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FACULTY OF SOCIAL, HUMAN AND MATHEMATICAL SCIENCES

Mathematical Sciences

Homotopy Decompositions of Gauge Groups over Real Surfaces

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ABSTRACT

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HOMOTOPY DECOMPOSITIONS OF GAUGE GROUPS OVER REAL SURFACES

by Michael William Thomas West

We study the homotopy types of gauge groups of principal $U(n)$-bundles associated to pseudo Real vector bundles in the sense of Atiyah [Ati66]. We provide satisfactory homotopy decompositions of these gauge groups into factors in which the homotopy groups are well known. Therefore, we substantially build upon the low dimensional homotopy groups as provided in [BHH10].

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Declaration of Authorship

I, Michael William Thomas West , declare that the thesis entitled *Homotopy Decompositions of Gauge Groups over Real Surfaces* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- none of this work has been published before submission

Signed:...

Date:..

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[1]Here, the word '*qualified*' is chosen carefully.

Chapter 1

Introduction

The calculation of the homotopy groups $\pi_*(X)$ of a topological space X plays a critical role in homotopy theory. However, their calculation continues to be very difficult, even for otherwise basic spaces such as spheres. The ideal scenario is to reduce the calculation of $\pi_*(X)$ to the calculation of other, easier to calculate, spaces. From the fact that

$$\pi_*(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_*(X_\alpha)$$

a standard technique is to decompose X, up to homotopy, as a product of other spaces. For example there is the classical result of Serre that localised away from 2 there is a homotopy equivalence

$$\Omega S^{2n} \simeq S^{2n-1} \times \Omega S^{4n-1}.$$

One therefore deduces that the calculation of odd torsion in the homotopy groups of even spheres is reduced to the calculation of odd torsion in the homotopy groups of odd spheres. Or without localisation the Hilton-Milnor theorem gives the homotopy equivalence

$$\Omega(\Sigma X \vee \Sigma Y) \simeq \Omega\Sigma X \times \Omega\Sigma(\bigvee_{j=0}^{\infty} X^{\wedge j} \vee Y).$$

The aim of this thesis is to provide satisfactory decompositions for particular examples of gauge groups. Due to their intimate ties with other fields, there has been a lot of interest in the topology of gauge groups and their classifying spaces. For instance, Donaldson [Don86] used gauge theory to introduce new restrictions on the intersection form of simply-connected differentiable 4-manifolds and also used gauge theory to introduce new polynomial invariants of such spaces [Don90]. As a result, this showed the existence of topological 4-manifolds with no smooth structure, as well as the existence of 4-manifolds with infinitely many non-diffeomorphic smooth structures.

The study of gauge groups exceeds these applications by having intimate ties with mathematical physics and algebraic geometry. In physics, gauge groups refer to certain internal symmetries of field theories and we point the reader to [CM94] for an exposition of how the topology of gauge groups fits into the picture. In algebraic geometry, the gauge group relates to moduli spaces of stable vector bundles, indeed this is the application that we will focus on. We point the reader to [BHH10], [LS13] and [Bai14] in which there are calculations of some of the topological invariants of the gauge groups studied in this thesis.

Unless otherwise stated, we will assume that all topological spaces are homotopy equivalent to CW-complexes.

1.1 Definitions and Notation

Consider a degree n polynomial with real coefficients

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = 0.$$

By the Fundamental Theorem of Algebra we know that the polynomial has exactly n roots over $\mathbb{C}$. However, over the field $\mathbb{R}$, we only know that the number of roots is congruent to $n \mod 2$. This is due to the fact that complex roots come in pairs, hence information about the complex case reveals information about the real case.

In [Ati66], Atiyah's idea was to try to compare real and complex K-theory in a similar fashion by providing a K-theory that catered for both. The idea was to take real vector bundles and 'complexify' them in some way. This new flavour of bundle, termed Real vector bundle, is associated with the Real principal bundles described in Chapter 4.

We highlight a natural way to think about this 'complexification'. In the complex case, it is very well known that to a smooth projective complex algebraic curve, one can associate a compact Riemann surface. This correspondence provides an equivalence between the category of smooth complex algebraic curves and the category of compact Riemann surfaces.

Alternatively, taking a smooth projective real algebraic curve Y, we can still associate Riemann surface using 'complexification', that is, extending Y to $\mathbb{C}$. This new space $X := Y \times_{\mathbb{R}} \mathbb{C}$ has a canonical antiholomorphic involution $\sigma \colon X \to X$ induced by complex conjugation. In general the pair (X, σ) with X a compact Riemann surface and σ an antiholomorphic involution will be called a *Real surface*. We can then form the category of Real surfaces where morphisms are continuous maps $f \colon X \to Y$ with the property

$$f\sigma_X = \sigma_Y f.$$

In this new setting, the correspondence described above provides an equivalence between the category of smooth projective real algebraic curves and the category of Real surfaces.

To a Real surface (X, σ) we associate the following triple $(g(X), r(X), a(X))$ where

- $g(X)$ is the genus of X;
- $r(X)$ is the number of path components of the fixed set X^σ;
- $a(X) = 0$ if X/σ is orientable and $a(X) = 1$ otherwise.

We note that the path components of X^σ are each homeomorphic to S^1. The following classification of Real surfaces was studied in [Wei83].

Theorem 1.1 Weichold. *Let (X, σ) and (X', σ') be Real surfaces then there is a isomorphism $X \to X'$ (in the category of Real surfaces) if and only if*

$$(g(X), r(X), a(X)) = (g(X'), r(X'), a(X')).$$

Furthermore, if a triple (g, r, a) satisfies one of the following conditions

1. *if $a = 0$, then $1 \leq r \leq g + 1$ and $r \equiv (g + 1) \mod 2$;*
2. *if $a = 1$, then $0 \leq r \leq g$;*

then there is a Real surface (X, σ) such that $(g, r, a) = (g(X), r(X), a(X))$. □

Therefore a Real surface (X, σ) is completely determined by its triple (g, r, a) which we call the *type* of the Real surface.

Let $\pi\colon P \to X$ be a principal $U(n)$-bundle over the underlying Riemann surface X of the Real surface (X, σ). A *lift* of σ is a map $\tilde{\sigma}\colon P \to P$ satisfying

1. $\sigma\,\pi = \pi\,\tilde{\sigma}$;
2. $\tilde{\sigma}(p \cdot g) = \tilde{\sigma}(p) \cdot \overline{g}$ for all $p \in P, g \in U(n)$;

where $\overline{g}$ represents the entry-wise complex conjugate of $g \in U(n)$. We remark that, due to Property 2 of a lift, the fixed point set $P^{\tilde{\sigma}}$ has the structure of a principal $O(n)$-bundle over the real points X^σ.

Let $\tilde{\sigma}$ be a lift then we say that $(P, \tilde{\sigma}) \to (X, \sigma)$ is a *Real principal* $U(n)$*-bundle* (or *Real bundle*) if $\tilde{\sigma}$ further satisfies

3. $\tilde{\sigma}^2(p) = p$ for all $p \in P$;

or if n is even we say that $(P,\tilde{\sigma}) \to (X,\sigma)$ is a *Quaternionic principal* $U(n)$*-bundle* (or *Quaternionic bundle*) if $\tilde{\sigma}$ satisfies

3′. $\tilde{\sigma}^2(p) = p \cdot (-I_n)$ for all $p \in P$.

where I_n represents the $n \times n$ identity matrix. We point the reader to Chapter 4 for some general theory on equivariant bundles.

The following two propositions originally appeared in [BHH10] but the reader can find their proofs in Section 4.5.

Proposition 1.2. *Let (X,σ) be a Real surface with r fixed components X_i for $1 \le i \le r$. Then Real principal $U(n)$-bundles $(P,\tilde{\sigma}) \to (X,\sigma)$ are classified by the first Stiefel-Whitney classes of the restriction to bundles $P_i \to X_i$ over the fixed components*

$$\omega_1(P_i) \in H^1(X_i,\mathbb{Z}/2) \cong \mathbb{Z}/2$$

and the first Chern classes of the bundle over X

$$c_1(P) \in H^2(X,\mathbb{Z}) \cong \mathbb{Z}$$

subject to the relation

$$c_1(P) \equiv \sum w_1(P_i) \mod (2).$$

Furthermore, given any such characteristic classes there is a Real principal $U(n)$-bundle that realises them.

We write

$$(c, w_1, w_2, \ldots, w_r) := (c_1(P), w_1(P_1), w_1(P_2), \ldots, w_1(P_r))$$

and we will refer to the tuple $(c, w_1, w_2, \ldots, w_r) \in \mathbb{Z} \times \prod_r \mathbb{Z}_2$ as the *class* of the Real principal $U(n)$-bundle $(P,\tilde{\sigma})$.

Proposition 1.3. *Quaternionic principal $U(n)$-bundles $(P,\tilde{\sigma}) \to (X,\sigma)$ are classified by their first Chern class which must be even. Furthermore, given any such Chern class there is a Quaternionic principal $U(n)$-bundle that realises it.*

Writing $c = c_1(P)$, we will therefore refer to $c \in 2\mathbb{Z}$ as the *class* of the Quaternionic principal $U(n)$-bundle $(P,\tilde{\sigma})$.

Let $(P,\tilde{\sigma}) \to (X,\sigma)$ be a Real or Quaternionic principal $U(n)$-bundle. An *automorphism* of $(P,\tilde{\sigma})$ is a $U(n)$-equivariant map $\phi\colon P \to P$ such that the following diagrams commute

$$\begin{array}{ccc} P & \xrightarrow{\phi} & P \\ \downarrow & & \downarrow \\ X & \xrightarrow{\mathrm{id}_X} & X \end{array} \qquad \text{and} \qquad \begin{array}{ccc} P & \xrightarrow{\phi} & P \\ \downarrow{\scriptstyle\tilde{\sigma}} & & \downarrow{\scriptstyle\tilde{\sigma}} \\ P & \xrightarrow{\phi} & P. \end{array}$$

Let $\mathrm{Map}(P, P)$ be endowed with the compact open topology. We then define the *(unpointed) gauge group* $\mathscr{G}(P, \tilde{\sigma})$ to be the subspace of $\mathrm{Map}(P, P)$ whose elements are automorphisms of $(P, \tilde{\sigma})$.

In order to provide homotopy decompositions for $\mathscr{G}(P, \tilde{\sigma})$, it will be convenient to provide decompositions for certain subspaces of the gauge group. Choose a basepoint $*_X$ of (X, σ) such that $\sigma(*_X) = *_X$ if $r > 0$. Then the *(single)-pointed gauge group* $\mathscr{G}^*(P, \tilde{\sigma})$ consists of the elements of $\mathscr{G}(P, \tilde{\sigma})$ that restrict to the identity above $*_X$.

Another pointed gauge group of interest was considered in [BHH10]. Let (X, σ) be a Real surface of type (g, r, a), then for each $1 \leq i \leq r$ choose a designated point $*_i$ contained in the fixed component X_i. Further if $a = 1$ choose another designated point that is not fixed by the involution. Then the $(r + a)$*-pointed gauge group* $\mathscr{G}^{*(r+a)}(P, \tilde{\sigma})$ consists of the elements of $\mathscr{G}(P, \tilde{\sigma})$ that restrict to the identity above these $(r + a)$ designated points of (X, σ).

We now present results of [BHH10] corresponding to the low dimensional homotopy groups of $\mathscr{G}(P, \sigma)$ and $\mathscr{G}^{*(r+a)}(P, \tilde{\sigma})$. We note that we only previously defined Quaternionic bundles with even rank and hence n is assumed to be even for the last row. We also note that these homotopy groups depend on the type of Real surface (g, r, a) but are independent of the choice of class of $(P, \tilde{\sigma})$.

Theorem 1.4 Biswas, Huisman, Hurtubise**.** *The low dimensional homotopy groups of the rank n gauge groups above a Real surface of type (g, r, a) are as follows*

Real	$\pi_0(\mathscr{G}^{*(r+a)}(P, \tilde{\sigma}))$	$\pi_0(\mathscr{G}(P, \tilde{\sigma}))$	$\pi_1(\mathscr{G}^{*(r+a)}(P, \tilde{\sigma}))$	$\pi_1(\mathscr{G}(P, \tilde{\sigma}))$
$n > 2$	$\mathbb{Z}^{g+a} \times (\mathbb{Z}_2)^r$	$\mathbb{Z}^g \times (\mathbb{Z}_2)^{r+1}$	$\mathbb{Z}$	$\mathbb{Z} \times (\mathbb{Z}_2)^r$
$n = 2$	$\mathbb{Z}^{g+a+r}$	$\mathbb{Z}^{g+r} \times \mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}^{r+1}$
$n = 1$	$\mathbb{Z}^{g+a}$	$\mathbb{Z}^g \times \mathbb{Z}_2$	0	0
Quat. *rank* $2n$	$\mathbb{Z}^{g+a}$	$\mathbb{Z}^g \times (\mathbb{Z}_2)^a$	$\mathbb{Z}$	$\mathbb{Z}$

Our aim to is expand on the results presented in Theorem 1.4, and the reader is directed to Appendix A for these results. In this appendix, we also highlight a discrepancy with the third and fourth columns of the table in Theorem 1.4.

1.2 Main Results for Real Bundles

In this section we present the main results pertaining to homotopy decompositions of gauge groups of Real principal $U(n)$-bundles. Their proofs can be found in Chapter 5, and the author has provided hyper-links on each result to ease navigation. We recall that the isomorphism class of a gauge group of a Real bundle $(P, \tilde{\sigma})$ depends on

1. the type of the underlying Real surface (g, r, a);
2. the isomorphism class of the bundle $(c, w_1, w_2, \ldots, w_r)$;

subject to the relations in Theorem 1.1 and Proposition 1.2. Therefore, to ease notation we will sometimes use the following

- $\mathscr{G}((g, r, a); (c, w_1, w_2, \ldots, w_r))$ to represent the unpointed gauge group of a Real bundle of class $(c, w_1, w_2, \ldots, w_r)$ over a Real surface of type (g, r, a);
- $\mathscr{G}^*((g, r, a); (c, w_1, w_2, \ldots, w_r))$ to represent the single-pointed gauge group of the Real bundle as above;
- $\mathscr{G}^{*(r+a)}((g, r, a); (c, w_1, w_2, \ldots, w_r))$ to represent the $(r + a)$-pointed gauge group of the Real bundle as above.

Our aim is to provide homotopy decompositions for these spaces and in doing so we will be able to significantly build upon the results in Theorem 1.4; the reader is directed to Appendix A for an extension of these results.

We first present the results relating to when gauge groups of different Real bundles have the same homotopy type. For $(r + a)$-pointed gauge groups this is always the case.

Proposition 1.5. *Let $(P, \tilde{\sigma})$ and (P', σ') be Real principal $U(n)$-bundles over a Real surface (X, σ) of arbitrary type (g, r, a), then there is a homotopy equivalence*

$$B\mathscr{G}^{*(r+a)}(P, \tilde{\sigma}) \simeq B\mathscr{G}^{*(r+a)}(P', \sigma').$$

However, this is not necessarily the case for the single-pointed and unpointed gauge groups, although we do have the following results.

Proposition 1.6. *For any c, c', w_1, w_1' there is a homotopy equivalence*

$$B\mathscr{G}^*((g, r, a); (c, w_1, w_2, \ldots, w_r)) \simeq B\mathscr{G}^*((g, r, a); (c', w_1', w_2, \ldots, w_r)).$$

Proposition 1.7. *Let the following be classifying spaces of rank n gauge groups. Then there are homotopy equivalences*

$$B\mathscr{G}((g, r, a); (c, w_1, w_2, \ldots, w_r)) \simeq B\mathscr{G}((g, r, a); (c + 2n, w_1, w_2, \ldots, w_r)).$$

Proposition 1.8. *Let n be odd then there are homotopy equivalences*

1. $B\mathscr{G}((g,r,a);(c,w_1,w_2,\ldots,w_r)) \simeq B\mathscr{G}((g,r,a);(c,\sum_{i=1}^{r} w_i,0,\ldots,0))$;
2. $B\mathscr{G}^*((g,r,a);(c,w_1,w_2,\ldots,w_r)) \simeq B\mathscr{G}^*((g,r,a);(c,\sum_{i=1}^{r} w_i,0,\ldots,0))$.

After discussion with the external examiner, it is apparent that there are stronger statements than Propositions 1.7 and 1.8 which have simpler and more conceptual proofs. We include the stronger statements here and the corresponding proofs immediately after those of Propositions 1.7 and 1.8.

Proposition 1.7 (Strong). *Let the following be gauge groups of rank n. Then there are isomorphisms of topological groups*

$$\mathscr{G}((g,r,a);(c,w_1,w_2,\ldots,w_r)) \cong \mathscr{G}((g,r,a);(c+2n,w_1,w_2,\ldots,w_r)).$$

Proposition 1.8 (Strong). *Let n be odd then there are isomorphisms of topological groups*

1. $\mathscr{G}((g,r,a);(c,w_1,w_2,\ldots,w_r)) \cong \mathscr{G}((g,r,a);(c,\sum_{i=1}^{r} w_i,0,\ldots,0))$;
2. $\mathscr{G}^*((g,r,a);(c,w_1,w_2,\ldots,w_r)) \cong \mathscr{G}^*((g,r,a);(c,\sum_{i=1}^{r} w_i,0,\ldots,0))$.

Due to Proposition 1.5, it is easy to state homotopy decompositions for $(r+a)$-pointed gauge group as seen in Theorem 1.9.

Theorem 1.9. *Let $(P,\tilde{\sigma})$ be of arbitrary class then there are integral homotopy decompositions*

Type	*Decompositions for $\mathscr{G}^{*(r+a)}(P,\tilde{\sigma})$*
$(g,0,1)$ *for g even*	$\mathscr{G}^*((0,0,1);0) \times \prod^{g} \Omega U(n)$
$(g,0,1)$ *for g odd*	$\mathscr{G}^*((1,0,1);0) \times \prod^{g-1} \Omega U(n)$
$(g,r,0)$	$\Omega^2(U(n)/O(n)) \times \prod^{(g-r+1)+(r-1)} \Omega U(n) \times \prod^{r-1} \Omega O(n)$
$(g,r,1)$ *for $g-r$ even*	$\mathscr{G}^*((1,1,1);(0,0)) \times \prod^{(g-r)+(r-1)+1} \Omega U(n) \times \prod^{r-1} \Omega O(n)$
$(g,r,1)$ *for $g-r$ odd*	$\mathscr{G}^*((1,1,1);(0,0)) \times \prod^{(g-r-1)+(r-1)+2} \Omega U(n) \times \prod^{r-1} \Omega O(n)$

However, in the single-pointed case we have to be a little more careful with regards to the class of the underlying Real bundle. For the cases where $\mathscr{G}^{*(r+a)}(P,\tilde{\sigma}) \neq \mathscr{G}^*(P,\tilde{\sigma})$, that is when $r+a>1$, we have the results in Theorem 1.10.

Theorem 1.10. *Let n be odd or let $(P,\tilde{\sigma})$ be of class $(c,w_1,0,\dots,0)$. Let $r+a>1$ then there are integral homotopy decompositions*

Type	*Decompositions for* $\mathscr{G}^*(P,\tilde{\sigma})$
$(g,r,0)$	$\Omega^2(U(n)/O(n)) \times \prod^{g-r+1} \Omega U(n) \times \prod^{r-1} \Omega O(n) \times \prod^{r-1} \Omega(U(n)/O(n))$
$(g,r,1)$ $g-r$ *even*	$\mathscr{G}^*((1,1,1);(0,0)) \times \prod^{g-r} \Omega U(n) \times \prod^{r-1} \Omega O(n) \times \prod^{r-1} \Omega(U(n)/O(n))$
$(g,r,1)$ $g-r$ *odd*	$\mathscr{G}^*((1,1,1);(0,0)) \times \prod^{(g-r-1)+1} \Omega U(n) \times \prod^{r-1} \Omega O(n) \times \prod^{r-1} \Omega(U(n)/O(n))$

The remaining cases seem to integrally indecomposable, however we will obtain the following localised homotopy decompositions.

Theorem 1.11. *Let $p\neq 2$ be prime and let n be odd, then there are the following p-local homotopy equivalences*

1. $\mathscr{G}^*((0,0,1);c) \simeq_p \Omega^2(U(n)/O(n)) \times \Omega(U(n)/O(n))$;
2. $\mathscr{G}^*((1,0,1);c) \simeq_p \Omega^2(U(n)/O(n)) \times \Omega(U(n)/O(n)) \times \Omega U(n)$;
3. $\mathscr{G}^*((1,1,1);(c,w_1)) \simeq_p \Omega^2(U(n)/O(n)) \times \Omega(U(n)/O(n)) \times \Omega O(n)$.

We move on to some integral homotopy decompositions for unpointed gauge groups. The reader is invited to compare the tables of Theorem 1.12 and Theorem 1.10.

Theorem 1.12. *Let $(P,\tilde{\sigma})$ be of class $(c,w_1,w_2,\dots,w_r)$ then there are integral homotopy decompositions*

1.

Type	*Decompositions for* $\mathscr{G}(P,\tilde{\sigma})$
$(g,r,0)$	$\mathscr{G}((r-1,r,0);(c,w_1,\dots,w_r)) \times \prod^{g-r+1} \Omega U(n)$
$(g,r,1)$ $g-r$ *even*	$\mathscr{G}((r,r,1);(c,w_1,\dots,w_r)) \times \prod^{g-r} \Omega U(n)$
$(g,r,1)$ $g-r$ *odd*	$\mathscr{G}((r+1,r,1);(c,w_1,\dots,w_r)) \times \prod^{g-r-1} \Omega U(n)$

2. $\mathscr{G}((2,1,1);(c,w_1)) \simeq \mathscr{G}((1,1,1);(c,w_1)) \times \Omega U(n)$.

Further, for $r \geq 1$ and when $(P, \tilde{\sigma})$ is of class $(c, w_1, 0, \ldots, 0)$ or n is odd, there are integral homotopy decompositions

(3)

Type	*Decompositions for* $\mathscr{G}(P, \tilde{\sigma})$
$(r-1, r, 0)$	$\mathscr{G}((0,1,0);(c,\Sigma w_i)) \times \prod^{r-1} \Omega O(n) \times \prod^{r-1} \Omega(U(n)/O(n))$
$(r, r, 1)$	$\mathscr{G}((1,1,1);(c,\Sigma w_i)) \times \prod^{r-1} \Omega O(n) \times \prod^{r-1} \Omega(U(n)/O(n))$
$(r+1, r, 1)$	$\mathscr{G}((2,1,1);(c,\Sigma w_i)) \times \prod^{r-1} \Omega O(n) \times \prod^{r-1} \Omega(U(n)/O(n))$

The remaining unfamiliar spaces in Theorem 1.12 seem to be integrally indecomposable, however localising at particular primes permits further decompositions.

Theorem 1.13. *Let n be a positive integer and let p be a prime with $p \nmid n$.*

1. *Let the following be gauge groups of rank n then there are p-local homotopy equivalences*

   (a) $\mathscr{G}((g,1,a);(c,0)) \simeq_p O(n) \times \mathscr{G}^*((g,1,a);(c,0))$;

   further if $p \neq 2$ and n is odd, then there are p-local homotopy equivalences

   (b) $\mathscr{G}((0,0,1);c) \simeq_p SO(n) \times \Omega^2(U(n)/SO(n))$;

   (c) $\mathscr{G}((1,0,1);c) \simeq_p SO(n) \times \Omega^2(U(n)/SO(n)) \times \Omega U(n)$.

2. *Let the following be gauge groups of rank p then there are p-local homotopy equivalences*

   (a) $\mathscr{G}((g,1,a);(c,0)) \simeq_p O(p) \times \mathscr{G}^*((g,1,a);(c,0))$;

   further if $p \neq 2$, then there are p-local homotopy equivalences

   (b) $\mathscr{G}((0,0,1);c) \simeq_p SO(p) \times \Omega^2(U(p)/SO(p))$;

   (c) $\mathscr{G}((1,0,1);c) \simeq_p SO(p) \times \Omega^2(U(p)/SO(p)) \times \Omega U(p)$.

1.3 Main Results for Quaternionic Bundles

We now focus on the results for homotopy decompositions of gauge groups of Quaternionic principal $U(2n)$-bundles, the proofs of these results are similar to the Real case but some details are provided in Section 5.4. To distinguish the notation of gauge groups from the Real case we will use a subscript Q, for example $\mathscr{G}_{\mathrm{Q}}(P, \tilde{\sigma})$. Recall that the isomorphism class of a gauge group of a Quaternionic bundle $(P, \tilde{\sigma})$ depends on

1. the type of the underlying Real surface (g, r, a);
2. the isomorphism class of the bundle c;

subject to the relations in Theorem 1.1 and $c \equiv 0 \mod 2$. Therefore, to ease notation we will sometimes use the following

- $\mathscr{G}_{\mathrm{Q}}((g,r,a);c)$ to represent the unpointed gauge group of a Quaternionic bundle of class c over a Real surface of type (g, r, a);
- ${\mathscr{G}_{\mathrm{Q}}}^*((g,r,a);c)$ to represent the single-pointed gauge group of the Quaternionic bundle as above;
- ${\mathscr{G}_{\mathrm{Q}}}^{*(r+a)}((g,r,a);c)$ to represent the $(r+a)$-pointed gauge group of the Quaternionic bundle as above.

Once again, our aim is to provide homotopy decompositions for these spaces and in doing so we will be able to significantly build upon the results in the last row of Theorem 1.4; the reader is directed to Appendix A.

We present results in the same order as we did in the Real case. In the Quaternionic case, the homotopy types of the pointed and $(r+a)$-pointed gauge groups are independent of the class of the bundle.

Proposition 1.14. *Let (X, σ) be a Real surface of fixed type (g, r, a). Let $(P, \tilde{\sigma})$ and (P', σ') be Quaternionic principal $U(2n)$-bundles over (X, σ), then there are homotopy equivalences*

1. $B{\mathscr{G}_{\mathrm{Q}}}^*(P,\tilde{\sigma}) \simeq B{\mathscr{G}_{\mathrm{Q}}}^*(P',\sigma')$;
2. $B{\mathscr{G}_{\mathrm{Q}}}^{*(r+a)}(P,\tilde{\sigma}) \simeq B{\mathscr{G}_{\mathrm{Q}}}^{*(r+a)}(P',\sigma')$.

For the unpointed case, we have an analogue of Proposition 1.7.

Proposition 1.15. *Let (X, σ) be a Real surface of fixed type (g, r, a) and let the following be the classifying spaces of gauge groups of Quaternionic bundles of rank $2n$. Then for any even integer c, there is a homotopy equivalence*

$$B\mathscr{G}_{\mathrm{Q}}((g,r,a);c)) \simeq B\mathscr{G}_{\mathrm{Q}}((g,r,a);c+4n).$$

We now present homotopy decompositions for pointed gauge groups in the Quaternionic case. The reader is invited to compare the following results to their Real analogues.

Theorem 1.16. *Let $(P, \tilde{\sigma})$ be a Quaternionic principal $U(2n)$-bundle of class c then there are integral homotopy decompositions*

Type	*Decompositions for* ${\mathscr{G}_{\mathrm{Q}}}^{*(r+a)}(P, \tilde{\sigma})$
$(g, 0, 1)$ *for* g *even*	${\mathscr{G}_{\mathrm{Q}}}^{*}((0,0,1);0) \times \prod^{g} \Omega U(2n)$
$(g, 0, 1)$ *for* g *odd*	${\mathscr{G}_{\mathrm{Q}}}^{*}((1,0,1);0) \times \prod^{g-1} \Omega U(2n)$
$(g, r, 0)$	$\Omega^2(U(2n)/Sp(n)) \times \prod^{g} \Omega U(2n) \times \prod^{r-1} \Omega Sp(n)$
$(g, r, 1)$ *for* $g - r$ *even*	${\mathscr{G}_{\mathrm{Q}}}^{*}((1,1,1);0) \times \prod^{g} \Omega U(2n) \times \prod^{r-1} \Omega Sp(n)$
$(g, r, 1)$ *for* $g - r$ *odd*	${\mathscr{G}_{\mathrm{Q}}}^{*}((1,1,1);0) \times \prod^{g} \Omega U(2n) \times \prod^{r-1} \Omega Sp(n)$

For the cases where ${\mathscr{G}_{\mathrm{Q}}}^{*(r+a)}(P, \tilde{\sigma}) \neq {\mathscr{G}_{\mathrm{Q}}}^{*}(P, \tilde{\sigma})$, that is when $r + a > 1$, we have the results in Theorem 1.17.

Theorem 1.17. *For $(P, \tilde{\sigma})$ of arbitrary class c, there are integral homotopy decompositions*

Type	*Decompositions for* ${\mathscr{G}_{\mathrm{Q}}}^{*}(P, \tilde{\sigma})$
$(g, r, 0)$	$\Omega^2(U(2n)/Sp(n)) \times \prod^{g-r+1} \Omega U(2n) \times \prod^{r-1} \Omega Sp(n) \times \prod^{r-1} \Omega(U(2n)/Sp(n))$
$(g, r, 1)$ $g - r$ *even*	${\mathscr{G}_{\mathrm{Q}}}^{*}((1,1,1);0) \times \prod^{g-r} \Omega U(2n) \times \prod^{r-1} \Omega Sp(n) \times \prod^{r-1} \Omega(U(2n)/Sp(n))$
$(g, r, 1)$ $g - r$ *odd*	${\mathscr{G}_{\mathrm{Q}}}^{*}((1,1,1);0) \times \prod^{g-r} \Omega U(2n) \times \prod^{r-1} \Omega Sp(n) \times \prod^{r-1} \Omega(U(2n)/Sp(n))$

Again, the remaining cases seem to integrally indecomposable, however we will obtain the following localised decompositions.

Theorem 1.18. *Let $p \neq 2$ be prime, then there are p-local homotopy equivalences*

1. ${\mathscr{G}_{\mathrm{Q}}}^{*}((0,0,1);0) \simeq_p \Omega^2(U(2n)/Sp(n)) \times \Omega(U(2n)/Sp(n))$;
2. ${\mathscr{G}_{\mathrm{Q}}}^{*}((1,0,1);0) \simeq_p \Omega^2(U(2n)/Sp(n)) \times \Omega(U(2n)/Sp(n)) \times \Omega U(2n)$;
3. ${\mathscr{G}_{\mathrm{Q}}}^{*}((1,1,1);0) \simeq_p \Omega^2(U(2n)/Sp(n)) \times \Omega(U(2n)/Sp(n)) \times \Omega Sp(n)$.

We now present homotopy decompositions for the unpointed case.

Theorem 1.19. *For $(P, \tilde{\sigma})$ of arbitrary class c, there are integral homotopy decompositions*

Type	*Decompositions for* $\mathscr{G}_{\mathrm{Q}}(P, \tilde{\sigma})$
$(g, 0, 1)$ *g even*	$\mathscr{G}_{\mathrm{Q}}((0,0,1);c) \times \prod^{g} \Omega U(n)$
$(g, 0, 1)$ *g odd*	$\mathscr{G}_{\mathrm{Q}}((1,0,1);c) \times \prod^{g-1} \Omega U(n)$
$(g, r, 0)$	$\mathscr{G}_{\mathrm{Q}}((0,1,0);c) \times \prod^{r-1} \Omega Sp(n) \times \prod^{r-1} \Omega(U(2n)/Sp(n)) \times \prod^{g-r+1} \Omega U(n)$
$(g, r, 1)$	$\mathscr{G}_{\mathrm{Q}}((1,1,1);c) \times \prod^{r-1} \Omega Sp(n) \times \prod^{r-1} \Omega(U(2n)/Sp(n)) \times \prod^{g-r} \Omega U(n)$

The remaining unfamiliar spaces in Theorem 1.19 seem to be integrally fundamental, however localising at a particular prime permits further decompositions.

Theorem 1.20. *Let n be a positive integer and let p be a prime such that $p \nmid 2n$. Let the following be gauge groups of a Quaternionic bundle of rank $2n$, then there are p-local homotopy equivalences*

1. $\mathscr{G}_{\mathrm{Q}}((g,1,a);c) \simeq_p Sp(n) \times B\mathscr{G}_{\mathrm{Q}}{}^*((g,1,a);c)$;
2. $\mathscr{G}_{\mathrm{Q}}((0,0,1);c) \simeq_p Sp(n) \times \Omega^2\big(U(2n)/Sp(n)\big)$;
3. $\mathscr{G}_{\mathrm{Q}}((1,0,1);c) \simeq_p Sp(n) \times \Omega^2\big(U(2n)/Sp(n)\big) \times \Omega U(2n)$.

1.4 Summary of Contents

Chapter 2 is dedicated to providing the background homotopy theory used throughout this thesis. As a consequence there are many unrelated sections and the flow of this chapter is independent from the rest of the thesis. Sections 2.1 and 2.2 were added for completeness; the aim is provide rigorous statements and proofs used later in the thesis. Section 2.3 follows the work of [Har61], introducing a map that has intimate ties with some of the results in Chapter 5.

Chapter 3 introduces gauge groups related to principal $U(n)$-bundles in the usual sense, that is, not Real nor Quaternionic bundles. In Section 3.1 we discuss in some detail the properties of principal G-bundles, their group of automorphisms (gauge groups) and how they relate to a certain type of mapping space. In Sections 3.2 and 3.3 we go on to discuss some decompositions in the case of principal $U(n)$-bundles over Riemann surfaces as considered by [The11]. The methods used in these sections are used markedly in Chapter 5.

Chapter 4 aims to provide general statements about equivariant bundle theory and how their associated gauge groups are related to mapping spaces in a similar way to the gauge groups of Chapter 3. We acknowledge that the generalised equivariant bundle theory of Sections 4.1–4.3 is well-known by experts but we have expanded on some of the proofs which arise as a generalisation of the non-equivariant case. From Section 4.4 we focus on the Real and Quaternionic bundles defined in the introduction and go on to prove the classification of such bundles as studied in [BHH10].

Chapter 5 provides the proofs for the statements in Sections 1.2 and 1.3. The proofs aim to be in the same order as stated in these sections.

Chapter 2

Tools in Homotopy Theory

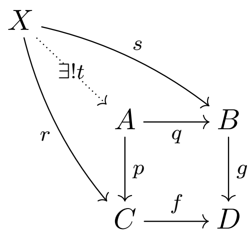
In the following sections we highlight various facts and techniques in homotopy theory that are used throughout this thesis. Unless otherwise stated, all spaces will be assumed to be homotopy equivalent to CW-complexes.

2.1 Homotopy Pullbacks

Some of the homotopy decompositions of gauge groups rely on the homotopy theory of pullbacks. In the category of topological spaces, a *(strict) pullback* of the diagram

$$C \xrightarrow{f} D \xleftarrow{g} B$$

is a space A along with maps $p\colon A \to C$ and $q\colon A \to B$ such that the following square commutes



(2.1)

and the following universal property holds. For every space X and maps $s\colon X \to B$ and $r\colon X \to C$ such that $fr = gs$ there exists a unique map $t\colon X \to A$ that makes the triangles in diagram (2.1) commute.

If a commuting square is a pullback then it will be decorated with a right angle as follows

$$\begin{array}{ccc} A & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow \ \lrcorner & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{f} & D. \end{array}$$

We define the *standard pullback* of the diagram $C \xrightarrow{f} D \xleftarrow{g} B$ to be the space

$$C \times_D B := \{(c, b) \in C \times B \mid f(c) = g(b)\}$$

along with that maps $p\colon C \times_D B \to C$ and $q\colon C \times_D B \to B$ given by

$$p(b, c) = c \text{ and } q(b, c) = b.$$

It is clear that the standard pullback is a pullback and therefore pullbacks exist. It is also clear that pullbacks are unique up to isomorphism. We now present some useful properties of pullbacks whose proofs can be found in [Sel97].

Proposition 2.1. *Let the following be a pullback square of pointed topological spaces*

$$\mathscr{D} := \begin{array}{ccc} A & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ C & \xrightarrow{f} & D \end{array}$$

then

1. *if g (or f) is a fibration then p (or q) is a fibration;*
2. *if g is a fibration and f is a homotopy equivalence then q is a homotopy equivalence.*
3. *if g and p are fibrations, then the restriction of q to $p^{-1}(*_C)$ is a homeomorphism onto $g^{-1}(*_D)$.*

Let the following be a commutative diagram of topological spaces

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

then

4. *if the right square is a pullback then the left square is a pullback if and only if the outside square is a pullback.* □

Consider the replacement of the diagram $C \xrightarrow{f} D \xleftarrow{g} B$ with $C' \xrightarrow{f'} D \xleftarrow{g'} B'$ where spaces are replaced up to homotopy equivalence and maps up to homotopy. One may hope that the pullbacks of these diagrams will have the same homotopy type, however the following example shows that this is not the case.

