UNIVERSITY OF SOUTHAMPTON

Deformation spaces and irreducible automorphisms of a free product

by

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in the

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ABSTRACT

FACULTY OF MATHEMATICS
SCHOOL OF SOCIAL, HUMAN, AND MATHEMATICAL SCIENCES

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The (outer) automorphism group of a finitely generated free group F_n, which we denote by $Out(F_n)$, is a central object in the fields of geometric and combinatorial group theory. My thesis focuses on the study of the automorphism group of a free product of groups. As every finitely generated group can be written as a free product of finitely many freely indecomposable groups and a finitely generated free group (Grushko's Theorem) it seems interesting to study the outer automorphism group of groups that split as a free product of simpler groups. Moreover, it turns out that many well known methods for the free case, can be used for the study of the outer automorphism group of such a free product. Recently, $Out(F_n)$ is mainly studied via its action on a contractible space (which is called Culler - Vogtmann space or outer space and we denote it by CV_n) and a natural asymmetric metric which is called the Lipschitz metric. More generally, similar objects exist for a general non-trivial free product. In particular, in this thesis we generalise theorems that are well known for CV_n and $Out(F_n)$ in the case of a finite free product, using the appropriate definitions and tools.

Firstly, in [30], we generalise for an automorphism of a free product, a theorem due to Bestvina, Feighn and Handel, which states that the centraliser in $Out(F_n)$ of an irreducible with irreducible powers automorphism of a free group is virtually infinite cyclic, where it is well known irreducible automorphisms form a (generic) class of automorphisms in the free case.

In [31], we use the previous result in order to prove that the stabiliser of an attractive fixed point of an irreducible with irreducible powers automorphism in the relative boundary of a free product, can be computed. This was already well known for the free case and it is a result of Hilion.

Finally, in [29] we prove that the Lipschitz metric for the general outer space is not even quasi-symmetric, but there is a 'nice' function that bounds the asymmetry. As an application, we can see that this metric is quasi-symmetric if it is restricted on the thick part of outer space. The result in the free case is due to Algom-Kfir and Bestvina.

Academic Thesis: Declaration of Authorship

I, Dionysios Syrigos, declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research.

Title of thesis: Deformation spaces and irreducible automorphisms of a free product.

I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
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   [30] D. Syrigos: Irreducible laminations for IWIP Automorphisms of a free product and Centralisers
   [31] D.Syrigos: Stabiliser of an Attractive Fixed Point of an IWIP Automorphism of a free product
   [29] D. Syrigos: Asymmetry of outer space of a free product

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Θα ήθελα να ευχαριστήσω τους φίλους μου, Στέλιο και Κωνσταντίνο, που έκαναν ευχάριστη την παρουσία μου στην Αγγλία και περάσαμε όμορφες στιγμές μαζί. Οι συζητήσεις που κάναμε με βοήθησαν να σκέφτομαι διαφορετικά.
Επίσης, θα ήθελα να ευχαριστήσω την οικογένεια μου και συγκεκριμένα τον πατέρα μου Σάκη, την μητέρα μου Στέλλα, τον αδερφό μου Βαγγέλη, τον παππού μου Διονύση και την γιαγιά μου Σούλα για την υποστήριξη, την βοήθεια και όλα όσα μου προσέφεραν όλα αυτά τα χρόνια. Είμαι ευγνώμον για όλα όσα μου έχουν δώσει, που συνεχίζουν να μου δίνουν και για το ότι ήταν πάντα δίπλα μου, στις δύσκολες και τις όμορφες στιγμές, χωρίς αυτούς δεν θα μπορούσα να βρίσκομαι εδώ.
Τέλος, θέλω να ευχαριστήσω την Δήμητρα για την καθημερινή στήριξη, την αγάπη και την υπομονή που δείχνει. Είναι ο κύριος λόγος που η ζωή μου μακριά από την Ελλάδα, συνεχίζει να είναι όμορφη και χωρίς εκείνη δεν θα μπορούσα να συνεχίσω να προσπαθώ. Είναι ένα διαρκές κίνητρο να βελτιώνομαι και να προσπαθώ για το καλύτερο. Σε ευχαριστώ.

Contents

Background

1 Introduction

In the first main chapter of this report we deal with the automorphisms of finitely generated free groups. The groups $Aut(F_n)$ and $Out(F_n)$, where F_n is the free group on n generators, are very basic and important for Combinatorial and Geometric Group theory. Initially, these groups were studied via combinatorial methods and for example these methods yield a finite presentation for $Aut(F_n)$ and $Out(F_n)$. More recently, $Out(F_n)$ is studied through its actions on nice spaces and in particular CV_n which is a contractible space, that is usually called 'Outer Space' or 'Culler-Vogtmann space', on which $Out(F_n)$ acts properly. We present different definitions of CV_n, its basic properties and the first applications of this action. Also, we describe three different natural topologies that can be used for the study of CV_n. In addition, we explain how CV_n can be seen as a metric space and we list some interesting properties of this (pseudo-) metric, which is called the Lipschitz metric. The geometry which is induced by this metric, it's the most recent way that has been used to study CV_n. The main problem of this (pseudo-) metric is that it fails to be symmetric. In fact, it is highly asymmetric, but we state a theorem of Algom-Kfir and Bestvina that explains the lack of the asymmetry in terms of a function. Finally, we list the definitions and the basic facts for a very powerful tool that can be used to study CV_n, which is the existence of train track representatives for a class of automorphisms, the irreducible automorphisms. As an application of the train track representatives, we can associate a set of laminations to any irreducible with irreducible powers automorphism and we can also study the set of attractive fixed points of an automorphism in the boundary of the free group.
In the second chapter, we deal with the automorphisms of general (finite) free products and some well known facts about them. Our approach is to describe the study of the groups $Aut(G)$ and $Out(G)$ (where G splits an a non-trivial finite free product) which is motivated by methods that are well established for finitely generated free groups. Firstly, we explain how we can obtain a generating set providing that we have generating sets for the free factors and the outer automorphisms of the factors. Then we will see how CV_n can be generalised in a much more general context using the notion of a deformation space. Also, we describe the main properties of the general space and the notions of

the Lipschitz metric and the train track representatives in the general case. Finally, we present how a classical theorem (Tits alternatives) for subgroups of $Out(F_n)$ can be generalised for subgroups of $Out(G)$.
Finally, in the last chapter we present the main results of the following papers: [29], [30] and [31]. All of them are generalisations in the case of a general free product of well known theorems for $Out(F_n)$ and CV_n. More precisely, in [29] we prove that in the general case the well known theorem about the asymmetry of the Lipschitz metric still holds. In [30], we study the theory of attractive laminations associated to irreducible automorphisms of a free product. As an application, in [31] we generalise in the free product case, a well known theorem about the stabiliser of an attractive fixed point of an irreducible automorphism in the boundary of a free group.

2 Free groups

2.1 Definitions and Basic Results

Let F be a group and let X be any arbitrary subset of F. We will define a free group using a universal property.

Definition 2.1. (Universal Property of Free groups)
We say that F is free on the set X and we write $F = F(X)$, if for any group H we have that any map $\phi : X \to H$ can be uniquely extended to a homomorphism $\psi : F \to H$ so that $\psi\iota = \phi$, where we denote by ι the natural inclusion from X into F.

$$\begin{array}{ccc} X & \xrightarrow{i} & F \\ & \overset{\phi}{\searrow} & \downarrow \psi \\ & & H \end{array}$$

Actually, given any set X, we can construct the group $F(X)$ as follows. A *word* in the alphabet X, is a finite sequence (possibly empty) of elements of $X \cup X^{-1}$, where X^{-1} is the set of formal inverses of elements of $x \in X$. A *reduced word* is a word without subwords of the form yy^{-1} for some $y \in X \cup X^{-1}$ and we let the trivial element 1 correspond to the empty word. Then we can define the group $F(X)$ as the set of all reduced words in X, where the operation is the concatenation of words and then reduction. It can be proved that:

Theorem 2.2. *(Normal form)*
Let F be a free group on a set X. Then every element of F can be uniquely written as a reduced word on X. Conversely, if G is generated by the set X and every element of G can be uniquely written as a reduced word on X, then G is free on X.

(Recall that we say that a group G is generated by a subset X of G, if every element $g \in G$ can be written as a word in $X \cup X^{-1}$ or equivalently if G is the smallest subgroup

of G that contains X.)
In both cases, the set X is called a basis of the free group $F(X)$. The cardinality of X is called the rank of $F(X)$. It can be proved that two free groups are isomorphic if and only if they have same rank.
Let's suppose that G is generated by the set X. Then using the universal property there is a homomorphism from the free group $F(X)$ to G, which is surjective. Therefore by the 1st isomorphisms theorem, G is isomorphic to the quotient of $F(X)$ over the kernel of that homomorphism. Conversely, let X be any set and let R be a subset of $F(X)$, then we say that G has the presentation $< X|R >$ if it is isomorphic to the group $F(X)/ << R >>$, where we denote by $<< R >>$ the intersection of all normal subgroups of $F(X)$ that contain R. R is called the relation set of the presentation. It is easy to see that the free group F on a set X has a presentation of the form $< X| >$, where the relation set is the empty set.

An important theorem for free groups is the following:

Theorem 2.3. *(Nielsen - Schreier)*
Let F be a free group and H be any subgroup of F, then H is free.

However, we can not say anything about the rank of the subgroup. It is possible to have infinitely generated subgroups of the free group on two generators.
Now we will describe a classical result, which states that a group F is free if and only if F acts freely on a tree.
Let's define the notion of a tree in the sense of Serre (see [27]). A *graph* B is a quintuple $(V, E, \iota, \tau, ^-)$, where V is the set of vertices, E is the set of oriented edges and $^- \colon E \to E$ sends an edge e to the edge $\bar{e}$ which has the inverse orientation. Moreover, ι, τ are functions from E to V, which send an edge to the its initial and terminal vertex, respectively. In additional, we have that $\bar{\bar{e}} = e, \bar{e} \neq e$ and $\iota(e) = \tau(\bar{e})$ for every $e \in E$. A path $p = e_1 e_2 ... e_n$ is a finite sequence of edges (elements of E), where $\tau(e_i) = \iota(e_{i+1})$, and we say that this path starts at $\iota(e_1)$ and finishes at $\tau(e_n)$. In additional, a path is *reduced* if it has no subpaths of the form $e\bar{e}$ for some $e \in E$. A path $p = e_1 ... e_n$ is called a *circuit* or a *loop*, if $\iota(e_1) = \tau(e_n)$. A graph is called *connected*, if for any two distinct vertices v_1, v_2 there is a path that starts at v_1 and finishes at v_2. A graph is called a *tree*, if it is connected and it has no reduced loops.
A action of a group G on a tree T is called *free*, if the stabilisers of all points are trivial. Moreover, it can be proved (for more details and proofs of these facts see [27]):

Theorem 2.4. *A group F is free if and only if F acts freely on a tree T.*

2.2 Automorphisms of Free Groups

Let's denote by F_n the free group with basis the set $X = \{x_1, ..., x_n\}$. Any automorphism of a free group can be uniquely described by considering its restriction in X, since any two automorphisms with the same restriction in X are equal. Conversely, every map from X to F_n induces a unique endomorphism of F_n (by the universal property of F_n), though all not of these are automorphisms as F_n possesses injective endomorphisms that are not surjective (F_n is not Co-Hopfian). However, as we see below free groups are Hopfian, which means that surjective endomorphisms are automorphisms. In addition, every automorphism sends a basis to some (possibly different) basis. The group of automorphisms of F_n, which we denote by $Aut(F_n)$, is a very well studied group. Moreover, we denote by $Inn(F_n)$ the group of inner automorphisms of F_n, namely $\psi \in Inn(F_n)$ if and only if there is some $x \in F_n$ such that $\psi(g) = xgx^{-1}$ for every $g \in F_n$. In fact, $Aut(G)$ and $Inn(G)$ can be defined for any group G and it can be proved that $Inn(G)$ is isomorphic to $G/Z(G)$, where we denote by $Z(G)$ the centre of G. Since $Z(F_n)$ is trivial for every $n > 1$, we have that in this case $Inn(F_n) \simeq F_n$. Moreover, it is not difficult to see that $Inn(F_n)$ is a normal subgroup of $Aut(F_n)$ and so we can define the quotient $Out(F_n) = Aut(F_n)/Inn(F_n)$, which is called the group of outer automorphisms.
The study of these groups is not new, as $Aut(F_n)$ and $Out(F_n)$ are basic objects of 'combinatorial group theory'. Initially, there were a lot of results proved using combinatorial methods.
Let's consider a (finite) family of automorphisms that generate the group $Aut(F_n)$. Firstly, for every $x_i \in X$ $(i = 1, ..., n)$, let's define α_i which sends x_i to x_i^{-1} and leaves unchanged the elements of $X - \{x_i\}$. Moreover, for every (distinct) $i, j \in \{1, ..., n\}$ we consider the automorphism $\beta_{i,j}$ that sends x_i to $x_i x_j$ and leaves unchanged the elements of $X - \{x_i\}$ (these maps clearly induce surjective endomorphisms, and hence automorphisms). The automorphisms that are induced by the α_i's and the $\beta_{i,j}$'s, are called elementary Nielsen transformations. Nielsen proved that $Aut(F_n)$ and $Out(F_n)$ are finitely generated (since the elementary Nielsen transformations generate $Aut(F_n)$, see [21] p. 131) and in fact they are finitely presented (Nielsen, see again [21] p.165). Later, a simpler presentation was given by Mc Cool ([24]).
A group G is called residually finite, if for every (non-trivial) $g \in G$ there is a finite group F and a homomorphism $f : G \to F$ such that $f(g)$ is not trivial in F. This is a very interesting group property, as we can study a residually finite group G using its finite index subgroups. It is a classical fact (see [2]) that if a group G is finitely generated and residually finite then the automorphism group of G, $Aut(G)$, is also residually finite. In particular, $Aut(F_n)$ is residually finite (since it is not very difficult to see that F_n is residually finite). However, we cannot conclude immediately that $Out(F_n)$ holds the same property, since there are examples of quotients of residually finite groups that are not residually finite, but Grossman ([14]) proved that $Out(F_n)$ is residually finite. Combining the facts above and the well known result that any finitely generated and

residually finite group is also Hopfian, which means that every surjective map from the group to itself it is also injective, we have that it holds that both $Aut(F_n)$ and $Out(F_n)$ are Hopfian.
Let's focus on the group $Out(F_2)$, which is interesting since it is isomorphic to $GL(2, \mathbb{Z})$ and therefore it is arithmetic, but also isomorphic to the mapping class group of a genus-1 torus with a single puncture. More general, for any n and by considering the abelianisation map from F_n to $\mathbb{Z}^n$, which induces a homomorphism from $Aut(F_n)$ to $GL(n, \mathbb{Z})$ and it is not difficult to see that every inner automorphisms belong to the kernel. As a consequence, we have a natural map from $Out(F_n)$ to $GL(n, \mathbb{Z})$ for which it can be proved that it is actually surjective (and also isomorphism for $n = 2$). Therefore we can see an analogy of $Out(F_n)$ with $GL(n, \mathbb{Z})$. Also, there is strong a connection with mapping class groups, since $Out(F_n)$ contains the mapping class group of a compact surface S with fundamental group F_n (see [33]). As we will see, much of the work for $Out(F_n)$ has been motivated by well known results for general linear groups and mapping class groups of compact surfaces.

2.3 Outer Space

We would like to investigate more properties for $Out(F_n)$, but sometimes the combinatorial methods are very complicated. However, there is a more recent topological approach for these groups.
These methods motivated by techniques which were used successfully for arithmetic and mapping class groups and they can be used in order to understand better the automorphism groups of finitely generated free groups. For example, given a surface S, the Teichmuller space of S is defined as the space of 'marked' Riemann surfaces, where a 'marking' is an isotopy class of homeomorphisms from S to S. The mapping class group $MCG(S)$, which is the quotient of the group of orientation preserving homeomorphisms of S modulo the group of homeomorphisms of S which are isotopic to the identity, acts on the Teichmuller space of S. Culler and Vogtmann constructed an 'Outer space', that we denote by CV_n, on which the group $Out(F_n)$ acts with a particular nice way and which has been motivated by the construction of the Teichmuller space using metric graphs instead of Riemann surfaces. More specifically, the points of this space, correspond to finite 'marked' metric graphs with fundamental group isomorphic to F_n. Moreover, $Out(F_n)$ acts on this contractible space with finite point stabilisers. As we will see, CV_n has a lot of interesting properties, and so we can obtain some new results for $Out(F_n)$ and different proofs of known results.

There are different ways to define CV_n, but firstly we will describe it as a space of 'marked' metric graphs. Let's denote by R_n the rose with n petals (a graph which has exactly 1 vertex and n loops, which we denote by $e_1, ..., e_n$) and then we can identify the free group $F_n =< x_1, ..., x_n >$ with the fundamental group $\pi_1(R_n)$ of R_n, in such way that each x_i corresponds to e_i, for $i = 1, ..., n$. Under this identification, every reduced word in F_n corresponds to a reduced edge-path loop based of R_n and vice versa. Thus we can see that we can correspond to each outer automorphism $\phi : F_n \to F_n$ a homotopy equivalence $f : R_n \to R_n$ (since we don't fix a base point). Now we are in position to define the marked metric graphs:

Definition 2.5. A pair (g, Γ) is called *marked metric graph*, if:

- Γ is a finite connected graph, where all vertices of Γ have valence ≥ 3
- $g : R_n \to \Gamma$ is a homotopy equivalence, called the 'marking' of Γ
- Each edge e of Γ is assigned a positive number which is called the length of e. Γ endowed with this metric, can be seen as a metric space via the path metric (the distance between two points of Γ is defined to be the minimum of the lengths of all paths connecting these points).

