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4. RESEARCH ON THE TEACHING AND LEARNING OF GEOMETRY

INTRODUCTION

The chapter provides a comprehensive review of recent research in geometry education, covering geometric and spatial thinking, geometric measurement, and visualization related to geometry, as well as encompassing theoretical developments and research into teaching and teacher development. Without doubt, the research of the *International Group for the Psychology of Mathematics Education* (PME) community in the field of geometry education has advanced since the first PME research handbook reviewed PME research over the 30 years from the inception of PME to 2005 (see Gutiérrez & Boero, 2006). In general, the emphasis of subsequent geometry education research has increasingly been on the use of technology (especially forms of dynamic geometry software) and how this impacts on geometry teaching and learners' geometrical thinking (especially on the teaching and learning of geometrical reasoning and proving), on teachers' geometric content knowledge, and on teacher development for geometry education. As such, studies examining the uses of forms of digital technology are addressed in every section of this chapter.

At same time, there has been continuing work related to spatial reasoning, geometric measurement, and visualization related to geometry. There has also been a continuing focus on the development of students' knowledge regarding understanding of geometric figures, definitions and inclusion relations, identification of shapes and language issues. In these studies, there are fewer examples of a furtherance of the Piagetian legacy, while use of the van Hiele model has continued alongside more recent developments in theory and methodology such as discursive, embodied, and eco-cultural perspectives (e.g. Ng, 2014; Owens, 2015). Thus, many research studies have focused on modes of understanding (visual, figural, conceptual), as well as on mental images and their manipulation, while employing new theoretical notions and methodologies.

The content of this chapter reflects the main emphases of research in geometry education as presented at PME conferences over the period 2005–2015. The synthesis is presented in the form of the following sections: spatial reasoning, geometric visualization, geometric measurement, geometric reasoning and proving, students' knowledge, teachers' knowledge and development, and teaching geometry and the design and use of geometric tasks.

A NOTE ON REVIEW METHODOLOGY

There are a number of well-established methods for conducting a research review (Cooper, Hedges, & Valentine, 2009). While a literature review is a vital part of every research report, the purpose of this research synthesis is to make explicit some of the connections and relations between individual studies that otherwise may not be so visible. As such, constructing this review involved the purposeful selection, review, analysis and synthesis of research on geometry education that was presented at annual PME conferences over the period 2005–2015, inclusive. Where appropriate, connection is made to work presented at PME conferences prior to 2005, as is connection to work published in relevant journals and books. The content of each set of PME proceedings from 2005 to 2015 was digitally – searched, and also hand-searched, to create a database of research reports. Each research report was reviewed and analysed, and this set of analyses used to develop the synthesis presented in this chapter.

SPATIAL REASONING

> Spatial reasoning has always been a vital capacity for human action and thought, but has not always been identified or supported in schooling. (Whiteley, Sinclair, & Davis, 2015, p. 3)

Previously in the field of spatial reasoning, spatial capability was examined essentially for its relation to mathematical learning, connected to cultural and teaching factors as well as to imagery and strategies for geometric measurement of area and volume (Owens & Outhred, 2006). There were also some studies about spatial problem-solving strategies in relevant tasks (e.g. Oikonomou & Tzekaki, 2005). However, there was limited specific interest in this capability *per se*, its meaning and definition, its role in curricula, its development in school.

A link between spatial capability and geometric thinking was made during earlier PME research on the use of technology in approaching geometry, such as the use of *Logo* (e.g. Edwards, 1994). More systematic research increased when the learning of space acquired a particular value. As Sack, Vazquez and Moral (2010, p. 113) have argued, spatial reasoning is now seen as a vital component of learners' successful mathematical thinking and problem solving. More recently, Sinclair and Bruce (2014) led a compendium of reports on projects that have focussed on spatial reasoning for young learners. This mapped out "the terrain of established research on spatial reasoning" by examining "the actualities and possibilities of spatial reasoning in contemporary school mathematics" through offering "examples of classroom emphases and speculations on research needs that might help to bring a stronger spatial reasoning emphasis into school mathematics" (p. 173). Much of this work is expanded upon by Davis and the Spatial Reasoning Study Group (2015).

Studies of Students' Knowledge Related to Spatial Capabilities

Earlier studies investigated connections between spatial capability and geometric thinking. In their research, Xistouri and Pitta-Pantazi (2006) examined connections between spatial capabilities (mental rotation and perspective-taking) and geometrical thinking related to symmetry, while Kalogirou, Elia and Gagatsis (2013) investigated how visualization and mental rotation might be related to geometrical figure apprehension (perceptual and operative) as proposed by Duval (1999). Using data from relatively large-scale samples of primary and secondary school students, these studies showed significant relations between spatial capabilities and performance in symmetry, perspective-taking capability as well as geometrical figure apprehension. More specifically, the results of the first study indicated that perspective-taking capability is more related to symmetry performance than spatial rotation, being thus a predictor of students' performance in reflective symmetry, while data from the second showed that spatial capability is "positively related to geometry achievement and problem solving" (Kalogirou et al., 2013, p. 134). By examining the data from the sample of secondary school students, the authors suggested that it is likely that, as students get older and receive more advanced teaching in geometry, they tend to use figures not just as spatial representations but as "semiotic representations of geometric objects" (p. 135).

In a study of primary students on spatial visualization and spatial orientation with net tasks (matching net cubes to cubes) and model tasks (finding top views of models), Diezmann and Lowrie (2009) found that students mainly used matching or matching-and-eliminating strategies. The researchers' concluded that the students' difficulties in visualizing and explaining their thinking might be due to the lack of prior experience and under-developed mental imagery.

In investigating the development of spatial reasoning in pre-school children, Tzekaki and Ikonomou (2009) invited 30 children, aged 4.5 to 6.5 years old, to observe, one by one, two-dimensional Lego configurations and retain their characteristics in order to reconstruct them, either by watching or from memory. The analyses of the children's reconstructions demonstrated a continuous improvement of their spatial thinking and provided interesting information about the spatial characteristics that children at this age retain mentally when they attempt to copy a spatial situation. More specifically, such children easily retain information related to the number and shape of bricks, or to their left-right placement (corresponding to their own orientation), but they encounter difficulties in finding relative positions that demand combining spatial information.

In order to investigate young children's spatial strategies from kindergarten to primary age, Reinhold, Beutler and Merschmeyer-Brüwer (2014) video-recorded task-based one-to-one clinical interviews with 22 pre-schoolers (aged 5 to 7) as each child was presented with a series of four tasks that involved 'buildings' made of glued cubes and drawings of 'buildings' (shown in a 'cavalier' perspective).

Using Thurstone's (e.g. 1950) framework of distinguishing three major spatial capability factors (spatial relations, visualization, and spatial orientation) and using previous research on cube building (e.g. Battista & Clements, 1996), they reported on the nature of pre-schoolers' building strategies in relation to their capabilities of enumerating the number of cubes in a three-dimensional cube building. While Reinhold et al. found that while students' paying attention to intended structural elements (counting in rows or columns) does not guarantee an awareness of the structure of the 'building', they could gain insight into structural elements and could change "trial and error building strategies into orientation in structural elements" (p. 87).

More specific research by Panorkou and Pratt (2009, 2011) explored how individuals experience and think about dimension. In their first study, in which a phenomenographic approach was implemented, two pairs of 10 years old students and 10 teachers were interviewed with questions related to their dimensional thinking. The findings formed a characterization of this thinking in a variety of ways: dimension as action; as state (involving location); material dimension (involving measuring or conceptions based on vision or touch); abstract dimension; and dimension as prototype or hierarchy (with relationships between dimensions). Continuing their study Panorkou and Pratt (2011) designed tasks using *Google Sketchup* and conducted a number of extended task-based interviews with 10 year-old students. They observed the students expressing various "situated abstractions" such as "polygons can be 'flat' (in a 2-D space) or 'coming out' (in a 3D space)" and "polygons that look flat in 3D can be disconnected" or "twisted" (pp. 342–343). They concluded that "a key idea about dimension seems to be that it in some sense depicts the level of capacity of the space" (p. 343).

Studies by Diezmann and Lowrie (2008), and by Lowrie, Diezmann and Logan (2011), focused on primary students' knowledge of maps of localities. In the first study a GLIM (Graphical Languages in Mathematics) test was administrated to a sample of 378 4th grade students, plus 98 students were interviewed using 12 items from the test. The results revealed key difficulties including interpreting vocabulary incorrectly, attending to incorrect foci on maps, and overlooking critical information. In the later study, information is encoded in the form of fixed attributes (marks and symbols) in a particular spatial orientation. Lowrie et al. (2011) examined the performance on six map items of 583 students of 2nd and 3rd grades, from metropolitan and non-metropolitan locations. The results showed significant performance differences in favour of metropolitan students on two of six map tasks. In trying to explain the differences, they speculated that metropolitan students might be more likely to be exposed to coordinate map systems than students in non-metropolitan areas and that "the additional requirement for students to locate information besides what was provided in the direct instructions proved challenging for non-metropolitan students" (p. 149).

Summarizing, research in the field of spatial capabilities indicates a low development of skills related to spatial orientation, spatial relations and transformations, as well as understanding of dimensions and localities. However, spatial experiences such as reconstruction of spatial configurations or cube building are likely to support progress of spatial abilities. This kind of research is significant because, as noted above, spatial reasoning, more than being an important component of human action and thought, is known to be closely connected to geometric thinking and development of geometric knowledge.

Teaching Proposals Improving Spatial Reasoning

A range of studies has aimed at improving spatial reasoning for different ages. In earlier research, Owens (2005) examined how pre-service teachers were using substantive communication about space mathematics in primary schools. A qualitative analysis of observations in their classroom showed that, teachers, after taking a large number of example lessons, worked systematically with their students' knowledge attempting to extend it, by providing effective challenges and questions. In general, working with spatial tasks in the classroom, games, toys or relevant software improve significantly different aspects of spatial capabilities and spatial thinking.

More recently, Highfield, Mulligan and Hedberg (2008) studied the case of two children exploring a *Bee-bot* programmable toy, a tool that enabled them to engage in transformational geometry. These two children demonstrated relational thinking to plan, program and manipulate the toy through a complex pathway and developed interesting problem-solving strategies.

Experimenting with teaching approaches, Chino, Morozumi, Arai, Ogihara, Oguchi and Miyazaki (2007) proposed a spatial geometry curriculum utilizing 3-D dynamic geometry software in lower secondary grades. The results, coming after comparing experimental with control groups as well as results of the national survey of Japan, identified positive effects regarding the construction of spatial figures by moving a plane figure and the explanation the students gave for a 3-D figure represented in 2-D. Hegedus (2013) reported on a multi-modal interactive environment where young learners were able not only to "click-drag-deform mathematic objects on a screen as in traditional dynamic geometry" but also experience "force feedback related to mathematical properties through the same device" (p. 33). Psycharis (2006) reported on how 13 year-olds dynamically manipulated geometrical figures involving ratio and proportion tasks, while Samper, Camargo, Perry and Molina (2012) reported a case study of implication and abduction in dynamic geometry.

Both Moustaki and Kynigos (2011) and Ferrara and Mammana (2014) have researched the spatial capability of much older students. In their research, Moustaki and Kynigos (2011) looked for instances in which students' visualization, construction and mathematical reasoning processes might contribute to the enhancement of those capabilities. They developed a '3-D Modelling & Cutting' microworld and used it with some 12th grade engineering students specializing in Programming Computer Numerical Control (CNC) Machines. The analysis showed that the students initially perceived the figures and shapes represented in the 2-D drawing in a "purely iconic way instead of a mathematical one" (p. 262). With greater experience, the students came to realise that they had been 'misled' by the static 2-D drawing and needed to use 3-D geometrical objects to specify spatial relationships among the component's parts that would not differentiate as they changed viewpoints.

For Ferrara and Mammana (2014), the visual challenge involved in the approach of spatial geometry was the use of 'flat' diagrams for geometrical figures. Using the dynamic geometry software *Cabri 3D*, they introduced a definitional 'analogy' between quadrilaterals and tetrahedra for 'edges' and 'faces'. Undergraduate mathematics students tackled two main tasks; introducing the medians for quadrilaterals and tetrahedra, and conjecturing about the properties that hold in both cases. These tasks, say the researchers, pushed the students towards a search for similarities and differences, invariants and changes, between the two figures. In this way the learners managed to "see in space" (p. 59) through the affordances offered by the dynamic geometry software.

With elementary-age children (in Grade 3), Sack, Vazquez and Moral (2010) and Sack and Vazquez (2011) reported on using 3-D models, 2-D conventional and semiotic (abstract) representations, verbal descriptions of figures, and tasks using *Geocadabra* (Lecluse, 2005) software by which a multi-cube structure can be viewed as 2-D conventional representations or as top, side and front views or numeric top-view grid coding. Working with different representations, the children had to calculate in multiple ways how many unit cubes were in relevant structures and connect the result to the sum of the numbers in the figures' top-view coding grid.

Summarizing the results of these studies, spatial tasks combining 2-D and 3-D geometric figures supported by relevant technological tools are likely to foster spatial-knowledge development and improve students' spatial reasoning, confirming, thus, the important role of technological environments in the development of spatial thinking.

GEOMETRICAL VISUALIZATION AND VISUAL THINKING

> Geometry comprises those branches of mathematics that exploit visual intuition (the most dominant of our senses) to remember theorems, understand proof, inspire conjecture, perceive reality, and give global insight. (Zeeman, quoted in Royal Society, 2001, p. 12)

In this section visualisation is taken to be the capacity to "represent, transform, generate, communicate, document, and reflect on visual information" (Hershkowitz, 1990, p. 75) and attention is paid to visual intuition. For both, there is some inevitable overlap with spatial reasoning. As such, some research reported in the section on spatial reasoning may also appear in this section, and vice versa.

