A framework for the analysis of vibrations of structures with uncertain attachments

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Abstract. Attachments affect the dynamic response of an assembled structure. When engineers are modelling structures, small attachments will often not be included in the “bare” model, especially in the initial design stages. The location of these attachments might be poorly known, yet they affect the response of the structure. This paper considers how attachments jointed to the structure at uncertain points, can be included in the dynamic model of a structure. Two approaches are proposed. In the time domain, a combination of component mode synthesis, characteristic constraint modes and modal analysis gives a computationally efficient basis for subsequent analysis using, for example, Monte Carlo simulation. The frequency domain approach is based on assembly of frequency response functions of bare structure and attachment. Numerical examples of a beam and a plate with a point mass added at an uncertain location are considered and predictions compared with experiment results.

1. Introduction

In the initial design stage in particular, a model of the structure is often incomplete, with there being various additional componentry of various forms, referred to here as attachments, being added later, or maybe not included at all in the numerical model. Attachments play an important role in complex structures, for example bundles of wires in cars or aircraft, small control and sensing boxes in engines etc. When engineers model structures, normally small attachments will not be included in the “bare” mode. The location and properties of attachments might be poorly known at the beginning of the design process, and they are often fitted later. These uncertain attachments produce uncertainties in the dynamic response of the assembled structures. However, they might affect change the behaviour of the structures significantly and be a detrimental influence on the response of the products, such as higher noise or vibration levels. This paper considers the case where the properties of the attachment are known, but the points on the main structure to which it is attached are uncertain.

The finite element method (FEM) is well established for vibration analysis (e.g. [1]). In practice, models might be large and computational cost significant. The multi-point constraint (MPC) [2] method can be used to consider attachments at any position of the model without re-meshing or modifying the model. MPC was used by De Alba Alvarez *et al.* [3] to consider variability in the position of a spot weld on a structure. The frequency domain method [4] is another method to consider the attachments. McMillan *et al.* [5] used the plate Green function with the frequency domain method to consider effect of the point masses on a simply supported plate.

However, when applying FEM to complex structures there may be a very large number of DOFs, which can be a disadvantage. Therefore model reduction methods have been developed to improve the efficiency of calculation. One of the commonly used model reduction approaches is component mode synthesis (CMS), which partitions a structure into components. Thus different components could be analysed independently and these components are assembled together to calculate the eigensolutions of the whole structure. According to the different assumptions regarding the interface between sub-structures, it can be classified as fixed interface (Craig-Bampton) [6], free interface [7], loaded interface [8] and hybrid interface [9]. Other approaches have also been combined with CMS to do further calculations. Singh and Suarez [10] developed a method combining CMS and dynamic condensation together, in which the models of components were reduced by dynamic condensation. Finally Characteristic Constraint (CC) modal analysis has been suggested to reduce the number of the interface DOFs [11].

Uncertainty and variability have significant influence on manufactured products due to geometrical, material property parameter variations, assembly tolerances, model inaccuracy etc. Generally variability involves these inherent variations of the physical properties or environmental condition, such as the variations of geometric and material properties, and manufacturing variability. Compared with variability, lack of knowledge of systems leads to uncertainty, for example the inaccurate model and the unknown physical characteristics in the design stage [12]. Note that uncertainty can be reduced by increasing the knowledge of the system, while it is hard to eliminate the influence of variability. Variability is normally assessed by probabilistic methods, in which a probability density function is applied to the variable parameters. Monte Carlo simulation (MCS) is one of the commonly used methods to investigate variability in the structures. An overview of Monte Carlo simulation can be found in [13, 14]. While this approach is effective for the analysis of complex structures with many variables, the simulation is computationally expensive, especially when the complex structures have many uncertain parameters. One of the possible solution methods is the importance sampling method [15]. It selects the sampling points non-uniformly based on another probability distribution rather than sampling in the whole region. It has been demonstrated that the requirement of the number of sampling points to meet a given confidence level is very much lower than that of simple MCS [16]. A number of studies have concerned uncertain structures, but most work focusses on the uncertainty and variability of the structure itself, such as geometry [17], loading [18], etc.

