Joint Pricing and Inventory Decisions for Substitutable Perishable Products under Demand Uncertainty

by

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ABSTRACT

The focus of this thesis is to develop demand uncertainty models for retailers making optimal pricing and inventory decisions on substitutable and perishable products. In particular we study three approaches for handling demand uncertainty models: (i) a stochastic programming approach for two substitutable and perishable products over a two period planning horizon; (ii) a stochastic programming approach for multiple substitutable and perishable products over multiple periods; (iii) a robust optimization approach for two substitutable and perishable products over a single period. The three models support decision makers in retailing to incorporate the future demand uncertainty and substitution between similar products into their pricing and inventory decisions.

In the context of a stochastic programming approach for two substitutable and perishable products problem over two periods, a stochastic dynamic programming model has been proposed in which the retailer aims at maximizing the total profit. The property of optimal solutions is analyse from which an efficient search algorithm is developed to obtain the optimal results. Numerical results are reported using a case study based on a high-street fashion company. The sensitivity of the models' parameters is also analysed to address the importance of data accuracy on decision variables and total profit. The benefits of considering pricing and inventory decisions simultaneously will be demonstrated and the total profit is observed to be significantly improved through the consideration of price substitution between substitutable products.

In the context of a stochastic programming approach for multiple substitutable and perishable products problem over multiple periods, two stochastic dynamic programming models are proposed in which the decision maker can employ multiple markdowns on the prices. An efficient search algorithm has been developed by analysing the property of the optimal solutions. The benefits of making joint pricing and inventory decisions, considering substitutions between similar products; and dividing selling periods into more periods have been quantified.

In the context of the robust optimization approach, we relax the assumption on the complete knowledge of the demand distribution from the stochastic dynamic programming models and develop a robust optimization model. The parameters of the demand function are assumed to belong to an uncertainty set, and our objective is to find the optimal ordering quantity and price which maximize the worst-case profit. We extend a Newsvendor model in the face of uncertainty to consider the optimal pricing and inventory decisions of a retailer. Numerical tests are presented based on a case study of the retailing branch of a solar panel manufacturer. The trade-off between uncertainty level and total profits is illustrated, the sensitivity of parameters is also analysed.
# Contents

Declaration of Authorship .............................................................. xv
Acknowledgements ............................................................................. xvii

1 Introduction ..................................................................................... 1
1.1 Background .................................................................................... 1
1.2 Definition of Substitution ................................................................. 1
1.3 Definition of Perishability ................................................................. 2
1.4 Motivation and Contributions ......................................................... 3
1.5 Demand Uncertainty on Joint Pricing and Inventory Decisions ........ 4
1.5.1 Stochastic Programming ............................................................. 5
1.5.2 Robust Optimization ................................................................. 6
1.6 Structure of the Thesis .................................................................... 7

2 Literature Review ............................................................................. 9
2.1 Inventory Management under Demand Uncertainty ....................... 10
2.1.1 Stable Products ......................................................................... 10
2.1.2 Perishable Products ................................................................. 10
2.2 Pricing Optimization under Demand Uncertainty ......................... 11
2.2.1 Stable Products ......................................................................... 11
2.2.2 Perishable Products ................................................................. 13
2.3 Joint Pricing and Inventory Decisions with Deterministic Demand .... 14
2.4 Joint Pricing and Inventory Decisions with Stochastic Programming ... 15
2.4.1 Stable Products ......................................................................... 15
2.4.2 Perishable Products ................................................................. 16
2.4.3 Multiple Substitutable Products .................................................. 18
2.4.3.1 Stable Products .................................................................. 19
2.4.3.2 Perishable Products ............................................................ 19
2.5 Joint Pricing and Inventory Decisions under Robust Optimization .... 20
2.5.1 Inventory ................................................................................. 21
2.5.2 Pricing ..................................................................................... 21
2.6 Conclusion ..................................................................................... 22

3 Joint Pricing and Inventory Decisions for Two Substitutable and Perishable Products Over Two Periods Lifetime: A Stochastic Programming Approach .................................................. 25
3.1 Introduction .................................................................................... 25
3.2 Model Formulation ....................................................................... 27
3.2.1 Assumptions and Notations ........................................ 27
3.3 Dynamic Programming Model ........................................ 28
3.4 Optimality Analysis .................................................. 30
  3.4.1 Optimal Discounted Price ....................................... 30
  3.4.2 Optimal Order Quantity ......................................... 32
3.5 Algorithm ............................................................ 36
3.6 Numerical Example .................................................... 39
  3.6.1 Data Specification ............................................... 39
  3.6.2 Benefits of Jointly Making Inventory and Pricing Decision .... 42
  3.6.3 Benefits of Considering Substitutions .......................... 43
  3.6.4 Parameter Sensitivity Analysis ................................. 45
    3.6.4.1 Optimal Decisions and Expected Profit Responsiveness
            under Increasing Demand Loss Rate .......................... 46
    3.6.4.2 Optimal Decisions and Expected Profit Responsiveness
            Under Increasing Demand Substitution Rate .................. 51
    3.6.4.3 Optimal Decisions and Expected Profit Responsiveness
            under Increasing Demand Intercept $b_2$ ...................... 58
3.7 Conclusion .......................................................... 63

4 Joint Pricing and Inventory Decisions for Two or More Substitutable
  and Perishable Products Over Multiple Periods Lifetime: A Stochastic
  Programming Approach .................................................. 65
  4.1 Introduction ........................................................ 65
  4.2 Two Substitutable Products ....................................... 66
    4.2.1 Problem Description ......................................... 66
    4.2.2 Model Formulation .......................................... 66
  4.3 Multiple Products .................................................. 73
    4.3.1 Model Formulation .......................................... 74
    4.3.2 Optimality Analysis ....................................... 75
  4.4 Algorithm .......................................................... 77
  4.5 Numerical Results .................................................. 79
    4.5.1 Benefits of Jointly Making Inventory and Pricing Decisions .... 80
    4.5.2 Benefits of Considering Substitution ........................ 82
    4.5.3 Benefits of Dividing Selling Periods into More Periods ........ 88
  4.6 Conclusion ........................................................ 89

5 A Robust Optimization Approach for Joint Pricing and Inventory De-
  cisions for Two Substitutable and Perishable Products .................. 91
  5.1 Introduction ........................................................ 91
  5.2 Robust Formulation ............................................... 92
  5.3 Robust Optimization with Budget of Uncertainty .................. 95
  5.4 Robust Optimization for Substitutable Products .................. 98
  5.5 Numerical Results ............................................... 104
    5.5.1 Single Product Robust Optimization Model .................... 105
    5.5.2 Two Substitutable Products Robust Optimization Model ....... 116
  5.6 Conclusion ........................................................ 132

6 Conclusion .......................................................... 133
## CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1 Introduction</td>
<td>133</td>
</tr>
<tr>
<td>6.2 Implications</td>
<td>135</td>
</tr>
<tr>
<td>6.2.1 Theoretical Implications</td>
<td>136</td>
</tr>
<tr>
<td>6.2.2 Practical Implications</td>
<td>136</td>
</tr>
<tr>
<td>6.3 Limitations</td>
<td>136</td>
</tr>
<tr>
<td>6.4 Future Work</td>
<td>137</td>
</tr>
</tbody>
</table>

References 139
List of Figures

3.1 Products flow .................................................. 26
3.2 Variables Description of 2 Substitutable Perishable Products over 2 Periods Lifetime problem .................................................. 27
3.3 Holding cost and backorder cost in a function of $z_{it}$ ......................... 31
3.4 Expected total profit $V_1$ ........................................ 35
3.5 Expected total profit in the second period .................................... 37
3.6 Total profit increase by optimization model ................................... 41
3.7 Total profit of three optimization models .................................... 42
3.8 The total profit $V_2$ from model 1 and model 2 in second period .............. 44
3.9 The percentage of differences between model 1 and model 2 in each run .... 44
3.10 The percentage of differences between model 1 and model 2 during each run in higher demand substitution cases ................................. 45
3.11 The impact of varying demand loss rate $a_{12}$ on optimal ordering quantities of both products .................................................. 46
3.12 The impact of varying demand loss rate $a_{12}$ on optimal prices of both products .................................................. 47
3.13 The impact of varying demand loss rate $a_{12}$ on expected total profit ....... 47
3.14 The impact of varying demand loss rate $a_{22}$ on optimal ordering decisions of both products .................................................. 48
3.15 The impact of varying demand loss rate $a_{22}$ on optimal prices of both products .................................................. 49
3.16 The impact of varying demand loss rate $a_{22}$ on expected total profit ....... 50
3.17 The impact of varying demand substitution rate $l_{12}$ on optimal ordering quantities of both products .................................................. 52
3.18 The impact of varying demand substitution rate $l_{12}$ on optimal prices of both products .................................................. 53
3.19 The impact of varying demand substitution rate $l_{12}$ on expected total profit 54
3.20 The impact of varying demand substitution rate $l_{22}$ on optimal ordering quantities of both products .................................................. 55
3.21 The impact of varying demand substitution rate $l_{22}$ on optimal prices of both products .................................................. 56
3.22 The impact of varying demand substitution rate $l_{22}$ on expected total profit 57
3.23 The impact of varying demand upper bound $b_{12}$ on optimal ordering quantities of both products .................................................. 59
3.24 The impact of varying demand upper bound $b_{12}$ on optimal prices of both products .................................................. 59
3.25 The impact of varying demand upper bound $b_{12}$ on expected total profit . 60
3.26 The impact of varying demand upper bound $b_{22}$ on optimal ordering quantities of both products 61
3.27 The impact of varying demand upper bound $b_{22}$ on optimal prices of both products 62
3.28 The impact of varying demand upper bound $b_{22}$ on expected total profit 63

4.1 Products flow 66
4.2 Total holding and backorder cost in period $t$ 70
4.3 Expected total profit in the first period 72
4.4 Differences of three cases applied in period 3 only on overall total profits 81
4.5 Differences on total profits of period 3 on all three cases 81
4.6 Benefits of joint pricing and inventory optimization by comparing the total profits of all three cases 82
4.7 Difference in total profits by applying model 1 and model 2 in period 3 only 84
4.8 Percentage of profit increase by applying model 2 in Period 3 85
4.9 Differences by applying model 1 and 2 in period 2 and 3 86
4.10 Percentage of profit increase by applying model 2 in period 3 with a higher demand substitution rate 87
4.11 Benefits of considering substitution with a higher demand substitution rate 88
4.12 Benefits of dividing one period into more periods 89

5.1 The graphical description of the function $f_i(\tilde{D}_i)$ 94
5.2 The graphical description of the function $f_i(D_i)$ 102
5.3 Trade-off between robustness and optimal profits 107
5.4 Trade-off between robustness and optimal price 108
5.5 Trade-off between robustness and optimal order quantity 109
5.6 The impact of the half-length for the range of demand lost rate ($\beta'$) on optimal total profits under two levels of uncertainty ($\Gamma = 0.5$ and $\Gamma = 1.5$) 111
5.7 The impact of the half-length for the range of demand loss rate ($\beta'$) on optimal price under two levels of uncertainty ($\Gamma = 0.5$ and $\Gamma = 1.5$) 112
5.8 The impact of the half-length for the range of demand loss rate ($\beta'$) on optimal order quantities under two levels of uncertainty ($\Gamma = 0.5$ and $\Gamma = 1.5$) 113
5.9 The impact of the half-length for the range of the constant value in demand ($\alpha'$) on optimal total profits under two levels of uncertainty ($\Gamma = 0.5$ and $\Gamma = 1$) 114
5.10 The impact of the half-length for the range of the constant value in demand ($\alpha'$) on optimal price under two levels of uncertainty ($\Gamma = 0.5$ and $\Gamma = 1$) 115
5.11 The impact of the half-length for the range of the constant value in demand ($\alpha'$) on optimal order quantities under two levels of uncertainty ($\Gamma = 0.5$ and $\Gamma = 1$) 116
5.12 Trade-off between robustness and optimal profits 118
5.13 The impact of varying $\Gamma_1$ and optimal price 119
5.14 The impact of varying $\Gamma_1$ on optimal order quantities 120
5.15 The impact of the half-length for the range of demand loss rate ($\beta'_1$) on total profits under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$) 121
5.16 The impact of the half-length for the range of demand loss rate ($\beta_1'$) on optimal prices of product 1 under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$) ............................................ 122
5.17 The impact of the half-length for the range of demand loss rate ($\beta_1'$) on optimal prices of product 2 under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$) ............................................ 123
5.18 The impact of the half-length for the range of demand loss rate ($\beta_1'$) on optimal order quantities of product 1 under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$) ............................................ 124
5.19 The impact of the half-length for the range of demand loss rate ($\beta_1'$) on optimal order quantities of product “2” under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$) ............................................ 125
5.20 The impact of $\alpha_1'$ on optimal total profits under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$) ............................................ 126
5.21 The impact of half-length for the range of demand substitution rate $l_2'$ on optimal total profits under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$) ............................................ 127
5.22 The impact of half-length for the range of demand substitution rate $l_2'$ on optimal prices of product “1” under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$) ............................................ 128
5.23 The impact of half-length for the range of demand substitution rate $l_2'$ on optimal prices of product “2” under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$) ............................................ 129
5.24 The impact of half-length for the range of demand substitution rate $l_2'$ on optimal order quantities of product “1” under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$) ............................................ 130
5.25 The impact of half-length for the range of demand substitution rate $l_2'$ on optimal order quantities of product “2” under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$) ............................................ 131
List of Tables

2.1 Legend for Classification System ........................................ 9
3.1 Optimal decision variables ............................................. 40
3.2 Summary of varying $b_{i2}$ on optimal decisions and total profit .......... 51
3.3 Summary of varying $t_{i2}$ on optimal decisions and profit ................. 58
3.4 Summary of varying $b_{i2}$ on optimal decisions and Profit ................ 63
4.1 Optimal decisions of this three-periods case ................................ 79
5.1 Input parameters for demand ............................................ 105
5.2 Given input parameters .................................................. 106
5.3 Summary of results on six solvers .................................... 106
5.4 Input parameters for demand ............................................ 117
5.5 Input parameters of costs .............................................. 117
Declaration of Authorship

I, Fei Fang, declare that this thesis entitled Joint Pricing and Inventory Decisions for Substitutable Perishable Products under Demand Uncertainty and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

• this work was done wholly or mainly while in the candidature for a research degree at this University;
• this work has been presented at 2013 RMPI conference and 2016 Informs International Conference;
• where any part of this thesis has previously been submitted for a degree or any other qualification at this University or other institution, this has been clearly stated;
• where I have consulted the published work of others, this is always clearly attributed;
• where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
• I have acknowledged all main sources of help;
• where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:..........................................................................................................................

Date:.............................................................................................................................
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To my parents, for their love and support
Chapter 1

Introduction

1.1 Background

Since the economic crisis in 2007, retailers have been facing ever-changing sets of challenges including lower customer confidence on spending, lower sales and squeezed profits, and have expended great effort to overcome them in order to remain competitive. In recent years, we have witnessed an increasing number applications of analytic models to support management decisions regarding inventory management/production planning (which determines the cost of satisfying demand) and pricing strategy (which influences that demand). Pricing and inventory management has long been, and will continue to be, the core capability for retailers.

The area of joint pricing and inventory decisions is an important one in inventory management. In recent years there has been a rapid growth of scholars working on this topic in an effort to improve retailers’ operations. Innovative pricing strategies have been successfully deployed in a variety of industries. For instance, dynamic pricing is employed by many firms incorporating inventory management to improve their operations. There are several factors leading to the growth of the employment of dynamic pricing strategies. Firstly, the collection and maintenance of customer data has taken great advantage of the information technology breakthrough. Secondly, the cost of price changes has significantly dropped. Thirdly, there is a rich body of academic literature which provides analytical models and algorithms for price optimization. Finally, decision support applications for complex optimization problems have been developed.

1.2 Definition of Substitution

In this thesis, we examine the impact of product substitution on optimal solutions under demand uncertainty. In the context of this work, we will investigate the strategy of
meeting customers’ demands by substituting products for other similar products. There are two types of substitution generally studied in the literature. Bitran et al. (2006) describes both as follows: One is inventory-driven substitution, is an approach of shifting to a substitute product to meet the demand if a product is out of stock; Price-driven substitution, is another approach of exploring the impact of the price differences for other similar products on the demand of one product. In our research, we only consider the optimal solutions by focusing on price-driven substitution under uncertainty.

Nowadays, many retailers increase product variety by introducing a group of similar products in order to obtain more market share in the competitive world. Accordingly, the challenge on how to manage them has been rising. Firstly, similar products are distinguished by different variants (e.g. brand, material, colour or design), and are often substitutable to a certain extent (Tang and Yin, 2007). The underlying demand for these similar products can be influenced by the price-driven substitution of other similar products. The total profit can be maximized through jointly considering making pricing decisions of these substitutable products to manipulate the demand. In this research, we will address this challenge by studying how to determine the prices for substitutable products in order to maximize the total profits.

1.3 Definition of Perishability

In the areas of pricing and inventory optimization literature, two types of products can be considered. One is a stable product, which will remain sellable indefinitely, the other being a perishable product, which will not be sellable due to its short shelf-life (Federguen et al., 1986). Some typical types of perishable products include: newspapers, magazines, seasonal products, fast fashion products, airlines tickets, hotel rooms, fresh food, seasonal clothing, and new cars.

The perishable products problem is usually modelled quite differently from that of stable products. Whereas stable products can be infinitely planned for and sold, the planning horizon for perishable products is very short. Furthermore, pricing discounts are usually given for many perishable products which approach the ends of their lifetimes.

Over the decades of the development of inventory management, there have been a lot of different classifications of perishability. Nahmias (1982) classify perishable products with fixed lifetime and random lifetime. He assumes that in a fixed lifetime case, the products will be stocked to meet the demand until they have to be discarded after a known fixed time. In the random lifetime case, it is difficult to determine the lifetime of some products in advance, therefore products will be discarded when they become spoiled; the time at which this happens may be uncertain. However, the lifetime is assumed to follow a known distribution. Examples of these type of products are fruit, vegetables and flowers. Raafat (1991) describes two concepts of deterioration: (1) products which are obsolete
at the end of the planning horizon, e.g., fashion products, airline tickets, car rental; (2) 
products which deteriorate throughout the planning horizon; either fixed shelf life or 
random shelf life is assumed. He further introduces the classification of perishability 
on the relation between time and utility as: (1) constant-utility, where utility remains 
the same until the end of the usage period, e.g., drugs; (2) increasing utility where 
utility increases in the planning horizon, e.g., wines; (3) decreasing utility over the 
planning horizon, e.g., fruits, vegetables and fresh foods. More recently, Ferguson and 
Koenigsberg (2007) give the categorization of perishability related to the level of quality 
perceived by consumers as: (1) perceived quality remains the same but products will be 
discarded after a known period, - examples include hotel rooms and airline tickets; (2) 
value decreases continuously over time until reaching zero, - examples include newspapers 
and weather forecasting; and (3) perceived quality deteriorates over the planning horizon 
but may not reach zero, - examples include fashion clothing and high technology products 
with short life cycle. Amorim et al. (2013) review the literature on the classification 
of perishability. They provide a unified framework to classify perishability composed 
of three classifying dimensions: physical product deterioration, authority limits, and 
customer value. They suggest that by considering these three factors perishability can 
be classified in a more accurate way than by considering just one factor. Physical product 
deterioration reflects whether or not products are subject to physical modifications or 
not over the planning periods. Authority limits represent the external regulations or 
conventions which may affect the perishability. Customer value corresponds to the 
willfulness of customers to pay for the products.

In this thesis, we consider perishable products subject to at least one of the following 
scenarios: (1) physical deterioration; (2) customer value reaches zero after a given time, 
also referred as obsolescence; (3) customer value decreases over the planning horizon. 
Examples of physical deterioration without customer value decreasing include human 
blood, drugs and gasoline. Examples of customer value decreasing without physical 
changes include newspapers, fast fashion clothes and high technology products with a 
short life cycle; examples of customer value reaching zero include airline tickets, car rental 
and hotel rooms. In Chapters 3 and 4, we model the problem that retailers orders and 
sells substitutable and perishable products in the 2 or more planning periods. The value 
decreases and reaches zero at the end of the planning periods, therefore we consider 
them as a perishable product problem. In Chapter 5, we consider the robust model 
that the value of the leftover products reaches zero at the end of the period, therefore, 
considered as a perishable product problem.

1.4 Motivation and Contributions

Traditionally, many researchers’ work has focused on pricing optimization as a tool to 
improve revenue management and inventory optimization to reduce the cost of inventory
management. The integration of pricing and inventory decisions to jointly, rather than individually, maximize the profits of a retail system can significantly improve profitability. Furthermore, due to the fast development of new technologies and rapidly changing fashion trends and markets, product value decreases more quickly than before, and more products can be categorised as perishable products. Researchers have focused on studying the joint pricing and inventory decisions for a single perishable product: Li et al. (2009), Chen and Sapra (2013), Chen et al. (2014) and Chung et al. (2015). However, nowadays, many retailers order and sell multiple similar products simultaneously, and thus there is a great need to investigate the substitution of similar perishable products by jointly considering pricing and inventory decisions, which may increase their profits.

Previous studies have employed stochastic programming to address demand uncertainty by assuming a complete knowledge of the probability distributions of data. However, in real-world applications, the use of stochastic programming is often limited by the dependency on the availability of probabilistic information, the shortage in operational research knowledge of practitioners, and computational challenges. Some researchers take the more recent approach to address demand uncertainty, that of robust optimization. This approach has been developed to study demand uncertainty without making any assumption on the distributions. The robust optimization approach is still not widely used in many retail, but it has a great potential to improve profitability in much the same way as stochastic programming has done in this area.

The intention of this thesis is to develop demand uncertainty models by stochastic programming and robust optimization, in order to study the joint pricing and inventory decisions problem for retailers making pricing and inventory decisions on substitutable and perishable products. The numerical results are based on case studies using real data. To the best of our knowledge, Chapters 3 and 4 are the first two works in the literature to study the joint pricing and inventory control problem across similar perishable products by considering holding and backorder costs. Chapter 5 is the first work that uses ideas of robust optimization in the context of joint pricing and inventory decisions of perishable products by considering price-substitution. More specific contributions are listed in the conclusion of each chapter.

1.5 Demand Uncertainty on Joint Pricing and Inventory Decisions

Over the past 10 years, customer preference is more uncertain than ever before (Dyer et al., 2014). The more unknowns we have relating to customer preference, the higher the level of demand uncertainty, which means that retailers have difficulty in accurately forecasting future customer demand. This imposes a considerable challenge on decision makers when determining pricing and making inventory decisions.
There are a number of factors which cause uncertain demand. The development of new products, limited data collected, a rising number of competitors due to globalisation and changes in customer needs and interests all result in more difficulty for retailers to plan future demand. These factors could, therefore, be a critical factor in the profitability of retailers. In joint pricing and inventory management, when demand cannot be accurately predicted, the inventory could be over-bought and profit would then be negatively impacted due to the cost of holding excess inventory and disposing of wasted products. Conversely, if less was bought to avoid waste, unsatisfied demand would cause high backorder costs or lost sales which may upset customers.

Demand uncertainty has been of great interest among researchers studying joint pricing and inventory management (Federgruen and Heching, 1999; Su, 2007; Aydin and Porteus, 2008; Chen et al., 2015). There are two popular types of approaches in addressing demand uncertainty: stochastic programming and robust optimization. In stochastic programming, uncertain parameters are assumed as a random variable, with known probability distribution, whereas in robust optimization, no known probability distribution of the uncertain parameters is assumed; rather, there is only an assumption of a range of possible values for the uncertain parameters, usually in the form of an interval centred around some nominal value.

The demand for a product can be dependent on many variables such as price, brand, design and quality. In this research, the demand is restricted to be dependent only on the price. We consider two approaches in our thesis; stochastic programming is studied in Chapters 3 and 4 whereas robust optimization is employed in Chapter 5. The uncertain demand considered in this thesis are backordered, lost sale is not considered here. We assume the scenario that if the product is out of stock, retailers record the sales order, and then place the order from supplier. Once they received it, they will inform customers to collect it. The new shipment is assumed to be available at a very short time. Lost sales are considered in the case that the customer does not wait for the product and changes their mind about purchasing the product, this demand is mostly not recorded.

1.5.1 Stochastic Programming

The first work employing stochastic programming to model data uncertainty can be traced back to Dantzig (1955), where the researcher models some random events which could occur at the later stages. In stochastic programming, uncertainty is addressed as random variables which follow probability distributions. These distributions are assumed to be known or can be estimated. The objective is to maximize or minimize the expectation of the objective function in such a way that the models can be solved analytically or numerically and analysed to provide some insights to decision makers.
In stochastic settings, the demand is usually formulated as a function of price \( p \) and random noise \( \xi \) which is independent of price \( p \), and is denoted as \( d(p, \xi) \). Generally, there are two ways in which random noise is formulated in the demand function: the additive demand case and multiplicative demand case. In the additive demand case, demand is assumed to be a deterministic function of price \( p \), plus the random noise \( \xi \) and represented as follows: \( d(p, \xi) = d(p) + \xi \). In the multiplicative demand case, demand is a deterministic function of price \( d(p) \) multiplied by a random noise \( \xi \). In both cases, the demand noise is non-negative, and usually normalized to have a unit mean.

These demand models assume that the demand for a product is dependent on its price. In our research in Chapters 3 and 4, we consider substitutable products, where the demand is dependent on its own price and all other substitutable products’ prices. Let \( P = (p_1, p_2, ..., p_n)' \) denote the price vector for \( n \) products, and \( D = (d_1, d_2, ..., d_n)' \) be the vector of demands. The demand model for multiple products in an additive demand case is therefore formulated as:

\[
D(P) = b' - AP + \xi',
\]

where \( b = (b_1, b_2, ..., b_n) \) is the vector of coefficients representing the upper bound of demand while all prices are set at zero, and \( A \) is an \( n \times n \) matrix representing the price sensitivity, where \( a_{ii} \) is often non-negative, thereby guaranteeing the demand loss for each unit of its own price increase, whereas \( a_{ij} \) is non-positive and represents the demand substitutable rate (or demand transfer rate) for transferring the demand for product \( i \) to product \( j \) at each unit \( p_i \) increase. \( \xi = (\xi_1, \xi_2, ..., \xi_n) \) is the demand noise vector.