In the first PME handbook, Owens and Outhred (2006) covered a good deal of research on visualization alongside findings concerning the use of imagery in mathematics in general, and also in spatial processing and geometric thinking. In relation to this, Presmeg (2006) summarised issues in visualization by first clarifying terms relating to semiotics (such as signifier, registers, iconic, indexical, or symbolic signs) and then explicitly examining imagery (mental images) and externally-presented inscriptions involving visualization. Presmeg explained that “both visual imagery and inscriptions are sign vehicles that are instantiations of visualization in mathematics, insofar as they depict the spatial structure of a mathematical object” (p. 22).

Visual Cognition of Geometrical Objects

A number of research studies have focused on ‘visual cognition’, defining it as a mental process (perceiving, recognizing, retaining in memory, etc.) that refers to the way an individual acquires and processes visual information. Usefully, Kalogirou, Elia and Gagatsis (2013) pointed to differences between terms such as visual perception and visualization. They suggested that visual perception, while one of the most important factors affecting the capability to recognize plane shapes, only provides a “direct access to the shape and never gives a complete apprehension of it” (pp. 129–130). On the contrary, they argued, visualization is “based on the production of a semiotic representation of the concept and gives at once a complete apprehension of any organization of relations”; as such, visualization in mathematics “requires specific training in order to grasp directly the whole configuration of relations and to handle the figure as a geometrical object” (p. 130).

Widder, Berman and Koichu (2014) have been searching for “a better understanding of the visual obstacles’ constituents, and the interaction between them” as that might be “the key to improve spatial geometry instruction” (p. 370). With data from testing high-attaining grade 12 students, their study confirmed “the existence of a prototype representing a cube” in that the overwhelming majority of the participants “drew the same normatively-positioned cube frequently used during spatial geometry instruction” (p. 375). While the prototypical use of normative drawings of cubes in spatial geometry instruction “may form a mental image meant to assist visualization”, at the same time Widder et al. argued that this “may not allow enough flexibility, and therefore hinder identification and manipulation of a 3-D geometrical situation in un-normative sketches” (Widder et al., 2014, p. 375).

Relevant to students’ visual cognition appears to be teachers’ capability in visualization in geometry. For example, Markovits, Rosenfeld and Eylon (2006) investigated 25 teachers’ performance in visual tasks along with their prior content knowledge and beliefs in the area of visual cognition. The results showed that the visual cognition of these teachers was limited, and their capabilities in visual estimation, free recall and graphical reproductions were close to those of 3rd grade students. Cohen (2008) examined pre-service and in-service teacher’s knowledge of mental images and their beliefs about geometrical straight lines and planes. Their findings revealed conflicting teacher beliefs between formal knowledge and mental images as well as typical misconceptions about lines and planes.

Sack and Vazquez (2008), based on a spatial operation capacity model (SOC) conducted an after-school teaching experiment with two groups of 3rd and 4th grade students. The authors found that the student's performance on standardized test items that use verbal visualization terms (for example, top, side and front views) "may be compromised by unconventional language use rather than lack of visual cognition" (p. 224).

Haj-Yahya and Hershkowitz (2013) aimed at "linking visualization, students' construction of geometrical concepts and their definitions, and students' ability to prove" (p. 409). With data from testing grade 10 students, they found that many of them knew the formal definitions of the various quadrilaterals but did not make use of the definitions when faced with tasks using forms of visual representation of shapes. In many cases, say Haj-Yahya and Hershkowitz, "students know the formal definition but do not make use of it when faced with a visual task representation|" (p. 415).

Chumachemko, Shvarts and Budanov (2014) were also interested in the development of visual perception. Focusing on the Cartesian coordinate system and, in particular, the "transformations of perception that are needed to approach this mathematical visual model" (pp. 313–314), they compared the eye movements of participants at three levels of mathematics competence and they confirm their hypothesis: when detecting a point on the Cartesian plane "the better participants are educated, the shorter are their gaze paths, and the more the number of their fixations is reduced, and the durations of their tasks solving become shorter" (p. 316).

Overall, research agrees in the existence of limits to visual cognition and how there are visual obstacles in different recognition processes both for students and for teachers. In some cases, the visual aspect might even distract students from their mental or relevant theoretical knowledge, a finding that needs further investigation.

Visualization in Reasoning and Problem Solving

Introducing notions of 'linking visual active representations' (LVAR) and 'reflective visual reaction' (RVR), the aim of a teaching experiment by Patsiomitou and Koleza (2008) was to explore the role of these notions in a dynamic geometry software environment. With data from 14 secondary school students, the results showed that prior knowledge played a significant role in parallel with LVAR and RVR as a shift from visual to formal proof led students to formulate "if …then" propositions and to move "between two successive 'Linking Visual Active Representations' only by means of mental consideration, without returning to previous representations to reorganize his/her thoughts" (p. 94).

When undergraduate students are reading a 'worked proof', research by Lin, Wu and Sommers (2012) found that visualization corresponds to "needing to keep spatial representations in their working memory and to look between proof and figures" (p. 151). By studying the eye-tracking movements of undergraduate students as they read geometry proofs of different difficulty levels, the researchers found evidence that "visual reception and visualization occur simultaneously" (p. 152).

In their studies with pre-service teachers, Torregrosa and Quesada (2008, 2009) focused on what they call configural reasoning in which discursive and operative apprehensions (Duval, 1999) are coordinated in order to solve a problem or generate a proof. They found that visual predominance tends to inhibit the visualisation of the configuration sucg that configurative reasoning and the proving process are not always interrelated.

In the same context of solving geometrical problems, Pitta-Pantazi and Christou (2009) investigated whether individuals' cognitive styles, measured in terms of object imagery, spatial imagery and verbal capability, were related to their mathematical creativity. Some 96 pre-service teachers answered the Object-Spatial Imagery and Verbal Questionnaire (OSIVQ) and were examined in a mathematical creativity test for their capabilities in area, shape, pattern, problem solving and number. The results showed significant connections between spatial imagery and cognitive style, on the one hand, and mathematical fluency, flexibility and originality (as components of creativity) on the other, but no connections of object imagery and verbal cognitive capability to any dimension of creativity.

In their study Ramfull and Lowrie (2015) examined the connections between students' cognitive style, visualization and mathematics performance. They examined 807 6th graders from Singapore schools with three instruments: the C-OSIVQ questionnaire for measures of cognitive styles, the Paper Folding Test for spatial visualization and the Mathematics Processing Instrument for problem solving performance. The results align with previous studies by indicating significant correlations between cognitive styles (mainly spatial imagery information processing) and spatial visualization and problem solving abilities.

It is apparent from all aforementioned studies that visualization is indispensable in proving and problem solving. Visual aids support students' and teachers' thinking and both appear to improve their visual imagery for the needs of a solution or a proof. However, the visual representations or process they develop are not always effective in solving or proving relevant tasks, but there is still limited research related to the connection of visuality (as defined at the beginning of this section) to creative developments. Studies with digital technologies, such as DGEs, are providing more evidence and are offering new possibilities in the visualization of geometric objects.

Visualizing in Geometry and Use of Gestures

> Humans make use not just of one communicative medium, language, but also of three mediums concurrently: language, gesture, and the semiotic resources in the perceptual environment (Roth, 2001, p. 9)

Research in geometry education has special interest in the role of gestures in mathematical communicating and thinking as an aspect of geometric visualization. In their Research Forum, Arzarello and Edwards (2005) examined gestures as a way of processing and communicating geometric ideas based on psychological, semiotic and psycholinguistics theoretical frameworks (Alibali, Kita, & Young, 2000; Bara & Tirassa, 1999; Peirce, 1955; Radford, 2003). Thus, they recorded the dynamic evolution in the use of gestures as pointed out by the social activity of the students in a geometric context and their discussion about solid shapes. They first analysed gestures and speech alongside written words and mathematical signs (c.f. Edwards, 2005). Later in the forum, Arzarello, Ferrara, Robutti and Paola (2005) extended this by examining relations between the use of gestures and the development of new ‘perceivable signs’. They recorded the progression of students’ solution during the construction of solids and examined the introduction of signs with gestures. At first, the students’ gestures had an iconic function presenting the solid they were describing. Gradually they became ‘indexes’ (in the sense of Pierce) in the communicative attempt of transferring knowledge to others and finally they acquired a symbolic function; thus their relation developed in a piece of theoretical knowledge.

Maschietto and Bartolini Bussi (2005) approached the study of the construction of mathematical meanings in terms of development of semiotic systems (gestures, speech in oral and written form, drawings) in a Vygotskian framework with reference to cultural artefacts. In their paper they presented a teaching experiment related to perspective drawing with 4th-5th grade students. The authors described how they analysed “the appropriation of an element of the mathematical model of perspective drawing (visual pyramid) through the development of gestures, speech and drawings, starting from a concrete experience with a Dürer’s glass to the interpretation of a new artefact as a concrete model of that mathematical object…” (p. 315). Analysis of the students’ protocols highlighted the parallel development of different semiotic systems (gestures, speech in oral and written form, drawings) and their mutual complementary enrichment. Research by Sack, Vazquez and Moral (2010), mentioned earlier, also reveals the use of gestures by young students.

In their research, Ng and Sinclair (2013) studied children’s use of gestures on spatial transformation tasks. They found that children used gestures “as multi-modal resources to communicate temporal relationships about spatial transformations” (p. 361). Subsequently, Ng (2014) reported on the interplay between language, gestures, dragging and diagrams in bilingual learners’ mathematical communications, when students rely on “gestures and dragging as multimodal resources to communicate about dynamic aspects of calculus” (p. 289). For more on high school students engaged in perceptual, bodily, and imaginary experiences while discussing about calculus concepts in a dynamic geometry environment, see Ferrara and Ng (2014).

GEOMETRIC MEASUREMENT

> Measurement plays a central role in reasoning about all aspects of our spatial environment. (Battista, 2007, p. 891)

In their review of earlier PME research, Owens and Outhred (2006) depicted the complexities of measurement principles and their teaching. Here, subsequent research is reviewed – first on length, then on area, volume, and angle.

Length

> An understanding of linear measure is imperative, as it provides the basis for length, area, and volume. (Cullen & Barrett, 2010, p. 281)

As Watson, Jones and Pratt (2013, p. 76) confirm, research has shown that when children measure lengths they can end up "applying a poorly-understood procedure rather than focusing on the correspondence between the units on the ruler (which may be seen erroneously as a counting device) and the length being measured". What is more, research by McDonough (2010, p. 294) reports "confusion regarding unit name, length, and relationships" when the object being measured is longer than the ruler.

Given the different ways that measurement tasks can be presented, Cullen and Barrett (2010) compared the strategies used by young children (aged 4–5 years, and 7–8 years) when engaged in measurement tasks that were presented either using *Geometer's Sketchpad* (GSP) software or as paper-and-pencil. Noting that measurement strategies include the endpoint strategy (where the child refers either to the right or left endpoint as the length of the object) and the point-to-the-middle-of-an-interval strategy, the researchers found that "linking the intervals on a ruler to iterable discrete objects, or to virtual representations of those objects, were both successful ways to motivate students to use the effective 'point to midpoint' strategy" (p. 287). They concluded that interval-identifying strategies should be beneficial when teaching students to measure the length of an object with a ruler.

Beck, Eames, Cullen, Barrett, Clements and Sarama (2014) investigated whether grade 6 children's knowledge of measurement related to their capability to use double number lines when solving problems involving proportional reasoning. Using ideas of hierarchic interactionalism, Beck et al. defined a series of 'levels' – the first two of which are Length-Unit-Relater-and-Repeater (LURR) level, where children "measure by repeating, or iterating, a unit, and understand the relationship between the size and number of units", and the Consistent-Length-Measurer (CLM) level, whereby children "see length as a ratio comparison between a unit and an object" and "use equal-length units, understand the zero point on the ruler, and can partition units to make use of units and subunits" (p. 106). They found that children at the LURR level relied on iterative strategies, while children at the CLM level could "partition and correctly attend to units along one scale but not yet coordinate units along two scales simultaneously" (pp. 111–112).

Future research could build on what is already known about the foundational ideas of measurement such as identical units, iteration and zero-point.

Area

Given that area measurement is known to pose further challenges for learners (see Watson, Jones, & Pratt, 2013, p. 76), Gonulates and Males (2011) analysed US primary school mathematics textbooks and found little variety in the ways in which knowledge was expressed. The researchers concluded that the textbooks did not provide opportunities for students to engage with conceptual knowledge of area.

Whether primary-age children might benefit from being taught a curriculum that integrates 2-D geometry with area measurement, compared with a curriculum that stressed numerical calculation of area, was studied by Huang (2011). Huang's conclusion was that integrating area measurement instruction with numerical strategies and geometric materials seemed to be "a promising approach to promoting children's conceptual understanding of area measurement" as well as their capacity to "explain geometric reasoning with measurement when solving problems" (pp. 47–48).

The development of different components of students' knowledge about area measurement was investigated by Frade (2005). Frade found that students aged 11 to 12 showed a concept of area as a physical geographic space while by age 12–13 this had evolved to them being able to use "the rectangle area formula adequately" and having "the 'know how' to solve a number of problems" (p. 327).

Area concepts continue to appear in the mathematics curriculum through to university. Cabañas-Sánchez and Cantoral-Uriza (2010) focused on how first-year university mathematics students could transform convex and non-convex polygons so that area was conserved. In analysing the arguments presented by the students, the researchers found that the students used both 'parallelism' (area between parallel lines is conserved) and relevant formulae to calculate areas.

Future research might develop further promising ways of promoting children's conceptual understanding of area.

Volume

Turning to 3-D measures, Watson, Jones and Pratt (2013, p. 76) note that these introduce "even more complexity, not only by adding a third dimension and thus presenting a significant challenge for students' spatial sense, but also in the very nature of the entity being measured". As noted above in the section on spatial reasoning, in research on how 8–9 year-old children solve 3-D tasks using the software *Geocadabra* (Lecluse, 2005), Sack and Vazquez (2011) concluded that "coding of rectangular array structures fosters children's understanding of the volume formula in concert with their emerging multiplication skills" (p. 95). Huang (2012) was similarly interested in how children would benefit from a curriculum that integrates geometry with volume measurement, as compared to teaching that stresses numerical calculations and application of the formula. By designing different week-long teaching sequences for two 5th grade classes (pupils aged 10–11), Huang found that each approach "facilitated the children's acquisition of the idea of volume measurement" and their capability to "solve different types of problems embedded with volume measurement concepts" (p. 361).