Uncertainty and variability in the physical locations of attachments have been considered much less frequently. One example is given in reference [3]. This paper considers how the effects of these attachments might be included in the model and how they affect the dynamic response. Typically a (possibly very large) finite element (FE) model of the bare structure will be known and the attachment connected at some unknown location(s) within a region, as illustrated in Figure 1. The next section describes two approaches to the numerical analysis, section 3 contains numerical examples and experimental results, while section 4 concludes.

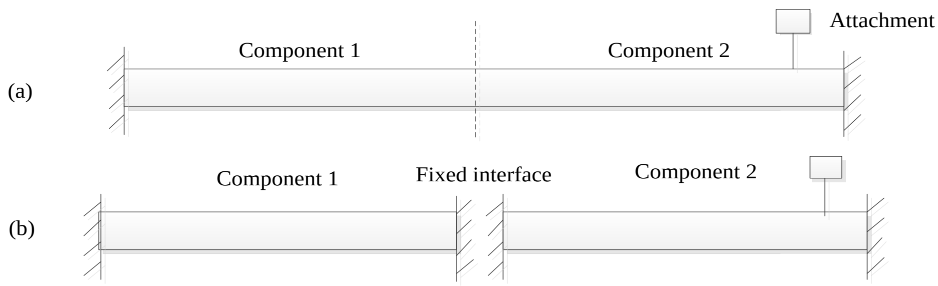


Figure 1 (a) A structure with two components and an attachment; (b) fixed interface model

2. Numerical methods

In this section two approaches are described, one based on a time domain, modal description using CMS: this involves assembly of the mass and stiffness matrices of the attachment and part of the structure. The second is a frequency domain approach (the frequency responses might follow from a modal model) with the assembly being performed in the frequency domain. In both cases the attachment is connected at some unknown point or points within a region of a structure. The location of the point might be described probabilistically (by a probability density function for example) or possibilistically (i.e. it lies at some points within the region) and predictions of the statistics or bounds of the response might be desired. The dynamic model of both bare structure and attachment are assumed to be known: uncertainty arises only in the points at which they are connected. In practice of course there might also be uncertainty in the dynamic models.

2.1. Component mode synthesis and characteristic constraint modes

Reviews of CMS can be found in [19]. The structure is divided into components connected at interfaces. The mass and stiffness matrices for each component are found. In the Craig-Bampton (CB) method, adopted here, the DOFs are the internal DOFs $\mathbf{x}_i$ of the component and the interface DOFs $\mathbf{x}_b$. The response is then described in terms of the fixed interface modes of the component and the constraint modes associated with unit displacement of each of the interface DOFs.

The general undamped equations of motion are

$$\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = \mathbf{F} \tag{1}$$

where **M** is the mass matrix, **K** is the stiffness matrix, **F** is the force vector and **x** is the vector of DOFs. Equation (1) can be partitioned as [20]

$$\begin{bmatrix} \mathbf{M}_{ii} & \mathbf{M}_{ib} \\ \mathbf{M}_{bi} & \mathbf{M}_{bb} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{x}}_i \\ \ddot{\mathbf{x}}_b \end{Bmatrix} + \begin{bmatrix} \mathbf{K}_{ii} & \mathbf{K}_{ib} \\ \mathbf{K}_{bi} & \mathbf{K}_{bb} \end{bmatrix} \begin{Bmatrix} \mathbf{x}_i \\ \mathbf{x}_b \end{Bmatrix} = \begin{Bmatrix} \mathbf{F}_i \\ \mathbf{F}_b \end{Bmatrix} \tag{2}$$

where *b* indicates the interface DOFs and *i* indicates the internal DOFs of the component. The $\mathbf{x}(t)$ is now transformed to a new set of coordinates by two terms: (1) vibration mode shapes $\mathbf{\Phi}_i$; (2) constraint modes $\mathbf{\Phi}_c$. The transformation can be written as

$$\mathbf{x} = \mathbf{\Phi}_i \boldsymbol{\eta}_i + \mathbf{\Phi}_c \mathbf{x}_b = \begin{bmatrix} \mathbf{\Phi}_{ii} & \mathbf{\Phi}_{ib} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\eta}_i \\ \mathbf{x}_b \end{Bmatrix} = \mathbf{\Psi}\mathbf{X} \tag{3}$$

where $\mathbf{\Phi}_{ii}$ is the reduced matrix of fixed interface mode shapes of internal DOFs, $\mathbf{\Phi}_{ib} = -\mathbf{K}_{ii}^{-1}\mathbf{K}_{ib}$, and $\mathbf{\Psi}$ is the transformation matrix. The resulting model of the component is smaller since only some of the fixed interface modes are retained.