Example 2.1. *Let X be a subspace of a space Y with inclusion $i\colon X \to Y$. Let D^2 be a 2-disk with $j\colon S^1 \to D^2$ the inclusion of its boundary. Consider the pullbacks*

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Map}^*(D^2, Y) \\ \big\downarrow & \lrcorner & \big\downarrow j^* \\ \Omega X & \xrightarrow{i_*} & \Omega Y \end{array} \qquad \textit{and} \qquad \begin{array}{ccc} A' & \longrightarrow & * \\ \big\downarrow & \lrcorner & \big\downarrow \\ \Omega X & \xrightarrow{i_*} & \Omega Y. \end{array}$$

Using the description of the standard pullback we see that

$$A' \cong \{(\gamma, *) \in \Omega X \times * \mid i_*(\gamma) = *\} = *$$

and

$$A \cong \{(\gamma, \delta) \in \Omega X \times \mathrm{Map}^*(D^2, Y) \mid i_*(\gamma) = j^*(\delta)\}.$$

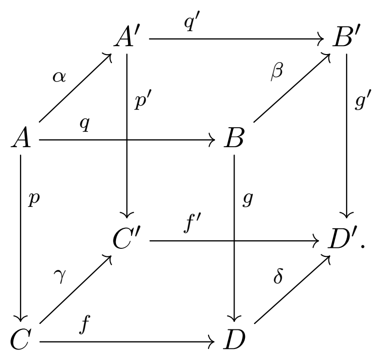
However the last space is homeomorphic to the subspace of $\mathrm{Map}^*(D^2, Y)$ *whose elements under i_* have image lying in X. In general we can see that $A \not\simeq A'$, in particular if X is nullhomotopic in Y then $A \simeq \Omega^2 Y$.*

Remark 2.2. This example was used to highlight that changing a diagram up to homotopy may yield a different limit (or colimit). The author admits that whilst this example may seem exotic for this purpose, it serves the dual purpose of introducing a format of pullbacks that is heavily used in Chapter 5.

Therefore, our aim is to provide a definition of a 'pullback' that is invariant when changes are made up to homotopy. Let the following be squares of topological spaces that commute up to homotopy

$$\mathscr{D} := \begin{array}{ccc} A & \xrightarrow{q} & B \\ p\big\downarrow & & \big\downarrow g \\ C & \xrightarrow{f} & D. \end{array} \qquad \text{and} \qquad \mathscr{E} := \begin{array}{ccc} A' & \xrightarrow{q'} & B' \\ p'\big\downarrow & & \big\downarrow g' \\ C' & \xrightarrow{f'} & D'. \end{array}$$

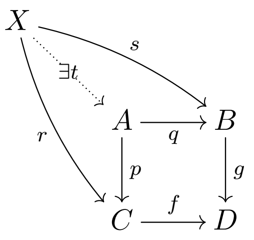
Then a *map of squares* from $\mathscr{D}$ to $\mathscr{E}$ is a homotopy commuting diagram



(2.2)

We say that $\mathscr{D}$ is *homotopy equivalent* to $\mathscr{E}$ if there exists such a map of squares with α, β, γ and δ homotopy equivalences. If $\mathscr{D}$ and $\mathscr{E}$ are homotopy equivalent, and further if $\mathscr{E}$ is a strict pullback and f' and g' are fibrations, then we say that $\mathscr{D}$ is a *homotopy pullback square*. Further, we say that the space A along with the maps p and q are a *homotopy pullback* of the diagram $C \xrightarrow{f} D \xleftarrow{g} B$.

Proposition 2.3. *The following 'universal' property is satisfied by homotopy pullback squares. With notation as above, let $\mathscr{D}$ be a homotopy pullback square. For a space X and maps $s\colon X \to B$ and $r\colon X \to C$ such that $fr \simeq gs$*



(2.3)

there exists $t\colon X \to A$ making the diagram commute up to homotopy. However t is not necessarily unique, even up to homotopy.

Proof. The homotopy pullback $\mathscr{D}$ slots into the homotopy commuting diagram (2.2) where the back square is a strict pullback, the maps f' and g' are fibrations and α, β, γ and δ are homotopy equivalences. Hence there is a homotopy

$$H\colon X \times I \to D'$$

such that $H_0 = f'\gamma r$ and $H_1 = g'\beta s$. However, using the homotopy lifting property of the fibration f', there is a homotopy $\tilde{H}\colon X \times I \to C'$ such that $\tilde{H}_0 = \gamma r$ and $H = f'\tilde{H}$. Therefore $f'\tilde{H}_1 = g'\beta s$ and by the pullback universal property of the back square, there is a unique map $t'\colon X \to A'$ such that $\tilde{H}_1 = p't'$ and $\beta s = q't'$.

For a homotopy inverse α^{-1} of α we set $t := \alpha^{-1}t'$ and check that the diagram (2.3) homotopy commutes

$$pt = p\alpha^{-1}t' \simeq \gamma^{-1}p't' = \gamma^{-1}\tilde{H}_1 \simeq_{\tilde{H}} \gamma^{-1}\tilde{H}_0 = \gamma^{-1}\gamma r \simeq r$$

for a homotopy inverse γ^{-1} of γ. One can similarly check that upper triangle homotopy commutes.

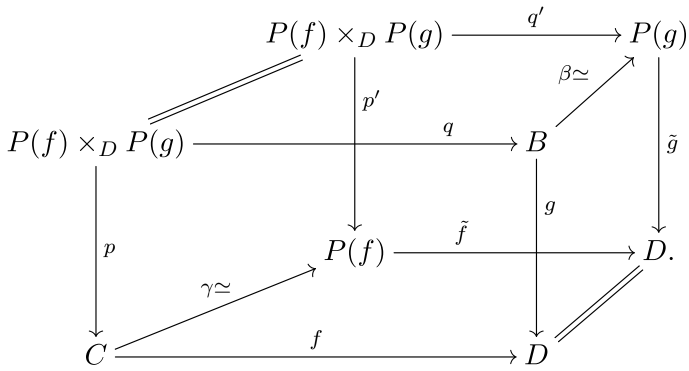
The choices of $\tilde{H}$ and α^{-1} mean that t is not necessarily unique, even up to homotopy. □

We remark that this universal property only required one of the maps f or g to be a fibration. With this in mind, we shall see that the definition of a homotopy pullback square can be relaxed so that a homotopy pullback square only has to be homotopy equivalent to a pullback square whose vertical or horizontal maps are fibrations. Firstly, it will be useful to construct a standard homotopy pullback squares in the same way we constructed one for strict pullbacks.

It is well known that any map $\alpha\colon X \to Y$ between topological spaces can be factored as

$$X \xrightarrow{\simeq} P(\alpha) \xrightarrow{\tilde{\alpha}} Y$$

where $\tilde{\alpha}$ is a fibration with fibre $F(\alpha)$. We apply this process to the maps f and g in diagram $C \xrightarrow{f} D \xleftarrow{g} B$ which we will use to obtain the following diagram



(2.4)

In the diagram the back square is the standard pullback which defines p' and q'. We define $p := p'\gamma^{-1}$ and $q := q'\beta^{-1}$ for γ^{-1} and β^{-1} homotopy inverses of γ and β and this makes the diagram homotopy commute. Therefore the front square is a homotopy pullback square and we refer to the space $P(f) \times_D P(g)$ with maps p and q as the *standard homotopy pullback* of the diagram $C \xrightarrow{f} D \xleftarrow{g} B$.

Proposition 2.4. *Let*

$$\mathscr{D} := \begin{array}{ccc} A & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{f} & D. \end{array}$$

be a homotopy commuting square of topological spaces. Then the following are equivalent

1. $\mathscr{D}$ *is a homotopy pullback square;*

2. $\mathscr{D}$ *is homotopy equivalent to a pullback square whose vertical arrows are fibrations;*

3. $\mathscr{D}$ *is homotopy equivalent to a pullback square whose horizontal arrows are fibrations.*

Furthermore, if A, B, C and D have the homotopy type of connected CW-complexes then the following are equivalent to the above

4. *there exists an induced map of homotopy fibres $F(q) \to F(f)$ which is a homotopy equivalence.*

5. *there exists an induced map of homotopy fibres $F(p) \to F(g)$ which is a homotopy equivalence.*

Proof. $(2 \Rightarrow 1)$ Suppose that the diagram $\mathscr{D}$ is homotopy equivalent to a pullback square

$$\mathscr{E} := \begin{array}{ccc} C' \times_{D'} B' & \xrightarrow{q'} & B' \\ {\scriptstyle p'}\downarrow\!\!\downarrow & \lrcorner & \downarrow\!\!\downarrow{\scriptstyle g'} \\ C' & \xrightarrow{f'} & D' \end{array}$$

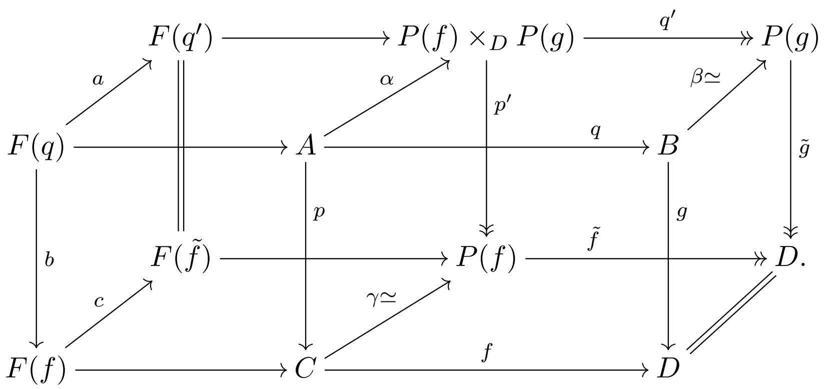
where double arrowheads denote fibrations. We will show that $\mathscr{E}$ is homotopy equivalent to a pullback square whose vertical and horizontal arrows are fibrations.

We now replace f' with a fibration to obtain the diagram

$$\begin{array}{ccccc} C' \times_{D'} B' & \xrightarrow{h} & P(f') \times_{D'} B' & \twoheadrightarrow & B' \\ \downarrow & & \downarrow\!\!\downarrow \;\lrcorner & & \downarrow\!\!\downarrow \\ C' & \xrightarrow{\simeq} & P(f') & \overset{\tilde{f}'}{\twoheadrightarrow} & D' \end{array}$$

where h is defined using the universal property of the right hand pullback square. Note that this implies that the outside square is just the diagram $\mathscr{E}$ and therefore is a pullback square. By Proposition 2.1 (4), this implies that the left hand square is also a pullback square and further that h is a homotopy equivalence by Proposition 2.1 (2). This shows that the outside square is homotopy equivalent to the right hand square and the result follows.

$(4 \Rightarrow 1)$ We replace the maps f and g by fibrations to obtain the following diagram



(2.5)

The back right square is a strict pullback and hence α is defined using the universal property. The left hand cube is a choice of induced maps on the (homotopy) fibres and by assumption we can choose b to be a homotopy equivalence. The back left vertical map is a homeomorphism by Proposition 2.1 (3) and the map c is a homotopy equivalence by the five-lemma. We conclude that a is a homotopy equivalence and again by the five-lemma that α is a homotopy equivalence. This shows that $\mathscr{D}$ is a homotopy pullback.

The implication $(1 \Rightarrow 4)$ is clear from diagram (2.5) and the five lemma. However, notice that connectivity of the spaces can be relaxed. Furthermore, we trivially have that $(1 \Rightarrow 2)$ and by symmetry we have the equivalences of the remaining statements. $\square$

In the following, we shall see that the homotopy type of the homotopy pullback is invariant when maps or spaces in $C \xrightarrow{f} D \xleftarrow{g} B$ are replaced up to homotopy. However, we will require some general properties of homotopy pullbacks.

Proposition 2.5. *Let the following be a homotopy commuting square of spaces having the homotopy type of connected CW-complexes*

$$\mathscr{D} := \begin{array}{ccc} A & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{f} & D \end{array}$$

and assume that f (or g) is a homotopy equivalence. Then q (or p) is a homotopy equivalence if and only if $\mathscr{D}$ is a homotopy pullback square.

Proof. This follows immediately from Proposition 2.4 and the five-lemma. □

Proposition 2.6. *Let the following be a commutative diagram of topological spaces*

$$\begin{array}{ccccc} X & \longrightarrow & Y & \longrightarrow & Z \\ \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' \end{array}$$

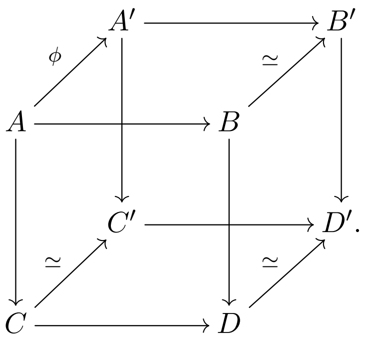
then if the right square is a homotopy pullback then the left square is a homotopy pullback if and only if the outside square is a homotopy pullback.

Proof. By replacing the relevant diagrams with strict pullbacks, the results follow from Proposition 2.1. □

Proposition 2.7. *Let the following be homotopy pullback squares of spaces with the homotopy type of connected CW-complexes*

$$\mathscr{D} := \begin{array}{ccc} A & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{f} & D. \end{array} \quad \text{and} \quad \mathscr{E} := \begin{array}{ccc} A' & \xrightarrow{q'} & B' \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle g'} \\ C' & \xrightarrow{f'} & D'. \end{array}$$

Assume that the diagrams $C \xrightarrow{f} D \xleftarrow{g} B$ and $C' \xrightarrow{f'} D' \xleftarrow{g'} B'$ are homotopy equivalent, that is, there exist homotopy equivalences making the following diagram homotopy *commute*



(2.6)

Then a map $\phi\colon A \to A'$ *induced by the universal property of* $\mathscr{E}$ *is a homotopy equivalence, in other words* $\mathscr{D}$ *and* $\mathscr{E}$ *are homotopy equivalent.*

Proof. Consider the front and right squares

$$\begin{array}{ccccc} A & \longrightarrow & B & \xrightarrow{\simeq} & B' \\ \downarrow & & \downarrow & & \downarrow \\ C & \longrightarrow & D & \xrightarrow{\simeq} & D'. \end{array}$$

Then by assumption the front square is a homotopy pullback and the right square is a homotopy pullback by Proposition 2.5. Therefore the outside rectangle of this diagram is a homotopy pullback by Proposition 2.6. By the homotopy commutativity of the cube, the outside rectangle of the left and back squares

$$\begin{array}{ccccc} A & \xrightarrow{\phi} & A' & \longrightarrow & B' \\ \downarrow & & \downarrow & & \downarrow \\ C & \xrightarrow{\simeq} & C' & \longrightarrow & D' \end{array}$$

is also a homotopy pullback. By assumption the back square is a homotopy pullback and hence the left hand square is by Proposition 2.6. We conclude that ϕ is a homotopy equivalence by Proposition 2.5. □

Corollary 2.8. *Let the following be a homotopy pullback square of spaces having the homotopy type of connected CW-complexes*

$$\mathscr{D} := \begin{array}{ccc} A & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{f} & D \end{array}$$

and assume that f *(or* g*) is a fibration. Then there is a homotopy equivalence*

$$A \simeq B \times_D C.$$

Proof. The assumption that f (or g) is a fibration implies that the strict pullback square

$$\begin{array}{ccc} B \times_D C & \xrightarrow{q} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow{f} & D \end{array}$$

is a homotopy pullback square. The result follows from Proposition 2.7. □

2.2 Obtaining Homotopy Decompositions

At the start of this thesis we noted that providing a homotopy decomposition of any space can be a powerful method for obtaining homotopy groups. However, it feels like a bit of logical leap to immediately expect gauge groups to decompose as a product. In this section we provide motivation for that expectation.

A product of two spaces $A \times B$ satisfies the following universal property:

$$\begin{array}{ccccc} & & X & & \\ & {\scriptstyle f_1}\swarrow & \downarrow{\scriptstyle f} & \searrow{\scriptstyle f_2} & \\ A & \xleftarrow[\pi_1]{} & A \times B & \xrightarrow[\pi_2]{} & B \end{array}$$

for any space X and maps $f_1\colon X \to A$ and $f_2\colon X \to B$ there is a unique map

$$f\colon X \to A \times B$$

making the above diagram commute. This universal property gives a bijection of sets

$$\mathrm{Map}(X, A \times B) \longleftrightarrow \mathrm{Map}(X, A) \times \mathrm{Map}(X, B).$$

Suppose f and f' in $\mathrm{Map}(X, A \times B)$ are sent to (f_1, f_2) and (f_1', f_2') under the bijection, then the universal property also implies that the following are equivalent

1. there exists homotopies such that $f_1' \sim f_1$ and $f_2' \sim f_2$;
2. there exists a homotopy such that $f' \sim f$.

We conclude that there is an bijection of homotopy sets $[X, A \times B]$ and $[X, A] \times [X, B]$. Dually, there is a bijection $[A \vee B, X]$ and $[A, X] \times [B, X]$. Since a lot of homotopy invariants take the form of these homotopy sets, it makes sense to decompose spaces as either a product or a wedge. We now motivate the expectation that topological groups should not decompose as a wedge.

Proposition 2.9. *Let F be a field, let X be a connected H-space and suppose $X \simeq A \vee B$ then either $\tilde{H}_1 := \tilde{H}_*(A; F)$ or $\tilde{H}_2 := \tilde{H}_*(B; F)$ is trivial.*

Proof. There is an induced H-space structure on A and B since they retract off X. As graded F-modules we can write

$$H_*(X;F) \cong F \oplus \tilde{H}_1 \oplus \tilde{H}_2.$$

It is well known that the homology of a connected H-space has the structure of a Hopf algebra. Furthermore, the positively graded Hopf algebra structure of $H_*(X;F)$ should split as a direct product as above. However, since

$$(\tilde{H}_1 \otimes \tilde{H}_1) \oplus (\tilde{H}_2 \otimes \tilde{H}_2) \neq (\tilde{H}_1 \oplus \tilde{H}_2) \otimes (\tilde{H}_1 \oplus \tilde{H}_2)$$

this is impossible unless either $\tilde{H}_1$ or $\tilde{H}_2$ is trivial. □

Taking motivation from the above, we seek methods to homotopy decompose a space as a product. The following results highlight examples of such methods used throughout this thesis.

Proposition 2.10. *Let $\Omega B \to F \xrightarrow{f} E$ be a principal fibration. If there is a map $s\colon E \to F$ such that $fs \simeq \mathrm{id}_E$, then there is a homotopy equivalence*

$$F \simeq \Omega B \times E.$$

Proof. We will show that the sequence

$$\Omega B \times E \xrightarrow{\mathrm{id}_{\Omega B} \times s} \Omega B \times F \to F$$

is a homotopy equivalence where the last arrow is the action of ΩB on F. Consider the homotopy commutative diagram, where the columns are homotopy fibrations

$$\begin{array}{ccccc}
\Omega B \times * & \longrightarrow & \Omega B \times \Omega B & \longrightarrow & \Omega B \\
\downarrow & & \downarrow & & \downarrow \\
\Omega B \times E & \xrightarrow{\mathrm{id}_{\Omega B} \times s} & \Omega B \times F & \longrightarrow & F \\
\downarrow & & \downarrow & & \downarrow \\
E & = & E & = & E.
\end{array}$$

Then if we consider the induced maps on homotopy groups then the top and bottom rows are the identity. The result follows. □

Corollary 2.11. *Let $\Omega B \to F \xrightarrow{f} E$ be a principal fibration induced by a map $E \xrightarrow{\partial} B$. If ∂ is nullhomotopic then there is a homotopy equivalence*

$$F \simeq \Omega B \times E.$$

Proof. This follows immediately from Proposition 2.10 since if ∂ is nullhomotopic then the identity map $\mathrm{id}_E\colon E \to E$ lifts to a section $s\colon E \to F$ as in the diagram

$$\begin{array}{ccccc} & & E & & \\ & \overset{s}{\swarrow} & \downarrow{\scriptstyle \mathrm{id}_E} & & \\ F & \longrightarrow & E & \xrightarrow{\partial} & B. \end{array}$$

□

Proposition 2.12. *Let the following be a principal fibration*

$$\Omega E \xrightarrow{j} M \times N \xrightarrow{f} F$$

such that there exist maps $l\colon M \to \Omega E$ and $g\colon F \to N$ with the properties

$$(M \xrightarrow{l} \Omega E \xrightarrow{j} M \times N \xrightarrow{p_1} M) \simeq \mathrm{id}_M \tag{2.7}$$

$$(N \hookrightarrow M \times N \xrightarrow{f} F \xrightarrow{g} N) \simeq \mathrm{id}_N \tag{2.8}$$

where p_1 is the projection onto the first factor. Let D be the homotopy fibre of $g\colon F \to N$ then there is an equivalence

$$\Omega E \simeq M \times \Omega D.$$

Proof. By property (2.8), the principal fibration $\Omega D \to \Omega F \to \Omega N$ has a section. Hence by Proposition 2.10 the following composition is a homotopy equivalence

$$\Omega N \times \Omega D \to \Omega F \times \Omega D \to \Omega F. \tag{2.9}$$

By restricting the action obtained from the following homotopy fibration

$$\Omega F \xrightarrow{\partial} \Omega E \xrightarrow{j} M \times N$$

we can induce an action $\alpha\colon \Omega E \times \Omega D \to \Omega E$. Analogous to proof of Proposition 2.10, we aim to show that the composition

$$M \times \Omega D \xrightarrow{l \times \mathrm{id}_{\Omega D}} \Omega E \times \Omega D \xrightarrow{\alpha} \Omega E$$

is homotopy equivalence.

The above composition then fits into the following diagram

$$\begin{array}{ccccc}
\Omega N \times \Omega D & \longrightarrow & \Omega F \times \Omega D & \longrightarrow & \Omega F \\
\downarrow{\scriptstyle * \times \mathrm{id}_{\Omega D}} & & \downarrow{\scriptstyle \partial \times \mathrm{id}_{\Omega D}} & & \downarrow{\scriptstyle \partial} \\
M \times \Omega D & \xrightarrow{l \times \mathrm{id}_{\Omega D}} & \Omega E \times \Omega D & \xrightarrow{\alpha} & \Omega E \\
\downarrow{\scriptstyle \mathrm{id}_M \times *} & & \downarrow{\scriptstyle j \times *} & & \downarrow{\scriptstyle j} \\
M \times N & = & (M \times N) \times * & = & M \times N
\end{array}$$

where the columns are homotopy fibrations. The right hand squares homotopy commute by properties of the action α.

The top left square commutes on the restriction to ΩD because both routes give the identity $\mathrm{id}_{\Omega D}$. For the factor ΩN, the upper route consists of the composition

$$\Omega N \hookrightarrow \Omega M \times \Omega N \xrightarrow{\Omega f} \Omega F \xrightarrow{\partial} \Omega E$$

which is nullhomotopic and hence the top left square homotopy commutes.

The bottom left square commutes on the restriction to ΩD. For the other factor, notice that the composition

$$\Omega E \xrightarrow{j} M \times N \xrightarrow{p_2} N$$

is nullhomotopic since $p_2 \simeq g \circ f$ by property (2.8). Using property (2.7), we conclude that the bottom left square homotopy commutes.

The top row is the same as the composition in (2.9) and hence is a homotopy equivalence. The induced maps on homotopy groups are then isomorphisms for the top and bottom rows. The result then follows from the five lemma. □

2.3 Homotopy Types of Classical Groups

Some of the results in this thesis are intimately linked with some of the results found in [Har61]. For completeness, we replicate some of those results here.

Theorem 2.13. *Let $p \neq 2$ be prime, then there are p-local homotopy equivalences*

1. $SU(2n) \simeq_p Sp(n) \times (SU(2n)/Sp(n))$;
2. $SU(2n+1) \simeq_p SO(2n+1) \times (SU(2n+1)/SO(2n+1))$.

From which we will deduce the following.

Corollary 2.14. *Let $p \neq 2$ be prime, then there is a p-local homotopy equivalence*

$$Sp(n) \simeq_p SO(2n+1).$$

To prove Theorem 2.13, we will show that for a prime $p \neq 2$ there exist p-local sections to the principal fibrations

$$Sp(n) \to SU(2n) \xrightarrow{q_0} SU(2n)/Sp(n);$$
$$SO(2n+1) \to SU(2n+1) \xrightarrow{q_1} SU(2n+1)/SO(2n+1).$$

The existence of these p-local sections essentially follows from the existence of involutions on $SU(m)$ with nice enough properties.

Let $\sigma_1 \colon SU(m) \to SU(m)$ be the involution induced by complex conjugation and for m even define $\sigma_0 = J^{-1}\sigma_1 J$ where

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

Note that $SU(2n+1)^{\sigma_1} = SO(2n+1)$ and $SU(2n)^{\sigma_0} = Sp(n)$. We will define the p-local sections

$$s_1 \colon SU(2n+1)/SO(2n+1) \to SU(2n+1);$$
$$s_0 \colon SU(2n)/Sp(n) \to SU(2n).$$

Let $K_1 = K_1(2n+1) := SO(2n+1)$ and let $K_0 = Sp(n)$, then for $i \in \{0,1\}$ define s_i to be the map

$$s_i(AK_i) := A\sigma_i(A)^{-1}.$$

The maps s_i are well defined since the fixed point set of σ_i is K_i. We will calculate the induced maps in cohomology

$$(q_i s_i)^* \colon H^*(SU(2n+i)/K_i) \to H^*(SU(2n+i)/K_i)$$

and show that they are isomorphisms localised a prime $p \neq 2$.

It is well known (see [Bor53] or [MT91] for example) that localised at a prime not equal to 2 we have

$$H^*(SU(2n+i)) \cong \Lambda[x_3, x_5, \ldots, x_{4(n+i)-1}]$$

and that

$$H^*(SU(2n+i)/K_i) \cong \Lambda[y_5, y_9, \ldots, y_{4(n+i)-3}] \tag{2.10}$$

such that $q_i^*(y_j) = x_j$ for $j \equiv 1 \mod 4$ and $5 \leq j \leq 4(n+i)-3$. We first calculate the induced map $(s_iq_i)^* \colon H^*(SU(2n+i)) \to H^*(SU(2n+i))$.

Lemma 2.15. *For each i, the map $(s_iq_i)^*$ sends generators of $H^*(SU(2n+i))$ as follows*

$$(s_iq_i)^*(x_j) = \begin{cases} 2x_j & \text{for } j \equiv 1 \mod 4; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Denote $m = 2n+i$ then notice that the map s_iq_i is homotopic to the composition

$$SU(m) \xrightarrow{\Delta} SU(m) \times SU(m) \xrightarrow{1\otimes\sigma_i} SU(m) \times SU(m) \xrightarrow{1\otimes -1} \\ \to SU(m) \times SU(m) \xrightarrow{\mu} SU(m)$$

where Δ denotes the diagonal map, -1 is the map that sends an element to its inverse and μ is the multiplication in $SU(m)$. Since each x_j is primitive, we obtain

$$\begin{aligned} (s_iq_i)^*(x_j) =& \Delta^*(1 \otimes \sigma_i^*)(1 \otimes -1^*)\mu^*(x_j) \\ =& \Delta^*(x_j \otimes 1 + 1 \otimes \sigma_i^*(-x_j)) \\ =& x_j - \sigma_i^*(x_j). \end{aligned}$$

It remains to calculate the element $\sigma_i^*(x_j)$, but since σ_0 is a conjugate of σ_1 it suffices to calculate $\sigma_1^*(x_j)$. The following is a principal bundle

$$SU(m-1) \xrightarrow{\alpha} SU(m) \xrightarrow{\beta} SU(m)/SU(m-1) \cong S^{2m-1}.$$

Re-notate each generator of $H^*(SU(m))$ by x_j^m, then by looking at the long exact sequence in cohomology arising from the Serre spectral sequence we can see that generators can be chosen such that

$$\alpha^*(x_j^m) = x_j^{m-1} \tag{2.11}$$

for $1 \leq j \leq 2m-3$. Again, from the long exact sequence in cohomology, we can choose a generator $y \in H^{2m-1}(S^{2m-1};\mathbb{Z})$ such that

$$\beta^*(y) = x_{2m-1}^m.$$

Therefore, we can study σ_1^* by inductively looking at the induced maps on S^{2k-1}. The map $\beta\colon SU(m) \to S^{2m-1}$ can be thought of as restricting to the last column of the matrix or more concretely

$$q(A) = Ae_m = (a_{1m}, a_{2m}, \ldots, a_{mm})^t.$$

The induced map of σ_1 on S^{2m-1} is therefore defined via

$$(a_{1m}, a_{2m}, \ldots, a_{mm})^t \mapsto (\overline{a_{1m}}, \overline{a_{2m}}, \ldots, \overline{a_{mm}})^t.$$

It is easy to see that this can be decomposed as a series of m reflections and is therefore a degree $(-1)^m$ map. Indeed, for $U(1) \cong S^1$, this can easily be seen to be the case since

$$\sigma_1(e^{i\theta}) = e^{-i\theta}.$$

We deduce that each map σ_i^* is defined as follows on the generators

$$\sigma_i^*(x_j^m) = \begin{cases} -x_j^m & \text{for } j \equiv 1 \mod 4 \\ x_j^m & \text{for } j \equiv 3 \mod 4. \end{cases}$$

This finishes the proof. □

It is not much more work to deduce Theorem 2.13.

Proof of Theorem 2.13. By Lemma 2.15 and equation (2.10), we deduce that for generators y_j of $H^*(SU(2n+i)/K_i)$ we have

$$(q_i s_i)^*(y_j) = s_i^*(x_j) = 2y_j.$$

Therefore, after localising at a prime $p \neq 2$, the map $q_i s_i$ induces an isomorphism on the cohomology of the simply connected space $SU(2n+i)/K_i$. Hence the result follows by Whitehead's theorem. □

We now provide the proof for Corollary 2.14 in which we assume that spaces and maps are localised at a prime $p \neq 2$.

Proof of Corollary 2.14. By Theorem 2.13 we have

$$SU(2n+1) \simeq_p SO(2n+1) \times SU(2n+1)/SO(2n+1)$$

then let $p_1 \colon SU(2n+1) \to SO(2n+1)$ be the projection onto the first factor. Now, we define $\gamma \colon Sp(n) \to SO(2n+1)$ to be the composition

$$Sp(n) \hookrightarrow SU(2n) \xrightarrow{\alpha} SU(2n+1) \xrightarrow{p_1} SO(2n+1)$$

where the two leftmost arrows are the canonical inclusions. It is well known that

$$H^*(SO(2n+1)) \cong H^*(Sp(n)) \cong \Lambda[z_3, z_7, \ldots, z_{4n-1}]$$

and by Theorem 2.13 and equation (2.11) the induced map γ^* in cohomology is an isomorphism. Finally $SO(2n+1)$ is p-locally homotopic equivalent to its simply connected cover $Spin(2n+1)$ and hence γ is a p-local homotopy equivalence. □

Chapter 3

Gauge Groups

3.1 Definitions and Basic Properties

Let P be a topological space and G be a topological group. A *right action* of G on P is a continuous map

$$\begin{aligned} P \times G &\to P \\ (p, g) &\mapsto p \cdot g \end{aligned}$$

for $p \in P$ and $g \in G$ such that

1. $(p \cdot g) \cdot h = p \cdot (gh)$ for all $p \in P$ and $g, h \in G$;
2. $p \cdot e = p$ for all $p \in P$ and e the identity of G.

A *left action* is defined similarly for a continuous $G \times P \to P$. We call the space P a *right (or left) G-space* or just a *G-space* when the context is clear. In general, we will use prefix terms with G to mean 'G-equivariant', for example G-maps will be G-equivariant maps and G-homotopies will mean G-equivariant homotopies.

Let X be a topological space, then we define a *principal G-bundle* over X to be a surjective map $\pi \colon P \to X$ together with an action $P \times G \to P$ such that

1. the map $P \times G \to P \times P$ given by $(p, g) \mapsto (p, p \cdot g)$ is a homeomorphism onto its image;
2. $X = P/G$ and π is the quotient map;
3. P is locally trivial. That is, for all $x \in X$ there exists an open neighbourhood U of x such that there is a homeomorphism $\varphi \colon \pi^{-1}(U) \to U \times G$ satisfying $\pi = \pi_2 \circ \varphi$

for π_2 the projection onto the second factor, and $\varphi(p\cdot g) = \varphi(p)\cdot g$ where the action on $U \times G$ is given by $(x, g) \cdot g' = (x, gg')$.

For a principal G-bundle $\pi\colon P \to X$, we call X the *base space*, P the *total space* and π the *projection*. We remark that property 1 of $\pi\colon P \to X$ implies that

$$p \cdot g \neq p \text{ for all } p \in P \text{ and } g \neq \mathrm{id} \in G. \tag{3.1}$$

If an action of G on P has property (3.1) we describe it as a *free* action. One can also see that properties 1 and 2 of $\pi\colon P \to X$ imply that for any point $x \in X$ we have $\pi^{-1}(x) = G$. We therefore call G the *fibre* of the bundle.

Given principal G-bundles P and P' over a space X, an *isomorphism* of such bundles is a G-equivariant map $\phi\colon P \to P'$ such that the diagram

$$\begin{array}{ccc} P & \xrightarrow{\phi} & P' \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\mathrm{id}} & X \end{array} \tag{3.2}$$

commutes. We denote isomorphism classes of principal G-bundles over Y by $Prin_G(Y)$.

The next few statements are classical properties of principal G-bundles. Proofs of these statements can be found in [Hus94, Ch.4] or proofs of equivariant analogues can be found in Chapter 4.

Proposition 3.1. *Let $\pi\colon P \to X$ be a principal G-bundle, let $f\colon Y \to X$ be a map and consider the strict pullback (see Section 2.1)*

$$\begin{array}{ccc} f^*(P) & \longrightarrow & P \\ \downarrow{\scriptstyle \pi'} & \lrcorner & \downarrow{\scriptstyle \pi} \\ Y & \xrightarrow{f} & X. \end{array}$$

Then the map $\pi'\colon f^*(P) \to Y$ is also a principal G-bundle. □

We therefore call $\pi'\colon f^*(P) \to Y$ the *pullback bundle* of P under f. We remark that the following two theorems require X and G to have numerable covers, which is automatic for the case where X and G are CW-complexes, see [Hus94, Se.4.12-13] for details.

Theorem 3.2. *Let $\pi\colon P \to X$ be a principal G-bundle, and let $f_0, f_1\colon Y \to X$ be maps from a space Y. Suppose that f_0 is homotopic to f_1 then their pullback bundles are isomorphic*

$$f_0^*(P) \cong f_1^*(P).$$ □

This theorem gives the existence of a well defined map

$$[Y, X] \to Prin_G(Y)$$

from the set of homotopy classes of maps from Y to X to isomorphism classes of principal G-bundles over Y. We say that a principal G-bundle is *universal* if the map above is a bijection for all Y.

Theorem 3.3. *Let G be a topological group, then a principal G-bundle*

$$EG \to BG.$$

is a universal bundle if and only if EG is weakly contractible. □

The existence of a universal bundle was shown in [Mil56] where an explicit construction of EG as the infinite join $\lim_{n\to\infty} G^{*n}$ is shown to have the appropriate properties. In particular, it is clear that EG has a right action induced by an action of G on itself, and BG is defined to be EG/G.

Let $\pi\colon P \to X$ be a principal G-bundle and recall the definition of isomorphism above diagram (3.2), then we define an *automorphism* of P to be an isomorphism $\phi\colon P \to P$. Now let the mapping space $\mathrm{Map}(P,P)$ be endowed with the compact open topology. We define the *gauge group* $\mathscr{G}(P)$ of P to be the set of all automorphisms of P endowed with the subspace topology from $\mathrm{Map}(P,P)$. It is clear that $\mathscr{G}(P)$ is a group via composition.

Proposition 3.4. *The gauge group $\mathscr{G}(P)$ can be identified with the space*

$$\mathrm{Map}_G(P,G) = \left\{ f\colon P \to G \mid f(p\cdot g) = g^{-1}f(p)g \right\}.$$

Moreover, if P is trivial or if G is abelian, then we have

$$\mathscr{G}(P) = \mathrm{Map}(X,G).$$

Proof. Let $\phi : P \to P$ be an element of $\mathscr{G}(P)$, then for all $p \in P$ we have $\pi(p) = \pi\phi(p)$. Hence p and $\phi(p)$ lie in the same fibre, and therefore for some $g_p \in G$ we have $\phi(p) = p\cdot g_p$. The desired map $f\colon P \to G$ is then defined as $f(p) = g_p$. For G-equivariance, the fact that the G-action on P is free and

$$(p\cdot g)\cdot g_{p\cdot g} = \phi(p\cdot g) = \phi(p)\cdot g = (p\cdot g_p)\cdot g$$

implies that $gg_{p\cdot g} = g_p g$ and so $f(p\cdot g) = g^{-1}f(p)g$.

Conversely, given $f \in \mathrm{Map}_G(P,G)$, we define $\phi \in \mathscr{G}(P)$ via

$$\phi(p) = p\cdot f(p).$$

Then, clearly ϕ covers the identity and it is easy to show that ϕ is G-equivariant using the equivariance of f.