While focusing on mass rather than volume, McDonough, Cheeseman and Ferguson (2012) developed a one-week teaching unit for 6–8 year olds. Through this they found that the children were capable of thinking constructively about the intricacies of mass measurement. In terms of comparing and ordering masses, they found that the children appeared to "draw on prior experiences and sometimes on visual cues, but with appearance-based comparison for mass not as likely a reliable strategy as it might be, say, for length" (p. 207).

These studies illustrate the continuing need for active research on the topic of volume, and for research on the related topics of mass and capacity.

Combinations of Measures

As well as studying individual measures, researchers have also conducted studies involving more than one measure. For example, Stephanou and Pitta-Pantazi (2006) analysed the answers that upper primary school students gave to area and perimeter tasks. They found that more than half of the students' answers were influenced "not so much by the specific context of a task (area or perimeter) or the presence of a diagram" but rather they were influenced "by the external features (change of one/both dimensions) of the task that trigger the intuitive rule 'if A then B, if not A then not B'" (p. 183). Huang (2010) also examined children's understanding of perimeter and area. The findings indicated that even where children (aged 8–9) had the computational capability to calculate perimeters, this did not necessary mean that they had complete comprehension of the meanings of multiplication and of the formula for area calculation.

Cullen, Miller, Barrett, Clements and Sarama (2011) compared three different unit-eliciting task structures for measurement comparison tasks. With a sample of children from grades 2–4, the researchers found that students were most successful with a task structure that asked "how much longer/bigger?" and were least successful with a task structure that asked "how many times longer/bigger?" (p. 249). What is more, in response to "how much longer/bigger?" the children tended to use an additive comparison while they tended to produce multiplicative comparisons in response to "how many times longer/bigger?" (ibid).

Research by Fernández and De Bock (2013, p. 297) focused on a frequently-investigated case of students' misuse of linearity; that of effect of an enlargement or reduction of a geometrical figure on its area or volume. Here, learners have the tendency to treat relations between length and area, or between length and volume, as linear instead of, respectively, quadratic and cubic – perhaps, the researchers suggest, because secondary school students struggle with the distinction between dimensionality and 'directionality' (an example of that latter being that while the perimeter of a square is one-dimensional, it has two 'directions' in the form of length and breadth). Analysis of the responses to a set of tasks by 13–14 year olds confirmed the preponderance of "linear" answers and also indicated that more than 20% of the students' answers were "directional" (ibid). The distinction between dimensionality and directionality was more a struggle for figures where the number of directions and dimensions coincided, such as when a square has two dimensions and also two directions.

Curry, Mitchelmore and Outhred (2006) surveyed 96 students of Grades 1–4 using tasks assessing understanding of the five measurement principles: the need for congruent units; the importance of using an appropriate unit; the need to use the same unit when comparing objects; the relationship between the unit and the measure; and the structure of the unit iteration. Their results showed that while some of these principles were found to be clearer to older children, a precise order of development was not evident. The researchers concluded that appropriate learning tasks could be ones that help focus students on "the reasons for using a fixed unit size, for not leaving gaps, for using multiplication in some contexts, for rejecting certain units and accepting others, and for the inverse principle" (p. 383).

Such suggestions can be compared to those of Owens and Kaleva (2008), who have studied the many differing indigenous communities of Papua New Guinea (PNG). In setting out to collect and analyse approaches to measurement for as many PNG language groups as possible, Owens and Kaleva generalise to say that PNG people "have a sense of area (tacit knowledge) developed through sleeping, gardening and house building in particular" and "are able to use this idea of area to make judgements such as the estimated amount of material needed for a house of a particular floor size"; likewise, PNG people "would visualise a garden by knowing its length" (p. 79). The researchers concluded "by making these points explicit, teachers can reduce the discontinuities in knowledge and hence build a firm basis for school mathematics" (ibid).

The issue of primary students' measurement estimates has been studied by Huang (2015) and by Ruwisch, Heid and Weiher (2015). Huang reported that good estimators tended to adopt multiple strategies and mental rulers more frequently than poor estimators, while Ruwisch and colleagues found that the children (and educators) that they studied gave better estimations for lengths than for capacities.

Angle

The measuring of angle is, according to Bryant (2009, p. 4), "another serious stumbling block for pupils". One problem, according to Bryant, is that turning 90 degrees (a 'dynamic' angle) appears very different to the corner of a book being 90 degrees (a 'static' angle). The study by Masuda (2009) confirms that learner difficulties range from grade 5 students having difficulty paying attention to an angle as one of the attributes of the shape (and distinguishing it from measuring a side of a shape) to grade 11 students being unclear about radians and degrees.

Kaur (2013) researched the ideas of elementary school children (aged 5–6) working on angle comparison using dynamic geometry software (DGS). Here, the children's gestures and motion played an important role in their decision-making on angle comparison tasks. In particular, the use of gestures, such as hands as the 'arms' of an angle, enabled the children to see the process of turning even in case of 'static' shapes. In this way, "embodied routines could be helpful in looking at dynamic thinking, especially in case of young children" (p. 151).

Dohrmann and Kuzle (2014) focused on the development from grade 5 to 10 of pupils' understanding of an angle of 1 degree. The results showed that many of the children's misconceptions were directly connected to the measuring tool, namely the set square, and to the way they tried to draw an angle of 1°. In the case of the set square, this tool was found to privilege a 'static', rather than 'dynamic', perspective on angle.

In shedding light on the meanings of angle in 3-D space held by 12-year-old students, Latsi and Kynigos (2011) used a specially-designed "Turtle Geometry with dynamic manipulation microworld" within a teaching experiment in which the children "addressed angle as a directed turn … in the context of noticing and understanding 3-D objects' spatial and geometrical properties" (p. 127). The researchers found that the students benefitted from experiencing "a vehicle of motion metaphor (e.g. flying the turtle)" (ibid). In this way the students came to use angle as "a spatial visualisation concept" (ibid).

In research by Tomaz and David (2011), the focus was on the definition of the bisector of an angle and measuring the angles formed by it. In the study, students aged 13–14 tackled the problem of finding the measure of an angle formed by the bisectors of two given adjacent angles. This "opened the possibilities to deepen their [the students'] understanding about the measure of angles" (p. 264). This illustrates, say the researchers, the "power of the visual representations for structuring and modifying the mathematical activity in the classroom" (p. 259).

While the difficulties that students encounter with the notion of angle are well known in the literature, these studies show how research is needed on fusing, rather than confusing, for learners the 'static' and 'dynamic' perspectives on angle.

GEOMETRICAL REASONING AND PROVING

> An important aspect of geometry is concerned with the development of deductive reasoning and proof. (Royal Society, 2001, p. 9)

Students' Developing Capabilities with Geometric Reasoning and Proving

Research continues to focus on the capabilities of students at different grade levels with geometric reasoning and proving. Investigating cognitive predictors of geometrical proof competence, Ufer, Heinze and Reiss (2008) proposed a model comprising three levels: basic calculations; one-step proofs; and multi-step proofs. With data from testing 341 students in grade 9, the research confirmed that while knowledge was an important predictor of geometric proof competence, other predictors were also significant. The authors concluded that "if a student does not understand the nature of mathematical proofs, or has no problem-solving strategies at hand, he or she will hardly be able to construct a proof in spite of the best geometric content knowledge" (p. 367). Such a conclusion was echoed by Yang, Lin and Wang (2007) in a study of students' capabilities when reading geometry proofs.

The issue of how geometrical proof competence is connected to the capability to define geometric concepts was studied by Silfverberg and Matsuo (2008). In data from testing 152 Japanese and 162 Finnish students at 6th and 8th grade on the definitions of quadrilaterals, the researchers found that in both countries the students' understanding of defining geometric concepts related to their "understanding of the class inclusion relations" (p. 263). In examining students' capabilities in making geometric generalizations, Yevdokimov (2008) found that the higher-attaining students could formulate generalized arguments. Antonini (2008) showed how students treated contradictions in geometric argumentations and proofs, indicating how proof by contradiction is not straightforward for learners. Ginat and Spiegel (2015) found an absence of the 'fluency' and 'flexibility' aspects of creativity in novices' geometry proofs.

Bieda (2011) reported on the aspects of proofs and non-proofs that were convincing to middle grade students. The analysis found that the students "valued the explanatory power of an argument when evaluating a proof for a true geometry statement that provided a diagram" (p. 153). In a study of the assumptions made by 10th grade students when proving geometric statements, Dvora and Dreyfus (2011) found that unjustified assumptions arose when students "misused theorems or assigned extraneous properties to geometric objects", and that unjustified assumptions were "made with the purpose of reaching a critical step in the proof" (p. 289).

Matos and Rodrigues (2011) investigated how the construction of geometric proof related to the social practice developed in the classroom, and, in particular, the role of geometric diagrams. The researchers concluded that diagrams played "an important role in the process of sharing and increasing the ownership of meaning of proof by highlighting the relevant properties" (p. 183). For an interesting analysis of geometric pictures, see Stenkvist (2012).

In proof problems involving 2-D representations of 3-D shapes, the diagram may not always help. For example, Jones, Fujita and Kunimune (2012) reported a study of lower secondary school pupils (aged 12–15) who tackled a 3-D geometry problem that used a particular diagram as a representation of the cube. The analysis showed how some of the students could "take the cube as an abstract geometrical object and reason about it beyond reference to the representation", while others needed to be offered "alternative representations to help them 'see' the proof" (p. 339). The influence of 3-D representations on students' level of 3-D geometrical thinking is reported by Kondo, Fujita, Kunimune and Jones (2013) and the follow-up paper by Kondo, Fujita, Kunimune, Jones and Kumakura (2014).

Attempting to deepen the ways in which visually-based geometric materials support students' generating of conjectures, Lin and Wu (2007) examined how 6th graders, still in the process of intuitive geometry, generated geometrical conjectures when geometrical conditions in diagrams were given. The analysis showed that students generated more related conjectures if they looked at one example, instead of two or three at the same time, and they generated more conjectures if the examples were conjunctive (that is, the example was the conjunction of the conditions given in the question with other conditions). Komatsu (2011) studied how grade 9 students generalized their conjecture through proving. After the students proved their conjecture and faced its counterexample, applying their proof to a boundary case between example and counterexample of their conjecture was found to be crucial.

Given that there can be a tension between the practical aspect of physically carrying out a geometrical construction and the theoretical aspect of constructing the related proof, Fujita, Jones and Kunimune (2010) studied the extent to which there might be 'cognitive unity' between students' geometrical constructions and their proving activities. The results suggested that while grade 9 students gained a much greater appreciation of how to use already-known facts to proceed with further investigations in geometry, the uniting of student conjecture production and proof construction was not automatic. As the authors concluded "further research is necessary to give a fuller answer to the matter of how, and to what extent, geometrical constructions encourage the uniting of student conjecture production and proof construction" (p. 15). In a follow-up report, the same authors reported two cases from grade 7 where the use of geometrical constructions enabled the students to shift "from relying on visual appearances or measurement to reasoning with properties of shapes" (Fujita, Kunimune, & Jones, 2014, p. 65).

A range of studies has examined students' proof and proving when using dynamic geometry software (DGS). Patsiomitou and Emvalotis (2010), for example, concluded from their study that "the dynamic manipulation of objects in the software led the students to construct the properties of figures" and this, in turn, helped the students classify the figures (see also, Patsiomitou, 2011). Baccaglini-Frank, Mariotti and Antonini, (2009) reported on different perceptions of invariants and generality of proof in dynamic geometry, while Baccaglini-Frank, Antonini, Leung and Mariotti (2011) reported on a study with upper secondary-age students (aged 16–18) that focused on constructing a proof by contradiction. The latter showed that "there can be a strong subjective element in the process of producing a geometrical proof (or a convincing argument) via the solver's conscious choices of construction and dragging in a DGS" (pp. 87–88). Olivero (2006) investigated the role of the DGS hide/show tool in the conjecturing and proving processes. While this facility offers students the possibility to focus on different elements during a geometric construction, the analysis confirmed that the visible elements on the screen guided the focus of the students and it was this that effected the construction of conjectures and the development of proofs. In a different approach, Leung and Or (2007) studied oral explanations and written proofs provided by secondary students working on construction tasks with DGS. The researchers concluded that writing up DGS proofs "may involve using mathematical symbols or expressions that transcend the usual semantic of a traditional mathematical symbolic representation (p. 183).

Fujita, Jones and Miyazaki (2011) and Miyazaki, Fujita and Jones (2014) reported on studies of a "web-based proof learning support environment" (p. 353) in which learners tackled geometrical congruency-based proof tasks by dragging sides, angles and triangles to cells of a flowchart-style proof while the web-based system automatically transfered figural to symbolic elements so that learners could concentrate on the logical and structural aspects of their proofs. From their research, the researchers argued that with this approach, alongside suitable guidance from the teacher on the structural aspects of a proof, students could "start bridging the gap in their logic and thereby begin to overcome circular arguments in mathematical proofs" (2011, p. 353).

Textbooks may, or may not, provide support for students' developing capabilities with geometric proving. Dolev and Even (2012) compared six 7th grade Israeli mathematics textbooks, examining the opportunities provided by the textbooks to justify and explain mathematical work about triangle properties. They found, compared with algebra, that all six textbooks included "considerably larger percentages of geometric tasks that required students to justify or explain their solutions" (p. 203). Miyakawa (2012) compared textbooks from France and Japan and found differences such as what gets called proof in the textbook, the form of proof used, and the functions of proof employed.

Given that definitions are integral to geometric proof, Okazaki (2013) found that for 5th grade pupils five situations should help: "(1) understanding the meaning of identifying geometric figures, (2) constructing examples from non-examples and justifying the constructions via comparisons, (3) recognizing equivalent combinations, (4) examining undetermined cases via counterexamples, and (5) conceiving figures as relations beyond the given actualities" (p. 409). Haj-Yahya, Hershkowitz and Dreyfus (2014) investigated 11th grade students' geometrical proofs through the lens of the students' definitions and found that the difficulties students had in understanding geometric definitions affected their understanding of the proving process and hence the capability to prove.

