} = K_{I_\ell} \subsetneq \cdots \subsetneq K_{I_1} = K$$

such that K_{I_i} is a triangulation of a $(d+1-i)$-sphere. Then we say that this is a *spherical filtration* of K of length ℓ.

Remark: Implicitly, $I_\ell \subsetneq \cdots \subsetneq I_1 = [n]$ and $|I_\ell| = d+3-\ell$.

Definition 4.11. For any triangulated sphere K, define the *filtration length* $\mathrm{filt}(K)$ to be the largest integer ℓ such that K admits a spherical filtration of length ℓ.

For example, $\mathrm{filt}(\partial\Delta^{d+1}) = 1$. Generally, there are the following bounds with respect to joins, connected sums, and cup product length.

Lemma 4.12. *If K and L are both triangulations of S^d, and σ is a d-face common to K and L, then*

$$\mathrm{filt}(K\#_\sigma L) \geq \max\{2, \mathrm{filt}(K), \mathrm{filt}(L)\}.$$

Proof. As in the proof of Lemma 4.5, a full subcomplex $Z_{\mathcal{I}}$ of $K\#_\sigma L$ satisfying $|Z_{\mathcal{I}}| \cong S^k$ for some $k < d$ must either be a full subcomplex of exactly one of K or L, or else $Z_{\mathcal{I}} = \partial\sigma$. Moreover, $K\#_\sigma L$ always has the length 2 spherical filtration $\partial\sigma \subsetneq K\#_\sigma L$. The lemma follows immediately. □

Lemma 4.13. *If K and L are any triangulated spheres, then*

$$\mathrm{filt}(K * L) \geq \mathrm{filt}(K) + \mathrm{filt}(L).$$

Proof. Let K and L be on vertex sets $[n]$ and $[m]$. Let $d = \dim K$, $d' = \dim L'$, $\ell = \mathrm{filt}(K)$, $\ell' = \mathrm{filt}(L)$, and take $\partial\Delta^{d+2-\ell} = K_{I_\ell} \subsetneq \cdots \subsetneq K_{I_1} = K$ and $\partial\Delta^{d'+2-\ell'} = L_{J_{\ell'}} \subsetneq \cdots \subsetneq L_{J_1} = L$ to be spherical filtrations of K and L.

Since $|K_{I_i} * L_{J_j}| \cong |K_{I_i}| * |L_{J_j}| \cong S^{d+1-i} * S^{d'+1-j} \cong S^{d+d'-i-j+3}$, and $|K_{I_i} * L_{J_j}|$ is a full subcomplex of $|K_{I_{i'}} * L_{J_{j'}}|$ when $i \leq i'$ and $j \leq j'$, we have a length $\ell+\ell'$ spherical filtration of $K * L$

$$\partial\Delta^{d+2-\ell} \subsetneq (K_{I_\ell} * L_{J_{\ell'}}) \subsetneq \cdots \subsetneq (K_{I_\ell} * L_{J_2}) \subsetneq (K_{I_\ell} * L_{J_1}) \subsetneq (K_{I_{\ell-1}} * L_{J_1}) \subsetneq \cdots \subsetneq (K_{I_1} * L_{J_1}) = K * L,$$

where $\partial\Delta^{d+2-\ell}$ is the full subcomplex of $K_{I_\ell} * L_{J_{\ell'}} = \partial\Delta^{d+2-\ell} * \partial\Delta^{d'+2-\ell'}$ on the vertices of $\partial\Delta^{d+2-\ell}$. Therefore $\mathrm{filt}(K * L) \geq \ell + \ell'$.

□

Lemma 4.14. *If K is a well-behaved triangulation of S^d, then* $\mathrm{filt}(K) \leq \mathrm{cup}(\mathcal{Z}_K)$.

Proof. Let $\ell = \mathrm{filt}(K)$ and $\partial\Delta^{d+1-\ell} = K_{I_\ell} \subsetneq \cdots \subsetneq K_{I_1} = K$ be a spherical filtration of length ℓ.