Actually, we would like to identify homothetic marked metric graphs. More precisely, let $(g, \Gamma), (h, \Delta)$ be two marked metric graphs, then we define $\sim$ such that $(g, \Gamma) \sim (h, \Delta) \Leftrightarrow$ $\exists$ a homothety $m : \Gamma \to \Delta$ (i.e. there exists some $\lambda > 0$ s.t. $d(m(x), m(y)) = \lambda d(x, y)$) with $m \circ g = h$. Then $\sim$ is an equivalence relation on the set of marked metric graphs. CV_n is defined as the space of equivalence classes of marked metric graphs up to $\sim$ and it is called Outer Space. Sometimes, it is more convenient to normalise the metric graphs by choosing as representatives the points with volume (i.e. the sum of the lengths of edges) 1. If we choose the normalised CV_n, we could change the homothety m above, with an isometry ($\lambda = 1$).
The next step is to describe the action of $Out(F_n)$ on CV_n. Intuitively, we can say that every automorphism leaves invariant the underlying graph and changes the marking. Namely, let $a \in Aut(F_n)$ and (g, Γ) be a marked metric graph then we can still denote by $a : R_n \to R_n$ the homotopy equivalence which corresponds to a. The automorphism a acts by the rule: $(g, \Gamma)a = (g \circ a, \Gamma)$. It's not difficult to see that $Inn(F_n)$ acts trivially (which means that (g, Γ) and $(g \circ \iota, \Gamma)$ are homothetic, for $\iota \in Inn(F_n)$), and so there is a well defined action of $Out(F_n)$ on CV_n. Note that we can see that the stabiliser of any marked graph (g, Γ) is isomorphic to the group of isometries of Γ, and in particular it is finite, since Γ is finite. Therefore the action of $Out(F_n)$ on CV_n is proper.

2.4 $\mathbb{R}$-trees

We would like to give an alternative definition of CV_n using free F_n - actions on trees, but firstly we will discuss some basic definitions and results for isometric G-actions on trees for a general group G. We will see these notions in a more general context, as later we will need non-necessarily free group actions.

We will start by defining the notion of an $\mathbb{R}$-tree. An $\mathbb{R}$-tree T is a non-empty metric space in which for any two points there is an arc that joins them and in which every arc is isometric to a closed interval in the real line.
We are interested only in simplicial $\mathbb{R}$-trees, which means that we suppose that the set of points of T which have valence more than 2, is discete and closed. Actually, every simpicial $\mathbb{R}$-tree can be constructed as below. Let X be a connected 1-dimensional simplicial complex without circles, then for any edge e of X we choose an embedding of e to $\mathbb{R}$. For any $x, y \in X$, there exists a unique arc from x to y and this arc can be subdivided into subarcs $A_1, ..., A_n$ each of which is contained in an edge of X. The length of A_i is defined to be equal to the length of its image in $\mathbb{R}$ (under the chosen embedding). Then the distance d from x to y is defined by the sum of the lengths of A_i's. Then the metric space (X, d) is a *simplicial* $\mathbb{R}$*-tree*.

Some interesting properties of the actions are defined below.

Definition 2.6. Let G be a group that acts on an $\mathbb{R}$-tree (T, d). We say that the action is:

1. *non-trivial*, if no point of T is fixed by the whole group.
2. *minimal*, if there is no proper G-invariant subtree.
3. *free*, if the stabiliser of every point of T is trivial.
4. *co-compact*, if the quotient graph G/T is compact (finite).
5. *isometric*, if $d(gx, gy) = d(x, y)$ for every $g \in G$.
6. *simplicial*, if whenever v is a vertex of T then gv is a vertex of T and whenever e is an edge of T then ge is an edge of T, for every $g \in G$.

As we are interested in isometric actions of a group G on $\mathbb{R}$-trees, we need the classification of isometries of $\mathbb{R}$-trees as every element of G induces an isometry of T. See [10] for more details.
We can classify the isometries, by studying the number (which is called the translation length of g in T) $\ell_T(w) := \inf_{p \in \Gamma} d(p, wp)$, for every $w \in G$. It can be proved that the infimum is always achieved for some $p \in T$ and so it is actually a minimum. So there are two cases:

Remark. *1)If $\ell_T(g) > 0$, then we say that g is a* ***hyperbolic*** *isometry (or a hyperbolic element). In this case it can be proved that there exists a unique g-invariant line, which is called axis of g and is denoted by $Ax_T(g)$ such that the restriction of the action of g to the line is just translation by (the positive number) $l_T(g)$ or*
2) If $\ell_T(g) = 0$, then g is called ***elliptic****. In this case, g fixes a non-empty subtree of T.*

2.5 CV_n using free actions on trees

In the first definition of CV_n, we have a finite graph together with a homotopy equivalence between the rose R_n, corresponding to some fixed basis $x_1, .., x_n$ of F_n, and Γ, so we can identify the fundamental group of the graph Γ with $F_n = \pi_1(R_n)$. Then we can consider the universal cover T of Γ which is an $\mathbb{R}$- tree (as we can lift the metric) and we get a minimal free action (by deck transformations) of F_n on T, via the marking g. Conversely, given any minimal free action of F_n by isometries on a simplicial $\mathbb{R}$ -tree A, we get that the quotient graph $\Gamma = F_n/A$ is a finite graph (by minimality), with a metric induced by the metric of A and the action induces a homotopy equivalence $g_\Gamma : R_n \to \Gamma$. Therefore we have a 1-1 relation between marked metric graphs and minimal free isometric F_n- actions on real trees. Moreover, we can see that the equivalence relation of marked metric graphs that we have defined, corresponds to actions which are the same up to F_n - equivariant homothety (i.e. there is a homothety $f : T \to S$ between the $\mathbb{R}$-trees T, S such that $f(gx) = gf(x)$ for every $g \in F_n$). The discussion above, implies that CV_n can be alternatively defined as the equivalence classes of minimal free actions of F_n on simplicial $\mathbb{R}$- trees up to F_n - equivariant homothety.
The action here can be naturally defined. Given an outer of automorphism $a \in Out(F_n)$ and a tree $T \in CV_n$, then we can define by $a(T)$ to be the element of CV_n with the same underlying tree with T but the action is given by twisting the F_n-actions, i.e. the new action will be given by the rule $g \star x := a(g)x$ (where the second is the action of T) for every $g \in F_n$ and for every $x \in T$.
Now we are in position to define a topology for CV_n. Let $T \in CV_n$ and $w \in F_n$. As we have seen, we can consider the translation length function $\ell_T : F_n \to \mathbb{R}$, where $\ell_T(w) := \inf_{p \in T} d(p, wp)$. As we have already mentioned above, it is well known that for a minimal action the infimum is realised and, in the case of the free actions, we have that for any non-trivial element $w \in F_n$, it is not equal to 0. In particular, every non-trivial element of F_n acts on T hyperbolically. Culler and Morgan (in [10]) proved that actually any minimal free action (in fact they proved something much more general) on an $\mathbb{R}$ - tree T is determined (up to conjugacy) by the associated length function ℓ_T. Therefore, if we denote by $\mathcal{C}$ the set of conjugacy classes of F_n, we get a map from CV_n to $\mathbb{R}^{\mathcal{C}}$ (i.e. all the maps from $\mathcal{C}$ to $\mathbb{R}$) sending an element T of CV_n to the corresponding translation length function. Hence we can see the projective space CV_n as a subset of the projective space $\mathbb{P}\mathbb{R}^{\mathcal{C}}$ (since every two homothetic marked graphs have two translation lengths functions which are positive scalar multiples of each other, this map is well defined) and we can give to CV_n the subspace topology, which is usually called the axes topology.

2.5.1 Gromov- Hausdorff topology

We will describe a second topology for CV_n, which is called Gromov - Hausdorff topology, that it can be defined for any space of G-trees (i.e $\mathbb{R}$ - trees with an isometric G-action)

and in particular for CV_n.
Let us describe this topology in general, on the set of G-trees which we denote it by $\mathcal{T}(G)$.

Definition 2.7. Let K, K' be two metric spaces. Let $\epsilon > 0$. An ϵ-approximation between K and K' is a relation $\mathcal{R}$ in $K \times K'$ that is onto (i.e. $pr_1(\mathcal{R}) = K$ and $pr_2(\mathcal{R}) = K'$) such that:
$\forall x, y \in K, \forall x', y' \in K', x\mathcal{R}x'$ and $y\mathcal{R}y' \Rightarrow |d(x,y) - d(x',y')| < \epsilon$

We will describe here the basis for our topology:

Definition 2.8. Let $T \in \mathcal{T}(G)$. Let also K be a compact subset of T, P be a finite subset of G and let ϵ be a positive number. Let's denote by $V_T(K, P, \epsilon)$ the set which consisted of the elements $T' \in \mathcal{T}(G)$, such that there exist a compact subset K' of T' and a closed ϵ-approximation $\mathcal{T}(G)$ between K and K', which is P-equivariant in the following sense:
$\forall x \in K, \forall x' \in K', \forall a \in P, ax \in K$ and $x\mathcal{R}x' \Rightarrow ax' \in K$ and $ax\mathcal{R}ax'$

The collection of $V_T(K, P, \epsilon)$ for $T \in \mathcal{T}(G)$, defines a neighbourhood basis at T for a topology on $\mathcal{T}(G)$ (see Paulin for more details, [25]), called the Gromov-Hausdorff topology. In particular, Gromov-Hausdorff topology for CV_n is the quotient topology that it is induced if we identify two G-trees that they are G-equivariently isometric.

2.6 Simplicial structure

CV_n has an additional important property, as it has a simplicial structure. More specifically, CV_n decomposes into a disjoint union of open simplices. We will use here the normalised representative of a point in CV_n, so we have a marked metric graph with volume 1, and the homothety between the graphs can be chosen to be an isometry here.
Let us describe this simplicial structure. For each marked graph (g, Γ) (if we ignore the metric), we can see that it corresponds to the open simplex of CV_n, consisting of all marked metric graphs with underlying graph given by Γ and the same marking $g : R_n \to \Gamma$, where the metric varies over all the (positive) lengths of the (finitely many) edges subject to the constraint that the volume remains equal to 1. More precisely, let $e_1, ..., e_k$ be the labelled edges of Γ then the set X of all marked metric graphs which have the same underlying graph and the same marking as Γ, while the metric varies as above, can be embedded in $\mathbb{R}^k$ using the map $\Gamma \to (L_\Gamma(e_1), ..., L_\Gamma(e_k))$, and actually X coincides with the open $k-1$ simplex of $\mathbb{R}^k$, $\{(x_1, ..., x_k) | x_i > 0, \sum x_i = 1\}$.
So we have a natural subdivision of CV_n into open simplices and as a consequence we have a third topology for CV_n, which is called the weak topology. Precisely, a subset U of CV_n is closed if and only if the intersection of U with all closed simplices is closed.
Even if we have three different topologies for CV_n, it can be proved that they all coincide

(see [9], note that the main recipient for the proof is the fact that the trees of CV_n are locally finite).
However, CV_n is not a simplicial complex, as there are simplices of CV_n which have some ideal faces (and some true faces), but CV_n fails to be a simplicial complex due to the presence of ideal faces. More precisely, if A is an open simplex corresponding to the underlying graph X, a face N of A is obtained by setting to zero the lengths of some of the edges of X. This topologically corresponds to collapsing such edges to a point. If the resulting graph has still (fundamental group with) rank n, then N can be seen as a simplex of CV_n, and in this sense it is a true face. On the other hand, if the rank decreases, then the new graph is not in CV_n and in this case we say that it is an ideal face of A. For example, let's suppose that $n = 2$ and let's consider the rose R_2 corresponding to the base a_1, a_2. Then the corresponding simplex is a 1-simplex of the form $\{(x_1, x_2)|x_1, x_2 > 0, x_1 + x_2 = 1\}$. But here if we collapse the edge corresponding to a_1 that means that we set x_1 to 0, then the induced 0-simplex does not belong to CV_n as it corresponds to a graph with one loop, and so its fundamental group is not isomorphic to F_2.
However, we can find a deformation retract of CV_n which has no ideal faces. An edge e of a graph Γ is called a separating edge if $\Gamma - e$ is disconnected.
There is a natural equivariant deformation retraction of Outer space onto the subspace, which is called Reduced Outer Space (see [9]), consisting of marked graphs (g, Γ) such that Γ has no separating edges, where the deformation proceeds by uniformly collapsing all separating edges in all marked graphs.
It is interesting to see some facts about the dimensions of CV_n, which lead us to results about the virtual cohomological dimension of $Out(F_n)$ as we will see in the next section. More precisely:

Proposition 2.9. *The maximum dimension of a simplex in CV_n is $3n - 4$.*

However, note that the fact that we can find not homeomorphic points of CV_n within arbitrarily small distance implies that CV_n is not a manifold, as for example the Teicmuller space of a compact manifold is. Moreover, the action of $Out(F_n)$ on CV_n is not co-compact, which means that $CV_n/Out(F_n)$ is not compact. On the other hand, the simplicial structure is locally finite and there are finitely many orbits of simplices.

2.7 Spine and Thick part

As we have seen the action of $Out(F_n)$ on CV_n is not co-compact, but we can find subspaces of CV_n that are co-compact. Firstly, we will define the spine of CV_n, which we denote by K_n. It is a useful subspace of CV_n, since for example it was used in the original proof of contractibility of the whole space (see [9]).

CV_n contains the complex K_n, which is an equivariant deformation retract of CV_n and whose quotient is compact. This spine K_n has the structure of a simplicial complex, and in fact it can be identified with the geometric realization of the partially ordered sets of open simplices (with the face relation) of CV_n. Any partially ordered set gives rise to a simplicial complex. In particular, k simplices are totally ordered chains of length $k+1$. Thus a vertex of CV_n is an equivalence class of marked graphs (g, Γ), considered without lengths on the edges. A set of vertices $\{(g_0, \Gamma_0), ..., (g_k, \Gamma_k)\}$ spans a k-simplex if (g_i, Γ_i) is obtained from (g_{i-1}, Γ_{i-1}) by collapsing a forest (subgraph of Γ_{i-1} which does not contain any loops) in Γ_{i-1}. It can be proved actually that K_n is a simplicial complex and as in the previous proposition, we can get that $dim K_n = 2n-3$.
Now let's define the thick part of Outer Space, which we denote by $CV_n(\epsilon)$, for some positive ϵ. One reason that the action fails to be co-compact is the fact that we can find elements of the free group with arbitrarily small translation length. Therefore we can define the subspace of CV_n, where $T \in CV_n(\epsilon)$ if and only if $\ell_T(g) \geq \epsilon$ for every non trivial $g \in F_n$. For sufficiently small ϵ, we have that $CV_n(\epsilon)$ is $Out(F_n)$ invariant and the action is co-compact.

2.8 Main Properties and Corollaries

In this section, we will recall the basic properties of the action and some interesting corollaries for $Out(F_n)$ which we can get using the action. All the details, the proofs and many more facts about CV_n can be found in [9] and in [32] which is a very interesting survey paper of K.Vogtmann.
Firstly, let's recall that CV_n and all its subspaces that we have defined are contractible and locally finite. In fact, it can be proved that K_n is contractible and that the contractibility of K_n implies that CV_n is also contractible. Also, the action of $Out(F_n)$ on CV_n is simplicial with finite point stabilisers. Moreover, the action is proper and if we are restricted on the spine K_n or on the thick part $CV_n(\epsilon)$ is co-compact.
We can get a lot of important properties using this action. Firstly, using the fact that $Out(F_n)$ acts co-compactly on a contractible complex (the spine, K_n) with finite stabilisers, we get that $Out(F_n)$ is finitely presented.
Moreover, using the well known Nielsen realisation theorem which states that every finite subgroup of $Out(F_n)$ fixes a point of CV_n, we can get that $Out(F_n)$ is virtually torsion free.
As a consequence of the dimension of K_n, we can obtain that the $vcd(Out(F_n)) \leq 2n-3$ (and we will see that the equality also holds). In fact, it has a stronger property than just finite virtual cohomological dimension. Namely, a group G is said to be WFL if every torsion-free finite index subgroup H has a free resolution of finite length with each term finitely generated over the group ring $\mathbb{Z}H$. In particular this implies that H has finite cohomological dimension. In our case, the quotient of the spine K_n by $Out(F_n)$ is finite, thus the quotient by any finite index subgroup H is also finite. Therefore the

chain complex for K_n means that we have a free resolution of finite length for H with each term finitely generated over $\mathbb{Z}H$, and so $Out(F_n)$ is WFL. Since the dimension of the simplicial complex K_n is $2n-3$, this chain complex has length $2n-3$, and this fact give us an upper bound for the virtual cohomological dimension of $Out(F_n)$. The same argument works for $Aut(F_n)$ and it can be proved that it is WFL, and the vcd of $Aut(F_n)$ is at most $2n-2$.

On the other hand the group $Aut(F_n)$ contains a free abelian subgroup of rank $2n-2$: and more precisely, if $F_n =< x_1, ..., x_n >$, then for $i > 1$ the automorphisms that a_i, b_i which are defined such that they leave invariant all the x_j's for $j \neq i$ and they satisfy the property $a_i(x_i) = x_i x_1, b_i(x_i) = x_1 x_i$. Then the automorphisms $a_i, b_i, i \in I$ form a basis for a free abelian subgroup of rank $2n-2$ of $Aut(F_n)$. Then because of the conjugation, their image in $Out(F_n)$ is of rank $2n-3$. Therefore, since the cohomological dimension of a free abelian group is equal to the dimension of a basis, we have a lower bound for vcd which is equal to the upper bound. As a consequence, we have the equalities $vcd(Out(F_n)) = 2n-3$ and $vcd(Aut(F_n)) = 2n-2$. Finally, as corollary of the previous results we can get that $Out(F_n)$ has finitely many conjugacy classes of finite subgroups.