In summary, stochastic programming is a mature area in operational research. There is a rich body of literature on modelling joint pricing and inventory management with stochastic programming. However, in reality, the application of stochastic programming is often limited by the dependency on the availability of probabilistic information on the uncertainty and its computational challenges. The need for an alternative, non-probabilistic approach has been rising in recent years.

### 1.5.2 Robust Optimization

Robust optimization is a more recent approach for addressing uncertainty, with most research on the topic having been conducted in the past 15 years. The first study employing robust optimization was suggested by Soyster (1973). He models uncertainty as uncertain parameters belonging to a known uncertainty set, and uses the worst case analysis as a tool for the treatment of uncertainty. However, as each uncertain parameter is required to be equal to its worst-case value, the model was believed to be too conservative for practical implementation. In the mid-1990s, some studies controlled
the conservatism of the approach by proposing a tractable reformulation. Since then, more studies have been employing robust optimization in statistics, operational theory, control theory, finance, logistics and manufacturing engineering.

Let us consider a single product case, in which the deterministic demand is formulated as:

\[
d = a - bp.
\]

If we consider the parameters \(a\) and \(b\) as uncertain parameters belonging to uncertainty sets, the uncertain demand in robust optimization is formulated as:

\[
\tilde{d} = \tilde{a} - \tilde{b}p,
\]

where \(\tilde{a}\) is the realization of uncertain parameter \(a\) and belongs to the uncertain set which is centred at its nominal value \(a\) and of half-length \(a'\) (a given parameter), \([a - a', a + a']\), whereas \(\tilde{b} \in [b - b', b + b']\).

In robust optimization, we are searching for a solution that is feasible for any realization of the uncertain data and maximizes the realized objective functions in the worst case. To guarantee the feasibility and optimality of the solution, worse-case instances of the parameters are taken into account. Therefore, The range of the uncertainty set is used to model the level of conservativeness we have on the uncertain parameters. If we make this uncertainty set too large, we over protect the model and this could downgrade the performance of the suggested solution and could even lead to infeasibility. We have therefore introduced a parameter called the ‘budget of uncertainty’ which served as the upper bound on the total amount of variation of the uncertain parameters. This parameter is an input of the model and allows us to control the level of conservativeness. The scaled deviations of parameters \(a\) and \(b\) are defined as \(u = \frac{\tilde{a} - a}{a'}\), \(w = \frac{\tilde{b} - b}{b'}\), where the scaled deviation \(u\) and \(w\) takes value in \([-1, 1]\). We then introduce the budget of uncertainty in the following sense: The total variation of the parameters cannot exceed the budget of uncertainty \(\Gamma\):

\[
|u| + |w| \leq \Gamma.
\]

By taking \(\Gamma = 0\), all the uncertain parameters equal their nominal value. By having \(\Gamma\) vary in \([0, 2]\), we can make a trade-off between the levels of robustness and performance. If \(\Gamma\) is assumed to be 2, this leads to very conservative cases and all coefficients equal their worst-case values.

\section{Structure of the Thesis}

This thesis contains 6 chapters. In Chapter 2, we review the current literature on this topic. Chapter 3 provides a stochastic dynamic programming framework to model the joint pricing and inventory problem for two substitutable and perishable products over
a two-period planning horizon. Chapter 4 models a joint pricing and inventory problem under stochastic demand uncertainty for two or more substitutable and perishable products over multiple periods. In Chapter 5 we extend the newsvendor model and take the robust optimization approach to model a joint pricing and inventory problem for two substitutable and perishable products. Chapter 6 provides the summary of our contributions and suggests directions for future research.
Chapter 2

Literature Review

This chapter reviews the relevant research in the areas of examination. Inventory management and pricing optimization individually will be briefly discussed first, followed by joint pricing and inventory control under three sections: deterministic demand, stochastic programming and robust optimization.

There is a vast pool of research on the topic of joint pricing and inventory decision problems currently in the literature. A classification table is provided in Table 2.1 to introduce the important factors of the papers being reviewed in this chapter.

<table>
<thead>
<tr>
<th>Elements</th>
<th>Descriptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Products</td>
<td>Stable/Perishable</td>
</tr>
<tr>
<td>Model</td>
<td>Dynamic programming/Mixed Integer Non-linear Programming</td>
</tr>
<tr>
<td>Demand</td>
<td>Deterministic/ Uncertain</td>
</tr>
<tr>
<td>Demand input parameters</td>
<td>Price, Time, Inventory</td>
</tr>
<tr>
<td>Length of Horizon</td>
<td>Single Period/ Multiple/ Infinite</td>
</tr>
<tr>
<td>Replenishment</td>
<td>Yes/No</td>
</tr>
<tr>
<td>Sales</td>
<td>Backorder/Lost Sales</td>
</tr>
<tr>
<td>Substitution</td>
<td>Yes(brands/ages)/No</td>
</tr>
</tbody>
</table>

This chapter is structured as follows: We carry out a brief literature review on Inventory Management and Pricing Optimization for stable products and perishable products respectively in Section 2.1 and 2.2. In Section 2.3, we review the deterministic demand models in the areas of joint pricing and inventory management. In Section 2.4, stochastic programming models are reviewed, stable products problem, perishable products problem and multiple substitutable products problem are reviewed accordingly. In Section 2.5, the literature of robust optimization is explored. Finally we summarise the differences between our work and existing literature in the conclusion.
2.1 Inventory Management under Demand Uncertainty

2.1.1 Stable Products

Inventory control is one of the most highlighted research areas in operations research. Scarf (1959) examines inventory control for stable items. He focuses on studying methods on exploring the prior distribution structure and the state-space reduction technique to reduce the complexity of the statistical inventory control problem. More recent work have come out by Parlar and Goyal (1984); they optimize ordering decisions for two substitutable products as the great importance of replenishment in inventory management. Wang and Parlar (1994) uses a game theory model for a single-period inventory problem where retailers try to determine the optimal order quantity. Products are also substitute with random demands.

More recently, Smith and Agrawal (2000) develop a probabilistic demand model for items in an assortment that captures the effects of substitution and a methodology for selecting item inventory levels so as to maximize total expected profit under given resource constraint. Schmitt et al. (2010) model a retailer whose supplier is subject to complete supply disruptions. They examine optimal base-stock inventory policies using infinite-horizon and periodic-review models for a single supplier whose single retailer is subject to stochastic disruptions. In this model, they combine discrete-event uncertainty (disruptions) and continuous sources of uncertainty (stochastic demand or supply yield), which have different impacts on optimal inventory setting. However, due to the complexity of discrete-event uncertainty and continuous source of uncertainty, they model both uncertainties individually.

2.1.2 Perishable Products

As it has been mentioned in the classification of perishability, perishable products are identified as having fixed lifetime and random lifetime. For joint inventory and pricing control for perishable products, the lifetime is mostly considered as fixed. Therefore, this section will extensively review papers on the inventory control for perishable products with fixed life time, whereas the papers on random lifetime perishable products inventory system will only be briefly reviewed.

The earliest model of the perishable inventory problem is carried out by Derman and Klein (1958), where the demand schedule is given. Only a few papers study inventory control problem for perishable products problem with demand uncertainty, as stochastic perishable inventory models are quite complex. The earlier description of the fixed-life perishability problem with the assumption of two-period lifetime is studied with backlogging and lost sales by Nahmias and Pierskalla (1973). Later, models which assumes goods perish all at once after more than two periods time is carried out by
Nahmias (1975). Due to the complexity of stochastic perishable inventory models, the optimal policy is not shown. The earlier work focuses on on the inventory replenishment policy under periodic reviews (Nahmias, 1982) and on the issuing policy in blood bank management (Prastacos, 1984). Ferguson et al. (2007) consider a variation of the economic order quantity (EOQ) model where cumulative holding cost is a nonlinear function of time by extending the model of Weiss (1982). In this model, they seek the optimal order strategy for perishable goods, such as milk and produce sold in small to medium size grocery stores where there are delivery surcharges due to infrequent ordering. Managers frequently apply markdown pricing strategy for such perishable goods. Their findings show the model improves the classic EOQ model significantly (Goyal and Giri, 2001).

A typical example of random lifetime products whose exact lifetime cannot be determined in advance is a fresh product whose time of spoilage is uncertain and for this reason the lifetime is assumed to be a random variable. The research of random lifetime perishable inventory systems is considerably more difficult than fixed lifetime perishable inventory systems. Jain and Silver (1994) propose a stochastic dynamic programming model for determining the optimal ordering policy for a perishable or potentially obsolete product so as to satisfy known time-varying demand over a specified planning horizon. They consider random lifetime perishability where, at the end of each discrete period, the total remaining inventory either becomes worthless or remains usable for at least the next period. They compare optimal and heuristic methods on a large set of test problems and analyse their performance as a function of various problem parameters. Moreover, a few Markovian (s, S) systems with zero lead time has also been studied by Kalpakam and Arivarignan (1988) and Liu (1990).

2.2 Pricing Optimization under Demand Uncertainty

2.2.1 Stable Products

There are more papers focusing on pricing of non-perishable products compared to perishable products. Traditional pricing strategies are based on applying a mark-up to the cost of the article, see Gabor and Granger (1964) and Monroe (1990).

More recently, many authors have developed new models on pricing strategies. Keller and Rady (1999) work in continuous space and continuous time with differential equations, and with a randomly-changing reward function. Depending on the model parameters, they identify two experimentation regimes, extreme and moderate. Bertsimas and De Boer (2005) present an optimization approach for the demand as a function of price, and dynamically setting the prices of products in an oligopoly environment in order to maximize expected revenue. In their model, the demand is not known in advance.
Ding et al. (2006) study a dynamic price discounts model to encourage backlogging of demand for customer classes denied immediate service. They distinguish customer classes by their contractual price and sensitivity to discounts when customers are assumed to arrive over several stages in a period. In this model, customers are served in class order, and the allocation of inventory to demand is determined by considering the current number of customers backlogged, as well as the current inventory position. Based on the results, they show that profits are highly influenced by the allocation of capacity.

Caldentey and Vulcano (2007) address a problem of an online seller selling a fixed initial inventory and facing a stochastic arriving stream of strategic consumers. Two alternative formulations are proposed, single-auction channel model (compete with third parties) and dual-channel model (monopolist). Then they consider an asymptotic version formulation of the problem in which the demand rate and the initial number of units grow proportionally large.

Popescu and Wu (2007) consider a dynamic pricing problem of a monopolist firm in a market with repeated interactions. Demand is sensitive to the firm’s pricing history. In this model, they prove that optimal pricing policies induce a perception of monotonic prices, whereby consumers always perceive a discount as a surcharge, relative to their expectations.

Ahn et al. (2007) study a joint manufacturing/pricing decision model, demand in each period is determined by the interaction of pricing decisions in the current and previous periods, whereas in most deterministic manufacturing decision models, demand is either known or induced by pricing decisions in the period that the demand is experienced.

Araman and Caldentey (2009) formulate the retailer’s problem as a Poisson intensity control problem and derive structural properties of an optimal solution which is used to propose a simple approximated solution. In their model, two cases are considered. One is where the retailer is constrained to sell the entire initial stock of the non-perishable product before a different assortment is considered. The other is where the retailer is able to stop selling the non-perishable product at any time to switch to a different menu of products.

More recent work is carried out by Png and Wang (2010); they consider two-part pricing on offering a service to risk-averse buyers under demand uncertainty. They discuss the applications of beach and ski resorts when the weather condition is uncertain, for this reason, the buyers are subject to demand uncertainty. In this paper, they provide guidance on applying two-part pricing strategy to buyers with insurance against demand uncertainty. Su (2010) researches a monopolist firm selling a fixed capacity subject to demand uncertainty. He specifies equilibrium prices and profits and analyses the long-run capacity decisions of the firm. Based on his results, several model extensions are explored highlighting the robustness.
Another relevant research is about production line design decisions under a personalized or group pricing strategy (Schön, 2010). They extend the approach of Chen and Hausman (2000) to determine an optimal product line under a personalized or group pricing strategy in markets with multiple heterogeneous consumers such that total profit is maximized.

### 2.2.2 Perishable Products

This part of literature is well known as revenue management. At the beginning, a fixed number of a perishable assets are given; retailers must sell them prior to the time at which they perish. The retailer can dynamically adjust the price both upward and downward. Demand is sensitive to price and items are disposed of at an outdating cost at the end of the horizon. An early close work done in these areas are the models carried out for airline seat and hotel problems as the availability can also seen to be perishable.

Gallego and Van Ryzin (1994) is the first to consider optimal dynamic pricing of inventories with stochastic demand over finite horizons based on the basic models carried out by Kincaid and Darling (1963). In 1997, Gallego and Van Ryzin extend this model to analyse the multi-product dynamic pricing problem. Chatwin (2000) employs a continuous-time dynamic programming model in which the demand arrives according to a Poisson process. Li (2001) develops an optimal pricing model for a monopolist, which uses a purchase restriction as a mechanism to segment the demand market for perishable, non-storable products such as airline seats and hotels. Then Li (2005) investigates the validity of the optimality results derived from his model carried out on 2001 by relaxing the following two assumptions: (a) the firm offers restricted units first and then unrestricted units later at higher price levels; (b) only one type of product is available during the whole selling process. Their results indicate the existence of an optimal policy that is sustainable even when all active prices are made available at the same time.

More recent work is done by Levin et al. (2009); they present a dynamic pricing model for oligopolistic firms which sells differentiated perishable products to multiple finite strategic consumers. Consumers are aware that the pricing is dynamic and may time their purchases accordingly. This model considers equilibrium price dynamics under different levels of competition. Demand is modelled as stochastically homogeneous; in this case, consumers within a segment have the same distributional and parametric characteristics. A finite number of consumers are assumed to be present in the market within the whole sale season. Nevertheless, they model the differences in product preference by consumer’s evaluations that are also interpreted to willingness to pay.

Zhou et al. (2009) consider a problem of dynamic pricing and warranty policies for products with fixed lifetime. They offer a warranty policy with product purchase. The
Chapter 2 Literature Review

product is repairable with known market entry and departure times. They seek to maximize the manufacturer’s expected profit by determining a joint dynamic pricing and warranty policy. In this model, they study customer’s purchase patterns under several different pricing strategies by the manufacturer and they discuss the optimal joint pricing and warranty strategies. The warranty length is assumed to be able to be adjusted once during the product lifetime. Based on their findings, a dynamic warranty policy heavily outperforms a fixed-length warranty policy.

Xu (2009) studies a joint pricing and product quality decision problem in a distribution channel, in which a manufacturer sells a product through a retailer. In this paper, they compare two distribution channel structures: the manufacturer sells the product directly to customers or the manufacturer sells the product through a retailer. Four commonly used customer demand functions are considered: Exponential demand function, uniform demand function, normal demand function and Gumbel demand function.

Pricing optimization has been a long focus to improve profits both in the literature and real world practice. The coordination of price decisions with other elements in the supply chain has been studied in a way that the coordination of these decisions means a approach to optimize the system rather than individual elements. This calls the development of studying joint pricing and inventory control problems, where such models are clearly very important for retailers.

2.3 Joint Pricing and Inventory Decisions with Deterministic Demand

Early research in joint pricing and inventory control for perishable products problems commonly assumed a deterministic demand function. Kunreuther and Schrage (1973) develop an algorithm for joint pricing and ordering decisions for a firm which produces a single product under deterministic demand. The price is set to be the same throughout the periods. Cohen (1977) studies a joint pricing and ordering decision problem with a continuous review policy and deterministic demand for an exponentially decaying product. It is fairly similar to the Economic Order Quantity (EOQ) method. The optimal ordering quantity is derived and is sensitive to the perishability and product price. Research by Rajan et al. (1992) investigates the relationship between the optimal pricing and ordering decisions for a monopolistic retailer selling perishable products. Abad (1996) extends Rajan et al.’s model (1992) by allowing partial backlogging of demand. They study this generalized model of dynamic pricing and lot-sizing of a retailer selling a single perishable product. They find that under the condition of high perishability of a product, the retailers may need to permit backlog demand in order to price the product reasonably.
More recently, Sana (2011) develops a deterministic economic order quantity model with price-sensitive demand for perishable products. Unlike other demand functions, they assume the demand follows a quadratic function of the selling price. They find the optimal ordering quantity and optimal sales prices analytically by maximizing the total profit. Avinadav et al. (2013) study a single perishable problem and formulate a model to decide the optimal pricing, order quantity and replenishment period. They analyse the properties of the variables and obtain an optimal policy. Demand follows a deterministic approach and is a function of price and time. They reduce this three-variable profit maximization problem into a single-variable problem, where the only variable is the duration of the replenishment period. Panda et al. (2013) also study an optimal pricing and lot-sizing problem for perishable inventory; the demand is deterministic and dependent on prices and time. The number of price changes and the replenishment cycle length are set to be the decision variables, the optimal solution is provided and verified. Dalfard and Nosratian (2014) present a constrained single-product pricing and inventory-production model for perishable products. Demand is deterministic and a decreasing function of price. They formulate a non-linear programming model and solve it by using a hybrid genetic algorithm and simulated annealing. In the numerical results obtained they compare the results of two methods and find that a hybrid genetic algorithm performs better. Qin et al. (2014) study a pricing and lot-sizing problem for perishable products where the deterioration rate of quality and physical quantity is considered to be time-dependent. Demand is deterministic and dependent on the quality of products, the selling price and the stock in hand. They discuss the theory of finding the optimal solution, develop an algorithm to find the optimal solution and present numerical results.

2.4 Joint Pricing and Inventory Decisions with Stochastic Programming

2.4.1 Stable Products

In the recent two decades, a significant amount of literature which addresses the joint pricing and inventory control problem exists. A lot of work has been undertaken regarding non-perishable products. The majority of the relevant studies address single product problems. Federgruen and Heching (1999) address the joint pricing and inventory replenishment problem under demand uncertainty. They study a single item, periodic review model, with independent demands in each period depending on the items’ prices. Backlogging is permitted. They address this problem in both finite and infinite horizons, with the objective of maximizing the total expected discounted profit or its time average value, assuming that price can either be adjusted arbitrarily (upward or downward) or that it can only be decreased. The base-stock-list-price policy is introduced and proved
to be optimal for both cases. When the inventory level drops below a base-stock level, list price should be charged and retailers order up to the base-stock level for that period; When inventory is above the basestock level, the retailer should order nothing and charge less than the list price.

More recently, Khouja (2006) apply delayed incentives in the form of mail-in cash rebates which is very popular among suppliers to retailers models. In this research, they formulate and solve models for jointly determining the optimal price, rebate face value and the optimal order quantity for a price and rebate sensitive deterministic demand. In the results, they show that offering rebates can significantly affect the pricing and inventory policy, as well as profits. Yin and Rajaram (2007) consider the joint pricing and inventory control problem for a single product over a finite horizon with periodic review. In this model, the demand distribution in each period is determined by an exogenous Markov chain. Pricing and order decisions are made at the beginning of each period. Backlogging is allowed in this model. Their findings indicate the existence of an optimal base-stock-list-price type feedback policy for the additive demand model. Chao et al. (2008) study a joint optimization problem of replenishment and pricing for a periodic-review inventory system with random supply capacity over a multi-horizon. Replenishment and pricing decisions are made in the beginning of each period. While making these decisions, firms only know the supplier’s available capacity in the current period. In their model, the random supply capacities for different periods are dependent. For this reason, several stochastic dependency structures are considered in this research for the supply capacity sequence, including the one-lag and the multi-lag dependencies. Xu (2009) considers an optimal dynamic decision-making problem for a retailer in a price-sensitive, multiplicative demand framework. In this model, lost sales, holding cost, fixed and variable procurement costs and salvage value are investigated. The structure of the retailer’s expected profit-maximizing dynamic inventory policy are characterized for both finite and infinite selling horizons. Other related works use stochastic programming in the presence of uncertain demand include Su (2007), Aydin and Porteus (2008), Zhang et al. (2010), Pal et al. (2014) and Chen et al. (2015).

2.4.2 Perishable Products

As mentioned earlier, for perishable products, if a product remains unused up to its life-time, it is considered to be out-dated and must be disposed of. The stocking policy for fixed lifetime products follows either First-In-First-Out (FIFO) or Last-In-First-Out (LIFO) rules. Under the assumption of First-In-First-Out (FIFO) rules, the first unit that become available on the shelf will be sold first, whereas Last-In-First-Out assumes the last unit become available on the shelf will be sold first.

In the literature of joint pricing and inventory control for perishable products, most studies assume that the inventory follows a first-in-first-out (FIFO) approach. Li et al.
(2009) extend Federgruen and Heching’s work (1999) to study joint pricing and inventory control under demand uncertainty to consider the perishable products problem. They address the joint pricing and inventory replenishment decisions problem over an infinite horizon. Demands in each period are independent and influenced by price. In particular, price is set to be a decision variable to maximize the total discounted profits. They analyse optimal solutions for a two or more periods lifetime problem. An optimal policy as a base-stock-list-price policy are introduced. Experiments are carried out to demonstrate the effectiveness of this policy. Chao et al. (2008) study a finite horizon, single product, period review model with joint pricing and inventory decisions. Backlogging is considered in this model. Demand is set to be distributions with random vectors and depends on the product price, as do most of the models with uncertain demands. However, unlike most assumptions of other works with fixed ordering cost, the ordering cost here is a convex function of the amount ordered. In the numerical results, they demonstrate how the total profit, order up to level, reorder point and optimal price change with respect to state and time.

More recently, Chen and Sapra (2013) consider a two-period lifetime perishable product case, with objective to find the optimal joint pricing and inventory decisions over a finite horizon. Demand is dependent on prices and an uncertain noise, and backlogging is permitted for unsatisfied demand. No substitution between old and new products is considered; instead, they consider the two cases of FIFO, LIFO and obtain insights on the structure of decision variables. Chen et al. (2014) consider a perishable product problem by analysing joint pricing and inventory decisions. Both backlogging and lost-sale cases are considered. In each period, the inventory is replenished with a lead time. Same price is charged for inventory of different ages. Li et al. (2015) consider a joint pricing and inventory control problem for perishable products under stochastic inventory system. Unlike most stochastic problems in this area which assume uncertainty in demand function, they assume the uncertainty for the inventory level. By solving a Hamilton–Jacobi–Bellman (HJB) equation, they find the optimal joint dynamic pricing and production schedule and the benefits of dynamic pricing is quantified. Chung et al. (2015) study a multiple period joint pricing and inventory problem with multiple price markups for products under a short life cycle. They formulate mixed integer nonlinear models for both buyers’ and suppliers’ expected profits and replenishment is allowed over time.

Under demand uncertainty, some researches also analyse the differences between old and new products for retailers who sell them simultaneously and capture the substitution between old and new order products, where products are not issued in a FIFO manner. For example, fresh milk and nearly expired milk sold simultaneously at different prices. In the current literature, there are a few papers addressing this substitution. Deniz et al. (2004) study a discrete-time supply chain for perishable goods with separate demand streams for items of different ages. They consider the substitution effect between old
and new products. Two practical replenishment policies are analysed according to order-up-to level policies based on total inventory quantities or new items in stock. In the results, they compare the substitution options (substitution from other brands or other ages) analytically in terms of the infinite horizon expected costs, providing conditions on cost parameters that determine which substitution is most profitable. Rujing (2007) studies a joint pricing and inventory control problem for a single perishable product of different ages. The substitution between products of different ages is captured in the uncertain demand function. Inventory is replenished at the beginning of each period. A dynamic programming model is investigated to find the optimal order quantity for the new product and optimal prices for old products.

Ferguson and Koenigsberg (2007) study a two-period model which captures the effect of substitution between new (higher quality) and old products (lower quality) on a firm’s simultaneous determination of pricing and inventory decisions. They analyse the firm’s optimal strategy and find conditions where it is better to carry all, some, or none of its leftover inventory. They demonstrate the benefits of having a second selling opportunity under which the firm can price new products higher and stock more items. Jia and Hu (2011) study dynamic ordering and pricing for a two-period lifetime perishable product supply chain with one supplier and one retailer. The demand in each period is dependent on the price and a random factor. At each period, the supplier decides the wholesale price and the retailer decides the order quantity and retail price. They find that the optimal pricing strategy for old products depends only on inventory whereas for new products it depends on wholesale price. They also find that the problem can be reduced to a one-period problem between the retailer and supplier. At the end of their study, they extend the case to infinite horizons and longer lifetime products. Sainathan (2013) studies a two-period shelf life perishable product problem by jointly considering pricing and ordering decisions over an infinite horizon. A product is “new” in the first period and becomes “old” in the subsequent periods. Both products can be substituted by each other. No holding cost, backorder or lost sale is considered in that model. Chintapalli (2014) addresses a joint pricing and inventory control problem for perishable products under price-sensitive and stochastic uncertain demand. He considers the cases of arbitrary lifetime and two-period lifetime. Holding cost and backorder cost are not considered in this study. He employs a myopic policy which makes optimal pricing and ordering decisions based on an initial inventory vector by maximizing the current period’s profit. No recursive equation for the stochastic dynamic program is formulated.

### 2.4.3 Multiple Substitutable Products

Currently in the literature, two types of substitution are studied within the subject of inventory control and pricing management: Inventory-based substitution and price-based substitution. In typical inventory-based substitution, customers choose an alternative
when their first choices are out of stock (see Kök and Fisher (2007), Yücel et al. (2009), Gao et al. (2012)).

Our research considers price-based substitution within multiple perishable products. Price-based substitution assumes that customer demand is determined by the prices of all the substitutable products.

2.4.3.1 Stable Products

Zhu and Thonemann (2009) consider the joint pricing and inventory control problem across two substitutable products. Demand is a function of two prices and an uncertain factor. They find an optimal pricing and inventory control policy which is different to the base-stock list-price policy. They quantify the benefits of managing two substitutable products together in the numerical results. This study is extended by Ye (2008) to consider three or more similar products.