Several studies have examined the ways in which high-attaining students compose or construct a proof, or create a definition, and how this might help in understanding the proving approach of the students in general because the characteristics of their approaches are very close to the mathematical proving or defining processes. Examples include Lee (2005) and Song, Chong, Yim and Chang (2006) who examined the constituents of proving that high-attaining students produce, Ryu, Chong and Song (2007) who researched their spatial visualization of solid figures, Lee, Kim, Na, Han and Song (2007) who researched their use of utilize induction, analogy, and imagery, and Lee, Ko and Song (2007) who studied the ways they define geometric objects. These researchers concluded that teachers need to draw explicit attention to the value of informal proofs and that for students to develop their sense of geometrical reasoning there needs to be extensive experience of conjecturing and then verifying. Kim, Lee, Ko, Park and Park (2009) built on this work in a study of how high-attaining students can become aware of unjustified assumptions in geometric constructions.

Teaching Proposals Improving Students' Performance in Geometric Proving

In looking to help students, Cheng and colleagues examined strategies such as reading-and-colouring (Cheng & Lin, 2006), the use of coloured flashcards to support geometric argumentation (Cheng & Lin, 2007), and step-by-step reasoning in two-step geometry proofs (Cheng & Lin, 2008). With the reading-and-colouring teaching approach entailing students using colours to show known and unknown information in proving tasks, teaching experiments with 9th grade students found that the approach helped students to see the necessary information for proving a statement. As a way of supporting geometry proof reasoning in slower students, Cheng and Lin (2008) developed a step-by-step reasoning strategy and found that this teaching strategy improved the students' proving process. Research by Kuntze (2008) confirmed that writing about geometrical proving can foster "the competency of solving geometrical proof tasks" (p. 295).

Huang (2005) investigated how a sample of teachers in Hong Kong and Shanghai taught Pythagoras' theorem. The findings showed both similarities and differences in terms of the approach to the justification of the theorem. Although teachers in both places emphasized the justification of the theorem by various activities, the following differences were noticeable: Hong Kong teachers were what they called "visual verification-orientated" while Shanghai teachers were "mathematical-proof-orientated" (p. 166). Moreover, compared with Hong Kong teachers, Shanghai teachers made more effort to encourage students to speak about and construct their own proofs. Zaslavsky, Harel and Manaster (2006) also investigated the teaching of the Pythagoras theorem, in particular how examples were used and how this enabled analysis of teacher mathematical and pedagogical knowledge that may support or inhibit student learning.

The role of the teacher is known to be crucial to students' developing capabilities with geometric proving. Dimmel and Herbst (2014) found that teachers had different views of the appropriate level of detail in a student's geometrical proof. Focused on classroom interaction, Miyakawa and Herbst's (2007) study of classroom geometrical proving found differences between what they called "installing theorems" and "doing proofs": in the former, "details may be excluded, and a theorem may be established without proof" while when 'doing proofs' the conclusion "cannot be used until proved" (p. 288).

In the same direction, Fuglestad and Goodchild (2009) examined teachers' knowledge about proof and its necessity, concluding that some teachers do not appear certain about the nature and the necessity of a proof. Attempting to support teachers' understanding of geometric reasoning and proof, Bayazit and Jakubowski (2008) proposed constructions with compass and straightedge, while De Bock and Greer (2008) proposed to pre-service teachers a challenging task (in this case, finding and proving which rectangles with sides of integral length have equal area and perimeter). Lei, Tso and Lu (2012) examined how reading comprehension of geometry proof might be influenced by worked-out examples. With data from 85 grade 8 students who were novices at deductive proof in geometry, they found that lower-attaining students tended to overlook the overall logical structure of proof by only repeating the steps from worked-out examples and that these students failed to apply related knowledge in proving.

Brockmann-Behnsen and Rott (2014) reported on a long-term study conducted in four 8th grade classes. Two of these classes served as control groups, with the mathematics lessons of the other two classes frequently enriched by structured argumentation and the training in the use of heuristics. In the post-test, the treatment groups obtained significantly better results than the control groups (who had no special training in heuristics and argumentation strategies). While not a controlled trial, Fielding-Wells and Makar (2015) describe a teaching unit with a class of 10–11 year-olds which included the task "Can a pyramid have a scalene face?" (p. 297). Through their analysis the researchers identified several benefits of argumentation for the learners.

STUDENTS' GEOMETRIC KNOWLEDGE

Owens and Outhred (2006, p. 85) pointed to the impact of Piaget on earlier research on student's knowledge about geometric figures. In subsequent research, evidence of the legacy has been much less. In contrast, the van Hiele model (ibid, pp. 86–89) continues to feature. More recent studies have employed various frameworks, including figure apprehension according to Duval (1999), and the notion of figural concept by Fischbein (1993). In addition, use of more general frameworks includes Sfard's (2008) commognition approach, as well as notions of embodiment (Gibbs, 2006).

The Piagetian Legacy and Use of the van Hiele Model

Examples of continuation of the Piagetian legacy in PME research include the study by Cullen et al. (2011) who used the Piagetian idea of the importance of comparison in measurement and Maier and Benz (2014) who used Piagetian notions of drawing skills in investigating how children aged between 4 and 6 drew different kinds of triangles. Examples of the use of the van Hiele model include research by, for example, Wu and Ma (2005), Wu and Ma (2006), Wu, Ma, Hsieh and Li (2007). Such studies of elementary school students confirm the outcomes of previous research that students tend to judge geometric figures by their appearance, with the circle the easiest and quadrilaterals the more difficult.

More recently, Guven and Okumus (2011) tested the van Hiele levels of 8th grade Turkish students together with their classification preferences (hierarchical or partitional) about relationships between some quadrilateral pairs. They found that "most of the students were at van Hiele level 2 before starting their high school education and the students generally chose partitional classification" (p. 473). For Kospentaris and Spyrou (2009), after examining data on the van Hiele levels of secondary school students, it was because of geometry teaching methods that such students barely surpass level 1. Patsiomitou and Emvalotis (2010) used the van Hiele levels in a study of the development of students' geometrical thinking through a guided-reinvention process with DGS. They found that students "developed their geometrical thinking processes and applied skills, reaching a higher level of abstraction" (p. 39)

Apprehension of Geometric Figures

According to Duval's (1999) theoretical framework, there are four different ways to organize and process visual aspects in geometric figures: perceptual apprehension (recognizing figures); sequential apprehension (perceiving their different parts); discursive apprehension (on the basis of statements, definitions, descriptions); and operative apprehension (modifying a figure or some of its element). A study by Elia, Gagatsis, Deliyianni, Monoyiou and Michael (2009) of various aspects of figure modification confirmed students' tendency to apply part-whole modifications rather than modifications referring to the position or orientation of a figure. In later research (Deliyianni, Michael, Monoyiou, Gagatsis, & Elia, 2011), the researchers aimed at confirming a composite theoretical model concerning middle and high school students' geometrical figure understanding. More recently, Kalogirou, Elia and Gagatsis (2013) investigated how two major components of spatial capability, that of visualization and mental rotation, might be related to geometrical figure apprehension (perceptual and operative) as proposed by Duval (1999). Statistical analysis indicated a moderate though significant correlation between spatial capability and geometrical figure apprehension.

Sinclair and Kaur (2011) found that kindergarten children were able to "develop an understanding of symmetry that showed awareness of the properties of reflectional symmetry through the behaviour of dynamic images" (p. 193). For Sinclair, Moss and Jones (2010) the focus was children aged 5 to 7 trying to decide whether two lines on a DGS screen that they know continue (but cannot see all of the continuation) would intersect, or not. They report that, in tackling this question, the children engaged in "aspects of deductive argumentation" (p. 191). Kaur and Sinclair (2014) reported part of a longitudinal study of the development of young children's geometric thinking (aged 7–8). They found that "during the teacher-led explorations and discussions with dynamic sketches, children's routines moved from description of tool-based informal properties to formal properties" (p. 415), as well as from particular to more general discourse about what is a triangle.

Knowledge of Definitions and Inclusion Relations

A study by Ubuz (2006) of secondary school students' definitions of polygons and quadrilaterals, and the ways these figures are presented in the textbooks, found that "figures (in textbooks) often provide an instantiation of a definition, not a general and rigorous proof" so that the students "focus on figural understanding to produce conceptual understanding" (p. 347).

The understanding of the inclusion relations between quadrilaterals has been the focus of a number of studies (Guven & Okumus 2011; Okazaki, 2009; Okazaki & Fujita, 2007; Silfverberg & Matsuo, 2008). Such studies confirmed that students' difficulties in understanding the inclusion relations differ from grade to grade and can be related to tacit properties and significant prototype phenomena. In their study of how Japanese and Finnish students were able to apply class inclusion and disjunctive classification, Silfverberg and Matsuo (2008) found that about half of the students could identify the inclusion of squares into rectangles, and of rectangles into parallelograms. Okazaki and Fujita (2007), grounding their research on Hershkowitz's (1990) theoretical frame of prototype phenomenon, obtained data from Japanese 9th graders and from Scottish pre-service primary teachers. They found that for Japanese students the prototype phenomenon appeared "strongly in squares and rectangles" and that such prototype images and implicit properties were "obstacles for the correct understanding of the rectangle/parallelogram and square/rectangle relations" (p. 47), while even though the pre-service teachers had a "relatively flexible images of parallelograms" the strongest prototype phenomenon appeared with squares.

The image of angles in a parallelogram or a rectangle appears to be an obstacle in understanding inclusion properties, as shown in the study by Ozakaki (2009). The simple identification of geometric figures does not necessarily allow students to approach inclusion relations as they remain with the tacit properties that they have in mind.

Matsuo (2007) recorded the differences in students' understanding of geometric quadrilaterals. The results revealed four ordered states in understanding relations: not distinguishing between two geometric figures; identifying both figures respectively; distinguishing or identifying figures based on their differences or similarities; and understanding the inclusion relation. Serow (2006) examined the development of triangle property relationships using the SOLO taxonomy (Biggs & Collis, 1982). The analysis revealed differences in the ways students understood the relationships among properties. As notted earlier, Haj-Yahya and Hershkowitz (2013) found that, when definitional statements about quadrilaterals were given verbally to 10th graders without any visual support, more students were able to identify and explain the inclusion relationships.

Identification of 2-D and 3-D Shapes

A number of studies have investigated the identification of shapes such as triangles through different grades (e.g. Horne & Watson, 2008), as well as the type of criteria that students use to identify geometric figures more generally (e.g. Sophocleous, Kalogirou, & Gagatsis, 2009). Such studies have confirmed that students develop the concept of shapes through experiences both inside and outside school and from holistic visual approaches to properties recognition. Horne and Watson (2008) tested students across seven consecutive grades on a task related to identification of triangles. While they found an improvement across grades 1 to 4, most students' errors concerned the inclusion, rather than the exclusion, of triangles. Maier and Benz (2014) studied young children's ideas of triangles by analysing their drawings. They found that children aged 3–11 mainly drew isosceles triangles (although the researchers were not sure whether the children were attempting to draw equilateral triangles with limited drawing skills). Moreover, they found that prototypical presentations were dominant not only for the first drawn triangle but also as varying triangles because "most children varied their triangles through area size" (p. 160).

The study by Sophocleous, Kalogirou and Gagatsis (2009) compared the criteria of figure recognition with solutions that 5th and 6th grade students proposed in creativity tasks with overlapping figures. Their results indicated that the more critical attributes of shapes the students could recognize, the better they performed in creativity tasks.

More recently, Arai (2015) investigated how instructional tasks change the ways first graders identify geometric figures. A questionnaire with instructional tasks was administrated to three groups of 69 students. In the first group the students have to find the number of sides and vertices of triangles, to draw figures and read definition of triangles, while in the second group they have only to find the number of sides and vertices of triangles and draw figures, and in the third group students read definition of triangles. While most of the students "used visual reasoning to identify triangles, and were noticeable influenced by prototype examples", there were signs that they could change their reasoning "after engaging in instructional tasks" (p. 55).

A number of studies have investigated students' knowledge of 3-D shapes. Wu, Ma and Chen (2006) investigated students of different grades and found that higher grade students had more sophisticated representations of 3-D shapes. In a later study, Ma, Wu, Chen and Hsieh (2009) examined students' drawings of solid cuboids and compared their results to those given by Mitchelmore (1978) two decades earlier. This indicated an improved distribution of the stages compared with that presented by Mitchelmore.

Nevertheless, research by Pittalis, Mousoulides and Christou (2009) has underlined that students have many difficulties in representing, identifying, or interpreting. With data from 40 students from 5th to 9th grade, the researchers identified four levels of sophistication in the representations: no proper drawings; coordination of front and side views; proper conventions of 3-D drawings with some errors; proper drawings.

Hatterman (2008) observed 15 university students, trained in 2-D DGEs (Euklid-DynaGeo and Cabri 3D), while they worked in groups on Archimedes Geo3D and Cabri 3D. The results showed that experiences in 2D-environments appeared insufficient when students work in 3-D space. The students had problems in justifying simple facts in 3D-environments and benefitted from access to 3-D models to solve given tasks. In their study, Leung and Or (2009) investigated perspective dragging in Cabri 3D and found that this helped students to identify and reason about geometric properties of 3D objects.

Language Issues in the Development of Geometrical Thinking

In research on language issues in the development of geometrical thinking, Leung and Park (2009) found that common names in geometric and in everyday language both support and prevent students' understanding of figures and their properties because the terms direct students to fix their attention on some special characteristics that are not always consistent with the definition of the figures. More recently, Ng (2014), as noted above, studied the "interplay between language, gestures, dragging and diagrams" (p. 290) in 12th grade bilingual learners' mathematical communications about various aspects of Calculus through geometrical dynamic sketches using DGS. The findings suggested that "bilingual learners utilised a variety of resources, including language, gestures and visual mediators in their mathematical communication – with gestures taking on a prevalent role" (p. 295).