For the second part, let $P = X \times G$ and then $\mathscr{G}(P) = \mathrm{Map}_G(X \times G, G)$. For all $x \in X$ and $g \in G$ we have $f(x, g) = f(x, e) \cdot g$. Therefore f is completely determined by what it does to elements (x, e), hence $\mathscr{G}(P) = \mathrm{Map}(X, G)$.

If G is abelian, then we have $f(p \cdot g) = f(p)$, and since $P/G = X$ we have that $\mathrm{Map}_G(P, G)$ descends to the required space. □

Theorem 3.5. *Let P be a principal G-bundle that is a compact CW-complex, there is a homotopy equivalence*

$$B\mathscr{G}(P) \simeq \mathrm{Map}(X, BG; P)$$

where $\mathrm{Map}(X, BG; P)$ denotes the connected component of $\mathrm{Map}(X, BG)$ containing a map $f\colon X \to BG$ such that P is the pullback bundle of EG under f.

Proof. Consider the space $\mathrm{Map}_G(P, EG)$ of G-equivariant maps from P to the total space of the universal bundle. By composing, there is a free action of $\mathscr{G}(P)$ on $\mathrm{Map}_G(P, EG)$. Let $f \in \mathrm{Map}_G(P, G)$ and $g \in \mathrm{Map}_G(P, EG)$, then with the identification shown in Proposition 3.4, the action of $\mathscr{G}(P)$ on $\mathrm{Map}_G(P, EG)$ corresponds to

$$(g \cdot f)(p) = g(p) \cdot f(p).$$

We deduce that $\mathrm{Map}_G(P, EG)/\mathscr{G}(P) = \mathrm{Map}_G(P, BG)$ where G acts on BG trivially. This trivial G-action implies that $\mathrm{Map}_G(P, BG)$ can be identified with the mapping space $\mathrm{Map}(X, BG; P)$. Then, the following is obviously a fibration

$$\mathscr{G}(P) \to \mathrm{Map}_G(P, EG) \to \mathrm{Map}(X, BG; P)$$

but one can arrange for it to be locally trivial (see [AB83]) and therefore it is actually a principal $\mathscr{G}(P)$-bundle.

It remains to show that $\mathrm{Map}_G(P, EG)$ is contractible, we adapt an argument of [Bai14] which in turn is an adaptation of [Dol63]. When P is a compact CW-complex we can deduce[1] that

$$\mathrm{Map}_G(P, EG) = \lim_{n\to\infty} \mathrm{Map}_G(P, G^{*n})$$

where G^{*n} is the n-fold topological join. It is therefore enough to show that for each n there exists an m such that the inclusion

$$i_n\colon \mathrm{Map}_G(P, G^{*n}) \to \mathrm{Map}_G(P, G^{*n+m})$$

is nullhomotopic. But i_n factors as

$$\mathrm{Map}_G(P, G^{*n}) \hookrightarrow \mathrm{Map}_G(P, G^{*n}) * \mathrm{Map}_G(P, G^{*m}) \to \mathrm{Map}_G(P, G^{*n+m})$$

[1] This is shown in a much more general setting in [Hov99, Prop 2.4.2]. The fact that P is compact is essential, and the fact that P is a CW-complex implies that P is Hausdorff and that for $m > n$ the injections $G * n \hookrightarrow G * m$ have closed image.

and the inclusion of a factor of a join is nullhomotopic. □

The *pointed gauge group*, $\mathscr{G}^*(P)$ is defined as the space of automorphisms of P that restrict to the identity over a basepoint of X. It should be noted that the arguments in Propositions 3.4 and 3.5 also hold for the pointed case and hence we obtain

$$B\mathscr{G}^*(P) \simeq \mathrm{Map}^*(X, BG; P).$$

By studying the mapping spaces above we will be able to recover some properties of the topology of the gauge groups $\mathscr{G}(P)$. We now describe two fibrations involving these mapping spaces that will be used throughout the thesis. First, we let

$$A \xrightarrow{f} X \xrightarrow{g} X/A$$

be a cofibration, then for any space Z the sequence

$$\mathrm{Map}^*(X/A, Z) \xrightarrow{g^*} \mathrm{Map}^*(X, Z) \xrightarrow{f^*} \mathrm{Map}^*(A, Z) \tag{3.3}$$

is a fibration. This follows from the duality of the homotopy lifting and extension properties.

If we instead applied the unpointed functor $\mathrm{Map}(-, Z)$ then we would not obtain a pointed fibration, however there is a fibration relating the unpointed and pointed mapping spaces. For spaces, X and Z we have the *evaluation* fibration

$$\mathrm{Map}^*(X, Z) \to \mathrm{Map}(X, Z) \xrightarrow{\mathrm{ev}} Z \tag{3.4}$$

where 'ev' evaluates a function at the basepoint of X. We can restrict these fibrations to a particular path component of $\mathrm{Map}(X, Z)$ (or $\mathrm{Map}^*(X, Z)$) which allows us to analyse the topology of the classifying space of the gauge groups.

Remark 3.6. By the pointed exponential law, there is a homotopy equivalence

$$\Omega\,\mathrm{Map}^*(X, BG) \simeq \mathrm{Map}^*(X, G).$$

In the unpointed case, the relevant equivalence does not hold in general, however it does hold for the cases in the second part of Proposition 3.4. In these cases, we use the evaluation fibration (3.4) to obtain the principal fibration

$$\mathrm{Map}^*(X, G) \to \mathrm{Map}(X, G) \xrightarrow{\Omega\,\mathrm{ev}} G.$$

There is a section $s\colon G \to \mathrm{Map}(X, G)$ defined by $s(g)(x) = g$ for all $g \in G, x \in X$. Furthermore, the multiplication on G induces a multiplication on $\mathrm{Map}(X, G)$, and therefore we obtain a homotopy equivalence

$$\mathrm{Map}(X,G) \simeq \mathrm{Map}^*(X,G) \times G.$$

3.2 Homotopy Types of Path-Components

Clearly the gauge group depends on the isomorphism class of the bundle P. However, there are cases where gauge groups are homotopy equivalent for non-isomorphic bundles. In the pointed case, the gauge groups behave particularly well in this manner. We take motivation from Theorem 3.5 and prove some general facts about the path components of spaces $\mathrm{Map}^*(X,Y)$.

Proposition 3.7. *Let X be a space and let Y be a homotopy-associative H-space with homotopy inverse. Then all the path components of $\mathrm{Map}^*(X,Y)$ are homotopy equivalent.*

Proof. The assumed H-space structure on Y endows the space $\mathrm{Map}^*(X,Y)$ with the structure of a homotopy-associative H-space with homotopy inverse. Therefore the set

$$\pi_0(\mathrm{Map}^*(X,Y)) = [X,Y]_*$$

has the structure of a group. Let α be an element in $[X,Y]_*$ and let $\mathrm{Map}^*(X,Y;\alpha)$ denote the path component of $\mathrm{Map}^*(X,Y)$ corresponding to α.

We fix an element $\tilde{f}$ in $\mathrm{Map}^*(X,Y;\alpha^{-1})$ and define

$$\Theta_{\tilde{f}}\colon\ \mathrm{Map}^*(X,Y;\alpha) \to \mathrm{Map}^*(X,Y;1)$$

to be the map that sends an element $f \in \mathrm{Map}^*(X,Y;\alpha)$ to

$$f + \tilde{f} := X \xrightarrow{\Delta} X \times X \xrightarrow{f\times\tilde{f}} Y \times Y \xrightarrow{m} Y$$

where Δ is the diagonal map and m is the multiplication in Y. Let $\tilde{g}\colon X \to Y$ be the map that sends x to $(\tilde{f}(x))^{-1}$. Now $\tilde{g}$ is in $\mathrm{Map}^*(X,Y;\alpha)$ so $\tilde{g}$ defines a map

$$\Theta_{\tilde{g}}\colon\ \mathrm{Map}^*(X,Y;1) \to \mathrm{Map}^*(X,Y;\alpha)$$

in the same way as $\Theta_{\tilde{f}}$. Notice that $\tilde{g} + \tilde{f} \simeq \tilde{f} + \tilde{g} \simeq c_{*_Y}$ where c_{*_Y} is the constant map onto $*_Y$. Using homotopy associativity, we deduce that $\Theta_{\tilde{g}}$ is a homotopy inverse to $\Theta_{\tilde{f}}$. □

It is clear that the above also holds if we require Y to be a space and X to be a homotopy-co-associative co-H-space with homotopy-inverse. Essentially this holds due to the co-multiplication on the space X, and more generally one should expect a similar result if there is a co-action on X by a co-H-space. For an n-dimensional CW-complex X with one n-cell, the pinch $X \to X \vee S^n$ is the co-action of the cofibration sequence

$$X^{n-1} \hookrightarrow X \to S^n$$

where X^{n-1} is the $(n-1)$-skeleton of X. This induces an action of $\pi_n(Y)$ on $\mathrm{Map}^*(X, Y)$ defined via

$$X \xrightarrow{\text{pinch}} X \vee S^n \xrightarrow{f \vee \alpha} Y \vee Y \xrightarrow{\text{fold}} Y \tag{3.5}$$

where $\alpha \in \pi_n(Y)$ and $f \in \mathrm{Map}^*(X, Y)$. This gives a homotopy equivalence between the path components containing f and the map (3.5).

Theorem 3.8. *Let X be an n-dimensional CW-complex and let P and P' be principal G-bundles over X. Suppose that there is an action of $\pi_n(BG)$ on $\mathrm{Map}^*(X, BG)$ defined as above and that there exists an element of $\pi_n(BG)$ that sends*

$$\mathrm{Map}^*(X, BG; P) \text{ to } \mathrm{Map}^*(X, BG; P').$$

Then $B\mathscr{G}^*(P) \simeq B\mathscr{G}^*(P')$. □

The unpointed case is more complicated, when $X = S^4$ and $G = SU(2)$ there are infinitely many homotopy types of the components of $\mathrm{Map}(X, BG)$, although there are only 6 homotopy types of $\mathscr{G}$(P) as we vary P, see [Kon91] and [Mas90]. In fact, it was shown in [CS00] that there are only ever finitely many homotopy types of the gauge group $\mathscr{G}$(P). We go on to consider the case where $G = U(n)$ and X is a Riemann surface.

3.3 Principal $U(n)$-bundles over a Riemann Surface

Let X be a Riemann surface of genus g, then principal $U(n)$-bundles over X are classified by $[X, BU(n)]_* \cong \mathbb{Z}$, which can be thought of as the first Chern class. Therefore for an integer c we denote the relevant gauge groups by $\mathscr{G}(X, U(n); c)$ and similarly we denote $\mathrm{Map}(X, BU(n); c)$ for the relevant components of the mapping spaces.

Since X is a 2-dimensional CW-complex, the preamble to Theorem 3.8 induces an action of $\pi_2(BU(n)) \cong \mathbb{Z}$ on $\mathrm{Map}^*(X, BU(n))$. It is easy to see that $d \in \pi_2(BU(n))$ sends elements in $\mathrm{Map}(X, BU(n); c)$ into $\mathrm{Map}(X, BU(n); c+d)$. We deduce that Theorem 3.8 gives homotopy equivalences between any two path components of $\mathrm{Map}^*(X, BU(n))$. Therefore the homotopy types of the pointed gauge groups do not depend on the isomorphism class of the bundle.

We will analyse some properties of path components in the unpointed case which were considered in [Sut92], but first we reduce to the case when $g = 0$ by following [The10].

Theorem 3.9. *There is a homotopy decomposition*

$$\mathscr{G}(X, U(n); c) \simeq \mathscr{G}(S^2, U(n); c) \times \prod_{i=1}^{2g} \Omega U(n).$$

Proof. Consider the cofibration sequence

$$S^1 \xrightarrow{f} \bigvee_{2g} S^1 \xrightarrow{i} X \xrightarrow{\text{pinch}} S^2 \xrightarrow{\Sigma f} \bigvee_{2g} S^2 \tag{3.6}$$

where f is the attaching map of the 2-cell in X. Explicitly, let $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ be the one-cells of X, then we glue the boundary of the 2-cell via

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_g\beta_g\alpha_g^{-1}\beta_g^{-1}.$$

It is then clear that Σf is trivial, since

$$\Sigma f = \Sigma\alpha_1\Sigma\beta_1\Sigma\alpha_1^{-1}\Sigma\beta_1^{-1} \cdots \Sigma\alpha_g\Sigma\beta_g\Sigma\alpha_g^{-1}\Sigma\beta_g^{-1}$$

and each expression on right hand side is an element of the abelian group $\pi_2(\bigvee_{2g} S^2)$. The triviality of Σf can also be deduced from the fact that it is the suspension of a sum of Whitehead products. Essentially, the theorem will follow due to this triviality.

Let $q\colon X \to S^2$ denote the pinch map from (3.6), then an exact sequence associated to this cofibration sequence is

$$[\bigvee_{2g} S^2, BU(n)]_* \xrightarrow{(\Sigma f)^*} [S^2, BU(n)]_* \xrightarrow{q^*} [X, BU(n)]_* \xrightarrow{i^*} [\bigvee_{2g} S^1, BU(n)]_* \cong 0$$

The triviality of Σf shows that q^* is an isomorphism. Therefore, when we restrict the induced map $\text{Map}(S^2, BU(n)) \xrightarrow{q^*} \text{Map}(X, BU(n))$ to the c-th component, the image is in $\text{Map}(X, BU(n); c)$.

We will restrict fibrations (3.3) and (3.4) to the c-th components. First consider the fibration (3.4), then a map $h\colon S^2 \to BU(n)$ and its image under q^* must agree on basepoints which implies that the right hand square in the following diagram commutes

$$\begin{array}{ccccccc}
U(n) & \xrightarrow{\partial_c} & \text{Map}^*(S^2, BU(n); c) & \longrightarrow & \text{Map}(S^2, BU(n); c) & \xrightarrow{\text{ev}} & BU(n) \\
\| & & \downarrow q^* & & \downarrow q^* & & \| \\
U(n) & \xrightarrow{\varphi_c} & \text{Map}^*(X, BU(n); c) & \longrightarrow & \text{Map}(X, BU(n); c) & \xrightarrow{\text{ev}} & BU(n).
\end{array}$$

The middle square obviously commutes because both vertical arrows are induced by the same map q which was shown to preserve path components of the mapping space. The homotopy commutativity of the left hand square follows and we deduce that φ_c factors through $\text{Map}^*(S^2, BU(n); c)$. The restriction of fibration (3.3) to the c-th component obtains the fibration

$$\mathrm{Map}^*(S^2, BU(n); c) \xrightarrow{q^*} \mathrm{Map}^*(X, BU(n); c) \to \mathrm{Map}^*(\bigvee_{2g} S^1, BU(n)). \tag{3.7}$$

Now, from the homotopy $\varphi_c \simeq q^* \circ \partial_c$ and the extension of fibration (3.7), we obtain a homotopy pullback diagram which defines the maps h and h'

$$\begin{array}{ccccc}
 & & \mathrm{Map}^*(\Sigma(X), BU(n); c) & = & \mathrm{Map}^*(\Sigma(X), BU(n); c) \\
 & & \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle (\Sigma i)^*} \\
\mathscr{G}(S^2, U(n); c) & \xrightarrow{h'} & \mathscr{G}(X, U(n); c) & \xrightarrow{h} & \mathrm{Map}^*(\bigvee_{2g} S^2, BU(n)) \\
\| & & \downarrow & & \downarrow{\scriptstyle (\Sigma f)^*} \\
\mathscr{G}(S^2, U(n); c) & \longrightarrow & U(n) & \xrightarrow{\partial_c} & \mathrm{Map}^*(S^2, BU(n); c) \\
 & & \downarrow{\scriptstyle \varphi_c} & & \downarrow{\scriptstyle q^*} \\
 & & \mathrm{Map}^*(X, BU(n); c) & = & \mathrm{Map}^*(X, BU(n); c).
\end{array}$$

Since Σf is trivial, the same is true for $(\Sigma f)^*$ and hence $(\Sigma i)^*$ has a right homotopy inverse s. Therefore, the map js is a right homotopy inverse to h, since $hjs \simeq (\Sigma i)^* s \simeq \mathrm{id}$ and hence the sequence

$$\mathscr{G}(S^2, U(n); c) \times \mathrm{Map}^*(\bigvee_{2g} S^2, BU(n); c) \xrightarrow{h' \times js} \\ \to \mathscr{G}(X, U(n); c) \times \mathscr{G}(X, U(n); c) \to \mathscr{G}(X, U(n); c)$$

is a homotopy equivalence. We remark that $\mathrm{Map}^*(\bigvee_{2g} S^2, BU(n)) \simeq \prod \Omega U(n)$ and this completes the proof. □

It should be remarked that the theorem does hold more generally than just for $U(n)$, see [The10]. From the theorem, we deduce that we need only study the gauge groups $\mathscr{G}(S^2, U(n); c)$. We now follow some results from [Sut92].

Proposition 3.10. *For any integer c, there is a homotopy equivalence*

$$\mathrm{Map}(X, BU(n); c) \simeq \mathrm{Map}(X, BU(n); c + n).$$

Proof. We first restrict to the case $X = S^2$. An element in $\mathrm{Map}(S^2, BU(n))$ is homotopic to a map that has image in the 2-skeleton of $BU(n)$ which is a copy of S^2. Therefore a map in $\mathrm{Map}(S^2, BU(n))$ is in the c-th component if and only if it is homotopic to a map $S^2 \to S^2$ of degree c.

Let $i\colon S^2 \to BU(n)$ be the inclusion of the bottom cell. In the following we will define a map $T\colon BU(n) \times BU(1) \to BU(n)$ which will be used to define

$$\theta\colon \operatorname{Map}(S^2, BU(n); c) \to \operatorname{Map}(S^2, BU(n); c+n)$$

that sends f to

$$S^2 \xrightarrow{\Delta} S^2 \times S^2 \xrightarrow{f\times i} BU(n) \times BU(1) \xrightarrow{T} BU(n)$$

with a natural homotopy inverse.

First we define $\tilde{T}\colon U(n)\times U(1) \to U(n)$ to be scalar multiplication on $U(n)$. The induced map $\tilde{T}_*\colon H_1(U(n)) \oplus H_1(U(1)) \to H_1(U(n))$ is given by

$$\tilde{T}_*(\alpha_n, 0) = \alpha_n \text{ and } \tilde{T}_*(0, \alpha_1) = n\alpha_n$$

where α_i are generators. This can be seen by the fact that the map $\tilde{T}\mid_{*\times U(1)}$ is the inclusion of the center of $U(n)$. The quotient of which is $PU(n)$, the projective unitary group and $\pi_1(PU(n)) \cong \mathbb{Z}_n$ which shows that the induced map on homotopy groups is multiplication by n. By the Hurewicz map, the map on homology groups is also multiplication by n.

We now set $T = B\tilde{T}$ and a homotopy inverse to θ is constructed in the same way but we replace $\tilde{T}$ with conjugate scalar multiplication.

For the general case, let $q\colon X \to S^2$ denote the pinch map that sends all of the 1-cells of X to a point. Then we define a homotopy equivalence $\tilde{\theta}$ in a similar way to θ but we replace i with the composition $i \circ q$. □

We deduce from Proposition 3.10 that there are a maximum of n homotopy types of the path components of $\operatorname{Map}(X, BU(n))$. By studying the homotopy groups of these components, Sutherland was able to distinguish some of these.

Proposition 3.11. *Suppose that* $\operatorname{Map}(X, BU(n); c) \simeq \operatorname{Map}(X, BU(n); d)$ *then the highest common factors* (c, n) *and* (d, n) *are equal.*

Proof. Again, we restrict to the case $X = S^2$. Consider the long exact sequence corresponding to the evaluation fibration

$$\pi_i(\operatorname{Map}^*(S^2, BU(n); c)) \to \pi_i(\operatorname{Map}(S^2, BU(n); c)) \to$$
$$\to \pi_i(BU(n)) \xrightarrow{(\partial_c)_*} \pi_{i-1}(\operatorname{Map}^*(S^2, BU(n); c)). \quad (3.8)$$

Let $\Omega U(n)_0$ denote the connected component of $\Omega U(n)$ containing the identity, then we have $\operatorname{Map}^*(S^2, BU(n); c) \simeq \Omega U(n)_0$ and the boundary map can be identified with a map

$$\pi_{i-1}U(n) \to \pi_i U(n).$$

In fact, if we let α, β be the generators of $\pi_1 U(n)$ and $\pi_{2n-1}U(n)$, we can choose this map such that

$$(\partial_c)_*\beta = \pm\langle\beta, c\alpha\rangle$$

where $\langle\ ,\ \rangle$ denotes the Samelson product. For details see [Sut92].

Using a corollary in [Bot60], this Samelson product is $(n-1)!c$ times a generator of the group $\pi_{2n}(U(n)) \cong \mathbb{Z}_{n!}$. Therefore the kernel of

$$\pi_{2n-1}U(n) \xrightarrow{(\partial_c)_*} \pi_{2n}U(n) \tag{3.9}$$

depends on (c, n). Hence, by sequence (3.8), so does $\pi_{2n}(\mathrm{Map}(S^2, BU(n); c))$.

Now let $q\colon X \to S^2$ be the pinch map, then there is a commutative diagram

$$\begin{array}{ccc} \pi_{2n-1}U(n) & \xrightarrow{(\partial_c)_*} & \pi_{2n}U(n) \\ \| & & \downarrow{\scriptstyle q^*} \\ \pi_{2n-1}U(n) & \xrightarrow{\partial} & \pi_{2n-1}(\mathrm{Map}^*(X, BU(n); c)). \end{array}$$

By the above, the kernel of ∂ depends on (c, n) and therefore by the evaluation fibration involving X, the group $\pi_{2n}(\mathrm{Map}(X, BU(n); c))$ also depends on (c, n). □

We state a partial converse to Proposition 3.11, the proof can be found in [Sut92].

Proposition 3.12. *Suppose that $(c, n) = (d, n)$, then the d-th and c-th components of* $\mathrm{Map}(X, BU(n))$ *are homotopy equivalent after completion at any prime p.* □

We highlighted the proof of Proposition 3.11, where the idea was to study the boundary map ∂_c in the evaluation fibration. The analysis of this map is crucial, for if ∂_c were trivial then we would obtain a homotopy decomposition of the fibration, recall Section 2.2. We will follow [The11] where this boundary map was studied for $G = U(n)$ and X a Riemann Surface.

3.3.1 Homotopy Decompositions of the Gauge Groups

In light of Theorem 3.9 we reduce our study to the gauge groups over S^2, and therefore define $\mathscr{G}(U(n); c) = \mathscr{G}(S^2, U(n); c)$. We first state the main theorem in [The11].

Theorem 3.13. *With the notation above, fix a prime p, then*

1. *if $q \neq p$ is a prime then there is a q-local homotopy decomposition*

$$\mathscr{G}(U(p); c) \simeq U(p) \times \Omega^2 U(p);$$

2. *if $p \mid c$, there is a p-local homotopy decomposition*

$$\mathscr{G}(U(p);c) \simeq \prod_{i=0}^{p-1} S^{2i+1} \times \prod_{j=1}^{p-1} \Omega^2 S^{2j+1};$$

3. *if $p \nmid c$ there is a p-local homotopy decomposition*

$$\mathscr{G}(U(p);c) \simeq \prod_{i=0}^{p-2} S^{2i+1} \times \prod_{j=2}^{p-1} \Omega^2 S^{2j+1} \times (S^1 \times \Omega^2 S^{2p+1}).$$

Therefore we reduce our study of the topology of $\mathscr{G}(X, U(p); c)$ to the study of spheres and $U(p)$. Notice that since p is prime, unless we have $c = mp$ for some integer m, we have that $(c, p) = 1$. Hence, the fact that the right hand side of the decompositions are independent of c agrees with Proposition 3.12.

We note that by [Ser53] there is a p-local homotopy equivalence

$$e\colon U(p) \to \prod_{i=0}^{p-1} S^{2i+1}. \tag{3.10}$$

Therefore we will be able to deduce Theorem 3.13 (1) and (2) if we prove the q (or p)-local triviality of the boundary map ∂_c in

$$\Omega^2 U(n) \to \mathscr{G}(U(n);c) \to U(n) \xrightarrow{\partial_c} \Omega U(n)_0.$$

For part (3), it is clear there needs to be a deeper analysis of this boundary map.

In a more general setting, the boundary map of the evaluation fibration was studied in [Lan73] and we state a lemma due to this paper.

Let $\mathrm{ev}'\colon \Sigma U(n) \to BU(n)$ be the evaluation map, that is the map

$$\Sigma\Omega BU(n) \to BU(n)$$
$$(t, \gamma) \mapsto \gamma(t)$$

precomposed with the homotopy equivalence $\Sigma U(n) \to \Sigma\Omega BU(n)$. Let $i\colon S^2 \to BU(n)$ be the inclusion of the bottom cell, and let $c\colon \Omega U(n)_0 \to \Omega U(n)_0$ be the c-th power map.

Lemma 3.14. *The adjoint of the map $U(n) \xrightarrow{\partial_c} \Omega U(n)_0$ is homotopic to the Whitehead product*

$$S^2 \wedge U(n) \xrightarrow{[ci, \mathrm{ev}']} BU(n). \qquad \square$$

We take the adjoints of the maps i and ev' to obtain

$$\bar{i}\colon S^1 \to U(n) \text{ and } \bar{\mathrm{ev}}'\colon U(n) \to U(n).$$

Adjoint to the Whitehead product $[ci, \mathrm{ev}']$ is the Samelson product $\langle c\bar{i}, \bar{\mathrm{ev}}'\rangle$ which is the composition

$$S^1 \wedge U(n) \xrightarrow{c\bar{i}\wedge\bar{\mathrm{ev}}'} U(n) \wedge U(n) \xrightarrow{k} U(n) \tag{3.11}$$

where $k(x, y) = xyx^{-1}y^{-1}$ is the commutator. The pointed homotopy set

$$[S^1 \wedge U(n), U(n)]_*$$

is given a group structure by the co-H structure of S^1 and therefore the Samelson product $c\langle\bar{i}, \bar{\mathrm{ev}}'\rangle$ is the same as (3.11). This equality clearly holds more generally, it follows from the fact the Samelson product is bilinear. We have deduced the following lemma.

Lemma 3.15. *There is a homotopy*

$$\partial_c \simeq c \circ \partial_1. \qquad \square$$

Proof of Theorem 3.13 (b). The above lemma gives $\partial_0 \simeq 0 \circ \partial_1$ and hence ∂_0 is nullhomotopic. This implies that there is an integral homotopy equivalence

$$\mathscr{G}(U(p); 0) \simeq U(p) \times \Omega^2 U(p).$$

Now our assumption is that $p \mid c$, hence by Proposition 3.10 there are homotopy equivalences

$$\mathscr{G}(U(p); c\alpha) \simeq U(p) \times \Omega^2 U(p)$$

for all $\alpha \in \mathbb{Z}$. Using the p-local equivalence (3.10) gives the required result. $\square$

We proceed to prove Theorem 3.13 (a).

Proposition 3.16. *The boundary map $U(n) \xrightarrow{\partial_1} \Omega U(n)_0$ has order n.*

Proof. It has already been noted that ∂_0 is nullhomotopic, now by Proposition 3.10 and Theorem 3.8 the following diagram homotopy commutes

$$\begin{array}{ccccc}
U(n) & \xrightarrow{\partial_0} & \mathrm{Map}^*(S^2, BU(n); 0) & \longrightarrow & \mathrm{Map}(S^2, BU(n); 0) \\
\downarrow \gamma & & \downarrow \simeq & & \downarrow \simeq \\
U(n) & \xrightarrow{\partial_n} & \mathrm{Map}^*(S^2, BU(n); n) & \longrightarrow & \mathrm{Map}(S^2, BU(n); n)
\end{array}$$

where γ is an induced map of fibres. Using the five lemma, we see that γ is a homotopy equivalence. Therefore, the left square implies that $\partial_n \simeq n \circ \partial_1$ is nullhomotopic, and hence n is a multiple of the order of ∂_1. However, recall (3.9) in Proposition 3.11

$$\pi_{2n-1} U(n) \xrightarrow{(\partial_1)_*} \pi_{2n-1} \Omega U(n)_0.$$

It was stated in this proposition that $(\partial_1)_*$ has image $(n-1)!$ times a generator of the group $\pi_{2n-1}U(n) \cong \mathbb{Z}_{n!}$ and therefore it must have order n. We deduce that the order of ∂_1 must be a multiple of n and the proposition follows. □

We immediately obtain the following corollary which appeared in [The11].

Corollary 3.17. *Let n be a positive integer and let q be prime such that $q \nmid n$. Then there is a there is a q-local homotopy decomposition*

$$\mathscr{G}(U(n); c) \simeq_q U(n) \times \Omega^2 U(n).$$

Proof. Let $q \nmid n$ be a prime, then n is a unit in $\mathbb{Z}_{(q)}$ and therefore localised at q the map ∂_1 has order 1, or in other words, ∂_1 is nullhomotopic. Since we have $\partial_c \simeq c \circ \partial_1$ the same is true for ∂_c. Therefore the principal homotopy fibration sequence

$$\Omega^2 U(n) \to \mathscr{G}(U(n); c) \to U(n)$$

has a section and the q-local homotopy decomposition follows. □

Proof of Theorem 3.13 (a). This follows from Corollary 3.17 if we take $n = p$. □

The proof of part (c) is harder due to the non-triviality of the boundary map in this case. We run through a sketch of the proof but the full details can be found in [The11].

Proof of Theorem 3.13 (c). Throughout this proof we assume that spaces and maps have been localised at p. By Lemma 3.15, the following diagram is homotopy commutative

$$\begin{array}{ccccc}
\mathscr{G}(U(p); 1) & \longrightarrow & U(p) & \xrightarrow{\partial_1} & \Omega U(p)_0 \\
\downarrow{\scriptstyle \phi} & & \| & & \downarrow{\scriptstyle c} \\
\mathscr{G}(U(p); c) & \longrightarrow & U(p) & \xrightarrow{\partial_c} & \Omega U(p)_0
\end{array}$$

where ϕ is an induced map of fibres. We assume that $p \nmid c$ and hence c is unit modulo p, therefore the map c is a homotopy equivalence. By the five-lemma ϕ is also a homotopy equivalence and we reduce our study to the space $\mathscr{G}(U(p); 1)$.

Now let $c\colon S^3\langle 3\rangle \to S^3$ be the map from the three-connected cover and let

$$\iota\colon S^{2p-1} \to \Omega S^3\langle 3\rangle$$

be the inclusion of the bottom cell[2]. We now define $\alpha = (\Omega c)\iota$. It was shown in [The11] that the homotopy fibre of α is $S^1 \times \Omega^2 S^{2p+1}$ and that there is a homotopy commutative

[2]We recall that we have localised at p.

diagram

$$\begin{array}{ccc}
U(p) & \xrightarrow{\partial_1} & \Omega U(p)_0 \\
\big\downarrow e & & \big\downarrow (\Omega e)_0 \\
\prod_{i=0}^{p-1} S^{2i+1} \xrightarrow{\text{proj}} S^{2p-1} \xrightarrow{\alpha} \Omega S^3 & \xrightarrow{\text{incl}} & \prod_{j=1}^{p-1} \Omega S^{2j+1}
\end{array}$$

where e is as in (3.10) and $(\Omega e)_0$ is the map Ωe restricted to the component containing the basepoint.

Both vertical arrows are homotopy equivalences, hence by the five-lemma, we will be able to deduce the homotopy type of $\mathscr{G}(U(p); 1)$ by calculating the homotopy fibre of the bottom horizontal arrow. But it can be shown that the homotopy fibre of the map $(\text{incl} \circ \alpha \circ \text{proj})$ is just the product of each of the homotopy fibres and the result follows. □

Chapter 4

Real Principal Bundles

The aim of this chapter is to provide a more comprehensive introduction to the objects defined in Chapter 1. In Section 4.1, we present a general background of equivariant bundle theory and our aim is to show that the classifying spaces of gauge groups of such bundles correspond to mapping spaces in a similar way to the non-equivariant case. We then restrict our attention to the Real and Quaternionic bundles of Chapter 1 with the ultimate aim to prove the classification results in Propositions 1.2 and 1.3.

4.1 Equivariant Principal Bundles

We recall that a *Real surface* is a pair (X, σ) where X is a closed orientable Riemann surface and $\sigma : X \to X$ is an antiholomorphic involution. More generally, for an arbitrary space X we define an *involution* to be a map $\sigma\colon X \to X$ with the property $\sigma^2 = \mathrm{id}_X$. The pair (X, σ) is then described as a *Real space* and a morphism (or map) between Real spaces (X, σ_X) and (Y, σ_Y) is a continuous map $f\colon X \to Y$ such that $\sigma_Y \circ f = f \circ \sigma_X$.

The category of Real spaces can be thought of as the category of $\mathbb{Z}_2$-spaces with $\mathbb{Z}_2$-maps between them. More generally, for a compact Lie group Γ, we can think of the category of Γ-spaces with Γ-maps between them and this is the context we will think of in this section. The intention of introducing this generality is to encourage the possibility of future projects where one could consider bundles endowed with actions other than $\mathbb{Z}_2$.

We will introduce the concept of generalised equivariant bundles as considered in [LM86] whilst highlighting the context we are interested in. Let Γ and G be compact Lie groups and let $\pi\colon P \to X$ be a principal G-bundle with X a Γ-space. We would like the action of Γ on X to lift appropriately to P. Initially, one may ask for the G-action on P to extend to a $\Gamma \times G$ action as studied in [Las82]. However, with this definition, an extension

$$0 \to G \to \Pi \xrightarrow{q} \Gamma \to 0 \tag{4.1}$$

is excluded as an example of such a bundle. Therefore, for a fixed extension Π, we define a *principal* (G,Π)-*bundle* to be the bundle $\pi\colon P \to X$ with the G-action on P extended to Π in such a way that π is a Π-map, with the Π-action on X understood to be via q.

If (4.1) is a split extension then Π is a semi-direct product and can be written as $\Gamma \times_\alpha G$ for some continuous homomorphism $\alpha\colon \Gamma \to Aut(G)$. Therefore, for $p \in P$ we require the action of Π to satisfy

$$\gamma \cdot (p \cdot g) = (\gamma \cdot p) \cdot \alpha(\gamma)(g)^{1}.$$

When $\Gamma = \mathbb{Z}_2$, G is a connected reductive affine algebraic group defined over $\mathbb{C}$ and $\alpha(1)$ corresponds to an antiholomorphic automorphism, we will call these *Real principal G-bundles* or just Real bundles. If $G = U(n)$ and $\alpha(1)$ is complex conjugation, we can see that these correspond to Real principal $U(n)$-bundles as introduced in Chapter 1.

We compare this definition of a Real bundle with the definition using Real space notation. Let (X,σ) be a Real space, G a complex Lie group and $\alpha(1)$ an antiholomorphic automorphism, then a *lift* of σ is a map $\tilde{\sigma}\colon P \to P$ satisfying

1. $\tilde{\sigma}^2(p) = p$ for all $p \in P$;
2. $\tilde{\sigma}(p \cdot g) = \tilde{\sigma}(p) \cdot \alpha(1)(g)$ for all $p \in P$ and $g \in G$.

More generally, we will say that $\pi\colon P \to X$ is a *pseudo Real principal G-bundle* or a pR-bundle if there exists $g_0 \in G$ such that the lift $\tilde{\sigma}$ satisfies

$$\tilde{\sigma}^2(p) = p \cdot g_0 \text{ for all } p \in P$$

in addition to Property 2. Of course, if we set $G = U(n)$, the automorphism $\alpha(1)$ to correspond to complex conjugation and $g_0 = -I_n$, we have the definition of Quaternionic bundles of Chapter 1.

For pR-bundles, it can be seen that the action on P can be described by the extension $\Pi \cong \mathbb{Z} \times_{\alpha\beta} G/\sim$ where $(n,g) \sim (n-2, g \cdot g_0)$ and the product on Π is given by

$$(n,g)(m,h) = (n+m, \alpha(\beta(n))(h) \cdot g)$$

for the quotient map $\beta\colon \mathbb{Z} \to \mathbb{Z}_2$. We let $Z_{\mathbb{R}}(G)$ denote the set containing elements in the center of G that are fixed by $\alpha(1)$.

Proposition 4.1. *Let $(P,\tilde{\sigma}) \to (X,\sigma)$ be a pseudo Real principal G-bundle then g_0 (as above) is in $Z_{\mathbb{R}}(G)$.*

[1] Compare with [tD87, 8.7f.].