Let $J_{i+1} = I_i - I_{i+1}$ for $i < \ell$. Since K is well-behaved, $|K_{J_{i+1}}| \simeq S^0$, so $|K_{J_{i+1}}| \cong D_i \coprod D_i'$ for some contractible subcomplexes D_i and D_i' of $|K_{I_i}|$. Take the inclusion induced by $K_{I_i} \hookrightarrow K_{I_{i+1}} * K_{J_{i+1}}$

$$\iota_i : |K_{I_i}| \hookrightarrow |K_{I_{i+1}} * K_{J_{i+1}}| \cong |K_{I_{i+1}}| * |K_{J_{i+1}}|,$$

and let $h_i : |K_{I_{i+1}}| * |K_{J_{i+1}}| \longrightarrow |K_{I_{i+1}}| * S^0$ be the join of the identity map on the left factor with the map collapsing D_i and D_i' to -1 and 1 in $S^0 = \{-1, 1\}$ on the right factor. Then h_i is a homotopy equivalence since D_i and D_i' are contractible. Notice $h_i \circ \iota_i$ is the quotient map that collapses D_i to a point and D_i' to another point. Since D_i and D_i' are disjoint contractible subcomplexes of $|K_{I_i}|$, $h_i \circ \iota_i$ is a homotopy equivalence. Therefore, ι_i is a homotopy equivalence.

Take the composite of inclusions

$$|K| = |K_1| \xhookrightarrow{\iota_1} |K_{I_2} * K_{J_2}| \xhookrightarrow{\iota_2'} |K_{I_3} * K_{J_3} * K_{J_2}| \xhookrightarrow{\iota_3'} \cdots \xhookrightarrow{\iota_{\ell-1}'} |K_{I_\ell} * K_{J_\ell} * \cdots * K_{J_2}|. \tag{8}$$

The i^{th} map ($i \geq 2$) in this composite

$$\iota_i' : |K_{I_i} * K_{J_i} * \cdots * K_{J_2}| \hookrightarrow |K_{I_{i+1}} * K_{J_{i+1}} * K_{J_i} * \cdots * K_{J_2}|$$

is the join of the homotopy equivalence ι_i and the identity $|K_{J_i} * \cdots * K_{J_2}| \xrightarrow{\mathbb{1}_i} |K_{J_i} * \cdots * K_{J_2}|$, and since each $|K_{J_j}| \simeq S^0$, then $\mathbb{1}_i$ is homotopy equivalent to the identity $S^{i-2} \xrightarrow{\mathbb{1}} S^{i-2}$. Each ι_i' (and ι_1) is therefore a homotopy equivalence, and then so is composite (8) (in fact, since $|K_{I_i}| \cong S^{d+1-i}$, ι_i and (8) are homotopy equivalent to homeomorphisms $S^{d+1-i} \xrightarrow{\cong} S^{d+1-i}$ and $S^d \xrightarrow{\cong} S^d$). Therefore (8) induces a non-trivial map on cohomology, and so the Hochster theorem implies there is a non-trivial length ℓ cup product in $H^+(\mathcal{Z}_K)$.

□

4.3. **Triangulated d-spheres for** $d = 0, 1, 2$**.** The only triangulated 0-sphere is S^0, and the only triangulated 1-sphere with $n \geq 3$ vertices is the n-gon. We will say that C is a *chordless cycle* in K if C is a full subcomplex K_I of K, and C is an m-gon for some $m \geq 3$. When K is a graph, this is the same as C being an induced cycle of G. K is said to be *chordal* if it contains no chordless cycles with 4 or more vertices.

Lemma 4.15. *If K is a triangulation of S^1 on vertex set $[n]$, then*

(i) $\mathrm{filt}(K) \geq 2$ *whenever K has at least 4 vertices.*

If K is a triangulation of S^2, then

(ii) $\mathrm{filt}(K) \geq 2$ *whenever K has a chordless cycle;*

(iii) $\mathrm{filt}(K) \geq 3$ *whenever K has a chordless cycle with at least 4 vertices.*

Proof. Let $|K| \cong S^1$. If K has at least $n \geq 4$ vertices, then K being an n-gon means we can take $I' \subset [n]$, $|I'| = 2$, such that $|K_{I'}| = S^0$. Then $S^0 \subsetneq K$ is a length 2 spherical filtration of K.

Let $|K| \cong S^2$, and $C = K_I$ be a chordless cycle for some $I \subset [n]$. We have $|K_I| \cong S^1$ and $K_I = \partial\Delta^2$ when K_I has 3 vertices, in which case $K_I \subsetneq K$ is a length 2 spherical filtration. Otherwise, when K_I has at least 4 vertices, $\mathrm{filt}(K) \geq 3$ since there is a spherical filtration $S^0 \subsetneq K_I \subsetneq K$ by part (i). □

Proposition 4.16. *Let K be a triangulated d-sphere, $d = 0, 1, 2$. Then*

$$1 \leq \mathrm{filt}(K) = \mathrm{cup}(\mathcal{Z}_K) = \mathrm{cat}(\mathcal{Z}_K) \leq d + 1.$$

In particular, letting $m = \mathrm{cat}(\mathcal{Z}_K)$, when $d = 1$ we have $m = 1$ iff $K = \partial\Delta^2$, and $m = 2$ iff K at least 4-vertices; and when $d = 2$ we have $m = 1$ iff $K = \partial\Delta^3$, $m = 2$ iff K has a chordless cycle but none with more than 3 vertices, and $m = 3$ iff K has a chordless cycle with at least 4 vertices.