2.9 Train Track Maps

A very useful remark for CV_n is that for every outer automorphism ϕ of $Out(F_n)$ can be represented by a homotopy equivalence $f : \Gamma \to \Gamma$ for every $\Gamma \in CV_n$. This means that we can see the outer automorphism as a homotopy equivalence between a graph with fundamental group F_n. However, if we would like to study the dynamics of the automorphism there are problems that arise by the cancellation of a topological representative. Bestvina and Handel (in [8]) during their study of the famous Scott Conjecture (which states that the rank of the fixed subgroup of every automorphism of a finitely generated free group is bounded by the rank of the free group), they introduced the train track representatives and even more general relative train track representatives of outer automorphisms. Such a map is a very important tool which allows us to control the cancellation and it can be used to investigate further the outer space and as a consequence $Out(F_n)$.

As we have seen before, every automorphism ϕ of F_n can be identified with a homotopy equivalence $f : \pi_1(R_n) \to \pi_1(R_n)$ and we say that f represents ϕ.

In general, we say that an outer automorphism ϕ of F_n is topologically represented by a homotopy equivalence $f : \Gamma \to \Gamma$, if there exists a marked metric graph (g, Γ) such that if $h : \Gamma \to R_n$ is chosen to be a homotopy inverse of g where the composition $hfg : R_n \to R_n$ induces an automorphism of $F_n = \pi_1(R_n)$ whose outer automorphism class is equal to ϕ.

Actually, it can be proved that for every $\phi \in Out(F_n)$ and for every $\Gamma \in CV_n$ there is a topological representative $f : \Gamma \to \Gamma$ of ϕ. We can also assume that every vertex is

sent to a vertex and every edge is sent to an edge-path, and even that f can be chosen to be piecewise linear. We will make these assumptions for our representatives without further mention.
However, let's consider the automorphism ϕ of F_3 with free basis a, b, c, which sends a to c, b to $c^{-1}a$ and c to $c^{-1}b$. In this case we can see that $\phi^2(c) = b^{-1}cc^{-1}a = b^{-1}a$. Therefore for the topological representative f of ϕ in the rose corresponding to the basis a, b, c, we have cancellation for some iteration of ϕ. So we cannot expect that we have always representatives of automorphisms in the rose, without cancellation when we iterate the representative. But we can give an alternative definition:

Definition 2.10. Let ϕ be an outer automorphism of F_n and $f : \Gamma \to \Gamma$ be a topological representative of ϕ. Then we say that f is a train track map, if for every edge e of Γ and for every k we have that $f^k(e)$ is reduced (there is no cancellation).

Example 1.
- *Let ϕ be the outer automorphism of F_2 that a is sent to aba and b to ba. Then ϕ induces a natural train track representative in the rose R_2 with marking corresponding the free basis a, b.*
- *In general, any positive automorphism of F_n (i.e. there is some basis $x_1, ..., x_n$ of F_n such that the images of the generators x_i's don't contain any x_i^{-1}) induces a topological representative on the rose, which is a train track map.*

However, unfortunately we cannot find train track representatives for any automorphism of F_n. But we can find such representatives for a class of automorphisms (this class is generic in terms of random walks, see more details in [26]).
A topological representative $f : \Gamma \to \Gamma$ of $\phi \in Out(F_n)$ is called irreducible, if there is no proper non-trivial (at least one of its connected components it is not just a vertex) f-invariant subgraph of Γ. More precisely:

Definition 2.11. An outer automorphism ϕ of F_n is called irreducible, if any topological representative $f : \Gamma \to \Gamma$ of ϕ where Γ has no valence one vertices and has no proper nontrivial f-invariant forests, is irreducible. Otherwise, ϕ is called reducible.

Moreover, ϕ is called irreducible with irreducible powers (simply IWIP) or fully irreducible, if any power ϕ^k of ϕ is irreducible. We can see IWIP automorphisms as the analogous object to the pseudo - Anosov homeomorphisms of the mapping class group of a compact surface. This is an additional motivation to study this class of automorphisms.
We gave a topological definition for irreducibility, but there is also an equivalent algebraic definition. More precisely:

Definition 2.12. An automorphism $\phi \in Out(F_n)$ is called irreducible, if there is no proper ϕ-invariant (up to conjugacy) free factor F' of F_n. Namely, there is no F' such that $F_n = F' * H$ where both F' and H are non trivial and $\phi(F') = gF'g^{-1}$ for some $g \in F_n$.

As we have already mentioned every irreducible automorphism can be represented by an (irreducible) train track map. The main idea of the proof is that we can start with a topological representative of an irreducible ϕ and after a finite sequence of foldings of some edges and clean-ups of the induced topological representatives. This procedure can be done for any automorphism, but the problem is that we don't know if this will stop. The irreducibility condition combined with the Perron - Frobenius theory is the reason that in this special case, this machinery allows us to obtain a train track representative. Even if it is not easy to prove directly that a given automorphism is irreducible, however there are some other criteria to prove irreducibility.

Example 2.
1. *Let ϕ be the outer automorphism of F_2 that sends a to ab and b to a. Then ϕ is irreducible and actually it can be proved that it is IWIP.*
2. *Let ϕ be the outer automorphism of F_2 that sends a to ab and b to b Then ϕ is reducible since for the free factor $< b >$ of F_2 we have that $\phi(< b >) =< b >$.*

However, it's not always possible to find train track representatives of an outer automorphism. For example, it can be proved that the reducible automorphism of F_3 which sends a to $abcb^{-1}c^{-1}$, b to bcb and c to cb has no train track representative. Therefore in order to study all the automorphisms we need a generalisation of train track representatives. Namely, we can define a relative train track representative $f : \Gamma \to \Gamma$.

Definition 2.13. Let $\phi \in Out(F_n)$. Then we say that a topological representative $f : \Gamma \to \Gamma$ of ϕ, is a relative train track map if:

- there is a stratification of Γ, $H_1 \subseteq H_2 \subseteq ... \subseteq H_k = \Gamma$ where $f(H_i) \subseteq H_i$ for every i and
- for every edge e of H_i and for every k, $f^k(e)$ has cancellation only into H_{i-1} (so there is no cancellation into $H_i - H_{i-1}$).

Bestvina and Handel proved in the same paper ([8]) that every outer automorphisms of a finitely generated free group can be represented by some relative train track map.

2.10 Lipschitz metric

As we have seen previously, the methods to study CV_n were mainly combinatorial and topological. Recently, there in a new metric theory for Outer space that it is based on a natural non-symmetric metric. Therefore there is a new resulting geometric point of view which allows us to get new information about $Out(F_n)$ as well as elegant new proofs and better understanding of older results. Moreover, this metric theory allow us to see new aspects of the analogy between Outer space and the Teichmuller space, as this metric is a generalisation of a well known metric for the Teichmuller space.

Francaviglia and Martino studied this metric systematically in [11]. Let's describe how we can define this metric. The main idea of this metric is to study how much the translation lengths of the group elements can be stretched between two elements of CV_n and then to take the maximum of these strething factors. However, since we are not in advance sure that this maximum is achieved, we have to take the supremum. More precisely, for any pair of marked metric graphs A, B we can define the maximal stretching factors as

$$\Lambda_R(A,B) = \sup_{1 \neq w \in F_n} \frac{l_B(w)}{l_A(w)}$$

and asymmetric pseudo -distance:

$$d_R(A,B) = \ln(\Lambda_R(A,B)) \tag{1}$$

Using the pseudo-distance we can define the metric

$$d(A,B) = d_R(A,B) + d_R(B,A) \tag{2}$$

Firstly, note that it can be proved that there is always some $w \in F_n$ s.t. $\Lambda_R(A,B) = \frac{l_B(w)}{l_A(w)}$. It is not very hard to see that d is a well defined metric for CV_n and d_R is an asymmetric metric for CV_n (but it is not true that $d_R(A,B) = d_R(B,A)$). In fact, it can be proved that it is a complete metric (actually, any closed ball is compact, see [11]). Also, we could give an alternative definition of the metric using Lipschitz maps and this is the reason that this metric is called Lipschitz metric and it can be defined in a more general context.

Let $(g, A), (h, B)$ be two elements of CV_n. We say that a map $\phi : A \to B$ is a difference of markings map if $\phi g \simeq h$. Here we will only consider Lipschitz piecewise linear maps (as there are always piecewise Lipschitz maps between any two graphs of CV_n) and we denote by $\sigma(\phi)$ the Lipschitz constant of ϕ. We can define the distance alternatively as:

$$d_R(A,B) = \ln \inf_{\phi} \sigma(\phi) \tag{3}$$

Then the metric d can be defined as before.

In order to prove that the infimum is actually minimum we need to recall the Arzelá-Ascoli theorem, which states that any sequence of L-Lipschitz maps between two compact metric spaces has a convergent subsequence. Since every $A \in CV_n$ is compact, this theorem implies that the infimum above is realised by some map ϕ and in fact (see [11]) in this case there exists a difference of markings ϕ such that $\Lambda_R(A,B) = \sigma(\phi)$, so we can see that the two definitions are equivalent. We could define the notion of optimality of a map, which are actually the maps that realise the infimum in the definition, however we will explain the details in a more general context. Note that all the notions that we have already defined using the description of CV_n as the space of marked metric graphs, we could define them if we see CV_n as the space of minimal free F_n- actions. For example, the train track representatives could be defined as topological representatives from a

tree T to itself. Similarly, for all the other notions. We will describe how we can define them in a more general context which also applies in this case.
As we have seen in the first definition of Λ_R, there is always some element of the free group $w \in F_n$ that is maximally stretched. In fact, there is a very useful lemma (sausage lemma, see [11]) which allows us to choose some g which realises the supremum above to be rather simple. More precisely, let $A, B \in CV_n$. Then there exists a loop γ corresponding to the group element g, such that $\frac{\ell_B(g)}{\ell_A(g)} = L_R(A, B)$ such that the loop γ in A is either a simple closed loop, either is an embedded bouquet of two circle or a 'dumbbell', which means a loop that it is consisted of two disjoint simple loops and an arc connecting them. In particular, there are finitely many choices of such a loop and therefore we need just to check the translation length of finitely many elements.

Example 3. *Let A be the rose corresponding to the base a, b and the length of each simple loop is $1/2$. Similarly, we can define B_ϵ to be the same marked graph as A, but the length of a is ϵ and the length of b is $1-\epsilon$, for some positive and sufficiently small ϵ.Then in order to check the distance of $d_R(A, B_\epsilon)$, from the sausage lemma we need just the loops a, b, ab of A. Then we can see that $\ell_{B_\epsilon}(a)/\ell_A(a) = 2\epsilon$, $\ell_{B_\epsilon}(b)/\ell_A(b) = 2(1-\epsilon)$ and $\ell_{B_\epsilon}(ab)/\ell_A(ab) = 1$. Therefore for small ϵ, $d_R(A, B_\epsilon) = \ln(2(1-\epsilon))$. However, similarly we can see that $d_R(B_\epsilon, A) = -\ln(2\epsilon)$. As a consequence, d_R is not symmetric and it is not even a quasi - isometry.*

However, even if d_R is not even a quasi-isometry, there is the following theorem of Algom-Kfir and Bestvina ([1]):

Theorem 2.14. *There is an $Out(F_n)$-invariant continuous, piecewise analytic function $\Psi : CV_n \to \mathbb{R}$ and constants $K, L > 0$ (depending only on n) such that for every $x, y \in CV_n$ we have that $d_R(y, x) \leq K d_R(x, y) + L[\Psi(y) - \Psi(x)]$.*

As a corollary, we get that if x, y belong in the same orbit, then we have that we need only the multiplicative constant and namely $d(y, x) \leq K d(y, x)$. Similarly, we need just the multiplicative constant if we are restricted to the thick part $CV_n(\epsilon)$ of CV_n.
Finally, this metric induces a metric topology for CV_n. We have seen that the simpilicial, the axes topology and the Gromov-Hausdorff topology, they all coincide. In [11], it was proved that the metric induces an equivalent topology, too.

2.11 Attractive Lamination of a free group and Tits alternative

A well known fact about the centraliser of an IWIP automorphism is the following:

Theorem 2.15. *Let $\phi \in Out(F_n)$ which is an IWIP. Then the centraliser $C_{Out(F_n)}(\phi)$ is a virtually infinite cyclic group.*

There are various proofs of the theorem above (for instance, see [4] or [20]). The first proof was by Bestvina, Feighn and Handel who constructed a homomorphism from $C_{Out_G}(\phi)$ to $\mathbb{R}$. The proof uses the notion of the attractive Lamination corresponding to an irreducible with irreducible powers outer automorphism ϕ. They associate an attractive Lamination to ϕ, which is a space of lines. This laminations is called attractive since it is constructed by choosing a train track representative of the outer automorphism, and then by iterating the image of some edge with some fixed point in the interior we get a bi-infinite line. The lamination is actually the equivalence class that it is induced by this line. It can be proved that the lamination does not depend on the chosen train track map. Using the maps between any two marked metric graphs, we have a lamination $\Lambda(H)$ corresponding to each marked metric graph H. Therefore we can define the collection (space) Λ of all the laminations when H varies over all the marked metric graphs H and the collection of the attractive laminations $\{\Lambda_\psi : \psi \text{ is IWIP } \}$ on which $Out(F_n)$ acts naturally. Then they constructed a stretching homomorphism from the $Out(F_n)$-stabiliser of the lamination to the positive real numbers. As a second step, they proved that the image of this homomorphism is discrete (using the Perron - Frobenius theory) and therefore we have a map from the stabiliser $Stab(\Lambda)$ of the lamination to $\mathbb{Z}$. Then they study the kernel of the homomorphism and actually they proved that every element of the kernel has finite order (there is the need to distinguish cases for irreducible and reducible automorphisms). Since $Out(F_n)$ is virtually torsion free, we get immediately that the kernel of the action is finite. As a consequence, $Stab(\Lambda)$ is a virtually infinite cyclic group. By the definition of the action, we have that the centraliser of ϕ is contained to $Stab(\Lambda)$ and so as corollary we get that $C(\phi)$ is virtually $\mathbb{Z}$.
These techniques are well established and they can be generalised in order to prove a lot of interesting results for CV_n and $Out(F_n)$. For example, the same authors in a series of papers they proved the Tits alternative for $Out(F_n)$, namely they proved that any subgroup of $Out(F_n)$ either contains a f.g. free abelian subgroup of finite index (virtually abelian) or it contains a free group on two generators. In that series of three papers, they associate in every automorphism a finite set of attractive laminations using the relative train track representatives instead of train track representatives in order to establish the Tits alternatives ([5], [6] and [7]). Namely, they proved:

Theorem 2.16. *Every finitely generated subgroup K of $Out(F_n)$ is either virtually abelian or it contains a subgroup of rank* 2.

Later, this result generalised for any subgroup, not just for finitely generated.

2.12 Attractive fixed points in the boundary of a free group

As we have already mentioned, Bestvina and Handel (1992) proved the Scott Conjucture. Since then, this result has been generalised and has been improved in many different directions. For example, Gaboriau, Jaeger, Levitt and Lustig (in [13], 1998) used the

boundary of a finitely generated free group in order to define the infinite fixed points of an automorphism. Firstly, given a finitely generated free group F_n with basis the finite set X, we can define the boundary ∂F_n (which is a Cantor set), as the set of infinite reduced words relative to the word length corresponding to X. It can be proved that the definition does not depend on the basis. Moreover, naturally F_n can be seen as a metric space using the metric induced by the word length and every $\phi \in Aut(F_n)$ induces a quasi-isometry of F_n. As we can define a natural metric d for ∂F_n, by the formula $d(X,Y) = e^{-|X \wedge Y|}$, where by $\wedge$ we denote the largest initial segment of X and Y, we can see that every automorphism induces a homeomorphism $\partial\phi$ of the boundary ∂F_n. Note that we could define ∂F_n, as the boundary of any tree of CV_n, which is well defined up to homeomorphism.
Now for $\phi \in Aut(F_n)$, let's denote by $Fix(\phi)$ the fixed subgroup of ϕ. As we have discussed, $Fix(\phi)$ is a free group of finite rank and therefore its boundary that it can be defined similarly as above, we get that $\partial Fix(\phi)$ embeds into ∂F_n. It is not difficult to see that actually $\partial Fix(\phi)$ is contained to $Fix(\partial\phi)$. We are in position to state the lemma that give us a characterisation of the infinite fixed points of $\partial\phi$. More precisely, a point of $\partial Fix(\phi)$ is said to be singular if it belongs to $\partial Fix(\phi)$ and it is said regular otherwise. For a regular point, we say that an infinite fixed point X of $\partial\phi$ is attractive, if there is a neighbourhood U of X in F_n such that the sequences $\phi^k(y)$ converge to X for all y in U. On the other hand, an infinite fixed point X of $\partial\phi$ is called repulsive, if it is attractive for $\partial\phi^{-1}$. Then (in [13]), it is proved that an infinite fixed point of $\partial\phi$ is either *singular* or *attractive* or *repulsive*.
We could also ask that if for given $X \in \partial F_n$ can we compute the subgroup $Stab(X) = \{\psi \in Out(F_n) : \partial\psi(X) = X\}$ of $Out(F_n)$. However, this question looks like very difficult to be answered in general. But there is a partial result by Hilion for attractive fixed points of an IWIP automorphism and more specifically:

Theorem 2.17. *[17] Let ϕ be an automorphism of F_n and X be an attractive fixed point of ϕ. Then $Stab(X)$ is infinite cyclic.*

The proof of this result heavily relies on the machinery of the attractive lamination corresponding to an IWIP automorphism.