Kim and Bell (2011) investigate the impact of price-based substitution on single-period joint pricing and production decisions. Demands for two substitutable products are assumed to be linear additive demand functions, which is a linear function plus a probability distribution. Nevertheless, the demand loss rate (substitution rate) is assumed to be the same for both products. They extend the above discussed work (Kim and Bell, 2014) by considering production capacity constraints and investigate the impact on joint pricing and production decisions. A detailed analytical procedure is presented for comparison with that of Kim and Bell’s (2011) model.

More recently, Yu (2015) considers a two substitutable products problem over a one-period lifetime under resources constraints. Demand is assumed to be linear in prices and subject to additive uncertainty. Optimal pricing and inventory policy are characterized in their research. This is the first research considering a multi-product joint pricing and inventory problem with product substitution and demand uncertainty.

2.4.3.2 Perishable Products

Dong et al. (2009) consider dynamic pricing and inventory control for a retailer selling multiple substitutable products in a short selling horizon. They employ the customer choice model which can be used to estimate the level of customer demand for alternative ‘service product’ in non-monetary terms by assuming a probability of arriving in each period. No holding cost, or backorder/lost sale is considered in this model. An optimal solution is presented and dynamic pricing is demonstrated with more profits achieved than static pricing in the numerical results.

Rasouli and Kamalabadi (2014) take a different approach to consider substitution from a rival firm. They develop a joint pricing and inventory control model for seasonal and
substitutable products. The two substitutable products belong to two rival firms. The objective is to decide the optimal price, order quantity and the number of planning periods for one firm. They propose a mixed integer non-linear model and find the total profit is a concave function of price. With this analysis, they derived a unique optimal solution.

2.5 Joint Pricing and Inventory Decisions under Robust Optimization

The book by Ben-Tal et al. (2009) contains most of the recent developments in robust optimization, along with some examples illustrated. There is a very limited amount of work on joint pricing and inventory control employing robust optimization. The first research considering robust optimization is conducted by Singh (1982), who studies a linear optimization problem to find a solution for all the data belonging to a convex set. In his model, he addresses uncertainty by taking a worst-case approach. However, his model yields a very conservative solution. Further steps have been taken in this direction to address the issue of over-conservatism by El Ghaoui et al. (1998), and Ben-Tal and Nemirovski (1998).

Adida and Perakis (2006) take a robust optimization approach to consider a dynamic pricing and inventory control for a make-to-stock manufacturing system. Demand $d_i(t)$ is assumed to be a linear function of the price $p_i(t)$,

$$d_i(t) = a_i(t) - b_i(t)p_i(t),$$

and there is uncertainty on the demand parameter $a_i(t)$. The nominal problem is a multiple-period nonlinear program for the multiple products. However, there is no substitution considered between these products. They formulate two robust optimization models and prove that the robust optimization is of the same level of complexity as the nominal problem and demonstrate how to employ a deterministic solution algorithm to the robust problem. This is the first paper introducing ideas of robust optimisation in a fluid model for dynamic pricing.

Adida and Perakis (2010a) take a robust optimization approach to study dynamic pricing and inventory control for a firm facing uncertainty in demand and competition from another firm. Multiple item cases are considered for this firm and each of them competes with a similar item from another company. They show the existence of a Nash equilibrium in continuous time in the solution structure and demonstrate how to compute a particular Nash equilibrium in the numerical results.

Adida and Perakis (2010b) then extend their research (Adida and Perakis, 2006) on a make-to-stock manufacturing system and formulate a variety of models to study joint pricing and inventory control problem for a multi-product capacitated case under demand uncertainty. They consider both stochastic and robust optimization approaches to
address demand uncertainty and formulate a dynamic programming model and mixed integer non-linear optimization model respectively. In the numerical results, they study the difficulty of all the models they propose and their performances.

2.5.1 Inventory

Some works in inventory management using robust optimization are related to our work. Bertsimas and Thiele (2006) propose a robust optimization approach to a multiple-period inventory problem; the demand belongs to a known set and a budget of uncertainty is introduced. They find that the optimal robust policy is qualitatively similar to the policy obtained by dynamic programming.

Vairaktarakis (2000) studies a newsvendor problem by robust optimization. He addresses uncertainty on demand through the use of two scenarios: interval and discrete. Several minmax formulations are developed for this multi-item newsvendor problem with a budget constraint. For the interval demand scenario, a linear time optimal algorithm is developed to solve it, whereas dynamic programming is employed to solve the discrete demand scenario.

Bienstock and Özay (2008) study the robust multi-stage inventory management problem and present algorithms for some general models to compute robust base-stock levels. Aharon et al. (2009), analyse and test an extension of the Affinely Adjustable Robust Counterpart (AARC) method to consider an inventory control problem in serial supply chains.

Song (2010) study a single product multiple period inventory control problem by robust optimization. They propose an approach to combine in a single step data fitting and inventory optimization by using histograms directly as input for the optimization model.

Cakmak (2012) investigates two approaches of modelling risk in inventory management: Risk-averse models and robust optimization models to consider a classical multi-period single-item inventory problem. A multi-period newsvendor model and dynamic robust models are formulated under robust optimization and resulting policies are analysed.

2.5.2 Pricing

Some other works on pricing management are also related to our work. Thiele (2006) investigates a single capacitated product pricing problem over a finite time horizon by robust optimization. She introduces a budget of resource consumption by uncertainty instead of a budget of uncertainty. budget of uncertainty constrains the number of uncertain parameters can reach their worst-case value in one period which has draw much attention in the literature, the budget of resource consumption by uncertainty she
proposed is limiting the amount of resources can be used by random part of cumulative demand across different periods. Demand is modelled as:

\[d_t(p_t) = \overline{d_t}(p_t) + z_t \delta_t,\]

where \(\overline{d_t}(p_t)\) is the average demand and dependent on the price, \(\delta_t \in [-\hat{\delta}_t, \hat{\delta}_t]\) and \(|z_t| \leq 1\). The budget of resource is introduced by:

\[|\sum_{t=0}^{T-1} z_t \delta_t| \leq \Delta\]

Perakis and Sood (2006) study a pricing problem of a number of retailers selling a single perishable product with fixed inventory. Competition between these retailers is addressed in the demand function. A budget of uncertainty is introduced for each retailer across their demand parameters. This is the first work addressing competition in a pricing problem by means of a robust optimization approach.

Lim and Shanthikumar (2007) study a single product dynamic pricing problem that accounts for errors in the model at the optimization stage. The equivalence between the robust pricing problem and the revenue management problem with an exponential utility function without uncertainty is presented.

2.6 Conclusion

In this research, we study a joint pricing and inventory decision for substitutable perishable products by stochastic dynamic programming and robust optimization. In Chapter 3 and 4, we consider stochastic dynamic programming models while the setting in this work share some of the elements in the existing work of Gilbert (2000), Zhu and Thonemann (2009), Ye (2008) and Rujing (2007), our work deviates from these existing work in the following ways.

- Our work differs from the work of Gilbert (2000), Zhu and Thonemann (2009) and Ye (2008) in the way that they study substitutable and non-perishable products while we are considering substitution among perishable products.

- Our work also differs from the work of Rujing (2007) in the way that she studies a single perishable product problem while we are considering two or more substitutable and perishable products with price substitution between them.

In Chapter 5, we take a robust optimization approach; it shares some elements in the existing work of Adida and Perakis (2006), Adida and Perakis (2010b) and Adida and
Perakis (2010a). The difference is we considering the substitution across similar products.

To the best of our knowledge, Chapter 3 and 4 are the first two works in the literature considering substitution for a joint pricing and inventory control problem across similar perishable products with holding cost and backorder cost. Chapter 5 is the first work employing robust optimization to consider joint pricing and inventory decisions for substitutable and perishable products.
Chapter 3

Joint Pricing and Inventory Decisions for Two Substitutable and Perishable Products Over Two Periods Lifetime: A Stochastic Programming Approach

3.1 Introduction

In the original Newsvendor problem, pricing is not one of the decision variables. The earliest research which includes pricing decisions in the Newsvendor problem is by Whitin (1955), where optimal pricing and ordering decisions are obtained simultaneously under the objective of maximizing the expected profit. Nowadays, the rapid development of information technology provides more opportunities to collect historical data for the application of advanced dynamic pricing and inventory control strategies.

In this chapter, we study the integration of the pricing and inventory control problem across two substitutable and perishable products, where we extend the original single period Newsvendor problem to two periods. We develop a dynamic programming model addressing this problem over a two-period planning horizon.

As shown in Figure 3.1, the retailer reviews an inventory and pricing policy for a two-period planning horizon. They order the quantity of ‘y’ products at the beginning of the first period; at this stage, no lead time is considered. Products will be sold at
a given price during the first period, and all leftover products with quantity ‘x’ will be sold at a discounted price in the second period. In this model, no replenishment is considered during the selling periods; rather, the products are sold out before the demand is satisfied. The length of each period is not necessarily equal; the length of the second period under discounted price is typically shorter than the first period under full price. This model can be best suited to some fast fashion products or seasonal products where no replenishment is considered during the selling horizon. At the end of the second period, all leftover products will be disposed of.

We study this extended Newsvendor problem by considering the substitution between two different products (different brands or products with similar utility). In the assumption, we have individual demand in each period. Demands in the first period are non-negative, independent and identically distributed with known probability distributions. Each demand in the second period is dependent on the selling prices of the two products, expressed as known stochastic demand functions. At the beginning of the first period, a retailer makes decisions on the order quantities for both products, and the leftover products will be carried over to the second period. At the beginning of the second period, prices are determined for both products. At the end of the second period, all leftover products will be disposed of. No resale value is considered in this model. This model is a good fit to the seasonal product problem, where there is limited selling horizon and no initial inventory is held by retailers. Furthermore, this model can also be used to consider airline ticket problems or hotel room sales problems without substitution between products on different days. In this case, the order quantity could be considered the quantity of tickets in each fare group, with no replenishment allowed during the two-period planning horizon.

In this thesis, the retailers’ objective is to maximize the total profit over the planning horizon. In this paper, we first analyse the properties of decision variables, affording us the ability to prove that the local optimum is also the global optimum in section 4. Then we run numerical tests; by analysing the properties of decision variables, we can simplify the numerical results in order to solve large-scale problems. We suggest that retailers manage two substitutable products concurrently in order to achieve higher profit.

Figure 3.1: Products flow


3.2 Model Formulation

3.2.1 Assumptions and Notations

In this thesis, we consider a retailer who orders, stocks, and sells two substitutable products.

![Figure 3.2: Variables Description of 2 Substitutable Perishable Products over 2 Periods Lifetime problem](image)

The lifetimes of these two products are assumed to be two periods and this constitutes the planning horizon. As shown in Figure 3.2, in the beginning of the first period, products are categorised as new with a given price $P_{11}, P_{21}$. All unsold products $x_{12}, x_{22}$, with holding costs $h$ accounted during the first period, will be carried to the second period and sold at a discounted price $P_{12}, P_{22}$. The inventory holding cost $h$ and backorder cost $\pi_{11}, \pi_{21}$ are considered in all periods. In retailers, while the products are out of stock, the unsatisfied demand will be backordered, in such a way that they place orders from supplier at a cost and once it arrives, they will inform customers to collected it.

The order quantities in the first period and pricing in the second period are the decision variables in this model. Let $i = 1, 2$ denote the two types of products, where $t = 1, 2$ denotes the periods. Products are deemed as new while $t = 1$, and old while $t = 2$. We apply $P_{it}$ to be the price of product $i$ in period $t$. Let $y_i$ be the order quantity of product $i$ at the beginning of the period 1 and let $x_{i2}$ be the leftover quantity of product $i$ being carried over to period 2. The notation is employed as follows:

- $t$: Period
- $i$: Types of products
- $P_{it}$: Retailing price of product $i$ in period $t$
- $D_{it}$: Demand of product $i$ in period $t$
- $h$: Holding cost.
- $\pi_{it}$: Backorder cost of product $i$ in period $t$
- $c_i$: ordering cost from suppliers of product $i$
- $a_{it}$: Demand loss rate per unit price increase of product $i$ to the demand of $i$ in period $t$
• \( l_{it} \): Demand substitution rate per unit price increase of product \( i \) to the demand of \( j \) in period \( t \)

• \( V_t \): Maximized Expected Profit in period \( t \)

• \( G_t \): Total profit in period \( t \)

Let \( y_1, y_2, P_{12}, P_{22} \) be the four decisions variables in this model. Demands in the first period are non-negative, independent and identically distributed with known probability distribution. The demands of both products in the second period depend on the retail prices, i.e. we assume that the demand of product \( i \) in the second period follows a linear demand function, such that \( D_{i2} = b_{i2} - a_{i2} P_{i2} + l_{j2} P_{j2} + \xi_{i2} \) where \( j = 1, 2 \) and \( j \neq i, a_{i2}, l_{j2} \geq 0 \). Note that the parameter \( a_{i2} \) represents demand loss per unit increase of demand of product \( i \) with respect to \( P_{i2} \). Let \( l_{j2} \) represent the demand substitution rate from demand of product \( j \) (demand increase per unit price increase) to demand of product \( i \) with respect to \( P_{j2} \). Parameter \( b_{i2} \) represents the demand upper bound while all products are charged at zero price. \( \xi_{i2} \) denotes the uncertainty in the demand model following a distribution.

For the demand loss rate and demand substitution rate, we make the assumption on that \( a_{i2}, a_{j2} \geq l_{i2}, l_{j2} \). We assume \( a_{i2} \geq l_{i2} \) to imply that the price change of product \( i \) affects its own demand more than the demand for the other product. In addition, we assume \( a_{i2} \geq l_{j2} \), which state that demand of product \( i \) is more sensitive to its own price change than that of the other product. The demand noise \( \xi_{i2} \) denotes the uncertainty in demand of product \( i \), which has a continuous density function \( f_{i2}(\xi_{i2}) \), and \( \xi_{i2} \) is bounded in \([-\xi_{i2}^{\min}, \xi_{i2}^{\max}]\). The expected value of \( \xi_{i2} \) is 0. Additionally, we assume \( b_{i2} > -\xi_{i2}^{\min} \) to ensure the positive upper bound of demands.

When employing this model, it is important to ensure the non-negativity of discounted price \( P_{i2} \). Therefore, we assume a lower bound \( \underline{P}_{i2} \) and upper bound \( \overline{P}_{i2} \leq P_{i1} \) for the optimum price \( P_{i2} \), where \( \overline{P}_{i2} < \frac{b_{i2} + l_{j2} + \xi_{i2}^{\min}}{a_{i2}} \). The inequality \( \overline{P}_{i2} < \frac{b_{i2} + l_{j2} + \xi_{i2}^{\min}}{a_{i2}} \) implies non-negativity of demand of product \( i \) in period 2.

### 3.3 Dynamic Programming Model

We formulate this problem as a dynamic programming model. Given the initial inventory level, the total profit in period 2 is:

\[
G_2(x_{12}, x_{22}, P_{12}, P_{22}) = \sum_{i \in \{1, 2\}} [p_{i2} \mathbb{E}D_{i2} - L_{i2}(x_{i2})],
\]

where \( D_{i2} = b_{i2} - a_{i2} P_{i2} + l_{j2} P_{j2} + \xi_{i2} \) is the demand in the second period for product \( i \) and \( L_{i2} \) is the holding cost and backorder cost for product \( i \) considered here and denoted by:

\[
L_{i2} = h \mathbb{E}(x_{i2} - D_{i2})^+ + \pi_{i2} \mathbb{E}(D_{i2} - x_{i2})^+
\]
The maximum expected profit in period 2 is:

$$V_2(x_{12}, x_{22}) = \max_{P_{12}, P_{22}} G_2(x_{12}, x_{22}, P_{12}, P_{22}).$$

The decision variables in period 2 are $P_{12}, P_{22}$, and $x_{12}, x_{22}$ are leftover quantities from period 1 and given in period 2.

The total profit over two periods is:

$$G_1(y_1, y_2) = \sum_{i \in \{1, 2\}} \{P_{i1}E[D_{i1}] - L_{i1}(y_i, D_{i1}) - c_i y_i] + \gamma \cdot [V_2(x_{12}, x_{22})].$$

The maximum profit over 2 periods is:

$$V_1 = \max_{y_1, y_2} G_1(y_1, y_2).$$

In period 1, the order quantities $y_1, y_2$ are the decision variables, whereas the $P_{11}, P_{21}$ are the known parameters.

We denote the expected inventory holding cost and backorder cost in period 1 by

$$L_{i1} = h E(y_i - D_{i1})^+ + \pi_{i1} E(D_{i1} - y_i)^+. $$

Constraints:

$$P_{i2} \leq P_{i} \leq \bar{P}_{i2}, \bar{P}_{i2} < \frac{b_{i2} + l_{j2} \cdot P_{j2} + \xi_{i2}^{min}}{a_{i2}} \leq P_{i1}. $$

$$x_{i2} = y_i - D_{i1}, \quad x, y, D \text{ are non-negative integers.}$$

In this model, the holding cost $h$ is money spent to keep and maintain a stock of goods in storage. It also represents opportunity cost, as the presence of the goods means that they are not being sold while that money could be deployed elsewhere. At the end of the second period, all leftover products are disposed of. $\gamma$ represents the stochastic discounted factor (pricing kernels). $\gamma$ is a random variable often used to compute the market price today, by discounting, state by state, the corresponding payoffs at a future date. It is a given parameter in our study. In the case where no products are left over from period 1, $x_{12}, x_{22} = 0$, all demand in the second period will be backordered. In real world practice, back-ordering can be done by retailers placing last minute orders at a higher cost to satisfy the demand.

In dynamic programming, the optimality functions are computed recursively backwards in periods, starting at period 2 and ending at period 1 according to Bellman’s equation. In this model, $V_2$ is the maximized expect profit of period 2, given the inventory level $x_{12}, x_{22}$. The value function in earlier period $V_1$ is objective function to be maximized here, is calculated by maximizing $G_1$. $G_1$ is the total expected profit in period 1 with $y_1, y_2$ products ordered and $x_{12}, x_{22}$ products leftover, plus the maximized expected profit.
in period 2 with inventory level \( x_{12}, x_{22} \). Since \( V_2 \) is calculated for needed states (different inventory levels \( x_{12}, x_{22} \)), the above operation yields \( V_1 \) for those states. Finally, the optimal values of the decision variables can be recovered, one by one, by tracking back the calculations already performed.

We firstly prove that, with a given inventory, total profit \( G_2 \) is a concave function of prices, by which we can find the global optimum. We then demonstrate a closed form solution of optimal initial inventory levels under the maximization of expected profit in the second period. Lastly, we derive a closed form solution of global optimal order quantities.

Retailers can use this optimal order quantity at the beginning of the first period and the optimal retail prices in the second period to achieve maximized total profit. We assume that the price for the first period is fixed (e.g. we use the list price for the item) and that we can only change the price used in the second period; with the given inventory levels in the second period, we can obtain optimum discounted prices to achieve the highest profit.

### 3.4 Optimality Analysis

#### 3.4.1 Optimal Discounted Price

In this section, we analyse the properties of optimum discounted prices in the second period. We show that, for given \( x_{12}, x_{22} \), total profit is a jointly concave function with respect to discounted price, and an optimum discounted price obtained under the maximization of the expected profit exists.

**Lemma 3.1.**

The total profit \( G_2(x_{12}, x_{22}, P_{12}, P_{22}) \) is jointly concave with respect to \( x_{12}, x_{22}, P_{12}, P_{22} \).

**Proof.** To derive the optimum conditions, we reformulate the expected profit of the second period as follows:

\[
G_2(P_{12}, P_{22}) = P_{12}(b_{12} - a_{12}P_{12} + l_{22}P_{22}) \leftarrow \text{revenue}\n+ P_{22}(b_{22} - a_{22}P_{22} + l_{12}P_{12}) \leftarrow \text{revenue}\n- L_{12} - L_{22}, \leftarrow \text{holding cost and backorder cost}
\]

where

\[
L_{12} = h\mathbb{E}(x_{12} - D_{12})^+ + \pi_{12}\mathbb{E}(D_{12} - x_{12})^+,\n\]

\[
L_{22} = h\mathbb{E}(x_{22} - D_{22})^+ + \pi_{22}\mathbb{E}(D_{22} - x_{22})^+,
\]
and

\[ D_{12} = b_{12} - a_{12}P_{12} + l_{22}P_{22} + \xi_{12}, \]
\[ D_{22} = b_{22} - a_{22}P_{22} + l_{12}P_{12} + \xi_{22}. \]

It is easy to prove that \( P_{12}(b_{12} - a_{12}P_{12} + l_{22}P_{22}) + P_{22}(b_{22} - a_{22}P_{22} + l_{12}P_{12}) \) is jointly concave to \( P_{12}, P_{22}. \)

To simplify the proof of Lemma 3.1 where \( G_2(x_{12}, x_{22}, P_{12}, P_{22}) \) is jointly concave with respect to \( x_{12}, x_{22}, \) we then let

\[ z_{12} = x_{12} - b_{12} + a_{12}P_{12} + l_{22}P_{22}, \]
\[ z_{22} = x_{22} - b_{22} + a_{22}P_{22} - l_{12}P_{12}. \]

The total holding cost and backorder cost can be rewritten as:

\[ L_{12} = hE(z_{12} - \xi_{12})^+ + \pi_{12}E(\xi_{12} - z_{12})^+, \]
\[ L_{22} = hE(z_{22} - \xi_{22})^+ + \pi_{22}E(\xi_{22} - z_{22})^+. \]

We find \( L_{i2} \) is a piecewise linear convex function with respect to \( z_{i2} \) as described graphically in the following figure:

Moreover, \( z_{i2} \) is a linear function to \( x_{i2}, P_{12} \) and \( P_{22}. \) \( x_{12}, x_{22} \) are separable, then we complete the proof that the total profit \( G_2(x_{12}, x_{22}, P_{12}, P_{22}) \) is jointly concave with respect to \( x_{12}, x_{22}, P_{12}, P_{22}. \)

From this lemma, the unique optimum solution \( P_{12}, P_{22} \) exists. Under the joint concavity, efficient algorithms such as the steepest ascent method can be applied to obtain the
optimum solution.

3.4.2 Optimal Order Quantity

In this section, we firstly prove that the optimum discounted price is a function of leftover products from the first period. Then we can show that the maximum total profit in the second period is a concave function of leftover products from the first period. Based on these two properties, we can further prove that total profit in the first period are a concave function with respect to order quantities.

Lemma 3.2.
Let \( P_{*2} \) be the optimal solution of the unconstrained optimization problem \( \max_{P_{12}, P_{22}} G_2(P_{12}, P_{22}) \), then \( P_{*2} \) is a non-increasing function of \( x_{12} \), and a non-decreasing function of \( x_{22} \).

Proof. As \( G_2(x_{12}, x_{22}, P_{12}, P_{22}) \) is jointly concave with respect to \( P_{12}, P_{22} \), given \( (x_{12}, x_{22}) \) is separable with \( (P_{12}, P_{22}) \), we take the first order derivatives of \( G_2(x_{12}, x_{22}, P_{12}, P_{22}) \) with respect to \( P_{12}, P_{22} \) respectively. For convenient notations, we let \( A = x_{12} - b_{12} + a_{12}P_{12} - l_{22}P_{22} \) and \( B = x_{22} - b_{22} + a_{22}P_{22} - l_{12}P_{12} \). The following must hold:

\[
0 = \frac{\partial G_2}{\partial P_{12}}(P_{*12}, P_{*22}) = \int_B^A (l_{12}P_{*22} + l_{12}h) f_{22}(\xi_{22}) d\xi_{22} \tag{3.1}
\]

\[
+ \int_{\xi_{22}^{min}}^{\xi_{22}^{max}} (l_{12}P_{*22} - l_{12}\pi_{12}) f_{22}(\xi_{22}) d\xi_{22}
\]

\[
+ \int_{\xi_{12}^{min}}^{\xi_{12}^{max}} (b_{12} - 2 \cdot a_{12}P_{*22} + l_{22}P_{*22} + \xi_{12} - h \cdot a_{12}) f_{12}(\xi_{12}) d\xi_{12}
\]

\[
+ \int_{\xi_{12}^{min}}^{\xi_{12}^{max}} (b_{12} - 2 \cdot a_{12}P_{*22} + l_{22}P_{*22} + \xi_{12} + \pi_{12}a_{12}) f_{12}(\xi_{12}) d\xi_{12}.
\]

Taking the first order derivative of \( G_2 \) with respect to \( P_{22} \), we have

\[
0 = \frac{\partial G_2}{\partial P_{22}}(P_{*12}, P_{*22}) = \int_A^B (l_{22}P_{*12} + l_{22}h) f_{12}(\xi_{12}) d\xi_{12} \tag{3.2}
\]

\[
+ \int_{\xi_{12}^{min}}^{\xi_{12}^{max}} (l_{22}P_{*12} - l_{22}\pi_{12}) f_{12}(\xi_{12}) d\xi_{12}
\]

\[
+ \int_{\xi_{22}^{min}}^{\xi_{22}^{max}} (b_{22} - 2 \cdot a_{22}P_{*22} + l_{12}P_{*12} + \xi_{22} - h\pi_{22}) f_{22}(\xi_{22}) d\xi_{22}
\]

\[
+ \int_{\xi_{22}^{min}}^{\xi_{22}^{max}} (b_{22} - 2 \cdot a_{22}P_{*22} + l_{12}P_{*12} + \xi_{22} + \pi_{22}a_{22}) f_{22}(\xi_{22}) d\xi_{22}.
\]
By taking the first order derivative of equations (3.1) and (3.2) with respect to \( x_{12} \), and by rearranging the terms, we obtain

\[
0 = -2a_{12} \frac{dP*_{12}}{dx_{12}} + \frac{dP*_{22}}{dx_{12}} (l_{12} + l_{22}) (l_{12} + l_{22}) + a_{12}(h + \pi_{12})(1 + a_{12} \frac{dP*_{12}}{dx_{12}} - l_{22} \frac{dP*_{22}}{dx_{12}})f_{12}(A) + l_{12}(h + \pi_{22})(a_{22} \frac{dP*_{22}}{dx_{12}} - l_{12} \frac{dP*_{12}}{dx_{12}})f_{22}(B),
\]

and

\[
0 = -2a_{22} \frac{dP*_{22}}{dx_{12}} + \frac{dP*_{12}}{dx_{12}} (l_{12} + l_{22}) + l_{22}(h + \pi_{12})(1 + a_{12} \frac{dP*_{12}}{dx_{12}} - l_{22} \frac{dP*_{22}}{dx_{12}})f_{12}(A) - a_{22}(h + \pi_{22})(a_{22} \frac{dP*_{22}}{dx_{12}} - l_{12} \frac{dP*_{12}}{dx_{12}})f_{22}(B).
\]

It is necessary and sufficient to prove that \( \frac{dP*_{12}}{dx_{12}} \leq 0, \frac{dP*_{22}}{dx_{12}} \geq 0 \).