Proof. The case $d = 0$ is immediate since S^0 is the only triangulated 0-sphere, and $\mathcal{Z}_{S^0} \cong S^3$. Since K is well-behaved by Lemma 4.4, we have by Lemma 3.3 and Lemma 4.14

$$1 \leq \mathrm{filt}(K) \leq \mathrm{cup}(\mathcal{Z}_K) \leq \mathrm{cat}(\mathcal{Z}_K) \leq d + 1,$$

so it remains to show that $\mathrm{cat}(\mathcal{Z}_K) \leq \mathrm{filt}(K)$.

Fix $d = 1$. Using Lemma 4.15, if K has at least 4 vertices, then $\mathrm{filt}(K) \geq 2$, so $\mathrm{cat}(\mathcal{Z}_K) \leq \mathrm{filt}(K)$ since $\mathrm{cat}(\mathcal{Z}_K) \leq d + 1 = 2$. Otherwise, if K has 3-vertices, then $K = \partial\Delta^2$ and $\mathcal{Z}_K \cong S^5$, so $\mathrm{filt}(K) = \mathrm{cat}(\mathcal{Z}_K) = 1$.

Now fix $d = 2$. By Lemma 4.15, $\mathrm{filt}(K) \geq 3$ whenever K has chordless cycles with at least 4 vertices, so $\mathrm{cat}(\mathcal{Z}_K) \leq \mathrm{filt}(K)$ since $\mathrm{cat}(\mathcal{Z}_K) \leq d + 1 = 3$. On the other hand, suppose K has chordless cycles, but none with more than 3 vertices. Then $\mathrm{filt}(K) = 2$ by Lemma 4.15, and the 1-skeleton $K^{(1)}$ is a chordal graph. The chordal property is closed under vertex deletion (taking full subcomplexes). Moreover, recall from [21] that chordal graphs have a *total elimination ordering*, that is, they can be built up one vertex v at a time in some order such that at each step the neighbours of v form a clique. Inducting on this ordering, one sees that chordal graphs are HMF-complexes, therefore $\mathrm{cat}(\mathcal{Z}_{K^{(1)}}) \leq 1$ by Proposition 3.10 (this also follows the main result in [35]). Using Lemma 3.3, we have $\mathrm{cat}(\mathcal{Z}_K) \leq \mathrm{cat}(\mathcal{Z}_{K^{(1)}}) + 1 \leq 2 = \mathrm{filt}(K)$. Otherwise, $K = \partial\Delta^3$ when K has no chordless cycles at all, so we have $\mathcal{Z}_K \cong S^7$ and $\mathrm{filt}(K) = \mathrm{cat}(\mathcal{Z}_K) = 1$.

□

Corollary 4.17. *Let K be a triangulated d-sphere, $d = 1, 2$, then*

$$\mathrm{cat}(\mathcal{Z}_{K^{(d-1)}}) \le \max\{1, \mathrm{cat}(\mathcal{Z}_K) - 1\} = \max\{1, \mathrm{filt}(K) - 1\}.$$

Proof. The last equality $\mathrm{filt}(K) = \mathrm{cat}(\mathcal{Z}_K)$ is from the previous proposition. Let $m = \mathrm{cat}(\mathcal{Z}_K)$. Using Proposition 4.16, the statement simplifies to $\mathrm{cat}(\mathcal{Z}_{K^{(0)}}) \le 1$ when $d = 1$, or $d = 2$ and $m = 1$. This is true since $\mathcal{Z}_{K^{(0)}}$ has the homotopy type of a wedge of spheres by [31], or by Proposition 3.10 (as $K^{(0)}$ is a collection of disjoint points). Fix $d = 2$. As in the proof of Proposition 4.16, $\mathrm{cat}(\mathcal{Z}_{K^{(1)}}) \le 1 < m$ when $m = 2$, since $\mathrm{filt}(K) = 2$. This last inequality also holds when $m = 3$ since $\mathrm{cat}(\mathcal{Z}_{K^{(1)}}) \le \dim K^{(1)} + 1 = 2$ is always true by Lemma 3.3. □

4.4. **Triangulated d-spheres for $d \ge 3$.**

Proposition 4.18. *Suppose $K \in \mathcal{C}_d$, $d \ge 0$. Then*

$$1 \le \mathrm{filt}(K) = \mathrm{cup}(\mathcal{Z}_K) = \mathrm{cat}(\mathcal{Z}_K) \le d + 1.$$