3 Free product of groups

3.1 Definitions and free products

Let G be a group and let $G_i, i \in I$ be a family of groups.

Definition 3.1. (Universal Property of Free products)
We say that G is the free product of the G_i's and we write $G = *_{i\in I} G_i$, if there are

homomorphisms $j_i : G_i \to G$ such that for every group H and for every family of homomorphisms $\phi_i : G_i \to H$, there is a unique homomorphism $\phi : G \to H$ so that $\phi j_i = \phi_i$ for every i.

$$\begin{array}{ccc} G_i & \xrightarrow{j_i} & G \\ & \searrow^{\phi_i} & \downarrow \phi \\ & & H \end{array}$$

Note that if we have presentations for each $G_i =< X_i|R_i >$ then we get a presentation of $G =< \sqcup X_i| \sqcup R_i >$, where the generators and the relations in G are just the disjoint unions of the generators and the relations of the G_i's, respectively.

We could give an alternative definition of a free product of groups. Namely, let G and G_i as above.
A *reduced word* $w = y_1 y_2 ... y_n$, $y_i \in G_j$ (not trivial) in G is a word where for every i, y_i, y_{i+1} do not belong to the same G_i.

Theorem 3.2. *(Normal form of free products)*
Let G be the free product of the groups G_i, $i \in I$. Then every element of G can be uniquely written as a reduced word. In particular, if the length of the reduced sequence for $w \in G$ is not 0, then w is not the trivial element of G.

Now let's describe the structure of the subgroups of a free product.

There is one more important theorem for subgroups of free products, the Kurosh Subgroup theorem:

Theorem 3.3. *Let $G = *_{i \in I} A_i$ be a free product of groups and H a subgroup of G. Then $H = *(H \cap A_i^g) * F$, where the g ranges over a set of double coset representatives in $A_i \backslash G/H$ for each $i \in I$ and where F is a free group.*

We can define the *Kurosh rank* of H with respect to the free product $*_{i \in I} A_i$ as the sum of the number of non-trivial factors $(H \cap A_i^g)$ and of the rank of F. Of course, it is possible that some of these numbers (or even both) is infinite. The Kurosh rank is denoted $\kappa - r_G(H)$ and it can be proved that it is independent of the double coset representatives (this was proved in Lemma 3.4 of [19]), so it is well defined, as it depends only on the subgroup.
There is also a description of a free product of groups in terms of group actions on trees. This is a special case of the very rich Bass- Serre theory, but in this thesis only the results that we use, will be referred. For much more details and proofs for the Bass-Serre Theory see [27].

Theorem 3.4. *A group G is a free product of groups if and only if G acts on a tree T with trivial edge stabilisers.*

Actually, every free factor G_i corresponds to a vertex stabiliser and conversely every vertex stabiliser is conjugate to some G_i.

3.2 Automorphisms of a free product

Let G be a finitely generated group, then it is well known (Grushko's theorem) that G can be written as a free product, $G = G_1 * ... * G_p * F_k$, where we denote by F_k the free group of rank k and where every G_i is a non-trivial freely indecomposable group which is not isomorphic to $\mathbb{Z}$. Moreover, the G_i's in this decomposition are unique up to conjugation and the free rank is well defined. Namely, if $G = H_1 * ... * H_r * F_m$ is another such decomposition, the number of factors is the same, $m = k$, $p = r$ and (after reordering) each H_i is conjugate in G to G_i. The number $r + m$ is called the Kurosh rank of G. This decomposition is called Grushko's decomposition. In this case, we have also that for every automorphism ψ of G, it holds that for every i there exists some j_i such that $\psi(G_i) = g_i G_{j_i} g_i^{-1}$. Actually, all the $Out(G_i)$'s and $Out(F_n)$ can be seen as subgroups of $Out(G)$, but actually the study of $Out(G)$ is much more complicated than the study of these subgroups.
Firstly, let us describe a generating set for $Out(G)$. Let's denote by X a free basis of F_k. We need the permutations of isomorphic factors, i.e. automorphisms that send each G_i to some isomorphic G_j or permute the set $X \cup X^{-1}$. In addition, we need the factor automorphisms, which are the automorphisms that are induced by an automorphism of some free factor G_i. Finally there are the Whitehead automorphisms which are the automorphisms of the form α_x, where x is a non-trivial element of some G_j or an element of $X \cup X^{-1}$, such that :either for every j we have that $\alpha_x(g) = x^{-k_j} g x^{k_j}$, where k_j is either 0 or 1, for every $g \in G_j$ or for every $s \in X$, $\alpha_x(s)$ is equal to on of s, sx, x^{-1} or $x^{-1}sx$. In fact, we don't need that every G_i is freely indecomposable and not isomorphic to $\mathbb{Z}$, but in that case these automorphisms generate a subgroup of $Out(G)$, and this subgroup is equal to $Out(G)$ if we further suppose that the free decomposition is the Grushko one. Finally, we can see that if every G_i and every $Out(G_i)$ is finitely generated, then this subgroup is also finitely generated.

3.3 Deformation Spaces

For this subsection, let's suppose that G is any finitely generated free group. We will associate to G a collection of spaces that each of them is a space of R-trees (and on each of these trees G acts minimally). Intuitively, we can say that each space corresponds to a collection of subgroups of G, which are exactly the subgroups of G that fix a point of every tree of this space. More precisely, let's denote by $\mathcal{T}(G)$ the set of G-equivariant homothety classes (exactly, as we did for the outer space) of G -actions on metric simplicial trees T, where we suppose that the action of G on T is non-trivial, minimal, simplicial, isometric and without inversions of edges. In addition, a subgroup H of G is called elliptic if it fixes a point in T. Note that if H is elliptic then every element of H is elliptic, but the inverse is not always true if we don't suppose that H is finitely generated.

Definition 3.5. Given a $\mathbb{R}$-tree T on which G acts as above, we can associate to T its deformation space $\mathcal{D} = \mathcal{D}(T) \subseteq \mathcal{T}(G)$, consisting of all $\mathbb{R}$-trees that have the same elliptic subgroups as T.

For example, we can see CV_n as a deformation space corresponding of the Cayley tree of the free group (for some fixed basis). In the previous chapter, we defined three topologies for CV_n. We can define the same topologies for any deformation space. In particular, the Gromov-Hausdorff topology, the axes topology and the weak topology (using the simplicial structure) can be defined in this general case. We will describe them again with more details in the case of a free product. But it is in general true that if T is locally finite, then all these topologies agree.
Also, Meinert (see [22]) studied the Lipshcitz metric (and the train track representatives for 'irreducible' automorphisms) for this general space.

3.4 Relative Outer Space for a free product

The definitions and the construction below are due to Vincent Guirardel and Gilbert Levitt, see [16].
In the case of a free group, we have described methods that apply for finitely generated free groups, here we are interested for groups with finite Kurosh rank, this is equivalent to the fact that the group admits a co-compact action on a $\mathbb{R}$-tree T with trivial edge stabilisers and indecomposable vertex stabilisers. For a group G we fix a decomposition $G = H_1 * ... * H_q * F_m$ (and we keep the notation for the rest of this chapter), such that G has finite Kurosh rank (however, it is not necessary to assume that the H_i's are not isomorphic to $\mathbb{Z}$).
As a special case, of particular interest to us, is the so-called deformation space $\mathcal{O} = \mathcal{O}(G, \{H_i\}_{i=1}^q, F_m)$ corresponding to a G-action to some $\mathbb{R}$-tree with vertex stabilisers conjugates to the H_i's and trivial edge stabilisers. Such a tree always exist, by Bass-Serre theory (Bass-Serre Tree, see [27] for more details).
However, for completeness, let's give the definition of $\mathcal{O}$, as a space o marked metric graph of groups, even if in this case it is more convenient for us to use the alternative definition.
We define a point of $\mathcal{O}$ as the equivalence class (up to homothety) of a finite graph of groups Γ with trivial edge groups such that:

- each edge is assigned a positive number (length).

- the marking, in this case, is an isomorphism from G to $\pi_1(\Gamma)$ (up to composition with an inner automorphism of G)
- for each $i \in \{1, ..., p\}$ there is a vertex v_i such that the vertex group G_{v_i} is conjugate to H_i in G, while all the other vertex groups are trivial
- every terminal vertex v (i.e. $\Gamma - \{v\}$ is connected) is equal to some v_i.

As in the free case, we can normalise and we can think that the sum of the lengths of all the edges of Γ is 1.
Now let us describe the action. In order the action to be well defined, we would like the automorphisms to respect the structure of the graph of groups, or equivalently the free product decomposition. However, this is not in general true, so we need to consider the subgroup $Aut(G, \mathcal{O})$ of automorphisms that preserve the set of conjugacy classes of the H_j's. More precisely, an automorphism a of G belongs to $Aut(G, \mathcal{O})$ if $a(H_i)$ is conjugate to one of the H_j's (possibly, different). Note that in the case of the Grushko decomposition $Aut(G) = Aut(G, \mathcal{O})$. We can define a natural action for every automorphism ϕ of $Aut(G, \mathcal{O})$, by leaving the underlying metric graph invariant and by changing the marking, exactly as in the free case. Also, since $Inn(G)$ acts trivially on $\mathcal{O}$, this action induces an action of $Out(G, \mathcal{O})$ on $\mathcal{O}$.

3.5 Definition as space of $\mathbb{R}$-trees

As in the classical case, there is a definition of the outer space via $\mathbb{R}$-trees. Note that in this case is easier to work with this definition. As we have already mentioned, G can be seen as a finite graph of groups with fundamental group isomorphic to F_m, trivial edge groups and vertex groups either trivial or isomorphic to some of the H_i's. Also, G acts on the Bass-Serre tree T corresponding to this graph of groups. The relative outer space $\mathcal{O}$ could be alternatively defined as the deformation space which can be associated to T. In order to summarise, we can see a point of $\mathcal{O} = \mathcal{O}(G, (H_i)_{i=1}^q, F_m)$ as an equivalence class up to equivariant isometry, of a simplicial, metric G-tree T, satisfying that:

- the G-action of T is minimal, with trivial edge stabiliser
- for each $i = 1, ..., q$ there is exactly one orbit of vertices with stabiliser conjugate to H_i, and they are called non-free vertices
- all the other vertices have trivial stabiliser and they are called free vertices.

Note that minimality of the G-action on T implies that every terminal vertex is a v_i (for some i) and under these assumptions, we have also that the action is co-compact.
As an application of the Bass-Serre theory, the two definitions are actually equivalent, as the quotient G/T can be seen as a marked metric graph of groups exactly as in the

first definition.
We are in position to define the action of $Out(G, \mathcal{O})$. As in the first definition we have to consider only the automorphisms that preserve the given free product decomposition of G. Namely, for $a \in Aut(G, \mathcal{O})$ and $T \in \mathcal{T}(G)$, the image of T under a is the G-tree with the same underlying metric tree as T, but the action is given by the rule $g \cdot_{a(T)} x := \phi(g) \cdot_T x$. As before, $Inn(G)$ acts trivially on $\mathcal{O}$ and so $Out(G, \mathcal{O}) = Aut(G, \mathcal{O})/Inn(G)$ naturally acts on $\mathcal{O}$. Only in the case of the Grushko decomposition of G, there is a natural action of $Out(G)$ on $\mathcal{O}$.

3.5.1 Simplicial Stucture and Topologies for $\mathcal{O}$

In the free case, we have seen that there are three natural topologies (weak topology, Gromov-Hausdorff and axes topology) which coincide. These topologies can be naturally defined for the general space, but here they are not all the same because the trees may be not locally finite. More generally, we can define these topologies for a general deformation space and the topologies coincide only if the deformation space is locally finite (see [15] and [22], for more details). Note that the elements of $\mathcal{O}$ are locally finite trees if and only if all the H_i's are finite.
Firstly, we have seen in the previous chapter that we can always define the Gromov - Hausdorff topology for $\mathcal{T}(G)$, which immediately induces a topology for $\mathcal{O}$.
Now let's describe the axes topology for $\mathcal{O}$. There is a natural map from the space $\mathcal{T}(G)$ to $\mathbb{R}^G$, mapping some G-tree T to $(\ell_T(g))_{g \in G}$, where by ℓ_T we denote the translation function. It is easy to see that if there is a G- equivariant isometry between two elements T, S of $\mathcal{T}(G)$, then they have the same image under the previous map. Therefore we can see that the restriction of this map factors through $\mathcal{O}$ and in fact it is true (by [10], theorem [3.7]) that this restriction is injective. Therefore $\mathcal{O}$ can be naturally seen as a subspace of $\mathbb{R}^G$ and so it carries the subspace topology. We usually call this topology, the axes topology, as in the free case.
Finally, we can describe a simplicial structure for $\mathcal{O}$, exactly as in the free case. Every marked graph of groups Γ can be seen as an open simplex, by ignoring the metric, of Γ, consisting of all marked metric graphs of groups with underlying graph given by Γ and the same marking, and where the metric varies over all the (positive) lengths of the (finitely many) edges, subject to the constraint that the volume remains equal to 1. More precisely, let $e_1, ..., e_k$ be the labelled edges of Γ then the set X of all marked metric graphs which have the same underlying graph and the same marking as Γ, while the metric varies as above, can be embedded in $\mathbb{R}^k$ using the map $\Gamma \to (L_\Gamma(e_1), ..., L_\Gamma(e_k))$, and actually Γ coincides with the open $k-1$ simplex of $\mathbb{R}^k$, $\{(x_1, ..., x_k) | x_i > 0, \sum x_i = 1\}$. As in the free case, this method implies that there is a natural subdivision of $\mathcal{O}$ into open simplices. The induced topology by the simplicial structure is called the weak topology. As in the free case, this complex is not a simplicial complex as there are missing faces,

i.e. faces that correspond to graph of groups that don't correspond to this free product decomposition but probably some other.
However these topologies don't coincide in general. Unlike CV_n the complex $\mathcal{O}$ is usually not locally finite (because there are trees in this space with vertices of infinite valence, here it is interesting to note that these vertices must be non-free vertices so in the orbit of some v_i, since the covering map between the tree and the graph of groups is locally homeomorphism on free vertices), and the weak topology is different from the equivariant Gromov-Hausdorff topology and axes topology (which coincide).
However, it can be proved([16]) that $\mathcal{O}$ is contractible on each of these topologies. Actually, this can be extended for any deformation space. (see [15]).

3.6 Lipschitz Metric

In this and in the next section, we follow the paper of S. Francaviglia and A. Martino, [12], which they study the Lipscitz metric for a general outer space. This can also be generalised for deformation spaces ([22]). In this report, we see the space $\mathcal{O}$, as the space of trees as it is easier to work using this definition.
We can define a metric for $\mathcal{O}$ analogous to the metric for CV_n, but we will use trees instead of marked metric graphs. In fact, we can do the same in the free case by considering the universal covers and the lifts of the corresponding maps. The main difference here is that in a general G-tree, there are non-trivial elliptic elements and in particular there are elements with translation length 0. As a consequence, in the definition we cannot take the supremum of stretching factors over all the non-trivial elements, but just for the hyperbolic elements of a given tree T. Since the elliptic subgroups are the same for every $S \in \mathcal{O}$, that means that the subset of hyperbolic elements depends only on $\mathcal{O}$. This is actually a direct generalisation of the method of the free case, as in CV_n the F_n-action is free and so all (non-trivial) elements are hyperbolic.
Let T be a G-tree and let's denote by $Hyp(T), Ell(T)$ the sets of hyperbolic and elliptic elements $g \in G$, respectively. If $T \in \mathcal{O}$ and $g \in Ell(T)$, then g fixes some vertex of T, then by definition there is $i \in \{1, ..., q\}$ such that $g \in g_i H_i g_i^{-1}$ and, conversely, if g lies in a G-conjugate of some H_i, $i \in \{1, ..., q\}$, that means that g fixes $g_i v_i$, which means that it is not hyperbolic (it is elliptic). Therefore, $g \in G$ is hyperbolic with respect to $T \in \mathcal{O}$ if and only if g is hyperbolic with respect to $S \in \mathcal{O}$. The set of hyperbolic elements of G for some (and hence for all) T in $\mathcal{O}$ is denoted by $Hyp(\mathcal{O})$.
Now we are in position to define the stretching factors and the Lipschitz metric. For any pair $A, B \in \mathcal{O}$ we define the right and left maximal stretching factors and the asymmetric (pseudo-) metric

$$\Lambda_R(A, B) = \sup_{g \in Hyp(\mathcal{O})} \frac{l_B(g)}{l_A(g)}$$

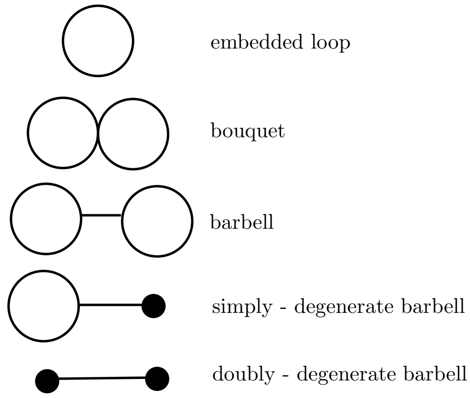

FIGURE 1: Projections of candidates

$$d_R(A,B) = \ln \Lambda_R(A,B)$$

Then if we are restricted to the space of normalised trees (with co-volume 1) d_R is a well defined asymmetric metric. As in the free case, the supremum is actually maximum. Similarly, the symmetric version of this metric can be defined by the rule

$$d(A,B) = d_R(A,B) + d_R(B,A)$$

It can be proved that d is actually a metric for $\mathcal{O}$, for more details see [12].
As for the induced metric topology it can be proved that it is the same as the axes topology. So we have actually two, probably, different topologies for $\mathcal{O}$.
We could also define the metric using the Lipschitz constants of all G- equivariant Lipschitz maps that preserve the structure (i.e. they permute the finitely many orbits of non-free vertices). There are always such maps and they are called $\mathcal{O}$-maps.([12]). For $A, B \in \mathcal{O}$, we could also define:

$$\Lambda_R(A,B) = \ln \inf_f \sigma(f) \tag{4}$$

where f varies over all the $\mathcal{O}$-maps between A and B.
One important remark here, is that even if all the notions can be generalised for the general case, some of them have much more complicated proofs due to the fact that the space may not be locally compact (because the trees may have some non-free vertices with infinite valence). For example, the infimum above is achieved by some $\mathcal{O}$-map, but the proof of this fact is much more complicated. In fact, here the Arzelá-Ascoli does not apply as $\mathcal{O}$ is not co-compact.
Also, there is a theorem which is analogous to the sausage lemma:

Theorem 3.6. *Let $A, B \in \mathcal{O}$. Then the minimal stretching factor $\Lambda_R(A,B)$ is realized by an element g such that the projection of $axis_A(g)$ to A/G is of the form:*

1. *Embedded simple loop*

2. *embedded figure-eight (a bouquet of two copies of S^1)*

3. *embedded barbel (two simple loops joined by a segment)*

4. *embedded singly degenerate barbell (a non-free vertex and a simple loop joined by a segment)*

5. *embedded doubly degenerate barbell (two non-free vertices joined by a segment)*

The loops and segments above may contain free and non-free vertices.

