Supplemental section: Emergence of correlated optics in one-dimensional waveguides for classical and quantum atomic gases

Janne Ruostekoski[1] and Juha Javanainen[2]

[1]*Mathematical Sciences, University of Southampton, Southampton SO17 1BJ, United Kingdom*
[2]*Department of Physics, University of Connecticut, Storrs, Connecticut 06269-3046*

In this supplemental section we present the basic one-dimensional (1D) electrodynamics in a waveguide and show how it can be solved using stochastic simulations. We introduce the basic relations for the transfer matrices. Moreover, we explain the suppression of light-induced correlations in the presence of inhomogeneous broadening and density fluctuations in terms of the transfer matrix correlations.

One-dimensional electrodynamics

Hierarchy of equations of motion

For instance, in situations where the atoms are strongly confined close to the centre of the waveguide or when an effective two-level system is obtained from the $J = 0 \to J' = 1$ system, we may consider the 1D scalar electrodynamics for the coupled system of atoms and light by renormalizing the fields, as $\pi\xi_\varrho^2 \mathbf{E}^+(\mathbf{r}) \to \tilde{E}^+(x)$, $\pi\xi_\varrho^2 \mathbf{D}_F^+(\mathbf{r}) \to \tilde{D}_F^+(x)$. The total electric field amplitude is the sum of the incident and the scattered fields

$$\epsilon_0 \tilde{E}^+(x) = \tilde{D}_F^+(x) + \frac{ik}{2}\int dx' \, e^{ik|x-x'|} P^+(x'), \quad (1)$$

where $k = \Omega/c$ and Ω is the frequency of the incident driving field. For the coherently scattered light we take the expectation values of Eq. (1) and assume that the monochromatic incident field is in a coherent state. Noting that $ie^{ik|x|}/(2k)$ is the the Green's function of the 1D Helmholtz differential operator,

$$(\nabla^2 + k^2)\langle \tilde{D}_F^+ \rangle = 0, \quad (\nabla^2 + k^2)\frac{i}{2k}e^{ik|x|} = -\delta(x), \, (2)$$

we can transform the integral equation (1) to a differential equation

$$(\nabla^2 + k^2)\langle \tilde{D}^+ \rangle = \nabla^2 \langle P^+ \rangle, \quad (3)$$

which is the 1D Maxwell's wave equation in a polarizable medium [1].

The atomic polarization acts as a radiation source in Eq. (1) and in second quantization is expressed as $P^+(x) = \mathcal{C}_{ge}\mathcal{D}\,\psi_g^\dagger(x)\psi_e(x)$ in terms of the atomic field operators for the electronic ground $[\psi_g(x)]$ and excited $[\psi_e(x)]$ states, the Clebsch-Gordan coefficient $\mathcal{C}_{ge}$, and the reduced dipole matrix element $\mathcal{D}$. In order to solve the coupled theory for atoms and light, we need the

Heisenberg's equations of motion for the atomic polarization operator $P^+(x)$. This can be derived from the interaction term between the atomic polarization and the electric displacement field $P(x)\tilde{D}(x)/\epsilon_0$. For the expectation value $P_1(x) = \langle P^+(x) \rangle$ we obtain

$$\dot{P}_1(x) =(i\delta - \gamma_t)P_1(x) + \frac{2i\gamma_w}{k}\rho\tilde{D}_F^+(x)$$
$$- \gamma_w \int dx' e^{ik|x-x'|} P_2(x;x'). \quad (4)$$

Here $\delta = \Omega - \omega_0$ denotes the detuning of the light frequency Ω from atomic resonance frequency ω_0 and the radiative inewidth is given by

$$\gamma_t = \gamma_l + \gamma_w, \quad (5)$$

where γ_l is the radiative loss rate from the waveguide and

$$\gamma_w = k\mathcal{D}^2/2\pi\xi_\rho^2\hbar\epsilon_0 \quad (6)$$

the decay rate into the waveguide, with the characteristic radial length scale ξ_ρ. Equation (4) is derived in the low light intensity limit, i.e., to first order of the amplitude D, and correspondingly keeping terms that include at most one of the fields $D, \psi_e, \psi_e^\dagger$ [2]. In the low light intensity limit the ground-state atom density ρ is not changed by the driving light, and has the same value as before the light enters the sample. Light-mediated dipole-dipole interactions induce correlations between the atoms that depend on their positions. Polarization depends on the two-atom correlation function

$$P_2(x_1;x_2) \equiv \langle \psi_g^\dagger(x_1)P^+(x_2)\psi_g(x_1) \rangle. \quad (7)$$