The structure in Figure 1 is divided into two components. Component 1 comprises the bare structure outside the region in which the attachment might be connected: the modal analysis of this component need be performed just once.

Component 2 comprises the bare structure within the connection region and the attachment itself. The mass and stiffness matrices for component 2 are found by assembling the parts associated with part of the bare structure and the attachment. Modal analysis of the assembled matrices yields the fixed interface modes.

Now, since the connection points are uncertain, repeated analysis will be required for a range of possible connection points: in the examples below a Monte Carlo simulation (MCS) is performed. This then involves re-assembling the two parts of component 2. This multiple re-assembly is expedited if MPCs are used, so that the connection points can lie anywhere in the FE mesh.

The analysis thus involves a single modal analysis for most of the bare structure, assembly and modal analysis multiple times for the attachment region, global modal analysis and response prediction.

2.1.1. Characteristic constraint modes

Characteristic constraint (CC) modes involve a further modal analysis of the partitions of the matrices associated with the constraint modes [11] and further reduction in the model size. This further improves computational efficiency. The constraint modes of the CMS matrices can be written as [11]

$$\mathbf{K}_c \phi^{bb} = \lambda \mathbf{M}_c \phi^{bb} \tag{4}$$

Solving this eigenproblem gives the eigenvectors. After these eigenvectors are truncated, only some of the selected modes are kept, i.e.

$$\underbrace{\mathbf{\Phi}^{bb}}_{N_m \times N_m} \rightarrow \underbrace{\hat{\mathbf{\Phi}}^{bb}}_{N_m \times N_{CC}} \tag{5}$$

where $\mathbf{\Phi}^{bb}$ is the full eigenvector matrix of the interface, $\hat{\mathbf{\Phi}}^{bb}$ is the reduced eigenvector matrix with CC modes, N_m is the number of interface DOFs and hence constraint modes and N_{CC} is the number of CC modes. The retained eigenvectors are used to transform the mass and stiffness matrices of the CB model. It reduces the size of the matrices and yields a reduced-order model (ROM). The transformation matrix can be written as

$$\mathbf{X} = \begin{bmatrix} \mathbf{I} & 0 & 0 \\ 0 & \mathbf{I} & 0 \\ 0 & 0 & \hat{\mathbf{\Phi}}^{bb} \end{bmatrix} \mathbf{Y} \tag{6}$$

where $\mathbf{X}$ contains the generalized coordinates of the CMS in (3) and $\mathbf{Y}$ the coordinates of the ROM.

2.2. Frequency domain approach

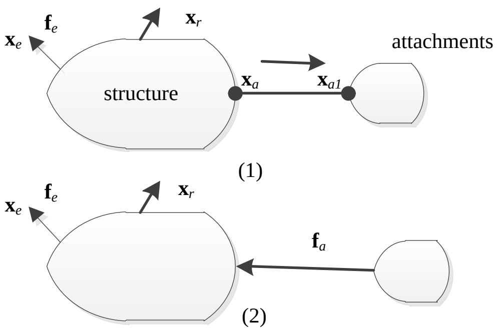


Figure 2 A structure with an attachment

In this approach the frequency responses of the bare structure and attachment are assembled. Excitation and response DOFs $\mathbf{x}_e$ and $\mathbf{x}_r$ on the structure in Figure 2 are identified. In terms of receptance the response of the bare structure without the attachments is

$$\mathbf{x}_r = \mathbf{H}_{re}\mathbf{f}_e \tag{7}$$

In Figure 2, $\mathbf{x}_a$ and $\mathbf{x}_{a1}$, are the response of the structure and the attachments respectively, and $\mathbf{f}_a$ is the interconnecting force. When the attachments are added, continuity and equilibrium imply that

$$\begin{gathered}\mathbf{x}_a = \mathbf{x}_{a1} \\ \mathbf{H}_{re}\mathbf{f}_e + \mathbf{H}_{ra}\mathbf{f}_a = -\mathbf{H}_a\mathbf{f}_a\end{gathered} \tag{8}$$