3.7 Train Track maps

It would be useful to have a notion of train track maps that represents an outer automorphism for free products as in the free case. Indeed, Francaviglia and Martino in [12], they constructed train track representatives and they proved that 'irreducible' automorphisms can be represented by a such a map. Again, we use the trees instead of the graphs. The topological representatives of an automorphism ϕ can be defined as an $\mathcal{O}$-map between the trees T and $\phi(T)$. More precisely,

Definition 3.7. Let $T \in \mathcal{O}$ and $\phi \in Out(G, \mathcal{O})$, then we say that a Lipschitz surjective map $f : T \to T$ represents ϕ, if for any $g \in G$ and $t \in T$ we have $f(gt) = \phi(g)(f(t))$.

In general, there are $\mathcal{O}$-maps between any two elements $T, S \in \mathcal{O}$. In particular, there are topological representatives of $\phi \in Out(G, \mathcal{O})$ in $T \in \mathcal{O}$ for any ϕ and every T. Moreover, we can choose them to be piecewise linear, as in the free case. We will assume that all the $\mathcal{O}$-maps are PL without any further mention, exactly as in the free case. Now we can define train track representatives of an automorphism:

Definition 3.8. Let $\phi \in Out(G, \mathcal{O})$. Then a topological representative $f : T \to T$ of ϕ is a train track representative, if for every edge $e \in T$ there is no cancellation in $f^k(e)$ for every k.

We could give a more general definition using the notion of a train track structure, but here we want to explain the main idea, even if we lose some technical details. As in the free case, it's not possible to have always train track representatives. We need also a notion of irreducibility:

Definition 3.9. We say $\phi \in Out(G, \mathcal{O})$ is $\mathcal{O}$-irreducible (or simply irreducible) if for any $T \in \mathcal{O}$ and for any $f : T \to T$ representing ϕ, if $W \subseteq T$ is a proper f-invariant subgraph then W/G is a union of trees each of which contains at most one non-free vertex.

Alternatively, we can define the irreducibility algebraically. We need the notion of free factor systems. In particular, a (finite) free factor system of G that is induced by a free factor decomposition $G = G_1 * ... * G_m * F_r$ is the set $\mathcal{G} = \{gG_jg^{-1} : g \in G\}$. We can compare two free factor systems by setting $\mathcal{G} \sqsubseteq \mathcal{H}$ if and only if for every G_i there is some H_j such that $G_i \leq gH_jg^{-1}$ for some $g \in G$. The minimal free factor decomposition is the Grushko free factor system and the maximal is the trivial free factor system $\mathcal{G} = \{G\}$. Also, for $\phi \in Out(G)$, a free factor system $\mathcal{G}$ is ϕ- invariant if ϕ permutes the conjugacy classes of the G_j's.
Every space $\mathcal{O}$, corresponds to some free factor system $\mathcal{G}$. Then ϕ is irreducible relative to $\mathcal{O}$, if $\mathcal{G}$ is a non-trivial ϕ-invariant free factor system. Moreover, ϕ is IWIP if ϕ^k is irreducible for every k. As in the free case every IWIP automorphism ϕ relative to $\mathcal{O}$, can be represented by a train track map. In fact, the representative can be chosen to be simplicial sending vertices to vertices, see [12] for much more details.

3.8 Tits Alternatives for free products

As we have seen Bestvina, Feighn and Handel proved a stronger version of the Tits alternatives for finitely generated subgroups of $Out(F_n)$, using the notion of the lamination lamination. The Tits alternative can be generalised for general free products, however in the classical form. More precisely, the classical Tits alternative is defined as:

Definition 3.10. We say that a group G satisfies the Tits alternative, if every subgroup of G either is virtually solvable or it contains a free subgroup of rank 2.

In this case, if we suppose that every G_i and every $Out(G_i)$ satisfies the Tits Alternative, then we get the theorem below:

Theorem 3.11 (Horbez, [18])**.** *Let G be a finitely generated group, and let $G = G_1 * ... * G_q * F_r$ be the Grushko decomposition of G. Let's suppose that for every $i = 1, ..., q$, both G_i and $Out(G_i)$ satisfy the Tits alternative. Then $Out(G)$ satisfies the Tits alternative.*

However, Horbez in his proof uses the actions of $Out(G)$ on different complexes (outer space and hyperbolic complexes) and their boundaries in order to establish this theorem. He does not generalise the train track and the laminations machineries that were used in the free case.

4 Methods and Results

In the previous chapters we have introduced the basic notions about the outer space of a free group and general deformation spaces (and especially the relative outer space corresponding to a fixed free product decomposition) of a group. In particular, as we have seen there are a lot of similarities between the two cases and we have a lot of common tools which are very useful (for example, train track maps and the Lipschitz metric). My research is focused on proving more similarities between these spaces. Moreover, these results for the spaces imply a lot of group theoretic properties of the outer automorphism group and individual automorphisms of a general free product, but also relative results for the classical case of a free group, as we can suppose that every free factor is a finitely generated free group.

For the sections below, we fix the notation below. Let $G = G_1 * ... * G_q * F_k$ be an arbitrarily non-trivial free product decomposition of G and then we denote by $\mathcal{O}$ the relative outer space corresponding to this free product decomposition. We will present the results of the following papers: [29], [30], [31].

4.1 Irreducible Lamination of an IWIP automorphism and Centralisers

In this section, we present the results of [30]. As we have seen in the second chapter, for every irreducible with irreducible power (IWIP) outer automorphism ϕ of F_n, we can construct an attractive lamination associated to ϕ. In ([30]), we generalise this construction for a relative IWIP automorphism of a general free product (of finite Kurosh rank). We use exactly the same method, but the presence of the outer automorphism groups if the free factors does not allow us to get the same result for the stabiliser of the lamination. In fact, there are examples which show that we cannot expect that this stabiliser is virtually infinite cyclic. However, after some natural assumptions (for example, if every G_i is finite) we can get the same result.

More precisely, let ϕ be an IWIP automorphism, then we can associate to ϕ a collection of lines, which is called attractive lamination and it can be constructed as in the free case using some train track representative $f : T \to T$ of ϕ, where $T \in \mathcal{O}$. Firstly, we can define the lamination in T-coordinates as the equivalence class $\Lambda_f(T)$ corresponding to the line ℓ that is defined by iterating the image of some edge e which contains some fixed point in the interior (we can always find such an edge), in fact, we get ℓ as the limit of large iterates $f^k(e)$ of f (which coincides with a pair of two distinct elements of the boundary of T). Then it can be proved that $\Lambda_f(T)$ does not depend on the chosen train track representative (every train track representative induces a unique such line, up to the equivalence relation). Since there are (optimal) $\mathcal{O}$-maps between any two elements T, S of $\mathcal{O}$, and namely if $f : T \to S$ is an optimal $\mathcal{O}$-map then we can define the lamination $\Lambda_f(S)$ in S- coordinates, as the reduced image $[f(\ell)]$ of the bi-infinite line ℓ, which we have already been defined in T-coordinates. Moreover, we can prove, that if there is a train track representative $h : S \to S$ of ϕ in S-coordinates, then $\Lambda_f(S) = \Lambda_h(S)$. Therefore we can define the set Λ_ϕ as the collection of all the laminations $\Lambda_f(A)$, where $A \in \mathcal{O}$. Here we cannot always define the action of $Out(G)$ on the set of all the attractive laminations $\mathcal{L} = \{\Lambda_\psi : \psi \text{ is IWIP relative to } \mathcal{O}\}$, but we can define an action of $Out(G, \mathcal{O})$ on $\mathcal{L}$, by the formula $\psi\Lambda_\phi = \Lambda_{\psi\phi\psi^{-1}}$. Actually, we would like to study the stabiliser $Stab(\Lambda_\phi)$ of the lamination $\Lambda_\phi = \Lambda$ of a given IWIP automorphism ϕ, which by definition it contains the (relative) centraliser of ϕ. The most important step for our result is to construct a stretching homomorphism from $Stab(\Lambda_\phi)$ to the positive real numbers. Assuming that $\psi \in Stab(\Lambda_\phi)$ and let $h : S \to S$ be a topological representative of ψ, the homomorphism can be constructed as the limit of the fraction of the lengths of the reduced image $[h(L)]$ of $h(L)$ over the length of L, when the length of the subpath L of ℓ is going to infinity. This give us a well defined unique number that does depend only on ψ and we can prove that the induced map is a homomorphism . As the next step, we can prove that the image of this homomorphism is discrete (using again the Perron - Frobenius theory), which means that we have a homomorphism from the stabiliser of the lamination $Stab(\Lambda_\phi)$ to $\mathbb{Z}$. Finally, we have to study the kernel of this homomorphism. In this step, there is a difference between the free and the general case. In our case, it's not always true that the automorphisms of the kernel have finite order. However, we can distinguish cases for the reducible and the irreducible case and for both cases it can be proved that they every automorphism of the kernel has a topological representative $f : T \to T$ which induces the identity on the quotient G/T for some $T \in \mathcal{O}$. This implies the following theorem:

Theorem 4.1.

There is a normal periodic subgroup A (i.e. all the elements of A have finite order) of $Stab(\Lambda)$, such that the group $Stab(\Lambda)/A$ has a normal subgroup $B = B_1/A$ isomorphic to subgroup of $\bigoplus_{i=1}^{q} Out(G_i)$ and $(Stab(\Lambda)/A)/B$ is isomorphic to $\mathbb{Z}$.

Moreover, in the case where $Out(G)$ is virtually torsion free, the theorem above can be improved and namely:

Theorem 4.2. *Let's also suppose that $Out(G)$ is virtually torsion free. Then $Stab(\Lambda)$ has a (torsion free) finite index subgroup K such that K/B' is isomorphic to $\mathbb{Z}$, where B' is a normal subgroup of K isomorphic to subgroup of $\bigoplus_{i=1}^{q} Out(G_i)$.*

Finally, if we also assume that every $Out(G_i)$ is finite then we have as an immediate corollary a generalisation of the original result:

Theorem 4.3. *If $Out(G)$ is virtually torsion free and every $Out(G_i)$ is finite, then $Stab(\Lambda)$ is virtually (infinite) cyclic.*

In fact, the results above give us as corollaries similar group-theoretic properties of the (relative) centraliser $C(\phi)$ of an IWIP automorphism ϕ in $Out(G, \mathcal{O})$. More precisely, we can summarise these results in the theorem above:

Theorem 4.4. *Let ϕ be an IWIP relative to $\mathcal{O}$. Then:*

1. *There is a normal periodic subgroup* A_1 *of* $C(\phi)$*, such that the group* $C(\phi)/A_1$ *has a normal subgroup* B_1 *isomorphic to a subgroup of* $\bigoplus_{i=1}^{q} Out(G_i)$ *and* $(C(\phi)/A_1)/B_1$ *is isomorphic to* $\mathbb{Z}$*.*

2. *Let's also suppose that* $Out(G)$ *is virtually torsion free. Then* $C(\phi)$ *has a (torsion free) finite index subgroup* K_1' *such that* K_1'/B_1' *is isomorphic to* $\mathbb{Z}$*, where* B_1' *is a normal subgroup of* K_1' *isomorphic to subgroup of* $\bigoplus_{i=1}^{q} Out(G_i)$*.*

3. *Finally, if we further suppose that every* $Out(G_i)$ *is finite, then* $C(\phi)$ *is virtually (infinite) cyclic.*

Finally, we give an example that we show that we cannot expect that the centraliser of an IWIP automorphism is virtually cyclic, as it can contain very big subgroups.

Example 4. *We fix the free product decomposition* $F_n = G_1 * < b_1 > * < b_2 >$*, where* b_i *are of infinite order and we denote by* $F_2 =< b_1 > * < b_2 >$ *the 'free part' and we consider the relative outer space* $\mathcal{O}(F_n, G_1, F_2) = \mathcal{O}$*. Then we define the outer automorphism* ϕ*, which satisfies* $\phi(a) = a$ *for every* $a \in G_1$*,* $\phi(b_1) = b_2 g_1, \phi(b_2) = b_1 b_2$ *for some non trivial element* $g_1 \in G_1$*, then we can see that* $\phi \in Out(G, \mathcal{O})$ *is an IWIP relative to* $\mathcal{O}$*. But in this case every factor automorphism of* G_1 *that fixes* g_1 *commutes with* ϕ *and therefore* $C(\phi)$ *contains the subgroup* A *of* $Aut(G_1)Inn(G)$ *that fixes* g_1*. So if* A *is sufficiently big, the relative centraliser is not virtually cyclic. Since we can change* G_1 *with any group (of finite Kurosh rank) and we can get automorphisms with arbitrarily big centralisers. For example, if* G_1 *is isomorphic to* F_3 *and* g_1 *an element of its free basis, we have that* $C(\phi)$ *contains a subgroup which is isomorphic to* $Aut(F_2)Inn(G)$*.*

4.2 Stabiliser of an Attactive Fixed Point of an IWIP automorphism

In this section, we describe the results of [31].

The main result is a generalisation of the Theorem 2.17. Firstly, we need some notion of boundary of the group. There are a lot of notions of boundary that could be used, but as we study the space $\mathcal{O}$ the most natural way is to use the (relative to the free factor decomposition) boundary of any tree T of $\mathcal{O}$ which is well defined up to homomorphism. Alternatively, we could define the boundary, relative to the free product decomposition, as the set of infinite words with respect to the free product length.