P_2 describes correlations between a ground-state atom at x_1 and the polarization at x_2. Similarly, the equation for P_2 is coupled to a three-atom correlation function

$$\dot{P}_2(x_1;x_2) =(i\delta - \gamma_t)P_2(x_1;x_2) - \gamma_w e^{ik|x_1-x_2|}P_2(x_2;x_1)$$
$$+ \frac{2i\gamma_w}{k}\rho(x_1,x_2)\tilde{D}_F^+(x_2)$$
$$- \gamma_w \int dx_3 \, e^{ik|x_2-x_3|} P_3(x_1,x_2;x_3), \quad (8)$$

where

$$P_3(x_1,x_2;x_3) \equiv \langle \psi_g^\dagger(x_1)\psi_g^\dagger(x_2)P^+(x_3)\psi_g(x_2)\psi_g(x_1) \rangle, \quad (9)$$

represents a polarization density at x_3, given that there are ground-state atoms at x_1 and x_2. The three-atom

correlation function P_3 in turn depends on four-atom correlations, which are coupled to five-atom correlations, etc., leading to the hierarchy of equations for the correlation functions that involve the polarization and an increasing number of ground-state atom densities.

The ground-state atom pair correlation function

$$\rho(x_1, x_2) \equiv \langle \psi_g^\dagger(x_1)\psi_g^\dagger(x_2)\psi_g(x_2)\psi_g(x_1) \rangle, \qquad (10)$$

in the low light-intensity limit is unaffected by the driving light and corresponds to the initial correlations between the atoms in the absence of the driving light.

Generally, we define normally ordered mixed polarization-atom density correlation functions and correlation functions of density for ground state atoms as

$$P_l(x_1, \cdots, x_{l-1}; x_l)$$
$$\equiv \langle \psi_g^\dagger(x_1) \cdots \psi_g^\dagger(x_{l-1}) P^+(x_l) \psi_g(x_{l-1}) \cdots \psi_g(x_1) \rangle,$$
$$(11)$$
$$\rho_l(x_1, \cdots, x_l) \equiv \langle \psi_g^\dagger(x_1) \cdots \psi_g^\dagger(x_l) \psi_g(x_l) \cdots \psi_g(x_1) \rangle,$$
$$(12)$$

with $l = 1, 2, \ldots$. The hierarchy of equations for the correlation functions may then be written in the following form

$$\dot{P}_l(x_1, \cdots, x_{l-1}; x_l) = (i\delta - \gamma_t) P_l(x_1, \cdots, x_{l-1}; x_l)$$
$$-\gamma_w \sum_{k=1}^{l-1} e^{ik|x_l - x_k|} P_l(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_l; x_k)$$
$$+ \frac{2i\gamma_w}{k} \rho_l(x_1, \cdots, x_l) \tilde{D}_F^+(x_l)$$
$$-\gamma_w \int dx_{l+1} \, e^{ik|x_l - x_{l+1}|} P_{l+1}(x_1, \cdots, x_l; x_{l+1}). \quad (13)$$

The terms in the sum on the right-hand side of Eqs. (13) represent recurrent scattering processes in which the l atoms at $x_1, \ldots, x_l$ repeatedly exchange photons. Such processes are the microsocopic mechanism for collective linewidths and line shifts. The integral stands for a process in which yet another atom shines its light on the atom at x_l.

Stochastic simulations

In the limit of low light intensity, the hierarchy of equations of motion for the correlation functions (13) represents coupled weakly excited linear atomic dipoles. Although the recurrent scattering can induce strong correlations between the atoms, in a two-level atomic ensemble the hierarchy can, nevertheless, be entirely understood by classical electrodynamics. Indeed, the hierarchy of equations for the correlation functions can be solved exactly by means of classical stochastic simulations where the positions of the atoms are sampled from a probabilistic ensemble that corresponds to the position correlations between the atoms in the absence of the driving light. The technique fully accounts for the recurrent scattering processes to all orders.

If the atoms are initially uncorrelated before the light enters the medium, as for classical atoms or for an ideal Bose-Einstein condensate, the stochastic position of each atom is sampled independently from the positions of the other atoms. For the zero-temperature fermionic position correlations (for an ideal gas of fermionic atoms or for a strongly interacting Tonks gas of bosonic atoms) the initial positions of the atoms are given by the joint probability distribution equal to the absolute square of the Fermi-Dirac many-atom wave function. The correlated positions may then be sampled using the Metropolis algorithm [3].

In each stochastic realization the optical response of a system of classical electrodynamics equations for point dipoles is solved exactly for a fixed set of atomic positions. The dynamics of the hierarchy of correlation functions (13) can then be reproduced to all orders by ensemble-averaging over many realizations of spatial positions that are stochastically sampled from the joint many-body probability distribution [4].