Equation (8) can be rewritten as [4]

$$\begin{gathered}\mathbf{x}_a = -\mathbf{H}_{aa}\mathbf{H}_a^{-1}\mathbf{x}_a + \mathbf{H}_{ae}\mathbf{f}_e \\ \mathbf{x}_r = \mathbf{H}_{re}\mathbf{f}_e - \mathbf{H}_{ra}\mathbf{H}_a^{-1}\mathbf{x}_a\end{gathered} \tag{9}$$

where $\mathbf{H}_a$ is the receptance matrix of the attachments.

The total response of the combined structure at position r can be expressed as

$$\mathbf{x}_r = \left[\mathbf{H}_{re}\mathbf{f}_e - \mathbf{H}_{ra}\mathbf{H}_a^{-1}\left[\mathbf{I} + \mathbf{H}_{aa}\mathbf{H}_a^{-1}\right]^{-1}\mathbf{H}_{ae}\right]\mathbf{f}_e \tag{10}$$

Note that the matrices being inverted are relatively small.

3. Numerical and experiment results

In this section two examples – a beam and a plate – are considered to demonstrate the above methods and to validate results against experimental measurements. In the two examples the attachment is a small mass, modelled as a point mass, attached within a region of an otherwise homogeneous structure.

3.1. Cantilever beam with added mass

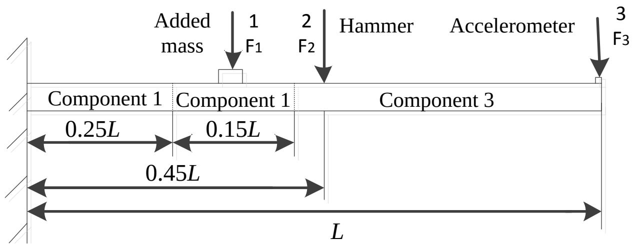
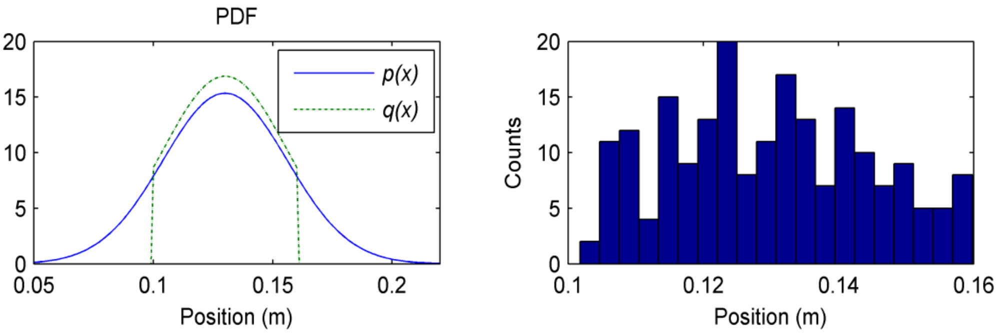


Figure 3 The beam with three components

The first example comprises a steel cantilever beam. The physical and geometrical properties are: cross-sectional area A=39.85$\times$2.01mm^2, density $\rho = 7.750\times10^{-3}$g/mm^3 and length L=400mm. The elastic modulus, estimated by minimising the mean-square error of the first 7 measured and predicted natural frequencies, was taken to be E=1.9982GPa. An impact hammer was used to excite the beam and the response measured by an accelerometer mounted at the tip. The mass of the accelerometer was included in the predictions. The frequency response between these points was measured both for the bare beam (to estimate E) and with a small mass of 47g added at some point between 0.25L and 0.4L from the clamped end.