More precisely, for every $T \in \mathcal{O}$, we can define the boundary $\partial_x T$ as the set of lines passing through a (fixed) base point $x \in T$. Firstly, it can be proved that $\partial_x T$ is independent of x, so we can omit x in the notation. Also, we can define a natural topology of the boundary. Let $p, q \in V(T) \cup \partial T$, we define the operation $\wedge$ as follows: $p \wedge q$ is the common initial subpath (starting from x) of the unique edge paths $[x, p], [x, q]$ that connect p, q with the base point x. We define the r neighbourhood of a point p in the boundary ∂T, as $V(p, r) = \{q \in \partial T|$ for any geodesic rays ℓ_1, ℓ_2 starting at x, with

$\ell_1 = p, \ell_2 = q$ we have $\liminf_{n\to\infty} |\ell_1(n) \wedge \ell_2(n)| \geq r\}$, where $|[p,x]| := d_T(p,x)$.
A topology for ∂T can be defined by setting the collection $\{V(p,r)|r \geq 0\}$ as a basis of neighbourhoods for any $p \in \partial T$. Moreover, this topology is metrisable and in particular, the metric for ∂T is given by $d(p,q) = e^{-|[x,p]\wedge[x,q]|}$ for $p, q \in \partial T$ (where we set $e^{-\infty} = 0$). Actually, it can be proved that any quasi-isometry $f : T \to S$, induces a homeomorphism between the boundaries $\partial T, \partial S$. In particular, since any $\mathcal{O}$-map $f : T \to S$ is a quasi-isometry, it can be extended to the boundary and so it induces a well defined homeomorphism between $\partial T, \partial S$, which we denote by $\partial f : \partial T \to \partial S$. Therefore it holds that:

Lemma 4.5. *Let $T, S \in \mathcal{O}$. Then ∂T is homeomorpic to ∂S.*

Note that in our case, if there is some G_i which is infinite, then ∂T is not compact, as it is in the free case (and more generally in the locally finite case), with respect to the metric topology. For example, if it is easy to see that a point of infinite valence, induces a sequence of (distinct) lines that they have constant distance between each other. Therefore we have a sequence in ∂T, which has no converging subsequence.
Alternatively, we can define $\partial G = \partial(G, \{G_1, ..., G_q\})$ as the set of infinite reduced words with respect to the free product length which is induced by the (fixed) free product decomposition. For any $A, B \in G \cup \partial(G, \{G_1, ..., G_q\})$, we denote by $A \wedge B$ the longest common initial subword of A, B. It is easy to see that the map $d(A, B) = e^{-|A\wedge B|}$, for $A \neq B$ and $d(A, A) = 0$ is a metric on the space $G \cup \partial G$. Similarly, G can be seen as a metric space using the free product length. Let's denote by X a basis of the free part F_k, then it is well known that every element g of G is written uniquely in the normal form, as a reduced word $g_1...g_n$ where $g_i \in G_j \cup X \cup X^{-1}$and every g_i, g_{i+1} do not belong to the same factor G_j. where the free product length of a word in G is the length of the unique normal form. Finally, it can be proved that any $\phi \in Aut(G, \mathcal{O})$ induces a quasi-isometry of G, and therefore exactly as in the free case it induces a homeomorphism of the boundary $\partial(G, \{G_1, ..., G_q\})$ which we denote by $\partial\phi$. As we have said, it can be proved that the two notions of the boundary are equivalent and more specifically:

Lemma 4.6. *Let $T \in \mathcal{O}$. Then ∂T is homeomorphic to $\partial(G, \{G_1, ..., G_q\})$.*

As we have a well defined notion of the relative boundary, it is possible to define the notion of infinite fixed points for an automorphism of $Aut(G, \mathcal{O})$. As in the free case, there is a classification of these fixed points. More precisely, for an automorphism $\phi \in Aut(G, \mathcal{O})$, as in the free case there is a classification of these points. Martino studied the attractive fixed points, and in particular he proved in his phd thesis ([23]), that:

Proposition 4.7. *Let $\phi \in Aut(G, \mathcal{O})$. A fixed point of $\partial\phi$ is :*

- *either singular*

- *or attractive*
- *or repulsive*

Now as we have seen, every notion in the theorem of Hilion can be generalised in the general case.
Our proof is similar to that of ([17]), but we use the generalisations of the results that were used by Hilion. As we mentioned, the main tool that he used in his proof is the attractive lamination associated to an IWIP automorphism by Bestvina, Feighn and Handel. In our proof, we use the constructions and the results presented in the previous subsection. The main step of the proof is the construction of a nice splitting of a given attractive fixed point X of an IWIP outer automorphism ϕ, using the notion of train track representatives, which matches the language of the attractive lamination. Then it's not difficult to relate the subgroup $Stab(X) = \{\psi \in Aut(G, \mathcal{O}) | \partial\psi(X) = X\}$ to the stabiliser of the attractive lamination $Stab(\Lambda_\phi)$. Then by applying 4.1 and a technical lemma, we conclude that:

Theorem 4.8. *If $X \in \partial(G, \{G_1, ..., G_q\})$ is an attractive fixed point of an IWIP automorphism ϕ, then $Stab(X)$ has a normal subgroup B isomorphic to a subgroup of $\bigoplus_{i=1}^{p} Out(G_i)$ and $Stab(X)/B$ is infinite cyclic.*

In our case, there are examples of X, as above, where $Stab(X)$ is not infinite cyclic. We can use essentially the same example as in the previous section. The main idea to construct a counter-example, is that there are attractive fixed words corresponding to an IWIP automorphism that contain even finitely many elements of the elliptic free factors. Moreover, if a composition of factor automorphisms of the G_i's fixes these words, then it stabilises the attractive fixed point. So if some G_i is sufficiently big, there are non-trivial automorphisms of G_i that fix these words. As a consequence, we can find arbitrarily large subgroups of $Stab(X)$. On the other hand, if we suppose that every $Out(G_i)$ is finite, then we have a similar result as in the free case. In particular, as an immediate corollary we get:

Corollary 4.9. *If $X \in \partial(G, \{G_1, ..., G_q\})$ is an attractive fixed point of an IWIP automorphism ϕ and every $Out(G_i)$ is finite, then $Stab(X)$ is virtually cyclic.*

4.3 Asymmetry of Outer space of a free product

Finally in this section, we will describe the results of [29].
As we have mentioned in the second chapter, the Lipschitz metric d_R for CV_n is asymmetric and actually it does not induce a quasi-isometry in general. But Algom-Kfir and Bestvina proved that there is a function that bounds this asymmetry. This result has interesting applications especially for IWIP outer automorphisms of F_n. Therefore it

seems interesting to see if this result can be generalised in the general case.
Firstly, note that in every non-trivial case the Lipschitz metric for the space $\mathcal{O}$ is still highly asymmetric. In fact, in [29] we follow their approach and we generalise their construction. For any $T \in \mathcal{O}$ we get an open simplex Σ_T and for every $\ell \in \Sigma_T$, we define the tangent space $T_\ell(\Sigma_T) = \Big\{\tau : E(T) \to \mathbb{R} | \sum_{e \in E(T)} \tau(e) = 0\Big\}$, where $E(T)$ is the finite set of orbits of edges in T. Initially we introduce an asymmetric Finsler norm on the tangent space of the relative Outer space that induces the asymmetric Lipschitz metric. This is the first attempt, but then we need to correct this norm in order to make it quasi-symmetric and this is the second step for our proof. As soon as we have the corrected norm, we can get our main result which explains the lack of quasi-symmetry in terms of a certain function and more specifically:

Theorem 4.10. *There is an $Out(G, \mathcal{O})$-invariant continuous, piecewise smooth function $\Psi : \mathcal{O} \to \mathbb{R}^+$ and constants $A, B > 0$ (depending only on the numbers r, q) such that for every $T, S \in \mathcal{O}$ we have $d_R(T, S) \leq A \cdot d_R(S, T) + B \cdot [\Psi(T) - \Psi(S)]$.*

As an application, we can prove that if we restrict the asymmetric metric d_R to the ϵ-thick part of the relative Outer space for $\epsilon > 0$, (which can be defined as in the free case, and namely, $\mathcal{O}(\epsilon)$ is the subspace of $\mathcal{O}$ of the points for which all hyperbolic elements have length bounded below by ϵ) then d_R is quasi-symmetric.

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Irreducible laminations for IWIP Automorphisms of a free product and Centralisers

Dionysios Syrigos

Abstract

For every free product decomposition $G = G_1 * ... * G_q * F_r$, where F_r is a finitely generated free group, of a group G of finite Kurosh rank, we can associate some (relative) outer space $\mathcal{O}$. In this paper, we develop the theory of (stable) laminations for (relative) irreducible with irreducible powers (IWIP) automorphisms. In particular, we examine the action of $Out(G, \mathcal{O}) \leq Out(G)$ (i.e. the automorphisms which preserve the set of conjugacy classes of G_i's) on the set of laminations. We generalise the theory of the attractive laminations associated to automorphisms of finitely generated free groups. The strategy is the same as in the classical case (see [1]), but some statements are slightly different because of the factor automorphisms of the G_i's.

As a corollary, we prove a generalisation of the fact that the centralisers of IWIP automorphisms are virtually cyclic. However, in our statement for the (relative) centraliser of a (relative) IWIP automorphism, the factor automorphisms of G_i's appear. As a direct corollary, if $Out(G)$ is virtually torsion free and every $Out(G_i)$ is finite, we prove that the centraliser of an IWIP is virtually cyclic. Finally we give an example which shows that we cannot expect that any centraliser of an IWIP is virtually cyclic, as in the free case.

Contents

1 Introduction

Let G be a group which splits as a free product $G = G_1 * ... * G_q * F_r$. Guirardel and Levitt in [11] constructed an outer space relative to any free product decomposition for a f.g. group and later Francaviglia and Martino in [10] noticed that the outer space $\mathcal{O} = \mathcal{O}(G, (G_i)_{i=1}^q, F_r)$ can be constructed for any group G of finite Kurosh rank. Let $Out(G, \mathcal{O})$ be the subgroup of $Out(G)$, which consists of the automorphisms which preserve the conjugacy classes of G_i's (note that in the case of the Grushko decomposition, $Out(G) = Out(G, \mathcal{O})$). We could define the notion of irreducibility using representatives of automorphisms between the elements of $\mathcal{O}$ which leave invariant subgraphs, but here it is less complicated to use the notion of free factor systems. More specifically, we say that an element $\phi \in Out(G, \mathcal{O})$ is irreducible relative to $\mathcal{O}$, if the corresponding free factor system $\mathcal{G} = \{[G_i] : 1 \leq i \leq q\}$ is a maximal proper, ϕ-invariant free factor system. Therefore we define the notion of an irreducible with irreducible powers (or simply IWIP) automorphism relative to $\mathcal{O}$, as in the special case where G is a finitely generated free group.

In this paper, we study IWIP automorphisms and in particular we show that we can define the stable (and unstable) lamination Λ associated to an IWIP, using exactly the same method as in the free case. In the classical case, it can be proved that the stabiliser of the lamination is virtually cyclic (see [1]). However, in the general case, the presence of the factor automorphisms of the G_i's, does not allow us to get the same statement and as we will see this is not true in general, but can prove the following generalisation:

Theorem 1.1. *Let ϕ be an IWIP relative to some relative outer space $\mathcal{O}$. Let's denote by $Stab(\Lambda_\phi) = Stab(\Lambda)$ the $Out(G, \mathcal{O})$ stabiliser of the stable lamination Λ.*

1. *There is a normal periodic subgroup A of $Stab(\Lambda)$, such that the group $Stab(\Lambda)/A$ has a normal subgroup B isomorphic to subgroup of $\bigoplus_{i=1}^{q} Out(G_i)$ and $(Stab(\Lambda)/A)/B$ is isomorphic to $\mathbb{Z}$.*
2. *Let's suppose that $Out(G)$ is virtually torsion free. Then $Stab(\Lambda)$ has a (torsion free) finite index subgroup K such that K/B' is isomorphic to $\mathbb{Z}$, where B' is a normal subgroup of K isomorphic to subgroup of $\bigoplus_{i=1}^{q} Out(G_i)$.*
3. *Finally, if we further suppose that every $Out(G_i)$ is finite, then $Stab(\Lambda)$ is virtually (infinite) cyclic.*

We can find the notion of laminations for a free group , in a lot of different forms and contexts in the literature and study of them implies important results (for example, see [1] [2], [3], [6], [7] and [12]), therefore it looks like interesting to generalise this notion in a more general context. In addition, further motivation is that we can find natural generalisations for a lot of facts about CV_n in the general case, for example in

[10], Francaviglia and Martino generalised a lot of tools like train track maps and the Lipschitz metric. But there are some recent papers that show we can also use further methods of studying $Out(Fn)$ for $Out(G)$ (where G is written as free product as above) such that the closure of outer space, the Tits alternatives for $Out(G)$, the hyperbolic complex corresponding to $Out(G)$ and the asymmetry of the outer space of a free product (see [14], [15], [16], [19] and [24]). Finally, the author as an application of the results of the present paper generalises a result of Hilion ([13]), about the stabiliser of attractive fixed point of an IWIP automorphism ([25]).

Given a group G and an element $g \in G$, a natural question is to study the centraliser $C(g)$ of g in G. In several classes of groups, centralisers of elements are reasonably well-understood and sometimes they are useful to the study of the group. For example, Feighn and Handel in [9] classified abelian subgroups in $Out(F_n)$ by studying centralisers of elements. Moreover, a well known result for an IWIP automorphism of a free groups (there are several proofs, see [1], [20] or [18]) states that their centralisers are virtually cyclic. Again, it's not true for a relative IWIP, but in the general case, we can obtain a generalisation of this result where the group of factor automorphisms is still appeared and namely:

Theorem 1.2. *Let ϕ be an IWIP as above. Let's denote by $C(\phi)$ the centraliser of ϕ in $Out(G, \mathcal{O})$.*

1. *There is a normal periodic subgroup A_1 of $C(\phi)$, such that the group $C(\phi)/A_1$ has a normal subgroup B_1 isomorphic to subgroup of $\bigoplus_{i=1}^{q} Out(G_i)$ and $(C(\phi)/A_1)/B_1$ is isomorphic to $\mathbb{Z}$.*
2. *Let's also suppose that $Out(G)$ is virtually torsion free. Then $C(\phi)$ has a (torsion free) finite index subgroup A'_1 such that A'_1/B_1 is isomorphic to $\mathbb{Z}$, where B_1 is a normal subgroup of A_1 isomorphic to subgroup of $\bigoplus_{i=1}^{q} Out(G_i)$.*
3. *Finally, if we further suppose that every $Out(G_i)$ is finite, then $C(\phi)$ is virtually (infinite) cyclic.*

In fact, we can get a stronger result for commensurators instead of centralisers.
Note that there are a lot of IWIP automorphisms that don't commute with the factor automorphisms of G_i's and in particular for them, the theorem above implies that their centralisers are virtually cyclic.
On the other hand, there are examples of IWIP automorphisms which have big centraliser (in particular, they are not virtually cyclic).

Example 1.3. We fix the free product decomposition $F_n = G_1 * < b_1 > * < b_2 >$, where b_i are of infinite order and we denote by $F_2 =< b_1 > * < b_2 >$ the "free part". Then in the corresponding outer space $\mathcal{O}(F_n, G_1, F_2)$, which we denote by $\mathcal{O}$. In each tree

$T \in \mathcal{O}$ there is exactly one non free vertex v_1 s.t $G_{v_1} = G_1$. Then we define the outer automorphism ϕ, which satisfies $\phi(a) = a$ for every $a \in G_1$, $\phi(b_1) = b_2 g_1, \phi(b_2) = b_1 b_2$ for some $g_1 \in G_1$, then we can see that $\phi \in Out(G, \mathcal{O})$ is an IWIP relative to $\mathcal{O}$. But then every factor automorphism of G_1 that fixes g_1 commutes with ϕ and therefore $C(\phi)$ contains the subgroup A of $Aut(G_1)Inn(G)$ that fixes g_1. So if A is sufficiently big, the relative centraliser is not virtually cyclic. Since we can change G_1 with any group (of finite Kurosh rank) and we can get automorphisms with arbitarily big centralisers. For example, if G_1 is isomorphic to F_3 and g_1 an element of its free basis, we have that $C(\phi)$ contains a subgroup which is isomorphic to $Aut(F_2)Inn(G)$.

Strategy of the proof: The paper is organized as follows:

In Section 2, we recall some preliminary definitions, facts and well known results about the outer space of a free product. In Section 3, we prove a useful technical lemma for $\mathcal{O}$-maps, more specifically we prove that every two such maps are equal except possibly two bounded (depends only on the map, not the path) paths near the endpoints. The next sections form the main part of this paper and we follow exactly the same approach as in [1]. In section 4, we define the lamination using train track representatives, and then we extend the notion to any tree. Also, we list some useful properties. In Section 5, we define the action of $Out(G, \mathcal{O})$ on the set of irreducible laminations. In Section 6 we define the notion of a subgroup which carries the lamination and then we prove that any such subgroup has finite index in the whole group. In Section 7, which is the most crucial for our arguments, we construct a homomorphism from the stabiliser to group of positive real numbers. Then in Section 8, we study the kernel of this homomorphism, and in particular, we prove that any element of the kernel is non-exponentially growing and in the reducible case it has a relative train track representative with a very good form restricted to the lower strata. Also, we prove the discreteness of the image which allows us to think the previous map, as a homomorphism from the stabiliser to the group of integers. Finally, in Section 9, we prove some useful lemmas and the main results.

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2 Preliminaries

2.1 Outer space and $\mathcal{O}$-maps

In this subsection we recall the definitions of outer space and some basic properties. For example, the existence of $\mathcal{O}$ - maps between any two elements of the space which is a very useful tool.
Everything in the present and the next subsection about the outer space, the $\mathcal{O}$ - maps and the train track representatives can be found in [10].
Let G be a group which splits as a finite free product of the following form $G = H_1 * ... * H_q * F_r$, where every H_i is non-trivial, not isomorphic to $\mathbb{Z}$ and freely indecomposable. We say that such a group has *finite Kurosh rank* and such a decomposition is called *Gruskho decomposition.* For example, every f.g. group admits a splitting as above (by the Grushko's theorem). We are interested only for groups which have finite Kurosh rank.
Now for a group G, as above, we fix an arbitary (non-trivial) free product decomposition $G = H_1 * ... * H_q * F_r$ (without the assumption that the H_i's are not isomorphic to $\mathbb{Z}$ or freely indecomposable), but we additionally suppose that $r > 0$. These groups admit co-compact actions on $\mathbb{R}$-trees (and vice-versa). It is useful that we can also apply the theory in the case that G is free, and the G_i's are certain free factors of G (relative free case).
We will define an outer space $\mathcal{O} = \mathcal{O}(G, (G_i)_{i=1}^{p}, F_r)$ relative to the free product decomposition (or relative outer space). The elements of the outer space can be thought as simplicial metric G-trees, up to G-equivariant homothety. Moreover, we require that these trees also satisfy the following:

- The action of G on T is minimal.
- The edge stabilisers are trivial.
- There are finitely many orbits of vertices with non-trivial stabiliser, more precisely for every H_i, $i = 1, ..., q$ (as above) there is exactly one vertex v_i with stabiliser H_i (all the vertices in the orbits of v_i's are called *non-free vertices*).
- All other vertices have trivial stabiliser (and we call them *free vertices*).