Specifically, in each stochastic realization of discrete atomic positions $\{x_1, x_2, \ldots, x_N\}$, we solve for the coupled set of point dipoles that account for the polarization density $\sum_j \mathfrak{P}^{(j)} \delta(x - x_j)$, where $\mathfrak{P}^{(j)}$ denotes the excitation dipole of the atom j. The coupled-dipole equations for the steady-state solution in the present system read

$$\mathfrak{P}^{(j)} = \alpha \tilde{D}_F^+(x_j) + \eta_\delta \sum_{l \neq j} e^{ik|x_j - x_l|} \mathfrak{P}^{(l)}, \qquad (14)$$

where each dipole amplitude is driven by the incident field and the scattered field from all the other $N-1$ dipoles. Once all $\mathfrak{P}^{(j)}$ are calculated, we may obtain in each realization the expression for the scattered light from the induced dipole excitations

$$\epsilon_0 \tilde{E}^+(x) = \tilde{D}_F^+(x) + \frac{ik}{2} \sum_l e^{ik|x - x_l|} \mathfrak{P}^{(l)}. \qquad (15)$$

Finally, we evaluate the ensemble average of the optical response over many stochastic realizations of atomic positions.

Truncation of hierarchy and "cooperative Lamb shift"

We may obtain a MFT solution to the transmitted light by ignoring recurrent scattering processes and any light-induced correlations. In the main text this was first done for N atoms by calculating the product of the N independent single-atom transmission results. Alternatively, we may obtain effective medium MFT solutions

directly from the hierarchy of equations for the correlation functions by means of an appropriate truncation of the hierarchy. The lowest-order recurrent scattering process where an atom pair is involved in the repeated photon exchanges appear in the equation for $P_2(x; x')$. We may ignore these, and any correlation effects between a ground-state atom x and a polarization at x', by factorizing the two-atom correlation function,

$$P_2(x_1; x_2) = \rho(x_1) P_1(x_2). \quad (16)$$

When Eq. (16) is substituted in the steady-state solution for P_1 [Eq. (4) in the main text], or to Eq. (4), we obtain a closed equation for P_1,

$$P_1(x) = \alpha \rho \tilde{D}_F^+(x) + \eta_\delta \rho(x) \int dx' e^{ik|x-x'|} P_1(x'). \quad (17)$$

The decorrelation approximation (16) is an effective medium theory where each atom interacts with the average behavior of its neighboring atoms and the position correlations due to the dependence of the dipole-dipole interactions on the relative interatomic separation $x_1 - x_2$ is ignored. The optical response that deviates from the solution of Eq. (17) therefore indicates a nonvanishing value for light-induced two-atom correlations

$$\delta P_2(x_1; x_2) \equiv P_2(x_1; x_2) - \rho(x_1) P_1(x_2). \quad (18)$$

In a 1D waveguide we can solve (17) for the atoms filling the region $x \in [0, L]$ by substituting $P_1(x) = P_+ \exp(ik'x) + P_- \exp(-ik'x)$, for $0 \le x \le L$. Here k' takes complex values [$\text{Im}(k') > 0$], representing a damped plane wave propagation. The comparison between Eq. (17) and the expectation value of Eq. (1) yields $\alpha \rho \epsilon_0 \langle \tilde{E}^+(x) \rangle = P_1(x)$. By means of the refractive index $n = k'/k$, we can then express the electric susceptibility as

$$\chi = n^2 - 1 = \alpha \rho. \quad (19)$$

Note that, unlike in the corresponding 3D system, the Lorentz-Lorenz local field correction is absent here.

In the low atom-density limit in a 3D system, the factorization analogous to Eq. (16) leads to the familiar expression for the "cooperative Lamb shift" (CLS) that was originally calculated by Friedberg, Hartmann and Manassah for atoms in a 3D slab geometry [5], and was recently experimentally verified in a hot atom vapor [6]. Here we calculate CLS and the corresponding resonance width for the 1D waveguide by solving Eq. (17), subject to Eq. (19). The low atom density expansion of MFT yields

$$\Delta_{\text{CLS}} = \frac{\gamma_w \rho}{2k} \left(1 - \frac{\sin 2Lk}{2Lk} \right) + \mathcal{O}[\gamma_w^2 \rho^2/(k^2 \gamma_t)]. \quad (20)$$

The oscillatory behavior results from the etalon interference effects of the sample thickness. Figure 3 in the main text shows example cases for the comparison between the CLS and the resonance shift of the full numerical solution. With carefully chosen parameter values the result is close to the CLS shift (20). However, when, for instance, the optical thickness of the sample is increased, the oscillatory behavior is absent in the full numerical solution and the CLS result qualitatively fails.

The resonance HWHM width Γ_{FHM} can also be derived from the effective continuous medium MFT model to the same order in the density expansion as CLS. We find

$$\Gamma_{\text{FHM}} = \gamma_t \sqrt{1 + \frac{\gamma_w \rho}{\gamma_t k} \left(\frac{1 + 2L^2 k^2 - \cos(2Lk)}{2Lk} \right)}. \quad (21)$$

For thin atomic ensembles we may also expand the square root,

$$\Gamma_{\text{FHM}} = \gamma_t + \frac{\gamma_w \rho}{k} \left(\frac{1 + 2h^2 k^2 - \cos(2hk)}{4hk} \right). \quad (22)$$

Transfer matrix solutions

Basic relations

In this section we provide a description for the light-induced correlations between the atoms in light propagation and how they depend on the density of the atoms and their thermal distribution. The analysis utilizes transfer matrix theory where we adapt theoretical methods of localization analysis in transport phenomena [7] that were originally developed for describing electric conductivity in 1D wires.