Proof. Let $\overline{g}$ denote $\alpha(1)(g)$ and let p be in P, then

$$p \cdot g_0^2 = (p \cdot g_0) \cdot g_0 = \tilde{\sigma}^2(\tilde{\sigma}^2(p)) = \tilde{\sigma}(\tilde{\sigma}^2(\tilde{\sigma}(p))) = \\ = \tilde{\sigma}(\tilde{\sigma}(p) \cdot g_0) = \tilde{\sigma}^2(p) \cdot \overline{g_0} = p \cdot g_0\overline{g_0}.$$

Since the action on P restricted to G is free, it follows that $\overline{g_0} = g_0$. Now let h be in G then

$$p \cdot g_0 h = \tilde{\sigma}^2(p) \cdot h = \tilde{\sigma}^2(p \cdot h) = p \cdot h g_0$$

where the second equality holds by Property 2 and the fact that $\overline{\overline{h}} = h$. It follows that g_0 is in the center of G. □

Remark 4.2. Let $(P, \tilde{\sigma})$ be a pR-bundle with $\tilde{\sigma}^2 = \mathrm{id}_P \cdot g_0$. We can alter the lift $\tilde{\sigma}$ by the action of an element $h \in Z(G)$ by setting $\tilde{\sigma}' = \tilde{\sigma} \cdot h$. Now

$$(\tilde{\sigma})'^2 = \tilde{\sigma} \cdot h \circ \tilde{\sigma} \cdot h = \mathrm{id}_P \cdot g_0(\alpha(1)(h))h.$$

The fact that h is in $Z(G)$ forces $g_0(\alpha(1)(h))h$ to be in $Z_{\mathbb{R}}(G)$ and we have defined a new pR-structure. If $g_0 = (\alpha(1)(h))h$ for some $h \in Z(G)$, in particular if $g_0 = h^2$ for some $h \in Z_{\mathbb{R}}(G)$, then we normalise the pR-structure to a Real one.

The aim of this section is to discuss the theory of principal (G, Π)-bundles, but the reader is encouraged to use pR-bundles as an ongoing example. Most of the content in this section is due to [LM86] or [tD87] but we elaborate on some of the proofs in the interest of completeness.

It is clear that a principal (G, Π)-bundle is locally trivial as a principal G-bundle but we would like to define what it means to be locally trivial as a (G, Π)-bundle. To do this, we discuss the local structure of a completely regular Γ-space Y.

Let y be a point in Y and denote the isotropy subgroup at y by Γ_y. We say an open Γ-set $U \subset Y$ is a *tube* about y if $y \in U$ and there exists a Γ-map $f\colon U \to \Gamma/\Gamma_y$. Notice that $A = f^{-1}(e\Gamma_y)$ is a Γ_y-subspace of U and therefore we can obtain a map $F\colon \Gamma \times_{\Gamma_y} A \to U$ defined via $[\gamma, a] \mapsto \gamma a$. The next proposition says that this is a homeomorphism.

Proposition 4.3. *Let $f\colon Y \to \Gamma/\Lambda$ be a Γ-map, let $A = f^{-1}(e\Lambda)$ and let the Γ-map $F\colon \Gamma \times_\Lambda A \to Y$ be defined by $[g, a] \mapsto ga$. Then F is always a bijection and if Γ is compact Hausdorff and Λ is a closed in G then F is a homeomorphism.*

Proof. For surjectivity, let $y \in Y$, then $f(y) = \gamma\Lambda$ for some $\gamma \in \Gamma$. Hence a preimage to y under F is the element $[\gamma, \gamma^{-1}y]$. Now, for injectivity, let $[g, a], [g', a']$ be in $\Gamma \times_\Lambda A$ with $ga = g'a'$. Then $g^{-1}g'\Lambda = e\Lambda$ and so $g^{-1}g' = \lambda$ for some $\lambda \in \Lambda$. Then $a = \lambda a'$ and

$$[g, a] = [g, \lambda a'] = [g\lambda, a'] = [g', a'].$$

Finally when Γ is compact, we note the action $\Gamma \times Y \to Y$ is a closed mapping (see [tD87, 3.1(iii)]) and since Λ is closed we find that F is also a closed mapping. We deduce that F is a continuous closed bijection and therefore a homeomorphism. □

For our purposes, we therefore think of a tube U as Γ-homeomorphic to $\Gamma \times_{\Gamma_y} A$. The following theorem highlights one of the key local properties of Γ-spaces. We recall that a space X is *completely regular* if for every closed subset $F \subseteq X$ and point $x \in X \setminus F$, there is map $f\colon X \to [0,1]$ such that $f(x) = 1$ and $f(y) = 0$ for every $y \in F$.

Theorem 4.4. *Let Γ be a compact Lie group and Y a completely regular Hausdorff Γ-space. Then there is a tube about each of its points.*

Proof. See Theorem 5.7 in [tD87]. □

We say that a Γ_y-invariant subspace $A_y \subset Y$ containing y is a *slice* if the map

$$\mu\colon \Gamma \times_{\Gamma_y} A_y \to Y,\ \ \mu[\gamma, v] = \gamma v$$

is an embedding onto a tube at y. It is clear that the subspace A above is an example of a slice. The following lemma starts to motivate how the local structure of a (G,Π)-bundle should look.

Lemma 4.5. *Let $\pi\colon P \to X$ be a principal (G,Π)-bundle. Let $x \in X$ and $p \in \pi^{-1}(x)$ with respective isotropy groups Γ_x and Π_p. Then the restriction of $q\colon \Pi \to \Gamma$ to Π_p is an isomorphism onto Γ_x and $\Pi_p \cap G = \{e\}$.*

Proof. For $v \in \Pi_p$, we have

$$q(v)\cdot x = q(v)\cdot \pi(p) = \pi(v \cdot p) = \pi(p) = x$$

and hence $q(v) \in \Gamma_x$. For surjectivity, if $\gamma \in \Gamma_x$ then there exists $w \in \Pi$ with $\gamma = q(w)$ and $w \cdot p = p \cdot g$ for some $g \in G$. But then $vg^{-1} \in \Pi_p$ and $q(vg^{-1}) = \gamma$. For injectivity we first note that $\Pi_p \cap G = \{e\}$ because the action of G on P is free. Now let $v, v' \in \Pi_p$ with $q(v) = q(v')$, then $v = v'h$ for some $h \in G$ but this implies $h \in \Pi_p$ and so $h = e$. □

We say that a principal (G,Π)-bundle $\pi\colon P \to X$ is *trivial* if it isomorphic to a bundle of the form $\Pi \times_H A \to \Gamma \times_\Lambda A$, where $H < \Pi$, $\Lambda < \Gamma$, $H \cap G = \{e\}$, the homomorphism $q\mid_H$ is an isomorphism and A is a Λ-space that is thought of as an H-space via q. It is clear that A is a Λ-slice as defined above.

We say that $\pi\colon P \to X$ is *locally trivial* if there is an open cover of X by Γ-sets $\{A_\beta\}$ such that $\pi\mid_{\pi^{-1}(A_\beta)}$ is trivial. Further, we say π is *numerable* if the base space has a cover satisfying local triviality and has a Γ-invariant partition of unity (PoU) subordinate to it; the interval $[0,1]$ is understood to be endowed with the trivial Γ-action.

Proposition 4.6. *Let $\pi\colon P \to X$ be a locally trivial principal (G, Π)-bundle with X paracompact and Hausdorff*[2]*. Then π is numerable.*

Proof. This follows from the non-equivariant case since the quotient map

$$\tilde{q}\colon X \to X/\Gamma$$

is open and the orbit space X/Γ is also paracompact and Hausdorff. The idea is that we can lift partitions of unity on X/Γ to Γ-invariant partitions of unity on X. Compare with [Las82, 1.12]. □

Proposition 4.7. *Let $\pi\colon P \to X$ be a principal (G, Π)-bundle with completely regular total space. Then $\pi\colon P \to X$ is locally trivial.*

Proof. The details can be found in [LM86, 3(ii)]. □

We will now start to discuss the usual properties of principal bundles in the equivariant context. The ultimate aim of this section will be to show the existence of a universal (G, Π)-bundle. Our method of attack is to follow the usual method, that is, to prove that for a locally trivial bundle $\pi\colon P \to X \times I$, there is an equivalence

$$P \simeq P\,|_{\pi^{-1}(X\times\{0\})} \times I. \tag{4.2}$$

We first study a specific case which will then be used to prove (4.2) in all generality. We recall that in the non-equivariant case, a locally trivial G-bundle $\pi\colon P \to X$ has an open cover $\{U_\beta\}$ of X such that each $\pi^{-1}(U_\beta)$ is G-homeomorphic to the product bundle $p_1\colon U_\beta \times G \to U_\beta$. It would be useful to extend this description to the equivariant case, but it becomes problematic due to the possible non-commutativity of the actions of G and Γ on P. Therefore, we briefly restrict our study to bundles where these actions do commute, that is principal (G, Π)-bundles where $\Pi = \Gamma \times G$. In this case, Bierstone introduced a condition for locally trivial bundles that is equivalent to our definition when X is completely regular, see [Las82, pp. 258-259].

Definition 4.8 Bierstone's Condition**.** We say that a principal $(G, \Gamma\times G)$-bundle satisfies *Bierstone's condition* if the following property holds. For every $x \in X$, there is a Γ_x-invariant neighbourhood U_x such that $\pi^{-1}(U_x)$, considered as a $(G, \Gamma_x \times G)$-bundle, is isomorphic to the bundle $U_x \times G$ where Γ_x acts via

$$\gamma(u, g) = (\gamma u, \rho_x(\gamma)g)$$

for $u \in U_x$, $\gamma \in \Gamma_x$ and for a homomorphism $\rho_x\colon \Gamma_x \to G$.

[2] We note that Γ-complexes (see Section 4.3) have these properties and that X/Γ is automatically given a CW-structure.

We note that the homomorphism ρ_x captures the information about the lift of the Γ-action. It is now easy to prove (4.2) for trivial bundles.

Proposition 4.9. *Let $\pi\colon P \to X \times I$ be a trivial principal $(G, \Gamma \times G)$-bundle. Then P satisfies (4.2).*

Proof. By the Bierstone condition, the bundle P is $(G, \Gamma \times G)$-equivalent to the bundle $(X \times I) \times G$. Since the action of $\Gamma \times G$ on this copy of I is trivial, there is an obvious equivalence $P \simeq P\,|_{\pi^{-1}(X\times\{0\})} \times I$. □

Using the usual methods, we can extend (4.2) to locally trivial $(G, \Gamma \times G)$-bundles over paracompact spaces and therefore prove the Corollary 4.10. However we will prove similar results in a more general setting and so we omit them for now. Let $\Lambda < \Gamma$, then we say a Γ-map has *equivariant homotopy lifting property (EHLP)* with respect to Λ-spaces if it satisfies the homotopy lifting property for all Λ-spaces and additionally the lifted homotopies can be chosen to be Λ-maps.

Corollary 4.10. *Let X be paracompact and let $\pi\colon P \to X$ be a locally trivial principal $(G, \Gamma \times G)$-bundle and let $\Lambda < \Gamma$ be a subgroup. Then π has EHLP for paracompact Λ-spaces.* □

We still require the next two lemmas to prove that (4.2) holds in general. For the following we will require that the following bundles enjoy the EHLP property for paracompact Λ-spaces

- Bundle 1 — the quotient bundle $\Gamma \to \Gamma/\Lambda$ where Λ acts on Γ via conjugation and on Γ/Λ via left multiplication;
- Bundle 2 — and $\Pi/H \xrightarrow{q} \Gamma/\Lambda$ where $H \cong \Lambda$ via q and Λ acts via left multiplication on both spaces.

We note that the groups $\Pi, H, \Lambda,$ and Γ as in a trivial (G, Π)-bundle should be the motivating example for Bundle 2. It is clear that the actions of Λ and G on Π/H do commute and so we immediately conclude that it is has EHLP. Now, the following diagram is a pullback of Λ-spaces

$$\begin{array}{ccc} \Pi & \longrightarrow & \Pi/H \\ \downarrow & & \downarrow \\ \Gamma & \longrightarrow & \Gamma/\Lambda. \end{array}$$

and (by generalising the usual arguments) we deduce that $\Pi \to \Lambda$ also has EHLP with respect to paracompact Λ-spaces.

It is shown in [Las82, p.266] that Bundle 1 also has EHLP, by showing that it is equivalent to a $(\Lambda \times \Lambda, \Lambda \times \Lambda \times \Lambda)$-bundle.

Lemma 4.11. *Let X be a Γ-space and let A be a Λ-space with the property that there is a Γ-homeomorphism*

$$\phi\colon X \times I \to \Gamma \times_\Lambda A.$$

Then X is Γ-homeomorphic to $\Gamma \times_\Lambda A_0$ where $A_0 = \phi^{-1}(A) \cap (X \times \{0\})$.

Proof. Let $f\colon X \to \Gamma/\Lambda$ be the composition

$$X \xrightarrow{i_0} X \times I \xrightarrow{\phi} \Gamma \times_\Lambda A \xrightarrow{p_1} \Gamma/\Lambda$$

where i_0 is the inclusion into $X \times \{0\}$ and p_1 is the projection onto the first factor. Notice that this is well-defined, that it is a Γ-map and that $f^{-1}(e\Lambda) = A_0$. Then by Proposition 4.3, we have $X = \Gamma \times_\Lambda A_0$. □

Lemma 4.12. *Let X, A, A_0 and ϕ be as in the previous lemma and let Λ act on Γ by conjugation $\gamma \cdot \lambda \mapsto \lambda\gamma\lambda^{-1}$ for $\lambda \in \Lambda, \gamma \in \Gamma$. Then there exists a Λ-map $\theta\colon A_0 \times I \to \Gamma$ such that*

1. *the map $\psi(a_0, t) = \theta(a_0, t)^{-1}\phi(a_0, t)$ is a Λ-homeomorphism and;*
2. *for all $[g, a_0] \in \Gamma \times_\Lambda A_0 = X$ we may write the Γ-homeomorphism ϕ as follows*

$$\phi([g, a_0], t]) = [g\theta(a_0, t), \psi(a_0, t)].$$

Proof. Define a Λ-map by $\phi'\colon A_0 \times I \to \Gamma/\Lambda$ by

$$\phi'(a_0, t) = p_1\phi(a_0, t).$$

Notice that $\phi'(A_0 \times \{0\}) = e\Lambda$ so there is an obvious lift of $\phi' |_{(A_0 \times \{0\})}$ to the constant map $A_0 \to \Gamma$ given by $a_0 \mapsto e$. By the comments before Lemma 4.11, the quotient map $q'\colon \Gamma \to \Gamma/\Lambda$ satisfies the EHLP for paracompact Λ-spaces and so ϕ' lifts to a Λ-map which is the required map $\theta\colon A_0 \times I \to \Gamma$.

Now, $\phi |_{A_0 \times I}$ is a Λ-homeomorphism onto its image but there is no guarantee that this lands in A. But since $\phi'(a_0, t) = q'\theta(a_0, t)$, the map $\psi(a_0, t) = \theta(a_0, t)^{-1}\phi(a_0, t)$ has image in A. The details for the fact that ψ is a Λ-homeomorphism can be found in the proof of [Las82, 2.8]. □

Corollary 4.13. *Let $\pi\colon P \to X \times I$ be a trivial principal (G, Π)-bundle, then there is a bundle isomorphism $P \simeq P |_{\pi^{-1}(X \times \{0\})} \times I$.*

Proof. By definition π is trivial if it is bundle isomorphic to a bundle of the form

$$\Pi \times_H A \to \Gamma \times_\Lambda A.$$

Now with $X = \Gamma \times_\Lambda A_0$ as in the previous lemmas, the bundle $P\mid_{\pi^{-1}(X\times\{0\})}$ is isomorphic to the bundle $\Pi \times_H A_0 \to \Gamma \times_\Lambda A_0$. Therefore to prove the statement we need to show that there is a Π-homeomorphism

$$\tilde{\phi}\colon (\Pi \times_H A_0) \times I \to \Pi \times_H A$$

covering $\phi\colon (\Gamma \times_\Lambda A_0) \times I \to \Gamma \times_\Lambda A$. Now by the preamble to Lemma 4.11, the bundle $\Pi \xrightarrow{q} \Gamma$ has the EHLP with respect to paracompact Λ-spaces where Λ acts on Π via q^{-1}. Therefore the map θ from Lemma 4.12 lifts to a Λ-map $\tilde{\theta}\colon A_0 \times I \to \Pi$. We therefore define $\tilde{\phi}$ by

$$\tilde{\phi}([g, a_0], t) = [g\tilde{\theta}(a_0, t), \tilde{\theta}(a_0, t)^{-1}\phi(a_0, t)]$$

which is a lift of ϕ and is a Π-homeomorphism. □

From herein, we adapt arguments from the non-equivariant setting. We point the reader to [Hus94] or [Sel97] for these arguments.

Theorem 4.14. *Let $\pi\colon P \to X \times I$ be a numerable principal (G, Π)-bundle, then it is isomorphic to the bundle $P\mid_{\pi^{-1}(X\times\{0\})} \times I \to (X \times \{0\}) \times I$.*

Sketch of Proof. We outline the non-equivariant method featured in [Hus94, 9.5f.] which generalises to this case.

1. By extending covers along the interval, one can shown that any numerable cover of $X \times I$ can be taken to be of the form $\{U_\beta \times I\}$ where $\{U_\beta\}$ is a numerable Γ-cover of X.

2. Construct Γ-invariant PoU $\lambda_\beta\colon X \to I$, where $\operatorname{supp}(\lambda_\beta) \subset U_\beta$.

3. We note that by Corollary 4.13, we have bundle isomorphisms

$$\phi_\beta\colon P\mid_{\pi^{-1}(U_\beta)\times\{0\}} \times I \to P\mid_{\pi^{-1}(U_\beta\times I)}.$$

   Now define maps $h_\beta\colon P\mid_{\pi^{-1}(U_\beta\times I)} \to P\mid_{\pi^{-1}(U_\beta\times I)}$ by

$$h_\beta(\phi_\beta(x, t)) = \phi_\beta(x, \min(\lambda_\beta(p), t))$$

   which we note are equivariant.

4. By picking a total ordering of the index set β, we define a Π-map

$$h\colon P \to P$$

   by sending a point $x \in X$ to the composition $h_{\beta_n} h_{\beta_{n-1}} \cdots h_{\beta_1}(x)$ for every time $x \in U_{\beta_i}$. The paracompactness of X makes the composition finite and hence makes h continuous. We note that h covers the map $r\colon X \times I \to X \times I$ given by $r(x,t) = (x,0)$.

5. Now this shows that P is (G,Π)-equivalent to the bundle $r^*(P)$ which is in turn (G,Π)-equivalent to the bundle $P\mid_{\pi^{-1}(X\times\{0\})} \times I$.

$\square$

Corollary 4.15. *Let $P \to X$ be a numerable principal (G,Π)-bundle and let*

$$f_0, f_1\colon Y \to X$$

be Γ-maps. Suppose that $f_0 \sim f_1$ under a Γ-homotopy then the pullback bundles are isomorphic as (G,Π)-bundles

$$f_0^*(P) \cong f_1^*(P).$$

Proof. Let H denote the Γ-homotopy between f_0 and f_1, and consider the pullback square

$$\begin{array}{ccc} H^*(P) & \longrightarrow & P \\ \downarrow & & \downarrow \\ Y \times I & \xrightarrow{H} & X. \end{array}$$

By Theorem 4.14 and the fact that $H^*(P)\mid_{Y\times\{0\}} = f_0^*(P)$, there is a (G,Π)-bundle isomorphism

$$\begin{array}{ccc} f_0^*(P) \times I & \xrightarrow{F} & H^*(P) \\ \downarrow & & \downarrow \\ Y \times I & = & Y \times I. \end{array}$$

We now restrict this square to $Y \times \{1\}$ and we recall that $H(-,1) = f_1$ to give the required isomorphism. $\square$

Let X and Y be Γ-spaces, and then let $[X,Y]_\Gamma$ denote Γ-equivariant homotopy classes of Γ-equivariant maps from X to Y. Further, let $Prin_{(G,\Pi)}(X)$ denote isomorphism classes of principal (G,Π)-bundles over X. The following corollary is an immediate consequence of the previous one.

Corollary 4.16. *Let $P \to Y$ be a principal (G,Π)-bundle. Then there is a well defined map*

$$[X,Y]_\Gamma \to Prin_{(G,\Pi)}(X).$$ $\square$

In the non-equivariant case, we recall that an explicit model of a universal G-bundle was given by the Milnor construction $\lim_{n\to\infty} G^{*n}$. Explicitly, as a set there is an equivalence

$$EG = \Big\{(g_0,t_0,g_1,t_1,\dots) \in (G\times I)^\infty \mid \sum t_i = 1 \text{ and almost all } t_i = 0\Big\} / \sim$$

where 'almost all' means the complement is at most finitely many, and

$$(g_0, t_0, g_1, t_1, \dots, g_n, 0, \dots) \sim (g_0, t_0, g_1, t_1, \dots, g_n', 0, \dots).$$

Suppose now that G is given a Π-action extending a right G-action, then there is an obvious Π-action on EG via

$$h \cdot (g_0, t_0, g_1, t_1, \dots, g_n, t_n, \dots) = (h \cdot g_0, t_0, h \cdot g_1, t_1, \dots, h \cdot g_n, t_n, \dots)$$

for $h \in \Pi$. Notice that $EG/G = BG$ endowed with a Γ-action. We denote the induced quotient map by $EG_\Pi \to BG_\Pi$ and note that it is a numerable principal (G, Π)-bundle. We will see in Theorem 4.18 that this bundle is a universal (G, Π)-bundle, but first we need the following lemma.

Lemma 4.17. *Let $\alpha_1 \colon EG_\Pi \to EG_\Pi$ be the map defined by*

$$\alpha_1(g_0, t_0, g_1, t_1, \dots, g_k, t_k, \dots) = (e, 0, g_0, t_0, e, 0, g_1, t_1, \dots, e, 0, g_k, t_k, e, 0, \dots)$$

and hence whose image lies in the odd factors of EG_Π. Then $\alpha_1 \simeq_\Pi \mathrm{id}_{EG_\Pi}$.

Proof. It suffices to show that the map $\beta \colon EG_\Pi \to EG_\Pi$ defined by

$$\beta(g_0, t_0, g_1, t_1, \dots, g_n, t_n, \dots) = (e, 0, g_0, t_0, g_1, t_1, \dots, g_n, t_n, \dots).$$

is Π-homotopic to id_{EG_Π}.

Let $I_n = [1 - (\frac{1}{2})^n, 1 - (\frac{1}{2})^{n+1}]$ and $L_n' \colon I_n \to I$ be defined by $L_n'(s) = 2^{n+1}s - 2^{n+1} + 2$, this is the linear map that takes the end points $1 - (\frac{1}{2})^n$ and $1 - (\frac{1}{2})^{n+1}$ to 0 and 1. Now define $L_n \colon I \to I$ by

$$L_n(s) = \begin{cases} 0 & \text{for } s \in [0, 1 - (\frac{1}{2})^n] \\ L_n'(s) & \text{for } s \in [1 - (\frac{1}{2})^n, 1 - (\frac{1}{2})^{n+1}] \\ 1 & \text{for } s \in [1 - (\frac{1}{2})^{n+1}, 1] \end{cases}$$

We now define a homotopy $H \colon EG \times I \to EG$ by

$$\begin{aligned} &H((g_0, t_0, g_1, t_1, \dots, g_n, t_n, \dots), s) = \\ &\quad = (g_0, t_0 L_0(s), g_0, t_0(1 - L_0(s)) + t_1(L_1(s)), g_1, t_1(L_1(s)) + t_2(L_2(s)), \dots). \end{aligned}$$

Notice that for $x = (g_0, t_0, g_1, t_1 \dots)$, we have

$$H(x, 0) = (g_0, 0, g_0, t_0, g_1, t_1, \dots) = (e, 0, g_0, t_0, g_1, t_1, \dots) = \beta(x)$$

and $H(x, 1) = \mathrm{id}_{EG}$. It is obvious that H is Π-equivariant. $\square$

Theorem 4.18. *Let X be a Γ-space and let $EG_\Pi \to BG_\Pi$ be as above. Then the map*

$$[X, BG_\Pi]_\Gamma \to Prin_{(G,\Pi)}(X)$$

is a bijection.

Proof. We adapt a proof in [Sel97]. For surjectivity, let $\pi\colon P \to X$ be a locally trivial numerable principal (G,Π)-bundle, it is enough to construct a Π-map $f\colon P \to EG_\Pi$. Let $\{A_i\}_{i=0}^\infty$ be a numerable open cover of X by Γ-sets and let $\phi_i\colon \pi^{-1}(A_i) \to \Pi \times_{H_i} B_i$ be a trivialisation where B_i are H_i-slices. Let $\{h_i\}_{i=0}^\infty$ be a Γ-equivariant PoU relative to $\{A_i\}_{i=0}^\infty$ and we define

$$f(p) = (\pi_1(\phi_0 p)\cdot e, h_0(\pi(p)), \pi_1(\phi_1 p)\cdot e, h_1(\pi(p)), \dots, \pi_1(\phi_i p)\cdot e, h_i(\pi(p)), \dots)$$

where π_1 is the projection to the first factor and for $g \in \Pi$ the notation $g \cdot e$ refers to the action of Π on the identity element e of G. This map is well defined, since if $p \notin A_i$ then $h_i(p) = 0$ and so the preceding element in G does not matter. One can deduce that the map f is Π-equivariant due to the way the actions are defined.

For injectivity, let $f_0\colon X \to BG_\Pi$ and $f_1\colon X \to BG_\Pi$ be Γ-maps and suppose that there is a isomorphism $\theta\colon f_0^*(EG_\Pi) \to f_1^*(EG_\Pi)$. We will build a Γ-equivariant homotopy $F\colon X\times I \to B$ between f_0 and f_1. It is enough to build a Π-map $\tilde{F}\colon f_0^*(EG_\Pi)\times I \to EG_\Pi$ that covers F.

Let $\tilde{f}_0\colon f_0^*(EG_\Pi) \to EG_\Pi$ and $\tilde{f}_1\colon f_1^*(EG_\Pi) \to EG_\Pi$ be induced by f_0 and f_1 respectively. We now define $\tilde{F}$ by

$$\tilde{F}(p,s) = (g_0, (1-s)t_0, g_0', st_0', g_1, (1-s)t_1, g_1', st_1', \dots)$$

where $\tilde{f}_0(p) = (g_0, t_0, g_1, t_1, \dots)$ and $\tilde{f}_1(p) = (g_0', t_0', g_1', t_1', \dots)$. Recall the map α_1 from Lemma 4.17 and similarly define

$$\alpha_0(g_0, t_0, g_1, t_1, \dots, g_k, t_k, \dots) = (g_0, t_0, e, 0, g_1, t_1, e, 0, \dots, e, 0, g_k, t_k, e, 0, \dots)$$

whose image is in the even factors of EG_Π. Notice that by the previous lemma

$$\tilde{F}(p,1) = \alpha_1\tilde{f}_1(p) \simeq_\Pi \tilde{f}_1(p) \quad\text{and}\quad \tilde{F}(p,0) = \alpha_0\tilde{f}_0(p) \simeq_\Pi \tilde{f}_0(p).$$

It is clear that $\tilde{F}$ is a Π-map and this concludes the proof. □

We end this section noting that if $\Gamma = \mathbb{Z}_2$ then the space BG is given an involution. This involution will be essential in the next few sections where we will analyse bundles and gauge groups in the category of Real spaces.

4.2 Gauge Groups over Real Spaces

The following sections will concentrate on Real and pseudo Real bundles, hence we set $\Gamma \cong \mathbb{Z}_2$ and as in the preamble to Proposition 4.1 we let Π be $\mathbb{Z}_2 \times_\alpha G$ for Real bundles or $\mathbb{Z} \times_{\alpha\beta} G/\sim$ for the pseudo Real case.

Let $(P, \tilde{\sigma}) \to (X, \sigma)$ be a pseudo Real principal G-bundle. In comparison with the definitions in Sections 1.2, 1.3 and 3.1, an *automorphism* of $(P, \tilde{\sigma})$ is an automorphism $\phi\colon P \to P$ of the principal G-bundle P with the additional property that

$$\begin{array}{ccc} P & \xrightarrow{\tilde{\sigma}} & P \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi} \\ P & \xrightarrow{\tilde{\sigma}} & P. \end{array}$$

The *Real gauge group*, $\mathscr{G}(P, \tilde{\sigma})$ is defined to be the group of all automorphisms of $(P, \tilde{\sigma})$. The *pointed Real gauge group*, $\mathscr{G}^*(P, \tilde{\sigma})$ is defined to be the subgroup of $\mathscr{G}(P, \tilde{\sigma})$ that restricts to the identity above the basepoint of X.

Proposition 4.19. *There is an identification of $\mathscr{G}(P, \tilde{\sigma})$ with the space*

$$\mathrm{Map}_\Pi(P, G) = \left\{ f\colon P \to G \;\middle|\; \begin{array}{c} f((n,g)\cdot p) = \\ \alpha\beta(n)(g^{-1}f(p)g) \end{array} \right\}.$$

Proof. We recall as in Proposition 3.4 that given $\phi \in \mathscr{G}(P, \tilde{\sigma})$, we define a map $f\colon P \to G$ to be $f(p) = g_p$ where $\phi(p) = p{\cdot}g_p$. Notice that $(n,g){\cdot}(p) = \tilde{\sigma}^n(p){\cdot}\alpha\beta(n)(g)$ and therefore

$$\begin{aligned} \tilde{\sigma}^n(p) \cdot (\alpha\beta(n)(g))g_{(n,g)\cdot p} = \phi((n,g)\cdot(p)) = (n,g)\cdot\phi(p) = \\ = \tilde{\sigma}^n(p\cdot g_p g) = \tilde{\sigma}^n(p)\cdot\alpha\beta(n)(g_p g) \end{aligned}$$

and therefore

$$(\alpha\beta(n)(g))g_{(n,g)\cdot p} = \alpha\beta(n)(g_p g).$$

We deduce that

$$f((n,g)\cdot p) = g_{(n,g)\cdot p} = \alpha\beta(n)(g^{-1}g_p g)$$

as required. The converse is similar to the proof of Proposition 3.4. □

We would like an equivariant analogue of Theorem 3.5. Recall that we naturally induced a $\mathbb{Z}_2$-action on BG_Π, from herein we write $BG = BG_\Pi$. Given the $\mathbb{Z}_2$-action, we let $\mathrm{Map}_{\mathbb{Z}_2}(X, BG)$ denote the space of $\mathbb{Z}_2$-equivariant maps from X to BG. Again, we write $\mathrm{Map}_{\mathbb{Z}_2}(X, BG; P)$ to mean a particular component of $\mathrm{Map}_{\mathbb{Z}_2}(X, BG)$ containing a map that induces $(P, \tilde{\sigma})$ via Theorem 4.18.

Theorem 4.20. *When P is a compact CW-complex, there are homotopy equivalences*

$$B\mathscr{G}(P,\tilde{\sigma}) \simeq \mathrm{Map}_{\mathbb{Z}_2}(X, BG; P)$$

and

$$B\mathscr{G}^*(P,\tilde{\sigma}) \simeq \mathrm{Map}^*_{\mathbb{Z}_2}(X, BG; P).$$

Proof. By a similar argument to Theorem 3.5, we obtain a principal bundle

$$\mathscr{G}(P,\tilde{\sigma}) \to \mathrm{Map}_{\Pi}(P, EG) \to \mathrm{Map}_{\mathbb{Z}_2}(X, BG; P)$$

which again can be shown to be universal. □

4.3 Equivariant Mapping Spaces

In light of Theorem 4.20 we would like to check that some of the features of non-equivariant bundles transfer to this context. The main result of interest of this section is Proposition 4.26 which is the equivariant analogue of the following lemma.

Lemma 4.21. *Let $A \to X$ be a cofibration and let Z be a space, then the restriction map is a fibration*

$$\mathrm{Map}^*(X,Z) \to \mathrm{Map}^*(A,Z)$$

with fibre $\mathrm{Map}^*(X/A, Z)$*.*

By a Γ-*complex* we mean a CW-complex X endowed with an action of Γ such that

1. the action permutes cells of X;
2. if $g \cdot \tau = \tau$ for some $g \in \Gamma$ and cell τ of X then $g \cdot x = x$ for all $x \in \tau$.

Throughout the following, we will let I denote the interval $[0,1]$ endowed with the trivial Γ-action.

Proposition 4.22. *Let X, Y and Z be Γ-complexes, let A be a sub Γ-complex of Z and let $f\colon A \to Y$ be a cellular Γ-map then*

1. *$Z \sqcup_f Y$ is a Γ-complex and;*
2. *$X \times I$ is a Γ-complex.*

Proof. For the cells of $Z \sqcup_f Y$, we take the union of the cells in Y and $Z - A$. This union of cells inherits the Γ-actions from Z and Y and it is clear that it is closed under the Γ-action since A is a sub Γ-complex.

The attaching maps are the same for cells that originated from Y and the 'interior' of $Z - A$; cells τ that originated from Z such that $\tau \cap A = \emptyset$. For the remaining cells we alter the attaching maps by using f in the appropriate places. It is easy to see that $Z \sqcup_f Y$ is a Γ-complex because the map f was cellular and a Γ-map.

The space $X \times I$ obviously inherits the structure of a Γ-complex. □

Notice that by setting $Z = X \times I$ and $A = X \times \{0\}$, we immediately obtain the following.

Corollary 4.23. *Let X and Y be Γ-complexes and let $f\colon X \to Y$ be a cellular Γ-map. Then the mapping cone M_f is a Γ-complex.*

An inclusion $g\colon X \to M_f$ satisfies the usual homotopy extension property (HEP) but to prove Proposition 4.26 we require a stronger notion. We say that a Γ-map $\alpha\colon A \to B$ has Γ-*homotopy extension property* (ΓHEP) with respect to a Γ-space C if it satisfies HEP and we can choose the extended homotopy to be a Γ-homotopy. We say that $\alpha\colon A \to B$ is a Γ-*cofibration* if it satisfies ΓHEP with respect to all Γ-spaces.

Therefore, the aim of this section is to show that the inclusion $g\colon X \to M_f$ has ΓHEP with respect to all Γ-spaces Z. In the above, we have shown that the pair (M_f, X) is a Γ-complex pair and we generalise [Hat02, Prop 0.16] to obtain the following.

Proposition 4.24. *Let (X, A) be a Γ-complex pair then (X, A) has ΓHEP.*

Proof. In the proof of [Hat02, Prop 0.16], the space $X \times \{0\} \cup A \times I$ was shown to be a deformation retract of $X \times I$. Since the Γ-action only permutes cells and does not non-trivially send a cell to itself then this deformation retract is Γ-equivariant. Hence there is a Γ-equivariant retract

$$r\colon X \times I \to X \times \{0\} \cup A \times I$$

and the result follows. □

We have shown the following.

Theorem 4.25. *Let Γ be an arbitrary discrete group, let X and Y be Γ-complexes and let $f\colon X \to Y$ be a Γ-map that is cellular. Then f can be factored through maps*

$$X \xrightarrow{g} M_f \xrightarrow{h} Y$$

where g is a Γ-cofibration and h is a Γ-homotopy equivalence. □

This is particularly useful when combined with the following. We say that a pointed Γ-space X has a *fixed basepoint* if $g \cdot *_X = *_X$ for all $g \in \Gamma$.

Proposition 4.26. *Let $i\colon A \hookrightarrow X$ be a pointed Γ-cofibration between locally compact Hausdorff Γ-spaces with fixed basepoints and let Z be Γ-space with fixed basepoint. Then the following is a (non-Γ) fibration*

$$\mathrm{Map}^*_\Gamma(X,Z) \xrightarrow{i^*} \mathrm{Map}^*_\Gamma(A,Z)$$

with fibre $\mathrm{Map}^*_\Gamma(X/A,Z)$.

To prove the proposition we first need a technical lemma.

Lemma 4.27. *Let X and Y be pointed Γ-spaces where X is locally compact Hausdorff and with fixed basepoint. Let Z be a pointed space and consider it as a Γ-space with the trivial Γ-action. Then there is a homeomorphism*

$$\phi\colon\ \mathrm{Map}^*(Z,\mathrm{Map}^*_\Gamma(X,Y)) \to \mathrm{Map}^*_\Gamma(Z\wedge X,Y)$$

Proof. Forgetting about Γ-equivariance, with the conditions on X there is a homeomorphism (see [Sel97, Thm 3.1.2])

$$\tilde{\phi}\colon\ \mathrm{Map}^*(Z,\mathrm{Map}^*(X,Y)) \to \mathrm{Map}^*(Z\wedge X,Y).$$

Let ϕ be the restriction of $\tilde{\phi}$ to $\mathrm{Map}^*(Z,\mathrm{Map}^*_\Gamma(X,Y))$, then ϕ is a homeomorphism onto its image so it suffices to show that

$$\mathrm{im}(\phi) = \mathrm{Map}^*_\Gamma(Z\wedge X,Y).$$

when Z is endowed with a trivial Γ-action.