Proof. Triangulations of $0, 1, 2$-spheres are well-behaved by Lemma 4.4, so iterating Lemmas 4.6 and 4.5, any $K \in \mathcal{C}_d$ is well-behaved. Then by Lemmas 3.3 and 4.14,

$$1 \le \mathrm{filt}(K) \le \mathrm{cup}(\mathcal{Z}_K) \le \mathrm{cat}(\mathcal{Z}_K) \le d + 1.$$

It remains to show that $\mathrm{cat}(\mathcal{Z}_K) \le \mathrm{filt}(K)$. The $d = 0, 1, 2$ case is Proposition 4.16.

Suppose $K = L_1 * \cdots * L_k \in \mathcal{C}_d$ for some $L_i \in \mathcal{C}_{d_i}$, $d_i \le 2$, and $d_1 + \cdots + d_k = d - k - 1$. Then $\mathcal{Z}_K = \mathcal{Z}_{L_1} \times \cdots \times \mathcal{Z}_{L_k}$, and so using Proposition 4.16 and iterating Lemma 4.13,

$$\mathrm{cat}(\mathcal{Z}_K) \le \sum_{i=1,\dots,k} \mathrm{cat}(\mathcal{Z}_{L_i}) = \sum_{i=1,\dots,k} \mathrm{filt}(L_i) \le \mathrm{filt}(L_1 * \cdots * L_k) = \mathrm{filt}(K).$$

Moreover, iterating Corollary 3.7, and using Corollary 4.17, when each $L_i \ne S^0$

$$\mathrm{cat}(\mathcal{Z}_{K^{(d-1)}}) \le \sum_{i=1,\dots,k} \mathrm{cat}(\mathcal{Z}_{L_i^{(d_i-1)}}) + k - 1 \le \sum_{i=1,\dots,k} \max\{1, \mathrm{filt}(L_i) - 1\} + k - 1,$$

and when each $L_i \ne \partial\Delta^{d_i+1}$, we have $\mathrm{filt}(L_i) > 1$, therefore

$$\mathrm{cat}(\mathcal{Z}_{K^{(d-1)}}) \le \left(\sum_{i=1,\dots,k} \mathrm{filt}(L_i)\right) - 1 \le \mathrm{filt}(L_1 * \cdots * L_k) - 1 = \mathrm{filt}(K) - 1, \tag{9}$$

the second inequality by iterating Lemma 4.13.

Suppose $K = K_1 \#_{\sigma_1} \cdots \#_{\sigma_{\ell-1}} K_\ell$ where each $K_i = L_{1,i} * \cdots * L_{k_i,i}$ is a join of the above form such that each $L_{j,i}$ is not the boundary of a simplex, and σ_i is a d-face common to K_i and K_{i+1} with $\sigma_i \cap \sigma_j = \varnothing$ when $i \ne j$. By Corollary 3.13 and inequality (9), we have

$$\begin{aligned}
\mathrm{cat}(\mathcal{Z}_K) &\le \max\{1, \mathrm{cat}(\mathcal{Z}_{K_1^{(d-1)}}), \dots, \mathrm{cat}(\mathcal{Z}_{K_\ell^{(d-1)}})\} + 1 \\
&\le \max\{1, \mathrm{filt}(K_1) - 1, \dots, \mathrm{filt}(K_\ell) - 1\} + 1 \\
&= \max\{2, \mathrm{filt}(K_1), \dots, \mathrm{filt}(K_\ell)\} \\
&= \mathrm{filt}(K_1 \#_{\sigma_1} \cdots \#_{\sigma_{\ell-1}} K_\ell) \\
&= \mathrm{filt}(K),
\end{aligned}$$

where the second last inequality follows from iterating Lemma 4.12.

□

4.5. **Proof of Theorem 4.2.** The following is an immediate consequence of Theorem 4.4 in [42] and the fact that *category weight* $cwgt()$ as defined there is bounded below by 1, and linearly below with respect to cup products.

Theorem 4.19 (Rudyak [42]). *If* $\mathrm{cat}(X) \le m$*, then*

(1) $\mathrm{cup}(X) \le m$*;*
(2) *Massey products* $\langle v_1, \ldots, v_k\rangle$ *vanish in* $H^*(X)$ *whenever* $v_i = a_1 \cdots a_{m_i}$ *and* $v_j = b_1 \cdots b_{m_j}$*, and* $m_i + m_j > m$*, for some odd* i *and even* j *and* $a_s, b_t \in H^+(X)$. □

By our remarks in Section 3.1, and by definition $\mathrm{cup}(X) = \mathrm{nill}\, H^*(X) - 1$, we obtain

Proposition 4.20. *If* $\mathrm{cat}(\mathcal{Z}_K) \le m$*, then* K *is* m*-Golod.* □

Now Theorem 4.2 follows from Propositions 4.16 and 4.18, and the fact that $\mathrm{cup}(\mathcal{Z}_K) \le m$ when K is m-Golod (by definition).