For fixed atomic positions the atomic excitations can be solved in the low light-intensity limit from Eq. (14). The total electric field amplitude is then obtained by substituting the solution into Eq. (15). In order to analyze the transmission of light by one atom we rewrite Eqs. (15) and (14) in the vicinity of the atom $l = 1$ as

$$\epsilon_0 \tilde{E}^+(x) = \epsilon_0 \tilde{E}_{\text{ext}}^+(x) + \frac{ik}{2} e^{ik|x-x_1|} \mathfrak{P}^{(1)}, \quad (23)$$

$$\mathfrak{P}^{(1)} = \alpha \epsilon_0 \tilde{E}_{\text{ext}}^+(x_1), \quad (24)$$

where $\tilde{E}_{\text{ext}}^+(x_1)$ represents the incident field plus the scattered fields by all the other atoms in the ensemble. We separate the external fields that propagate from the negative x direction ($x_- < x_1$) to the atom and those that propagate from the positive direction ($x_+ > x_1$) to the atom

$$\tilde{E}_{\text{ext}}^+(x_1) = \tilde{E}_{\text{ext},-}^+(x_-) e^{ik(x_1-x_-)} + \tilde{E}_{\text{ext},+}^+(x_+) e^{-ik(x_1-x_+)} \quad (25)$$

Similarly, we separate the total field $\tilde{E}^+(x)$ into the field propagating in the positive x direction *away* from the atom, $\tilde{E}_+^+(x)$, and the field propagating in the negative x

direction away from the atom, $\tilde{E}_-^+(x)$. Substituting these into Eqs. (23) and (24) and separating the components of the two propagation directions, we find

$$\tilde{E}_+^+ = \tilde{E}_{\text{ext},-}^+ + \eta_\delta(\tilde{E}_{\text{ext},+}^+ + \tilde{E}_{\text{ext},-}^+), \qquad (26)$$

$$\tilde{E}_-^+ = \tilde{E}_{\text{ext},+}^+ + \eta_\delta(\tilde{E}_{\text{ext},+}^+ + \tilde{E}_{\text{ext},-}^+)\,. \qquad (27)$$

We solve these equations for the field amplitudes for the region $x > x_1$ in terms of the amplitudes for $x < x_1$. The fields at x_1+ and x_1- are related by

$$\begin{bmatrix} \tilde{E}_+^+ \\ \tilde{E}_{\text{ext},+}^+ \end{bmatrix} = \mathcal{T} \begin{bmatrix} \tilde{E}_{\text{ext},-}^+ \\ \tilde{E}_-^+ \end{bmatrix}, \qquad (28)$$

$$\mathcal{T}(\delta) = \begin{bmatrix} \frac{(2\eta_\delta+1)}{\eta_\delta+1} & \frac{\eta_\delta}{\eta_\delta+1} \\ -\frac{\eta_\delta}{(\eta_\delta+1)} & \frac{1}{(\eta_\delta+1)} \end{bmatrix}, \qquad (29)$$

where $\mathcal{T}$ is the transfer matrix for this problem. For a homogeneously broadened system we may consider all the detunings δ_j of the light from the atomic resonance to be equal. For an inhomogeneously broadened thermal gas these fluctuate according to the Doppler broadening that results from the thermal Maxwell-Boltzmann distributed atomic velocities.

For a single atom the transmission and reflection amplitudes, $t^{(1)}$ and $r^{(1)}$ respectively, follow directly from the transfer matrix. On the right side of the atom there is only the transmitted wave, call its amplitude $\tilde{E}_t^+$, whereas on the left we have the incoming and reflected waves, $[\tilde{E}_i^+, \tilde{E}_r^+]^T$. These satisfy

$$\begin{bmatrix} \tilde{E}_t^+ \\ 0 \end{bmatrix} = \mathcal{T} \begin{bmatrix} \tilde{E}_i^+ \\ \tilde{E}_r^+ \end{bmatrix}. \qquad (30)$$

We obtain

$$t^{(1)}(\delta) = \frac{\tilde{E}_t^+}{\tilde{E}_i^+} = \frac{(\gamma_w - \gamma_t) + i\delta}{i\delta - \gamma_t}, \qquad (31)$$

$$r^{(1)}(\delta) = \frac{\tilde{E}_r^+}{\tilde{E}_i^+} = \eta_\delta = \frac{\gamma_w}{i\delta - \gamma_t}, \qquad (32)$$

where we write

$$\eta_\delta = \sqrt{R^{(1)}}\zeta, \quad \zeta = e^{i\varphi}, \quad \varphi = \arctan(\delta/\gamma_t). \qquad (33)$$