Let f be in $\mathrm{Map}^*(Z,\mathrm{Map}^*_\Gamma(X,Y))$ then $\phi(f)\colon Z\wedge X\to Y$ is a well-defined Γ-map since X and Z have fixed basepoints. Explicitly, $\phi(f)$ must satisfy

$$\phi(f)(z,g\cdot x) = g\cdot\phi(f)(z,x) \text{ for all } x\in X,\ z\in Z \text{ and } g\in\Gamma$$

and hence $\phi(f)$ is in $\mathrm{Map}^*_\Gamma(Z\wedge X,Y)$.

Conversely, let $\tilde{\phi}^{-1}$ be the inverse homeomorphism to $\tilde{\phi}$. Then for $h\in\mathrm{Map}^*_\Gamma(Z\wedge X,Y)$ the map $\tilde{\phi}^{-1}(h)$ must satisfy

$$g\cdot\tilde{\phi}^{-1}(h)(z)(x) = \tilde{\phi}^{-1}(h)(z)(g\cdot x) \text{ for all } x\in X,\ z\in Z \text{ and } g\in\Gamma$$

and the result follows. □

Proof of Proposition 4.26. Let Y be an arbitrary pointed space then we wish to prove that the pointed homotopy $\tilde{F}$ exists as in the following commutative diagram

$$\begin{array}{ccc} Y \times \{0\} & \xrightarrow{f} & \mathrm{Map}^*_\Gamma(X, Z) \\ \big\downarrow & \overset{\tilde{F}}{\nearrow} & \big\downarrow i^* \\ Y \times I & \xrightarrow{F} & \mathrm{Map}^*_\Gamma(A, Z). \end{array}$$

Considering Y and I as trivial Γ-spaces, we use Lemma 4.27 to consider the Γ-maps

$$\phi(f)\colon Y \wedge X \to Z \quad \text{and} \quad \phi(F)\colon (Y \times I) \wedge A \to Z.$$

Now F is a pointed homotopy so the domain in $\phi(F)$ descends to the space $(Y \rtimes I) \wedge A$ where $(Y \rtimes I) = (Y \times I)/\sim$ with $(*_Y, t) \sim (*_Y, t')$ for all t, t' in I. We have that the space $(Y \rtimes I) \wedge A$ is the same as $(Y \wedge A) \rtimes I$. Now because i is a Γ-cofibration, the map

$$Y \wedge A \xrightarrow{\mathrm{id}_Y \wedge i} Y \wedge X$$

is also a Γ-cofibration. Therefore there is an extension of $\phi(F)$ to a pointed Γ-homotopy

$$H\colon (Y \wedge X) \rtimes I \to Z$$

and it is clear that $\tilde{F} = \phi^{-1}(H)$ gives the needed result. □

The first part of the following lemma follows directly from Lemma 4.27 with Z a copy of S^1 endowed with the trivial action. The proof of the second part is very similar.

Lemma 4.28. *Let X, Y be as in Lemma 4.27 then there are equivalences*

1. $\Omega\,\mathrm{Map}^*_\Gamma(X, Y) \cong \mathrm{Map}^*_\Gamma(\Sigma X, Y)$
2. $\mathrm{Map}^*_\Gamma(X, \Omega Y) \cong \mathrm{Map}^*_\Gamma(\Sigma X, Y)$ □

These results will be of particular interest when $\Gamma \cong \mathbb{Z}_2$ in the coming sections but first we show that Real surfaces admit a $\mathbb{Z}_2$-CW structure.

4.4 Real Surfaces as $\mathbb{Z}_2$-complexes

We recall that Real surfaces are classifie, up to involution-equivariant homeomorphism, by their types, that is, the triple (g, r, a) where

- g is the genus of X;
- r is the number of path components of the fixed set X^σ;

- $a = 0$ if X/σ is orientable and $a = 1$ otherwise.

subject to the relations in Theorem 1.1. We will follow [BHH10] and define a CW-structure for each type of X that has nice properties under the involution. In the following, $\Sigma_{p,q}$ will denote a Riemann surface of genus p with q open discs removed.

Type $(g, 0, 1)$**.** We first study the case where g is even. We can think of X as two copies of $\Sigma_{g/2,1}$ glued along their boundary components; each a copy of S^1. The involution restricted to S^1 is the antipodal map and extends to swap the two copies of $\Sigma_{g/2,1}$.

We give a CW-structure of X as follows, let X^0 be 2 zero-cells; $*$ and $\sigma(*)$. There are $2g+2$ one-cells

$$\alpha_1, \ldots, \alpha_{g/2}, \beta_1, \ldots, \beta_{g/2}, \gamma \text{ and}$$
$$\sigma(\alpha_1), \ldots, \sigma(\alpha_{g/2}), \sigma(\beta_1), \ldots, \sigma(\beta_{g/2}), \sigma(\gamma).$$

The boundaries of α_i, β_i are glued to $*$ and the boundaries of $\sigma(\alpha_i), \sigma(\beta_i)$ are glued to $\sigma(*)$. One end of γ is glued to $*$ and the other to $\sigma(*)$, whilst the same is done for $\sigma(\gamma)$ with the opposite orientation. There are 2 two-cells glued on, one with attaching map

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_{g/2}\beta_{g/2}\alpha_{g/2}^{-1}\beta_{g/2}^{-1}\gamma\sigma(\gamma)$$

and the other with the same attaching map but with α_i, β_i replaced with $\sigma(\alpha_i), \sigma(\beta_i)$ and $\gamma\sigma(\gamma)$ replaced with $\sigma(\gamma)\gamma$.

As the notation suggests, the involution swaps cells that differ by σ. In particular, this is a σ-equivariant CW-structure and hence descends to a CW-structure of X/σ.

Now assume that g is odd and let $g' = (g-1)$. We see that X can be thought of as two copies of $\Sigma_{g'/2,2}$ glued along their boundaries; two copies of S^1 in X. The involution swaps these copies of S^1 but reverses orientations, and it extends to X to swap the two copies of $\Sigma_{g'/2,2}$.

There are 2 zero-cells, $*$ and $\sigma(*)$ and $2g$ one-cells

$$\alpha_1, \ldots, \alpha_{g'/2}, \beta_1, \ldots, \beta_{g'/2}, \gamma, \delta \text{ and}$$
$$\sigma(\alpha_1), \ldots, \sigma(\alpha_{g'/2}), \sigma(\beta_1), \ldots, \sigma(\beta_{g'/2}), \sigma(\gamma), \sigma(\delta)$$

where $\alpha_i, \beta_i, \sigma(\alpha_i), \sigma(\beta_i), \gamma, \sigma(\gamma)$ are glued as before but the boundary of δ is glued to $*$ and $\sigma(\delta)$ to $\sigma(*)$. Now there are 2 two-cells, one with boundary map

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_{g'/2}\beta_{g'/2}\alpha_{g'/2}^{-1}\beta_{g'/2}^{-1}\delta\gamma\sigma(\delta)\gamma^{-1}$$

and the other glued equivariantly. The cells δ and $\sigma(\delta)$ correspond to the copies of S^1 above and here γ is a cell joining these copies of S^1.

Type $(g, r, 0)$**.** Let the involution fix r circles and let $g' = (g - r + 1)/2$, then X/σ is a $\Sigma_{g',r}$ and X can be thought of as two copies of $\Sigma_{g',r}$ glued along the r boundary components.

In this case, the basepoint is preserved under σ, however X^0 is given r zero-cells; one for each fixed component. The one cells are then

$$\alpha_1, \ldots, \alpha_{g'}, \beta_1, \ldots, \beta_{g'}, \gamma_2, \ldots, \gamma_r, \delta_1, \ldots, \delta_r \text{ and}$$
$$\sigma(\alpha_1), \ldots, \sigma(\alpha_{g'}), \sigma(\beta_1), \ldots, \sigma(\beta_{g'}), \sigma(\gamma_2), \ldots, \sigma(\gamma_r)$$

where α_i, β_i are as before and γ_i joins the basepoint to the i-th fixed component which is represented by δ_i. One of the 2 two-cells has attaching map

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_{g'}\beta_{g'}\alpha_{g'}^{-1}\beta_{g'}^{-1}\delta_1\gamma_2\delta_2\gamma_2^{-1}\cdots\gamma_r\delta_r\gamma_r^{-1}$$

and we again define the other one equivariantly.

Type $(g, r, 1)$ **for** $r > 0$**.** Let the involution fix r circles. We first consider the case where $g \equiv r \mod 2$. Let $g' = (g - r)/2$, then X can be thought of as two copies of $\Sigma_{g',r+1}$ glued along the boundary components. The involution fixes the first r of these components whilst restricting to the antipodal map on the extra copy of S^1.

Now, X^0 is given $(r + 2)$ zero-cells $*_i$; one for each fixed component and two for the extra S^1. The one cells are then

$$\alpha_1, \ldots, \alpha_{g'}, \beta_1, \ldots, \beta_{g'}, \gamma_2, \ldots, \gamma_{r+1}, \delta_1, \ldots, \delta_r, \delta \text{ and}$$
$$\sigma(\alpha_1), \ldots, \sigma(\alpha_{g'}), \sigma(\beta_1), \ldots, \sigma(\beta_{g'}), \sigma(\gamma_2), \ldots, \sigma(\gamma_{r+1}), \sigma(\delta)$$

where α_i, β_i are as before and γ_i joins the basepoint to the i-th boundary circle. Each fixed component is represented by δ_i and δ joins $*_{r+1}$ to $*_{r+2}$ and therefore $\delta\sigma(\delta)$ represents the extra copy of S^1. One of the 2 two-cells has attaching map

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_{g'}\beta_{g'}\alpha_{g'}^{-1}\beta_{g'}^{-1}\delta_1\gamma_2\delta_2\gamma_2^{-1}\cdots\gamma_r\delta_r\gamma_r^{-1}\gamma_{r+1}\delta\sigma(\delta)\gamma_{r+1}^{-1}$$

and we again define the other one equivariantly.

For the case where, $g \equiv r + 1 \mod 2$, we let $g' = (g - r - 1)/2$. Now X can be thought of as two copies $\Sigma_{g',r+2}$ glued along the boundary components. Again, the involution fixes r of these components, whilst swapping the final two copies of S^1 but reversing orientation.

Again X^0 is given $(r+2)$ zero-cells; one for each fixed component and one for each of the extra two copies of S^1. The one cells are then

$$\alpha_1, \dots, \alpha_{g'}, \beta_1, \dots, \beta_{g'}, \gamma_2, \dots, \gamma_{r+2}, \delta_1, \dots, \delta_{r+1} \text{ and}$$
$$\sigma(\alpha_1), \dots, \sigma(\alpha_{g'}), \sigma(\beta_1), \dots, \sigma(\beta_{g'}), \sigma(\gamma_2), \dots, \sigma(\gamma_{r+2}), \sigma(\delta_{r+1})$$

where α_i, β_i are as before and γ_i joins the basepoint to the i-th boundary circle. Each fixed component is represented by δ_i for $i \leq r$, and δ_{r+1} and $\sigma(\delta_{r+1})$ represent the extra copies of S^1. One of the 2 two-cells has attaching map

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1} \cdots \alpha_{g'}\beta_{g'}\alpha_{g'}^{-1}\beta_{g'}^{-1}\delta_1\gamma_2\delta_2\gamma_2^{-1} \cdots \gamma_{r+1}\delta_{r+1}\gamma_{r+1}^{-1}\gamma_{r+2}\sigma(\delta_{r+1})\gamma_{r+2}^{-1}$$

and we again define the other one equivariantly.

4.5 Real Principal $U(n)$-bundles over Real Surfaces

We now restrict our study to Real principal $U(n)$-bundles over Real surfaces, reserving pseudo Real bundles for later study. We first analyse how the bundles are classified using Theorem 4.18. This classification was originally studied in [BHH10]. We endow the group $U(n)$ with an action of $\mathbb{Z}_2 \times_\alpha U(n)$ where $\alpha(1)$ is induced by complex conjugation. We will use the notation $EU(n) \to BU(n)$ to mean the universal bundle with the Real structure induced by this action as given in the preamble to Theorem 4.18.

For a Real map $f: Y \to Z$ and a fixed point y of σ_Y we note that

$$\sigma_Z(f(y)) = f(\sigma_Y(y)) = f(y).$$

We conclude that Real maps send fixed points to fixed points. In light of Theorems 4.18 and 4.20 we will be studying Real maps of the form $f: (X, \sigma) \to (BU(n), \sigma_{BU(n)})$. Therefore fixed points X^σ are sent to fixed points of $\sigma_{BU(n)}$ which is the subspace $BO(n) \subset BU(n)$.

Throughout the next two sections we will let S_0 denote the bottom cell of $BU(n)$ which is homeomorphic to a copy of S^2. We denote the meridian of S_0 by S_1 which is a copy of S^1 and corresponds to the bottom cell of $BO(n) \subset BU(n)$. We are ready to prove Proposition 1.2 and restate here for convenience.

Proposition 1.2. *Let (X, σ) be a Real surface with r fixed components X_i for $1 \leq i \leq r$. Then Real principal $U(n)$-bundles $(P, \tilde{\sigma}) \to (X, \sigma)$ are classified by the first Stiefel-Whitney classes of the restriction to bundles $P_i \to X_i$ over the fixed components*

$$w_1(P_i) \in H^1(X_i, \mathbb{Z}/2) \cong \mathbb{Z}/2$$

and the first Chern classes of the bundle over X

$$c_1(P) \in H^2(X, \mathbb{Z}) \cong \mathbb{Z}$$

subject to the relation

$$c_1(P) \equiv \sum w_1(P_i) \mod (2). \tag{4.3}$$

Furthermore, given any such characteristic classes there is a bundle satisfying them.

Remark 4.29. Notice that the right hand side of (4.3) is 0 if there are no fixed points. We see that the classification is a lot simpler in this case, that is, we only require $c_1(P)$ be even. The reader should compare this to the classification of $U(n)$-bundles over Riemann surfaces which are characterised by even and odd Chern classes.

Proof. By Theorem 4.18, isomorphism classes are in bijection with the elements of the set $[X, BU(n)]_{\mathbb{Z}_2}$. We first study the case where there are no fixed points of σ.

Since S_0 is a two-cell, given a map $f\colon X \to BU(n)$ we can contract all the one-cells of X to a point. This implies that we can choose another representation f' of $[f]$ in $[X, BU(n)]_{\mathbb{Z}_2}$ that factors through $(S^2 \vee S^2, \mathrm{sw})$ where sw swaps the factors. There is no restriction on how we map the first copy of S^2, without loss of generality we assume it wraps c times S_0. This then completely determines the mapping of the other copy of S^2. Since the involution is orientation reversing we obtain a map of X that wraps $2c$ times round S_0, which shows the result for $r = 0$.

We now let r be greater than 0. Each fixed component $X_i = S^1$ is mapped into $BO(n)$ and such maps are classified by $\pi_1(BO(n)) = \mathbb{Z}_2$.

For relation (4.3) we let $f\colon X \to BU(n)$ be a Real map that is non-trivial on s fixed components, X_i for $1 \leq i \leq s$. The rest of the one-cells can be contracted to a point. Now we can choose another map f' representing $[f]$ that sends each X_i homeomorphically onto S_1. This forces one of the two-cells of X to wrap s times round the top hemisphere of S_0, and the other cell wraps s times round the bottom hemisphere of S_0. The overall effect of this alters the Chern class in the corresponding bundle by $+s$. Combining with the case $r = 0$, we find that X wraps around this bottom cell $2c + s$ times. This completes the proof. □

4.5.1 Pseudo-Real Bundles over Real Surfaces

We apply a similar analysis to the previous section but transfer to the setting of pseudo Real principal $U(n)$-bundles. Let $\pi\colon (P, \tilde{\sigma}) \to (X, \sigma)$ be a pR-bundle with $\tilde{\sigma}^2 = \mathrm{id}_P \cdot g_0$. We have already shown that we require $g_0 \in Z_{\mathbb{R}}(U(n))$. The center of $U(n)$ consists of elements $e^{ti} \cdot I_n$ for $t \in [0, 2\pi)$; a copy of $U(1) \subset U(n)$. Further the only points in

$U(1)$ fixed by complex conjugation are I_n and $-I_n$, hence these elements are the only possibilities for g_0.

We dealt with the case I_n above and so we focus on the case $-I_n$. Thus far, we have referred to bundles with the property $\tilde{\sigma}^2 = \mathrm{id}_P \cdot (-I_n)$ as Quaternionic bundles. We note that we have not restricted to Quaternionic bundles of even rank as we did in Chapter 1. However, in Chapter 5 we shall see that decompositions for gauge groups of Real bundles will generalise more naturally if we restrict to the even rank case. That said, in the coming section we first consider the classification of Quaternionic bundles of odd rank and then we consider the even case, in which, we will also motivate the term 'Quaternionic'.

In this section, we will once again denote the bottom cell of $BU(n)$ by S_0.

Proposition 4.30. *If n is odd then Quaternionic bundles only exist over Real spaces without fixed points.*

Proof. Let n be odd and suppose that the is a fixed point $\pi(p)$ of σ. Notice that $\tilde{\sigma}(p) = p \cdot g_p$ for some $g_p \in U(n)$. Clearly g_p must satisfy $g_p^2 = -I_n$ but also $\alpha(1)(g_p) = g_p$ since

$$p \cdot g_p g_p = \tilde{\sigma}(p \cdot g_p) = \tilde{\sigma}(p) \cdot \alpha(1)(g_p) = p \cdot g_p \alpha(1)(g_p).$$

This is impossible since $\det(-I_n) = -1$ implying $\det(g_p) = \pm i$ and therefore g_p cannot be fixed by the antiholomorphic map $\alpha(1)$. □

We therefore need only analyse isomorphism classes of bundles over type $(g, 0, 1)$ Real surfaces. As in the Real case, we follow the classification as in [BHH10].

Proposition 4.31. *The Quaternionic bundles of odd rank over a Real surface of genus g are classified by their first Chern class which satisfy*

$$c_1(P) \equiv g + 1 \mod (2).$$

Proof. Recall the CW decompositions of Real surfaces of type $(g, 0, 1)$. We first consider the case where g is even.

We note that up to homotopy the only involution on S_0 that has no fixed points is the antipodal map. We choose a basepoint x on the equator of this bottom cell and we homotope the one-cells α_i, β_i onto x. Therefore $\sigma(\alpha_i), \sigma(\beta_i)$ are mapped to $\sigma(x)$.

The one-cells $\gamma, \sigma(\gamma)$ require a bit more attention. We map γ to one half of the equator and hence $\sigma(\gamma)$ to the other half. Just as in the Real case, this increases or decreases the Chern class on the bundle by 1. We are free to wrap one of the two cells of the Real surface c times round S_0 and, as in the Real case, this forces the other cell to wrap around a further c times. Putting these together shows that the Chern class of such a bundle must be odd.

The odd genus case is quite similar, we homotope the one cells $\alpha_i, \beta_i, \delta$ onto x and $\sigma(\alpha_i), \sigma(\beta_i), \sigma(\delta)$ onto $\sigma(x)$. Again, γ and $\sigma(\gamma)$ are forced onto the equator. However, neither of the two-cells are glued onto both γ and $\sigma(\gamma)$, hence the Chern class is not altered by this mapping of the one-cells. Therefore, we are left with mapping the two-cells into S_0, and therefore we require an even Chern class. It is clear that all such bundles can be obtained. □

In the case $n = 2m$, there is a natural way of inducing a Quaternionic structure on the universal bundle $EU(2m) \to BU(2m)$. The aim is to define a $(\mathbb{Z} \times_{\alpha\beta} U(2m)/\sim)$-action on $U(2m)$. We first define the homomorphism $\alpha\colon \mathbb{Z}_2 \to Aut(U(2m))$.

There is a useful correspondence between the quaternionic numbers $\mathbb{H}$ and a particular subset of 2×2 matrices with complex coefficients which obeys addition and multiplication:

$$a + bi + cj + dk = (a + bi) \cdot 1 + (c + di)j \longleftrightarrow e + fj \longleftrightarrow \begin{pmatrix} e & f \\ -\overline{f} & \overline{e} \end{pmatrix}.$$

Notice also that quaternionic conjugation is given by conjugate transposition of the matrix. The correspondence gives an embedding of $Sp(m)$ in $U(2m)$ with each 2×2 block of an element of $U(2m)$ corresponding to a quaternionic number. We define a map on each 2×2 block via

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \overline{d} & -\overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix}$$

and notice that this fixes quaternionic numbers. We remark that we can extend this map to $U(2m)$ and that this corresponds to the involution σ_0 from Section 2.3. We define $\alpha(1)$ to be this automorphism.

We will now define how the element $(1, I_n)$ acts. First, we define a map

$$\begin{array}{ccc} \rho\colon M_2(\mathbb{C}) & \to & M_2(\mathbb{C}) \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \mapsto & \begin{pmatrix} -\overline{d} & \overline{c} \\ -\overline{b} & \overline{a} \end{pmatrix} \end{array}$$

that has the properties $\rho^2(A) = A \cdot (-I_2)$ and $\rho(AB) = \rho(A) \cdot \alpha(1)(B)$. Using this we can define a map on $U(2m)$ with the same properties and by the comments before Theorem 4.18 this induces a pR-structure on the universal bundle. Notice that the induced involution $\sigma_{BU(2m)}$ fixes the subspace $BSp(m)$.

Proposition 1.3. *Let (X, σ) be a Real surface, then Quaternionic principal $U(2m)$-bundles $(P, \tilde{\sigma}) \to (X, \sigma)$ are classified by their first Chern class which must be even.*

Furthermore, given any such Chern class there is a Quaternionic principal $U(2m)$-bundle that realises it.

Proof. As noted, the fixed points of $\sigma_{BU(2m)}$ is $BSp(m)$ which is 3-connected. Hence for any type of Real surface, all of the one-cells can be homotoped to a point p in S_0. We are then left with the usual situation of mapping the 2 two-cells in equivariantly. We conclude that the Chern class must be even. □

Chapter 5

Results

We will find that obtaining homotopy decompositions of gauge groups of Quaternionic bundles will follow a similar method to that of the Real case. However for the sake of clarity, the first part of this chapter restricts to the study of gauge groups of Real bundles. We then reserve the Quaternionic case until later but we only elaborate on where the proofs differ from the Real case.

For convenience, we recall the main objects of interest and their classification. Let $(P, \tilde{\sigma}) \to (X, \sigma)$ be a Real principal $U(n)$-bundle over a Real surface, then we aim to provide homotopy decompositions of the following spaces

- the unpointed gauge group $\mathscr{G}(P, \tilde{\sigma})$;
- the (single)-pointed gauge group $\mathscr{G}^*(P, \tilde{\sigma})$;
- the $(r+a)$-pointed gauge group $\mathscr{G}^{*(r+a)}(P, \tilde{\sigma})$.

From Section 1.1 we recall that the isomorphism class of these gauge groups depend on

1. the type (g, r, a) of the underlying Real surface (X, σ) where
   - g is the genus of X;
   - r is the number of path components of the fixed set X^σ;
   - $a = 0$ if X/σ is orientable and $a = 1$ otherwise;
2. the isomorphism class $(c, w_1, w_2, \ldots, w_r)$ of the bundle $(P, \tilde{\sigma})$ where
   - c is the first Chern class of the underlying principal $U(n)$-bundle P;
   - each w_i is the first Stiefel-Whitney class of the restriction to the bundle $P_i \to X_i$ over each fixed component X_i;

subject to the relations in Theorem 1.1 and Proposition 1.2.

Taking motivation from the non-equivariant case, we look to decompose the above gauge groups by studying the corresponding mapping spaces as in Theorem 4.20. We first provide such a correspondence for the $(r+a)$-pointed gauge group.

Let (X,σ) be a Real surface of type (g,r,a), then we recall that to define $\mathscr{G}^{*(r+a)}(P,\tilde{\sigma})$ we chose $(r+a)$ designated points as follows. For each $1 \leq i \leq r$ we chose a designated point $*_i$; one for each of the fixed components X_i. Further if $a=1$ we chose another designated point $*_{r+1}$ that is not fixed by the involution.

Now let $A := \amalg_{i=1}^{r+1} *_i \amalg \sigma(*_{r+1})$ and let $\mathrm{Map}^{*(r+a)}_{\mathbb{Z}_2}(X,BG)$ denote the subspace of $\mathrm{Map}_{\mathbb{Z}_2}(X,BG)$ whose elements send A to $*_{BG}$. Let $\overline{X}$ denote the cofibre of $A \hookrightarrow X$, then it is clear that Theorem 4.20 extends to this case. For convenience we rewrite the entire theorem.

Theorem 5.1. *With notation as above and with the involution on $BU(n)$ induced from complex conjugation, there are homotopy equivalences*

1. $B\mathscr{G}(P,\tilde{\sigma}) \simeq \mathrm{Map}_{\mathbb{Z}_2}(X,BU(n);P)$;
2. $B\mathscr{G}^*(P,\tilde{\sigma}) \simeq \mathrm{Map}^*_{\mathbb{Z}_2}(X,BU(n);P)$;
3. $B\mathscr{G}^{*(r+a)}(P,\tilde{\sigma}) \simeq \mathrm{Map}^{*(r+a)}_{\mathbb{Z}_2}(X,BU(n);P) \simeq \mathrm{Map}^*_{\mathbb{Z}_2}(\overline{X},BU(n);P)$;

where on the right hand side we pick the path component of $\mathrm{Map}_{\mathbb{Z}_2}(X,BU(n))$ that induces $(P,\tilde{\sigma})$.

Finally we highlight some notational conventions. As in Chapter 1, we will sometimes use

$$\mathscr{G}((g,r,a);(c,w_1,\dots,w_r))$$

to denote a gauge group of a bundle of class $(c,w_1,\dots,w_r)$ over a Real surface of type (g,r,a). Similar notation is used for the pointed gauge groups.

There are a number of $\mathbb{Z}_2$-spaces that will often appear, here we provide a dictionary:

- (X,id) - any space X with the trivial involution;
- $(X\vee X,\mathrm{sw})$ - the wedge $X\vee X$ equipped with the involution that swaps the factors;
- $(S^n,-\mathrm{id})$ - the sphere S^n equipped with the antipodal involution;
- (S^n,he) - the sphere S^n equipped with the involution that reflects along the equator.

5.1 Equivalent Components of Mapping Spaces

All of the results in the coming section will be exact restatements of those found in Section 1.2. Therefore to ease navigation, the following four propositions use the same numbering as in that section.

Proposition 1.5. *Let $(P, \tilde{\sigma})$ and (P', σ') be Real principal $U(n)$-bundles over a Real surface (X, σ) of arbitrary type (g, r, a), then there is a homotopy equivalence*

$$B\mathscr{G}^{*(r+a)}(P, \tilde{\sigma}) \simeq B\mathscr{G}^{*(r+a)}(P', \sigma').$$

Proof. Using the notation of Theorem 5.1, we wish to study the components of the mapping space $\mathrm{Map}^*_{\mathbb{Z}_2}(\overline{X}, BU(n))$. We take motivation from the non-equivariant case, and study the actions of $\pi_2(BU(n))$ and $\pi_1(BO(n))$ on $[\overline{X}, BU(n)]_{\mathbb{Z}_2}$. Recall that in the non-equivariant case the action of $\pi_2(BU(n))$ was defined via

$$X \xrightarrow{\text{pinch}} X \vee S^2 \xrightarrow{f \vee \alpha} BU(n) \vee BU(n) \xrightarrow{\text{fold}} BU(n) \tag{5.1}$$

with $\alpha \in \pi_2(BU(n))$ and $f \in \mathrm{Map}^*(X, BU(n))$. We now consider the equivariant case when $r = 0$. Let S^1 be the loop that is pinched together under the map $\overline{X} \to \overline{X} \vee S^2$. Due to equivariance, we are also forced to pinch the loop $\sigma(S^1)$ producing an extra factor of S^2 and the action becomes

$$\overline{X} \xrightarrow{\text{pinch}} \overline{X} \vee S^2 \vee \sigma(S^2) \xrightarrow{f \vee \alpha \vee \overline{\alpha}} BU(n) \vee BU(n) \vee BU(n) \xrightarrow{\text{fold}} BU(n).$$

where $\overline{\alpha} = \sigma_{BU(n)}\alpha$. Since σ and $\sigma_{BU(n)}$ are both orientation reversing, the action of $\alpha \in \pi_2(BU(n)) \cong \mathbb{Z}$ alters the class $[f]$ by 2α. Hence for $2c \in [\overline{X}, BU(n)]_{\mathbb{Z}_2} \cong 2\mathbb{Z}$, this action gives homotopy equivalences

$$\mathrm{Map}^*_{\mathbb{Z}_2}(X, BU(n); 2c) \simeq \mathrm{Map}^*_{\mathbb{Z}_2}(X, BU(n); 2c + 2\alpha).$$

In particular this gives the required homotopy equivalences for the case when $r = 0$.

When $r > 0$, the path components of $\mathrm{Map}^*_{\mathbb{Z}_2}(\overline{X}, BU(n))$ are classified by the tuple

$$(c, w_1, w_2, \ldots, w_r) \in \mathbb{Z} \times \prod_r \mathbb{Z}_2$$

subject to $c \equiv \sum_{i_1}^r w_i \mod 2$. We wish to construct an action of $\pi_1(BO(n))$ to alter each w_i. For $\beta \in \pi_1(BO(n))$, we note that the inclusion of the image of β into $BU(n)$ is nullhomotopic, so there is an extension $\beta' \colon D^2 \to BU(n)$ of β. Now, consider (S^2, he) and denote the fixed equator by E, the upper hemisphere by U and the lower hemisphere by L. We can extend β to a map $\tilde{\beta} \colon (S^2, \mathrm{he}) \to BU(n)$ where

$$\tilde{\beta}\mid_U = \beta' \text{ and } \tilde{\beta}\mid_L = \sigma_{BU(n)}\beta'$$

and therefore $\tilde{\beta}\mid_E = \beta$. By the last paragraph of the proof of Proposition 1.2, we deduce that the class $[\tilde{\beta}] \in \mathbb{Z} \times \mathbb{Z}_2$ is $(0,0)$ if β is trivial or $(\pm 1, 1)$ otherwise.

Let $(S^1, \mathrm{he}) \hookrightarrow \overline{X}$ be an inclusion such that the fixed points of (S^1, he) are mapped to the i-th fixed component X_i of $\overline{X}$. As in Equation (5.1) we apply the pinch map to this copy of (S^1, he) in $\overline{X}$, and hence produce a factor of (S^2, he). Now the action becomes

$$\overline{X} \xrightarrow{\text{pinch}} \overline{X} \vee (S^2, \mathrm{he}) \xrightarrow{f \vee \tilde{\beta}} BU(n) \vee BU(n) \xrightarrow{\text{fold}} BU(n).$$

For $\tilde{\beta}$ of class $(\pm 1, 1)$, we conclude that this action gives a homotopy equivalence between the components $(c, w_1, w_2, \ldots, w_r)$ and $(c \pm 1, w_1, \ldots, w_i + 1, \ldots, w_r)$. We can also apply the action of $\pi_2(BU(n))$ as before and combining these actions gives a homotopy equivalence between all the components of $\mathrm{Map}^*_{\mathbb{Z}_2}(\overline{X}, BU(n))$. □

We cannot provide such an extensive result for the single pointed gauge groups, due to the 'unpointed' fixed circles. However, choosing $*_1$ (see preamble to Theorem 5.1) as the basepoint obtains the following result.

Proposition 1.6. *For any c, c', w_1, w_1' there is a homotopy equivalence*

$$B\mathscr{G}^*((g,r,a);(c, w_1, w_2, \ldots, w_r)) \simeq B\mathscr{G}^*((g,r,a);(c', w_1', w_2, \ldots, w_r)). \qquad \square$$

We cannot hope to use the actions of π_1 and π_2 on the unpointed mapping space due to the lack of basepoint. But again, we take motivation from the non-equivariant case to provide some equivalences between components.

Proposition 1.7. *Let the following be classifying spaces of rank n gauge groups. Then there are homotopy equivalences*

$$B\mathscr{G}((g,r,a);(c, w_1, w_2, \ldots, w_r)) \simeq B\mathscr{G}((g,r,a);(c+2n, w_1, w_2, \ldots, w_r)).$$

Proof. Let i be the composition

$$(X, \sigma) \xrightarrow{q} (S^2 \vee S^2, \mathrm{sw}) \xrightarrow{\iota \vee \bar{\iota}} BU(1) \vee BU(1) \xrightarrow{\text{fold}} BU(1)$$

where q collapses the 1-skeleton of X, the map ι is the inclusion of the bottom cell of $BU(1)$ and $\bar{\iota} = \sigma_{BU(1)}\iota$.

The rest of the proof is almost identical to that of Proposition 3.10 but we sketch the proof again. We let $T\colon BU(n) \times BU(1) \to BU(n)$ be induced from scalar multiplication and we define

$$\theta\colon \mathrm{Map}_{\mathbb{Z}_2}(X, BU(n); (c, w_1, \ldots, w_r)) \to \mathrm{Map}_{\mathbb{Z}_2}(X, BU(n); (c+2n, w_1, \ldots, w_r))$$

to be the map that sends f to the composition

$$X \xrightarrow{\Delta} X \times X \xrightarrow{f\times i} BU(n) \times BU(1) \xrightarrow{T} BU(n).$$

As in Proposition 3.10, the map θ has a natural homotopy inverse given by replacing T with a map induced by conjugate scalar multiplication. □

Proposition 1.8. *Let n be odd then there are homotopy equivalences*

1. $B\mathscr{G}((g,r,a);(c,w_1,w_2,\ldots,w_r)) \simeq B\mathscr{G}((g,r,a);(c,\sum_{i=1}^{r} w_i,0,\ldots,0))$
2. $B\mathscr{G}^*((g,r,a);(c,w_1,w_2,\ldots,w_r)) \simeq B\mathscr{G}^*((g,r,a);(c,\sum_{i=1}^{r} w_i,0,\ldots,0))$.

Proof. To motivate the general case, we first restrict to the case when X is of type $(1,2,0)$, that is, a torus Σ_1 with orbit space a cylinder. Previously we defined a map

$$T\colon BU(n) \times BU(1) \to BU(n)$$

induced from the scalar product. In this new setting, we let T' be the restriction of T to $BU(n) \times BO(1)$. Our aim is to define homotopy equivalences

$$\theta\colon \operatorname{Map}_{\mathbb{Z}_2}(\Sigma_1, BU(n);(c,w_1,w_2)) \to \operatorname{Map}_{\mathbb{Z}_2}(\Sigma_1, BU(n);(c,w_1+1,w_2+1)).$$

In a similar fashion to the proof of Proposition 1.7, the map θ will send f to the composition

$$\theta(f)\colon \Sigma_1 \xrightarrow{\Delta} \Sigma_1 \times \Sigma_1 \xrightarrow{f\times i} BU(n) \times BO(1) \xrightarrow{T'} BU(n)$$

where i is to be defined.

We want to be considering how this map alters the Stiefel-Whitney class of the associated bundle. Hence we restrict T' to the map $T_2\colon BO(n) \times BO(1) \to BO(n)$ and we consider the induced map on first homology groups $(T_2)_*\colon \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{Z}_2$ which sends generators $\alpha \in H_1(BO(n))$ and $\beta \in H_1(BO(1))$ as follows

$$(T_2)_*(\alpha,0) = \alpha \text{ and } (T_2)_*(0,\beta) = \alpha.$$

The first equality is obvious, and the second equality comes from the fact that scalar multiplication of -1 on a matrix A in $O(n)$ switches the sign of the determinant if n is odd.

We now need to provide the map i, we first note that

$$[\Sigma_1, BO(1)]_{\mathbb{Z}_2} \cong [\Sigma_1/\sigma, BO(1)] \cong [\Sigma_1/\sigma, K(\mathbb{Z}_2,1)] \cong H^1(\Sigma_1/\sigma;\mathbb{Z}_2) \cong \mathbb{Z}_2.$$

We choose i so that one of the fixed circles is sent nontrivially to β (as represented as an element of $\pi_1(BO(1))$). This forces the other fixed circle to be sent to the same class.

We note that the composition as defined does not alter the Chern class of the associated bundle and so we have defined θ as above.

To extend to cases $r > 2$, we are only required to adjust the map i. We have

$$[X, BO(1)]_{\mathbb{Z}_2} \cong [X/\sigma, BO(1)] \tag{5.2}$$

but for some $r' \geq r-1$ we have $X/\sigma \simeq \vee_{r'} S^1$ under a deformation retract. In particular, we can choose this deformation retract to fix the fixed components X_i for $2 \leq i \leq r$ and we assume that the first $r-1$ copies of S^1 in $\vee_{r'} S^1$ are these components X_i. Now, for each $2 \leq i \leq r$ choose a representative $\tilde{w}_i \colon S^1 \to BO(1)$ of $w_i \in \pi_1(BO(1))$ and then we let i be induced by (5.2) via the composition

$$X/\sigma \xrightarrow{\simeq} \bigvee_{r'} S^1 \xrightarrow{\bigvee_{i=1}^{r-1} \tilde{w}_{i+1} \vee *} \bigvee_{r'} BO(1) \xrightarrow{\text{fold}} BO(1) \tag{5.3}$$

where $*$ is the constant map onto the basepoint. Due to the properties of T', this defines a map

$$\theta \colon \operatorname{Map}_{\mathbb{Z}_2}(X, BU(n); (c, w_1, w_2, \ldots, w_r)) \to \operatorname{Map}_{\mathbb{Z}_2}(X, BU(n); (c, \sum w_i, 0, \ldots, 0)).$$

Finally, we need to provide a homotopy inverse to θ. We note that we can extend θ to a map

$$\Theta \colon \operatorname{Map}_{\mathbb{Z}_2}(X, BU(n)) \to \operatorname{Map}_{\mathbb{Z}_2}(X, BU(n))$$

which is defined exactly in the same way as θ. Explicitly, the map Θ sends a map g to the composition

$$\Theta(g) \colon X \xrightarrow{\Delta} X \times X \xrightarrow{f \times i} BU(n) \times BO(1) \xrightarrow{T'} BU(n)$$

where T' is as above and i is the composition in (5.3).