(a) Normal and truncated PDFs *p(x)* and *q(x)* (b) Sample distribute

Figure 4 (a) A plot of *p*(x) and *q*(x), and (b) the sampling positions

In the numerical predictions a finite element model [1] was developed in Matlab using thin beam elements and MPCs and frequency domain methods when including the added mass. In the CMS approach the beam is divided into three components as shown in Figure 3. The added mass is located on component 2. Each component is divided into 20 elements and 10 fixed interface modes of each component are retained.

The position of the added mass was assume to be a random variable with a Gaussian distribution, the mean and standard deviation being 0.13m and 0.026m respectively, truncated so as to lie within region 2. A MCS was performed using the importance sampling method [15]. A scaled normal distribution $N\left[x\middle|0.13, 0.026^2\right]$ was selected as the sampling distribution, with parameters chosen such that $p(x) < kq(x)$, $0.1\text{m} < x < 0.16\text{m}$ and

$$p(x) = \frac{1}{\sqrt{2\pi(0.026)^2}} e^{\frac{-(0.13-x)^2}{2(0.026)^2}};\quad q(x) = \frac{1.1}{\sqrt{2\pi(0.026)^2}} e^{\frac{-(0.13-x)^2}{2(0.026)^2}} \tag{11}$$

200 sampling positions were selected and the relevant FRFs are predicted. The truncated PDF and the distribution of the sample are shown in Figure 4.

The measured accelerances of the beam (36 random positions), and those predicted (200 positions) by MPC (with CMS) and the frequency domain approach (with CMS), are shown in Figure 5 and Figure 6 respectively. The effect of the mass is to decrease the natural frequencies of the bare beam, the effects depending on the location of the mass. Both numerical approaches predict the spread of responses well. Errors are larger at higher frequencies, perhaps because the mass moment of inertia

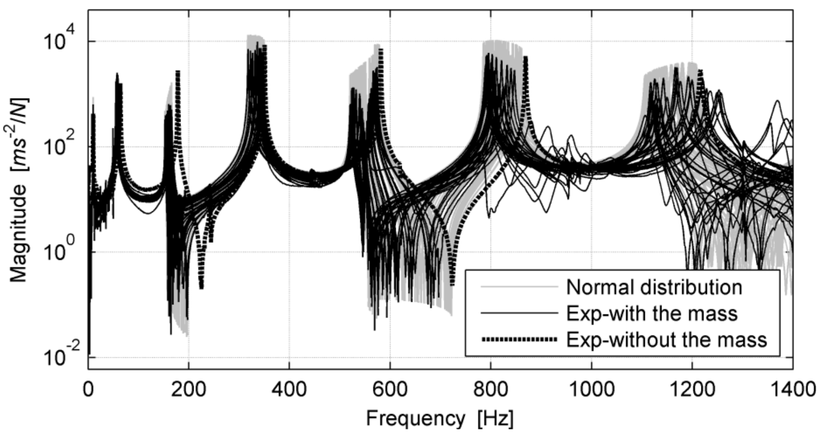


Figure 5 Measured FRFS and predicting using CMS and MPCs

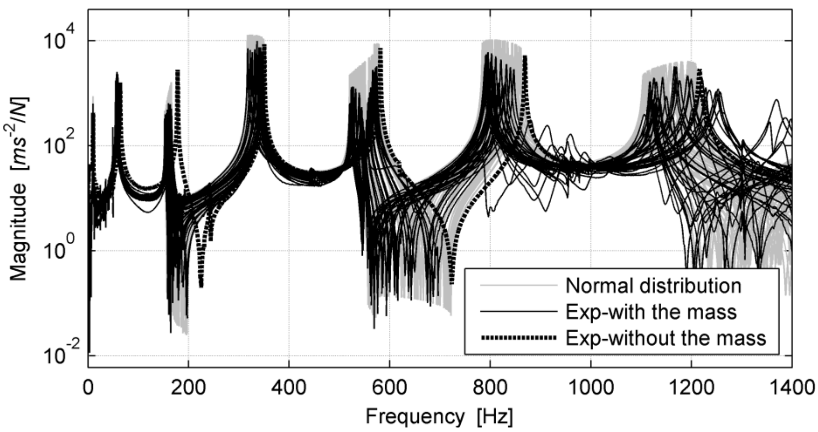


Figure 6 Measured FRFS and predicting using the frequency domain method and CMS

was neglected in the numerical models.