Rearranging (3.3) and (3.4), by letting \( M = -(h + \pi_{12})f_{12}(A), N = -(h + \pi_{22})f_{22}(B) \) we have

\[
\frac{dP*_{12}}{dx_{12}} = \frac{-a_{12}M - l_{22}N(l_{12} + l_{22} - a_{12}l_{22}M - a_{22}l_{12}N)}{(-2a_{12} + a_{12}^2 M + l_{12}^2 N)(-2a_{22} + a_{22}^2 N + l_{22}^2 M) - (l_{12} + l_{22} - a_{12}l_{22}M - a_{22}l_{12}N)^2}.
\]

Therefore, it is clear that

\[
-a_{12}M - l_{22}N(l_{12} + l_{22} - a_{12}l_{22}M - a_{22}l_{12}N) \geq 0
\]

\[
-2a_{22} + a_{22}^2 N + l_{22}^2 M \leq 0.
\]

According to Lemma 3.1,

\[
(-2a_{12} + a_{12}^2 M + l_{12}^2 N)(-2a_{22} + a_{22}^2 N + l_{22}^2 M) - (l_{12} + l_{22} - a_{12}l_{22}M - a_{22}l_{12}N)^2 \geq 0.
\]

We then can have \( \frac{dP*_{12}}{dx_{12}} \leq 0 \), similarly, we can have \( \frac{dP*_{22}}{dx_{22}} \geq 0 \).

Likewise, we can prove that \( \frac{dP*_{22}}{dx_{22}} \leq 0, \frac{dP*_{22}}{dx_{12}} \geq 0 \).
Lemma 3.3.
P_{i2}^*$ is a non-increasing function of $x_{i2}$, and a non-decreasing function of $x_{j2}$.

Proof. We have

$$P_{i2}^* = \min \{ \max(P_{i2}, P_{i2}^*), \overline{P}_{i2} \}$$

It is easy to prove that the minmax of a non-decreasing function is also a non-decreasing function, and that the minmax of a non-increasing function is also a non-increasing function. We therefore conclude that $P_{i2}^*$ is a non-increasing function of $x_{i2}$, and a non-decreasing function of $x_{j2}$.

Thus, we obtain

$$\frac{dP_{i2}^*}{dx_{i2}} \leq 0, \quad \frac{dP_{i2}^*}{dx_{j2}} \geq 0.$$ 

While the inventory level of product ‘1’ increases, the optimum price of that product will decrease and the optimum price of product ‘2’ will increase. In this case, to match the inventory level, the demand for product ‘1’ will increase. While the demand for product ‘2’ stays at the same level, the maximum total profit can be achieved.

Lemma 3.4.
The maximum expected profit $V_2(x_{12}, x_{22})$ is joint concave with respect to $x_{12}, x_{22}$.

Proof. From Lemma 3.1, we proved that the total profit $G_2(P_{12}, P_{22})$ is jointly concave with respect to $x_{12}, x_{22}, P_{12}, P_{22}$. In this Lemma, we prove the concavity of $V_2(x_{12}, x_{22})$ according to the Concavity Preservation Under Maximization (Proposition B-4 in Hayman and Sobel (1984)):

Let $X$ be a nonempty set with $A_x$ a nonempty set for each $x \in X$. Let $C = \{(x, y) : y \in A_x, x \in X\}$, let $J$ be a real-valued function on $C$, and define

$$f(x) = \inf \{J(x, y) : y \in A_x\}, x \in X$$

If $C$ is a convex set and $J$ is a convex function on $C$, then $f$ is a convex function on any convex subset of $X^* = \{X : X \in X, f(x) > -\infty\}$.

In our model, we are maximizing

$$V_2(x_{12}, x_{22}) = \max_{P_{12}, P_{22}} G_2(x_{12}, x_{22}, P_{12}, P_{22})$$

over a convex set, and $G_2$ has been proved to be a concave function, for this reason, $V_2(x_{12}, x_{22})$ is a concave function.

From Lemma 3.2, we can obtain the optimum pricing policy. The optimum discounted price during the second period is determined by the initial inventory levels of both
products. From Lemma 3.3, the optimum initial inventory levels exist to determine the optimum order quantity in the first period. With the concavity of $V_t(x_{12}, x_{22})$, we can utilise efficient searching algorithms to obtain the numerical results. From Lemma 3.2, we can obtain the optimum pricing policy. The optimum discounted price during the second period is defined by the initial inventory levels of both products. From Lemma 3.3, the optimum initial inventory levels exist to determine the optimum order quantity in the first period. With the concavity of $V_t(x_{12}, x_{22})$, we can employ efficient searching algorithms to obtain the numerical results.

**Theorem 3.5.** $G_1(y_1, y_2)$ is jointly concave with respect to $y_1, y_2$

**Proof.**

$$G_1(y_1, y_2) = \sum_{i \in \{1, 2\}} \{[P_i \mathbb{E}D_{i1} - L_{i1}(y_i, D_{i1}) - c_i y_i] + \gamma \mathbb{E}[V_2(x_{12}, x_{22})]\},$$

where $x_{i2} = y_i - D_{i1}$.

We firstly let

$$J_1(y_1, y_2) = \sum_{i \in \{1, 2\}} [P_i \mathbb{E}D_{i1} - L_{i1}(y_i, D_{i1}) - c_i y_i].$$

Figure 3.4 describes what $J_1$ looks like under the assumption $\pi_{i1} \geq c_i$. We claim it is a piece-wise linear concave function by the following property:

$$J_1(\lambda y_i + (1 - \lambda)y'_i) \geq \lambda J_1(y_i) + (1 - \lambda)J_1(y'_i)$$

![Figure 3.4: Expected total profit $V_1$](image)
We next prove that $V_2$ is jointly concave with respect to $y_1, y_2$. We formulate the following hessian matrix:

$$\text{det}(H_Y) = \begin{vmatrix} \frac{\partial^2 V_2}{\partial y_1} & \frac{\partial^2 V_2}{\partial y_1 \partial y_2} \\ \frac{\partial^2 V_2}{\partial y_2} & \frac{\partial^2 V_2}{\partial y_2 \partial y_1} \end{vmatrix}.$$  

Because $y_1$ and $y_2$ are separable, we have

$$\frac{\partial^2 V_2}{\partial y_1 \partial y_2} = \frac{\partial^2 V_2}{\partial y_2 \partial y_1} = 0.$$  

By applying chain rules, we obtain:

$$\frac{\partial^2 V_2}{\partial y_1^2} = \frac{\partial^2 V_2}{\partial x_1^2},$$  

$$\frac{\partial^2 V_2}{\partial y_2^2} = \frac{\partial^2 V_2}{\partial x_2^2}.$$  

With Lemma 3.4, $V_2(x_1, x_2)$ is joint concave with respect to $x_1, x_2$ and we have

$$\frac{\partial^2 V_2}{\partial x_1^2} \leq 0.$$  

From Theorem 3.5, the optimum order quantities $y_1, y_2$ exist and under the joint concavity of $G_1(y_1, y_2)$ with respect to $y_1, y_2$, efficient algorithms such as gradient search can be employed to obtain the optimum results.

**Remark 3.6** (Mistakes in the Literature). Rujing (2007) studies a single perishable product problem, in page 32, she formulates $x_2$ as $x_2 = (y_1 - D_1)^+$, she claims $V_2$ is a concave with respect to $y_1$. In fact, $V_2$ looks like that in Figure 3.5:

Clearly, it is not a concave function.

**Remark 3.7** (Mistakes in the Literature). Zhu and Thonemann (2009) study a multiple non-perishable product problem. In their assumption, $L_{it} = E(h(z_{it} - \xi_{it})^+ + E\pi_{it}(\xi_{it} - y_{it})^+)$ is twice differentiable in $z_{it}$. In fact, we found it is a piece-wise linear convex function, and it is not differentiable while $z_{it} = \xi_{it}$.

### 3.5 Algorithm

In this search algorithm, we start the search at a larger scale interval from lower bound to upper bound. With the optimal results, we repeat the search with a smaller scale interval on the neighbourhood of the optimal results to get a more specific result. With
this search algorithm, we can solve the problem much faster. In order to obtain the optimal values of \( y_1, y_2, p_{12}, p_{22} \) in a short time, the following algorithm is developed based on the results from the Lemma 3.1-3.4 and Theorem 3.5 we developed in this chapter.
The Pseudocode are presented as following:

**Data:** $c_i, h, a_{i2}, l_2, p_{i1}, \pi_{it}, \xi_{i2}$

**for** $n = 1:1000$ **do**

| 1000 values for $\xi$ from a normal distribution $N(\mu, \theta)$

**end**

Calculate the expectation of demand noise $\xi$

**for** $k = 1 : n$ **do**

Calculate $y_{11}^k, I_y^k, y_{12}^k, y_{22}^k$

**for** $y_1 = y_{11}^k : I_y^k : y_{12}^k$ **do**

**for** $y_2 = y_{21}^k : I_y^k : y_{22}^k$ **do**

**for** $m = 1 : n$ **do**

Calculate $P_{12}^m, I_P^m, \bar{P}_{12}^m, \bar{P}_{22}^m$

**for** $P_{12} = P_{12}^m : I_P^m : \bar{P}_{12}^m$ **do**

| Calculate $D_{12}, D_{22}$

| Calculate $G_2(P_{12}, P_{22})$

**end**

Solve $V_2(x_{12}, x_{22}) = \max_{P_{12}, P_{22}} G_2(P_{12}, P_{22})$

set $P_{12}^*, P_{22}^* = \operatorname{argmax}_{P_{12}, P_{22}}$ $V_2(x_{12}, x_{22})$

**end**

Set $G_1(y_1, y_2) = \phi(y_1, y_2) + V_2(x_{12}, x_{22})$

**end**

**end**

Set $V_1 = \max_{y_1, y_2} G_1(y_1, y_2)$

Set $y_1^*, y_2^* = \operatorname{argmax}_{y_1, y_2} G_1(y_1, y_2)$

**end**

**Algorithm 1:** Simulation Framework of Developed Algorithm

In this algorithm, $y_{11}^k, y_{12}^k$ represent the starting point for the search, $y_{21}^k, y_{22}^k$ represent the finishing point, $I_y^k$ is the intervals of the search. This algorithm is based on the Theorem 3.5, where the expected total profit is the concave function. We start the search from a large interval, after we find the optimal solution, we then search the neighbourhood from a smaller interval. The process will be repeated until we find the solution to places of decimals. Similarly, $p_{i1}^m$ represent the starting point for the search, $p_{i2}^m$ represent the finishing point, $I_P^m$ is the intervals of the search. This search is developed from Lemma 3.1 and the algorithm behind it is same to the search of optimal prices.
3.6 Numerical Example

To illustrate the algorithm, the following numerical example is presented. The results are obtained by Matlab R2014a. We collected data from a UK high-street fashion company. We targeted their winter collection, where products are sold at full price at the first period (for 1-2 months). Discounts are given starting from one week before Christmas until 2 weeks after Christmas, with all leftover products being donated to charity or given as a free gift to customers, no resale value is considered. We targeted classic jeans where there are two different wash finishes. All the other styles are the same. This is an ideal example for us in order to consider the price substitution between them.

3.6.1 Data Specification

The case we worked on is the sale a new design of Jeans in the next winter collection (2015-2016). They assume the new design has the same customer preference as an old version as they have a lot of similarities in the design. We collected data from the sales of the old version with two different washes (stone washed and fabric dyed) in the last winter sales (2014-2015). The stone washed jeans are deemed to be product 1 and fabric dyed are deemed to be product 2.

We are provided with the ordering cost from supplier, sold quantity and price for each wash in the full price selling period (Nov-Dec). This will be used to assume the order quantity and selling price in the first period in our model, where we have $c_1 = £26$, $c_2 = £28$, $P_{11} = £130$, $P_{12} = £140$. Holding costs and backorder costs are based on assumptions by the revenue manager. In the second period, holding costs and backorder costs are assumed to be the same in the first period. We are also provided six sets of observations for the second period, in each observation, we have the following variables: the price of two products with different washes; the sold quantities for them. Other information we have in the data is the ordering cost, the serial number, the description of the products, the store name and so on. With these six observations, we are able to employ linear regression to predict the parameters in the demand function.

We ran the linear regression models for all observations of the variables: price of product 1, sold quantity of product 1, price of product 2, sold quantity of product 2. SAS was employed to obtain the following estimated parameters for each variables: $a_{12} = -7.5$, $b_{12} = 789$, $l_{12} = 1.2$; $a_{22} = -1.6$, $b_{12} = 105$, $l_{12} = 0.6$; The error terms in the second period follow normal distributions with mean as 0 and standard deviation as 81 and 30 respectively. Standard deviation were obtained to be the Root MSE in the regression model by SAS. Root MSE represents a frequently used measure of the differences between values (sample and population values) predicted by a model or an estimator and the values actually observed. Therefore, Demand function for the product ‘1’ is formulated...
as:

\[ D_{12} = 789 - 7.5 * P_{12} + 0.6 * P_{22} + \xi_{12}, \]
\[ D_{22} = 105 - 1.6 * P_{22} + 1.2 * P_{12} + \xi_{22}. \]

Backorder costs are assumed to be 52 for both product ‘1’ and ‘2’, demand in the first period follows a normal distribution with mean as ‘156’, ‘56’ and standard deviation as 20, 10 respectively based on the assumption by revenue manager. In this research, the model was solved in Matlab 2014a in less than 1 minute, the algorithm we proposed provided us global optimal results in a very efficient way. This algorithm also has the potential to solve very large scale cases in just a few minutes. As the demand noise is relatively high in the demand function of product ‘1’, we run the model for 50 times and take the average value, the optimal results can be analysed as follows:

<table>
<thead>
<tr>
<th>( y_1 )</th>
<th>( y_2 )</th>
<th>( P_{12} )</th>
<th>( P_{22} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>439</td>
<td>115</td>
<td>74.6</td>
<td>84.3</td>
</tr>
</tbody>
</table>

The maximized total profit is 39208 pounds. A significant increase in total profit has been witnessed with the optimization model we presented. That is a huge improvement on profitability, since this high-street fashion company is facing substantial competition from other brands nowadays. Notice that, the maximized total profit are the potential expected total profit in the future sales. All the parameters are obtained from analysing historical data. The actual sales could be slightly different depending on changes of the customer preference for these products and the difference between the old and new products. Moreover, the overall comparison are limited to the errors in the parameters prediction. To have a better comparison, more data observation is needed to have an more accurate prediction of parameters.

Figure 3.6 compares the total profit the company has from their current strategy with the expected total profit from the optimization model we study. Our study indicates that, under the optimization model and compared with the policy they currently employ, this high-street fashion company can have a considerable profit increase of 17%. We next will quantify the benefits of jointly considering pricing and ordering policy by comparing the total profits.
Figure 3.6: Total profit increase by optimization model
3.6.2 Benefits of Jointly Making Inventory and Pricing Decision

Under the pricing optimization model, we take the average order quantity from historical data, which are 128 and 57 respectively, as a given parameter. Under the inventory optimization model, the average discounted prices at 93 pounds and 90 pounds respectively are given as input parameters.

Figure 3.7 compares the expected total profit under the pricing optimization, inventory optimization and joint pricing and inventory optimization models. It shows that the expected total profit in bar 3 under the joint pricing and inventory optimization model we are proposing achieves the highest profit, approximately a 74% increase compared with the inventory optimization displayed, and a 12% increase compared with the pricing optimization presented.

![Figure 3.7: Total profit of three optimization models](image-url)
3.6.3 Benefits of Considering Substitutions

In this experimentation, we will quantify the benefits of managing substitutable products together. We will quantify the differences on total profit of the following two models:

1. Without considering substitution, demand is assumed as: \( D_i = \beta_i - \alpha_i \times P_i + \xi_i', \forall i. \)
2. With considering substitution, demand is assumed as: \( D_i = b_i - a_i \times P_i + l_j \times P_j + \xi_i, \forall i \neq j. \)

In model 1, we assume the demand of a product depends only on its own price; this policy is considered the independent policy. In model 2, the price substitution has been taken account between products ‘1’ and ‘2’; this policy is the optimal policy we are proposing. Due to the limited size of the observation we have, to obtain a more reliable results; we will take the following steps to demonstrate the benefits.

1. We assume there is a “true” multivariate model: \( D_i' = b_i' - a_i' \times P_i' + l_j' \times P_j' + \xi_i', \forall i \neq j. \)
2. With the parameters on the true multivariate model, we generate 1000 random \( P, P_j \), then simulate 1000 observations of \( D_i, D_j, P, P_j. \)
3. From the 1000 observations, we estimate the new parameters by applying a linear regression on model 1 and 2.
4. We run optimization models of 1 and 2 respectively.
5. We quantify the benefits of considering substitution by analysing the difference on the expected total profit of models 1 and 2.

Figure 3.8 compares the expected profit under the two policies. We only look at the differences on the total profit in the second period, which is the only period applying two different policies; this gives us a clear overview of how much more profit can be yielded by applying model 2. We run each model 100 times, considering the large demand noise we have in both models. On average, model 2 achieves 3% more profit than model 1.

Figure 3.9 traces the differences between the two models on each run. It is clear that in the 100 times run, model 2 always achieves higher profit than model 1.
Figure 3.8: The total profit $V_2$ from model 1 and model 2 in second period

Figure 3.9: The percentage of differences between model 1 and model 2 in each run
Finally, we consider a higher demand substitution case when $l_{12} = l_{22} = 1.5$. We then compare the percentage of profit increasing from model 1 to model 2 in each run in the following figure:

Figure 3.10 reveals that in the higher demand substitution cases, model 2 yields much more profit than model 1, an average of 6%.

In summary, our numerical results indicated that, under the independent policy optimized by model 2, compared with the optimal policy on model 1, retailers suffer a considerable profit loss. In this case, managing substitutable products jointly can achieve approximately 3% increase in comparison with managing them individually. In the higher demand substitution case where the substitution is higher, more benefits can be obtained by managing substitutable products jointly.

### 3.6.4 Parameter Sensitivity Analysis

In an optimal pricing and inventory control problem, understanding the sensitivity of parameters is crucial for retailers. In this section, we will investigate the impact of different parameters on the pricing, ordering decisions and profitability of this retailer.
3.6.4.1 Optimal Decisions and Expected Profit Responsiveness under Increasing Demand Loss Rate

High demand loss rate indicates that customers are more sensitive to the price of the product. In the high street fashion company, the design, service, material, and product display are regarded as the major factors affecting demand loss rate. In joint pricing and inventory control for a single product problem, we have learned the demand loss rate has a considerable impact on pricing, and ordering decisions. In this section, we extend this study to illustrate the impact of demand loss rate on substitutable products.

Figure 3.11 shows that by increasing the demand loss rate \( a_{12} \) of product ‘1’, where consumers are more sensitive to the price of product ‘1’, optimal ordering quantities for product ‘1’ and ‘2’ decrease as the demands for both products decrease.

![Figure 3.11: The impact of varying demand loss rate \( a_{12} \) on optimal ordering quantities of both products](image)

Figure 3.12 shows that by increasing the demand loss rate \( a_{12} \) of product ‘1’, which implies the consumers are more sensitive to the price of product ‘1’, we observe that, to achieve an optimal profit, both optimal prices for ‘1’ and ‘2’ decrease. Figure 3.13 illustrates that the expected total profit decrease under an increasing demand loss rate \( a_{12} \). This is reasonable, as with decreasing optimal order quantities and optimal prices, the expected total profit decrease by up to 70%.
Figure 3.12: The impact of varying demand loss rate $a_{12}$ on optimal prices of both products.

Figure 3.13: The impact of varying demand loss rate $a_{12}$ on expected total profit.
With an increasing demand loss rate for product ‘2’, we observe an increasing optimal order quantity for product ‘1’ and a decreasing optimal order quantity for product ‘2’ (Figure 3.14). This result indicates that while demand for the substitutable products increases, it is different from the observation on \( a_{12} \). The reason behind this is that product ‘2’ is highly substitutable by product ‘1’, where more lost demand will be added to product ‘1’. Figure 3.15 compares the optimal price adjustments under increasing \( a_{22} \); we observe a slight decrease for product ‘1’ and a considerable decrease on product ‘2’. That is due to the weak substitution of product ‘1’ by product ‘2’. The decreasing expected profit are shown in figure 3.16.

![Figure 3.14: The impact of varying demand loss rate \( a_{22} \) on optimal ordering decisions of both products](image)
Figure 3.15: The impact of varying demand loss rate $a_{22}$ on optimal prices of both products.
Figure 3.16: The impact of varying demand loss rate $a_{22}$ on expected total profit
We summarize our findings about the impact of demand intercept on overall results and performance in Table 3.2, where the notation of $\uparrow$ (↓) indicates an increase (decrease) in the overall results.

<table>
<thead>
<tr>
<th></th>
<th>$a_{12}$</th>
<th>$a_{22}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order quantity $y_1$</td>
<td>↓</td>
<td>↑</td>
</tr>
<tr>
<td>Order quantity $y_2$</td>
<td>↓</td>
<td>↓</td>
</tr>
<tr>
<td>Retail price $P_{12}$</td>
<td>↓</td>
<td>↓</td>
</tr>
<tr>
<td>Retail price $P_{22}$</td>
<td>↓</td>
<td>↓</td>
</tr>
<tr>
<td>Expected Profit</td>
<td>↓</td>
<td>↓</td>
</tr>
</tbody>
</table>

**Table 3.2:** Summary of varying $b_{i2}$ on optimal decisions and total profit

### 3.6.4.2 Optimal Decisions and Expected Profit Responsiveness Under Increasing Demand Substitution Rate

The substitution between the two products is caused by product differentiation; in some research, they are denoted by cross-price effect(Zhu and Thonemann (2009), Niu et al. (2010)). The substitution is represented by demand substitution rate $l$. A higher $l$ indicates that products are considered to be more substitutable (less differentiated). In the fashion industry, product material, colour, and crafts are considered to be the major factors affecting the differentiation. In this section, we illustrate the optimal decisions responsiveness under different demand substitution rate.

Figure 3.17 shows that while product ‘2’ becomes more substitutable, more lost demand of product ‘1’ will substitute to product ‘2’, the optimal order quantity of product ‘1’ will decrease, whereas the optimal order quantity of product ‘2’ will increase. We also observe increased total demands for these two products, which is reasonable as more lost demand is substituted from one product to the other.
Figure 3.17: The impact of varying demand substitution rate $l_{12}$ on optimal ordering quantities of both products.
Figure 3.18 compares the optimal prices under an increasing demand substitution rate of product ‘1’, with both of them increasing. They become very close to each other while \( l_{12} = l_{22} \).

**Figure 3.18: The impact of varying demand substitution rate \( l_{12} \) on optimal prices of both products**

Figure 3.19 graphically describes the increasing expected profit under an increasing demand substitution rate \( l_{12} \). This reasonably reveals that by increasing \( l_{12} \), more lost demand of product ‘1’ will be added to the demand of product ‘2’; the total demands increase and, as a result, the profit increases.
Figures 3.20, 3.21 and 3.22 show the responsiveness of the overall results under an increasing demand substitution rate $l_{12}$, where product ‘1’ is more substitutable by product ‘2’. No significant difference is found when comparing with an increasing in $l_{12}$. 
Figure 3.20: The impact of varying demand substitution rate $l_{22}$ on optimal ordering quantities of both products.
Figure 3.21: The impact of varying demand substitution rate $l_{22}$ on optimal prices of both products.
Figure 3.22: The impact of varying demand substitution rate $l_{22}$ on expected total profit
We summarize our findings about the impact of demand substitution rate on overall results and performance in Table 3.3, where the notation of ↑ (↓) indicates an increase (decrease) in the overall results.

<table>
<thead>
<tr>
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<th>$l_{12}$ ↑</th>
<th>$l_{22}$ ↑</th>
</tr>
</thead>
<tbody>
<tr>
<td>Order quantity $y_1$</td>
<td>↓</td>
<td>↑</td>
</tr>
<tr>
<td>Order quantity $y_2$</td>
<td>↑</td>
<td>↓</td>
</tr>
<tr>
<td>Retail price $P_{12}$</td>
<td>↑</td>
<td>↑</td>
</tr>
<tr>
<td>Retail price $P_{22}$</td>
<td>↑</td>
<td>↑</td>
</tr>
<tr>
<td>Expected profit</td>
<td>↑</td>
<td>↑</td>
</tr>
</tbody>
</table>

Table 3.3: Summary of varying $l_{ij}$ on optimal decisions and profit

### 3.6.4.3 Optimal Decisions and Expected Profit Responsiveness under Increasing Demand Intercept $b_{ij}$

Demand intercept $b_{ij}$ represents the demand while all prices are 0. In this section, we investigate the impact of demand intercept $b_{12}$ on optimal decisions and expected profit.

Figure 3.23 compares the optimal ordering quantity under an increasing $b_{12}$; both ordering quantity is increasing generally as the demands are increasing. Figure 3.24 reveals the impact of $b_{12}$ on optimal prices; both optimal prices are increasing. From Figures 3.23 and 3.24 we can conclude that the expected profit will increase; this is evidenced by the Figure 3.25.
Figure 3.23: The impact of varying demand upper bound \( b_{12} \) on optimal ordering quantities of both products.