Here φ denotes the phase factor associated with the reflection. The transmission and reflection can be described in terms of the single-atom power transmission and reflection coefficients $T^{(1)} = |t^{(1)}|^2$ and $R^{(1)} = |r^{(1)}|^2$,

$$T^{(1)}(\delta) = \frac{(\gamma_t - \gamma_w)^2 + \delta^2}{\gamma_t^2 + \delta^2}, \quad R^{(1)}(\delta) = \frac{\gamma_w^2}{\gamma_t^2 + \delta^2}\,. \qquad (34)$$

Stochastic atomic positions and detunings

In order to analyze the suppression of light-induced correlations and the emergence of the MFT response, we first consider the case of two atoms at x_1 and x_2. Here we also need to consider the propagation phases of the light from x' to x, which are governed by the matrix

$$\Phi(x,x') = \begin{bmatrix} e^{ik(x-x')} & 0 \\ 0 & e^{-ik(x-x')} \end{bmatrix} \qquad (35)$$

for both the right- and left-propagating waves.

When considering amplitude transmission, ordinarily one is comparing the properties of the light between some fixed points before and after the sample, call them x_0 and x_3. We are then back to the same problem as in the case of a single atom, except that the composite transfer matrix has to include the two atoms and the propagation phases:

$$\mathcal{T}_{03} = \Phi(x_3,x_2)\mathcal{T}(\delta_2)\Phi(x_2,x_1)\mathcal{T}(\delta_1)\Phi(x_1,x_0). \qquad (36)$$

We obtain the two-atom transmission amplitude

$$t_{12}^{(2)} = \frac{t_2^{(1)}t_1^{(1)}}{1 - \sqrt{R_1^{(1)}R_2^{(1)}}\zeta_1\zeta_2\xi_{12}}e^{ik(x_3-x_0)}\,. \qquad (37)$$

Here $\xi_{12} = \exp(2ikx_{12})$ (where $x_{12} = x_2 - x_1$) is a propagation phase associated with a back-and-forth trip of the light between atoms 1 and 2, and ζ_1 and ζ_2 are phase factors upon reflection from each atom. The overall phase factor for propagation from x_0 to x_3 is trivial, and is henceforth omitted.

We can write Eq. (37) as a geometric series expansion

$$t_{12}^{(2)} = t_2^{(1)}t_1^{(1)} + t_2^{(1)}\sqrt{R_1^{(1)}R_2^{(1)}}e^{i\phi}t_1^{(1)}$$
$$+ t_2^{(1)}\sqrt{R_1^{(1)}R_2^{(1)}}e^{i\phi}\sqrt{R_1^{(1)}R_2^{(1)}}e^{i\phi}t_1^{(1)} + \dots, \qquad (38)$$

with

$$\phi = \varphi_1 + \varphi_2 + 2kx_{12}\,. \qquad (39)$$

The interpretation of Eq. (38) is straightforward: the sum is over all the repeated photon exchanges between the two atoms, and each term $(R_1^{(1)}R_2^{(1)})^{1/2}$ represents one recurrent scattering event for the photon.

Next we calculate the ensemble averages for the transmitted light for the composite two-atom system. For the transmission amplitude we have

$$\langle t_{12}^{(2)}(x_{12},\delta_1,\delta_2)\rangle_{x_{12},\delta_1,\delta_2} = \Big\langle \frac{t_2^{(1)}t_1^{(1)}}{1 - \sqrt{R_1^{(1)}R_2^{(1)}}\zeta_1\zeta_2\xi_{12}} \Big\rangle. \qquad (40)$$

The subscript indicates the averaging over the ensemble of interatomic separations x_{12}, and the detunings of the atoms 1 and 2, δ_1 and δ_2, respectively. The detuning δ_j appears in the single-atom transmission and reflection amplitudes $t_j^{(1)}$ and $[R_j^{(1)}]^{1/2}$ (in fact $R_j^{(1)} \to 0$ with $\delta_j \to \infty$), as well as in the phase ζ_j. Similarly, we can

calculate the expectation value of the optical thickness for the composite two-atom system

$$\langle D_{12}^{(2)}\rangle = -\langle \ln T_{12}^{(2)}\rangle = -\Big\langle \ln \Big(\frac{T_2^{(1)}T_1^{(1)}}{\big|1 - \sqrt{R_1^{(1)}R_2^{(1)}}e^{i\phi}\big|^2}\Big)\Big\rangle, \tag{41}$$

where $T_{12}^{(2)} = |t_{12}^{(2)}|^2$ denotes the transmission coefficient for the intensity of the two-atom system.