3.2. Free suspended plate with added mass

The second example comprises a 500 mm x 300mm steel plate, 1.62mm thick, as shown in Figure 7. The density and Poisson's ratio were taken as $7.5201\times10^{-3}\,\mathrm{g/mm^3}$ and 0.33 respectively. The

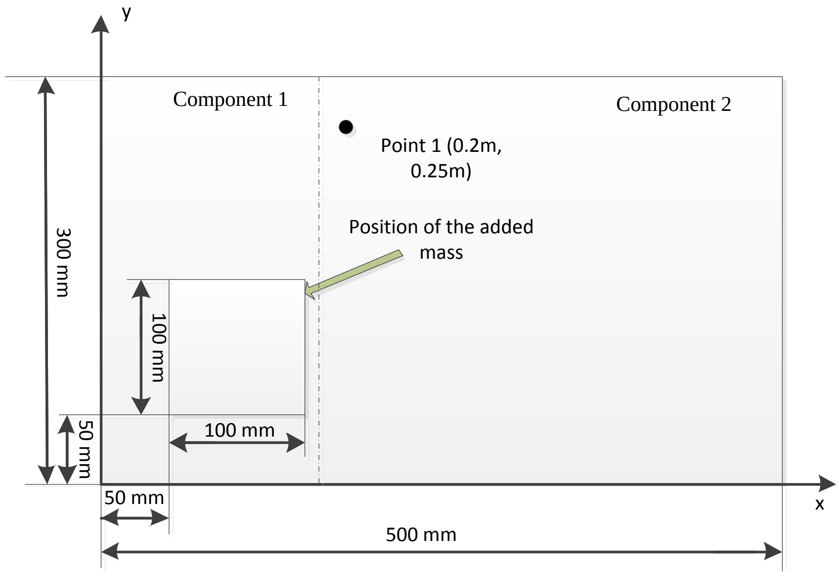


Figure 7 The profile of the plate divided into two components

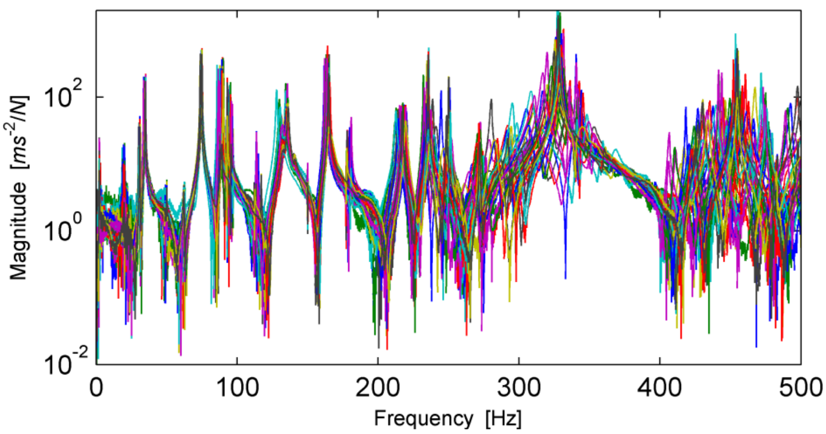


Figure 8 The measured FRFs of the plate with the added mass

Young's modulus was estimated by minimising the mean-square error in the first 13 measured and predicted natural frequencies and taken to be 1.748GPa.

In the experiment the plate was suspended by two elastic cords. An impact hammer and a laser vibrometer were used to measure the input accelerance at point 1. The mass of 156g was attached at 35 randomly selected points within the square of side 100mm shown. The variable points were selected around the central position of the square. Figure 8 shows the measured accelerances. There is a higher spread in the resonance peaks at higher frequencies.