Figure 3.24: The impact of varying demand upper bound \( b_{12} \) on optimal prices of both products.
Figure 3.25: The impact of varying demand upper bound $b_{12}$ on expected total profit.
Figure 3.26 illustrate the impact of $b_{22}$ on optimal ordering quantities, we find the optimal ordering quantity for product ‘1’ decreas while it increase for product ‘2’. We find a slight decrease on substitutable products’ optimal ordering quantities, unlike the impact of $b_{12}$; the reason for this is that product ‘2’ is highly substitutable by product ‘1’, therefore less demand will be added to demand of product ‘1’, and this causes a slight decrease. In Figure 3.27, the optimal prices are all increasing where the optimal prices for product ‘2’ shows a sharp increase.

Figure 3.26: The impact of varying demand upper bound $b_{22}$ on optimal ordering quantities of both products
Figure 3.27: The impact of varying demand upper bound $b_{22}$ on optimal prices of both products
Figure 3.28 presents an increasing trend when increasing $b_{22}$. We can conclude that under increasing demand intercept $b_i$, the expected profit increase.

![Figure 3.28: The impact of varying demand upper bound $b_{22}$ on expected total profit](image)

We summarize our findings about the impact of demand intercept on overall results and performance in Table 3.4, where the notation of ↑ (↓) indicates an increase (decrease) in the overall results.

<table>
<thead>
<tr>
<th></th>
<th>$b_{12}$</th>
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<tbody>
<tr>
<td>Order quantity $y_1$</td>
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<td>↓</td>
</tr>
<tr>
<td>Order quantity $y_2$</td>
<td>↑</td>
<td>↑</td>
</tr>
<tr>
<td>Retail price $P_{12}$</td>
<td>↑</td>
<td>↑</td>
</tr>
<tr>
<td>Retail price $P_{22}$</td>
<td>↑</td>
<td>↑</td>
</tr>
<tr>
<td>Expected Profit</td>
<td>↑</td>
<td>↑</td>
</tr>
</tbody>
</table>

Table 3.4: Summary of varying $b_{i2}$ on optimal decisions and Profit

### 3.7 Conclusion

In this study, we extend the original Newsvendor model by considering the pricing as a decision variable. We determine the order quantity for the first period and pricing for the second period with the objective of maximizing the total profit over the two periods. To the best of our knowledge, this is the first research studying joint pricing and inventory control problem for substitutable perishable products across brands with
holding cost and backorder cost considered. The total profit in the first period is jointly concave with respect to the order quantity. The expected profit in the second period is proved to be jointly concave with respect to given inventory levels. The optimum price in the second period is presented as a function of given inventory levels. We also find a few mistakes present in the literature.

By analysing these properties, we find the global optimum for the decision variables. We also develop an efficient search algorithm which solves the problem faster under these properties. In the case of large-scale problems, this algorithm makes solving large-scale problems possible on a personal computer. In the numerical results, we quantify the benefits of jointly making ordering and pricing decisions. We also illustrate the benefits of managing substitutable products together; our results indicate that the expected total profit can be increased in this case. Finally, we analyse the impacts of parameters in demand functions on optimal decisions, which suggests the great importance of the accuracy of these parameters. Future works can be done by extending this model to consider a multiple periods problem or multiple substitutable products problem as we see in Chapter 4.
Chapter 4

Joint Pricing and Inventory Decisions for Two or More Substitutable and Perishable Products Over Multiple Periods Lifetime: A Stochastic Programming Approach

4.1 Introduction

The model in this chapter is an extension of the model presented in Chapter 3, where retailers order, stock and sell perishable products over a multiple period time horizon, which fits perishable products with higher retail price or longer lifetime better than the first model. Similar to Chapter 3, retailers place orders at the beginning of the first period, assuming that there is no replenishment to be considered during multiple periods. Markdown pricing policy will be employed to determine the discounted price of the second and subsequent periods.

In this chapter, we extend the traditional newsvendor model to multiple periods, with the order quantity and retail prices being the decision variables in this model. The remainder of this chapter is organized as follows: In section 4.2, we formulate the joint pricing and inventory control problem as a dynamic program. In section 4.3, we derive the properties of the decision variables. We extend the problem further to consider multiple substitutable products in section 4.4, and in section 4.5, we present numerical results, followed by concluding remarks in section 4.6.
4.2 Two Substitutable Products

4.2.1 Problem Description

In this model, decision makers make ordering decisions \( y \) at the beginning of the first period. In that period, the products are assumed to be sold at a given price; usually this will be a recommended price given by suppliers. After the first period, products will be sold at discounted prices. The lengths of each period are not necessarily equal and all leftover products \( (x_2, x_3) \) will be carried over to the next period. At the end of the last period, all leftover products will be disposed of. The ordering quantities and discounted price after the first period are the decision variables in this model. Figure 4.1 demonstrates the products flow for multiple periods as follows:

Demand for the products in each period is assumed to be individual and backlogged. Demand in the first period is known and assumed to follow a normal distribution. After that period, demand is price-dependent with the addition of an error to consider the uncertainty in each period. The objective is to maximize the total profit over the multiple period planning horizon. In this chapter, we analyse the optimum solution-structure of the problem and find retailers can significantly improve profit by managing these substitutable products simultaneously.

4.2.2 Model Formulation

Let index \( i,j=1,2 \) denote the two similar products, and let \( t = 1...T \) denote the periods. The notation is employed as follows:

- \( y_i \): Order quantity for new product \( i \) at the beginning of the planning periods
- \( P_{it} \): Retailing price at period \( t \) for product \( i \)
- \( x_{it} \): Leftover products at period \( t \) for product \( i \)
- \( D_{it} \): Demand at period \( t \) for product \( i \)
- \( h_t \): Holding cost at period \( t \)
Chapter 4 Joint Pricing and Inventory Decisions for Two or More Substitutable and Perishable Products Over Multiple Periods Lifetime: A Stochastic Programming Approach

- $\pi_{it}$: Backorder cost at period $t$ for product $i$
- $c_i$: Products ordering cost from supplies at beginning for product $i$
- $a_{it}$: Demand loss rate per unit price increase at period $t$ for product $i$
- $l_{it}$: Demand substitution amount per unit price increase at period $t$ from product $i$
- $V_t(x_{it})$: Maximized Expected Profit at period $t$ given $x_{it}$
- $G_t(x_{it}; P_{it})$: Total profit

The demands of either product in period $t$ ($t \geq 2$) depend on the retail prices of both products. We assume that the demand of product $i$ in period $t \geq 2$ follows a linear demand function such that $D_{it} = b_{it} - a_{it}P_{it} + l_{jt}P_{jt} + \xi_{it}$ where $j \neq i$, $a_{it}, l_{jt} \geq 0$. Here, the parameter $a_{it}$ represents demand loss per unit increase of its price $P_{it}$ and $l_{jt}$ represents the demand substitution; i.e. the demand increase for product $i$ for each unit increase in price of its substitutable product. We assume that $a_{it} \geq l_{jt}$ to imply that the price change of product $i$ affects its own demand more than that of the other product. Therefore, demand for product $i$ is more sensitive to its own price change than that of the other group. The total demand $D_{it} + D_{jt}$ gets smaller while either product price increases. The stochastic demand function $\xi_{it}$ denotes the market uncertainty of product $i$, which has a continuous density function $f_{it}(\xi_{it})$ bounded in $[\xi_{it}^{\min}, \xi_{it}^{\max}]$. The expected value for $\xi_{it}$ is 0. Additionally, we assume $b_{it} > \xi_{it}^{\min}$ to assure the positive upper bound of demands. When employing this model, it is important to ensure the non-negativity of the discounted profit $P_{it}$. Therefore, we assume a lower bound $P_{it}^{\min}$ and upper bound $P_{it}^{\max} \leq P_{i,1}$ for the optimum price $P_{it}$, where $P_{it}^{\max} < (b_{it} + l_{it}P_{it} + \xi_{it}^{\min})/a_{it}$. This inequality implies non-negativity of demand for product $i$ in period 2. The lower bound $P_{it}^{\min}$ depends only on problem parameters which are strictly positive. We define $y_1, y_2$ as the ordering decisions made by decision makers at the beginning of the first period; $P_{it}, P_{jt}$, $\forall t = 2...T$, are the prices set at each period.

We formulate this joint pricing and inventory control problem as a dynamic programming model. The total profit in period $T$ is:

$$G_T(x_{1T}, x_{2T}, P_{1T}, P_{2T}) = \sum_{i \in \{1, 2\}} \{ P_{iT}E[D_{iT}] - L_{iT}(x_{iT}, D_{iT}) \}. $$

The decision maker sets the price $P_{1T}, P_{2T}$ in order to maximize the profit in period $T$:

$$V_T(x_{1T}, x_{2T}) = \max_{P_{iT}, P_{2T}} G_T(x_{1T}, x_{2T}, P_{1T}, P_{2T}).$$

The total profit in period $t = 2...T - 1$ is:

$$G_t(x_{1t}, x_{2t}, P_{1t}, P_{2t}) = \sum_{i \in \{1, 2\}} \{ P_{it}E[D_{it}] - L_{it}(x_{it}, D_{it}) \} + \gamma E[V_{t+1}(x_{it+1}, x_{2t+1})],$$
where for convenience, we denote the expected inventory holding cost and backorder cost by

\[
L_{i1} = h \mathbb{E}(y_i - D_{i1})^+ + \pi_i \mathbb{E}(D_{i1} - y_i)^+.
\]

\[
L_{it} = h \mathbb{E}(x_{it} - D_{it})^+ + \pi_{it} \mathbb{E}(D_{it} - x_{it})^+, \forall t = 2, ... T.
\]

and where \(V_t(x_{1t}, x_{2t})\) is formulated as:

\[
V_t(x_{1t}, x_{2t}) = \max_{P_{1t}, P_{2t}} G_t(x_{1t}, x_{2t}, P_{1t}, P_{2t}).
\]

Here \(x_{it+1}\) follows dynamic programs \(x_{it+1} = x_{it} - D_{it}, \forall t = 2, ... T\). The total profit in period 1 is:

\[
G_1(y_1, y_2) = \sum_{i \in \{1, 2\}} [P_{i1} \mathbb{E}D_{i1} - L_{i1}(y_i, D_{i1}) - c \cdot y_i] + \gamma \mathbb{E}[V_2(x_{12}, x_{22})].
\]

The maximum profit over multiple periods is:

\[
V_1 = \max_{y_1, y_2} \{\mathbb{E}[G_1(y_1, y_2)]\}.
\]

Constraints:

\[
x_{i2} = y_i - D_{i2}
\]

\[
p_{it}^{\min} \leq p_{it} \leq p_{it}^{\max}
\]

\(y, D\) are nonnegative integers.

The dynamic programs has a simple solution. If we know the value of \(V_{t+1}(x_{1t+1}, x_{2t+1})\) for each state \(x_{1t+1}, x_{2t+1}\), we could compute \(V_t(x_{1t}, x_{2t})\) for each state \(x_{1t}, x_{2t}\). Therefore, we start from \(V_T(x_{1T}, x_{2T})\), then step backwards in periods to compute all value functions.

**Theorem 4.1.** The following statements hold:

1. The expected profit \(G_t(x_{1t}, x_{2t}, P_{1t}, P_{2t})\) is jointly concave with respect to \((x_{1t}, x_{2t}, P_{1t}, P_{2t})\), \(\forall t = 2, ... T\).

2. The maximum expected profit \(V_t(x_{1t}, x_{2t})\) is jointly concave with respect to \(x_{1t}, x_{2t}\), \(\forall t = 2, ... T\).

In order to prove this, we need results from the following lemmas: (Lemmas 4.2-4.5)

**Lemma 4.2.** If \(V_{t+1}(x_{1t+1}, x_{2t+1})\) is jointly concave with respect to \(x_{1t+1}, x_{2t+1}\), then the expected profit \(G_t(x_{1t}, x_{2t}, P_{1t}, P_{2t})\) is jointly concave with respect to \(x_{1t}, x_{2t}, P_{1t}, P_{2t}\).
Proof. The total profit is formulated as: \( G_t(x_{1t}, x_{2t}, P_{1t}, P_{2t}) = J_t(x_{1t}, x_{2t}, P_{1t}, P_{2t}) + V_{t+1}(x_{1t+1}, x_{2t+1}) \), where \( J_t(x_{1t}, x_{2t}, P_{1t}, P_{2t}) \) is formulated as:

\[
J_t(x_{1t}, x_{2t}, P_{1t}, P_{2t}) = \sum_{i \in \{1, 2\}} [P_i \mathbb{E} D_{it} - L_{it}(x_{it}, D_{it})].
\]

By expanding \( J_t(x_{1t}, x_{2t}, P_{1t}, P_{2t}) \), we have

\[
J_t(x_{1t}, x_{2t}, P_{1t}, P_{2t}) = P_{1t}(b_{1t} - a_{1t}P_{1t} + l_{2t}P_{2t}) + P_{2t}(b_{2t} - a_{2t}P_{2t} + l_{1t}P_{1t}) - L_{1t} - L_{2t},
\]

where

\[
L_{1t} = h\mathbb{E}(x_{1t} - D_{1t})^+ + \pi_{1t}\mathbb{E}(D_{1t} - x_{1t})^+,
\]

\[
L_{2t} = h\mathbb{E}(x_{2t} - D_{2t})^+ - \pi_{2t}\mathbb{E}(D_{2t} - x_{2t})^+.
\]

The proof is completed through the following steps:

1. \( J_t(P_{1t}, P_{2t}, x_{1t}, x_{2t}) \) is jointly concave with respect to \( P_{1t}, P_{2t}, x_{1t}, x_{2t} \).
2. \( V_{t+1}(x_{1t+1}, x_{2t+1}) \) is jointly concave with respect to \( P_{1t}, P_{2t}, x_{1t}, x_{2t} \).

Firstly, we prove \( J_t(P_{1t}, P_{2t}, x_{1t}, x_{2t}) \) is jointly concave with respect to \( P_{1t}, P_{2t}, x_{1t}, x_{2t} \). It is easy to prove that \( P_{1t}(b_{1t} - a_{1t}P_{1t} + l_{2t}P_{2t}) + P_{2t}(b_{2t} - a_{2t}P_{2t} + l_{1t}P_{1t}) \) is jointly concave with respect to \( P_{1t}, P_{2t} \). We then let

\[
z_{1t} = x_{1t} - b_{1t} + a_{1t}P_{1t} - l_{2t}P_{2t},
\]

\[
z_{2t} = x_{2t} - b_{2t} + a_{2t}P_{2t} - l_{1t}P_{1t}.
\]

The total holding cost and backorder cost can be rewritten as:

\[
L_{1t} = h\mathbb{E}(z_{1t} - \xi_{1t})^+ + \pi_{1t}\mathbb{E}(\xi_{1t} - z_{1t})^+,
\]

\[
L_{2t} = h\mathbb{E}(z_{2t} - \xi_{2t})^+ + \pi_{2t}\mathbb{E}(\xi_{2t} - z_{2t})^+.
\]

We find that \( L_{it} \) is a piecewise convex function with respect to \( z_{it} \) as shown in the following figure. Therefore, we claim \( L_{it} \) is jointly concave with respect to \( P_{1t}, P_{2t}, x_{1t}, x_{2t} \) as \( z_{1t} \) is a linear function of \( P_{1t}, P_{2t}, x_{1t} \), \( z_{2t} \) is a linear function of \( P_{1t}, P_{2t}, x_{2t} \) and \( x_{1t}, x_{1t} \) is separable. In summary, \( J_t(P_{1t}, P_{2t}, x_{1t}, x_{2t}) \) is now proved to be jointly concave with respect to \( P_{1t}, P_{2t}, x_{1t}, x_{2t} \).

Finally, we prove \( V_{t+1}(x_{1t+1}, x_{2t+1}) \) is jointly concave with respect to \( P_{1t}, P_{2t}, x_{1t}, x_{2t} \). Similarly, we know \( x_{1t+1} \) is a linear function of \( P_{1t}, P_{2t}, x_{1t} \), where as \( x_{2t+1} \) is a linear function of \( P_{1t}, P_{2t}, x_{2t} \). \( x_{1t+1}, x_{2t+1}, x_{1t}, x_{2t} \) are separable. For this reason,
with the assumption here which is that $V_{t+1}(x_{1t+1}, x_{2t+1})$ is jointly concave with respect to $x_{1t+1}, x_{2t+1}$, we find that $V_{t+1}(x_{1t+1}, x_{2t+1})$ is jointly concave with respect to $P_{1t}, P_{2t}, x_{1t}, x_{2t}$.

In conclusion, with the above two steps, we find that $G_t$ is jointly concave with respect to $(x_{1t}, x_{2t}, p_{1t}, p_{2t})$.

**Lemma 4.3.** Assuming that $V_{t+1}(x_{1t+1}, x_{2t+1})$ is jointly concave with respect to $x_{1t+1}, x_{2t+1}$, the maximum expected profit $V_t(x_{1t}, x_{2t})$ is jointly concave with respect to $x_{1t}, x_{2t}$.

To prove this lemma, we need the following proposition:

**Proposition 1** (Proposition B-4 (Concavity reservation Under Maximization) in Hayman and Sobel (1984)). Let $X$ be a nonempty set with $A_x$ a nonempty set for each $x \in X$. Let $C = \{(x, y) : y \in A_x, x \in X\}$, let $J$ be a real-valued function on $C$, and define

$$f(x) = \inf \{J(x, y) : y \in A_x\}, x \in X$$

If $C$ is a convex set and $J$ is a convex function on $C$, then $f$ is a convex function on any convex subset of $X^* = \{X : X \in X, f(x) > -\infty\}$.

We are now ready to prove Lemma 4.3.

**Proof.** From T, we proved that the total profit $G_t(x_{1t}, x_{2t}, P_{1t}, P_{2t})$ is jointly concave with respect to $x_{1t}, x_{2t}, P_{1t}, P_{2t}$, thus the concavity of $V_t(x_{1t}, x_{2t})$ follows directly from the ‘Concavity reservation Under Maximization’ in Proposition 1.
From Lemma 4.2, for a given initial inventory level, there are optimal prices which can be determined to maximize the total profit. In Lemma 4.3, it implies that by maximizing the initial inventory level, the maximized expected profit can be achieved. Therefore, we can develop efficient searching algorithms to obtain the numerical results in a short time.

**Lemma 4.4.** The total profit $G_T(x_{1T}, x_{2T}, P_{1T}, P_{2T})$ is jointly concave with respect to $x_{1T}, x_{2T}, P_{1T}, P_{2T}$.

**Proof.** In the last period, the model can be presented as the following:

$$G_T(x_{1T}, x_{2T}, P_{1T}, P_{2T}) = \sum_{i \in \{1, 2\}} [P_{iT} \cdot \mathbb{E}D_{iT} - L_{iT}(x_{iT}, D_{iT})],$$

where

$$L_{iT} = h(x_{iT} - \mathbb{E}D_{iT}) + \pi_{iT}(\mathbb{E}D_{iT} - x_{iT})^+.$$

It is easy to prove $\sum_{i \in \{1, 2\}} [P_{iT}\mathbb{E}D_{iT}]$ is jointly concave with respect to $P_{1T}, P_{2T}$. We then let $z_{iT} = x_{iT} - b_{iT} + a_{iT}P_{iT} - l_{iT}P_{JT}$ and find $L_{iT}$ is a piecewise convex function with respect to $z_{iT}$, and $x_{1T}, x_{2T}$ are separable. We now complete the proof that the total profit in the last period $G_T(x_{1T}, x_{2T}, P_{1T}, P_{2T})$ is jointly concave with respect to $x_{1T}, x_{2T}, P_{1T}, P_{2T}$.

**Lemma 4.5.**

The maximum expected profit $V_T(x_{1T}, x_{2T})$ is jointly concave with respect to $x_{1T}, x_{2T}$.

**Proof.** The maximized profit in period $T$ is:

$$V_T(x_{1T}, x_{2T}) = \max_{P_{1T}, P_{2T}} G_T(x_{1T}, x_{2T}, P_{1T}, P_{2T}).$$

According to the Concavity Preservation Under Maximization in Proposition 1, and $G_T$ is jointly concave with respect to $(x_{1T}, x_{2T}, P_{1T}, P_{2T})$, we claim that $V_T(x_{1T}, x_{2T})$ is jointly concave with respect to $x_{1T}, x_{2T}$.

We are now ready to prove Theorem 4.1.

**Proof.** We will use the induction method and results from Lemmas 4.2-4.5 to prove this theorem.

First, according to Lemmas 4.4 & 4.5, the statements hold for $t = T$. Suppose the statements also hold for $t = k + 1$, we will also prove they hold for $t = k$. Indeed, by applying Lemma 4.2, we have $G_k(x_{1k}, x_{2k}, P_{1k}, P_{2k})$ is concave. Similarly, by applying Lemma 4.3, we have $V_k(x_{1k}, x_{2k})$ is concave. Thus, the statements hold for $t = k$. This completes the proof.
Theorem 4.6. $G_1(y_1, y_2)$ is jointly concave with respect to $y_1, y_2$.

Proof.

$$G_1(y_1, y_2) = \sum_{i \in \{1, 2\}} \left\{ [P_i E D_{i1} - L_{i1}(y_i, D_{i1}) - c_i y_i] + \gamma E[V_2(x_{12}, x_{22})] \right\}$$

where $x_{i2} = y_i - D_{i1}$.

We firstly let

$$J_1(y_1, y_2) = \sum_{i \in \{1, 2\}} [P_i E D_{i1} - L_{i1}(y_i, D_{i1}) - c_i y_i]$$

Figure 4.3 describes what $J_1$ looks like under the assumption $\pi_i 1 \geq c_i$. We claim it is a piece-wise linear concave function by the following property:

$$J_1(\lambda y_i + (1 - \lambda) y'_i) \geq \lambda J_1(y_i) + (1 - \lambda) J_1(y'_i)$$

We next prove that $V_2$ is jointly concave with respect to $y_1, y_2$. By applying chain rules, we obtain:

$$\frac{\partial^2 V_2}{\partial y_1^2} = \frac{\partial^2 V_2}{\partial x_{12}^2},$$

$$\frac{\partial^2 V_2}{\partial y_2^2} = \frac{\partial^2 V_2}{\partial x_{22}^2}.$$}

With Lemma 4.5, $V_2(x_{12}, x_{22})$ is joint concave with respect to $x_{12}, x_{22}$, we have

$$\frac{\partial^2 V_2}{\partial x_{12}^2} \leq 0.$$
Furthermore, with the fact that $y_1$ and $y_2$ are separable, we complete the proof that $G_1(y_1, y_2)$ is jointly concave with respect to $y_1, y_2$.

From Theorem 4.6, the optimum order quantities $y_1, y_2$ exist and under the joint concavity of $G_1(y_1, y_2)$ with respect to $y_1, y_2$, efficient algorithms such as gradient search can be employed to obtain optimum results.

### 4.3 Multiple Products

We let

- $P_t = (p_{1t}, ..., p_{nt})'$ = retail price in period $t$.
- $X_t = (x_{1t}, ..., x_{nt})'$ = initial inventory level in period $t$.
- $Y = (y_1, ..., y_n)'$ = order quantity in the first period.

All products are substitutable, this means that the demand for product $i$ not only depends on its own price, but on the price of all other substitutable products. We let $D_t = (d_{1t}, ..., d_{nt})'$ represent the demand in period $t$, where $D_t = b_t' - A_t P_t + \xi_t'$. $b_t = (b_{1t}, ..., b_{nt})$ is the constant vector representing that the expected demand when all prices are set to zero. The matrix $A_t$ models the substitution relationship between $n$ products, and $a_{iit}$ is the demand $i$ loss rate per unit price increase of product $i$, $a_{ijt}$ is the demand $j$ increase rate per unit price increase of product $i$, or the price substitution rate. The structure of matrix $A$ has the following properties:

1. $a_{iit} > 0, a_{ijt}, a_{jit} \leq 0, \forall i \neq j$.
2. $A_t$ is a diagonal dominant matrix, or

$$a_{iit} \geq \sum_{j,j \neq i} |a_{ijt}| \quad \text{and} \quad a_{iit} \geq \sum_{j,j \neq i} |a_{jit}|.$$ 

The inequality $a_{iit} > 0$ means that if we increase the price of product $i$, the demand of that product will decrease. $a_{ijt} \leq 0$ means for every unit price increase of product $i$, the demand of product $j$ will increase by $a_{ijt}$. $a_{jit} \leq 0$ means the demand of product $i$ is more sensitive to its own price than the prices of other products. The diagonal dominant matrix $A_t$ also guarantees that when all the prices are increased, the total demand will be decreased. The ordering cost $c = (c_1, ..., c_n)$ is charged at the beginning of the first period. All leftover products will be charged a holding cost $h = (h_1, ..., h_3)$. All unsatisfied demand will be backlogged at a cost of $\pi_t = (\pi_{1t}, ..., \pi_{nt})$. At the beginning of the first period, the decision makers will decide the order quantity $Y$ they place, and the pricing $P_t, \forall t = 2...T$ they set.
4.3.1 Model Formulation

We formulate this joint pricing and inventory control problem as a dynamic programming model. The total profit in the last period $T$ is:

$$G_T(X_T, P_T) = EP_T \cdot D_T - L_T(X_T, D_T).$$

The maximum profit in period $T$ is:

$$V_T(X_T) = \max_{P_T} G_T(X_T, P_T).$$

The total profit in period $t = 2, ... T - 1$ is:

$$G_t(X_t, P_t) = EP_t \cdot D_t - L_t(X_t, D_t) + \gamma EV_{t+1}(X_{t+1}),$$

where the expected inventory holding costs and backorder costs are:

$$L_t = h E(X_t - D_t)^+ + \pi_t E(D_t - X_t)^+,$$

and the maximum profit $V_t(X_t)$ in period $t = 2, ... T - 1$ is formulated as:

$$V_t(X_t) = \max_{P_t} G_t(X_t, P_t).$$

Here, $X_{t+1}$ follows

$$X_{t+1} = X_t - D_t, \quad \forall t = 2, ... T.$$

The total profit in period 1 is:

$$G_1(Y) = EP_1 \cdot D_1 - L_1(X_1, D_1) - cY + \gamma EV_2(X_2),$$

where the expected inventory holding costs and backorder costs are:

$$L_1 = h E(Y - D_1)^+ + \pi_1 E(D_1 - Y)^+,$$

and here the initial inventory level at the second period follows:

$$X_2 = Y - D_1.$$

The maximum profit over multiple periods is:

$$V_1 = \max_Y G_1(Y).$$
Constraints:

\[ y_{it}, d_{it} \text{ are nonnegative integers.} \]

### 4.3.2 Optimality Analysis

**Theorem 4.7.** The following statements hold:

1. The expected profit \( G_t(X_t, P_t) \) is jointly concave with respect to \((X_t, P_t)\), \(\forall t = 2...T\).

2. The maximum expected profit \( V_t(X_t) \) is jointly concave with respect to \(X_t\), \(\forall t = 2...T\).

In order to prove this, we need the results from the following lemmas (lemma 4.8-4.11)

**Lemma 4.8.** Assuming that \( V_{t+1}(X_{t+1}) \) is jointly concave with respect to \(X_{t+1}\), then the expected profit \( G_t(X_t, P_t) \) is jointly concave with respect to \(X_t, P_t\).