5. Further Applications

5.1. **Fullerenes.** A *fullerene* P is a simple 3-polytope all of whose 2-faces are pentagons and hexagons. These are mathematical idealisations of physical fullerenes - spherical molecules of carbon such that each carbon atom belongs to three carbon rings, and each carbon ring is either a pentagon or hexagon.

The authors in [16] have shown that the cohomology ring of moment-angle complexes is a complete combinatorial invariant of fullerenes, while Buchstaber and Erokhovets [9, 10] show that the finer details of their cohomology encode many interesting properties of fullerenes. For example, if P^* is the dual of P, then the bigraded Betti numbers of $\mathcal{Z}_{\partial P^*}$ count the number of k*-belts* in P. Here, a k-belt of a simple polytope such as P is a sequence of 2-faces $(F_1, \ldots, F_k)$ such that $F_k \cap F_1 \neq \varnothing$, $F_i \cap F_{i+1} \neq \varnothing$ for $1 \le i \le k-1$, and all other intersections are empty. Notice that the k-belts of P correspond to full subcomplexes of ∂P^* that are n-gons. But since fullerenes can have no 3-belts [9, 10], ∂P^* must only have n-gons as full subcomplexes for $n \ge 4$. Moreover, since ∂P^* is a triangulated 2-sphere that is not a boundary of the 2-simplex, it must have at least one such n-gon as a full subcomplex. Thus, $\mathrm{filt}(\partial P^*) = 3$, so by Theorem 4.2, we have the following description.

Theorem 5.1. *For a fullerene* P*,* $\mathrm{cat}(\mathcal{Z}_{\partial P^*}) = 3$ *and* ∂P^* *is* 3*-Golod.* □

Now, using Rudyak's result stated as Theorem 4.19, we relate Massey product to the Lusternik-Schnirelmann category of moment-angle complexes and describe conditions for vanishing of Massey products.

Theorem 5.2. *For a fullerene* P*, all Massey products consisting of decomposable elements in* $H^+(\mathcal{Z}_{\partial P^*})$ *vanish.* □

5.2. **Neighbourly Complexes.** For any finite simply-connected CW-complex X, let

$$\mathrm{hd}(X) = \max\left\{\max\left\{i \,\middle|\, \tilde{H}^i(X)\otimes\mathbb{Q} \neq 0\right\}, \max\left\{i \,\middle|\, Torsion(\tilde{H}^{i-1}(X)) \neq 0\right\}\right\}$$

and

$$\mathrm{hc}(X) = \min\left\{i \,\middle|\, \tilde{H}^{i+1}(X) \neq 0\right\}.$$

These coincide with the dimension and connectivity of X up to homotopy equivalence. It is well known (c.f. [36]) that X satisfies

$$\mathrm{cat}(X) \le \frac{\mathrm{hd}(X)}{\mathrm{hc}(X)+1}. \tag{10}$$

A version of the Hochster formula also holds for real moment-angle complexes, namely,

$$H^*(\mathbb{R}\mathcal{Z}_K) \cong \bigoplus_{I\subseteq[n]} \tilde{H}^*(\Sigma|K_I|). \tag{11}$$

Thus,

$$\mathrm{hd}(\mathbb{R}\mathcal{Z}_K) = 1 + \max\{\,\mathrm{hd}(|K_I|) \mid I \subseteq [n]\,\} \le 1 + \dim K$$

and

$$\mathrm{hc}(\mathbb{R}\mathcal{Z}_K) = 1 + \min\{\,\mathrm{hc}(|K_I|) \mid I \subseteq [n]\,\},$$

and using the inequality $\mathrm{cat}(\mathcal{Z}_K) \le \mathrm{cat}(\mathbb{R}\mathcal{Z}_K)$ from Corollary 3.5,

Proposition 5.3. *For any simplicial complex K,*

$$\mathrm{cat}(\mathcal{Z}_K) \le \frac{\mathrm{hd}(\mathbb{R}\mathcal{Z}_K)}{\mathrm{hc}(\mathbb{R}\mathcal{Z}_K)}.$$

Comparing the Hochster formula for $H^*(\mathbb{R}\mathcal{Z}_K)$ to the Hochster formula for $H^*(\mathcal{Z}_K)$ in Section 3.1, one sees that the inequality $\frac{\mathrm{hd}(\mathbb{R}\mathcal{Z}_K)}{\mathrm{hc}(\mathbb{R}\mathcal{Z}_K)} \le \frac{\mathrm{hd}(\mathcal{Z}_K)}{\mathrm{hc}(\mathcal{Z}_K)}$ usually holds, with the disparity between these two often being very large. In such case, the bound in Proposition 5.3 is an improvement over what one would get by applying (10) directly to $X = \mathcal{Z}_K$.