The quotient G/T is a finite graph of groups. We could also define the outer space as the space of "marked metric graph of groups" using the quotients instead of the trees, but we won't use this point of view because here it is easier to work using the trees. However, we use the quotients when the statements in this context are less complicated.
We would like to define a natural action of $Out(G)$ on $\mathcal{O}$, but this is not possible since it not always the case that the automorphisms preserve the structure of the trees (i.e.

they don't send non-free vertices to non-free vertices). However, we can describe here the action of a specific subgroup of $Out(G)$ (namely, the automorphisms that preserve the decomposition or equivalently the structure of the trees) on $\mathcal{O}$.
Let $Aut(G, \mathcal{O})$ be the subgroup of $Aut(G)$ that preserve the set of conjugacy classes of the G_i 's. Equivalently, $\phi \in Aut(G)$ belongs to $Aut(G, \mathcal{O})$ iff $\phi(G_i)$ is conjugate to one of the G_j 's. The group $Aut(G, \mathcal{O})$ admits a natural action on a simplicial tree by "changing the action", i.e. for $\phi \in Aut(G, \mathcal{O})$ and $T \in \mathcal{O}$, we define $\phi(T)$ to be the element with the same underlying tree with T, the same metric but the action is given by $g * x = \phi(g)x$ (where the action in the right hand side is the action of the G-tree T). Now since the set of inner automorphisms of G, $Inn(G)$ acts trivially on $\mathcal{O}$ we can define $Out(G, \mathcal{O}) = Aut(G, \mathcal{O})/Inn(G)$ which acts on $\mathcal{O}$ as above. Note that in the case of the Grushko decomposition we have $Out(G) = Out(G, \mathcal{O})$.
We say that a map between trees $A, B \in \mathcal{O}$, $f : A \to B$ is an **$\mathcal{O}$- map**, if it is a G-equivariant, Lipschitz continuous, surjective function. Note here that we denote by $Lip(f)$ the Lipschitz constant of f.
It is very useful to know that there are such maps between any two trees. This is true and, additionally, by their construction they coincide on the non - free vertices (and in section 3, we prove that every two such maps "almost" coincide). More specifically, by [10], we get:

Lemma 2.1. *For every pair $A, B \in \mathcal{O}$; there exists a $\mathcal{O}$-map $f : A \to B$. Moreover, any two $\mathcal{O}$-maps from A to B coincide on the non-free vertices.*

Let $f : A \to A$ be a simplicial (sending vertices to vertices and edges to edge-paths) $\mathcal{O}$-map, where $A \in \mathcal{O}$. Then f induces a map (here we denote by Df the map which sends every edge e to the first edge of the edge path $f(e)$) on the set of turns, sending every turn (e_1, e_2) to the turn $(Df(e_1), Df(e_2))$. Then as usually, we say that the turn (e_1, e_2) is *legal*, if for every k the turn $(Df^k(e_1), Df^k(e_2))$ is non-degenerate. This induces a pre-train track structure on the set of edges at each vertex. But there are also different pre-train track structures and one of which we will use later, therefore we need the general definition.

Definition 2.2.
1. A **pre-train track structure** on a G-tree T is a G-invariant equivalence relation on the set of germs of edges at each vertex of T. Equivalence classes of germs are called **gates**.
2. A **train track structure** on a G-tree T is a pre-train track structure with at least two gates at every vertex.
3. A **turn** is a pair of germs of edges emanating from the same vertex. A **legal turn** is called a turn for which the two germs belong to different equivalent classes. A **legal path**, is a path that contains only legal turns.

A pre-train track structure induced by some $\mathcal{O}$ - map is not always a train track structure, but there are some $\mathcal{O}$ - maps (we call them optimal maps) which induce train track structures. But firstly we need the notion of PL maps (which corresponds to piecewise linear homotopy equivalence in the free case). We call a map between two elements of the outer space **PL**, if it is piecewise linear and $\mathcal{O}$-map. We denote by $A_{max}(f)$ the subgraph of A consisting on those edges e of A for which $S_{f,e} = Lip(f)$ (i.e. the set of edges which are maximally stretched by f). Note that A_{max} is G-invariant and that in literature the set A_{max} is often referred to as *tension graph.*

As we have seen in the discussion above, for every map there is an induced structure. More specifically, if $A, B \in \mathcal{O}$ and $f : A \to B$ is a PL-map, then **the pre-train track structure induced by** f on A is defined by declaring germs of edges to be equivalent if they have the same *non-degenerate* f-image (so if two maps that are collapsed by f, they are not equivalent).

We are now in position to define optimal maps:

Definition 2.3. Let $A, B \in \mathcal{O}$. A PL-map $f : A \to B$ is not optimal at v, if A_{max} has only one gate at v for the pre-train track structure induced by f. Otherwise, f is **optimal at** v. The map f is **optimal**, if it is optimal at all vertices.

Remark. A PL-map $f : A \to B$ is optimal if and only if the pre-train track structure induced by f is a train track structure on A_{max}. In particular, if $f : A \to B$ is an optimal map, then at every vertex v of A_{max} there is a legal turn in A_{max}.

Note also that by [10], every PL-map is optimal at non-free vertices and for every $A, B \in \mathcal{O}$ there exists an optimal map from A to B. Therefore we can always choose our $\mathcal{O}$ - maps to be optimal and we will use optimal maps without further mention.

2.2 Relative Automorphisms

We denote by $Out(G, \{G_i\}^t)$ the subgroup of $Out(G, \mathcal{O})$ made of those automorphisms that act as a conjugation by an element of G on each G_i. Since the G_i's are free factors of G, each subgroup G_i is equal to its normalizer in G. Therefore, any element of $Out(G, \mathcal{O})$ (i.e. that preserves the conjugacy class of the G_i's) induces a well-defined outer automorphism of G_i. Therefore there is a natural homomorphism $Out(G, \{G_i\}^t) \to Out(G_i)$ and by taking the product over all groups G_i, we get a (surjective) homomorphism $Out(G, \mathcal{O}) \to \bigoplus_{i=1}^{p} Out(Gi)$, with kernel exactly $Out(G, \{G_i\}^t)$.

2.3 Train Track Maps and Irreducibility

In this section we will define the notion of a "good" representative of an outer automorphism. It is a generalisation of train track representatives of automorphisms of free

groups, but as we have already mentioned we work in the trees instead of their quotients. For more details for this approach see [10, 23]. As we have seen there are representatives of every outer automorphism (i.e. $\mathcal{O}$-maps from A to $\phi(A)$), but sometimes we can find representatives with better properties. These maps, which are called *train track maps*, are very useful and every irreducible automorphism has such a representative (we can choose it to be simplicial, as well).
For $T \in \mathcal{O}$ we say that a Lipschitz surjective map $f : T \to T$ **represents** ϕ if for any $g \in G$ and $t \in T$ we have $f(gt) = \phi(g)(f(t))$. (In other words, if it is an $\mathcal{O}$-map from T to $\phi(T)$.) We give below the definition of a train track map representing an outer automorphism. We are interested for these maps because we can control their cancellation (it is not possible to avoid it).

Definition 2.4. If $T \in \mathcal{O}$ then a PL-map $f : T \to T$, which representing ϕ, is a train track map if there is a train track structure on T so that

1. f maps edges to legal paths (in particular, f does not collapse edges)
2. If $f(v)$ is a vertex, then f maps inequivalent germs at v to inequivalent germs at $f(v)$.

In the free case, an automorphism ϕ is called *irreducible*, if it there is no ϕ-invariant free factor up to conjugation (or equivalently the topological representatives of ϕ haven't non-trivial proper invariant subgraphs). In our case we know that the G_i's are invariant free factors, but we don't want to have "more invariant free factors". More precisely, we will define the irreducibility of some automorphism *relative* to the space $\mathcal{O}$ or to the free product decomposition.

Definition 2.5. We say $\Phi \in Out(G, \mathcal{O})$ is *$\mathcal{O}$-irreducible* (or simply irreducible) if for any $T \in \mathcal{O}$ and for any $f : T \to T$ representing Φ, if $W \subseteq T$ is a proper f-invariant G-subgraph then G/W is a union of trees each of which contains at most one non-free vertex.

We can also give an alternative algebraic definition, but we need the notion of a free factor system. Suppose that G can be written as a free product, $G = G_1 * G_2 * ... G_p * G_\infty$. Then we say that the set $\mathcal{A} = \{[G_i] : 1 \leq i \leq p\}$ is a **free factor system for** G, where $[A] = \{gAg^{-1} : g \in G\}$ is the set of conjugates of A.
Now we define an order on the set of free factor systems for G. More specifically, given two free factor systems $\mathcal{G} = \{[G_i] : 1 \leq i \leq p\}$ and $\mathcal{H} = \{[H_j] : 1 \leq j \leq m\}$, we write $\mathcal{G} \sqsubseteq \mathcal{H}$ if for each i there exists a j such that $G_i \leq gH_jg^{-1}$ for some $g \in G$. The inclusion is strict, and we write $\mathcal{G} \sqsubset \mathcal{H}$, if some G_i is contained strictly in some conjugate of H_j. We can see $\{[G]\}$ as a free factor system and in fact, it is the maximal (under $\sqsubseteq$) free factor system. Any free factor system that is contained strictly to $\mathcal{G}$ is called **proper**. Note also that the Grushko decomposition induces a free factor system, which is actually

the minimal free factor system (relative to $\sqsubseteq$). A more detailed discussion for the theory of free factor systems can be found in [12].
We say that $\mathcal{G} = \{[G_i] : 1 \leq i \leq p\}$ is ϕ - **invariant** for some $\phi \in Out(G)$, if ϕ preserves the conjugacy classes of G_i's. We are only interested for free factor systems that G_∞ is a finitely generated free group. In particular, we suppose that $G = G_1 * G_2 * ... G_p * G_\infty$, and $G_\infty = F_k$ for some f.g. free group F_k. In each free factor system $\mathcal{G} = \{[Gi] : 1 \leq i \leq k\}$, we associate the outer space $\mathcal{O} = \mathcal{O}(G, (G_i)_{i=1}^p, F_k)$ and any $\phi \in Out(G)$ leaving $\mathcal{G}$ invariant, will act on $\mathcal{O}$ in the same way as we have described earlier.

Definition 2.6. Let $\mathcal{G}$ be a free factor system of G which is Φ- invariant for some $\Phi \in Out(G)$. Then Φ is called *irreducible relative to* $\mathcal{G}$, if $\mathcal{G}$ is a maximal (under $\sqsubseteq$) proper, Φ-invariant free factor system.

The next lemma confirms that the two definitions of irreducibility are related.

Lemma 2.7. *Suppose $\mathcal{G}$ is a free factor system of G with associated space of trees $\mathcal{O}$, and further suppose that $\mathcal{G}$ is ϕ-invariant. Then ϕ is irreducible relative to $\mathcal{G}$ if and only if ϕ is $\mathcal{O}$-irreducible.*

Moreover, one interesting fact is that for an irreducible automorphism we can give a characterisation of train track maps using the axes of hyperbolic elements. More specifically, if ϕ is irreducible, then for a map f representing $\phi \in Out(G, \mathcal{O})$, to be a train track map is equivalent to the condition that there is $g \in G$ (hyperbolic element) so that $L = axis_T(g)$ (the axis of g) is legal and $f^k(L)$ is legal $k \in \mathbb{N}$.

Now let's give the definition of an irreducible automorphism with irreducible powers relative to $\mathcal{O}$, which are the automorphisms that we will study.

Definition 2.8. An outer automorphism $\phi \in Out(G, \mathcal{O})$ is called **IWIP** (*irreducible with irreducible powers* or *fully irreducible*), if every ϕ^k is irreducible relative to $\mathcal{O}$.

The next theorem is very important since we can always choose representatives of irreducible automorphisms with nice properties, as in the free case. It generalises the well known theorem of Bestvina and Handel (see [4]) . In particular, we can apply it on every power of some $IWIP$.

Theorem 2.9 (Francaviglia- Martino)**.** *Let $\phi \in Out(G, \mathcal{O})$ be irreducible. Then there exists a (simplicial) train track map representing ϕ.*

The discussion above implies that we can always find an optimal train track representative of an irreducible $\phi \in Out(G, \mathcal{O})$. This map has the property that the image of every legal path (in particular, of edges) is stretched by a constant number $\lambda \geq 1$ which depends only on ϕ.

We close this subsection with an interesting remark.

Remark. Every outer automorphism $\phi \in Out(G)$ is irreducible relative to some appropriate space (or relative to some free product decomposition).

2.4 Bounded Cancellation Lemma

Let $T, T' \in \mathcal{O}$ and $f : T \to T'$ be an $\mathcal{O}$- map. If we have a concatenation of legal paths ab where the corresponding turn is illegal, then it is possible to have cancellation in $f(a)f(b)$. But the cancellation is bounded, with some bound that depends only on f and not on a, b . In particular, we can define the bounded cancellation constant of f (let's denote it $BCC(f)$) to be the supremum of all real numbers N with the property that there exist A, B, C some points of T with B in the (unique) reduced path between A and C such that $d_{T'}(f(B), [f(A), f(C)]) = N$ (the distance of $f(B)$ from the reduced path connecting $f(A)$ and $f(C)$), or equivalently is the lowest upper bound of the cancellation for a fixed $\mathcal{O}$-map.
The existence of such number is well known, for example a bound has given in [14]:

Lemma 2.10. *Let $T \in \mathcal{O}$, let $T' \in \mathcal{O}$, and let $f : T \to T'$ be a Lipschitz map. Then $BCC(f) \leq Lip(f)qvol(T)$, where $qvol(T)$ the quotient volume of T, defined as the infimal volume of a finite subtree of T whose G-translates cover T.*

We can also, exactly as in the free case, define a critical constant, C_{crit} corresponding to a train track map.
Let's suppose that f is train track map with expanding factor λ (for example, a train track representative of some IWIP ϕ). If we take a, b, c legal paths and abc is a path in the tree, and let's denote $l = length(b)$ the length of the middle segment. If we suppose further that satisfies $\lambda l - 2BCC(f) > l$, then iteration and tightening of abc will produce paths with the length of the legal leaf segment corresponding to b to be arbitarily long. This is equivalent to require that $l > \frac{2BCC(f)}{\lambda - 1}$, and we call the number $C_{crit} = \frac{2BCC(f)}{\lambda - 1}$, *the critical constant for f.*
For every C that exceeds the critical constant there is $m > 0$ such that b, as above, has length at least C then the length of the legal leaf segment of $[f^k(abc)]$ corresponding to b is at least $m\lambda^k length(b)$.
Therefore we can see that any path which contains a legal segment of length at least C_{crit}, has the property that the lengths of reduced f-iterates of the path are going to infinity.

2.5 N-periodic paths

A difference between the free and the general case is that it is not always the case that there are finitely many orbits of paths of a specific length (if there are non-free vertices with infinite stabiliser), but it is true that there are finitely many paths that have different projection in the quotient. Therefore the role of Nielsen periodic paths play the N-periodic paths that we define below. Note that if $h : S \to S$, we say that a point $x \in S$ is h-periodic, if there are $g \in G$ and some natural k s.t. $h^k(x) = gx$.

Definition 2.11. 1. Two paths p, q in $S \in \mathcal{O}$ are called *equivalent*, if they project to the same path in the quotient G/S. In particular, their endpoints $o(p), o(q)$ and $t(p), t(q)$ are in the same orbits, respectively.

2. Let $h : S \to S$ be a representative of some outer automorphism ψ, let p be a path in S and let's suppose that the endpoints of p are h - periodic (with period k), then we say that a path p in S is *N-periodic* (with period k), if the paths $[h^k(p)], p$ are equivalent.

Geometric and non-Geometric automorphisms: We will define here some notions for automorphisms that have been motivated by the properties of geometric and non-geometric automorphisms, respectively. The terminology also comes from the free case. In that case, we say that ϕ is geometric if it can be represented as a (pseudo-Anosov) homeomorphism of a punctured surface. It is well known that for the non-geometric case there is an integer m such that it is impossible to concatenate more than m indivisible Nielsen paths for every map f which represents ϕ. We will generalise this property in order to give our definitions, using the notion of an indivisible N-periodic path as in the free case. In particular:

Definition 2.12. We say that some ϕ has the *NGC* property , if it is impossible to concatenate more than m indivisible N-periodic paths for every $\mathcal{O}$-map f which represents ϕ. Otherwise, we say that ϕ has the *GC* property.

2.6 Relative train-track maps

Having good representatives of outer automorphisms, is very useful. If our automorphism is irreducible, it is possible to find train track representatives, as we have seen. But even in the reducible case we can find relative train track representative. The existence of such maps it follows from [10] or [5].

That we have is that every automorphism can be represented as an $\mathcal{O}$-map $f : T \to T$ such that T has a filtration $T_0 \subseteq T_1 \subseteq ... \subseteq T_k = T$ by f-invariant G-subgraphs, where T_0 contains every non- free vertex, we denote by $H_r = cl(T_r - T_{r-1})$ and we suppose that the transition matrix (it can be defined as in the free case but we count orbits of edges) of every H_r is irreducible (or zero matrix) so we can correspond in every H_r some PF eigenvalue (let's denote it λ_r) . In addition, f has some train track properties (such as mixed turns are legal and the map is r-legal). There is a very interesting corollary that we will use: for every edge-path a in H_r, the reduced image of a, $[f(a)]$, can be written

as a concatenation of non-degenerate edge-paths in T_{i-1} and H_i with the first and the last contained in H_i.
For such a, we can distinguish between two cases for the strata: if there exists some edge of e in H_r such that $[f(e)]$ contains at least two copies (orbits) of e, then we say that the stratum is *exponentially growing* and we can see the r-lengths of images of edges in H_r expands by $\lambda_r > 1$ and in particular the lengths of reduced f-iterates of edges in H_r are going to infinity (using the train track properties). Otherwise, the stratum called non-exponentially growing and the map f (if we ignore the lower strata) is just a permutation of edges of the same length. An automorphism is called *exponentially growing* if some representative has at least one exponentially growing stratum. In other case, it is called *non-exponentially growing* automorphism.