If the atomic positions are random, the propagation factor ξ_{12} fluctuates according to the relative positions between the atoms 1 and 2. For a homogeneously-broadened gas we set all the detunings equal $\delta_1 = \delta_2$. Consequently, the amplitudes $t_1^{(1)} = t_2^{(1)}$ and $[R_1^{(1)}]^{1/2} = [R_2^{(1)}]^{1/2}$, and the phase factors $\varphi_1 = \varphi_2$ are constant. For an inhomogeneously broadened thermal gas the detunings are Doppler broadened as a result of the thermal Maxwell-Boltzmann distribution of the atomic velocities. We then take δ_1 and δ_2 to be independent Gaussian distributed random variables, where the Gaussian distribution $f(\delta)$ is given by

$$f(\delta) = \frac{1}{\sqrt{2\pi}\Delta\omega}e^{-\delta^2/(2\Delta\omega)}, \tag{42}$$

with $\Delta\omega \equiv k\sqrt{k_B T/m}$ determined in terms of the wavenumber of light k, the temperature of the atoms T, the mass of the atoms m, and the Boltzmann constant k_B. The ensemble average of the transmitted light amplitude in a composite two-atom system then reads

$$\langle t_{12}^{(2)}(x_{12},\delta_1,\delta_2)\rangle_{\delta_1,\delta_2}$$
$$= \int_{-\infty}^{\infty} d\delta_1 d\delta_2 \frac{f(\delta_1)f(\delta_2)t_2^{(1)}(\delta_2)t_1^{(1)}(\delta_1)}{1 - \sqrt{R_1^{(1)}(\delta_1)R_2^{(1)}(\delta_2)}e^{i\phi(\delta_1,\delta_2)}} \tag{43}$$

Provided that the width of the Gaussian distribution satisfies

$$k\sqrt{k_B T/m} \gg \gamma_t, \tag{44}$$

the contribution of the second term in the denominator in Eq. (40) vanishes. We then find that the transmission amplitudes of the atoms 1 and 2 decouple

$$\langle t_{12}^{(2)}\rangle_{\delta_1,\delta_2} = \langle t_1^{(1)}\rangle_{\delta_1}\langle t_2^{(1)}\rangle_{\delta_2}. \tag{45}$$

The optical thickness of the two-atom system similarly decouples to the optical thicknesses of the individual atoms,

$$\langle D_{12}^{(2)}\rangle_{\delta_1,\delta_2} = \langle D_2^{(1)}\rangle_{\delta_2} + \langle D_1^{(1)}\rangle_{\delta_1}. \tag{46}$$

The emergence of MFT is a general consequence of inhomogeneous broadening, and is not necessarily due to thermal atomic motion. For quantum dots or circuit resonators it can result, e.g., from fabrication imperfections [8].

The decoupling (45) [or (46)] can be achieved even in a homogenously broadened sample when δ_j are constant, if the propagation phases $2kx_{12}$ fluctuate sufficiently [$\Delta(x_{12}) \gg \pi/k$, where $(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2$]. In a homogeneously broadened two-atom system, $\delta_1 = \delta_2 = \delta$, we have for completely randomly distributed $2kx_{12}$

$$\langle t_{12}^{(2)}(x_{12},\delta,\delta)\rangle_{x_{12}} = \frac{1}{2\pi}\int_0^{2\pi} d\phi \frac{t_2^{(1)}t_1^{(1)}}{1 - \sqrt{R_1^{(1)}R_2^{(1)}}e^{i\phi}}$$
$$= t_2^{(1)}t_1^{(1)}. \tag{47}$$

If the phases $2kx_{12}$ are approximately evenly distributed over the entire interval $[0, 2\pi)$, the second term in the denominator in Eq. (47) is of no consequence. Analogously, we find

$$\langle D_{12}^{(2)}(x_{12},\delta,\delta)\rangle_{x_{12}} = \langle D_1^{(1)}\rangle_{x_{12}} + \langle D_2^{(1)}\rangle_{x_{12}}. \tag{48}$$

The two-atom results (45)–(48) may be recursively generalized to N atoms. Whichever is the cause of strong averaging that eliminates recurrent scattering, we have

$$\langle t_{1,2,...,N}^{(N)}\rangle = \langle t_1^{(1)}\rangle\langle t_2^{(1)}\rangle \ldots \langle t_N^{(1)}\rangle, \tag{49}$$

$$\langle D_{1,2,...,N}^{(N)}\rangle = \langle D_1^{(1)}\rangle + \langle D_2^{(1)}\rangle + \ldots + \langle D_N^{(1)}\rangle, \tag{50}$$

where $D^{(1)} = -\ln T^{(1)}$ and $T^{(1)}$ is given by Eq. (34).

Owing to the total reflection of light by a resonant atom whenever $\gamma_w = \gamma_t$, the optical thickness $D = -\ln T$ diverges on resonance. The divergent optical thickness can be cured if we consider a Poisson-distributed number of atoms in a waveguide, and average over the atom number. We find for the MFT result for the optical thickness with the average atom number $\bar{N}$

$$D(\bar{N}) = -\ln T^{(N)} = \frac{(2\gamma_t - \gamma_w)\bar{N}\gamma_w}{\gamma_t^2 + \delta^2}. \tag{51}$$

The expression is finite, since there is a nonvanishing probability of having zero atoms in the waveguide, so that the average transmission is never zero.