**Proof.** The total profit of \( G_t(X_t, P_t) \) in period \( t \) is formulated as:

\[
G_t(X_t, P_t) = E P_t \cdot D_t - L_t(X_t, D_t) + \gamma E V_{t+1}(X_{t+1}).
\]

Firstly, we claim that the total profit \( G_t \) in period \( t \) is jointly concave with respect to \( P_t \).

In order to prove \( G_t \) is jointly concave with respect to \( P_t \), the total profit \( G_t \) is rewritten as:

\[
G_t = E P_t \cdot D_t - L_t(Z_t) + \gamma E V_t(X_{t+1}),
\]

where

\[
L_t(Z_t) = h E (Z_t - \xi_t^t)^+ + \pi_t E (\xi_t^t - Z_t)^+.
\]

It is easy to prove that \( E P_t \cdot D_t \) is a joint concave function to \( P_t \). \( L_t(p_t) \) is a piece-wise linear convex function to \( Z_t \).

At last, we claim \( E V_t(X_{t+1}) \) is a concave function with respect to \( P_t \). It is easy to prove because of the fact that \( V_t(X_{t+1}) \) is assumed to be concave with respect to \( X_{t+1} \), and \( X_{t+1} \) is a linear function of \( P_t \). Hence, we complete the proof that \( G_t \) is jointly concave with respect to \( P_t \).

\[
\square
\]

**Lemma 4.9.** Assuming that \( V_{t+1}(X_{t+1}) \) is jointly concave with respect to \( X_{t+1} \), then the maximum expected profit \( V_t(X_t) \) is jointly concave with respect to \( X_t \).
Proof. In Lemma 4.8, we proved that the total profit $G_t(X_t, P_t)$ is jointly concave with respect to $X_t, P_t$. According to proposition 1, the concavity of $V_{t+1}(X_{t+1})$ is preserved under maximization of $G_t(X_t, P_t)$. Hence, we claim that $V_t(X_t)$ is jointly concave with respect to $X_t$. \hfill \Box

**Lemma 4.10.**

The total profit $G_T(X_T, P_T)$ is jointly concave with respect to $X_T$ and $P_T$.

Proof. In the last period, the total profit $G_T$ is modelled as:

$$G_T(X_T, P_T) = EP_T \cdot D_T - LT(Z_T),$$

where

$$Z_T = X_T - b'_t + A_t P_t.$$  

$EP_T \cdot D$ is jointly concave to $P_T$, and $LT$ is a piece-wise linear convex function to $Z_T$. In addition, $Z_T$ is linear to $X_T$ and $P_T$. Hence, we complete the proof that $G_t(X_t, P_t)$ is jointly concave with respect to $(X_t, P_t)$. \hfill \Box

**Lemma 4.11.**

The maximum expected profit $V_T(X_T)$ is jointly concave with respect to $X_T$.

Proof. According to the Concavity Preservation Under Maximization (Proposition B-4 in Hayman and Sobel (1984)), and we already know that $G_T$ is jointly concave with respect to $(X_T, P_T)$, we claim that $V_T(X_T)$ is jointly concave with respect to $X_T$. \hfill \Box

We are now ready to prove Theorem 4.7.

**Theorem 4.12.** $G_1(Y)$ is jointly concave with respect to $Y$.

Proof. The total profit $G_1$ is modelled as

$$G_1(Y) = EP_1 \cdot D_1 - L_1(X_1, D_1) - cY + \gamma EV_2(X_2).$$
It is easy to prove $G_1$ is a concave function to $Y$. We firstly claim that $J_1(Y) = \mathbb{E}P_1 \cdot D_1 - L_1(X_1, D_1) - cY$ is a piece-wise linear concave function to $Y$ as $\pi_1 \geq c$. Moreover, we know that $X_2$ is linear to $Y$; for this reason, we know that $G_1(Y)$ is jointly concave with respect to $Y$. 

### 4.4 Algorithm

With the property of the decision variables analysed earlier in this chapter, we develop a search algorithm to find the optimal results in a short time. We start the search from a lower to an upper bound for decision variables with a large intervals. With the optimal results obtained, we repeat the search from the neighbourhood of the best results to find a more specific best result. Let index $i, j=1,2$ denote the two similar products, and let $t = 1..T$ denote the periods. The reminder of the notation is presented as follows:

- $y_i$: Order quantity for new product $i$ at the beginning of the planning periods
- $x_{it}$: Leftover products at period $t$ for product $i$
- $P_{it}$: Retailing price at period $t$ for product $i$
- $D_{it}$: Demand at period $t$ for product $i$
- $h_t$: Holding cost at period $t$
- $\pi_{it}$: Backorder cost at period $t$ for product $i$
- $c_i$: Products ordering cost from supplies at beginning for product $i$
- $a_{it}$: Demand loss rate per unit price increase at period $t$ for product $i$
- $l_{it}$: Demand substitution rate per unit price increase at period $t$ from product $i$
- $V_t(x_{it})$: Maximized Expected Profit at period $t$ given $x_{it}$
- $G_t(x_{it}; P_{it})$: Total profit
In order to obtain the optimal values of $y_1$, $y_2$, $P_{1t}$, $P_{2t}$ in a short time, the following algorithm has been developed based on the theorems we proved in this chapter.

**Data:** $c, h, a, l, p_{11}, π, σ$

for $k = 1 : 1000$ do
  1000 values for $ξ$ from a normal distribution $N(μ, θ)$
end

Calculate the expectation of demand noise

for $m = 1 : M$ do
  for $t = T : 2$ do
    Calculate $x_{it}$, $y_{it}$, and intervals $I_{m}$.
    for $x_{it} = x_{it} : I_{m} : y_{it}$ do
      for $n = 1 : N$ do
        Calculate $P_{it}$, $y_{it}$, and intervals $I_{P_{it}}$.
        for $P_{it} = P_{it} : I_{P_{it}} : \overline{P_{it}}$ do
          Calculate $D_{it}$.
          Calculate $G_{it}(P_{it})$.
        end
        Solve $V_{t}(x_{it}) = \max_{P_{it}} G_{t}(P_{it})$.
        Set $P_{it}^* = \arg\max_{P_{it}}$.
      end
    end
  end
  Calculate $y_{i}$, $\overline{y_{i}}$.
  for $y_{i} = y_{i} : I_{y_{i}} : \overline{y_{i}}$ do
    Calculate $G_{1}(y_{1}, y_{2}) = φ(y_{1}, y_{2}) + V_{2}(x_{12}, x_{22})$.
  end
  Solve $V_{1} = \max_{y_{i}} G_{1}(y_{i})$.
  Set $y_{i}^* = \arg\max_{y_{i}}$.
end

**Algorithm 2:** Simulation Framework of the Developed Algorithm

In this algorithm, $y_{i}$ represent the starting point for the search, $\overline{y_{i}}$ represents the finishing point, and $I_{y}$ are the intervals of the search. This algorithm is based on Theorem 4.12, where the expected total profit is a concave function. We start the search from a large interval, after we find the best solution, we then search the neighbourhood from a smaller interval. The process will be repeated until we find the solution to places of decimals. Similarly, $P_{it}$, $x_{it}$ represent the starting point for the search, $\overline{P_{it}}$ represent the finishing point, $I_{P_{it}}$ is the intervals of the search. This search is developed from Theorem 4.7 for optimal prices and the record of leftover quantity.
4.5 Numerical Results

In the numerical results, we carry on the case study for the UK high-street fashion brand. In this chapter, they order, store and sell multiple substitutable products over multiple periods. We study the case that they place orders on new season products at the beginning of the first period, and determine the retailer price at the second and third periods.

In this study, we suggest decision makers make pricing and inventory decisions simultaneously. In this section, we will demonstrate how the profitability of a firm can be improved by making pricing and inventory decisions together. Furthermore, we study a joint pricing and inventory problem, considering substitution between two similar products; therefore, in the numerical results, we will quantify the benefits of considering substitutions between these two products. We will also analyse the sensitivity of parameters in demand functions on decision variables to address the great importance of data accuracy in real world applications.

The results are obtained by Matlab R2014a. The data is from their winter collection; they currently have two selling periods for winter collections. During the first period, products are sold at full price which is determined by their global office. One week before Christmas they start the second selling period in which products are sold at a discounted price determined by their UK office. The orders are placed at the beginning of the first period, and the UK office makes the decisions regarding these. Two weeks after Christmas, the second selling period finishes, all leftover products are taken off the shelves and either donated to charity or given as a free gift to staff or customers. We continue using the data for the first and second periods in this numerical example. For the third period, due to the limitation of the data we have collected, we make some assumptions. We assume there is a third selling period and the parameter for this period is $a_{13} = -7.5$, $b_{13} = 500$, $l_{13} = 1.2$; $a_{23} = -1.6$, $b_{23} = 60$, $l_{23} = 0.6$. The demand noise in the third period also follows normal distributions with mean as 0 and standard deviation as 40 for both products.

With the algorithm we developed in this chapter, the overall results are obtained within 90 seconds. We run Matlab 100 times, with the average maximized total profit being $42646$. In the Table 4.1, the optimal decision of order quantity and pricing are presented with the total profit being closest to 42646.

<table>
<thead>
<tr>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$P_{12}$</th>
<th>$P_{22}$</th>
<th>$P_{13}$</th>
<th>$P_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>608</td>
<td>136</td>
<td>73</td>
<td>83.4</td>
<td>51.7</td>
<td>63.5</td>
</tr>
</tbody>
</table>

Table 4.1: Optimal decisions of this three-periods case.
4.5.1 Benefits of Jointly Making Inventory and Pricing Decisions

In this study, in order to increase profit, we suggest decision makers jointly make inventory and pricing decisions. This is evidenced by our experimentation in this section in the following three cases:

1. Inventory Optimization
2. Pricing Optimization
3. Joint pricing and inventory optimization

Due to the lack of the data in the third period, we generate the data by means of the following steps:

- We assume there is a “true” multivariate model: \[ D'_i = b'_i - a'_i \cdot P'_i + b'_j \cdot P'_j + \xi'_i, \forall i \neq j. \]
- With the parameters on the true multivariate model, we generate 1000 random \( P_i, P_j \), then simulate 1000 observations of \( D'_i, D'_j, P'_i, P'_j \).
- We take the average of \( P'_{i3}, P'_{j3} \) as a known parameter to consider inventory optimization.
- We take the average of \( D'_{i3}, D'_{j3} \) as a known parameter on order quantity to consider pricing optimization.

We also develop two scenarios to demonstrate the benefits of jointly make pricing and inventory decisions in detail. In the first scenario, we assume all three cases take different approach in period 3 only; In the second scenario, we assume 3 cases apply different strategies in all three periods.

Figure 4.4 illustrates the maximized total profit under cases 1, 2, and 3 respectively. In case 1, we consider the case in which the manager only makes optimal inventory decisions, and the retailer prices are the average prices we have from the generated data. In case 2, the manager only makes optimal pricing decisions, and the order quantities are the average quantities we obtained from the generated data. In case 3, the manager makes joint pricing and inventory decisions; it is clear that in jointly making pricing and inventory decisions, more profit can be obtained on average.

For an in-depth examination of the benefits of jointly making pricing and inventory decisions, we investigate the total profit in period 3, which is the only period applying different strategies for the three cases. Figure 4.5 indicates the maximized total profit in all cases. We clearly find that by jointly making pricing and inventory decisions, case 1 inventory optimization can increase profit by 6%, whereas case 2 pricing optimization is able to improve profitability by 3%.
Figure 4.4: Differences of three cases applied in period 3 only on overall total profits

Figure 4.5: Differences on total profits of period 3 on all three cases
Chapter 4 Joint Pricing and Inventory Decisions for Two or More Substitutable and Perishable Products Over Multiple Periods Lifetime: A Stochastic Programming Approach

Figure 4.6 represents the second scenario where, in periods 2 and 3, managers make pricing and inventory decisions independently, representing cases 1 and 2. We can observe further great differences in the three cases. In case 1, manager makes optimal inventory decisions for periods 2 and 3, the retailer price for period 2 is from the average of raw data, whereas the retailer price for period 3 is from the average of the generated data. By comparing case 1 and 3, we find there is a 9% profit loss for this retailer. In case 2, manager makes optimal pricing decisions for periods 2 and 3, where the ordering quantities are from the raw data in period 2 and the generated data in period 3. A significant profit loss of about 38% can be revealed.

4.5.2 Benefits of Considering Substitution

In this study, we recommend that managers jointly make decisions on substitutable products. We will quantify the benefits of this recommendation by comparing it with making decisions on substitutable products independently.
1. Without considering substitution, demand is assumed as: \( D_i = \beta_i - \alpha_i * P_i + \xi_i \), \( \forall i \).

2. Considering substitution, demand is assumed as: \( D_i = b_i - a_i * P_i + l_j * P_j + \xi_i \), \( \forall i \neq j \).

In model 1, the price substitution has been taken into account between products 1 and 2; this policy is the optimal policy we are proposing. In model 2, we assume that the demand of a product depends only on its own price; this policy reflected is the independent policy. Due to the limited size of the observations we have, and in order to obtain more reliable results, we will take the following steps in our experimentation:

- We assume there is a “true” multivariate model: \( D'_i = b'_i - a'_i * P'_i + l'_j * P'_j + \xi'_i \), \( \forall i \neq j \).
- With the parameters on the true multivariate model, we generate 1000 random \( P'_i, P'_j \), then simulate 1000 observations of \( D'_i, D'_j, P'_i, P'_j \).
- From the 1000 observations, we estimate the new parameters by applying linear regression on models ‘1’ and ‘2’.
- We run optimization models ‘1’ and ‘2’ respectively.
- We quantify the benefits of considering substitution by comparing the difference of the expected total profit of models ‘1’ and ‘2’.

Firstly, we assume for cases 1 and 2 the same decisions are being made in periods 1 and 2; we only examine the differences of making joint or independent decisions in the third period. We will repeat the four steps 100 times and illustrate the results through the following box plots:

Figure 4.7 displays the benefits of jointly making decisions on substitutable products in period 3. We find that model 2 achieves slightly higher overall profit than model 1 on average over 100 runs. As we only employ models 1 and 2 in the third period, the overall results are only slightly increased.
Figure 4.7: Difference in total profits by applying model 1 and model 2 in period 3 only.
Figure 4.8 demonstrates the differences of models 1 and 2 in period 3 over 100 runs; on average, model 2 obtains 1.8% more profit than model 1. We also find in almost 81 out of 100 runs, model 2 achieves higher profit than model 1. However, in the other 19 runs, model 1 obtains higher profit.

**Figure 4.8:** Percentage of profit increase by applying model 2 in Period 3
Secondly, we assume that models 1 and 2 make different decisions in the second and third selling periods. We then examine the differences between the impacts of joint and independent policies on total profit. Figure 4.9 expresses benefits of model 2. We find that model 2 achieves a higher profit than model 1; on average by 1%. If we only look at the total profit in the second and third periods, we find that by considering substitution, the profitability can be improved by 3%.

![Figure 4.9: Differences by applying model 1 and 2 in period 2 and 3](image)

Finally, we conduct experimentation to discover how much we can benefit by managing substitutable products together while the demand substitution rate is higher. In this study, we suggest that under higher substitution, more profit can be generated by considering substitution, this is evidenced by the figure 4.10 and 4.11.

Figure 4.10 demonstrates the case in which we increase the demand substitution rates $l_1$ and $l_2$ to 1.5, in all 100 runs. Model 2 yields more profit than model 1. On average, model 2 yields 6% more profit than model 1, as illustrated in Figure 4.11.
Figure 4.10: Percentage of profit increase by applying model 2 in period 3 with a higher demand substitution rate
In this section, we quantify the benefits of managing substitutable products together for this high street fashion industry. We further investigate the fact that the greater the substitution which exists, the higher the benefits which will be created by managing them together.

4.5.3 Benefits of Dividing Selling Periods into More Periods

For this high street fashion company, as the selling period approaches its end, consumers expect a larger discount; this is a typical phenomenon in the fast fashion industry, as fashion trends change enormously in a short period. To better satisfy the custom demand, we suggest retailers divide the total selling length into more periods. Thus, we have the following two cases encompassing the same total selling length:

1. Two Selling Periods
2. Three Selling Periods

In the model of case 1, products are sold at the same prices in periods 2 and 3; for this reason, these periods 2 and 3 are assumed to be a single selling period. In case 2, products are sold at different prices in period 2 and 3.
Figure 4.12 shows the differences between case 1 and 2 on total profit. It quantifies the fact that by dividing the second period into two small periods, the demand can be better satisfied, and the total profit can be increased significantly by 5%.

\[ \text{Expected Profit in Period 1 (Pounds)} \]

![Box plot showing expected profit in period 1](image)

**Figure 4.12: Benefits of dividing one period into more periods**

### 4.6 Conclusion

In this chapter, we extend the work conducted in Chapter 3 to study multiple period joint pricing and order decisions makings for multiple substitutable products. This is a more general case where there is usually more than one discounted period for fast fashion industries or other perishable products retailers, where the values of products decrease quickly. This extension has strong practical implications. At the beginning of the first period, managers make ordering decisions, with no replenishment being considered. At the beginning of each subsequent period, discounted retail prices are offered to better satisfy the demand. To the best of our knowledge, this is the first work in the literature studying joint pricing and inventory control problem across similar perishable products over multiple periods by considering holding cost and backorder cost.

With the properties derived in this chapter, we find that after period 1, the total profit is jointly concave with respect to the retail price, and the maximized total profit is a joint
concave function of initial inventory levels. We also find that the total profit in period 1 is a joint concave function of the initial order quantities. In the numerical results, we demonstrate that by jointly making pricing and inventory decisions, the retailers can benefit significantly. If managers consider substitutions between similar products, a larger total profit can be yielded, and the greater the substitution is, the higher the profit generated is. Finally, we conduct the experimentation to show that if selling periods are divided into more sales horizons, higher total profit can be achieved.
Chapter 5

A Robust Optimization Approach for Joint Pricing and Inventory Decisions for Two Substitutable and Perishable Products

5.1 Introduction

In the study of joint pricing and inventory control, it is assumed that in the stochastic programming approach, retailers maximize their profits by setting prices and determining inventory levels with complete knowledge of the probability distributions on modelling uncertainty. However, in the real world, the application of stochastic programming is often limited by the dependency on the availability of probabilistic information (e.g. the distribution of the demand). More recently, a new optimization approach has been developed that takes into account the influence of data uncertainties on the quality and feasibility of the model without making any assumption on its distributions; it considers an uncertainty set and looks for an optimal solution that is feasible for any realizations within that set. This alternative approach, known as robust optimization, can be used in a wide range of applications, with the added advantage of being intuitive and easy to understand.

In this chapter, we consider a joint pricing and inventory problem in a single period. We are considering an extended newsvendor model where decision makers can obtain parameters on the latest data they collected and therefore make optimal decisions. This model can be employed for perishable products problem like food, seasonal products, fast fashion industry, and high-tech products where the product becomes obsolete after a given time (the life cycle). We then extend this problem to consider a budget of
uncertainty to avoid over conservatism. In the third model, we consider the substitutable and perishable products problem. To the best of our knowledge, this is the first work in the literature to take a robust optimization approach to study the substitutable and perishable products problem.

With respect to other research in inventory and pricing models, we consider demand $D$ is a linear function on price, $D = \alpha - \beta p$, where $\alpha$ is the demand upper bound and $\beta$ is the demand loss rate per unit price increase. Holding cost is considered and demand will be backlogged to meet customers’ needs. The notation employed is as follows:

- $y$: Order Quantity
- $p$: Retailing price
- $h$: Holding cost
- $q$: Backorder cost
- $c$: order cost
- $\alpha$: Demand upper bound
- $\beta$: Demand loss rate per unit price increase

In the notations, $y$ and $p$ are the two decision variables we need to determine, whereas $h$, $q$, $c$, $\alpha$, $\beta$ are the parameters we can obtain from analysing historical data. The deterministic model for finding the optimal order and pricing decisions is formulated as:

$$
\max_{p,y} \quad [p(\alpha - \beta p) - h(y - \alpha + \beta p)^+ - q(\alpha - \beta p - y)^+ - yc]
$$

$$s.t. \quad 0 \leq p \leq \frac{\alpha}{\beta},$$

$$0 \leq p \leq \overline{p}, 0 \leq y \leq \overline{y}$$

where $p(\alpha - \beta p)$ is the revenue achieved from selling products, $h(y - \alpha + \beta p)^+$ is the total holding cost for unsold products, $q(\alpha - \beta p - y)^+$ is the backorder cost for unsatisfied demand and $yc$ is the total order cost. At the end of the period, the value of the product obsolete as we are studying a perishable product problem.

### 5.2 Robust Formulation

In real-world applications, there is a possibility that small uncertainty in demand (including factors such as the weather, market capacity) can change optimal solutions from a practical viewpoint. The importance of robustness in practical applications has been addressed by a number of studies (Soyster (1973), Ben-Tal and Nemirovski (1998)). In this study, we will develop a robust approach to solve joint pricing and order problems
with uncertain data.

In the robust optimization model, we aim to protect the system against unknown but bounded disturbances $\alpha$ and $\beta$:

$$\tilde{\alpha} \in [\alpha - \alpha', \alpha + \alpha'], \quad \tilde{\beta} \in [\beta - \beta', \beta + \beta']$$

where $\alpha$, $\beta$ are nominal values while $\alpha'$, $\beta'$ are some given half-length of ranges for the realization. We formulate a general robust optimization model to find a solution that is feasible for any realization of the uncertain data and maximize the realized objective functions.

$$\max_{p,y} \min_{\tilde{\alpha}, \tilde{\beta} \in \mathcal{F}} \quad p(\tilde{\alpha} - \tilde{\beta}p) - h(y - \tilde{\alpha} + \tilde{\beta}p)^+ - q(\tilde{\alpha} - \tilde{\beta}p - y)^+ - yc$$

(5.1)

$$p \leq \frac{\tilde{\alpha}}{\tilde{\beta}},$$

(5.2)

$$0 \leq p \leq \bar{p}, 0 \leq y \leq \bar{y},$$

(5.3)

$$0 \leq y \leq \bar{y},$$

(5.4)

Here, we would like to find the price $p$ and order quantity $y$ in the outer max problem to maximize the worst possible profit given the parameter $\tilde{\alpha}$ and $\tilde{\beta}$ belonging to the uncertainty set $\mathcal{F}$. The “worst case” here is modelled through the min operator.

This problem corresponds to a situation in which decision makers determine price $p$ and quantity $y$ at the beginning, in such a way that the total profit is maximized even for the worst realization of uncertain parameters. Holding costs will be considered and demand will be backordered subject to a cost. Since we are considering a perishable product problem, at the end of the planning horizon, the value of the product obsolete with no salvage value considered here. The uncertainty, $\tilde{\alpha}$ and $\tilde{\beta}$, always acts so as to minimize the profit, hence this problem solved by decision maker corresponds to a worst-case scenario: a maximization of the minimum problem. Notice that if all the parameters were deterministic, the problem would be a piecewise concave objective function that we would want to maximize. It has linear constraints, with 2 decision variables.

In order to solve the robust counterpart problem, we first solve the inner problem:

$$\min_{\tilde{\alpha}, \tilde{\beta} \in \mathcal{F}} \quad p(\tilde{\alpha} - \tilde{\beta}p) - h(y - \tilde{\alpha} + \tilde{\beta}p)^+ - q(\tilde{\alpha} - \tilde{\beta}p - y)^+ - yc$$

(5.5)

Let $\alpha = \alpha' - \bar{\alpha}, \bar{\alpha} = \alpha' + \bar{\alpha}', \beta = \beta' - \bar{\beta}, \bar{\beta} = \beta' + \bar{\beta}'$, we have the following proposition:
Proposition 5.1. Given $p$ and $y$, the formulation (5.1) equals the following linear optimization problem:

$$\min \left[ f(p, y; \bar{D}), f(p, y; \underline{D}) \right]$$

where $f(p, y; D) = p \cdot D - h(y - D)^+ - q(D - y)^+ - yc$, $\bar{D} = \alpha - \beta p$ and $\underline{D} = \alpha - \beta p$.

Proof. In the demand function, we model the demand as $D = \alpha - \beta \cdot p$. We observe that $\alpha$ and $\beta$ are independent from each other, with $D$ increasing on $\alpha$ but decreasing on $\beta$. Let $\tilde{D} = \tilde{\alpha} - \tilde{\beta} p$, then for every given $p$, we have

$$\tilde{D} \in [\alpha - \beta p, \alpha - \beta p]$$

Function (5.1) can be equivalently rewritten as:

$$\min_{\tilde{D}} \left( p \tilde{D} - h(y - \tilde{D})^+ - q(\tilde{D} - y)^+ - y(c - ky), \quad \forall \tilde{D} \in [\alpha - \beta p, \alpha - \beta p]. \right. \tag{5.6}$$

In the model assumption, backorder cost $q$ is assumed to be no less than $p$. The right-hand side in the above is a piecewise function of $\tilde{D}$, where

$$f(p, y; \tilde{D}) = \begin{cases} 
(p + h)\tilde{D} - (h + c)y, & \tilde{D} \leq y \\
(p - q)\tilde{D} - (c - q)y, & \tilde{D} \geq y
\end{cases}$$

For each given $p$ and $y$, while $\tilde{D} \leq y$, $f(p, y; \tilde{D})$ is increasing on $\tilde{D}$; whereas while $\tilde{D} \leq y$, it is decreasing on $\tilde{D}$, as illustrated in the figure below.