In the non-equivariant setting, we provided a homotopy inverse to θ by replacing T with a map representing the conjugate tensor product. In this setting, since T' is the restriction to $BU(n) \times BO(1)$, a homotopy inverse of θ is given by the restriction of Θ to the path component $\operatorname{Map}_{\mathbb{Z}_2}(X, BU(n); (c, \sum w_i, 0, \ldots, 0))$.

It is clear that we can apply the same method to the pointed mapping spaces. □

Proof of Proposition 1.7 (Strong). Let $\pi \colon (P, \tilde{\sigma}) \to (X, \sigma)$ be a Real principal $U(n)$-bundle of class $(c, w_1, w_2, \ldots, w_r)$ over a Real surface of type (g, r, a). The idea will be to tensor P with a Real $U(1)$-bundle $\pi_Q \colon (Q, \tau) \to (X, \sigma)$ of class $(2, 0, \ldots, 0)$. The proof works for the same underlying reason as in the proof of Proposition 1.7, but it detects information that is missed by the diversion through homotopy theory.

Using the inclusion of the centre $U(1) \hookrightarrow U(n)$, there is a $U(1)$-action on $(P, \tilde{\sigma})$. In the principal bundle setting, the tensor of $(P, \tilde{\sigma})$ and (Q, τ) is the pullback

$$\begin{array}{ccc} (\Delta^*(P \times_{U(1)} Q), \Delta^*(\tilde{\sigma} \times \tau)) & \longrightarrow & (P \times_{U(1)} Q, \tilde{\sigma} \times \tau) \\ \downarrow & & \downarrow \tilde{\pi} \\ (X, \sigma) & \xrightarrow{\quad \Delta \quad} & (X, \sigma) \times (X, \sigma). \end{array}$$

where Δ is the diagonal map and $\tilde{\pi} = \pi \times \pi_Q$. Using a similar method to the proof of Proposition 1.7, we calculate that the class of the pullback $(\Delta^*(P \times_{U(1)} Q), \Delta^*(\tilde{\sigma} \times \tau))$ is $(c + 2n, w_1, w_2, \ldots, w_r)$.

We then define

$$\Theta \colon \mathscr{G}(P, \tilde{\sigma}) \to \mathscr{G}(\Delta^*(P \times_{U(1)} Q), \Delta^*(\tilde{\sigma} \times \tau))$$

to be the map that sends $\phi \colon P \to P$ to $\Delta^*(\phi \times \mathrm{id})$. Then an inverse to Θ is defined in the same way as Θ, except that we replace the inclusion $U(1) \hookrightarrow U(n)$ with the conjugate inclusion defined via

$$a \mapsto \begin{pmatrix} \overline{a} & 0 & \cdots & 0 \\ 0 & \overline{a} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{a} \end{pmatrix}.$$

□

Proof of Proposition 1.8 (Strong). Let $\pi \colon (P, \tilde{\sigma}) \to (X, \sigma)$ be a Real principal $U(n)$-bundle of class $(c, w_1, w_2, \ldots, w_r)$ over a Real surface of type (g, r, a). The statement is proven using the same method as Proposition 1.7 (Strong), except that we tensor with a Real $U(1)$-bundle $(\tilde{Q}, \tilde{\tau})$ of class $(0, \sum_{i=2}^{r} w_i, w_2, \ldots, w_r)$. If n is odd, the class of the pullback $(\Delta^*(P \times_{U(1)} \tilde{Q}), \Delta^*(\tilde{\sigma} \times \tilde{\tau}))$ is then $(c, \sum_{i=1}^{r} w_i, 0, \ldots, 0)$.

An isomorphism $\Theta \colon \mathscr{G}(P, \tilde{\sigma}) \to \mathscr{G}(\Delta^*(P \times_{U(1)} \tilde{Q}), \Delta^*(\tilde{\sigma} \times \tilde{\tau}))$ is then defined in the same way as for Proposition 1.7 (Strong). □

5.2 Pointed Gauge Groups

In the following analysis, it will be necessary to distinguish the following types of Real surfaces

0. $r = 0$ ($\Rightarrow a = 1$)

1. $r > 0$ and $a = 0$

2. $r > 0$ and $a = 1$.

Generally we will analyse the gauge groups in order of ease. We therefore will first analyse the $(r+a)$-pointed gauge group and then the single pointed gauge group. Our results for the single pointed gauge groups will then be used to analyse the unpointed case.

5.2.1 Integral Decompositions

In the non-equivariant case, we recall that the attaching map of a surface $f\colon S^1 \to \vee_{2g} S^1$ is a sum of Whitehead products and hence we deduced that Σf was nullhomotopic. Now in the equivariant case, we still see Whitehead products appearing in the attaching maps of Section 4.4. Hence we obtain an analogous result for the equivariant case.

We will use the notation as defined in Section 4.4 and furthermore we require the following notation in this section. Let g' denote the number of one-cells of X which are of the form α_i, β_i in X. Explicitly

$$g' = \begin{cases} (g-r+1) & \text{when } a=0; \\ (g-r) & \text{when } a=1 \text{ and } g-r \text{ even}; \\ (g-r-1) & \text{when } a=1 \text{ and } g-r \text{ odd.} \end{cases}$$

Proposition 5.2. *Let $X_{\alpha\beta} = \vee S^1$ be the 1-cells $\alpha_i, \sigma(\alpha_i), \beta_i, \sigma(\beta_i)$ in the decomposition of (X,σ). Then the map μ in the $\mathbb{Z}_2$-cofibration sequence*

$$X_{\alpha\beta} \hookrightarrow X \to X' \xrightarrow{\mu} \Sigma(X_{\alpha\beta})$$

is $\mathbb{Z}_2$-nullhomotopic.

Proof. We recall that the attaching map of one of the two-cells in a Real surface of type $(g,r,0)$ is

$$\alpha_1\beta_1\alpha_1^{-1}\beta_1^{-1}\cdots\alpha_{g'}\beta_{g'}\alpha_{g'}^{-1}\beta_{g'}^{-1}\delta_1\gamma_2\delta_2\gamma_2^{-1}\cdots\gamma_r\delta_r\gamma_r^{-1}.$$

Note that the attaching map involving the cells α_i and β_i is a sum of Whitehead products. The idea is to collapse the rest of the cells.

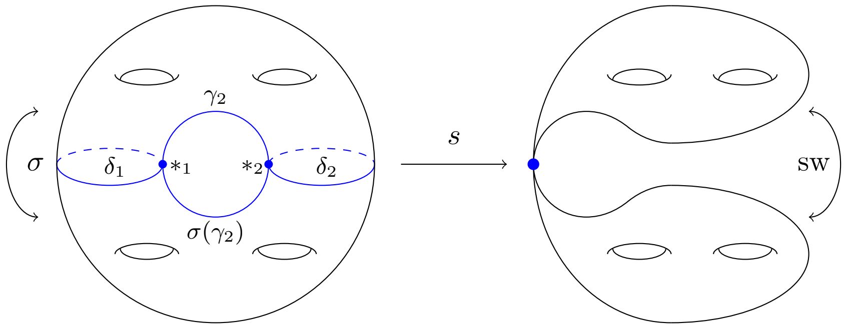
Now in the general case, let X be a type (g,r,a) Real surface, let $\Sigma_{g'/2}$ be a Riemann surface of genus $g'/2$ and let

$$s\colon X \to (\Sigma_{g'/2} \vee \Sigma_{g'/2}, \mathrm{sw})$$

be the map that collapses the 1-skeleton of X other than the cells $\alpha_i, \sigma(\alpha_i), \beta_i$ and $\sigma(\beta_i)$.

An example for the map s is illustrated in Figure 5.1. Note that four of the 'holes' are undisturbed by s; these correspond to the one-cells of the form $\alpha_i, \sigma(\alpha_i), \beta_i$ and $\sigma(\beta_i)$.

Figure 5.1: For a type $(5, 2, 0)$ Real surface, the map s collapses the one-cells coloured in blue: δ_1, δ_2, γ_2 and $\sigma(\gamma_2)$.



There is a commutative diagram

$$\begin{array}{ccccccc}
X_{\alpha\beta} & \longrightarrow & X & \longrightarrow & X' & \xrightarrow{\mu} & \Sigma(X_{\alpha\beta}) \\
\| & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} & & \| \\
X_{\alpha\beta} & \longrightarrow & (\Sigma_{g'/2} \vee \Sigma_{g'/2}, \mathrm{sw}) & \longrightarrow & (S^2 \vee S^2, \mathrm{sw}) & \xrightarrow{\Sigma f \vee \overline{\Sigma f}} & \Sigma(X_{\alpha\beta})
\end{array}$$

where the rows are $\mathbb{Z}_2$-cofiber sequences, s' is an induced map on cofibers and f is the attaching map of the Riemann surface $\Sigma_{g'/2}$. The $\mathbb{Z}_2$-triviality of μ therefore follows from the triviality of Σf. □

We immediately deduce the following theorem which contributes a lot of results to Theorems 1.9 and 1.10.

Theorem 5.3. *With notation as above, there are homotopy equivalences*

1. $\mathscr{G}^*(P, \tilde{\sigma}) \simeq \mathscr{G}^*((g - g', r, a); (c, w_1, \ldots, w_r)) \times \prod_{g'} \Omega U(n)$;
2. $\mathscr{G}^{(r+a)*}(P, \tilde{\sigma}) \simeq \mathscr{G}^{(r+a)*}((g - g', r, a); (c, w_1, \ldots, w_r)) \times \prod_{g'} \Omega U(n)$.

Proof. Follows from Theorem 5.1, Lemma 4.28 and Proposition 5.2. □

We note that for Real surfaces of type $(g, 0, 1)$, Theorem 5.3 leaves only types $(0, 0, 1)$ and $(1, 0, 1)$ to consider. The gauge groups of these types seem to be integrally indecomposable and so we leave their analysis until later.

5.2.2 Case: $r > 0$, $a = 0$

Although we restrict to the case $a = 0$, we will see that many of the methods in this section will also transfer to the case when $a = 1$.

We use the same notation as in the preamble to Theorem 5.1. The involution σ fixes r circles which we denote $\amalg_{i=1}^{r} X_i$. The r designated points $*_i$ are chosen such that $*_i \in X_i$ and we choose $*_1$ as the basepoint of X. We note that each $*_i$ is fixed by σ.

Due to Theorem 5.3 we restrict to the case when (X, σ) is of type $(r-1, r, 0)$. Using Theorem 5.1 and Lemma 4.28 and their notation we obtain the equivalences

$$\mathscr{G}^{*r}(P, \tilde{\sigma}) \simeq \mathrm{Map}^*_{\mathbb{Z}_2}(\Sigma(\overline{X}), BU(n)) \text{ and,}$$
$$\mathscr{G}^{*}(P, \tilde{\sigma}) \simeq \mathrm{Map}^*_{\mathbb{Z}_2}(\Sigma(X), BU(n)).$$

The aim of this section is to prove Theorems 1.9 and 1.10 for types $(g, r, 0)$ which is restated below.

Theorem 5.4. *Let $(P, \tilde{\sigma})$ be a Real bundle of class $(c, w_1, \ldots, w_r)$ over a Real surface (X, σ) of type $(r-1, r, 0)$. Then*

1. *there is a homotopy equivalence*

$$\mathscr{G}^{*r}(P, \tilde{\sigma}) \simeq \Omega^2(U(n)/O(n)) \times \prod_{r-1} \Omega O(n) \times \prod_{r-1} \Omega U(n);$$

2. *if $w_i = 0$ for all $i > 1$ or if n is odd then there is a homotopy equivalence*

$$\mathscr{G}^{*}(P, \tilde{\sigma}) \simeq \Omega^2(U(n)/O(n)) \times \prod_{r-1} \Omega O(n) \times \prod_{r-1} \Omega(U(n)/O(n)).$$

We first recall the $\mathbb{Z}_2$-CW decomposition of X. The 0-skeleton of X consists of r zero-cells labeled $*_i$ for $1 \le i \le r$. The one-cells consist of

$$\gamma_2, \ldots, \gamma_r, \sigma(\gamma_2), \ldots, \sigma(\gamma_r), \delta_1, \ldots, \delta_r$$

where each δ_i is a loop at $*_i$. The cells γ_j and $\sigma(\gamma_j)$ both connect $*_1$ to $*_j$ and are swapped under the involution.

There are 2 two-cells in X, the first has the attaching map

$$\delta_1 \gamma_2 \delta_2 \gamma_2^{-1} \gamma_3 \delta_3 \gamma_3^{-1} \cdots \gamma_r \delta_r \gamma_r^{-1} \tag{5.4}$$

and the other cell is glued on equivariantly. In the following X_γ will be the sub-complex of the one-cells of X that are denoted by either γ_i or $\sigma(\gamma_i)$.

Proposition 5.5. *Let (X, σ) be as above, then in the $\mathbb{Z}_2$-cofibration sequence*

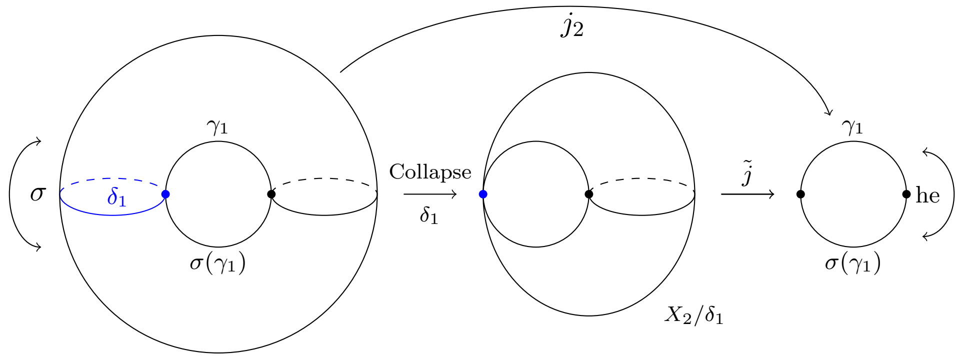
$$X_\gamma \xrightarrow{\iota} X \to \tilde{X} \xrightarrow{\mu'} \Sigma(X_\gamma)$$

there is a left $\mathbb{Z}_2$-homotopy inverse to ι. In particular μ' is $\mathbb{Z}_2$-nullhomotopic.

Proof. We will use induction on r; the number of fixed circles of X. Let X_r denote a Real surface of type $(r-1, r, 0)$ and let $(X_r)_\gamma$ be the sub-complex of X_r with one-cells denoted by either γ_i or $\sigma(\gamma_i)$. We aim to define left homotopy inverses $j_r \colon X_r \to (X_r)_\gamma$ of ι for each r.

Note that the space $(X_r)_\gamma$ is the wedge $\bigvee_{r-1}(S^1, \text{he})$ and hence the first non-trivial case is when $r = 2$. In this case, one can see that X_2 is the product $(S^1, \text{id}) \times (S^1, \text{he})$. We define j_2 to be the projection onto the second factor and Figure 5.2 illustrates this map.

Figure 5.2: The map j_2 projects to the factor (S^1, he) and j_2 factors through X_2/δ_1.



For $r = l$, we assume that j_l exists. For $r = l+1$, we first use a map j'_{l+1} that collapses a copy of $(S^1 \vee S^1, \text{sw})$ in X_{l+1} such that the image is homeomorphic to $X_l \vee X_2/\delta_1$ where X_2/δ_1 is a copy X_2 with the 1-cell δ_1 collapsed. The map j'_{l+1} is illustrated in Figure 5.3.

Figure 5.3: Collapse a copy of $(S^1 \vee S^1, \text{sw})$ to obtain the wedge $X_2/\delta_1 \vee X_l$.

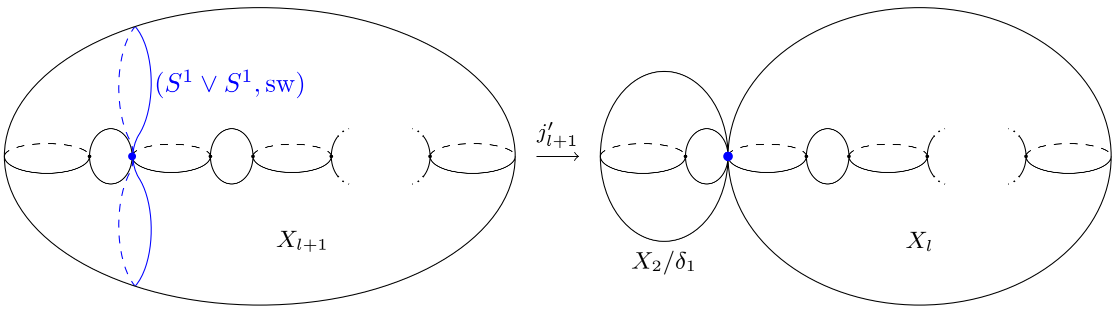


Figure 5.2 also shows that j_2 factors through the space X_2/δ_1. We therefore define j_{l+1} to be the composition

$$X_{l+1} \xrightarrow{j'_{l+1}} X_2/\delta_1 \vee X_l \xrightarrow{\tilde{j} \vee j_l} (X_{l+1})_\gamma$$

where $\tilde{j}$ is defined in Figure 5.2. □

As an easy consequence of Proposition 5.5 we obtain the following homotopy equivalences

$$\Sigma X \simeq \Sigma\tilde{X} \vee \Sigma X_\gamma \text{ and } \Sigma\overline{X} \simeq \Sigma\tilde{X} \vee \Sigma\overline{X}_\gamma.$$

In the following, we shall see that the factors ΣX_γ and $\Sigma\overline{X}_\gamma$ give the factors $\prod^{r-1}\Omega U(n)$ and $\prod^{r-1}\Omega(U(n)/O(n))$ respectively in Theorem 5.4 and that the factor $\Sigma\tilde{X}$ produces the factors $\Omega^2(U(n)/O(n)) \times \prod_{r-1}\Omega O(n)$. However, the map j_r automatically induces a map

$$\mathrm{Map}^*_{\mathbb{Z}_2}(X_\gamma, BU(n)) \to \mathrm{Map}^*_{\mathbb{Z}_2}(X, BU(n); (0,0,\ldots,0))$$

hence we only obtain a splitting on the level of mapping spaces in this trivial case.

We now restrict to this trivial case for the rest of this section. For the other cases, Proposition 1.5 will then give results for Theorem 5.4 (1) and Propositions 1.6 and 1.8 will give results for Theorem 5.4 (2). We provide further decompositions at the level of the Real surface to continue the proof of Theorem 5.4.

Proposition 5.6. *Let X_δ be the 1-cells in $\tilde{X}$ denoted by $\delta_2,\ldots,\delta_r$ then in the $\mathbb{Z}_2$-cofibration*

$$X_\delta \xrightarrow{\iota'} \tilde{X} \to (S^2,\mathrm{he}) \xrightarrow{\mu''} \Sigma(X_\delta)$$

the map μ'' is $\mathbb{Z}_2$-nullhomotopic.

Proof. The space $\tilde{X}$ is the quotient of a type $(r-1,r,0)$ Real surface with the one-cells denoted by $\gamma_2,\ldots,\gamma_r$ collapsed to a point. We see that the attaching map in Equation 5.4 becomes

$$\delta_1\delta_2\cdots\delta_r$$

and therefore conclude that $\tilde{X}$ is a sphere (S^2,he) with r of its fixed points identified.

Let U denote the upper 'hemisphere' of $\tilde{X}$; it is homeomorphic to a disc with r of its boundary points identified and notice that $\tilde{X} = U \cup \sigma(U)$. Now, there is a deformation retract $H\colon U\times I \to U$ of U onto the wedge $\bigvee_{i=2}^r \delta_i$. Therefore, we define a left inverse to the map ι' via

$$x \mapsto \begin{cases} H(x,1) & \text{for } x\in U\\ H(\sigma_{\tilde{X}}(x),1) & \text{for } x \in \sigma_{\tilde{X}}(U).\end{cases}$$

and the result follows. □

We deduce that

$$\Sigma\tilde{X} \simeq \Sigma X_\delta \vee \Sigma(S^2,\mathrm{he}).$$

The factor $\Sigma X_\delta = \bigvee^{r-1}(S^1,\mathrm{id})$ provides the factor $\prod^{r-1}\Omega O(n)$ for both cases in Theorem 5.4. We now show that the spaces $\Sigma(S^2,\mathrm{he})$ and ΣX_γ provide the other factors.

Lemma 5.7. *There are homotopy equivalences*

1. $\mathrm{Map}^*_{\mathbb{Z}_2}(\Sigma X_\gamma, BU(n)) \simeq \prod_{r-1} \Omega(U(n)/O(n))$;
2. $\mathrm{Map}^*_{\mathbb{Z}_2}(\Sigma \overline{X}_\gamma, BU(n)) \simeq \prod_{r-1} \Omega U(n)$.

Proof. The space $\Sigma(X_\gamma)$ is the same as the wedge $\bigvee_{r-1} \Sigma(S^1, \mathrm{he})$. Looking at the r-pointed case, the 0-skeleton of $\Sigma(X_\gamma)$ is collapsed and the space $\Sigma\overline{(X_\gamma)}$ becomes the wedge $\Sigma \bigvee_{r-1}(S^1 \vee S^1, \mathrm{sw})$. This shows part (2) of the lemma.

For part (1), we recall Example 2.1. The space $\mathrm{Map}^*_{\mathbb{Z}_2}((S^1, \mathrm{he}), BU(n))$ fits into the following pullback diagram

$$\begin{array}{ccc} \mathrm{Map}^*_{\mathbb{Z}_2}((S^1, \mathrm{he}), BU(n)) & \xrightarrow{\tilde{u}} & \mathrm{Map}^*(D^1, BU(n)) \\ \downarrow{\scriptstyle \tilde{r}} & & \downarrow{\scriptstyle r} \\ \mathrm{Map}^*_{\mathbb{Z}_2}((S^0, \mathrm{id}), BU(n)) & \xrightarrow{u} & \mathrm{Map}^*(S^0, BU(n)). \end{array}$$

Here $\tilde{r}$ restricts to the fixed points of (S^1, he) and $\tilde{u}$ restricts to the upper hemisphere of (S^1, he) and then forgets about equivariance. Since

$$\mathrm{Map}^*_{\mathbb{Z}_2}((S^0, \mathrm{id}), BU(n)) \simeq BO(n)$$

the map u is just the inclusion $BO(n) \hookrightarrow BU(n)$ and hence the homotopy fibre of u is $U(n)/O(n)$. Since r is a fibration, the square is also a homotopy pullback by Proposition 2.4. We note that the space $\mathrm{Map}^*(D^1, BU(n))$ is contractible and so the result follows. □

Lemma 5.8. *There is a homotopy equivalence*

$$\mathrm{Map}^*_{\mathbb{Z}_2}((S^2, \mathrm{he}), BU(n); (0,0)) \simeq \Omega(U(n)/O(n))_0$$

where $\Omega(U(n)/O(n))_0$ denotes the connected component of $\Omega(U(n)/O(n))$ containing the basepoint.

Proof. There is a similar pullback as in Lemma 5.7

$$\begin{array}{ccc} \mathrm{Map}^*_{\mathbb{Z}_2}((S^2, \mathrm{he}), BU(n)) & \xrightarrow{\tilde{u}} & \mathrm{Map}^*(D^2, BU(n)) \\ \downarrow{\scriptstyle \tilde{r}} & & \downarrow{\scriptstyle r} \\ \mathrm{Map}^*_{\mathbb{Z}_2}((S^1, \mathrm{id}), BU(n)) & \xrightarrow{u} & \mathrm{Map}^*(S^1, BU(n)). \end{array}$$

This time the map u is homotopic to the inclusion $O(n) \hookrightarrow U(n)$ and so the homotopy fibre of u is $\Omega(U(n)/O(n))$. The space $\mathrm{Map}^*(D^2, BU(n))$ is contractible and so there is an equivalence

$$\mathrm{Map}^*_{\mathbb{Z}_2}((S^2, \mathrm{he}), BU(n)) \simeq \Omega(U(n)/O(n))$$

and the result follows. □

Proof of Theorem 5.4. This follows from Propositions 5.5, 5.6 and Lemmas 5.7 and 5.8. □

5.2.3 Case: $r > 0$, $a = 1$

We use the techniques and notation of the previous section. In particular, let $(P, \tilde{\sigma})$ be a bundle of class $(0, 0, \ldots, 0)$ over a Real surface (X, σ) of type $(g, r, 1)$. We first note that by Proposition 5.2 we can restrict to the cases

$$g = r \quad \text{or} \quad g = r + 1. \tag{5.5}$$

With these cases in mind, the main aim will be to prove the following theorem which is a restatement of Theorems 1.9 and 1.10 for Real surfaces of type $(g, r, 1)$.

Theorem 5.9. *For the notation as above and g as in (5.5), there are homotopy equivalences*

1. $\mathscr{G}^*(P, \tilde{\sigma}) \simeq \mathscr{G}^*((g - r + 1, 1, 1); (0, 0)) \times \prod_{r-1} \Omega O(n) \times \prod_{r-1} \Omega(U(n)/O(n))$;
2. $\mathscr{G}^{*r+1}(P, \tilde{\sigma}) \simeq \mathscr{G}^{*2}((g - r + 1, 1, 1); (0, 0)) \times \prod_{r-1} \Omega O(n) \times \prod_{r-1} \Omega U(n)$.

We note that after we have proven the above theorem, the only cases we have left to analyse will be gauge groups over Real surfaces of type $(2, 1, 1)$ and type $(1, 1, 1)$.

For the proof of the theorem, we will essentially follow the methods of the previous section. Let X_γ denote the sub-complex of X consisting of the 1-cells denoted by either γ_i or $\sigma(\gamma_i)$ for $2 \leq i \leq r$.

Proposition 5.10. *Let (X, σ) be as above, then in the $\mathbb{Z}_2$-cofibration sequence*

$$X_\gamma \xrightarrow{\kappa} X \to \tilde{X} \xrightarrow{\nu} \Sigma(X_\gamma)$$

the map ν is $\mathbb{Z}_2$-nullhomotopic.

Proof. We define a left inverse to κ. First in X collapse the cells

$$\gamma_{r+1}, \sigma(\gamma_{r+1}), \delta_{r+1}, \sigma(\delta_{r+1})$$

and the cells $\gamma_{r+2}, \sigma(\gamma_{r+2})$ if they exist. We are left with a space $\mathbb{Z}_2$-homeomorphic to a Real surface of type $(r - 1, r, 0)$, we now use the map j_r as defined in the proof of Proposition 5.5. □

The proof of the next proposition is identical to that of Proposition 5.6 except we exchange (S^2, he) for a Real surface X' of type either $(2, 1, 1)$ or $(1, 1, 1)$.

Proposition 5.11. *Let X_δ be the 1-cells in $\tilde{X}$ denoted by $\delta_2, \dots, \delta_r$ then in the $\mathbb{Z}_2$-cofibration*

$$X_\delta \xrightarrow{\kappa'} \tilde{X} \to X' \xrightarrow{\nu'} \Sigma(X_\delta)$$

the map ν' is $\mathbb{Z}_2$-nullhomotopic. □

Proof of Theorem 5.9. This follows from Lemma 5.7 and Propositions 5.11 and 5.10. □

From Theorem 5.9, we reduce our study to the gauge groups

$$\mathscr{G}^*((1,1,1);(0,0)) \text{ and } \mathscr{G}^*((2,1,1);(0,0));$$
$$\mathscr{G}^{*2}((1,1,1);(0,0)) \text{ and } \mathscr{G}^{*2}((2,1,1);(0,0)).$$

The following theorem provides the remaining integral homotopy decompositions that we can obtain for these gauge groups. The theorem contributes to results in the last two rows of Theorem 1.9 and the last row in Theorem 1.10.

Theorem 5.12. *There are integral homotopy equivalences*

1. $\mathscr{G}^{*2}((1,1,1);(0,0)) \simeq \mathscr{G}^*((1,1,1);(0,0)) \times U(n)$;
2. $\mathscr{G}^{*2}((2,1,1);(0,0)) \simeq \mathscr{G}^*((1,1,1);(0,0)) \times U(n) \times U(n)$;
3. $\mathscr{G}^*((2,1,1);(0,0)) \simeq \mathscr{G}^*((1,1,1);(0,0)) \times U(n)$.

We analyse the structure of a type $(2,1,1)$ Real surface X'.

Proposition 5.13. *Let X'_γ be the 1-cells $\gamma_{r+1}, \gamma_{r+2}, \sigma(\gamma_{r+1}), \sigma(\gamma_{r+2})$ of a type $(2,1,1)$ Real surface X', then in the $\mathbb{Z}_2$-cofibration*

$$X'_\gamma \xrightarrow{\kappa''} X' \to X'/X'_\gamma \xrightarrow{\nu''} \Sigma(X'_\gamma)$$

the map ν'' is $\mathbb{Z}_2$-nullhomotopic.

Proof. We define a left inverse to κ''. In X' collapse the cell δ_1 and then collapse a copy of $(S^1 \vee S^1, \mathrm{sw})$ so that $X'/\sim$ is the wedge $((\Sigma_1/\sim) \vee (\Sigma_1/\sim), \mathrm{sw})$ where $(\Sigma_1/\sim)$ is a torus with δ_1 collapsed. We now project to $(S^1 \vee S^1, \mathrm{sw})$ as we did in the proof of Proposition 5.5, in fact, the left inverse is similar to the map j_3 from this proposition. □

In the following we show that the space X'/X'_γ is $\mathbb{Z}_2$-homotopy equivalent to a $(1,1,1)$ Real surface (X,σ). We first recall the $\mathbb{Z}_2$-decomposition of (X,σ). The 0-skeleton X^0 is given 3 zero-cells $*_i$ for $1 \le i \le 3$. The one cells are then

$$\delta_1, \delta, \sigma(\delta), \gamma_2, \sigma(\gamma_2)$$

where the fixed circle is represented by δ_1 and δ joins $*_2$ to $*_3$ and therefore $\delta\sigma(\delta)$ represents the copy of $(S^1, -\mathrm{id})$. The 1-cell γ_2 joins $*_1$ to $*_2$ and $\sigma(\gamma_2)$ joins $*_1$ to $*_3$. One of the 2 two-cells has attaching map

$$\delta_1\gamma_2\delta\sigma(\delta)\gamma_2^{-1}$$

and we define the other one equivariantly.

On the other hand, the space X'/X'_γ has an induced $\mathbb{Z}$-complex structure as follows. There is 1 zero-cell $*$, to which we attach the one-cells

$$\delta'_1, \delta' \text{ and } \sigma(\delta').$$

There are 2 two-cells, one of which is attached to the above 1-skeleton via

$$\delta_1\delta'\,\sigma(\delta')$$

and the other is glued equivariantly. However, the sub-complex given by $\gamma_2 \cup \sigma(\gamma_2)$ of (X, σ) is $\mathbb{Z}_2$-contractible and we see that (X, σ) is homotopy equivalent to $\mathbb{Z}_2$-complex structure of X'/X'_γ.

Proof of Theorem 5.12 (2) and (3). By Proposition 5.13, we obtain the following homotopy equivalences

$$\Sigma X' \simeq \Sigma X'_\gamma \vee \Sigma X'/X'_\gamma$$
$$\Sigma\overline{X}' \simeq \Sigma\overline{X}'_\gamma \vee \Sigma X'/X'_\gamma.$$

In the first case the factor $\Sigma X'_\gamma$ is the same as the space $(S^1 \vee S^1, \mathrm{sw})$. We see that collapsing the 0-skeleton of $\Sigma X'_\gamma$ provides $\bigvee_2(S^1 \vee S^1, \mathrm{sw})$ and hence this corresponds to the factor $\Sigma\overline{X}'_\gamma$ in the second equivalence. The result follows. □

Proof of Theorem 5.12 (1). We use the $\mathbb{Z}_2$-structure provided after Proposition 5.13. In this 2-pointed case, we identify the three 0-cells $*_1, *_2, *_3$ to produce $\overline{X}$. Let

$$X_\gamma = \gamma_2 \cup \sigma(\gamma_2)$$

and let $\overline{X}_\gamma$ be the image in the quotient $\overline{X}$. There is a left inverse to the inclusion

$$\overline{X}_\gamma \hookrightarrow \overline{X}$$

using a similar map to j_2 in the proof of Proposition 5.5. Therefore there is a homotopy equivalence

$$\Sigma\overline{X} \simeq \Sigma\overline{X}_\gamma \vee \Sigma(\overline{X}/\overline{X}_\gamma)$$

but by the comments after Proposition 5.13 the factor $\Sigma(\overline{X}/\overline{X}_\gamma)$ is $\mathbb{Z}_2$-homotopy equivalent to the suspension of a Real surface of type $(1,1,1)$. This finishes the proof. □

5.2.4 Non-integral Decompositions

By the previous sections, we have reduced our study of the pointed gauge groups to those over Real Surfaces of the following types

$$(0,0,1),\ (1,0,1) \text{ and } (1,1,1).$$

These spaces seem fundamental in some way and for the single-pointed case we do not obtain any further integral decompositions.

However, one may expect these spaces to become easier to examine when we choose to invert 2 since the involution has order 2 and the 2-torsion in $O(n)$ vanishes. This turns out to be the case and we will find that localising at a prime $p \neq 2$ will prove particularly fruitful.

Case: $(0,0,1)$

We fix notation, let $(S^2, -\mathrm{id})$ be a Real surface of type $(0,0,1)$. By Proposition 1.6, all of the pointed gauge groups over (S^2, he) are homotopy equivalent, so we assume that (P,σ) is of class 0. In this section, we aim to prove the following theorem which is a restatement of Theorem 1.11 (1).

Theorem 5.14. *Let $p \neq 2$ be prime and let n be odd, then there is a p-local homotopy equivalence*

$$\mathscr{G}^*(P,\tilde{\sigma}) \simeq_p \Omega(U(n)/O(n)) \times \Omega^2(U(n)/O(n)).$$

Let $u\colon B\mathscr{G}^*(P,\tilde{\sigma}) \to \mathrm{Map}^{*2}(D^2, BU(n))$ be the map that restricts to the upper hemisphere of $(S^2, -\mathrm{id})$ and forgets about equivariance considering the image as landing in the space $\mathrm{Map}^{*2}(D^2, BU(n))$. Let

$$r\colon B\mathscr{G}^*(P,\tilde{\sigma}) \to \mathrm{Map}^*_{\mathbb{Z}_2}((S^1 \vee S^1, \mathrm{sw}), BU(n))$$

be the map restricting to the 1-skeleton of $(S^2, -\mathrm{id})$. These maps fit into the following pullback

$$\begin{array}{ccc} B\mathscr{G}^*(P,\tilde{\sigma}) & \xrightarrow{\quad u \quad} & \mathrm{Map}^{*2}(D^2, BU(n)) \\ \Big\downarrow r & & \Big\downarrow r' \\ \mathrm{Map}^*_{\mathbb{Z}_2}((S^1 \vee S^1, \mathrm{sw}), BU(n)) & \xrightarrow{\ u'\ } & \mathrm{Map}^*(S^1 \vee S^1, BU(n)) \end{array} \tag{5.6}$$

where r' restricts to the 1-skeleton and u' forgets about equivariance.

Note that u' is homotopic to the map $\overline{\Delta}\colon U(n) \to U(n) \times U(n)$ where $\overline{\Delta}(\alpha) = (\alpha, \overline{\alpha})$, and the map r' is homotopic to the map $\Delta^{-1}\colon U(n) \to U(n) \times U(n)$ where $\Delta^{-1}(\alpha) = (\alpha, \alpha^{-1})$. Let Q be the strict pullback of $\overline{\Delta}$ and Δ^{-1} as in the following diagram

$$\begin{array}{ccc} Q & \xrightarrow{\pi_2} & U(n) \\ {\scriptstyle\pi_1}\big\downarrow & & \big\downarrow{\scriptstyle\Delta^{-1}} \\ U(n) & \xrightarrow{\overline{\Delta}} & U(n) \times U(n). \end{array}$$

By the comments at the beginning of Section 2.1, it is unclear if there is an equivalence

$$Q \overset{?}{\simeq} B\mathscr{G}^*(P, \tilde{\sigma})$$

and in fact this is not the case, but we will see that Q retracts off $B\mathscr{G}^*(P, \tilde{\sigma})$ after inverting the prime 2.