Summarizing, studies related to students' geometric knowledge keep attracting the interest of research on the teaching and learning of geometry, with older or newer approaches related to identification of 2-D or 3-D geometric figures. As a significant number of relevant studies have been accumulated in this field, a careful and systematic record of the findings and subsequent conclusions related to students' geometric knowledge might be imperative.

TEACHERS' GEOMETRIC KNOWLEDGE AND DEVELOPMENT

> Teaching geometry well involves [the teacher] knowing how to recognise interesting geometrical problems and theorems, appreciating the history and cultural context of geometry, and understanding the many and varied uses to which geometry is put. (Jones, 2002, p. 122)

Given that the nature and extent of teachers' knowledge affects the quality of their teaching (e.g. Ball & Bass, 2003), a number of studies have focused on examining pre-service and in-service teachers' knowledge of geometry – and on ways of developing this knowledge.

Geometric Knowledge of Teachers

Fujita and Jones (2006) reported on the geometric knowledge of Scottish pre-service primary teachers and the ways that these pre-service teachers defined and classified quadrilaterals. Based on the ideas of concept definition and concept image introduced by Tall and Vinner (1981), and of figural concept initiated by Fischbein (1993), Fujita and Jones (p. 130) distinguished what they called the individuals' "personal figural concept" (coming from personal experiences) from the "formal figural concept" (as defined in geometry). Almost 160 pre-service primary teachers in the first year of their studies were examined in questions related to quadrilateral properties, and 124 pre-service teachers in the third year of their studies were examined about quadrilateral relationships. Analysis of the first group's answers showed that there was a gap between figural concepts and definitions provided. Similarly, the analysis of the answers of the second group indicated a weak understanding of the hierarchical relationship of quadrilaterals.

For Tatsis and Moutsios-Rentzos (2013), their focus was the capability of pre-service primary school teachers to interpret and evaluate verbal information related to 2-D geometrical objects. The researchers found, in contrast with their conjecture, that the pre-service teachers mostly showed a stronger positive evaluation of the geometrical descriptions, followed by weaker positive evaluations of the topological descriptions. These, say the researchers, were accompanied by "relatively negative evaluations for everyday descriptions" (p. 270).

While the above studies focus on pre-service elementary teachers, Silfverberg and Joutsenlahti (2014) studied pre-service elementary and secondary teachers' notions of angles in a plane. They found that some of their respondents "interpreted an angle as a line consisting of two line segments, some consisting of two rays, and some as a region defined by these elements" (p. 190). What is more, interpretations differed as to "whether an angle continues outside the part shown in the drawing in the direction determined by the angle, or not" (ibid). Moore-Russo and Mudaly (2011) reported on a study of South African secondary school teachers' knowledge of gradient (or slope). Based on data from nine free-response test items completed by 251 practicing teachers pursing qualifications to teach grades 10–12 mathematics, their findings suggested that understanding of gradient of these teachers varied greatly, with many of the teachers lacking "even a basic understanding of this important concept" (p. 241).

In a similar vein, research by Son (2006) investigated pre-service primary and secondary teachers' conceptions of reflective symmetry and compared these with their teaching strategies. Based on the van Hiele model, the results showed that the pre-service teachers had a limited understanding of reflective symmetry and confused symmetry with rotation. Their deficiencies directed them to use procedural teaching approaches in their attempt to help students' understanding of symmetry and symmetrical constructions. Comparable results in the study of Van der Sandt (2005) showed that when secondary pre-service teachers did not adequately control the geometric subject matter, their deficiencies had implications in their classroom teaching. Paksu (2009) found that pre-service elementary teachers' self-efficacy in geometry was related to many factors such as their van Hiele geometric thinking level, their attitude towards geometry, and their attainment in geometry. Chiang and Stacey (2015) focused on in-service primary school teachers in Taiwan. In line with much existing research, they found the teachers lacked some basic geometric knowledge.

In a diagnostic test of the knowledge of both pre-service and in-service teachers about triangles, Alatorre and Saiz (2009) found both figural and conceptual misconceptions. These included the idea that the base of a triangle is necessarily horizontal (with the rest of the figure above it) and the height necessarily vertical and/or drawn from the highest point, the idea that triangles must necessarily be isosceles, that altitudes need to be internal, the idea that each triangle has only one base and one height, confusing the height with the median, the use of right-angled triangles terminology with non-right-angled ones, various misconceptions about the Pythagorean Theorem and its applications, and errors with the formula for the area of a triangle. Subsequently, Alatorre, Flores and Mendiola (2012) studied in-service primary teachers' reasoning and argumentation about triangle inequality. Their findings suggested that reasoning and argumentation "are not part of many [primary] teachers' professional practice" (p. 9).

Given the consistent findings of problems with teacher knowledge, more research could focus on how the geometrical knowledge of pre-service, and in-service, teachers could be improved. Existing PME research on this topic is addressed in the next section.

Teacher Development for Geometry Education

Various research studies have shown that the geometrical knowledge of pre-service, and in-service, teachers can be improved not only by the appropriate education (for instance, González & Guillén, 2008, proposed an *Initial Competence Model* for teachers for the teaching of geometric solids), but also by the use of technologies such as dynamic geometry environments (e.g. Haja, 2005). In the study by Haja, pre-service secondary teachers were studied for their problem-solving capabilities while they were undertaking geometrical constructions using DGS. The researcher applied a "knowledge-in-action design that expected them to "…apply their content knowledge to understand the given problem, construct the dynamic figures, make conjectures, verify the conjectures, and solve similar problems" (p. 82). Using open-ended tasks for which the pre-service teachers had to find a solution with the dynamic software, the evidence showed that they met the expectations of the knowledge in action design.

In a similar vein, Presmeg, Barrett and McCrone (2007) designed a course that included geometric constructions that the pre-service teachers could tackle both by using DGS and by traditional tools. These two different modes of representation of geometric concepts could, according to Duval's (1999) framework, support the pre-service teachers' constructions of generalized geometric knowledge. According to researchers' approach, the property of DGS sketches to stay together when the mouse moves points or lines, and the distinction between variant and invariant properties, were the two concepts that were more related to the development by the pre-service teachers of geometric generalizations. Moreover, collaborative discussions and sharing meanings were amongst the main factors for participants' accomplishments. Similarly, in a study conducted by Olvera, Guillén and Figueras (2008), the fostering of communities of practice of in-service primary teachers was found to improve their approaches in the teaching of solid geometry. Alqahtani and Powell (2015) studied teams of middle and high school in-service teachers during a semester-long professional development course in which the teachers participated in a collaborative online dynamic geometry environment. The researchers found that through this online dynamic geometry environment the teachers interacted to notice variances and invariances of objects and relations in geometrical figures and to solve open-ended geometry problems. For Morgan and Sack (2011) in their research with pre-service teachers, the van Hiele model remained "a useful framework to describe the evolving shape-building activities" (pp. 249–250).

Martignone's (2011) research provides examples of tasks for teachers involving artefacts (such as ruler and compasses) and how teachers can succeed in implementing such tasks in their classrooms. Lavy and Shriki (2012) studied how the skills of pre-service secondary school mathematics teachers in evaluating geometrical proofs could be improved through peer assessment of each other's proofs. The outcome was that the engagement of the pre-service teachers in peer assessment, both as assessors and as those being assessed, "contributed to the development of [their] assessment skills" (p. 41). By comparing the first and the second assessment tasks conducted by the pre-service teachers, the researchers found that the pre-service teachers developed their capabilities "to select a proper criteria list and assign a reasonable numerical weight to each criterion" (ibid).

Cirillo (2011) provides a case study of a beginning secondary school teacher working to improve the way of teaching geometrical proof and proving during their first three years of teaching. Given that the beginning teacher had a strong mathematics background, the study illustrated how content knowledge "is not necessarily sufficient preparation to teach proof" (p. 247). The study by Hähkiöniemi (2011) provides an account of how an experienced teacher was given the opportunity to try a pre-planned unit for high school students on approximating the area under a curve that was enriched with DGS-based tasks and how this raised the teachers' awareness of different teaching methods as well as the benefits and challenges of using such methods.

In general, the studies on teachers' geometric knowledge, and their pre-service and in-service education, indicate that attention needs to be given to how to build teachers' understanding of common 2-D and 3-D objects (e.g. triangles, quadrilaterals, or angles) with consequent implications for their teaching. In investigating approaches that improve teachers' geometrical education, relevant research confirms the effectiveness of general approaches (e.g. community in practice or peer assessment) but also the use of technological tools in geometric problem solving or proving.

TEACHING GEOMETRY AND GEOMETRIC TASKS

> Tasks shape the learners' experience of the subject and their understanding of the nature of mathematical activity. (Watson & Ohtani, 2015, p. 3)

Teaching Interventions

Of the various studies of geometry teaching, some entail genetic approaches involving historical, logical and epistemological, psychological and socio-cultural aspects (e.g. Safuanov, 2007) and some feature ethno-mathematical and humanist approaches valuing cultural and scientific heritage (e.g. Chorney, 2013; Gooya & Karamian, 2005), as well as the use of art work as a creative tool to approach geometric figures (Pakang & Kongtaln, 2007). On top of this, there have been studies related to the teachers' choices regarding the use of diagrams and examples (Zodik & Zaslavsky, 2007) and studies emphasizing algebraic approaches to solving geometrical problems (Dindyal, 2007). How students make sense of the 'figured world' of the geometry classroom was explored by Aaron (2008), while Ding and Jones (2006) investigated geometry teaching at the lower secondary school level in Shanghai, China.

Gal, Lin and Ying (2006) observed five different 9th grade classes aiming at investigating the factors and class characteristics that influenced students' low achievement. The findings suggested that the low achievers were provided with less learning opportunities. Similarly, Soares (2010) studied a 4th grade geometry class that was co-taught by two teachers with "different and complementary perspectives" (p. 201), one trying to encourage the students to solve challenging problems, and the other managing situations in which novel tasks are introduced. This combination of skills made for successful teaching. Both Hähkiöniemi (2011), as noted above, and

Hollebrands, Cayton and Boehm (2013) reported on the types of pivotal teaching moments, and related teacher actions, which can arise in a technology-intensive geometry classroom.

Geometric Tasks

For some studies, the design of geometric tasks was integral to the research. The research reported by Fujita, Jones and Kunimune (2010), Fujita, Jones and Miyazaki (2011), and Komatsu (2011), all relied on well-designed tasks. In Fujita, Jones and Kunimune (2010), the task was “how to construct the largest square within a given triangle ABC” (p. 12). The conclusion of the teaching experiment was that this task could be used to “encourage students’ mathematical arguments, reasoning and proof” (p. 15). In Fujita, Jones and Miyazaki (2011), the tasks were integral to the design of a “web-based proof learning support environment” (p. 353). In the tasks, learners tackled proof problems by dragging sides, angles and triangles to cells of the flow-chart proof and the web-based system automatically transferred figural to symbolic elements so that the learners could concentrate on logical and structural aspects of proofs. The task included both ordinary proof problems such as prove the base angles of an isosceles triangles are equal (the researchers call these closed problems) and problems by which students construct different proofs by changing premises under certain given limitations (which the researchers called open problems). Each time the learners selected a next step in their flow-chart proof, the web-based system checked for any error via a database of possible next steps. If there was an error, the learners received feedback in accordance with the type of error.

The study by Komatsu (2011) utilised a task concerning a small triangle placed on top of a larger one and the change in length of two segments after rotation of one triangle around a common point. For the students, the task was deliberately ambiguous as they were unclear what the ‘two segments’ meant, but it was this ambiguity that made the task interesting as it resulted in the students. It was also the boundary case between example and counterexample that played a crucial role.

Aspinwall and Unal (2005) conducted a teaching experiment called geo-arithmetic with pre-service secondary mathematics teachers. Their results confirmed that implementing a variety of different representational systems helped the pre-service teachers to translate from one to another. Other studies have examined geometrical tasks involving toys, machines or other tools, the use of which appear to support problem-solving processes and advancements in understanding (e.g. the use of *Bee-bots* by Highfield, Mulligan, & Hedberg, 2008, mentioned above).

Using DGS, the dynamic manipulation of geometric objects by ‘dragging’ is commonly referred to as the ‘drag mode’ (Hölzl, 1996; Jones, 1996). This is when an object in an on-screen diagram is ‘dragged’, the diagram is modified yet all the geometric relations used in its construction are preserved. This function supports teaching tasks that provide different apprehensions to the viewing of geometric objects and support of dynamic representations that enrich internal thought of students (Xu & Tso, 2009). A range of studies continues to explore the affordances of dynamic geometry 'dragging' environments. For example, Chan (2012) studied a university mathematics teacher who, while an accomplished mathematician, was unfamiliar with DGS. Chan found that initially the mathematician considered the software "a computational tool for the system of Euclid's Elements" but while working on explorative tasks, the mathematician experienced "the powerfulness of dragging and developed a new understanding towards DGS" (p. 297). The affordance of dragging for geometrical problem solving was a feature of the research of Jacinto and Carreira (2013). Here, 14 year-olds tackling a problem involving a rectangular lawn and a triangular flowerbed used 'dragging' to check or verify their solution. Similarly, Leung and Or (2009), investigating perspective dragging in Cabri 3D, showed that this function helps students' identification and reasoning about geometric properties of 3D objects.

Certainly, dragging in 3D software presents some differences compared to the manipulation of 3D physical models. Hattermann (2008, 2010) focused on the drag-mode of the 3D digital environment and underlined its importance in explaining that it transforms the static figures of geometry to dynamic objects. However, the use of this function is not so apparent to students who need encouragement to implement it and appreciate its advantages. In their study with 13–14 year old students, Lee and Leung (2012) confirm that, while generating more examples is the central affordance of dragging, generating such examples becomes possible for the student "only when prompted" (p. 66). Building on this and related studies, Leung (2014) proposes four principles for task design in dynamic geometry, while Sollervall (2012) reports on the design of spatial coordination tasks that make use of mobile technologies.