![Figure 5.1: The graphical description of the function $f_i(\tilde{D}_i)$](image)

Therefore, we can get a deterministic equivalent problem, and we solve a function:

$$\min \left[ f(p, y; \bar{D}), f(p, y; \underline{D}) \right].$$

With proposition 5.1, the overall optimization problem will be formulated as:

$$\max_{p, y} \min \left[ f(p, y; \bar{D}), f(p, y; \underline{D}) \right].$$
Let $r = \min[f(p, y; \mathcal{D}), f(p, y; \mathcal{D})]$, then the optimization problem can be equivalently formulated as:

\[
\begin{align*}
\max_{p, y, r} & \quad r \\
\text{s.t.} & \quad r \leq f(p, y; \mathcal{D}) \\
& \quad r \leq f(p, y; \mathcal{D}) \\
& \quad \alpha - \alpha' - (\beta + \beta')p \geq 0 \\
& \quad 0 \leq p \leq \bar{p}, 0 \leq y \leq \bar{y}.
\end{align*}
\]

We further expand the above problem to obtain the full optimization equivalent problem as:

\[
\begin{align*}
\max_{p, y, r} & \quad r \\
\text{s.t.} & \quad r \leq p(\alpha + \alpha' - (\beta - \beta')p) - h(y - \alpha - \alpha' + (\beta - \beta')p)^+ \\
& \quad - q(\alpha + \alpha' - (\beta - \beta')p - y)^+ - yc \\
& \quad r \leq p(\alpha - \alpha' - (\beta + \beta')p) - h(y - \alpha + \alpha' + (\beta + \beta')p)^+ \\
& \quad - q(\alpha - \alpha' - (\beta + \beta')p - y)^+ - yc \\
& \quad \alpha - \alpha' - (\beta + \beta')p \geq 0 \\
& \quad 0 \leq p \leq \bar{p}, 0 \leq y \leq \bar{y}.
\end{align*}
\]

We can replace each of the $(\cdot)^+$ component by two constraints (by which the coefficients are positive). Each of these constraints are of convex form and quadratic form. Therefore, we obtain a problem with a linear objective with a non-linear constrain; it is a convex non-linear program which can be solved by robust optimization solvers.

### 5.3 Robust Optimization with Budget of Uncertainty

In some cases with a large range of uncertain values, it may seem overly conservative to protect the solution against a worst-case scenario. In the mid-1990s, research conducted by address the issue of over-conservatism by El Ghaoui et al. (1998) and Ben-Tal and Nemirovski (1998) addressed the issue of overconservatism by introducing a constraint called “budget of uncertainty”. In such case, the uncertain parameters are restricted in order to remove the most unlikely outcomes from consideration and yields tractable mathematical programming problems. In this study, we followed this approach to employ a budget of uncertainty (deemed as $\Gamma$) which allows us to design a level of flexibility of choosing the trade-off between robustness and performance.
In this study, we formulate the budget of uncertainty as taken across parameters:

\[ |u| + |w| \leq \Gamma \]

where

\[ u = \frac{\hat{\alpha} - \alpha}{\alpha'}, \quad w = \frac{\hat{\beta} - \beta}{\beta'} \]

We have the following three cases:

1. If \( \Gamma = 0 \), all the uncertain parameters are equal to their nominal values; there is no protection against uncertainty.

2. If \( \Gamma = 2 \), this robust optimization problem is completely protected against uncertainty, which yields a very conservative solution.

3. If \( \Gamma \in (0, 2) \), decision-makers make a trade-off between the level of conservativeness and the performance.

We extended the work in Section 5.2, to consider a single perishable product problem, where retailer makes inventory and pricing decisions over a single period. Demand will be backordered and all leftover products will be charged holding cost. At the end of the period, the value of this product will obsolete, no salvage cost will be considered here. Therefore, the robust optimization with a budget of uncertainty is formulated as

\[
\max_{p,y} \min[f(p, y; \overline{D}), f(p, y; \underline{D})] \\
\text{s.t.} \\
f(p, y; \overline{D}) = p\overline{D} - h(y - \overline{D})^+ - q(\overline{D} - y)^+ - yc, \\
f(p, y; \underline{D}) = p\underline{D} - h(y - \underline{D})^+ - q(\underline{D} - y)^+ - yc, \\
\overline{D} = \max_{|u| + |w| \leq \Gamma} \left[ \alpha + u\alpha' - (\beta + w\beta')p \right], \\
\underline{D} = \min_{|u| + |w| \leq \Gamma} \left[ \alpha + u\alpha' - (\beta + w\beta')p \right], \\
\min_{|u| + |w| \leq \Gamma} \left[ \alpha + u\alpha' - (\beta + w\beta')p \right] \geq 0, \\
0 \leq y \leq \overline{y}, 0 \leq p \leq \overline{p}.
\]
The overall optimization problem can be equivalently formulated as follows:

\[
\begin{align*}
\max_{p, r, \bar{r}} \quad & r \\
\text{s.t.} \quad & r \leq pD - h(y - D) + q(D - y) + yc, \\
& r \leq p\bar{D} - h(y - \bar{D}) + q(\bar{D} - y) + yc, \\
& \bar{D} = \max_{u + |w| \leq 1} [\alpha + u\alpha' - (\beta + u\beta')p], \\
& D = \min_{u + |w| \leq 1} [\alpha + u\alpha' - (\beta + u\beta')p], \\
& \min_{|u| + |w| \leq 1} [\alpha + u\alpha' - (\beta + u\beta')p] \geq 0, \\
& 0 \leq y \leq \bar{y}, 0 \leq p \leq \bar{p},
\end{align*}
\]

where, \( \bar{D} \) and \( D \) can be obtained from the following scenarios:

Case (1):
If \( \alpha' \geq \beta'p \), then

\[
\bar{D} = \alpha - \beta p + \begin{cases} 
\alpha' + \beta'p, & \Gamma = 2 \\
\alpha' + (\Gamma - 1)\beta'p, & 1 \leq \Gamma < 2 \\
\Gamma\alpha', & \Gamma < 1
\end{cases}
\]

\[
D = \alpha - \beta p - \begin{cases} 
(\alpha' + \beta'p), & \Gamma = 2 \\
(\alpha' + (\Gamma - 1)\beta'p), & 1 \leq \Gamma < 2 \\
\Gamma\alpha', & \Gamma < 1
\end{cases}
\]

Case (2):
If \( \alpha' \leq \beta'p \), then

\[
\bar{D} = \alpha - \beta p + \begin{cases} 
\alpha' + \beta'p, & \Gamma = 2 \\
(\Gamma - 1)\alpha' + \beta'p, & 1 \leq \Gamma < 2 \\
\Gamma\beta'p, & \Gamma < 1
\end{cases}
\]

\[
D = \alpha - \beta p - \begin{cases} 
(\alpha' + \beta'p), & \Gamma = 2 \\
((\Gamma - 1)\alpha' + \beta'p), & 1 \leq \Gamma < 2 \\
\Gamma\beta'p, & \Gamma < 1
\end{cases}
\]

At this stage, we can write the deterministic equivalent problem by including the above 2 cases, and will solve the following two problems separately:
Chapter 5 A Robust Optimization Approach for Joint Pricing and Inventory Decisions for Two Substitutable and Perishable Products

P1:

\[
\begin{align*}
\max_{p, r_1} & \quad r_1 \\
\text{s.t.} & \quad r_1 \leq p(\overline{D} - h(y - D))^+ - q(\overline{D} - y)^+ - yc, \\
& \quad r_1 \leq pD - h(y - D)^+ - q(D - y)^+ - yc, \\
& \quad \overline{D} = \alpha - \beta p + \min(1, \Gamma)\alpha' + \min(1, (\Gamma - 1)^+)\beta' p, \\
& \quad D = \alpha - \beta p - \min(1, \Gamma)\alpha' - \min(1, (\Gamma - 1)^+)\beta' p, \\
& \quad \alpha - \beta p - \min(1, \Gamma)\alpha' - \min(1, (\Gamma - 1)^+)\beta' p \geq 0, \\
& \quad \alpha' \geq \beta' p, \\
& \quad 0 \leq y \leq \overline{y}, 0 \leq p \leq \overline{p}.
\end{align*}
\]

P2:

\[
\begin{align*}
\max_{p, r_2} & \quad r_2 \\
\text{s.t.} & \quad r_2 \leq p(\overline{D} - h(y - D))^+ - q(\overline{D} - y)^+ - yc, \\
& \quad r_2 \leq pD - h(y - D)^+ - q(D - y)^+ - yc, \\
& \quad \overline{D} = \alpha - \beta p + \min(1, \Gamma)\beta' p + \min(1, (\Gamma - 1)^+)\alpha', \\
& \quad D = \alpha - \beta p - \min(1, \Gamma)\beta' p - \min(1, (\Gamma - 1)^+)\alpha', \\
& \quad \alpha - \beta p - \min(1, \Gamma)\beta' p - \min(1, (\Gamma - 1)^+)\alpha' \geq 0, \\
& \quad \alpha' \leq \beta' p, \\
& \quad 0 \leq y \leq \overline{y}, 0 \leq p \leq \overline{p}.
\end{align*}
\]

Then the optimal total profits will be:

\[r = \max(r_1, r_2)\]

Now, the deterministic equivalent problem is a non-linear problem with a linear objective function and with convex quadratic constraints. It is easy to solve it by using non-linear solvers.

5.4 Robust Optimization for Substitutable Products

In this section, we consider two substitutable and perishable products problem denoted by \(i \in \{1, 2\}\), where the demand is dependent on both products’ prices. The demand
function is formulated as follows:

\[ D_i = \alpha_i - \beta_i p_i + l_j p_j, \forall i \neq j, i, j \in \{1, 2\}. \]

We firstly introduce a deterministic model where retailer makes pricing and inventory decisions for two substitutable and perishable products over a single period lifetime. Holding costs and backordering cost will be considered. At the end of the period, the value of these two products obsolete as we are considering perishable products problem. Therefore, this deterministic model is formulated as follows:

\[
\begin{align*}
\max_{p_i, y_i} \quad & \sum_i [p_i D_i - h(y_i - D_i)^+ - q_i (D_i - y_i)^+ - y_i c_i] \\
\text{s.t.} \quad & D_i = \alpha_i - \beta_i p_i + l_j p_j, \forall i \neq j, \\
& 0 \leq p_i \leq \overline{p}_i, \\
& 0 \leq y_i \leq \overline{y}_i, \\
& \beta_i \geq l_i, l_j, \forall i \neq j, \\
& D_i \geq 0.
\end{align*}
\]

In the robust optimization model, we aim to protect the system against unknown but bounded disturbances \( \alpha_i, \beta_i \) and \( l_i \):

\[ \tilde{\alpha}_i \in [\alpha_i - \alpha_i', \alpha_i + \alpha_i'], \]
\[ \tilde{\beta}_i \in [\beta_i - \beta_i', \beta_i + \beta_i'], \]
\[ \tilde{l}_i \in [l_i - l_i', l_i + l_i'], \]

where \( \alpha_i, \beta_i \) and \( l_i \) are the nominal function, \( \tilde{\alpha}_i, \tilde{\beta}_i \) and \( \tilde{l}_i \) are denoted to be the realization. We assume that the realization belongs in an interval centred around the nominal function with length \( \alpha_i', \beta_i' \) and \( l_i' \). To adjust the level of conservatism in protecting the solution against a worst-case scenario all the time, we introduce the “budget of uncertainty” denoted by \( \Gamma_i \). The constraints with “budget of uncertainty” can be written as follows:

\[ |u_i| + |w_i| + |v_j| \leq \Gamma_i, \]

where

\[ u_i = \frac{\tilde{\alpha}_i - \alpha_i}{\alpha_i'}, \quad w_i = \frac{\tilde{\beta}_i - \beta_i}{\beta_i'}, \quad v_j = \frac{\tilde{l}_j - l_j}{l_j'}. \]
We formulate a general robust optimization model to seek a solution that is feasible for any realization of data constrained by the “budget of uncertainty”.

\[
\begin{align*}
\max_{p_i, y_i} & \quad \sum_i \left[ p_i \tilde{D}_i - h(y_i - \tilde{D}_i)^+ - q_i(\tilde{D}_i - y_i)^+ - y_i c_i \right] \\
\text{s.t.} & \quad \tilde{D}_i = \tilde{\alpha}_i - \tilde{\beta}_i p_i + \tilde{l}_j p_j, \forall i \neq j, \\
& \quad 0 \leq p_i \leq \overline{p}_i, \\
& \quad 0 \leq y_i \leq \overline{y}_i, \\
& \quad \tilde{\beta}_i \geq \tilde{l}_i, \tilde{l}_j, \forall i \neq j, \\
& \quad \tilde{D}_i \geq 0 
\end{align*}
\]

This problem corresponds to a situation in which decision makers determine the prices \((p_1, p_2)\) and order quantities \((y_1, y_2)\) at the beginning by maximizing the total profits. In a robust approach, we aim to determine the feasible decision variables to ensure that the total profit is maximized even for the worst realizations of uncertain parameters. As a result of the uncertain parameters involved in this model, it becomes difficult to solve. If all the parameters are deterministic, the problem will be a piecewise concave objective function that we want to maximize, and which has linear constraints, with 4 decision variables. In order to obtain the deterministic equivalent problem, we solve the function as follows:

\[
\begin{align*}
\min_{\tilde{\alpha}_i, \tilde{\beta}_i, l_i \in F} & \quad \sum_i \left[ p_i \tilde{D}_i - h(y_i - \tilde{D}_i)^+ - q_i(\tilde{D}_i - y_i)^+ - y_i c_i \right] \\
\text{s.t.} & \quad \tilde{D}_i = \tilde{\alpha}_i - \tilde{\beta}_i p_i + \tilde{l}_j p_j, \forall i \neq j, \\
& \quad \tilde{\beta}_i \geq \tilde{l}_i, \tilde{l}_j, \forall i \neq j, \\
\end{align*}
\]

The uncertainty, \(\alpha_i, \beta_i, l_i \forall i\) here always acts to minimize the objective function; for this reason, the problem solved by decision maker corresponds to a worst-case scenario: a
maximisation of a minimum problem as follows:

$$\max_{p_i, y_i} \min_{\alpha_i, \beta_i, l_i \in F} \sum_i \left[ p_i \tilde{D}_i - h(y_i - \tilde{D}_i)^+ - q_i(\tilde{D}_i - y_i)^+ - y_i c_i \right]$$  \hspace{1cm} (5.7)

s.t. \hspace{1cm} \tilde{D}_i = \alpha_i - \beta_i p_i + \tilde{l}_i p_j, i \neq j, \hspace{1cm} (5.8)

\hspace{1cm} 0 \leq p_i \leq \overline{p}, \hspace{1cm} (5.9)

\hspace{1cm} 0 \leq y_i \leq \overline{y}, \hspace{1cm} (5.10)

\hspace{1cm} \tilde{\beta}_i \geq \tilde{l}_i, \hspace{1cm} (5.11)

\hspace{1cm} \tilde{\beta}_i \geq \tilde{l}_j, \forall i \neq j, \hspace{1cm} (5.12)

\hspace{1cm} \tilde{D}_i \geq 0. \hspace{1cm} (5.13)

The above max-min problem is very difficult to solve. We start with the inner problem to obtain a deterministic equivalent problem.

**Proposition 5.2.** Given \( p_i \) and \( y_i \), the inner problem (5.7) equals to the following linear optimization problem:

$$\min \left[ f_1(D_1), f_1(D_1) \right] + \min \left[ f_2(D_2), f_2(D_2) \right]$$

where \( f_i(D_i) = p_i \ast D_i - h(y_i - D_i)^+ - q_i(D_i - y_i)^+ - y_i c_i \).

**Proof.** The inner problem (5.7) is:

$$\min_{\alpha_i, \beta_i, l_i \in F} \sum_i \left[ p_i \tilde{D}_i - h(y_i - \tilde{D}_i)^+ - q_i(\tilde{D}_i - y_i)^+ - y_i c_i \right]$$

In the budget of uncertainty formulation, we know that \( \tilde{\alpha}_1, \tilde{\beta}_1, \tilde{l}_2 \) are independent from \( \tilde{\alpha}_2, \tilde{\beta}_2, \tilde{l}_1 \), then for each given \( p_i \) and \( y_i \), it is clear that \( D_1 \) and \( D_2 \) are separable; for this reason, the above problem can be re-written as:

$$\min_{\alpha_1, \beta_1, l_2 \in F_1} f_1(\tilde{D}_1) + \min_{\alpha_2, \beta_2, l_2 \in F_2} f_2(\tilde{D}_2)$$

where \( f_i(\tilde{D}_i) = p_i \ast \tilde{D}_i - h(y_i - \tilde{D}_i)^+ - q_i(\tilde{D}_i - y_i)^+ - y_i c_i \). It is easy to prove that \( f_i \) is a piece-wise linear function of \( D_i \) as illustrated:

It is now clear that

$$\min_{\alpha_i, \beta_i, l_i \in F_i} f_i(\tilde{D}_i) = \min[f_1(D_1), f_1(D_1)].$$

Hence, the inner problem (6) now can be written as:

$$\min[f_1(D_1), f_1(D_1)] + \min[f_2(D_2), f_2(D_2)].$$

We now complete the proof of proposition 2.
Chapter 5 A Robust Optimization Approach for Joint Pricing and Inventory Decisions
for Two Substitutable and Perishable Products

Figure 5.2: The graphical description of the function \( f_i(D_i) \)

Under this approach, we will determine the robust counterpart of \( \overline{D_i} \) and \( \underline{D_i} \). Secondly, in order to solve the problem, we have to determine the realization of uncertain parameters \( \tilde{\beta}_i, \tilde{l}_i, \tilde{l}_j, \forall i \neq j \) that is least favourable to satisfy the constraint. Constraints (5.11)-(5.13) consist of expressions involving uncertain parameters subject to a lower bound. The robust counterpart is obtained through the minimum of the expression satisfying the lower bound. Hence, with proposition 2, the overall optimization problem can be re-formulated as follows:

\[
\max_{\bar{p}_i, \overline{\beta_i}} \min[f_1(\overline{D_i}), f_1(\overline{D_i})] + \min[f_2(\overline{D_i}), f_2(\overline{D_i})]
\]

\[
f_i(D_i) = p_i \times D_i - h(y_i - D_i)^+ - q_i(D_i - y_i)^+ - y_i c_i,
\]

\[
\overline{D_i} = \max_{|u_i| + |w_i| + |v_j| \leq \Gamma_i} [\alpha_i + u_i \alpha_i' - (\beta_i + w_i \beta_i') p_i + (l_j + v_j l_j') p_j],
\]

\[
\underline{D_i} = \min_{|u_i| + |w_i| + |v_j| \leq \Gamma_i} [\alpha_i + u_i \alpha_i' - (\beta_i + w_i \beta_i') p_i + (l_j + v_j l_j') p_j],
\]

\[
\min_{|u_i| + |v_j| \leq \Gamma_i} [\beta_i + u_i \beta_i' - (l_i + v_i l_i')] \geq 0,
\]

\[
\min_{|u_i| \leq \min(\Gamma_i, 1), |v_i| \leq \min(\Gamma_i, 1)} [\beta_i + u_i \beta_i' - (l_i + v_i l_i')] \geq 0,
\]

\[
0 \leq p_i \leq \overline{p}_i,
\]

\[
0 \leq y_i \leq \overline{y}_i.
\]

Solving the above problem directly is very complicated. Therefore, we now analyse constraints (5.16)-(5.20) to simplify the overall optimization problem. Constraint (5.16)
can be solved by maximizing a linear function with linear constraints:

\[(5.16) \Leftrightarrow \alpha_i - \beta_i p_i + l_j p_j + \max_{|u_i| + |w_i| + |v_j| \leq \Gamma_i} [u_i \alpha_i' - w_i \beta_i' p_i + v_j l_j' p_j] \]

\[\Leftrightarrow \alpha_i - \beta_i p_i + l_j p_j + \min(\Gamma_i, 1) \max(\alpha_i', \beta_i' p_i, l_j' p_j) + \min((\Gamma_i - 2)^+, 1) \min(\alpha_i', \beta_i' p_i, l_j' p_j) \]

\[+ \min((\Gamma_i - 1)^+, 1) \min\{\max(\alpha_i', \beta_i' p_i), \max(\alpha_i', l_j' p_j), \max(\beta_i' p_i, l_j' p_j)\} \]

Similarly, the deterministic counterpart of (16) is:

\[(5.17) \Leftrightarrow \alpha_i - \beta_i p_i + l_j p_j - \min_{|u_i| + |w_i| + |v_j| \leq \Gamma_i} [u_i \alpha_i' - w_i \beta_i' p_i + v_j l_j' p_j] \]

\[\Leftrightarrow \alpha_i - \beta_i p_i + l_j p_j - \min(\Gamma_i, 1) \max(\alpha_i', \beta_i' p_i, l_j' p_j) + \min((\Gamma_i - 2)^+, 1) \min(\alpha_i', \beta_i' p_i, l_j' p_j) \]

\[+ \min((\Gamma_i - 1)^+, 1) \min\{\max(\alpha_i', \beta_i' p_i), \max(\alpha_i', l_j' p_j), \max(\beta_i' p_i, l_j' p_j)\} \]

Constraint (5.18) is:

\[(5.18) \Leftrightarrow \alpha_i - \beta_i p_i + l_j p_j - \min_{|u_i| + |w_i| + |v_j| \leq \Gamma_i} [u_i \alpha_i' - w_i \beta_i' p_i + v_j l_j' p_j] \geq 0 \]

\[\Leftrightarrow - \min(\Gamma_i, 1) \max(\alpha_i', \beta_i' p_i, l_j' p_j) - \min((\Gamma_i - 2)^+, 1) \min(\alpha_i', \beta_i' p_i, l_j' p_j) \]

\[+ \alpha_i - \beta_i p_i + l_j p_j \geq 0 \]

Constraints (5.19) and (5.20) are respectively

\[(5.19) \Leftrightarrow \min_{|u_i| \leq \min(\Gamma_i, 1), |v_j| \leq \min(\Gamma_j, 1)} [\beta_i + u_i \beta_i' - (l_i + v_j l_i')] \geq 0 \]

\[\Leftrightarrow \beta_i - l_i - \min(\Gamma_i, 1) \beta_i' - \min(\Gamma_j, 1) l_i' \geq 0 \]

\[(5.20) \Leftrightarrow \min_{|u_i| + |v_j| \leq \Gamma_j, |u_i|, |v_j| \leq 1} [\beta_i + u_i \beta_i' - (l_j + v_j l_j')] \geq 0 \]

\[\Leftrightarrow \beta_i - l_j - \min(\Gamma_i, 1) \max(\beta_i', l_j') - \min((\Gamma_j - 1)^+, 1) \min(\beta_i', l_j') \geq 0 \]
Now, we are ready to write the deterministic counterpart of this robust model.

$$\max_{p_i, y_i, r} r$$

s.t.  
$$r \leq f_1(D_1^r) + f_2(D_2^r),$$
$$r \leq f_1(D_1^e) + f_2(D_2^e),$$
$$r \leq f_1(D_1^l) + f_2(D_2^l),$$
$$r \leq f_1(D_1^s) + f_2(D_2^s),$$
$$f_i(D_i) = p_i \ast D_i - h(y_i - D_i)^+ - q_i(D_i - y_i)^+ - y_i c_i,$$
$$\text{max} = \alpha_i - \beta_i p_i + l_j p_j$$
$$\text{min} = \beta_i p_i + l_j p_j$$
$$\text{max} = \beta_i p_i + l_j p_j$$
$$\text{min} = \beta_i p_i + l_j p_j$$
$$\text{max} = \beta_i p_i + l_j p_j$$
$$\text{min} = \beta_i p_i + l_j p_j$$

We obtain a deterministic equivalent problem which is a linear problem with quadratic constraints. It can also be separated to a few sub-problems. In this deterministic equivalent problem, we deal with a single-layered maximization problem (linear objective and convex quadratic constraints) and we no longer have the max-min problem which is very difficult to solve.

5.5 Numerical Results

In the numerical study, we consider a case study from the European branch of a solar panel factory which orders and sells solar panels over a single period. The decision makers have to make plans on order quantity and pricing decisions at the beginning of that period. The solar modules will shipped to warehouse of this European branch in Netherlands. In the solar panel market, most companies are struggling with cash flows; the assume a penalty for leftover products in order to prevent over ordering to
have a better cash flow. The salvage cost is lower than the holding cost. In this case, we can assume the holding cost is the difference between penalty and salvage cost. It is equivalent to the perishable products problem where leftover products have no value after a certain time (obsolete). Therefore, we consider the solar panel problem here as a perishable product.

Meanwhile, the market changes dramatically and there are continuous changes in customer interests and needs, and improvements in the technology development. Therefore there is an urgent need to update the demand parameters with the latest data. Furthermore, as a result of the unstable market, the parameter in the demand is ideally within a range rather than a certain nominal value. A robust optimization model is developed to deal with the robustness against uncertainty. Our goal is to solve the robust optimization model and understand the relationship among the objective value, decision variables and robust parameters.

We consider two numerical examples to illustrate our results. In the first part of the numerical results, we solve the single product robust model with budget of uncertainty. We consider the case in which the aforementioned European branch orders and sells a version of solar panels which has no other versions in this company coming close to it in terms of efficiency. Therefore, no price substitution is considered in this case. In the second part, we aim to analyse the case that, currently, the European branch sells a special version manufactured both in China and Turkey, with the prices set to be different due to the differences in the quality and costs. Therefore, these two products are considered to be substitutable. We aim to find the optimal order and pricing decisions for both products.

5.5.1 Single Product Robust Optimization Model

We collected real data from “QSAR” which are only this European branch can order from the factory in Shanghai and it is the most advanced solar panel in terms of efficiency on energy conversion. The data is from sales in the 4th quarter of 2014. We apply our models to help decision makers in the first quarter of 2016. No pricing substitution with other versions is considered here. The data we collected are containing: countries, name of sales, volume (Kilowatt), contract value (Pounds), name of client, version type.