Consider, for instance, the case of k-neighbourly complexes. A simplicial complex K on vertex set $[n]$ is said to be k-*neighbourly* if every subset of k or less vertices in $[n]$ is a face of K. In this case $H_i(K_I) = 0$ for $i \le k-2$ and each $I \subseteq [n]$, so $\mathrm{hc}(\mathbb{R}\mathcal{Z}_K) \ge k-1$. Therefore we have the following result.

Theorem 5.4. *If K is k-neighbourly, $\mathrm{cat}(\mathcal{Z}_K) \le \frac{1+\dim K}{k}$. In particular, K is $\left(\frac{1+\dim K}{k}\right)$-Golod.* □

Corollary 5.5. *Suppose K is $\lceil \frac{n}{2} \rceil$-neighbourly. If $K \ne \Delta^{n-1}$, then $\dim K \le n-2$. Thus, $\mathrm{cat}(\mathcal{Z}_K) \le 1$ ($\mathcal{Z}_K$ is a co-H-space), K is 1-Golod and therefore all Massey products in $H^*(\mathcal{Z}_K)$ are trivial.* □

5.3. **Simplicial Wedges.** We recall the *simplicial wedge* construction defined in [41, 2]. Let K be a simplicial complex on vertex set $\{v_1, ..., v_n\}$, and for any face $\sigma \in K$, define the *link* of σ in K the subcomplex of K given by

$$\mathrm{link}_K(\sigma) = \{\tau \in K \mid \tau \cap \sigma = \varnothing, \tau \cup \sigma \in K\}.$$

By *doubling* a vertex v_i in K, we obtain a new simplicial complex $K(v_i)$ on vertex set

$$\{v_1, \ldots, v_{i-1}, v_{i1}, v_{i2}, v_{i+1}, \ldots, v_n\}$$

defined by

$$K(v_i) = (v_{i1}, v_{i2}) * \mathrm{link}_K(v_i) \cup_{\{v_{i1},v_{i2}\} * \mathrm{link}_K(v_i)} \{v_{i1}, v_{i2}\} * K \backslash \{v_i\},$$

where (v_{i1}, v_{i2}) is the 1-simplex with vertices $\{v_{i1}, v_{i2}\}$. One can of course iterate this construction by reapplying the doubling operation to successive complexes, and the order of vertices on which this is done is irrelevant. To this end, take any sequence $J = (j_1, \ldots, j_n)$ of non-negative integers, let u_j be the j^{th} vertex in the sequence $v_1, v_{12}, \ldots, v_{1j_1}, v_2, \ldots, v_n, v_{n2}, \ldots, v_{nj_n}$ and $N = j_1 + \cdots + j_n$, and define

$$K(J) = K_N$$

where $K_{j+1} = K_j(u_{j+1})$ and $K_0 = K$. In algebraic terms, the Stanley-Reisner ideal of $K(J)$ is obtained from the Stanley-Reisner ideal of K by replacing each vertex v_i by $v_{i1}, v_{i2}, \ldots, v_{ij_i}$ in each monomial. This construction arises in combinatorics (see [41]) and has the important property that if K is the boundary of the dual of d-polytope, then $K(J)$ is the boundary of the dual of a $(d+N)$-polytope.

Theorem 5.6. *For any J, $\mathrm{cat}(\mathcal{Z}_{K(J)}) \le \mathrm{cat}(\mathcal{Z}_K)$.*

Proof. Let (D^J, S^J) be the sequence of pairs $((D^{2j_1+2}, S^{2j_1+1}), \ldots, (D^{2j_n+2}, S^{2j_n+1}))$. By [2, 30], there is a homeomorphism

$$\mathcal{Z}_{K(J)} = (D^2, S^1)^{K(J)} \cong (D^J, S^J)^K,$$

and by Proposition 3.4, $\mathrm{cat}((D^J, S^J)^K) \le \mathrm{cat}((D^2, S^1)^K) = \mathrm{cat}(\mathcal{Z}_K)$. □

This result becomes algebraically useful when a good bound on $\text{cat}(\mathcal{Z}_K)$ is known. For instance, there are many examples of complexes K for which $\text{cat}(\mathcal{Z}_K) = 1$, duals of sequential Cohen Macaulay and shellable complexes, and chordal flag complexes to name a few [28, 33]. In each of these examples $\text{cat}(\mathcal{Z}_{K(J)}) \leq 1$, so $K(J)$ is Golod. Generally, $K(J)$ is at least $(1 + \dim K)$-Golod since $\text{cat}(\mathcal{Z}_K) \leq 1 + \dim K$, even though $\dim K(J) - \dim K$ can be arbitrarily large.