In the numerical predictions Kirchhoff plate theory [1] was used in Matlab, and the frequency domain method and MPCs were applied respectively to include the added mass. The moment of inertias of the added mass was ignored in simulation. Two 10×10 meshes were used to construct the FE model of component 1 and component 2. A CB model of the plate was generated and the first 20 modes were retained for each component. In addition, there were 33 constraint modes, which were truncated to 10 modes. The positions of the added mass were assumed to obey the normal distribution $N\left[(x,y)\middle|0.1,0.02^2\right]$, with the mean hence being in the centre of the square. The importance sampling method was used to sample 100 points in the $100\text{mm}\times100\text{mm}$ square area.

The measured and predicted results are shown in Figure 9 and 10. The agreement is very good at lower frequencies, below about 170Hz. The agreement is less good at higher frequencies, with there being greater dispersion in the predicted results.

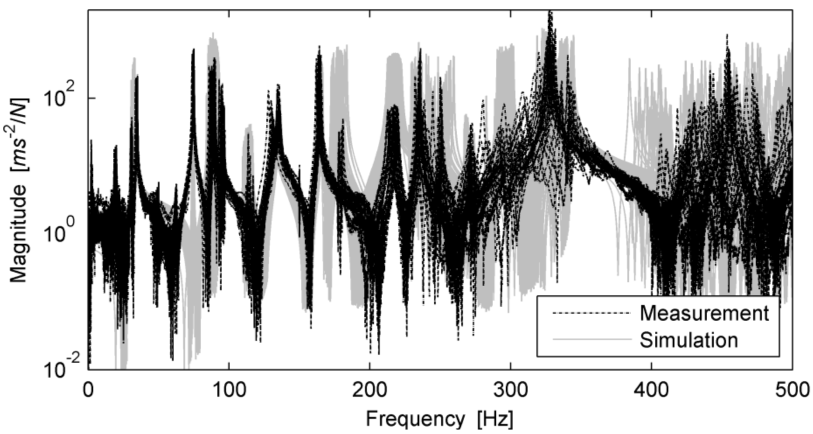


Figure 9 Measured FRFs and predicting using CMS and MPCs

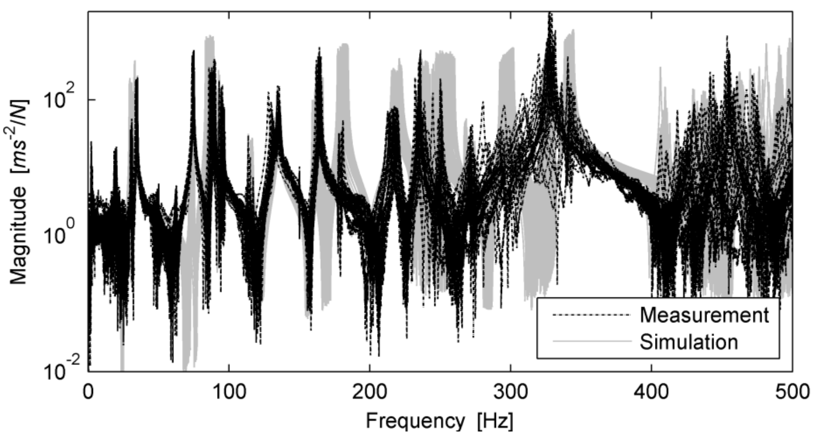


Figure 10 Measured FRFs and predicting using CMS and the frequency domain method

4. Concluding remarks

The paper discussed the influences of attachments located at uncertain points on the vibrational response of the structure. Two approaches were suggested. The first involves taking the mass and stiffness matrices of the attachment and the region of the structure to which it might be connected, and assembling them with the mass and stiffness matrices for the remainder of the structure. The resulting time domain model is then solved. To make this approach efficient, the model reduction method, Craig-Bampton CMS, with CC modes was proposed to increase the efficiency of calculation. Furthermore, MPCs were used so that the attachment can be connected at any location without the need to remesh the structure.

The second approach involves calculating the frequency response for both regions and assembling them using the method in chapter 8 of [4]. The application of the methods was validated experimentally on a cantilever beam and a free suspended plate with a small mass attached at a random position.

It was seen that the case of an added mass decreases the natural frequencies, and the effect depends on the locations of the added mass. The predicted results are consistent with the results of experiments especially at lower frequencies.

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