We analyse the data of volume and contract value by linear regression models in SAS to obtain the following input parameters for demand formulation:

<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>$\alpha'$</td>
<td>$\beta$</td>
</tr>
<tr>
<td>38751</td>
<td>3000</td>
<td>56</td>
</tr>
</tbody>
</table>
In the model assumption, $a'$ and $b'$, which represent the half-length of the allowed range for parameters $a$ and $b$, must satisfy $0 \leq a' \leq \frac{1}{2}a$, $0 \leq b' \leq \frac{1}{2}b$. The rest of the parameters are assumed by the EU branch as follows:

**Table 5.2: Given input parameters**

<table>
<thead>
<tr>
<th>$h$</th>
<th>$q$</th>
<th>$c$</th>
<th>$\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>600</td>
<td>300</td>
<td>1.5</td>
</tr>
</tbody>
</table>

Here, we assume the total order cost is a linear function of order quantity, similarly to the inventory control problem. With the analysis of our model, we are able to solve using several solvers in AMPL. All solvers achieve the same results in less than 20 seconds.

<table>
<thead>
<tr>
<th></th>
<th>LOQO</th>
<th>filter</th>
<th>IPOPT</th>
<th>KNOTRO</th>
<th>MINOS</th>
<th>SNOPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iterations</td>
<td>90</td>
<td>40</td>
<td>24</td>
<td>15</td>
<td>39</td>
<td>40</td>
</tr>
<tr>
<td>Evaluations</td>
<td>206</td>
<td>N/A</td>
<td>176</td>
<td>24</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>Optimal</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
<td>local</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>Total Profit</td>
<td>139837</td>
<td>139837</td>
<td>139837</td>
<td>139837</td>
<td>139837</td>
<td>139837</td>
</tr>
</tbody>
</table>

**Table 5.3: Summary of results on six solvers**

In these numerical results, we let the budget of uncertainty represent the robustness of our model. The higher the budget of uncertainty is, the higher level of uncertainty this model has. Our goal is to understand the relationship between the optimal values and the budget of uncertainty (robustness) as well as the length of the allowed range of parameters $a$ and $b$.

Figures 5.3, 5.4 and 5.5 represent the trade-off between robustness and optimal profits and decisions. In Figure 5.3, as the budget of uncertainty increases, the optimal total profits decrease; this illustrates the trade-off between optimality (high optimal objective value) and robustness (high level of uncertainty). In the upper and lower bounds of demand function, we have:

$$\min(1, \Gamma)\max(a', \beta'p) + \min(1, (\Gamma - 1)^+\min(a', \beta'p)).$$

While $\Gamma = 0$, this is a deterministic model and while $\Gamma = 2$, this is a very conservative case which yields the lowest objective value. Notice that the slope of the line changes at the point $\Gamma = 1$, the reason for this being that while $\Gamma < 1$, in the demand function, only the uncertain parameter which is the minimum of $(a', b')$ reflects protection against uncertainty up to the degree of $\Gamma$; the other uncertain parameter is equal to the nominal value. Whereas when $1 \leq \Gamma < 2$, the minimum of $(a', b')$ is completely incorporated in the demand function, the other uncertain parameter is partly incorporated.
Figures 5.4 and 5.5 illustrate the impact of increasing the budget of uncertainty on optimal decisions. In Figure 5.4, we find that while $\Gamma < 1$, the increasing budget of uncertainty comes with decreasing optimal prices, whereas while $\Gamma > 1$, the optimal price increases. The reason for this is also in the different scenarios in the demand function; the optimization models change afterwards, and therefore, the impact on the optimal results is also observed in a different way.

Figure 5.5 illustrates the trade-off between robustness and optimal order quantity. As the level of conservativeness increases, the optimal order quantity decreases.
Figure 5.4: Trade-off between robustness and optimal price
Figure 5.5: Trade-off between robustness and optimal order quantity
Figure 5.6, 5.7 and 5.8 illustrate the impact of the half-length for the range of demand loss rate on optimal results. In the model formulation, we have different scenarios for the function
\[
\min(1, \Gamma) \max(\alpha', \beta'p) + \min(1, (\Gamma - 1)^+) \min(\alpha', \beta'p),
\]
therefore, we analyse the impact under two levels of uncertainty while \( \Gamma = 0.5 \) and \( \Gamma = 1.5 \), where we obtain two different functions for the deterministic equivalent optimization problem. In Figure 5.6, for both scenarios, the slope of the two lines changes while \( \beta' \approx 6.2 \), due to the fact that the result of \( \min(\alpha', \beta'p) \) changes at that point. We then have a different deterministic equivalent optimization problem. Furthermore, the differences between the two scenarios is that while \( \Gamma = 0.5 \), in the lower and upper bounds of the demand function in the deterministic equivalent problem,
\[
\min(1, \Gamma) \max(\alpha', \beta'p) + \min(1, (\Gamma - 1)^+) \min(\alpha', \beta'p) = 0.5 * \alpha'.
\]
Therefore, the increase on \( \beta' \) does not affect the upper and lower bounds of the demand function, and the objective values remain unchanged. In other cases, the increase of \( \beta' \) expands the range of the uncertain parameters, where in robust optimization, we are maximizing the worst case scenarios. For this reason, the increase of \( \beta' \) always leads to a decrease in the optimal total profits.
Figure 5.6: The impact of the half-length for the range of demand lost rate ($\beta'$) on optimal total profits under two levels of uncertainty ($\Gamma = 0.5$ and $\Gamma = 1.5$).
In Figures 5.7 and 5.8, we also find that, other than the case where $b' \leq 6.5$ and $\Gamma = 0.5$, the increase of $\beta'$ always results in the decrease of the optimal price and the optimal order quantities. Notice that there is a big change at $\beta' \approx 6.2$, in which case the result of minimum and maximum $(\alpha', \beta'p)$ changes cause a different result in the upper and lower bounds of demand formulation for the deterministic optimization problem.

Figure 5.7: The impact of the half-length for the range of demand loss rate ($\beta'$) on optimal price under two levels of uncertainty ($\Gamma = 0.5$ and $\Gamma = 1.5$)
Figures 5.9, 5.10 and 5.11 illustrate the impact of varying $\alpha'$ on optimal results under two levels of uncertainty. While $\Gamma = 0.5$, by varying $\alpha'$ up to 3000, $\alpha' < \beta'p$ always stands, and the function in demand becomes:

$$\min(1, \Gamma)\max(\alpha', \beta'p) + \min(1, (\Gamma - 1)^+)\min(\alpha', \beta'p) = 0.5 * \beta'p.$$  

Therefore, the optimal results will stay the same with varying $\alpha'$. In the case of $\Gamma = 1.5$, we observe that an increasing $\alpha'$ results in the decrease of optimal total profits and optimal order quantity, whereas the optimal price increase. By increasing $\alpha'$, for each given order quantity and price, the upper and lower bounds of demand are expanded, therefore the optimal total profit which is a piece-wise concave function of demand are decreased.
Figure 5.9: The impact of the half-length for the range of the constant value in demand ($\alpha'$) on optimal total profits under two levels of uncertainty ($\Gamma = 0.5$ and $\Gamma = 1$).
Figure 5.10: The impact of the half-length for the range of the constant value in demand ($\alpha'$) on optimal price under two levels of uncertainty ($\Gamma = 0.5$ and $\Gamma = 1$)
In this part of the numerical study, we have demonstrated the results of the single product robust optimization model and investigated the impact of varying uncertainty parameters on optimal results. In the next part of the numerical study, we then illustrate the optimal results of the two substitutable products robust optimization model, where price-substitution is considered.

5.5.2 Two Substitutable Products Robust Optimization Model

To demonstrate the results of the two substitutable products robust optimization model, we analyse a case in which the European branch of the solar panel manufacturer introduced at the beginning of this chapter orders and prices an M156 version of its panels both in a Chinese factory and a Turkish factory. Due to the differences in cost and quality, the European branch price these two products differently. As the prices in the solar panel industry are highly impacted by the dynamics of the market, every month
they make decisions using the latest data they have. At the beginning of each month, the decision makers have to make plans for the production and decide the prices for both products. The data we collected consists of: countries, name of sales, volume, contract value, name of client, version type, and the unit price of the substitutable version. We run linear regression models of the following variables: contract value of product 1, volume of product 1, contract value of product 2, volume of product 2. We then obtained the following input parameters:

<table>
<thead>
<tr>
<th>Table 5.4: Input parameters for demand</th>
</tr>
</thead>
<tbody>
<tr>
<td>α</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>1 CHN</td>
</tr>
<tr>
<td>2 TR</td>
</tr>
</tbody>
</table>

The levels of uncertainty $\Gamma$ are assumed to be 1.5, and we will conduct the sensitivity study of it later in this section. Other parameters we input are:

<table>
<thead>
<tr>
<th>Table 5.5: Input parameters of costs</th>
</tr>
</thead>
<tbody>
<tr>
<td>c</td>
</tr>
<tr>
<td>---</td>
</tr>
<tr>
<td>1 CHN</td>
</tr>
<tr>
<td>2 TR</td>
</tr>
</tbody>
</table>

The optimal results are obtained by the solver “LOQ” in less than 20 seconds. We then experiment with varying $\Gamma_1$ to illustrate the trade-off between robustness and performance in the next part.

Figure 5.12 illustrates the trade-off between the robustness of one product and the optimal total profits. We observe that as the budget increases, the model becomes more robust, and the optimal total profits decrease. Similarly, like the single product robust model, while $\Gamma_1 > 1$ we observe that the slope of the lines changes as in the upper and lower bounds of the demand function. The budget impacts the results of the following function:

$$\min(\Gamma_1, 1)\max(\alpha_i', \beta_i'p_i, l'jp_j) + \min((\Gamma_1 - 2)^+, 1)\min(\alpha_i', \beta_i'p_i, l'jp_j)$$

$$+ \min((\Gamma_1 - 1)^+, 1)\min[\max(\alpha_i', \beta_i'p_i), \max(\alpha_i', l'jp_j), \max(\beta_i'p_i, l'jp_j)]$$
It is not very clear that there is a slope change at the point where $\Gamma_1 = 2$. At this point due to the lower substitution rate, by varying $\Gamma_1$, the extra part of the function $\min((\Gamma_1 - 2)^+, 1)l^2_2p_2$ does not affect the overall objective results much, which can be evidenced from Figures 5.13 and 5.14.
Figures 5.13 and 5.14 demonstrate the impact of varying budget $\Gamma_1$ on optimal prices and order qualities. The optimal decisions regarding product 2 only have slight changes which will result in few differences on total profits. The optimal price and order quantity of product 1 are observed to vary greatly, which results in a huge decrease in total profits in Figure 5.12. After $\Gamma_1 = 2.6$, the lower-bound of demand of product 1 becomes 0. All the profits are from selling product 2.
Figure 5.14: The impact of varying $\Gamma_1$ on optimal order quantities
Figure 5.15 demonstrates the impact of varying $\beta_1'$ on total profits under three levels of uncertainty.

![Figure 5.15: The impact of the half-length for the range of demand loss rate ($\beta_1'$) on total profits under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$)](image)

Similar to the single product robust optimization model, by varying demand loss rate $\beta_1'$, the optimal total profits are observed to decrease except in the cases of $\Gamma_1 = 0.5$ and $\Gamma_1 = 1.5$) while $\beta_1 < 1$. The reason for this is that under that circumstance, $\beta_1' p_1 < \alpha_1' < \beta_1' p_1$ are not taken account in the lower and upper bounds of demand, and varying $\beta_1'$ does not result in any changes in the optimal results. This is also evidenced by Figures 5.16-5.19. For all other cases, by varying $\beta_1'$, the ranges of the uncertain parameters are expanded, and the total profits are decreased as robust optimization is maximizing the worst-case scenarios.

Figures 5.16-5.19 represent the impact of varying $\beta_1$ on optimal decisions on both products. We observe that by varying $\beta_1$, there are no significant differences on the optimal decisions on product 2. For product 1, by increasing $\beta_1$, for all three cases, optimal prices decrease, and optimal order quantities are observed to be increasing and decreasing in different ranges of $\beta_1$. In the case of $\Gamma_1 = 2.5$, while $\beta_1 > 3.3$, the lower bound of demand of product 1 is 0, and all the profits are from selling product 2.
Figure 5.16: The impact of the half-length for the range of demand loss rate ($\beta_1'$) on optimal prices of product 1 under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$)
Figure 5.17: The impact of the half-length for the range of demand loss rate ($\beta'_1$) on optimal prices of product 2 under three levels of uncertainty ($\Gamma_1 = 0.5, \Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$).
Figure 5.18: The impact of the half-length for the range of demand loss rate ($\beta_1'$) on optimal order quantities of product 1 under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$)
Figure 5.19: The impact of the half-length for the range of demand loss rate ($\beta'_1$) on optimal order quantities of product “2” under three levels of uncertainty ($\Gamma_1 = 0.5, \Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$)
Figure 5.20 demonstrates the impact of $\alpha'_1$ on optimal total profits under three levels of uncertainty. In the cases of $\Gamma = 0.5$ and $\Gamma = 1.5, \alpha'_1 \leq 500$,

$$
\min(\Gamma_i, 1)\max(\alpha'_i, \beta'_i p_i, \ell'_j p_j) + \min((\Gamma_i - 2)^+, 1)\min(\alpha'_i, \beta'_i p_i, \ell'_j p_j)
+ \min((\Gamma_i - 1)^+, 1)\min[\max(\alpha'_i, \beta'_i p_i), \max(\alpha'_i, \ell'_j p_j), \max(\beta'_i p_i, \ell'_j p_j)]
- \min(\Gamma_i, 1)\beta'_i p_i + \min((\Gamma_i - 1)^+, 1)\ell'_j p_j,
$$

where $\alpha'_1$ is not included in the upper and lower bounds of demand function, therefore, varying $\alpha'_1$ doesn’t change the optimal objective value. In other cases, increased $\alpha'_1$ expands the range of uncertainty parameter $\alpha_1$. In robust optimization, while we are maximizing the worst-case scenario, ultimately the increase of $\alpha'_1$ results in the decrease of optimal total profits.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure520.png}
\caption{The impact of $\alpha'_1$ on optimal total profits under three levels of uncertainty ($\Gamma_1 = 0.5, \Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$)}
\end{figure}
Figure 5.21 graphically describes the impact of $l'_2$ on optimal total profits under three levels of uncertainty. In the case where $\Gamma_1 = 0.5$ and $\Gamma_1 = 1.5, l'_2 \leq 1.1$, we observe there are no changes by varying $l'_2$, the reason for this being that in these cases, in the following function:

$$\min(\Gamma_i, 1)\max(\alpha'_i, \beta'_i p_i, l'_j p_j) + \min((\Gamma_i - 2)^+, 1)\min(\alpha'_i, \beta'_i p_i, l'_j p_j)$$

$$+ \min((\Gamma_i - 1)^+, 1)\min[\max(\alpha'_i, \beta'_i p_i), \max(\alpha'_i, l'_j p_j), \max(\beta'_i p_i, l'_j p_j)]$$

$$= \min(\Gamma_i, 1)\beta'_i p_i + \min((\Gamma_i - 1)^+, 1)\alpha'_i,$$

Varying $l'_2$ doesn’t impact the upper and lower bounds of demand, and therefore, the overall results are not impacted.

This is also evidenced by Figure 5.22-5.25 where the optimal solutions are not affected in these cases. In all other cases, we observe that the increase of demand substitution
rate $l'_2$ results in the decrease of the optimal total profits.

Figure 5.22: The impact of half-length for the range of demand substitution rate $l'_2$ on optimal prices of product “1” under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$)
Figure 5.23: The impact of half-length for the range of demand substitution rate $l_2'$ on optimal prices of product “2” under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$)
Figure 5.24: The impact of half-length for the range of demand substitution rate $l_2'$ on optimal order quantities of product “1” under three levels of uncertainty ($\Gamma_1 = 0.5$, $\Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$)
Figures 5.22-5.25 demonstrate the effect of increasing $l_2'$ on optimal prices, and optimal order quantities. While $\Gamma_1 = 0.5$ and $\Gamma_1 = 1.5, l_2' \leq 1.1$, the $l_2'$ is not included in the function, the upper and lower bounds of demand, and there are no changes on these optimal solutions. In the case of $l_2' \geq 1.1, \Gamma_1 = 2.5$, there are some unusual movements in the optimal prices and optimal order quantities of product “1”. We notice that the lower-bound of demand “1” is 0 as bounded by the constraints. Other than that, it is clear that by increasing $l_2'$, the optimal prices of product “2” decrease in all cases, whereas there are no significant differences in product “1”. We also find an increase of $l_2'$ leads to the increase of optimal order quantities of product “2” and the decrease of optimal order quantities of product “1”.

Figure 5.25: The impact of half-length for the range of demand substitution rate $l_2'$ on optimal order quantities of product “2” under three levels of uncertainty ($\Gamma_1 = 0.5, \Gamma_1 = 1.5$ and $\Gamma_1 = 2.5$)
5.6 Conclusion

In this chapter, we study an extended news­vendor case in which the demand changes dramatically with the market, and retailers make optimal decisions with the latest data. We formulate three single period robust models to incorporate uncertainty in a pricing and inventory control problem. The first model is the very conservative model where the overall robust optimization problem is completely protected against uncertainty. In the second model, we introduce the “budget of uncertainty” to consider a less conservative case in which decision makers can make a trade-off between the level of conservativeness and performance. In the third model, we consider two-products model with price-substitution between them. The main contributions of this chapter are the following:

1. We formulate three models for pricing and inventory control of perishable products over a single period.

2. We address uncertainty in demand via robust optimization. To the best of our knowledge, this is the first work that uses ideas of robust optimization in the context of joint pricing and inventory control of perishable products considering price-substitution.

3. We analyse our models and present a deterministic equivalent problem which can solve this robust optimization problem via several non-linear solvers.

4. We illustrate our results through some numerical examples. We demonstrate that these robust optimization models allow us to manage the trade-off between performance and conservativeness. We also conduct a sensitivity study of some parameters to investigate the impact on overall optimal results.
Chapter 6

Conclusion

This thesis investigates a joint pricing and inventory control problem for substitutable perishable products under demand uncertainty. We consider stochastic programming for addressing uncertain demand in Chapters 3 and 4, and robust optimization for modelling uncertain demand in Chapter 5. In the final chapter, we summarise the main findings of each chapter and discuss the implications of our research for the theory and practice of operational research. In addition, we present a discussion of the limitations of our models, and suggest potential direction for future research.

6.1 Introduction

In this thesis, we study the joint pricing and inventory control problem for substitutable perishable products under stochastic programming and robust optimization. Chapters 3 and 4 employ stochastic programming to address demand uncertainty, whereas Chapter 5 consider robust optimization. The main focus for this research is perishability and substitution. The perishable products we consider become obsolete in value after a given time, where the physical status remains the same. With substitution, we consider the retailer orders and sells two similar products where they can improve their profits subject to the rate of substitution.

In the stochastic programming of Chapters 3 and 4, we consider pricing and inventory control decisions jointly for substitutable perishable products. While the setting in this work shares some of the elements in the existing work of Zhu and Thonemann (2009), Ye (2008) and Rujing (2007), our work deviates from these existing studies in the following ways:

- Our work differs from that of Zhu and Thonemann (2009) and Ye (2008), as they study competitive non-perishable products, while we consider competition among perishable products.
• Our work also differs from that of Rujing (2007), as she studies a single perishable product problem, while we consider two competitive perishable products which can substitute for each other.

To the best of our knowledge, our works in Chapters 3 and 4 are the only two works in the literature which study the joint pricing and inventory control problem across similar perishable products by considering holding costs and backorder costs. In the robust optimization of Chapter 5, we share some of the elements of the existing work of Adida and Perakis (2006), Adida and Perakis (2010b) and Adida and Perakis (2010a), expanding this research in order to consider price-substitution. To the best of our knowledge, ours is the first work that uses ideas of robust optimization in the context of joint pricing and inventory decisions of perishable products by considering price-substitution.

In Chapter 3, we propose a stochastic dynamic programming model for a two-period two-substitutable perishable products problem. The decision of the prices in the second period and the order quantities are made by managers with the objective of maximizing the total profit. We analyse the properties of the decision variables, and find a small number of common mistakes in similar previous studies. Having analysed these properties of decision variables, we develop an efficient search algorithm which finds the global optimum faster than the conventional search algorithm for inventory management. The data we use for our analysis comes from “Miss Sixty”, which is a fast fashion brand. Firstly, we find that our optimization model can improve the retailer’s profit significantly by 17%. Secondly, we graphically describe the benefits of jointly making pricing and inventory decisions. Thirdly, we describe graphically the benefits of managing substitutable products together, which can achieve a profit increase of 3% compared to managing them independently. Finally, we investigate the impacts of parameters in demand function on optimal results.

In Chapter 4, we expand the joint pricing and inventory control problem from the previous chapter. This extension has strong practical implications, in that most perishable products are often offered through multiple stages of discounts. The order quantity at the beginning of the first period and all the retail prices after that period are the decision variables in this model. After analysing the properties of the optimal solutions, we develop an efficient search algorithm to find the global optimum. Again we use data from fast fashion brand “Miss Sixty”’s winter collection. We demonstrate that with the algorithm we develop we are able to obtain global optimum within just one minute. We then show that:

1. By jointly making pricing and inventory decisions, “Miss Sixty” can significantly improve profits.
2. By managing substitutable products together, they can achieve a 1.8% increase compared with managing them independently.

3. The total profit decreases if there are fewer discount periods.

In Chapter 5, we consider one period robust optimization problems. With limited data collected and dramatically changing customer interests, retailers have to make optimal decisions with the latest data available. Furthermore, it is noted that it is difficult to assume the distribution in the stochastic demand function. We then develop three single-period robust optimization models incorporating demand uncertainty. The first model is a very conservative one in which the optimal solution is completely protected against uncertainty. In the second model, we introduce the “budget of uncertainty”, by which decision makers can make a trade-off between conservativeness and performance. Those first two models investigates a single product problem, whereas in the third model we study a two substitutable perishable products problem where price-substitution is considered in the demand function. To the best of our knowledge, this is the first research which uses the ideas of robust optimization for joint pricing and inventory control for a substitutable perishable products problem. We also analyse our models and show how to reformulate these into deterministic equivalent problems which can then be solved through the use of number of non-linear solvers. In the numerical results, we demonstrate that these robust optimization models allow the trade-off between robustness and performance to be managed. Finally, we conduct a sensitivity study of some parameters in the demand function to investigate the impact they have on overall optimal results.

The importance of our research both to academic research and real world practice can be summarised as follows:

- We fill the gap in the academic literature by considering joint pricing and inventory decisions for substitutable and perishable products.
- We build both stochastic dynamic programming models and robust optimization models, and address the complexity of the model and provide solution algorithms.
- We collect real world data to assess the performance of our algorithm and quantify the benefits of our optimization model.
- We quantify the benefits of jointly making pricing and inventory decisions, along with the benefits of considering substitution for similar perishable products.

6.2 Implications

This section provides the theoretical and practical implications of the research, addresses the industries which could be affected by this research, and advises the future decision makers in this area of ways in which they can benefit from our findings.
6.2.1 Theoretical Implications

This research not only provides the first work in stochastic dynamic programming, but is also the first work in robust optimization in the discipline of joint pricing and inventory control for substitutable and perishable products through considering holding and back-order costs. We found some mistakes in previous studies’ assumption for the stochastic dynamic programming model and so the closed form solution based on this assumption should not stand any more. Additionally, we propose a structure of quantifying the benefits of considering substitution. This has made significant contributions not only in the joint pricing and inventory discipline, but also in pricing optimization research.

6.2.2 Practical Implications

There are also practical implications from our research can be highlighted for managers in this area. Traditionally in retail, inventory and pricing decisions have been made by managers in different departments. Our research indicates that by jointly making pricing and inventory decisions rather than separately that total profits can be significantly improved. We also recommend that decision makers make joint pricing and inventory decisions by considering substitution between similar products, where we have demonstrated that total profits can improve. The greater the substitution between two similar products, the more profits can be achieved. Furthermore, we suggest that retailers keep a good record of their orders and sales of products in order to be able to more accurately predict demand.

6.3 Limitations

This research has offered some remarkable contributions in studying joint pricing and inventory decision problems, and the case study has provided an excellent method to assess the performance of the model and quantify the benefits of proposed strategies. However there are some limitations to our study.

1. The parameters of the numerical results in Chapters 3 and 4 are based on the historical data of similar older products. The actual sales could be slightly different due to the differences between these products.

2. Organizational barriers in jointly making pricing and inventory decisions. The inventory decisions are likely to be made by inventory, manufacturing or supply chain executives, whereas pricing is likely to be determined by revenue, sales or marketing departments. Therefore, some potential organizational barriers are likely in coordinating pricing and inventory decisions.
3. The lack of historical data to help find the appropriate parameters in modelling demand-price function. In many industries, such data can only be collected from actual sales data, making experimentation with different sets of pricing unlikely. This would result in a shortage of observations in parameter estimation which may have a negative impact on such modelling.

4. The lack of experience in determining the most appropriate model among deterministic models, stochastic models and robust optimization models.

6.4 Future Work

Scientific progress has been made regarding stable products. However more research on perishable products needs to be conducted to meet the remaining challenges and provide opportunities in the real world. There are some directions which can be suggested for future work for stochastic programs and robust optimization models.

In stochastic models:

- Optimal policies have been developed for stable products, however an optimal policy for substitutable perishable products needs to be identified.
- More appropriate demand models should be developed to investigate customer behaviour on demands.
- More empirical experimentation with other data can be investigated to quantify the benefits of considering substitution.
- Instead of considering backorder, lost sales can be considered to study the differences of these two models. The revenue obtained with backorder will be $D_t \times P_t$, whereas in lost sales cases, it is $P_t \times \min(D_t, y_t)$.
- Lead time can be incorporated into stochastic models for multiple-period stochastic dynamic programming models.

In robust optimization models studying the integrated pricing and inventory (production) problem:

- A multi-period multiple substitutable perishable products problem can be explored.
- Other ways to model the budget of uncertainty can be developed.
References