Notice $K(J)$ is a triangulation of a $(d + N)$-sphere whenever K is a triangulation of a d-sphere. Combining Theorem 5.6 and Proposition 4.20, the range of spheres for which Theorem 4.2 holds generalises as follows.

Corollary 5.7. *Let K be be any triangulated d-sphere for $d = 0, 1, 2$, or $K \in \mathcal{C}_d$ when $d \geq 3$, and let $m = \text{filt}(K)$ (equivalently $m = \text{cup}(\mathcal{Z}_K)$). Then $\text{cat}(\mathcal{Z}_{K(J)}) \leq m$ and $K(J)$ is m-Golod.* □

References

1. A. Bahri, M. Bendersky, F. R. Cohen, and S. Gitler, *The polyhedral product functor: a method of decomposition for moment-angle complexes, arrangements and related spaces*, Adv. Math. **225** (2010), no. 3, 1634–1668. MR 2673742 (2012b:13053)
2. ———, *Operations on polyhedral products and a new topological construction of infinite families of toric manifolds*, Homology Homotopy Appl. **17** (2015), no. 2, 137–160. MR 3426378
3. I. V. Baskakov, V. M. Bukhshtaber, and T. E. Panov, *Algebras of cellular cochains, and torus actions*, Uspekhi Mat. Nauk **59** (2004), no. 3(357), 159–160. MR 2117435 (2006b:57045)
4. P. Beben and J. Grbić, *Configuration spaces and polyhedral products*, ArXiv e-prints (2014).
5. ———, *$\frac{n}{3}$-neighbourly moment-angle complexes and their unstable splittings*, ArXiv e-prints (2015).
6. Alexander Berglund, *Homotopy invariants of Davis-Januszkiewicz spaces and moment-angle complexes.*
7. Alexander Berglund and Michael Jöllenbeck, *On the Golod property of Stanley-Reisner rings*, J. Algebra **315** (2007), no. 1, 249–273. MR 2344344 (2008f:13036)
8. Frédéric Bosio and Laurent Meersseman, *Real quadrics in $\mathbf{C}^n$, complex manifolds and convex polytopes*, Acta Math. **197** (2006), no. 1, 53–127. MR 2285318 (2007j:32037)
9. V. M. Buchstaber and N. Erokhovets, *Construction of fullerenes*, ArXiv e-prints (2015).
10. ———, *Truncations of simple polytopes and applications*, Proceedings of the Steklov Institute of Mathematics **289** (2015), 104–133.
11. Victor M. Buchstaber and Taras E. Panov, *Torus actions and their applications in topology and combinatorics*, University Lecture Series, vol. 24, American Mathematical Society, Providence, RI, 2002. MR 1897064 (2003e:57039)
12. ———, *Toric topology*, Mathematical Surveys and Monographs, vol. 204, American Mathematical Society, Providence, RI, 2015. MR 3363157
13. Octav Cornea, Gregory Lupton, John Oprea, and Daniel Tanré, *Lusternik-Schnirelmann category*, Mathematical Surveys and Monographs, vol. 103, American Mathematical Society, Providence, RI, 2003. MR 1990857 (2004e:55001)
14. Michael W. Davis and Tadeusz Januszkiewicz, *Convex polytopes, Coxeter orbifolds and torus actions*, Duke Math. J. **62** (1991), no. 2, 417–451. MR 1104531 (92i:52012)
15. Graham Denham and Alexander I. Suciu, *Moment-angle complexes, monomial ideals and Massey products*, Pure Appl. Math. Q. **3** (2007), no. 1, Special Issue: In honor of Robert D. MacPherson. Part 3, 25–60. MR 2330154 (2008g:55028)
16. F. Fan, J. Ma, and X. Wang, *B-Rigidity of flag 2-spheres without 4-belt*, ArXiv e-prints (2015).
17. Yves Felix, Steve Halperin, and Jean-Claude Thomas, *Lusternik-Schnirelmann category of skeleta*, Topology Appl. **125** (2002), no. 2, 357–361. MR 1933583 (2004c:55025)
18. Yves Félix and Daniel Tanré, *Rational homotopy of the polyhedral product functor*, Proc. Amer. Math. Soc. **137** (2009), no. 3, 891–898. MR 2457428 (2009i:55011)
19. Ralph H. Fox, *On the Lusternik-Schnirelmann category*, Ann. of Math. (2) **42** (1941), 333–370. MR 0004108 (2,320f)
20. M. Franz, *The integral cohomology of toric manifolds*, Tr. Mat. Inst. Steklova **252** (2006), no. Geom. Topol., Diskret. Geom. i Teor. Mnozh., 61–70. MR 2255969 (2007f:14050)
21. D. R. Fulkerson and O. A. Gross, *Incidence matrices and interval graphs*, Pacific J. Math. **15** (1965), 835–855. MR 0186421 (32 #3881)
22. T. Ganea, *Lusternik-Schnirelmann category and strong category*, Illinois J. Math. **11** (1967), 417–427. MR 0229240 (37 #4814)