The map r' in diagram (5.6) is a fibration, and hence this diagram is a homotopy pullback. Therefore, there is an induced homotopy commuting diagram

$$\begin{array}{cccc} Q & & & \\ & \overset{\tilde{\pi}}{\searrow} & \xrightarrow{\pi_2} & \\ & B\mathscr{G}^*(P, \tilde{\sigma}) & \xrightarrow{u} & U(n) \\ {\scriptstyle\pi_1}\searrow & \big\downarrow{\scriptstyle r} & & \big\downarrow{\scriptstyle\Delta^{-1}} \\ & U(n) & \xrightarrow{\overline{\Delta}} & U(n) \times U(n) \end{array} \tag{5.7}$$

where we have replaced the pullback square (5.6) with a homotopy equivalent[1] square. Lemma 5.15 hints at a possible left homotopy inverse to $\tilde{\pi}$, namely the composition

$$B\mathscr{G}^*(P, \tilde{\sigma}) \xrightarrow{r} U(n) \xrightarrow{q} U(n)/O(n)$$

where q is the quotient map.

Lemma 5.15. *The pullback Q is homeomorphic to $U(n)/O(n)$.*

Proof. We first show that Q is the space $\{A \in U(n) \mid A \text{ is symmetric}\}$ and then we show that this is homeomorphic to $U(n)/O(n)$. The space Q consists of the elements (α, β) of $U(n) \times U(n)$ such that $\alpha = \beta$ and $\overline{\alpha} = \beta^{-1} = \overline{\beta}^t$, exactly the symmetric matrices of $U(n)$.

We define a map $f\colon U(n) \to Q$ by $f(A) = AA^t$. Now for $A \in U(n)$ and $W \in O(n)$ we have

$$(AW)(AW)^t = AWW^tA^t = AA^t$$

and so f induces a map $f'\colon U(n)/O(n) \to Q$.

[1] This homotopy equivalence is in the sense of Section 2.1.

The map f' is injective. Let $A, B \in U(n)$ and suppose $AA^t = BB^t$, then

$$I_n = B^{-1}AA^tB^{t^{-1}} = (B^{-1}A)(B^{-1}A)^t$$

for $I_n \in U(n)$ the identity matrix. Therefore $B^{-1}A \in O(n)$ and so $AO(n) \equiv BO(n)$.

The map f' is surjective. Let A be in Q then due to the Autonne–Takagi factorisation (see [You61]), there is a unitary matrix P such that $A = PDP^t$ where D is a diagonal matrix with real entries. Let $\sqrt{D}$ be a diagonal (hence symmetric) matrix in $U(n)$ such that $\sqrt{D}^2 = D$, we note that any other matrix $\sqrt{D}'$ of such a form has the property that $\sqrt{D}' = \sqrt{D}I_\pm$ where $I_\pm$ is a matrix in $O(n)$ of the form

$$\begin{pmatrix} \pm1 & 0 & \cdots & 0 \\ 0 & \pm1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \pm1 \end{pmatrix}.$$

We have $A = P\sqrt{D}\sqrt{D}P^t = P\sqrt{D}(P\sqrt{D})^t$ and therefore $f'((P\sqrt{D})O(n)) = A$.

The map f' is therefore a continuous bijection, and since $U(n)/O(n)$ is compact and Q is Hausdorff it is a homeomorphism. □

The above diagram and Lemma 5.15 give the following composition

$$\varphi\colon U(n)/O(n) \xrightarrow{f'} Q \xrightarrow{\tilde{\pi}} B\mathscr{G}^*(P,\tilde{\sigma}) \xrightarrow{r} U(n) \xrightarrow{q} U(n)/O(n) \tag{5.8}$$

for q the quotient map. From the properties of π_1 we see that φ is homotopic to a map that sends an element $AO(n)$ to $AA^tO(n)$. For odd n, we showed in Section 2.3 that the related map

$$\begin{aligned} SU(n)/SO(n) &\to SU(n)/SO(n) \\ ASO(n) &\mapsto AA^tSO(n) \end{aligned} \tag{5.9}$$

is a homotopy equivalence when localised at a prime $p \neq 2$. Our aim is to show that the same is true for φ.

Lemma 5.16. *For a prime $p \neq 2$, there is an p-local homotopy equivalence*

$$U(n)/O(n) \simeq_p U(n)/SO(n).$$

Proof. Consider the following pullback diagram where the downward arrows represent taking universal covers

$$\begin{array}{ccccc} U(n)/SO(n) & \longrightarrow & BSO(n) & \longrightarrow & BU(n)\ . \\ \downarrow & & \downarrow & & \| \\ U(n)/O(n) & \longrightarrow & BO(n) & \longrightarrow & BU(n) \\ \downarrow & & \downarrow & & \\ K(\mathbb{Z}_2,1) & = & K(\mathbb{Z}_2,1) & & \end{array}$$

The result immediately follows. □

We now show that $U(n)/SO(n)$ further decompositions into the product

$$SU(n)/SO(n) \times S^1.$$

The map $BSO(n) \to BU(n)$ factors through $BSU(n)$. Hence, we obtain the following commutative diagram which defines the maps i and j

$$\begin{array}{ccccc} & & U(n) & = & U(n) \\ & & \downarrow & & \downarrow f \\ SU(n)/SO(n) & \xrightarrow{i} & U(n)/SO(n) & \xrightarrow{j} & S^1 \\ \| & & \downarrow & & \downarrow \\ SU(n)/SO(n) & \longrightarrow & BSO(n) & \longrightarrow & BSU(n) \\ & & \downarrow & & \downarrow \\ & & BU(n) & = & BU(n). \end{array} \tag{5.10}$$

It is not too much more work to show the following lemma.

Lemma 5.17. *There is a homotopy equivalence*

$$\eta\colon SU(n)/SO(n) \times S^1 \xrightarrow{\simeq} U(n)/SO(n).$$

Proof. There is a right inverse l to the map f and there is an action of $U(n)$ on $U(n)/SO(n)$, hence the composition

$$\eta\colon S^1 \times SU(n)/SO(n) \xrightarrow{l\times i} U(n) \times U(n)/SO(n) \xrightarrow{\text{‘action’}} U(n)/SO(n)$$

is the required homotopy equivalence. □

Let φ be the composition in equation (5.8) and then define

$$s\colon U(n)/SO(n) \to U(n)/SO(n)$$

to be the composition

$$U(n)/SO(n) \xrightarrow{\simeq} U(n)/O(n) \xrightarrow{\varphi} U(n)/O(n) \xrightarrow{\simeq} U(n)/SO(n)$$

Our aim is to show that s restricts to the factors $SU(n)/SO(n)$ and S^1 in a nice enough way.

Lemma 5.18. *There exist maps $s'': SU(n)/SO(n) \to SU(n)/SO(n)$ and $s': S^1 \to S^1$ such that the following is a homotopy commuting square*

$$\begin{array}{ccc} SU(n)/SO(n) \times S^1 & \xrightarrow{s''\times s'} & SU(n)/SO(n) \times S^1 \\ \downarrow \eta & & \downarrow \eta \\ U(n)/SO(n) & \xrightarrow{\quad s \quad} & U(n)/SO(n). \end{array}$$

Furthermore, s'' is homotopic to the map

$$ASO(n) \mapsto AA^tSO(n)$$

and s' is homotopic to the map $x \mapsto x^2$.

Proof. Let $\tilde{s}: SU(n)/SO(n) \times S^1 \to SU(n)/SO(n) \times S^1$ be the composition

$$SU(n)/SO(n) \times S^1 \xrightarrow{\eta} U(n)/SO(n) \xrightarrow{s} U(n)/SO(n) \xrightarrow{\eta^{-1}} SU(n)/SO(n) \times S^1$$

for a homotopy inverse η^{-1} of η. Let $\iota: SU(n)/SO(n) \to SU(n)/SO(n) \times S^1$ and $\kappa: S^1 \to SU(n)/SO(n) \times S^1$ be the inclusions. We note that ι is homotopic to

$$SU(n)/SO(n) \xrightarrow{i} U(n)/SO(n) \xrightarrow{\eta^{-1}} SU(n)/SO(n) \times S^1$$

where i is as in diagram (5.10). By the way the homotopy equivalences are defined in Lemmas 5.16 and 5.17, we see that the composition si is homotopic to

$$BSO(n) \mapsto BB^tSO(n) \text{ for } B \in SU(n)$$

and hence the image of this map lands in the image of i. We deduce that $\tilde{s}\iota$ has image in $SU(n)/SO(n)$ and we define

$$s'' = \tilde{s}\iota.$$

Similarly $\tilde{s}\kappa$ has image in S^1 and we define $s' = \tilde{s}\kappa$. We see that s'' is homotopic to a map defined via $ASO(n) \mapsto AA^tSO(n)$ and that s' is homotopic to the map $x \mapsto x^2$. □

We immediately obtain the following homotopy commuting diagram where the rows are homotopy fibrations

$$\begin{array}{ccccc} SU(n)/SO(n) & \xrightarrow{i} & U(n)/SO(n) & \longrightarrow & S^1 \\ \downarrow{\scriptstyle s''} & & \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s'} \\ SU(n)/SO(n) & \xrightarrow{i} & U(n)/SO(n) & \longrightarrow & S^1 \end{array} \tag{5.11}$$

By Lemma 5.18, the map s'' is homotopic to the map in equation (5.9) and hence it is a p-local equivalence when n is odd and $p \neq 2$ is a prime. We note that s' is also a p-local equivalence. Finally, the spaces in (5.11) are connected, hence s is also a p-local equivalence and we deduce the following proposition.

Proposition 5.19. *With notation as in equation (5.8), let F be the homotopy fibre of $qr\colon B\mathscr{G}^*(P,\tilde{\sigma}) \to U(n)/O(n)$. Then for n odd and for any prime $p \neq 2$, there is a p-local homotopy equivalence*

$$\mathscr{G}^*(P,\tilde{\sigma}) \simeq_p \Omega(U(n)/O(n)) \times \Omega F. \qquad \square$$

Therefore, to prove Theorem 5.14 it only remains to identify the fibre F.

Proposition 5.20. *For any prime $p \neq 2$, there is a p-local homotopy equivalence*

$$F \simeq_p \Omega(U(n)/O(n)).$$

Proof. Since qr is defined as a composition, there is a homotopy commutative diagram

$$\begin{array}{ccccc} F & \longrightarrow & B\mathscr{G}^*(P,\tilde{\sigma}) & \xrightarrow{qr} & U(n)/O(n) \\ \downarrow & & \downarrow{\scriptstyle r} & & \| \\ O(n) & \longrightarrow & U(n) & \xrightarrow{q} & U(n)/O(n) \end{array}$$

where the left square is a homotopy pullback square by Proposition 2.4. The map r is a fibration since it is induced by $i\colon (S^1,-\mathrm{id}) \hookrightarrow (S^2,-\mathrm{id})$; the inclusion of the meridian copy of $(S^1,-\mathrm{id})$ into $(S^2,-\mathrm{id})$. By Corollary 2.8, the space F is homotopy equivalent to the relative mapping space

$$\mathrm{Map}^*_{\mathbb{Z}_2}\Big(\big((S^2,-\mathrm{id}),(S^1,-\mathrm{id})\big),\big(BU(n),BO(n)\big);0\Big).$$

We will associate another pullback square with this description of F. There is a map $T\colon F \to \mathrm{Map}^*_{\mathbb{Z}_2}((S^2,-\mathrm{id}),(BU(n),\mathrm{id});0)$ given by

$$T(f)(x) = \begin{cases} f(x) & \text{for } x \text{ in the upper hemisphere including equator;} \\ f(-\mathrm{id}(x)) & \text{for } x \text{ in the lower hemisphere excluding equator.} \end{cases}$$

Let $i\colon (S^1, -\mathrm{id}) \hookrightarrow (S^2, -\mathrm{id})$ be defined as above, then i induces the following pullback diagram

$$\begin{array}{ccc} F & \xrightarrow{T} & \mathrm{Map}^*_{\mathbb{Z}_2}((S^2, -\mathrm{id}), (BU(n), \mathrm{id}); 0) \\ \big\downarrow{\scriptstyle i^*} & & \big\downarrow{\scriptstyle i^*} \\ O(n) & \hookrightarrow & U(n). \end{array}$$

There is a homeomorphism

$$\mathrm{Map}^*_{\mathbb{Z}_2}((S^2, -\mathrm{id}), (BU(n), \mathrm{id}); 0) \cong \mathrm{Map}^*(\mathbb{R}P^2, BU(n); 0)$$

but for a prime $p \neq 2$ the space $\mathbb{R}P^2$ is p-locally contractible. Therefore, p-locally, we have identified the space F as the fibre of the inclusion $O(n) \to U(n)$ and the result follows. □

Proof of Theorem 5.14. Use Propositions 5.19 and 5.20. □

Case: $(1, 0, 1)$

Let (T, τ) be a Real surface of type $(1, 0, 1)$ and since all pointed gauge groups over (T, τ) are homotopy equivalent, we restrict to the case where $(P, \tilde{\sigma})$ is a bundle of class 0 over (T, τ). We will use similar techniques to the even genus case to obtain the following theorem, which is a restatement of Theorem 1.11 (2).

Theorem 5.21. *For any prime $p \neq 2$ and n odd, there is a p-local homotopy equivalence*

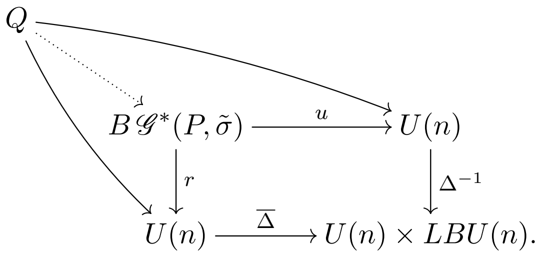
$$\mathscr{G}^*(P, \tilde{\sigma}) \simeq_p \Omega(U(n)/O(n)) \times \Omega^2(U(n)/O(n)) \times \Omega U(n).$$

Proof. Let $u\colon B\mathscr{G}^*(P, \tilde{\sigma}) \to \mathrm{Map}^*(C, BU(n))$ be the map that forgets about equivariance and restricts to the upper half of (T, τ) which is homeomorphic to a cylinder C. Let i be the inclusion of the boundary circles of C, then i induces a pullback

$$\begin{array}{ccc} B\mathscr{G}^*(P, \tilde{\sigma}) & \xrightarrow{u} & \mathrm{Map}^*(C, BU(n)) \\ \big\downarrow{\scriptstyle r} & & \big\downarrow{\scriptstyle r'} \\ \mathrm{Map}^*_{\mathbb{Z}_2}((S^1 \vee S^1, \mathrm{sw}), BU(n)) & \xrightarrow{u'} & \mathrm{Map}^*(S^1 \amalg S^1, BU(n)) \end{array} \tag{5.12}$$

where $r' = i^*$ and r is the restriction to the one-skeleton of (X, σ).

In a similar fashion to the way we obtained diagram (5.7), we replace (5.12) with a homotopy equivalent square and obtain the diagram

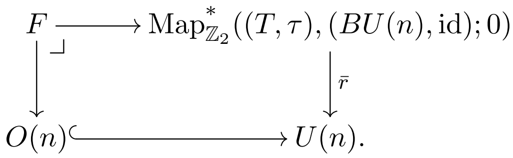


Here $LBU(n)$ denotes the free loop space of $U(n)$ and Q is the strict pullback of the diagram

$$U(n) \xrightarrow{\overline{\Delta}} U(n) \times LBU(n) \xleftarrow{\Delta^{-1}} U(n)$$

and hence Q is again the symmetric matrices in $U(n)$. We deduce that $U(n)/O(n)$ also p-locally retracts off $B\mathscr{G}^*(P,\tilde{\sigma})$.

It is clear that, as in the even case, there is similar description for the fibre F of the map $B\mathscr{G}^*(P,\tilde{\sigma}) \to U(n)/O(n)$. The space F fits into the following pullback diagram



We note that if we let K be a Klein bottle then there is an homeomorphism

$$\mathrm{Map}^*_{\mathbb{Z}_2}((T,\tau),(BU(n),\mathrm{id});0) \cong \mathrm{Map}^*(K,BU(n);0)$$

The map $\bar{r}$ is induced by the inclusion $S^1 \hookrightarrow K$ which on fundamental groups induces the quotient

$$\begin{aligned} \mathbb{Z} &\to \mathbb{Z} \times \mathbb{Z}_2 \\ a &\mapsto (0,[a]_2) \end{aligned}$$

onto the right factor. We see that for a prime $p \neq 2$, the map $\bar{r}$ is p-locally nullhomotopic and we obtain

$$\Omega F \simeq_p \Omega^2(U(n)/O(n)) \times \Omega\,\mathrm{Map}^*(K,BU(n);0).$$

Now for $p \neq 2$ prime we have a p-local homotopy equivalence $K \simeq_p S^1$ because K is a $K(\mathbb{Z} \times \mathbb{Z}_2, 1)$. Therefore, the space $\Omega\,\mathrm{Map}^*(K,BU(n);0)$ is homotopy equivalent to $\Omega U(n)$ when localised away from 2 and Theorem 5.21 follows. □

Case: $(1,1,1)$

Let (X, σ) be a Real surface of type $(1, 1, 1)$. For convenience we choose $(P, \tilde{\sigma})$ to be a bundle of class $(0, 0)$ over (X, σ). We use a very similar method to the previous sections to prove the following theorem, which is a restatement of Theorem 1.11 (3).

Theorem 5.22. *For any prime $p \neq 2$, there is a p-local homotopy equivalence*

$$\mathscr{G}^*(P, \tilde{\sigma}) \simeq_p \mathscr{G}^*((S^2, -\mathrm{id}); 0) \times \Omega O(n).$$

Proof. We first recall the $\mathbb{Z}_2$-decomposition of (X, σ). The 0-skeleton X^0 is given 3 zero-cells $*_i$ for $1 \leq i \leq 3$. The one cells are then

$$\delta_1, \delta, \sigma(\delta), \gamma_2, \sigma(\gamma_2)$$

where the fixed circle is represented by δ_1 and δ joins $*_2$ to $*_3$ and therefore $\delta\sigma(\delta)$ represents the copy of $(S^1, -\mathrm{id})$. The 1-cell γ_2 joins $*_1$ to $*_2$ and $\sigma(\gamma_2)$ joins $*_1$ to $*_3$. One of the 2 two-cells has attaching map

$$\delta_1 \gamma_2 \delta \sigma(\delta) \gamma_2^{-1}$$

and we define the other one equivariantly.

Since the subspace $\gamma_2 \cup \sigma(\gamma_2)$ is $\mathbb{Z}_2$-contractible, we amend the above decomposition to have only 3 one-cells $\delta_1, \delta, \sigma(\delta)$ and amend the attaching map to

$$\delta_1 \delta\, \sigma(\delta).$$

We obtain a pullback similar to that of the previous section

$$\begin{array}{ccc} B\mathscr{G}^*(P, \tilde{\sigma}) & \xrightarrow{\quad u \quad} & \mathrm{Map}^{*3}(D^2, BU(n)) \\ \big\downarrow r & & \big\downarrow r' \\ \mathrm{Map}^*_{\mathbb{Z}_2}((S^1, \mathrm{id}) \vee (S^1 \vee S^1, \mathrm{sw}), BU(n); w_1) & \xrightarrow{\ u'\ } & \mathrm{Map}^*(S^1 \vee S^1 \vee S^1, BU(n)) \end{array}$$

where r is the restriction to the one-skeleton of (X, σ) and u restricts to one of the two-cells and forgets about equivariance.

In a similar fashion to the way we obtained diagram (5.7), we obtain the diagram

$$\begin{array}{ccccc} O(n) & & & & \\ & \overset{f_3}{\searrow} & & \overset{f_2}{\searrow} & \\ & & B\mathscr{G}^*(P, \tilde{\sigma}) & \xrightarrow{\quad u \quad} & U(n) \times U(n) \\ & \underset{f_1}{\searrow} & \big\downarrow r & & \big\downarrow r' \\ & & SO(n) \times U(n) & \xrightarrow{\ u'\ } & U(n) \times U(n) \times U(n) \end{array} \tag{5.13}$$

where f_1, f_2 and f_3 are to be defined momentarily.

The map $r' \colon U(n) \times U(n) \to U(n) \times U(n) \times U(n)$ is the map

$$r'(A, B) = (B^{-1}A^{-1}, A, B)$$

and the map $u' \colon SO(n) \times U(n) \to U(n) \times U(n) \times U(n)$ is the map

$$u'(C, D) = (C, D, \overline{D}).$$

We can therefore define maps $f_1 \colon O(n) \to SO(n) \times U(n)$ and $f_2 \colon O(n) \to U(n) \times U(n)$ by

$$f_1(X) = (X^{-2}, X) \text{ and } f_2(Y) = (Y, Y)$$

such that $u'f_1 = r'f_2$. Since 5.13 is a homotopy pullback, there exists a map

$$f_3 \colon O(n) \to B\mathscr{G}^*(P, \tilde{\sigma})$$

such that the composition

$$\chi \colon O(n) \xrightarrow{f_3} B\mathscr{G}^*(P, \tilde{\sigma}) \xrightarrow{r} O(n) \times U(n) \xrightarrow{p_1} O(n)$$

sends an element X to X^{-2}. Then observe that χ has image lying in $SO(n)$ and therefore when χ is restricted to $SO(n)$, it is the inverse of the H-space squaring map. We conclude that restriction of χ to $SO(n)$ is a p-local homotopy equivalence for $p \neq 2$ and therefore $SO(n)$ retracts off $B\mathscr{G}(P, \tilde{\sigma})$.

The map $p_1 r$ is just the restriction to the fixed points of the involution. Hence the fibre of this map is the space $B\mathscr{G}^*((0, 0, 1); 0)$ which we have already studied. We finish by noting that $\Omega SO(n)$ and $\Omega O(n)$ are homeomorphic. □

5.3 Unpointed Gauge Groups

In the last section, we showed that certain trivialities of the attaching map of the top cells of X led to homotopy decompositions in the pointed case. We will see that these decompositions somewhat extend to the unpointed case. however the unpointed case needs to be treated with care due the non-equivalence of the components.

5.3.1 Integral Decompositions

Let (X, σ) be a Real surface of type (g, r, a). In the following proposition g' will denote the number of α_i and β_i cells in the description of (X, σ) in Section 4.4. Explicitly

$$g' = \begin{cases} (g-r+1) & \text{when } a = 0; \\ (g-r) & \text{when } a = 1 \text{ and } g-r \text{ even}; \\ (g-r-1) & \text{when } a = 1 \text{ and } g-r \text{ odd}. \end{cases}$$

We now present Proposition 5.23 which is a restatement of Theorem 1.12 (1).

Proposition 5.23. *There are homotopy equivalences*

$$\mathscr{G}((g,r,a);(c,w_1,\ldots,w_r)) \simeq \mathscr{G}((g-g',r,a);(c,w_1,\ldots,w_r)) \times \prod_{g'} \Omega U(n).$$

Proof. In essence, we follow the proof of Theorem 3.9. For convenience, we write $(c,\overline{w}) := (c,w_1,\ldots,w_r)$. Let $X_{\alpha\beta} = \bigvee_{g'}(S^1 \vee S^1, \mathrm{sw})$ be sub-complex of X represented by $\alpha_i, \sigma(\alpha_i), \beta_i, \sigma(\beta_i)$. Recall the $\mathbb{Z}_2$-cofibration sequence of Proposition 5.2

$$X_{\alpha\beta} \hookrightarrow X \xrightarrow{q} X' \xrightarrow{\mu} \Sigma(X_{\alpha\beta}).$$

Then the map q induces the following diagram

$$\begin{array}{ccccccc} \Omega B & \xrightarrow{\partial_{(c,\overline{w})}} & \mathrm{Map}^*_{\mathbb{Z}_2}(X', BU(n);(c,\overline{w})) & \longrightarrow & \mathrm{Map}_{\mathbb{Z}_2}(X', BU(n);(c,\overline{w})) & \xrightarrow{ev} & B \\ \| & & \downarrow q^* & & \downarrow q^* & & \| \\ \Omega B & \xrightarrow{\varphi_{(c,\overline{w})}} & \mathrm{Map}^*_{\mathbb{Z}_2}(X, BU(n);(c,\overline{w})) & \longrightarrow & \mathrm{Map}_{\mathbb{Z}_2}(X, BU(n);(c,\overline{w})) & \xrightarrow{ev} & B \end{array}$$

where

$$B = \begin{cases} BU(n) \text{ if } r = 0; \\ BO(n) \text{ otherwise.} \end{cases}$$

The fact that $\varphi_{(c,\overline{w})} = q^*\partial_{(c,\overline{w})}$ obtains the following diagram which defines the maps h and h'

$$\begin{array}{ccccc}
 & & \mathrm{Map}^*(\Sigma(X), BU(n); (c,\overline{w})) & = & \mathrm{Map}^*_{\mathbb{Z}_2}(\Sigma X, BU(n); (c,\overline{w})) \\
 & & \downarrow & & \downarrow{\scriptstyle (\Sigma i)^*} \\
\mathscr{G}(g-g') & \xrightarrow{h'} & \mathscr{G}((g,r,a);(c,\overline{w})) & \xrightarrow{h} & \mathrm{Map}^*_{\mathbb{Z}_2}(\Sigma(X_{\alpha\beta}), BU(n)) \\
\| & & \downarrow & & \downarrow{\scriptstyle \mu^*} \\
\mathscr{G}(g-g') & \longrightarrow & \Omega B & \xrightarrow{\partial_{(c,\overline{w})}} & \mathrm{Map}^*_{\mathbb{Z}_2}(X', BU(n); (c,\overline{w})) \\
 & & \downarrow{\scriptstyle \varphi_{(c,\overline{w})}} & & \downarrow{\scriptstyle q^*} \\
 & & \mathrm{Map}^*_{\mathbb{Z}_2}(X, BU(n); (c,\overline{w})) & = & \mathrm{Map}^*_{\mathbb{Z}_2}(X, BU(n); (c,\overline{w}))
\end{array}$$

and in which $\mathscr{G}(g-g') := \mathscr{G}((g-g',r,a);(c,\overline{w}))$. By Proposition 5.2 the map μ^* is trivial. Hence there is a section to the map $(\Sigma i)^*$, so there is also a section to h and the result follows. □

The quotient map q in Proposition 5.23 induced an isomorphism on π_0 between

$$\mathrm{Map}^*_{\mathbb{Z}_2}(X, BU(n); (c,\overline{w})) \text{ and } \mathrm{Map}^*_{\mathbb{Z}_2}(X', BU(n); (c,\overline{w})).$$

However, for a fixed cell δ_i of (X,σ), the quotient map $\tilde{q}\colon X \to X/\delta_i$ automatically induces the map

$$\mathrm{Map}_{\mathbb{Z}_2}(X/\delta_i, BU(n)) \xrightarrow{q^*} \mathrm{Map}_{\mathbb{Z}_2}(X, BU(n); 0)$$

hence the requirement for $w_i = 0$ in Theorem 1.12 (3). Whilst there is an equivalence

$$\mathrm{Map}^*_{\mathbb{Z}_2}(X, BU(n); (c,0)) \simeq \mathrm{Map}^*_{\mathbb{Z}_2}(X, BU(n); (c,1))$$

there is not necessarily an equivalence in the unpointed case in general. Hence, there is not enough information to guarantee the commutativity of the diagram needed to induce a homotopy decomposition.

Omitting such non-trivialities allows further splittings; let X_1 be a subset of the 1-cells of X such that

1. if there is a fixed cell $\delta_i \subset X_1$ then $w_i = 0$ and;
2. for appropriate components the induced map

$$g^*\colon\ \mathrm{Map}^*_{\mathbb{Z}_2}(\Sigma X_1, BU(n); (\overline{w})) \to \mathrm{Map}^*_{\mathbb{Z}_2}(X/X_1, BU(n); (c,\overline{w}))$$

   is $\mathbb{Z}_2$-nullhomotopic.

Under these assumptions, it is clear that the methods in the previous proposition would yield another homotopy decompositions.

Proof of Theorem 1.12 (2) and (3). The above conditions apply to the 1-cells considered in Propositions 5.5, 5.10 and 5.13 for bundles of arbitrary type.

Additionally the conditions are satisfied by the 1-cells considered in Propositions 5.6 and 5.11 for bundles of type $(c, w_1, 0, \dots, 0)$. When n is odd we can take advantage of Proposition 1.8 to obtain the table in Theorem 1.12 (3). We have now finished the proof of Theorem 1.12. □

5.3.2 Analysing the Boundary Map

Let $(P, \tilde{\sigma})$ be a Real bundle of class $(c, w_1, \dots, w_r)$ over a Real surface (X, σ) of type (g, r, a). Let

$$B = \begin{cases} BO(n) & \text{if } r > 0 \\ BU(n) & \text{otherwise} \end{cases}$$

and reconsider the evaluation fibration

$$\Omega B \xrightarrow{\partial_P} \mathrm{Map}^*_{\mathbb{Z}_2}(X, BU(n); P) \to \mathrm{Map}_{\mathbb{Z}_2}(X, BU(n); P) \to B \tag{5.14}$$

The analysis of the map ∂_P is absolutely crucial, if it is (q-locally) trivial then we will be able to reduce the unpointed case to the pointed one. Indeed, we recall Theorem 3.13 in which there were such trivialities for the non-equivariant case.

Theorem 3.13. *Fix a prime p, then*

1. *if $q \neq p$ is a prime then there is a q-local homotopy decomposition*

$$\mathscr{G}(U(p); d) \simeq U(p) \times \Omega^2 U(p);$$

2. *if $p \mid d$, there is a p-local homotopy decomposition*

$$\mathscr{G}(U(p); d) \simeq \prod_{i=0}^{p-1} S^{2i+1} \times \prod_{j=1}^{p-1} \Omega^2 S^{2j+1};$$

3. *if $p \nmid d$ there is a p-local homotopy decomposition*

$$\mathscr{G}(U(p); d) \simeq \prod_{i=0}^{p-2} S^{2i+1} \times \prod_{j=2}^{p-1} \Omega^2 S^{2j+1} \times (S^1 \times \Omega^2 S^{2p+1}).$$

For parts (1) and (2), the local triviality of the boundary map

$$U(n) \xrightarrow{\partial_d} \mathrm{Map}^*(S^2, BU(n); d) \tag{5.15}$$

provides the needed results. For part (3), it was a more delicate matter of calculating the homotopy fibre of the ∂_d.

We will find in the coming section that Theorem 3.13 provides some similar results for gauge groups of bundles of class $(2d, 0, 0, \ldots, 0)$. Note that the following proposition immediately implies Theorem 1.13 (1) for the cases $q \neq p$ and $q = p \mid d$.

Proposition 5.24. *Fix $d \in \mathbb{Z}$ and let ∂_d be the boundary map in (5.15). Let*

$$\partial_P \colon \Omega B \to B\mathscr{G}^*((g, r, a); (2d, 0, \ldots, 0))$$

be the boundary map of the evaluation fibration as in (5.14). If ∂_d is (q-locally) trivial then

1. *if $r > 0$ then ∂_P is (q-locally) trivial;*
2. *if $r = 0$ then the composition*

$$O(p) \hookrightarrow U(p) \xrightarrow{\partial_P} B\mathscr{G}^*((g, 0, a); (2d))$$

is (q-locally) trivial.

Proof. The key will be to compare both maps to another evaluation boundary map involving the $\mathbb{Z}_2$-space $Y = (S^2 \vee S^2, \mathrm{sw})$. We note that components of $\mathrm{Map}^*_{\mathbb{Z}_2}(Y, BU(n))$ are classified by even integers.

Let $S^2 \xrightarrow{i_1} S^2 \vee S^2 = Y$ be the inclusion onto the left factor, and note that this is not a $\mathbb{Z}_2$-map. The following diagram commutes

$$\begin{array}{ccccccc}
O(n) & \xrightarrow{\overline{\partial}_{2l}} & \mathrm{Map}^*_{\mathbb{Z}_2}(Y, BU(n); 2l) & \longrightarrow & \mathrm{Map}_{\mathbb{Z}_2}(Y, BU(n); 2d) & \longrightarrow & BO(n) \\
\downarrow & & \downarrow{\scriptstyle i_1^*} & & \downarrow & & \downarrow \\
U(n) & \xrightarrow{\partial_d} & \mathrm{Map}^*(S^2, BU(n); d) & \longrightarrow & \mathrm{Map}(S^2, BU(n); d) & \longrightarrow & BU(n).
\end{array} \tag{5.16}$$

Now, there is an inverse to i_1^* which sends a map f in $\mathrm{Map}^*(S^2, BU(n); d)$ to the composition

$$S^2 \vee S^2 \xrightarrow{f \vee f} BU(n) \vee BU(n) \xrightarrow{\mathrm{id} \vee \sigma_{BU(n)}} BU(n) \vee BU(n) \xrightarrow{\mathrm{fold}} BU(n)$$

which is $\mathbb{Z}_2$-equivariant because the involution on $S^2 \vee S^2$ swaps the factors. Note that the map induced on the unpointed mapping spaces does not have an inverse because the basepoint of Y must land in $BO(n)$. We conclude that if ∂_d is q-locally trivial then so is $\overline{\partial}_{2d}$.

Let $q\colon X \to Y$ be the map that collapses the 1-skeleton of the Real surface (X, σ). We obtain the following commutative diagram

$$\begin{CD}
O(n) @>{\overline{\partial}_{2d}}>> \mathrm{Map}^*_{\mathbb{Z}_2}(Y, BU(n); 2d) @>>> \mathrm{Map}_{\mathbb{Z}_2}(Y, BU(n); 2d) @>>> BO(n) \\
@V{f}VV @V{q^*}VV @VVV @VVV \\
\Omega B @>{\partial_P}>> \mathrm{Map}^*_{\mathbb{Z}_2}(X, BU(n); P) @>>> \mathrm{Map}_{\mathbb{Z}_2}(X, BU(n); P) @>>> B
\end{CD}$$

The map f is an equivalence if $r > 0$ and is the inclusion $O(n) \hookrightarrow U(n)$ otherwise. Since $\overline{\partial}_{2d}$ is (q-locally) trivial, the result follows. $\square$

The above proposition allows us to transfer Theorem 3.13 (1) and (2) to the equivariant setting and we now apply the same treatment for Theorem 3.13 (3).

Proposition 5.25. *Let $p \nmid d$ and let $(P, \tilde{\sigma})$ be a Real principal $U(p)$-bundle of class $(2d, 0, \ldots, 0)$ over a Real surface of type (g, r, a). Let*

$$\partial_P \colon \Omega B \to B\mathscr{G}^*((g, r, a); (2d, 0, \ldots, 0))$$

be the boundary map of the evaluation fibration. Then

1. *if $r > 0$ then ∂_P is p-locally trivial;*
2. *if $r = 0$ then the composition*

$$O(p) \hookrightarrow U(p) \xrightarrow{\partial_P} B\mathscr{G}^*((g, 0, a); (2d))$$

is p-locally trivial.

Proof. We assume that that $p \nmid d$ is a prime and that all spaces and maps are localised at p. Let $Y = (S^2 \vee S^2, \mathrm{sw})$ be as above, then there is a homotopy commuting diagram

$$\begin{array}{ccccccc}
O(p) & & \xrightarrow{\quad\overline{\partial}_{2d}\quad} & & & & \mathrm{Map}^*_{\mathbb{Z}_2}(Y, BU(n); 2d) \\
\big\downarrow{\scriptstyle i} & & & & & & \big\downarrow{\scriptstyle \simeq} \\
U(p) & & \xrightarrow{\quad\partial_d\quad} & & & & \Omega U(p)_0 \\
\big\| & & & & & & \big\uparrow{\scriptstyle d} \\
U(p) & & \xrightarrow{\quad\partial_1\quad} & & & & \Omega U(p)_0 \\
\big\downarrow{\scriptstyle e} & & & & & & \big\downarrow{\scriptstyle (\Omega e)_0} \\
\prod_{i=0}^{p-1} S^{2i+1} & \xrightarrow{\mathrm{proj}} & S^{2p-1} & \xrightarrow{\alpha} & \Omega S^3 & \xrightarrow{\mathrm{incl}} & \prod_{j=1}^{p-1} \Omega S^{2j+1}
\end{array} \tag{5.17}$$

where the bottom square is from diagram (5.16) and the top two squares are from the proof of Theorem 3.13 (3).