The simplicity of the full solutions (49) and (50) is remarkable. Even though the calculation incorporates all the recurrent scattering processes where the photons are repeatedly exchanged between the same atoms, in the final result they have precisely canceled out. The exact cooperative response of the atomic ensemble coincides with the approximate MFT analysis in which each atom in a row simply passes on the same fraction of light that it would if there were no other atoms present. The MFT solution neglects all the light-induced correlations between the atoms. These correlations are suppressed by the fluctuating detunings of the light from the atomic resonances or phase factors associated with each atom in the scattering processes. In particular, the exact solution and the MFT result coincide whenever these fluctuations are sufficiently strong. We find that the first condition is true
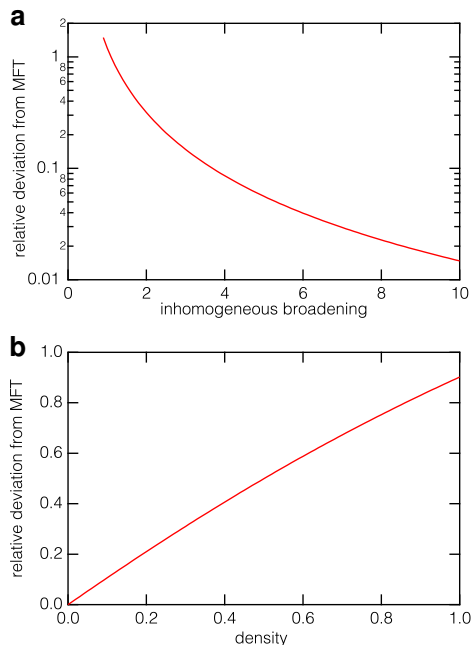
FIG. 1. Relative deviation of exact two-atom transmission amplitude from the corresponding MFT result. (a) The deviation r [Eq. (52)] as a function of the temperature, which is expressed as the corresponding inhomogeneous broadening in units of the linewidth of the atomic transmission $k\sqrt{k_B T/m}/\gamma_t$. This panel is for the extreme case when the fixed back-and-forth propagation phase between the two atoms is a multiple of 2π. (b) The deviation r for a homogeneously broadened two-atom system with $\delta/\gamma_t = 0.1$ as a function of the dimensionless density ρ/k. There are no losses from the fiber in either case, so we have $\gamma_w = \gamma_t$.

whenever the Doppler broadening considerably exceeds the resonance linewidth of the atoms. The fluctuations in the propagation phase factors due to the relative atomic positions, on the other hand, are sufficiently strong for the MFT to be valid if the atom density is low enough.

In order to illustrate the cancelation of the recurrent scattering, we first calculate the effect of thermal broadening on the two-atom correlation function. In Fig. 1(a) we show the relative deviations r of the exact two-atom result from the two-atom MFT result as a function of temperature. Here we have defined

$$r = \left| \frac{\langle t_{12}^{(2)} \rangle_{\delta_1, \delta_2} - \langle t_2^{(1)} \rangle_{\delta_1} \langle t_1^{(1)} \rangle_{\delta_2}}{\langle t_2^{(1)} \rangle_{\delta_1} \langle t_1^{(1)} \rangle_{\delta_2}} \right| . \tag{52}$$

For simplicity, we consider constant relative atomic positions by setting $2kx_{12} = 0$ in Fig. 1. The MFT result corresponds to the case where the light-induced correlations between the two atoms are ignored.

Next we analyze the effect of the atom density on the cancelation of the effect of recurrent scattering in a homogeneously-broadened system. We may estimate the distribution of the relative positions of adjacent atoms in

a classical, initially uncorrelated atomic ensemble. Let us assume that there is a number of atoms $n \gg 1$ distributed over the interval Δx for $x > x_1$ where x_1 is the position of the atom 1. The probability that there are no atoms on the interval $[x_1, x_1 + \delta x]$ is then $(1 - \delta x/\Delta x)^n \to e^{-\rho \delta x}$, when $n \to \infty$, where the density $\rho = n/\Delta x$. In the limit of large n, we may replace n and Δx by N and L, respectively. The $1/e$ width of the probability distribution of no atoms adjacent to the atom 1 for $x > x_1$ is $\Delta(x - x_1) \simeq 1/\rho$. The condition for the sufficient fluctuation of the propagation phases $2kx_{12}$ for Eqs. (49) and (50) to be valid is then approximately given by

$$\rho/k \ll 1/\pi . \tag{53}$$

We can then analytically calculate the expectation value of the two-atom transition amplitude for a homogeneously broadened atom pair by using the distribution