23. W. J. Gilbert, *Some examples for weak category and conilpotency*, Illinois J. Math. **12** (1968), 421–432. MR 0231375
24. Samuel Gitler and Santiago López de Medrano, *Intersections of quadrics, moment-angle manifolds and connected sums*, Geom. Topol. **17** (2013), no. 3, 1497–1534. MR 3073929
25. E. S. Golod, *Homologies of some local rings*, Dokl. Akad. Nauk SSSR **144** (1962), 479–482. MR 0138667 (25 #2110)
26. J. González, B. Gutiérrez, and S. Yuzvinsky, *The higher topological complexity of subcomplexes of products of spheres—and related polyhedral product spaces*, ArXiv e-prints (2015).
27. J. Grbic and S. Theriault, *Higher Whitehead products in toric topology*, ArXiv e-prints (2010).
28. Jelena Grbić, Taras Panov, Stephen Theriault, and Jie Wu, *The homotopy types of moment-angle complexes for flag complexes*, Trans. Amer. Math. Soc. **368** (2016), no. 9, 6663–6682. MR 3461047
29. Jelena Grbić and Stephen Theriault, *The homotopy type of the complement of a coordinate subspace arrangement*, Topology **46** (2007), no. 4, 357–396. MR 2321037 (2008j:13051)
30. ———, *The homotopy type of the polyhedral product for shifted complexes*, Adv. Math. **245** (2013), 690–715. MR 3084441
31. E. Grbich and S. Terio, *Homotopy type of the complement of a configuration of coordinate subspaces of codimension two*, Uspekhi Mat. Nauk **59** (2004), no. 6(360), 203–204. MR 2138475 (2005k:55023)
32. Melvin Hochster, *Cohen-Macaulay rings, combinatorics, and simplicial complexes*, Ring theory, II (Proc. Second Conf., Univ. Oklahoma, Norman, Okla., 1975), Dekker, New York, 1977, pp. 171–223. Lecture Notes in Pure and Appl. Math., Vol. 26. MR 0441987 (56 #376)
33. K. Iriye and D. Kishimoto, *Topology of polyhedral products and the Golod property of Stanley-Reisner rings*, ArXiv e-prints (2013).
34. ———, *Fat wedge filtrations and decomposition of polyhedral products*, ArXiv e-prints (2014).
35. ———, *Golodness and polyhedral products for two dimensional simplicial complexes*, ArXiv e-prints (2015).
36. I. M. James, *On category, in the sense of Lusternik-Schnirelmann*, Topology **17** (1978), no. 4, 331–348. MR 516214 (80i:55001)
37. L. Katthän, *A non-Golod ring with a trivial product on its Koszul homology*, ArXiv e-prints (2015).
38. I. Limonchenko, *Families of minimally non-Golod complexes and their polyhedral products*, ArXiv e-prints (2015).
39. ———, *Moment-angle manifolds, 2-truncated cubes and Massey operations*, ArXiv e-prints (2015).
40. Dennis McGavran, *Adjacent connected sums and torus actions*, Trans. Amer. Math. Soc. **251** (1979), 235–254. MR 531977 (82c:57026)
41. J. Scott Provan and Louis J. Billera, *Decompositions of simplicial complexes related to diameters of convex polyhedra*, Math. Oper. Res. **5** (1980), no. 4, 576–594. MR 593648
42. Yuli B. Rudyak, *On category weight and its applications*, Topology **38** (1999), no. 1, 37–55. MR 1644063 (99f:55007)
43. Graham Hilton Toomer, *Lusternik-Schnirelmann category and the Moore spectral sequence*, Math. Z. **138** (1974), 123–143. MR 0356037

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