We recall the map d-th power map $d\colon \Omega U(p)_0 \to \Omega U(p)_0$ is a homotopy equivalence because $p \nmid d$. Furthermore, the maps e and $(\Omega e)_0$ are homotopy equivalences provided in [Ser53]. Now for $p \neq 2$ prime, there is a p-local homotopy equivalence

$$SO(p) \simeq_p \prod_{i=1}^{\frac{p-1}{2}} S^{4i-1}$$

and furthermore the inclusion $O(p) \hookrightarrow U(p)$ is in fact the inclusion of these factors into $\prod_{i=0}^{p-1} S^{2i+1}$. We conclude that the composition

$$\chi\colon O(p) \hookrightarrow U(p) \to \prod_{i=0}^{p-1} S^{2i+1} \xrightarrow{\text{proj}} S^{2p-1} \tag{5.18}$$

is nullhomotopic and therefore so is $\overline{\partial}_{2d}$.

For $p = 2$, the space $O(2)$ is homeomorphic to $S^1 \amalg S^1$. Since χ in (5.18) has target space S^3, we conclude that χ and hence $\overline{\partial}_{2d}$ are nullhomotopic in this case too. The result then follows in a similar way to the last paragraph of Proposition 5.24. □

Proof of Proposition 1.13 (1a) *and* (2a). Corollary 3.17 and Proposition 5.24 immediately obtain part (1a). The analysis of the boundary map in Theorem 3.13 (2) and Proposition 5.24 obtains part (2a) for $p \mid d$. Finally, Proposition 5.25 provides the remaining cases when $p \nmid d$ in part (2a). □

5.3.3 Case: $(0, 0, 1)$

We restrict to analysing gauge groups above Real surfaces of type $(0, 0, 1)$. Fix an even integer c then we wish to analyse the boundary map ∂_c of the evaluation fibration.

For a $\mathbb{Z}_2$-space A, let $\overline{\Delta}\colon A \to A \times A$ be the composition

$$A \xrightarrow{\Delta} A \times A \xrightarrow{\text{id} \times \sigma_A} A \times A. \tag{5.19}$$

Let $u\colon B\mathscr{G}^*((0, 0, 1); c) \to U(n)$ be the map that restricts to the upper hemisphere of $(S^2, -\text{id})$ and forgets about equivariance except at $*$ and $\sigma(*)$. These are the same maps as in equation (5.7) and they fit into the following commutative diagram

$$\begin{array}{ccccccc}
U(n) & \xrightarrow{\partial_c} & B\mathscr{G}^*((0,0,1);c) & \longrightarrow & B\mathscr{G}((0,0,1);c) & \longrightarrow & BU(n) \\
\downarrow{\scriptstyle \overline{\Delta}} & & \downarrow{\scriptstyle u} & & \downarrow & & \downarrow{\scriptstyle \overline{\Delta}} \\
U(n) \times U(n) & \xrightarrow{\zeta} & U(n) & \longrightarrow & \text{Map}(D^2, BU(n)) & \xrightarrow{\text{ev}_2} & BU(n) \times BU(n)
\end{array}$$

where ev_2 evaluates at two antipodal points on the boundary of D^2 and ζ is defined via this diagram.

Since D^2 is contractible, the map ev_2 is homotopic to the diagonal map

$$\Delta\colon BU(n) \to BU(n) \times BU(n).$$

Therefore, the map ζ is homotopic to the map defined by $(A, B) \mapsto AB^{-1}$. Let $f\colon U(n) \to U(n)$ be defined as $f(A) = AA^t$, this is the same map as in Lemma 5.15. We conclude that $u\partial_c \simeq f$.

Localising the map f at a prime $p \neq 2$ yields the map

$$f'\colon SO(n) \times U(n)/SO(n) \to U(n)/SO(n)$$

which is an equivalence when we restrict to the factor $U(n)/SO(n)$. We have shown the following proposition.

Proposition 5.26. *Let n be odd, then localised at a prime $p \neq 2$ the following composition is a homotopy equivalence*

$$U(n)/SO(n) \hookrightarrow U(n) \xrightarrow{\partial_c} B\mathscr{G}^*((0,0,1);c) \xrightarrow{u} U(n) \to U(n)/SO(n). \qquad \square$$

With this proposition, we now have enough ammunition to prove the rest of Theorem 1.13.

Proof of Theorem 1.13 $(1b)$ *and* $(2b)$. We first prove part $(1b)$. Localise at a prime $p \neq 2$ such that $p \nmid n$. Reconsider the fibration sequence

$$\mathscr{G}((0,0,1);c) \to SO(n) \times U(n)/SO(n) \xrightarrow{\partial_c} B\mathscr{G}^*((1,0,0);c).$$

By Proposition 5.26 the factor $U(n)/SO(n)$ retracts off $B\mathscr{G}^*((1,0,0);c)$ and by Proposition 5.24 (2) the factor $SO(n)$ retracts off $\mathscr{G}((0,0,1);c)$. We now use Proposition 2.12 to obtain the required homotopy decomposition. The proof of part $(2b)$ is similar. $\square$

5.3.4 Case: $(1,0,1)$

We now analyse unpointed gauge groups above a Real surface (T,τ) of type $(1,0,1)$. We use a similar method to the $(0,0,1)$ case and adopt some of its notation.

As in the proof of Theorem 5.21, let $u'\colon B\mathscr{G}^*((1,0,1);c) \to \mathrm{Map}^*(C, BU(n))$ be the map that forgets about equivariance and restricts to the upper half of (T,τ) which is homeomorphic to a cylinder C. Let $\overline{\Delta}$ be as in (5.19) then we obtain the following diagram

$$\begin{array}{ccccccc}
U(n) & \xrightarrow{\partial_c} & B\mathscr{G}^*((1,0,1);c) & \longrightarrow & B\mathscr{G}((1,0,1);c) & \longrightarrow & BU(n) \\
\downarrow{\scriptstyle \overline{\Delta}} & & \downarrow{\scriptstyle u'} & & \downarrow & & \downarrow{\scriptstyle \overline{\Delta}} \\
U(n)\times U(n) & \xrightarrow{\zeta'} & \mathrm{Map}^{*2}(C, BU(n)) & \longrightarrow & \mathrm{Map}(C, BU(n)) & \xrightarrow{\mathrm{ev}_2} & BU(n)\times BU(n)
\end{array}$$

where ev_2 is another double evaluation map; viewing C as a sub-complex of (T,τ), the map ev_2 evaluates at the basepoint $*_1$ and its image under the involution $\tau(*_1)$. Again, the map ζ' is defined via this diagram.

As in the previous case, we aim to study the homotopy type of the map $\overline{\Delta}\zeta'$. However, it is not immediately clear on the homotopy type of the 'boundary' map ζ'. We note that $C \simeq S^1$ under a deformation retract fixing $*_1$ and taking $\tau(*_1)$ to $*_1$. Therefore, if we let $LU(n)$ be the free loop space of $U(n)$, we deduce that there is a homotopy commutative diagram

$$\begin{array}{ccc}
\mathrm{Map}(C, BU(n)) & \xrightarrow{\mathrm{ev}_2} & BU(n)\times BU(n) \\
\downarrow{\scriptstyle \simeq} & & {\scriptstyle \Delta}\uparrow \\
LBU(n) & \xrightarrow{\mathrm{ev}} & BU(n)
\end{array}$$

where ev evaluates at the basepoint $*_1$ and Δ is the diagonal map. Given that $\Delta\,\mathrm{ev}$ is a composition, we obtain the following homotopy commutative diagram

$$\begin{array}{ccccc}
 & & U(n)\times U(n) & = & U(n)\times U(n) \\
 & & \downarrow{\scriptstyle \zeta'} & & \downarrow{\scriptstyle \tilde{\zeta}} \\
U(n) & \xrightarrow{h'} & \mathrm{Map}^{*2}(C, BU(n)) & \xrightarrow{h} & U(n) \\
\| & & \downarrow & & \downarrow{\scriptstyle *} \\
U(n) & \longrightarrow & LBU(n) & \xrightarrow{\mathrm{ev}} & BU(n) \\
 & & \downarrow{\scriptstyle \Delta\,\mathrm{ev}} & & \downarrow{\scriptstyle \Delta} \\
 & & BU(n)\times BU(n) & = & BU(n)\times BU(n)
\end{array}$$

defining the maps h and h'.

By the triviality of the middle right vertical, there is a right homotopy inverse i to h and a left inverse q to h'. Therefore the space $\mathrm{Map}^{*2}(C, BU(n))$ is homotopy equivalent to the product $U(n)\times U(n)$.[2] Therefore the homotopy type of

$$\zeta' \colon U(n)\times U(n) \to \mathrm{Map}^{*2}(C, BU(n))$$

[2]Of course, this can be seen directly by studying the homotopy type of C.

can be determined by studying $q\zeta'$ and $h\zeta'$. It is clear that $q\zeta' \sim *$ and $h\zeta' \sim \tilde{\zeta}$. However, $\tilde{\zeta}$ is the same as the map $\zeta\colon U(n)\times U(n) \to U(n)$ in Case $(0,0,1)$ and therefore it homotopic to the map $(A,B) \mapsto AB^{-1}$.

We conclude that ζ' is homotopic to a map

$$\begin{aligned} U(n) \times U(n) &\to U(n) \times U(n) \\ (A,B) &\mapsto (I_n, AB^{-1}). \end{aligned}$$

Proof of Theorem 1.13 $(1c)$ *and* $(2c)$. We first prove part $(1c)$. Let $p \neq 2$ be a prime with $p \nmid n$. Then localised at p, in the same way as Proposition 5.26 we see that the factor $U(n)/SO(n)$ in

$$U(n) \simeq_q U(n)/SO(n) \times SO(n)$$

retracts off $B\mathscr{G}^*((1,0,1);c)$ via

$$U(n)/SO(n) \hookrightarrow U(n) \xrightarrow{\partial_c} B\mathscr{G}^*((1,0,1);c) \xrightarrow{u'} U(n) \to U(n)/SO(n).$$

Additionally by Propositions 5.24 (2), the factor $SO(n)$ retracts off the gauge group $\mathscr{G}((1,0,1);c)$. We use Proposition 2.12 to obtain the required homotopy decomposition. The proof of part $(2c)$ is similar. □

5.4 The Quaternionic Case

From herein we restrict to the Quaternionic case. For convenience, we recall the main objects of interest and their classification. Let $(P,\tilde{\sigma}) \to (X,\sigma)$ be a Quaternionic principal $U(2n)$-bundle over a Real surface, then we aim to provide homotopy decompositions of the following spaces

- the unpointed gauge group $\mathscr{G}_Q(P,\tilde{\sigma})$;
- the (single)-pointed gauge group $\mathscr{G}^*_Q(P,\tilde{\sigma})$;
- the $(r+a)$-pointed gauge group $\mathscr{G}^{*(r+a)}_Q(P,\tilde{\sigma})$.

From Section 1.1 we recall that the isomorphism class of these gauge groups depend on

1. the type (g,r,a) of the underlying Real surface (X,σ) where
    - g is the genus of X;
    - r is number of path components of the fixed set X^σ;
    - $a = 0$ if X/σ is orientable and $a = 1$ otherwise;

2. the isomorphism class c of the bundle $(P, \tilde{\sigma})$ where

   - c is the first Chern class of the underlying principal $U(2n)$-bundle P;

subject to the relations in Theorem 1.1 and Proposition 1.3.

Again, our method of attack will be to study some mapping spaces related to these gauge groups. In fact these mapping spaces are the same as in the Real case except $BU(2n)$ is endowed with an involution so that

$$(EU(2n), \sigma_{EU(2n)}) \to (BU(2n), \sigma_Q)$$

is a universal Quaternionic bundle. Therefore since a lot of the results in the Real case come from geometric properties of (X, σ), we will see that these results transfer to the Quaternionic setting without too much hassle. Furthermore, since $(BG)^{\sigma_Q} = BSp(n)$ we will see that a number of results will be easier to prove due to the high connectivity of $BSp(n)$.

For the $\mathbb{Z}_2$-space $(BU(2n), \sigma_Q)$ as above, we write

$$\mathrm{Map}_{\mathrm{Q}}(X, BU(2n)) := \mathrm{Map}_{\mathbb{Z}_2}(X, BU(2n))$$

to distinguish from the Real case and use similar notation for the pointed cases. Now let $\overline{X}$ be as in the preamble to Theorem 5.1, and we recall Theorem 4.20 for the Quaternionic case.

Theorem 5.27. *Let $(P, \tilde{\sigma})$ be a Quaternionic principal $U(2n)$-bundle of class c over a Real surface (X, σ) of type (g, r, a). Then there are homotopy equivalences*

1. $B\mathscr{G}_{\mathrm{Q}}(P, \tilde{\sigma}) \simeq \mathrm{Map}_{\mathrm{Q}}(X, BU(2n); P)$;
2. $B\mathscr{G}_{\mathrm{Q}}{}^{*}(P, \tilde{\sigma}) \simeq \overset{*}{\mathrm{Map}}_{\mathrm{Q}}(X, BU(2n); P)$;
3. $B\mathscr{G}_{\mathrm{Q}}{}^{*(r+a)}(P, \tilde{\sigma}) \simeq \mathrm{Map}_{\mathrm{Q}}{}^{*(r+a)}(X, BU(2n); c) \simeq \overset{*}{\mathrm{Map}}_{\mathrm{Q}}(\overline{X}, BU(2n); P)$.

where on the right hand side we pick the path component of $\mathrm{Map}_{\mathrm{Q}}(X, BU(n))$ that induces $(P, \tilde{\sigma})$.

As with the Real case, we sometimes write $\mathscr{G}_{\mathrm{Q}}((g, r, a); c)$ to mean the gauge group of a quaternionic bundle of class c over a Real surface (X, σ) of type (g, r, a). We use similar notation for the pointed cases. We now sketch the proofs for the results in Section 1.3.

Proposition 1.14. *Let (X, σ) be a Real surface of fixed type (g, r, a). Let $(P, \tilde{\sigma})$ and (P', σ') be quaternionic principal $U(2n)$-bundles over (X, σ), then there are homotopy equivalences*

1. $B\mathscr{G}_{\mathrm{Q}}{}^{*}(P,\tilde{\sigma}) \simeq B\mathscr{G}_{\mathrm{Q}}{}^{*}(P',\sigma')$;
2. $B\mathscr{G}_{\mathrm{Q}}{}^{*(r+a)}(P,\tilde{\sigma}) \simeq B\mathscr{G}_{\mathrm{Q}}{}^{*(r+a)}(P',\sigma')$;

Proof. We use the action of $\pi_2(BU(2n))$ on $[(X,\sigma),(BU(2n),\sigma_Q)]_{\mathbb{Z}_2}$ as presented in the proof of Proposition 1.5. □

As in the Real case, the lack of $\pi_2(BU(2n))$ action means that we cannot provide an analogue for $B\mathscr{G}_{\mathrm{Q}}(P,\tilde{\sigma})$. However, we now prove the Quaternionic analogue of Proposition 1.7.

Proposition 1.15. *Let (X,σ) be a Real surface of fixed type (g,r,a) and let the following be the classifying spaces of gauge groups of Quaternionic bundles of rank $2n$. Then for any even integer c, there is a homotopy equivalence*

$$B\mathscr{G}_{\mathrm{Q}}((g,r,a);c)) \simeq B\mathscr{G}_{\mathrm{Q}}((g,r,a);c+4n).$$

Proof. We will define a map

$$\theta\colon \mathrm{Map}_{\mathrm{Q}}(X,BU(2n);c) \to \mathrm{Map}_{\mathrm{Q}}(X,BU(2n);c+4n)$$

that sends a f to the composition

$$X \xrightarrow{\Delta} X\times X \xrightarrow{f\times i} BU(2n)\times BU(1) \xrightarrow{T} BU(2n).$$

Here, $BU(1)$ is seen as a subspace of $BU(2n)$ and hence $BU(1)$ is endowed with the involution induced by complex conjugation; turning T into $\mathbb{Z}_2$-map.

With this in mind, define $j\colon (S^2\vee S^2,\mathrm{sw}) \to BU(1)$ to be the inclusion of the bottom cell on the left hand factor S^2 and then define it equivariantly on the right hand factor. Then let i be the composition

$$i\colon (X,\sigma) \xrightarrow{q} (S^2\vee S^2,\mathrm{sw}) \xrightarrow{j} BU(1)$$

where q is the quotient map collapsing the 1-cells. This defines θ and a homotopy inverse to θ is given by replacing i with $\sigma_{BU(1)}i$. □

We sketch the proofs for the results related to homotopy decompositions of the gauge groups.

(Proof of 1.16 and 1.17). The proof is similar to those in Sections 5.2.1–5.2.3 except that in this case $BU(2n)^{\sigma_Q} = BSp(n)$. We recall that decompositions involving fixed circles in the Real case needed to be handled delicately, but this does not occur in the Quaternionic case due to the high connectivity of $BSp(n)$. □

Localised at a prime $p \neq 2$ and for n odd, we obtained a p-local decomposition in the Real case due to the fact that the p-local homotopy equivalence

$$\begin{aligned} U(n)/O(n) &\to U(n)/O(n) \\ AO(n) &\mapsto AA^t O(n) \end{aligned}$$

factored through $B\mathscr{G}^*((0,0,1);0)$. In Section 2.3, we presented a similar map involving $U(2n)/Sp(n)$ and we shall see that this map also factors through the Quaternionic analogue of this gauge group. The following is a restatement of Theorem 1.18 (1).

Proposition 5.28. *Let $p \neq 2$ be prime, then there is a p-local homotopy equivalence*

$$\mathscr{G}_{\mathrm{Q}}{}^*((0,0,1);0) \simeq_p \Omega^2(U(2n)/Sp(n)) \times \Omega(U(2n)/Sp(n)).$$

We will follow the proof and use the notation of Theorem 5.14. Recall from 2.3 that the involution σ_0 on $U(2n)$ is defined via $\sigma_0(A) = J^{-1}\overline{A}J$ where $\overline{A}$ denotes complex conjugation and

$$J = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

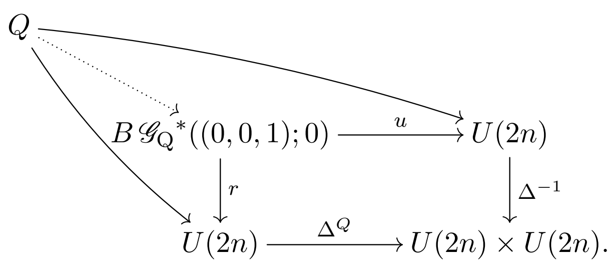
Let

$$u\colon B\mathscr{G}_{\mathrm{Q}}{}^*((0,0,1);0) \to \mathrm{Map}^{*2}(D^2, BU(2n))$$

be the map that restricts to the upper hemisphere of $(S^2, -\mathrm{id})$ and forgets about equivariance considering the image as landing in $\mathrm{Map}^{*2}(D^2, BU(2n))$. Let

$$r\colon B\mathscr{G}_{\mathrm{Q}}{}^*((0,0,1);0) \to \mathrm{Map}^*_{\mathbb{Z}_2}((S^1 \vee S^1, \mathrm{sw}), BU(2n))$$

be the map restricting to the 1-skeleton of $(S^2, -\mathrm{id})$. We obtain a similar homotopy commuting diagram to diagram (5.7)



where Δ^Q is the map $A \mapsto (A, \sigma_0 A)$. Here, Q is the strict pullback of the diagram

$$U(2n) \xrightarrow{\Delta^Q} U(2n) \times U(2n) \xleftarrow{\Delta^{-1}} U(2n)$$

and $B\mathscr{G}_Q{}^*((0,0,1);0)$ is the homotopy pullback of the same diagram. Once again, we aim to show that Q retracts off $B\mathscr{G}_Q{}^*((0,0,1);0)$.

Lemma 5.29. *The pullback Q is homeomorphic to $U(2n)/Sp(n)$.*

Proof. This is essentially the same proof as Lemma 5.15, but for the sake of clarity we include the details in this case.

The space Q consists of the elements (α, β) of $U(2n) \times U(2n)$ such that $\alpha = \beta$ and $J^{-1}\overline{\alpha}J = \beta^{-1}$. Hence $\alpha J = J\alpha^t$ and

$$(\alpha J)^t = J^t \alpha^t = -J\alpha^t = -\alpha J.$$

We conclude that $A \in U(2n)$ is in Q if and only if AJ is skew-symmetric.

We define a map $f\colon U(2n) \to U(2n)$ by $f(A) = A\sigma_0(A)^{-1}$. Note that $J^2 = -\operatorname{id}_{2n}$ and so

$$(f(A)J)^t = (AJ^{-1}A^tJJ)^t = -(AJ^{-1}A^t)^t = AJA^tJJ = -(AJ^{-1}A^tJ)J = -f(A)J$$

and so we redefine f to have image in Q.

Now for $A \in U(2n)$ and $W \in Sp(n)$ we have

$$f(AW) = AW\sigma_0(W)^{-1}\sigma_0(A)^{-1} = AWW^{-1}\sigma_0 A = A\sigma_0(A)^{-1} = f(A)$$

and so f induces a map $f'\colon U(2n)/Sp(n) \to Q$.

The map f' is injective. Let $A, B \in U(2n)$ and suppose $A\sigma_0(A)^{-1} = B\sigma_0(B)^{-1}$, then

$$\operatorname{id}_{2n} = B^{-1}A\sigma_0(A)^{-1}\sigma_0(B) = (B^{-1}A)\sigma_0(B^{-1}A)^{-1}.$$

Therefore $B^{-1}A \in Sp(n)$ and so $ASp(n) \equiv BSp(n)$.

The map f' is surjective. Let A be in Q then AJ is skew-symmetric and hence due the Youla Lemma [You61], there is a unitary matrix P such that $AJ = PJP^t$. Therefore

$$A = PJP^tJ^{-1} = P(J^{-1}\overline{P}J)^{-1} = f'(PSp(n))$$

The map f' is therefore a continuous bijection, and since Q is Hausdorff and $U(2n)/Sp(n)$ is compact, f' is a homeomorphism. □

Similar to the map in (5.8), we obtain the following composition

$$\varphi\colon U(2n)/Sp(n) \xrightarrow{f'} Q \to B\mathscr{G}_{\mathrm{Q}}{}^*((0,0,1);0) \xrightarrow{r} U(2n) \xrightarrow{q} U(2n)/Sp(n) \tag{5.20}$$

where q is the quotient map. The map φ sends an element $ASp(n)$ to $A\sigma_0(A)^{-1}Sp(n)$. In Section 2.3, we showed that the related map

$$\begin{aligned} s'\colon SU(2n)/Sp(n) &\to SU(2n)/Sp(n) \\ ASp(n) &\mapsto A\sigma_0(A)^{-1}Sp(n) \end{aligned} \tag{5.21}$$

is a homotopy equivalence when localised at a prime $p \neq 2$.

It is clear that there are analogue statements to Lemmas 5.17 and 5.18 and Proposition 5.20.

Lemma 5.30. *There is a homotopy equivalence*

$$\eta\colon U(2n)/Sp(n) \times S^1 \xrightarrow{\simeq} U(2n)/Sp(n).$$ □

Lemma 5.31. *There exist maps $s''\colon SU(n)/SO(n) \to SU(n)/SO(n)$ and $s'\colon S^1 \to S^1$ such that the following is a homotopy commuting square*

$$\begin{array}{ccc} SU(2n)/Sp(n) \times S^1 & \xrightarrow{s''\times s'} & SU(2n)/Sp(n) \times S^1 \\ \downarrow{\scriptstyle\eta} & & \downarrow{\scriptstyle\eta} \\ U(2n)/Sp(n) & \xrightarrow{s} & U(2n)/Sp(n). \end{array}$$

Furthermore, s'' and s' are p-local equivalences. □

Proposition 5.32. *Let F be the homotopy fibre of the composition*

$$B\mathscr{G}_{\mathrm{Q}}{}^*((0,0,1);0) \xrightarrow{r} U(2n) \xrightarrow{q} U(2n)/Sp(n)$$

then for any prime $p \neq 2$, there is a p-local homotopy equivalence

$$F \simeq_p \Omega(U(2n)/Sp(n)).$$ □

Proof of Proposition 5.28. For a prime $p \neq 2$, we have shown that there is a p-local section to the principal homotopy fibration

$$\Omega^2(U(2n)/Sp(n)) \to \mathscr{G}_{\mathrm{Q}}{}^*((0,0,1);0) \xrightarrow{\Omega(qr)} \Omega(U(2n)/Sp(n))$$

and the result follows. □

Proof of Theorem 1.18 (2) and (3). These follow using the same proofs as Theorems 5.21 and 5.22. □

In the unpointed case, the theorems involving integral decompositions follow immediately from the Real case.

Proof of Theorem 1.19. The results presented in Section 5.2.1 do not depend on the fixed point set of the involution on $BU(n)$ and hence Theorem 1.19 follows immediately. □

We proceed to prove the Quaternionic analogues of Section 5.3.2. Let

$$B = \begin{cases} BSp(n) & \text{if } r > 0 \\ BU(2n) & \text{otherwise} \end{cases}$$

and recall the evaluation fibration

$$\Omega B \xrightarrow{\partial_P} \mathrm{Map}^*_Q(X, BU(n); P) \to \mathrm{Map}_Q(X, BU(n); P) \to B. \tag{5.22}$$

The following proposition can be proven using the same method as Proposition 5.24.

Proposition 5.33. *Fix $d \in \mathbb{Z}$ and let ∂_d be the boundary map in (5.15). Let*

$$\partial_P : \Omega B \to B\mathscr{G}_{\mathrm{Q}}{}^*((g, r, a); 2d)$$

be the boundary map of the evaluation fibration as in (5.22). If ∂_d is (q-locally) trivial then

1. *if $r > 0$ then ∂_P is (q-locally) trivial;*
2. *if $r = 0$ then the composition*

$$Sp(n) \hookrightarrow U(2n) \xrightarrow{\partial_P} B\mathscr{G}_{\mathrm{Q}}{}^*((g, r, a); 2d)$$

is (q-locally) trivial. □

Proof of Theorem 1.20 (1). Let p be a prime such that $p \nmid 2n$. Then $2n$ is a unit mod p and hence by Corollary 3.17 the map ∂_{2n} is p-locally trivial. The result then follows from Proposition 5.33. □

Proof of Theorem 1.20 (2) and (3). The proof is similar to the proofs of Theorem 1.13 (1*b*) and (1*c*). We do require that $p \neq 2$ but this is automatic with the assumption that $p \nmid 2n$. □

Appendices

Appendix A

Tables of Homotopy Groups

In this appendix we present some of the homotopy groups of the discussed gauge groups that can be obtained from the homotopy decompositions presented in Sections 1.2 and 1.3. We will only present the homotopy groups of the trivial components, that is,

- $(c, w_1, \dots w_r) = (0, 0, \dots, 0)$ for Real bundles and;
- $c = 0$ for Quaternionic bundles;

with the understanding that results can be obtained for different components using Propositions 1.5, 1.6, 1.7, 1.8, 1.14 and 1.15.

We first recall the status of the calculation of the homotopy groups before this thesis, that is, we present Table A.1 which is a restatement of the table in Theorem 1.4.

Table A.1: Results of [BHH10] – the low dimensional homotopy groups of rank n gauge groups above a Real surface of type (g, r, a). The entries in blue disagree with the author's results.

Real	$\pi_0(\mathscr{G}^{*(r+a)}(P,\tilde{\sigma}))$	$\pi_0(\mathscr{G}(P,\tilde{\sigma}))$	$\pi_1(\mathscr{G}^{*(r+a)}(P,\tilde{\sigma}))$	$\pi_1(\mathscr{G}(P,\tilde{\sigma}))$
$n > 2$	$\mathbb{Z}^{g+a} \times (\mathbb{Z}_2)^r$	$\mathbb{Z}^g \times (\mathbb{Z}_2)^{r+1}$	$\mathbb{Z}$	$\mathbb{Z} \times (\mathbb{Z}_2)^r$
$n = 2$	$\mathbb{Z}^{g+a+r}$	$\mathbb{Z}^{g+r} \times \mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}^{r+1}$
$n = 1$	$\mathbb{Z}^{g+a}$	$\mathbb{Z}^g \times \mathbb{Z}_2$	0	0
Quat. rank $2n$	$\mathbb{Z}^{g+a}$	$\mathbb{Z}^g \times (\mathbb{Z}_2)^a$	$\mathbb{Z}$	$\mathbb{Z}$

From the results in Sections 1.2 and 1.3, we can see that our homotopy decompositions usually contain factors involving $U(n)$, $O(n)$ and $Sp(n)$. Due to Bott periodicity, it is easy to calculate some of the higher homotopy groups for high rank gauge groups. We present such results in Tables A.2 and A.3 where η is defined via

$$\eta = \eta(g, r, a) = \begin{cases} 1 & \text{if } r > 0 \text{ and } a = 1; \\ 0 & \text{otherwise.} \end{cases}$$

Some of the results in Table A.2 are a consequence of localised homotopy equivalences and hence may provide incomplete information. To highlight these localised results we use the following notation

- groups surrounded by $(-)_p$ are understood to have come from p-local homotopy equivalences where p and the rank n of the gauge groups satisfy the requirements of Theorems 1.11 and 1.13.

Table A.2: Homotopy groups for high rank gauge groups of Real bundles, that is, the homotopy groups π_i when the rank $n > i + 2$. The results in blue correspond to the top row in Table A.1.

	$\mathscr{G}^{*(r+a)}(P, \tilde{\sigma})$	$\mathscr{G}(P, \tilde{\sigma})$
π_{8j}	$\mathbb{Z}^{g-1} \times \mathbb{Z}_2^{r-1} \times (\mathbb{Z}^{1+a})_p \times (\mathbb{Z}_2^{1+\eta})_p$	$\mathbb{Z}^{g-1} \times \mathbb{Z}_2^{r-1} \times (\mathbb{Z})_p \times (\mathbb{Z}_2^{1+\eta})_p$
π_{8j+1}	$(\mathbb{Z}_2^{1+a})_p$	$\mathbb{Z}_2^{r-1} \times (\mathbb{Z}_2^{2+\eta})_p$
π_{8j+2}	$\mathbb{Z}^{g+r-2} \times (\mathbb{Z}^{1+\eta})_p \times (\mathbb{Z}_2^{a})_p$	$\mathbb{Z}^{g-1} \times \mathbb{Z}_2^{r-1} \times (\mathbb{Z})_p \times (\mathbb{Z}_2^{\eta})_p$
π_{8j+3}	$(\mathbb{Z})_p$	$(\mathbb{Z}^2)_p$
π_{8j+4}	$\mathbb{Z}^{g-1} \times (\mathbb{Z}^{1+a})_p$	$\mathbb{Z}^{g-1} \times (\mathbb{Z})_p$
π_{8j+5}	0	0
π_{8j+6}	$\mathbb{Z}^{g+r-2} \times (\mathbb{Z}^{1+\eta})_p$	$\mathbb{Z}^{g-1} \times (\mathbb{Z}^{1-\eta})_p$
π_{8j+7}	$\mathbb{Z}_2^{r-1} \times (\mathbb{Z})_p \times (\mathbb{Z}_2^{\eta})_p$	$\mathbb{Z}_2^{r-1} \times (\mathbb{Z}^2)_p \times (\mathbb{Z}_2^{\eta})_p$

Similarly, some of the results in Table A.3 are a consequence of localised homotopy equivalences and hence may provide incomplete information. To highlight these localised results we use the following notation

- groups surrounded by $(-)_p$ are understood to have come from p-local homotopy equivalences where p is prime and the rank $2n$ of the gauge groups satisfy the requirements of Theorems 1.18 and 1.20.

Table A.3: Homotopy groups for high rank gauge groups of Quaternionic bundles, that is, the homotopy groups π_i when the rank $2n > \frac{i+1}{4}$. The results in blue correspond to the bottom row in Table A.1.

	$\mathscr{G}_{\mathrm{Q}}{}^{*(r+a)}(P,\tilde{\sigma})$	$\mathscr{G}_{\mathrm{Q}}(P,\tilde{\sigma})$
π_{8j}	$\mathbb{Z}^{g-1} \times (\mathbb{Z}^{1+a})_p$	$\mathbb{Z}^{g-1} \times (\mathbb{Z})_p$
π_{8j+1}	0	0
π_{8j+2}	$\mathbb{Z}^{g+r-2} \times (\mathbb{Z}^{1+\eta})_p$	$\mathbb{Z}^{g-1} \times (\mathbb{Z}^{1-\eta})_p$
π_{8j+3}	$\mathbb{Z}_2^{r-1} \times (\mathbb{Z})_p \times (\mathbb{Z}_2^{\eta})_p$	$\mathbb{Z}_2^{r-1} \times (\mathbb{Z}^2)_p \times (\mathbb{Z}_2^{\eta})_p$
π_{8j+4}	$\mathbb{Z}^{g-1} \times \mathbb{Z}_2^{r-1} \times (\mathbb{Z}^{1+a})_p \times (\mathbb{Z}_2^{1+\eta})_p$	$\mathbb{Z}^{g-1} \times \mathbb{Z}_2^{r-1} \times (\mathbb{Z})_p \times (\mathbb{Z}_2^{1+\eta})_p$
π_{8j+5}	$(\mathbb{Z}_2^{1+a})_p$	$\mathbb{Z}_2^{r-1} \times (\mathbb{Z}_2^{2+\eta})_p$
π_{8j+6}	$\mathbb{Z}^{g+r-2} \times (\mathbb{Z}^{1+\eta})_p \times (\mathbb{Z}_2^{a})_p$	$\mathbb{Z}^{g-1} \times \mathbb{Z}_2^{r-1} \times (\mathbb{Z})_p \times (\mathbb{Z}_2^{\eta})_p$
π_{8j+7}	$(\mathbb{Z})_p$	$(\mathbb{Z}^2)_p$

Due to the properties of Bott periodicity, Table A.3 is a translation of Table A.2. We note that additional calculations can be made for the lower rank cases. We point the reader to [Mim95, Section 3.2] where explicit homotopy groups of some of the relevant factors can be found.

We note that the author's results disagree with the $\mathbb{Z}$-summands coloured in blue in Table A.1. In the pointed case, this $\mathbb{Z}$-summand arises in [BHH10] by studying a fibration arising from restricting the gauge group to the 1-skeleton of the Real surface.

For example, the corresponding fibration for a type $(g, r, 0)$ Real surface is

$$\Omega^2 U(n) \to \mathscr{G}^*(P,\tilde{\sigma}) \to \prod^{g} \Omega U(n) \times \prod^{r} \Omega O(n)$$

and we obtain the exact section

$$0 \to \pi_2(\mathscr{G}^*(P,\tilde{\sigma})) \xrightarrow{\nu} \mathbb{Z}^{g+r} \to \mathbb{Z} \xrightarrow{\mu} \pi_1(\mathscr{G}^*(P,\tilde{\sigma})) \to 0.$$

The claim in [BHH10] is that the map μ can be thought in terms of the classification of bundles over $S^2 \wedge X$. Further since μ is induced by a map that collapses the one skeleton of X, the map μ is essentially providing an identification of the second Chern class, and hence is an isomorphism.

The author agrees that this argument holds in the non-equivariant case. Indeed, if we consider X as a Riemann surface we obtain that $S^2 \wedge X$ is a wedge of spheres and then

μ is induced by a map that collapses all but the top copy of S^4.

However, we now demonstrate that $\pi_1(\mathscr{G}^*(P,\tilde{\sigma}))$ cannot contain a $\mathbb{Z}$-summand, at least for the type $(0,1,0)$ case. We assume that $\pi_1(\mathscr{G}^*(P,\tilde{\sigma}))$ contains a $\mathbb{Z}$-summand, and that subsequently the map μ is an isomorphism. Therefore ν is an isomorphism and we recall that it is induced by the map r' which restricts to the 1-skeleton of X. The map r' fits into the following commutative diagram[1]

$$\begin{array}{ccc} \mathscr{G}^*(P,\tilde{\sigma}) & \xrightarrow{\quad u' \quad} & \Omega\,\mathrm{Map}^*(D^2, BU(n)) \\ \downarrow{\scriptstyle r'} & & \downarrow{\scriptstyle r} \\ \Omega\,\mathrm{Map}^*_{\mathbb{Z}_2}((S^1,\mathrm{id}), BU(n); 0) & \xrightarrow{\ u\ } & \Omega\,\mathrm{Map}^*(S^1, BU(n)) \end{array}$$

where u' is the map that forgets about equivariance and restricts to the upper hemisphere of X.

Now u is homotopic to the inclusion $\Omega O(n) \hookrightarrow \Omega U(n)$ and hence by assumption the induced map

$$u_*\nu = (ur')_* \colon \mathbb{Z} \cong \pi_2(\mathscr{G}^*(P,\tilde{\sigma})) \to \pi_2(\Omega U(n)) \cong \mathbb{Z}$$

is multiplication by 2. But ru' is nullhomotopic because it factors through the contractible space $\Omega\,\mathrm{Map}^*(D^2, BU(n))$ and we obtain a contradiction. We conclude that μ cannot be an isomorphism.

It remains to show that the other blue entries in Table A.1 cannot contain $\mathbb{Z}$-summands. However, these entries were obtained from the calculation in the pointed case and therefore we argue that these cannot contain $\mathbb{Z}$-summands either.

[1] Recall from Example 2.1 that the diagram is actually a pullback.

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