$$g(\phi) = \frac{\rho}{2k} e^{-\rho\phi/2k}, \quad 0 \le \phi < \infty , \tag{54}$$

for the back-and-forth propagation phase $\phi = 2k(x - x_1)$. But, if we wish to calculate the average of a 2π periodic function $F(\phi)$, by virtue of the specific form of $g(\phi)$ we have

$$\begin{aligned}
\langle F(\phi) \rangle &= \int_0^\infty d\phi \, g(\phi) F(\phi) \\
&= \sum_{m=0}^\infty \int_{2\pi m}^{2\pi(m+1)} d\phi \, g(\phi) F(\phi) \\
&= \sum_{m=0}^\infty e^{-m\pi\rho/k} \int_0^{2\pi} d\phi \, g(\phi) F(\phi) \\
&= \int_0^{2\pi} d\phi \, \tilde{g}(\phi) F(\phi) \tag{55}
\end{aligned}$$

where $\tilde{g}$ is simply the probability density g normalized over the interval $[0, 2\pi)$,

$$\tilde{g}(\phi) = \frac{\rho/k}{2(1 - e^{\pi\rho/k})} e^{-\rho\phi/2k}, \quad 0 \le \phi < 2\pi . \tag{56}$$

We therefore find that

$$\begin{aligned}
\langle t_{12}^{(2)}(x_{12}, \delta, \delta) \rangle_{x_{12}} &= \int_0^{2\pi} d\phi \, \frac{\tilde{g}(\phi) t_2^{(1)} t_1^{(1)}}{1 - \sqrt{R_1^{(1)} R_2^{(1)}} e^{i\phi}} \\
&= \frac{[\delta - i(\gamma_w - \gamma_t)]^2}{(\delta + i\gamma_t)^2} \, {}_2F_1(1, \frac{i\rho}{2k}; 1 + \frac{i\rho}{2k}; -\frac{\gamma_w^2}{(\delta + i\gamma_t)^2}),
\end{aligned} \tag{57}$$

where ${}_2F_1$ denotes the hypergeometric function. In Fig. 1(b) we show the relative deviations r [defined analogously to Eq. (52), but with the averaging now with respect to x_{12}] of the exact two-atom result from the two-atom MFT result as a function of the density of the

atoms. The exact two-atom result (57) qualitatively differs from the MFT value when the density approaches to $\rho \simeq k$.

The uneven distribution of the propagation phases at high atom densities can also be illustrated by simple numerical examples. For instance for a density $\rho/k \simeq 0.06$ there is a less than 9% chance for the propagation phases to satisfy $2kx_{12} < \pi$. However, for the density $\rho/k \simeq 2$ this probability is increased to 96%, indicating that the phases are then expected to accumulate substantially more in the interval $[0, \pi)$ than in $[\pi, 2\pi)$, leading to a notably uneven phase distribution. In such a case the averaging resulting in the MFT response of Eqs. (50) and (51) is no longer valid, and the light transmission is affected by light-induced correlations between the atoms.

Waveguide with a strong loss rate

We can also consider the limit $\gamma_w/\gamma_t \ll 1$. The two-atom transmission is then to leading order in γ_w/γ_t

$$T_{12}^{(2)} \simeq T_1^{(1)} T_2^{(1)} \left(1 - \mathcal{O}(\gamma_w^2/\gamma_t^2)\right) , \qquad (58)$$

and to the lowest order we again recover the MFT response. Figure 2 in the main text illustrates how the exact numerical solution approaches the MFT result as the loss rate of the photons from the waveguide is increased.

———————

[1] M. Born and E. Wolf, *Principles of Optics*, 7th ed. (Cambridge University Press, Cambridge, UK, 1999); J. D. Jackson, *Classical Electrodynamics*, 3rd ed. (Wiley, New York, 1999).
[2] J. Ruostekoski and J. Javanainen, Phys. Rev. A **55**, 513 (1997).
[3] D. M. Ceperley, Rev. Mod. Phys. **67**, 279 (1995).
[4] J. Javanainen, J. Ruostekoski, B. Vestergaard, and M. R. Francis, Phys. Rev. A **59**, 649 (1999).
[5] R. Friedberg, S. R. Hartmann, and J. T. Manassah, Physics Reports **7**, 101 (1973).
[6] J. Keaveney, A. Sargsyan, U. Krohn, I. G. Hughes, D. Sarkisyan, and C. S. Adams, Phys. Rev. Lett. **108**, 173601 (2012).
[7] P. W. Anderson, D. J. Thouless, E. Abrahams, and D. S. Fisher, Phys. Rev. B **22**, 3519 (1980); R. Landauer, Philosophical Magazine **21**, 863 (1970); M. V. Berry and S. Klein, European Journal of Physics **18**, 222 (1997).
[8] S. D. Jenkins and J. Ruostekoski, Phys. Rev. B **86**, 205128 (2012).