In a different vein, Martignone and Antonini (2009) introduced pantographs for geometrical transformations. They presented a classification scheme efficient to analyse the interaction between a subject and the machine, and the processes involved. Subsequently, Martignone (2011) presented and discussed some examples of tasks for teachers that involved geometrical 'machines'; that is "reconstructions of tools belonging to the historical phenomenology of mathematics from ancient Greece to 20th century" (p. 193) such as curve drawers and pantographs. The teachers tackled tasks such as constructing an isosceles triangle and then later they adapted the tasks for their classroom.

Wu, Wong, Cheng and Lien (2006) designed a learning environment named *InduLab* that gave 4th grade students the possibility to discover the rules of triangle construction and thus approach the angle sum property. Lew and Yoon (2013) used a developing affordance of certain software to link geometry and algebra and reported how "constructing the solutions of quadratic equation offers an alternative approach that gives students an opportunity to connect algebra (quadratic equation) and geometry (construction)" (p. 255). Their study showed how understanding of the mathematics of geometric similarity connects quadratic equation with geometric construction.

Finally, Choy, Lee and Mizzi (2015) studied how textbooks support the teaching of the topic of gradient in Germany, Singapore, and South Korea. By examining textbooks in terms of “contextual (educational factors), content, and instructional variables” (p. 169), they concluded that the textbook ‘signature’ of each country is ‘unique’.

Summarizing, several studies have, to date, focused explicitly on geometric task design and entailed the use of technology. As the teaching of geometry is a multidimensional challenge, there is scope for more research on geometry teaching and tasks.

CLOSING REMARKS

Research on spatial reasoning has analysed different components, including perspective taking, rotation and mental transformation. Findings emerging from these investigations, mainly from tests or task-based interviews, both on younger and older ages, have concerned students’ and teachers’ capabilities in spatial understanding and processing. These capabilities improve over the age-range, but some individuals still retain vague conceptions of dimensions or space, and thus face spatial situations (even maps) with strategies that tend to be rather non-elaborated. These shortcomings are attributed to the lack of appropriate education and are generally improved by teaching proposals, especially when appropriate tasks and technological tools are implemented.

In terms of geometrical visualisation and visual thinking, there is evidence that, even though the role of visual process is particularly important in the learning and teaching of space and geometry, the number of investigations related to the visualizing capabilities of either students or teachers, or proposals for teaching interventions, has been somewhat limited. One reason for this could be the greater range of studies conducted in earlier years. Nevertheless, there remains a considerable interest in investigating visual processes in geometrical proving and problem solving, as well as a special concern about the use of gesture as an aspect of visualization.

In contrast to the somewhat limited development of research on geometrical visualisation and visual thinking, research continues to search for ways to improve the learning and teaching of geometric measurement. Using tests and interviews to examine conceptions about measurement of length, area or volume both on young and older students, research indicates low achievement and confusions regarding different aspects such as units, partition or iteration. Appropriate teaching proposals and relevant activities appear to improve measurement understanding.

Research into the teaching and learning of geometrical reasoning and proving continues apace, spurred by the increasing availability and sophistication of computer software. Studies with tests or interviews, mainly on secondary students, are attempting to connect proving processes to other capabilities or social practices and to identify predictors of proving skill. There is a special research interest in teaching proposals or use of relevant software with encouraging results regarding students' development in argumentation, generalization and proving. However, these results are only parts of a wide field of investigation. Constituting an important component of mathematical activity, geometric reasoning and proving requires further research in several under-researched issues.

Studies of students' geometric knowledge continue to form a main thrust in research on the teaching and learning of geometry, mainly based on the van Hiele model, Duval's figure apprehension framework, or other approaches related to identification of 2-D or 3-D geometric figures. Such research focuses on many of the key geometric ideas in the curriculum, and attempt to find connections with other mathematical issues (like spatial reasoning, visualization, proving or use of language). A systemization of the results in this field might be needed.

Paralleling the studies of students' geometric knowledge are studies of teachers' geometric knowledge and studies of teacher development for geometry education, indicating important figural and conceptual misunderstandings. Based on the same frameworks as with students, researching teachers' knowledge across different geometric ideas mainly indicates low understanding of geometry subject matter. This fact raises the need for an improvement of teachers' education and attracts the interest of several studies with proposals including relevant tasks, geometric software or teaching approaches.

Another rich vein of research in geometry education is that focusing on the teaching of geometry and the design and use of classroom tasks, especially the use of technology. Even so, research with proposals for appropriate teaching tasks remains somewhat limited and would benefit from further systematic investigation.

Some topics of research are under-represented. For example, there seems limited research explicitly on the topics of congruency and similarity, and little on transformation geometry. Research on analytic/coordinate geometry is also limited, as is research on vector geometry. On the positive side, research in geometry education is embracing the use of more recent discursive, embodied and eco-cultural perspectives, and is also employing new methods such as eye-tracking.

As research develops further, the affordance of digital technologies is enriching approaches to geometric and spatial teaching and learning by providing new ways of apprehension and representation, new manipulation and processes, wider and deeper conceptual understanding and linking of different meanings and treatments.

In general, results concerning the better understanding of how space and geometry are comprehended by students but also related to the development of effective teaching approaches, give opportunities for an enhanced access to relevant concepts and procedures. Moreover, the improvement of teachers' geometrical knowledge as well as their awareness of appropriate teaching methods, including the use of digital technology, develops the overall image. As mentioned previously, throughout the research effort, the systematization of findings and methods continues to be of great importance.

REFERENCES

Aaron, W. R. (2008). Academic identities of geometry students. *Proceedings of PME 32 and PME-NA 30*, *2*, 1–8.

Alatorre, S., & Saiz, M. (2009). Triangles' prototypes and teachers' conceptions. *Proceedings of PME 33*, *2*, 25–32.

Alatorre, S., Flores, P., & Mendiola, E. (2012). Primary teachers' reasoning and argumentation about the triangle inequality. *Proceedings of PME 36*, *2*, 3–10.

Alibali, M. W., Kita, S., & Young, A. (2000). Gesture and the process of speech production: We think, therefore we gesture. *Language and Cognitive Processes, 15*, 593–613.

Alqahtani, M., & Powell, A. (2015). Co-action and dynamic geometry knowledge. *Proceedings of PME 39*, *2*, 17–24.

Antonini, S. (2008). Indirect argumentations in geometry and treatment of contradictions. *Proceedings of PME 32 and PME-NA 30*, *2*, 73–80.

Arai, M. (2015). Japanese first grader's concept formation of geometric figures: Focusing on viewpoint changes while identifying figures. *Proceedings of PME 39*, *2*, 49–56.

Arzarello, F., & Edwards, L. (2005). Gesture and the construction of mathematical meaning. *Proceedings of PME 29*, *1*, 123–154.

Arzarello, F., Ferrara, F., Robutti, O., & Paola, D. (2005). The genesis of signs by gestures. *Proceedings of PME 29*, *2*, 73–80.

Aspinwall, L., Shaw, K. L., & Unal, H. (2005). How series problems integrating geometric and arithmetic schemes influenced prospective secondary teachers pedagogical understanding. *Proceedings of PME 29*, *2*, 89–96.

Baccaglini-Frank, A., Antonini, S., Leung, A., & Mariotti, M. A. (2011). Reasoning by contradiction in dynamic geometry. *Proceedings of PME 35*, *2*, 81–88.

Baccaglini-Frank, A., Mariotti, M. A., & Antonini, S. (2009). Different perceptions of invariants and generality of proof in dynamic geometry. *Proceedings of PME 33, 2,* 89–96.

Ball, D. L., & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.), *Proceedings of the 2002 annual meeting of the Canadian mathematics education study group* (pp. 3–14). Edmonton, Canada: CMESG/GDEDM.

Bara, B. G., & Tirassa, M. (1999). A mentalist framework for linguistic and extralinguistic communication. In S. Bagnara (Ed.), *Proceedings of the 3rd European conference on cognitive science* (pp. 285–290). Rome: IP-CNR.

Battista, M. T. (2007). The development of geometric and spatial thinking. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 843–908). Charlotte, NC: Information Age.

Battista, M. T., & Clements, D. H. (1996). Students' understanding of three-dimensional rectangular arrays of cubes. *Journal for Research in Mathematics Education, 27*(3), 258–292.

Bayazit, N., & Jakubowski, E. (2008). The use of geometric constructions to document pre-service mathematics teachers' geometric reasoning. *Proceedings of PME 32 and PME-NA 30, 1*, 238.

Beck, P. S., Eames, C. L., Cullen, C. J., Barret, J. E., Clements, D. H., & Sarama, J. (2014). Linking children's knowledge of length measurement to their use of double number lines. *Proceedings of PME 38 and PME-NA* 36, *2*, 105–112.

Bieda, K. N. (2011). Middle grades students' emerging beliefs about argumentation. *Proceedings of PME 35*, *2*, 153–160.

Biggs, J. B., & Collis, K. F. (1982). *Evaluating the quality of learning: The SOLO taxonomy*. New York, NY: Academic Press.

Brockmann-Behnsen, D., & Rott, B. (2014). Fostering the argumentative competence by means of a structured training. *Proceedings of PME 38 and PME-NA 36, 2,* 193–200.

Bryant, P. (2009). Understanding space and its representation in mathematics. In T. Nunes, P. Bryant, & A. Watson (Eds.), *Key understandings in mathematics learning*. London, UK: Nuffield Foundation.

Cabañas-Sánchez, M. G., & Cantoral-Uriza, R. (2010). Exploring the notions of comparison, conservation and measurement of the area in university students. A study through their arguments. *Proceedings of PME 34*, *2*, 241–248.

Chan, Y.-C. (2012). A mathematician's double semiotic link of a dynamic geometry software. *Proceedings of PME 36, 2,* 91–98.

Cheng, Y.-H., & Lin, F.-L. (2006). Using reading and colouring to enhance incomplete prover's performance in geometry proof. *Proceedings of PME 30, 2,* 289–296.

Cheng, Y.-H., & Lin, F.-L. (2007). The effectiveness and limitation of reading and colouring strategy in learning geometry proof. *Proceedings of PME 31, 2,* 113–120.

Cheng, Y.-H., & Lin, F.-L. (2008). A study on left behind students for enhancing their competence of geometry argumentation. *Proceedings of PME 32 and PME-NA 30, 2,* 305–312.

Chiang, P.-C., & Stacey, K. (2015). Geometric concepts of two-dimensional shapes by primary school teachers in Taiwan. *Proceedings of PME 39, 2,* 161–168.

Chino, K., Morozumi, T., Arai, H., Ogihara, F., Oguchi, Y., & Miyazaki, M. (2007). The effects of 'spatial geometry curriculum with 3-D DGS' in lower secondary school mathematics. *Proceedings of PME 31, 2,* 137–144.

Chorney, S. (2013). A post-humanist perspective on a geometric learning situation. *Proceedings of PME 37, 2,* 177–184.

Choy, B. H., Lee, M. Y., Mizzi, A. (2015). Textbook signature: An exploratory study of the notion of gradient in Germany, Singapore, and South Korea. *Proceedings of PME 39, 2,* 169–176.

Chumachemko, D., Shvarts, A., & Budanov, A. (2014). The development of the visual perception of the Cartesian coordinate system: An eye tracking study. *Proceedings of PME 38 and PME-NA 36, 2,* 313–320.

Cirillo, M. (2011). "I'm like the Sherpa guide": On learning to teach proof in school mathematics. *Proceedings of PME 35, 2,* 241–248.

Cohen, N. (2008). How do a plane and a straight line look like? Inconsistencies between formal knowledge and mental images. *Proceedings of PME 32 and PME-NA 30, 2,* 345–352.

Cooper, H., Hedges, L. V., & Valentine, J. C. (Eds.). (2009). *The handbook of research synthesis and meta-analysis* (2nd ed.). New York, NY: Russell Sage Foundation.

Cullen, C. J., & Barrett, J. E. (2010). Strategy use indicative of an understanding of units of length. *Proceedings of PME 34, 2,* 281–288.

Cullen, C. J., Miller, A. L., Barrett, J. E., Clements, D. H., & Sarama, J. (2011). Unit-eliciting task structures: Verbal prompts for comparative measures. *Proceedings of PME 35, 2,* 249–256.

Curry, M., Mitchelmore, M., & Outhred, L. (2006). Development of children's understanding of length, area, and volume measurement principles. *Proceedings of PME* 30, *2,* 377–384.

Davis, B., & the Spatial Reasoning Study Group. (2015). *Spatial reasoning in the early years*. New York, NY: Routledge.

De Bock, D., & Greer, B. (2008). The Isis problem: An instrument for examining future mathematics teachers' ideas about proof. *Proceedings of PME 32 and PME-NA 30, 1,* 337.

Deliyianni, E., Michael, P., Monoyiou, A., Gagatsis, A., & Elia, I. (2011). A composite model of students' geometrical figure understanding. *Proceedings of PME 35, 2,* 257–266.

Diezmann, C. M., & Lowrie, T. (2008). Assessing primary students' knowledge of maps. *Proceedings of PME 32 and PME-NA 30, 2,* 414–421.

Diezmann, C., & Lowrie, T. (2009). Primary students' spatial visualization and spatial orientation: An evidence base for instruction. *Proceedings of PME33, 2,* 417–424.

Dimmel, J. K., & Herbst, P. G. (2014). What details do geometry teachers expect in students' proofs? A method for experimentally testing possible classroom norms. *Proceedings of PME 38 and PME-NA 36, 2,* 393–400.

Dindyal, I. (2007). Teaching high school geometry: Perspectives from two teachers. *Proceedings of PME 31, 1,* 214.

Ding, L., & Jones, K. (2006). Students' geometrical thinking development at grade 8 in Shanghai. *Proceedings PME 30,* 1, 382.

Dohrmann, C., & Kuzle, A. (2014). Unpacking children's angle 'Grundvorstellungen': The case of distance 'Grundvortsellung' of 1° angle. *Proceedings of PME 38 and PME-NA 36, 2,* 409–416.

Dolev, S., & Even, R. (2012). Justifications and explanations in Israeli 7th grade math textbooks. *Proceedings of PME 36, 2,* 203–210.

Duval, R. (1999). Representation, vision and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. *Proceedings of PME 23, 1*, 3–26.

Dvora, T., & Dreyfus, T. (2011). Unjustified assumptions in geometry. *Proceedings of PME 35, 2*, 289–296.

Edwards, L. D. (1994). Making sense of a mathematical microworld: A pilot study from a logo project in Costa Rica. *Proceedings of PME 14, 2*, 235–242.

Edwards, L. D. (2005). The role of gestures in mathematical discourse: Remembering and problem solving. *Proceedings of PME* 29, *1*, 135–138.

Elia, I., Gagatsis, A., Deliyianni, E., Monoyiou, A., & Michael, S. (2009). A structural model of primary school students' operative apprehension of geometrical figures. *Proceedings of PME 33, 3*, 1–8.

Fernández, C., & de Bock, D. (2013). Does the confusion between dimensionality and 'directionality' affect students' tendency towards improper linear reasoning? *Proceedings of PME* 37, *2*, 297–304.

Ferrara, F., & Mammana, M. F. (2014). Seeing in space is difficult: An approach to 3-D geometry through a DGE. *Proceedings of PME 38 and PME-NA 36, 3*, 57–64.

Ferrara, F., & Ng, O.-L. (2014). Mathematical activities in a social learning framework. *Proceedings of PME 38 and PME-NA 36, 3*, 65–72.

Fielding-Wells, J., & Makar, K. (2015). "If it doesn't have an apex it's not a pyramid": Argumentation as a bridge to mathematical reasoning. *Proceedings of PME 39, 2*, 297–304.

Fischbein, E. (1993). The theory of figural concepts. *Educational Studies in Mathematics, 24*(2), 139–162.

Frade, C. (2005). The tacit-explicit nature of students' knowledge: A case study on area measurement. *Proceedings of PME* 29, *2*, 321–328.

Fuglestad, A. B., & Goodchild, S. (2009). I thought it was a proof. *Proceedings of PME 33, 1*, 379.

Fujita, T., & Jones, K. (2006). Primary trainee teachers' understanding of basic geometrical figures in Scotland. *Proceedings of PME 30, 3*, 129–136.

Fujita, T., Jones, K., & Kunimune, S. (2010). Students' geometrical constructions and proving activities: A case of cognitive unity. *Proceedings of PME 34*, *3*, 9–16.

Fujita, T., Jones, K., & Miyazaki, M. (2011). Supporting students to overcome circular arguments in secondary school mathematics: The use of the flowchart proof learning platform. *Proceedings of PME 35, 2*, 353–360.

Fujita, T., Kunimune, S., & Jones, K. (2014). The value of geometrical constructions: Discovering, reasoning and proving in geometry. *Proceedings of PME 38 and PME-NA 36, 6*, 65.

Gal, H., Lin, F.-L., & Ying, J.-M. (2006). The hidden side in Taiwanese classrooms: Through the lens of PLS in geometry. *Proceedings of PME 30, 3*, 145–152.

Gibbs, R. W. (2006). *Embodiment and cognitive science*. Cambridge, UK: Cambridge University Press.

Ginat, D., & Spiegel, H. (2015). On the absence of fluency and flexibility in novices' geometry proofs. *Proceedings of PME 39*, *2*, 329–336.

Gonulates, F., & Males, L. (2011). Textual expression of area measurement in elementary curricula: Illuminating opportunities to learn. *Proceedings of PME 35, 2*, 441–448.

González, E., & Guillén, G. (2008). Analysing a teaching model of the geometry of the solids in pre-service teachers' education. *Proceedings of PME 32, 1*, 342.

Gooya, Z., & Karamian, A. (2005). Teaching geometry in two secondary classrooms in Iran, using ethnomathematics approach. *Proceedings of PME 29, 1*, 240.

Gutiérrez, A., & Boero, P. (Eds.). (2006). *Handbook of research on the psychology of mathematics education. Past, present and future*. Rotterdam, The Netherlands: Sense Publishers.

Guven, B., & Okumus, S. (2011). 8th Grade Turkish students' van Hiele levels and classification of quadrilaterals. *Proceedings of PME 35, 2*, 473–480.

Hähkiöniemi, M. (2011). Teacher's experiences of a GeoGebra-enriched inquiry mathematics teaching unit. *Proceedings of PME 35*, *3*, 1–8.

Haja, S. (2005). Investigating the problem solving competency of pre service teachers in dynamic geometry environment. *Proceedings of PME 29, 3*, 81–88.

Haj-Yahya, A., & Hershkowitz, R. (2013). When visual and verbal representations meet the case of geometrical figures. *Proceedings of PME 37*, *2*, 409–416.

Haj-Yahya, A., Hershkowitz, R., & Dreyfus, T. (2014). Investigating students' geometrical proofs through the lens of students' definitions. *Proceedings of PME 38 and PME-NA 36, 3,* 217–224.
Hattermann, M. (2008). The dragging process in three dimensional dynamic geometry environments (DGE). *Proceedings of PME 32 and PME-NA 30, 3,* 129–136.
Hattermann, M. (2010). A first application of new theoretical terms on observed dragging modalities in 3-D dynamic-geometry-environments. *Proceedings of PME 34, 3,* 57–64.
Hegedus, S. J. (2013). Investigating how young children make sense of mathematical objects in a multimodal environment: A phenomenological approach. *Proceedings of PME 37, 3,* 33–40.
Hershkowitz, R. (1990). Psychological aspects of learning geometry. In P. Nesher & J. Kilpatrick (Eds.), *Mathematics and cognition* (pp. 70–95). Cambridge, UK: Cambridge University Press.
Highfield, K., Mulligan, J., & Hedberg, J. (2008). Early mathematics learning through exploration with programmable toys. *Proceedings of PME 32 and PME-NA 30, 3*, 169–176.
Hollebrands, K., Cayton, C., & Boehm, E. (2013). Pivotal teaching moments in a technology-intensive secondary geometry classroom. *Proceedings of PME 37, 3*, 73–80.
Hölzl, R. (1996). How does 'dragging' affect the learning of geometry? *International Journal of Computers for Mathematical Learning, 1*(2), 169–187.
Horne, M., & Watson, K. (2008). Developing understanding of triangle. *Proceedings of PME 32 and PME-NA 30, 3,* 177–184.
Huang, H.-M. E. (2010). The relationships between children's comprehension of multiplication and their understanding of perimeters and area measurement of rectangles. *Proceedings of PME 34*, *3*, 113–120.
Huang, H.-M. E. (2011). Fostering children's mathematical understanding of area measurement. *Proceedings of PME 35*, *3*, 41–48.
Huang, H.-M. E. (2012). An exploration of computer-based curricula for teaching children volume measurement concepts. *Proceedings of PME 36*, *2*, 315–322.
Huang, H.-M. E. (2015). Children‘s performance in estimating the measurements of daily objects. *Proceedings of PME 39, 3*, 73–80.
Huang, R. (2005). Verification or proof: Justification of Pythagoras' theorem in Chinese mathematics classrooms. *Proceedings of PME 29, 3*, 161–168.
Jacinto, H., & Carreira, S. (2013). Beyond-school mathematical problem solving: A case of students-with-media. *Proceedings of PME 37, 3*, 105–112.
Jones, K. (1996). Coming to know about dependency within a dynamic geometry environment. *Proceedings of PME 24, 3*, 145–151.
Jones, K. (2002). Issues in the teaching and learning of geometry. In L. Haggarty (Ed.), *Aspects of teaching secondary mathematics: Perspectives on practice* (pp. 121–139). London, UK: Routledge.
Jones, K., Fujita, T., & Kunimune, S. (2012). Representations and reasoning in 3-D geometry in lower secondary school. *Proceedings of PME 36*, *2*, 339–346.
Kalogirou, P., Elia, I., & Gagatsis, A. (2013). The relationship between visualization, spatial rotation, perceptual and operative apprehension. *Proceedings of PME 37, 3*, 129–136.
Kaur, H. (2013). Children's dynamic thinking in angle comparison tasks. *Proceedings of PME 37*, *3,* 145–152.
Kaur, H., & Sinclair, N. (2014). Young children's thinking about various types of triangles in a dynamic geometry environment. *Proceedings of PME 38 and PME-NA 36, 3,* 409–417.
Kim, J.-H., Lee, K.-H., Ko, E.-S., Park, M.-S., & Park, M.-M. (2009). Are gifted students aware of unjustified assumptions in geometric constructions? *Proceedings of PME 33, 3,* 337–345.
Komatsu, K. (2011). How do students generalize a conjecture through proving? The importance of boundary cases between example and counterexample. *Proceedings of PME 35*, *3*, 89–96.
Kondo, Y., Fujita, T., Kunimune, S., & Jones, K. (2013). Students' level of 3D geometrical thinking: The influence of 3D representations. *Proceedings of PME 37, 5*, 90.
Kondo, Y., Fujita, T., Kunimune, S., Jones, K., & Kumakura, H. (2014). The influence of 3D representations on students' level of 3D geometrical thinking. *Proceedings of PME 38 and PME-NA 36, 4,* 25–32.
Kospentaris, G., & Spyrou, P. (2009). Exploring students' strategies in comparison of area tasks. *Proceedings of PME 33*, *1*, 409.

Kuntze, S. (2008). Fostering geometrical proof competency by student-centred writing activities. *Proceedings of PME 32 and PME-NA 30, 3*, 289–296.

Latsi, M., & Kynigos, C. (2011). Meanings about dynamic aspects of angle while changing perspectives in a simulated 3-D space. *Proceedings of PME 35, 3*, 121–128.

Lavy, I., & Shriki, A. (2012). Engaging prospective teachers in the assessment of geometrical proofs. *Proceedings of PME 36, 3*, 35–42.

Lecluse, A. (2005). *Geocadabra* [Computer software]. Retrieved from http://www.geocadabra.nl

Lee, A., & Leung, A. (2012). Students' understanding of geometric properties experienced in a dynamic geometry environment. *Proceedings of PME 36, 3*, 59–66.

Lee, K., Kim, M., Na, G., Han, D., & Song, S. (2007). Induction, analogy, and imagery in geometric reasoning. *Proceedings of PME 31, 3*, 145–152.

Lee, K. H. (2005). Mathematically-gifted students' geometrical reasoning and informal proof. *Proceedings of PME 29, 3*, 241–248.

Lee, K. H., Ko, E., & Song, S. (2007). The analysis of activity that gifted students construct definition of regular polyhedral. *Proceedings of PME 31, 3*, 153–160.

Lei, K.-H., Tso, T.-Y., & Lu, F.-L. (2012). The effect of worked-out examples with practice on comprehending geometry proof. *Proceedings of PME 36, 3*, 75–82.

Leung, A. (2014). Principles of acquiring invariant in mathematics task design: A dynamic geometry example. *Proceedings of PME 38 and PME-NA 36, 4*, 89–96.

Leung, A., & Or, A. (2009). Cognitive apprehensions in Cabri 3-D environment. *Proceedings of PME 33, 1*, 417.

Leung, A., & Or, C. M. (2007). From construction to proof: Explanations in dynamic geometry environment. *Proceedings of PME 31, 3*, 177–184.

Leung, F., & Park, K. (2009). The influence of language on the conception of geometric figures. *Proceedings of PME 33, 1*, 418.

Lew, H.-C., & Yoon, O. (2013). Study on the learning of quadratic equation through proportion in a dynamic environment. *Proceedings of PME 37, 3*, 249–256.

Lin, M.-L., & Wu, C.-J. (2007). Uses of examples in geometric conjecturing. *Proceedings of PME 31, 3*, 209–216.

Lin, T.-W., Wu, C.-J., & Sommers, S. (2012). The influence of reading figures in geometry proof on eye movement. *Proceedings of PME 36, 3*, 147–152.

Lowrie, T., Diezmann, C., & Logan, T. (2011). Primary students' performance on map tasks: The role of context. *Proceedings of PME 35, 3*, 145–152.

Ma, H.-L., Wu, D.-B., Chen, J. W., & Hsieh, K.-J. (2009). Mitchelmore's development stages of the right rectangular prisms of elementary school students in Taiwan. *Proceedings of PME 33, 4*, 57–62.

Maier, A. S., & Benz, C. (2014). Children's conceptual knowledge of triangles manifested in their drawings. *Proceedings of PME 38 and PME-NA 36, 4*, 154–161.

Markovits, Z., Rosenfeld, S., & Eylon, B.-S. (2006). Visual cognition: Content knowledge and beliefs of pre-school teachers. *Proceedings of PME 30, 4*, 145–152.

Martignone, F. (2011). Tasks for teachers in mathematics laboratory activities: A case study. *Proceedings of PME 35, 3*, 193–200.

Martignone, F., & Antonini, S. (2009). Exploring the mathematical machines for geometrical transformations: A cognitive analysis. *Proceedings of PME 33, 4*, 105–112.

Maschietto, M., & Bartolini Bussi, M. G. (2005). Meaning construction through semiotic means: The case of the visual pyramid. *Proceedings of PME 29, 3*, 313–320.

Masuda, Y. (2009). Exploring students' understanding of angle measure. *Proceedings of PME 33, 1*, 424.

Matos, J. F., & Rodrigues, M. (2011). Proof in classroom social practice. *Proceedings of PME 35, 3*, 177–184.

Matsuo, N. (2007). Differences of students' understanding of geometric figures based on their definitions. *Proceedings of PME 31, 1*, 264.

McDonough, A. (2010). Foundational ideas of measure: Exploring length understandings. *Proceedings of PME 34, 3*, 289–296.

McDonough, A., Cheeseman, J., & Ferguson, S. (2012). Insights into children's understandings of mass measurement. *Proceedings of PME 36, 3,* 201–208.
Mitchelmore, M. C. (1978). Developmental stages in children's representation of regular solid figures. *The Journal of Genetic Psychology, 133,* 229–239.
Miyakawa, T. (2012). Proof in geometry: A comparative analysis of French and Japanese textbooks. *Proceedings of PME 36, 3,* 225–232.
Miyakawa, T., & Herbst, P. (2007). The nature and role of proof when installing theorems: The perspective of geometry teachers. *Proceedings of PME 31, 3,* 281–288.
Miyazaki, M., Fujita, T., & Jones, K. (2014). Functions of open flow-chart proving in introductory lessons of formal proving. *Proceedings of PME 38 and PME-NA 36, 4,* 225–232.
Moore-Russo, D., & Mudaly, V. (2011). South African teachers' common content knowledge of gradient. *Proceedings of PME 35, 3,* 241–248.
Morgan, S., & Sack, J. (2011). Learning creatively with giant triangles. *Proceedings of PME 35, 3,* 249–256.
Moustaki, F., & Kynigos, C. (2011). Engineering students' visualization and reasoning processes while interacting with a 3-D digital environment. *Proceedings of PME 35, 3,* 257–264.
Ng, O.-L. (2014). The interplay between language, gestures, dragging and diagrams in bilingual learners' mathematical communication. *Proceedings of PME 38 and PME-NA 36, 4,* 289–296.
Ng, O.-L., & Sinclair, N. (2013). Gestures and temporality: Children's use of gestures on spatial transformation tasks. *Proceedings of PME 37, 3,* 361–368.
Oikonomou, A., & Tzekaki, M. (2005). Improving spatial representations in early childhood. *Proceedings of PME 29, 1,* 268.
Okazaki, M. (2009). Process and means of reinterpreting tacit properties in understanding the inclusion relations between quadrilaterals. *Proceedings of PME 33, 4,* 249–236.
Okazaki, M. (2013). Identifying situations for fifth graders to construct definitions as conditions for determining geometric figures. *Proceedings of PME 37, 3,* 409–416.
Okazaki, M., & Fujita, T. (2007). Prototype phenomena and common cognitive paths in the understanding of the inclusion relations between quadrilaterals in Japan and Scotland. *Proceedings of PME 31, 4,* 41–48.
Olivero, F. (2006). Hiding and showing construction elements in a dynamic geometry software: A focusing process. *Proceedings of PME 30, 4,* 273–280.
Olvera, F., Guillén, G., & Figueras, O. (2008). Solid geometry in elementary school in Mexico. *Proceedings of PME 32, 1,* 298.
Owens, K. (2005). Substantive communication of space mathematics in upper primary school. *Proceedings of PME 29, 4,* 33–40.
Owens, K. (2015). A cultural perspective on visuospatial reasoning: An overview of 40 years of research. *Proceedings of PME 39, 1,* 193.
Owens, K., & Kaleva, W. (2008). Cases studies of mathematical thinking about area in Papua New Guinea. *Proceedings of PME 32 and PME-NA 30, 4,* 73–89.
Owens, K., & Outhred, L. (2006). The complexity of learning geometry and measurement. In A. Gutiérrez, & P. Boero (Eds.), *Handbook of research on the psychology of mathematics education: Past, present and future* (pp. 83–116). Rotterdam, The Netherlands: Sense Publishers.
Pakang, J., & Kongtaln, P. (2007). Creative thinking and mathematics learning about circle of ninth-grade students through art work. *Proceedings of PME 31, 1,* 271.
Paksu, A. D. (2009). Factors that predict geometry self-efficacy of pre-service elementary teachers. *Proceedings of PME 33, 1,* 368.
Panorkou, N., & Pratt, D. (2009). Mapping experience of dimension. *Proceedings of PME 33, 4,* 281–288.
Panorkou, N., & Pratt, D. (2011). Using Google sketchup to research children's experience of dimension. *Proceedings of PME 35, 3,* 337–344.
Patsiomitou, S. (2011). Theoretical dragging: A non-linguistic warrant leading to 'dynamic' propositions. *Proceedings of PME 35, 3,* 361–368.
Patsiomitou, S., & Emvalotis, A. (2010). The development of students' geometrical thinking through a DGS reinvention process. *Proceedings of PME 34, 4,* 33–40.

Patsiomitou, S., & Koleza, E. (2008). Developing students' geometrical thinking through linking representations in a dynamic geometry environment. *Proceedings of PME 32 and PME-NA 30, 4*, 89–96.
Peirce, C. S. (1955). Logic as semiotic: The theory of signs. In J. Buchler (Ed.), *Philosophical writings of Peirce* (Chapter 7). New York, NY: Dover.
Pittalis, M., Mousoulides, N., & Christou, C. (2009). Levels of sophistication in representing 3-D shapes. *Proceedings of PME 33*, 4, 385–392.
Pitta-Pantazi, D., & Christou, C. (2009). Mathematical creativity and cognitive styles. *Proceedings of PME 33*, *4*, 377–384.
Presmeg, N. (2006). A semiotic view of the role of imagery and inscriptions in mathematics teaching and learning. *Proceedings of PME 30, 1*, 19–34.
Presmeg, N., Barrett, J., & McCrone, S. (2007). Fostering generalization in connecting registers of dynamic geometry and Euclidean constructions. *Proceedings of PME 31*, *4*, 81–88.
Psycharis, G. (2006). Dynamic manipulation schemes of geometrical constructions: Instrumental genesis as an abstraction process. *Proceedings of PME 30, 4*, 385–392.
Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical Thinking and Learning*, *5*(1), 37–70.
Ramfull, A., & Lowrie, T. (2015). Cognitive style, spatial visualization and problem solving performance: Perspectives from grade 6 students. *Proceedings of PME 39*, *4*, 57–64.
Reinhold, S., Beutler, B., & Merschmeyer-Brüwer, C. (2013). Pre-schoolers count and construct: Spatial structuring and its relation to building strategies in enumeration-construction tasks. *Proceedings of PME 37*, *4*, 81–88.
Roth, W.-M. (2001). Gestures: Their role in teaching and learning. *Review of Educational Research*, *71*(3), 365–392.
Royal Society. (2001). *Teaching and learning geometry 11–19*. London: Royal Society.
Ruwisch, S., Heid, M., & Weiher, D. F. (2015). Measurement estimation in primary school: Which answer is adequate. *Proceedings of PME 39*, *4*, 113–120.
Ryu, H., Chong, Y., & Song, S. (2007). Mathematically-gifted students' spatial visualization ability of solid figures. *Proceedings of PME 31*, *4*, 137–144.
Sack, J., & Vazquez, I. (2008). Three-dimensional visualization: Children's non-conventional verbal representations. *Proceedings of PME 32 and PME-NA 30, 4*, 217–224.
Sack, J., & Vazquez, I. (2011). Development of a learning trajectory to conceptualize and represent volume using top-view coding. *Proceedings of PME 35*, *4*, 89–96.
Sack, J., Vazquez, I., & Moral, R. (2010). Elementary children's 3-D visualization development: Representing top-views. *Proceedings of PME 34*, *4*, 113–120.
Safuanov, I. S. (2007). Genetic approach to teaching geometry. *Proceedings of PME 31*, *4*, 145–152.
Samper, C., Camargo, L., Perry, P., & Molina, Ó. (2012). Dynamic geometry, implication and abduction: A case study. *Proceedings of PME 36*, *4*, 43–50.
Serow, P. (2006). Triangle property relationships. *Proceedings of PME 30, 5*, 89–96.
Sfard, A. (2008). *Thinking as communicating: Human development, the growth of discourses, and mathematizing*. Cambridge, UK: Cambridge University Press.
Silfverberg, H., & Joutsenlahti, J. (2014). Prospective teachers' conceptions about a plane angle and the context dependency of the conceptions. *Proceedings of PME 38 and PME-NA 36*, *5*, 185–192.
Silfverberg, H., & Matsuo, N. (2008). Comparing Japanese and Finnish 6th and 8th graders' ways to apply and construct definitions. *Proceedings of PME 32 and PME-NA 30, 4*, 257–264.
Sinclair, N., & Bruce, C. D. (2014). Spatial reasoning for young learners. *Proceedings of PME 38 and PME-NA 36*, *1*, 173–203.
Sinclair, N., & Kaur, H. (2011). Young children's understanding of reflectional symmetry in a dynamic geometry environment. *Proceedings of PME 35*, *4*, 193–200.
Sinclair, N., Moss, J., & Jones, K. (2010). Developing geometric discourse using DGS in K-3. *Proceedings of PME 34*, *4*, 185–192.
Soares, E. S. (2010). A geometry class conducted by two teachers with different and complementary perspectives. *Proceedings of PME 34*, *4*, 201–208.

Sollervall, H. (2012). From Euclid to GPS: Designing an innovative spatial coordination activity with mobile technologies. *Proceedings of PME 36, 4*, 107–114.
Son, J.-W. (2006). Investigating pre-service teachers' understanding and strategies on a student's errors of reflective symmetry. *Proceedings of PME 30, 5*, 145–152.
Song, S., Chong, Y., Yim, J., & Chang, H. (2006). Mathematically-gifted 6th grade Korean students' proof level for a geometric problem. *Proceedings of PME 30*, *1*, 334.
Sophocleous, P., Kalogirou, P., & Gagatsis, A. (2009). Creative ability and criteria in recognizing geometric figures. *Proceedings of PME 33, 5*, 153–160.
Stenkvist, A. (2012). Comparing realistic geometric pictures. *Proceedings of PME 36, 4*, 123–130.
Stephanou, L., & Pitta-Pantazi, D. (2006). The impact of the intuitive rule 'if A then B, if not A then not B' in perimeter and area tasks. *Proceedings PME 30, 5*, 177–184.
Tall, D. O., & Vinner, S. (1981). Concept image and concept definition in mathematics, with special reference to limits and continuity. *Educational Studies in Mathematics*, *12*(2), 151–169.
Tatsis, K., & Moutsios-Rentzos, A. (2013). Pre-service teachers describe geometrical figures: The `broken phone' revisited. *Proceedings of PME 37, 4*, 265–272.
Thurstone, L. L. (1950). *Some primary mental abilities in visual thinking*. Chicago, IL: University of Chicago Press.
Tomaz, V. S., & David, M. M. (2011). Classroom activity promoting students' learning about the use of drawings in geometry. *Proceedings of PME 35*, *4*, 257–264.
Torregrosa, G., & Quesada, H. (2008). The coordination of cognitive processes in solving geometric problems requiring formal proof. *Proceedings of PME 32 and PME-NA 30, 4*, 321–328.
Torregrosa, G., & Quesada, H. (2009). Factors limiting configural reasoning in geometrical proof. *Proceedings of PME 33*, *1*, 477.
Tzekaki, M., & Ikonomou, A. (2009). Investigating spatial representations in early childhood. *Proceedings of PME 33, 5*, 241–248.
Ubuz, B. (2006). Student conceptions and textbook messages: Polygons. *Proceedings of PME 30*, *1*, 347.
Ufer, S., Heinze, A., & Reiss, K. (2008). Individual predictors of geometrical proof competence. *Proceedings of PME 32 and PME-NA 30*, *4*, 361–368.
van der Sandt, S. (2005). The state and impact of geometry pre-service preparation: Possible lessons from South Africa. *Proceedings of PME 29, 1*, 288.
Watson, A., & Ohtani, M. (2015). Themes and issues in mathematics education concerning task design. In A. Watson & M. Ohtani (Eds.), *Task design in mathematics education: An ICMI study* (pp. 3–15). Berlin: Springer.
Watson, A., Jones, K., & Pratt, D. (2013). *Key ideas in teaching mathematics: Research-based guidance for ages 9–19*. Oxford, UK: Oxford University Press.
Whiteley, W., Sinclair, N., & Davis, B. (2015). What is spatial reasoning? In B. Davis & the Spatial Reasoning Study Group (Eds.), *Spatial reasoning in the early years* (pp. 3–14). New York, NY: Routledge.
Widder, M., Berman, A., & Koichu, B. (2014). Dismantling visual obstacles to comprehension of 2-D sketches depicting 3-D objects. *Proceedings of PME 38 and PME-NA 36, 5*, 369–376.
Wu, C.-J., Wong, W.-K., Cheng, Y.-H., & Lien, Y.-W. (2006). Generating and evaluating geometry conjectures with self-directed experiments. *Proceedings of PME 30, 5*, 401–408.
Wu, D.-B., & Ma, H.-L. (2005). A study of the geometric concepts of the elementary school students who are assigned to the van Hiele level one. *Proceedings of PME 29, 4*, 329–336.
Wu, D.-B., & Ma, H.-L. (2006). The distributions of van Hiele levels of geometric thinking among 1st through 6th graders. *Proceedings of PME 30, 5*, 409–416.
Wu, D.-B., Ma, H.-L., & Chen, D.-C. (2006). The developmental stages of representations of simple regular space figures of elementary school students. *Proceedings of PME 30, 1*, 430.
Wu, D.-B., Ma, H.-L., Hsieh, K.-J., & Li, Y.-F. (2007). A study of the concept of solid geometry of elementary students from the 4th grades to 6th grades in the central region of Taiwan. *Proceedings of PME 31*, *1*, 295.
Xistouri, X., & Pitta-Pantazi, D. (2006). Spatial rotation and perspective taking abilities in relation to performance in reflective symmetry tasks. *Proceedings of PME 30, 5*, 425–432.

Xu, S.-Y., & Tso, T.-Y. (2009). The interaction among dragging actions, dynamic representations and thought experiments in dynamic geometry environment. *Proceedings of PME 33*, *1*, 491.
Yang, K.-L., Lin, F.-L., & Wang, Y.-T. (2007). Reading strategies for comprehending geometry proof. *Proceedings of PME 31*, *1*, 333.
Yevdokimov, O. (2008). Making generalizations in geometry: Students' views on the process. A case study. *Proceedings of PME 32 and PME-NA 30*, *4*, 441–448.
Zaslavsky, O., Harel, G., & Manaster, A. (2006). A teacher's treatment of examples as reflection of her knowledge-base. *Proceedings of PME 30*, *5*, 457–464.
Zodik, I., & Zaslavsky, O. (2007). Is a visual example in geometry always helpful? *Proceedings of PME 31*, *4*, 265–272.

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