2.7 Graph of Groups and Subgroups

We will recall only some facts for the graph of groups. For more about graph of groups and their subgroups, see [22].
In the special case that we are interested, a graph of groups can be defined as a finite connected graph X (let call Γ the underlying graph) for which in every vertex v we correspond some (vertex) group G_v. We call *non -free* the vertices for which the corresponding group is non-trivial. Then the fundamental group of X, $\pi_1(X)$ is the free product of $\pi_1(\Gamma)$ (which is a f.g. free group) and the vertex groups.
We will use a specific kind of subgroups of $\pi_1(X)$. Let γ be a loop in $v_0 \in V(\Gamma)$. Then starting from v_o and following the path of γ we meet some non-free vertices (we can return back also, but we have always follow γ). So we can read words of a fixed form, and this process produces words of the fundamental group (we can see it as the group which it consists of all the words constructed as above but without fixing some loop γ). In fact, the set of all such words is a subgroup of $\pi_1(X)$, which corresponds to γ.

3 Every two $\mathcal{O}$ - maps coincide

In [10] it has been proved the existence of $\mathcal{O}$-maps. We will prove that even if in the construction of such maps there is a lot of freedom, the reduced images of all of them coincide, up to bounded error. As a consequence we obtain that their lengths are comparable.

Theorem 3.1. *Let $f, h : A \to B$ be $\mathcal{O}$ - maps. Then there exists a positive constant C (which depends only on f, h and A), so that for every path L in A, then $[f(L)]$ and $[h(L)]$ are equal, except possibly some subpaths near their endpoints which their lengths are bounded by C.*

Proof. Firstly, we suppose that there is at least one non-free vertex which we denote it by v. Then we have that $f(v) = h(v)$. If $L = [a, b]$ is an edge - path, then in distance at most $vol(A/G)$, we can find vertices of the form $g_1 v, g_2 v$ near a, b respectively such that $[a, b] \subseteq [g_1 v, g_2 v]$. Then $[f(L)]$ is contained in $[f(g_1 v), f(g_2 v)]$, except possibly some segments near a, b of length at most $C' = vol(A/G)Lip(f)$. Similarly, we apply the same argument for $[h(g_1 v), h(g_2 v)]$ and we get a constant $C'' = vol(A/G)Lip(h)$. Therefore since $[h(g_1 v), h(g_2 v)] = [f(g_1 v), f(g_2 v)]$, we get $[f(L)] = [h(L)]$ except possibly some segments near a, b which are bounded by $C = max(C', C'')$ (by definition depends only on $Lip(f), Lip(h), vol(G/A)$)
If there are no non- free vertices, we are in the free case and the result is well known. □

Note also that it is not difficult to see that every $\mathcal{O}$-map is a quasi-isometry.

4 Laminations

We follow exactly the same approach as in [1] and some of the proofs are essentially the same, but since in this context the definitions have adjusted appropriately, we give detailed proofs for the convenience of the reader. On the other hand, there are a lot of technical issues which are not appeared in the free case and they are addressed separately. In this section we define the notion of the lamination associated to an IWIP. Firstly, we use the train track maps to define the lamination in a specific tree and the existence of $\mathcal{O}$-maps between any two trees allows us to generalise it for every tree.

4.1 Construction of the lamination and properties

Let $\phi \in Out(G, \mathcal{O})$ be an (expanding) irreducible automorphism, with irreducible powers and $f : A \to A$ for some $A \in \mathcal{O}$ be a train track map which represents ϕ (so it satisfies $f(gx) = \phi(g)f(x)$). We can also suppose that f expand the length of the edges by a uniform factor $\lambda > 1$ (this can be done if we choose an optimal train track that represents f, as we have already seen).
By changing f with some iterate, if necessary, we can suppose that there is $x \in A$ which is a periodic point ($f^k(x) = x$, for some k), in the interior of some edge (in general there exists x s.t. $f^k(x) = gx$ since the quotient is finite, but we can change the space A, changing isometrically the action, with $\phi_g(A)$ and there the requested property holds).
Now let U some ϵ-neighbourhood, for some small ϵ (we want the neighbourhood to be contained in the interior of the edge) and then there is some $N > 0$ s.t. $f^N(U) \supset U$.
We can choose an isometry $\ell : (-\epsilon, \epsilon) \to U$ and extend it to the unique isometry $\ell : \mathbb{R} \to A$ s.t. $\ell(\lambda^N t) = f^N(\ell(t))$ and then we say that the bi-infinite line ℓ is *obtained by iterating a neighbourhood* of x.

Definition 4.1.
- We say that two isometric immersions $A : [a, b] \to A$ and $B : [c, d] \to A$, where $a, b, c, d \in \mathcal{R}$ are equivalent, if there exists an isometry $q : [a, b] \to [c, d]$ s.t the triangle commutes ($Bq = A$). (This relation is an equivalence relation on the set of isometric immersions from a finite interval to A).
- If P is an equivalence class and we choose a representative of that class $\gamma : [a, b] \to A$, we can define $f(P)$ as the equivalence class of $f\gamma : [a, b] \to A$, pulled tight and scaled so it is an isometric immersion.
- A leaf segment of an isometric immersion $\mathbb{R} \to A$ is the equivalence class of the restriction to a finite interval.

Let ℓ be an isometric immersion, then we correspond the G-set I_ℓ (of the leaf segments of ℓ) to ℓ. We can also define an equivalence relation on the set of isometric immersions from $\mathbb{R}$ to A.

Definition 4.2. Let ℓ, ℓ' be two isometric immersions from $\mathbb{R}$ to A, then we say that they are equivalent if $I_\ell = GI_{\ell'}$. Namely, we say that they are *equivalent* if for every leaf segment P of ℓ there is an element $g \in G$ and Q a leaf segment of ℓ' s.t. $P = gQ$ and vice versa (or equivalently every l.s. of ℓ is mapped by some g to a l.s. of ℓ')

Remark. Here note that it is obvious that if $\ell(t) = g\ell'(t)$ (ℓ, ℓ' are in the same orbit), then ℓ and ℓ' are equivalent.

We will prove that if we construct any other line by iterating a neighbourhood of any other periodic point (here we mean that there is k and $g \in G$ s.t. $f^k(x) = gx$) then it is equivalent with ℓ.

Lemma 4.3. *Let $y \in A$, be any other f-periodic point in the interior of some edge of A and ℓ' is the obtained by iterating of some neighborhood of y. Then ℓ and ℓ' are equivalent.*

Proof. We will show that any l.s. of ℓ is mapped by some element of G to a l.s. of ℓ', then the converse follows by symmetry.
Since f represents an irreducible automorphism (and the same holds for every power of f), ℓ' contains some orbit of every edge, so in particular if x is contained in the interior of the edge e we have that there exists some $g \in G$, s.t. $gx \in ge \subseteq \ell'$. So there is an isometry $\psi : (-\epsilon, \epsilon) \to (a - \epsilon, a + \epsilon)$ with the property $\ell(t) = g\ell'(\psi(t))$.
Let N' be a natural number s.t. $\ell'(\lambda^{N'} t) = f^{N'}(\ell(t))$ and then for any $t \in U$ (U as in the definition) we have that $\ell(\lambda^{kNN'} t) = f^{kNN'}(\ell(t)) = f^{kNN'}(g\ell'(\psi(t))) = \phi^{kNN'}(g) f^{kNN'}(\ell'(\psi(t))) = \phi^{kNN'}(g)\ell'(\lambda^{kNN'}\psi(t))$.
But since every prechosen interval is contained in some interval of the form $\lambda^{kNN'}(-\epsilon, \epsilon)$ for large k, we have that for every l.s. of ℓ is mapped by some $\phi^{kNN'}(g) \in G$ to some l.s. of ℓ'. □

We are now in position to define the stable lamination corresponding to A.

Now the **stable lamination** in A-coordinates $\Lambda = \Lambda_f^+(A)$ is the equivalence class of isometric immersions from $\mathbb{R}$ to A containing some (and by previous lemma any) immersion obtained as above (by iterating a neighborhood of a periodic point). We call the immersions representing Λ **leaves** of Λ and the leaf segments (l.s.) of some leaf of Λ **leaf segments** of Λ (by definition of the equivalence relation, every leaf of Λ contains some orbit of every l.s. of Λ).
Note that the every leaf of the lamination project to the same bi-infinite path in the quotient.
We will list some useful properties of the stable lamination.

Proposition 4.4. *1. Any edge of A is a leaf segment of Λ.*

2. *Any f-iterate of a leaf segment is a leaf segment.*

3. *Any subsegment of a leaf segment is a leaf segment.*

4. *Any leaf segment is a subsegment of a sufficiently high iterate of an edge.*

5. *For any leaf segment P there is a leaf segment P' such that $f(P') = P$.*

6. *Let a be a segment which is the period of the axis of some hyperbolic element which crosses k edges (counted with multiplicity). Then any f-iterate of a (pulled tight) can be written as concatenation of less or equal k leaf segments.*

Proof. 1. This is clear by the proof of the previous lemma, since f represents an irreducible automorphism and this implies that every ℓ contains orbits of every edge, so if ge is contained in ℓ then e is contained in $g^{-1}\ell$ which is equivalent to ℓ thus is a leaf of Λ, and as consequence e is leaf segment of a leaf therefore it is l.s. of Λ.

2. Firstly, we note that if x is f-periodic then $f(x)$ is f- periodic with the same period(in fact every $f^m(x)$ is periodic) and let's denote ℓ' the isometric immersion constructed as above, so if P is a l.s. of ℓ, then $f(P)$ is a l.s. of ℓ' but since ℓ, ℓ' are equivalent by lemma, we have that ℓ' is a leaf of Λ and therefore $f(P)$ is a l.s. of Λ. So we can do it for every iterate of f.

3. This is obvious, since we restrict the isometric immersion to the subsegment and it is a l.s. of a leaf of Λ and as a consequence a l.s. of Λ.

4. We have that f expands the length of every edge by λ, but we can use for representative the isometric immersion constructed as above (by iterating a periodic neighborhood) and the edge in which the periodic point belongs, then by construction of ℓ every l.s. is contained in an high iterate of this edge. For any other

representative ℓ' now we can translate ℓ as above (by some element $g \in G$) to have a common segment that contain the prechosen l.s. and the proof reduced to the first case.

5. Let P be a l.s. of Λ. By (iv) we have that there exists some iterate of an edge and so by ℓ an iterate of a l.s. P'' s.t. P is contained in $f^m(P'')$ and since iterates of l.s. are l.s. and subsegments are l.s. as well, we have that there is P' subsegment of $f^{m-1}(P'')$ with the property $P = f(P')$

6. This is obvious since edges are l.s. and f-iterates of l.s. are l.s..

□

We note that (ii) implies that $f^k(\ell)$ is a leaf of the lamination, for every k.

Definition 4.5. We say that a sequence a_i of isometric immersions $[0,1]_i \to A$ (where the metric on $[0,1]_i$ is scalar multiple of the standard part which depends on i), (weakly) converges to Λ,
if for every $L > 0$ the ratio,

$$\frac{m(\{x \in [0,1]_i | \text{the L- nbhd of x is a leaf segment}\})}{m([0,1]_i)}$$

converges to 1.

Proposition 4.6. *Suppose that a is a segment in A which corresponds to the period of the axis of some hyperbolic element, which is not N-periodic. Then the sequence (of tightenings of $f^i(a)$), $[f^i(a)]$ weakly converges to Λ.*

Note that such hyperbolic elements always exist. For example the basis elements of the free group, are not N-periodic by definition of irreducibility.

Proof. Suppose that a can be written as a concatenation of k l.s. then we have $k-1$ illegal turns (we don't count the endpoints) and since f is a train track map we have that the number of illegal turns in $[f^k(a)]$ is non-increasing so it contains less than or equal to $k-1$ l.s.. Therefore if the lengths of reduced iterates of a is bounded, and since there are finitely many inequivalent paths with length less than or equal to a specific number, we have that a is N-preperiodic and therefore periodic because a corresponds to a group element, which leads to a contradiction to the hypothesis. Therefore some $[f^i(a)]$ contains arbitarily long legal segments ($> C_{crit}$), and since the length of $[f^j(a)]$ expands for large j, we have that there are finitely many L-nbds contain points without the requested property (of the endpoints of the concatenation of l.s. so at most k) and the measure of these is at most $2Lk$, as a consequence the ratio converges to 1. □

Definition 4.7. An isometric immersion $l : \mathbb{R} \to A$ is quasiperiodic (qp), if for every $L > 0$ there exists $L' > 0$ s.t. for every l.s. P of ℓ of length L and for every l.s. Q of length L' there is $g \in G$ s.t. $gP \subseteq Q$ (P is mapped by g to a subsegment of Q).

Proposition 4.8. *Every leaf of Λ is quasiperiodic.*

Proof. We will first prove it for some ℓ which has constructed by iterating neighbourhood of a periodic point.
We first verify it for leaf segments Π that consists of only two edges.
If we choose $L_0 > 2max_e(len(e))$, then if a l.s. P has length $\geq L_0$, then it contains a subleaf segment which is an edge. Then there is N(we can also choose it to be multiple of k) s.t. f^N restricted to any edge crosses some orbit of every turn that they crossed by leaves of $\Lambda_f^+(A)$. So in particular for the chosen Π the iterate of f takes the orbit of that turn, so there exists $g \in G$ such that $\Pi \subseteq gf^N(P)$.
Now if P' is any l.s. of length $\lambda^N L_0$, then $P' = f^N(P)$ for some P l.s. of length L_0 and therefore $\Pi \subseteq gP'$.

For the general case, let $L > 0$ be given, then there is $M > 0$ (we choose it to have the property $\lambda^{-M}L < 2min(len(e)))$ s.t. any l.s. of length $\leq \lambda^{-M}L$ is a subsegment of a two-edge l.s. as above and let $L' = \lambda^{M+N}L_0$.
So let P be a l.s. of length L and P' be a l.s. of length L'. Then by the properties we have that $P = f^M(\Pi)$ where Π is contained to a l.s. as in the special case(by the choice of M, since Π has length $\lambda^{-M}L$), and similarly $P' = f^M(\Pi')$ for a l.s. Π' of length $\lambda^N L_0$. By the special case we have that $\Pi \subseteq g\Pi'$ and this implies that $P = f^M(\Pi) \subseteq \Phi^M(g)f^M(\Pi') = \Phi^M(g)P'$. Since ℓ is $\Phi^M(g)$-invariant, we have the requested property.
For any other equivalent isometric immersion ℓ', if we have P l.s. of length L and Q l.s. of length L' then we can find an isometric immersion ℓ like the first case with Q as common segment. Then by the equivalence there exists $g_1 \in G$ s.t. g_1P is l.s. of ℓ, and by quasiperiodicity of ℓ, there is g_2 s.t. $g_2P \subseteq Q$ and g_2P is a l.s. of ℓ', so we have that ℓ' is quasiperiodic. □

4.2 Lamination in every tree

Suppose that $f : A \to A$ and $\Lambda_f^+(A)$ as above and $B \in \mathcal{O}$. Then we know that there exists an optimal map(in particular $\mathcal{O}$ -map) $\tau : A \to B$. Then for any immersion $\ell : \mathbb{R} \to A$ we denote by $\tau(\ell) : \mathbb{R} \to B$ the unique (up to precomposition by an isometry of $\mathbb{R}$) pulled tight to be the isometric immersion corresponding to $\tau\ell$.

Lemma 4.9.
- *If $\ell, \ell' : \mathbb{R} \to A$ are equivalent leaves, then $\tau(\ell), \tau(\ell')$ are equivalent.*
- *If ℓ is quasiperiodic, then $\tau(\ell)$ is quasiperiodic.*

Proof. Every optimal map τ by [10], can be factored as the composition of a homeomorphism and a finite sequence of folds. We have just to prove that the lemma is true for homeomorphism and folds.
Firstly, let suppose that τ is homeomorphism. In particular $[\tau(\ell)] = \tau(\ell)$ and the same holds for ℓ' as well.
Let P' is a l.s. of $\tau(\ell)$, then there is some l.s. of ℓ P s.t. $P' = \tau(P)$, so there is a translation of P by some element of the group, gP which is contained in ℓ', therefore $\tau(gP) = gP'$ is contained in $\tau(\ell')$. The converse follows by symmetry and so $\tau(\ell)$ and $\tau(\ell')$ are equivalent.
Suppose now that ℓ is quasiperiodic, fix a $L > 0$ let P' l.s. of $\tau(\ell)$ of length L. Then there is a l.s. P of length at most K (by Bounded Cancellation Lemma there exists such K which doesn't depend on P but only on L) s.t. $P' = \tau(P)$. Then we can define $L'' = L'Lip(\tau)$, where L' is the constant corresponding by quasiperiodicity to K and we have that if we choose any Q' l.s. of $\tau(\ell)$ of length L'' then there exists a l.s. Q of ℓ of length at least L' s.t. $\tau(Q) = Q'$. Then Q contains orbits of any l.s. of length at most K, in particular it contains some translation of P for some $g \in G$ and therefore as above Q' contains some translation of P'. So $\tau(\ell)$ is quasiperiodic.
We suppose that τ is an equivariant isometric simple fold of some segments starting from the same point v and has the same τ - image, let call them a, b and c be the corresponding segment in the quotient.
For the first statement, we note that is obvious for a l.s. of $[\tau(\ell)]$ which don't contain some orbit of c, since there τ is the identity. On the other hand, if P' is l.s. of $\tau(\ell)$ which contains some orbits of c, then there exists P which contain the same number of orbits as the folded turn and $[\tau(P)] = P'$ (it is concatenation of the segments before and after the folds). Since ℓ, ℓ' are equivalent we have that we can find $g \in G$ s.t. gP is contained in ℓ', then $[\tau(gP)]$ is a a l.s. of $[\tau(\ell')]$. But $[\tau(gP)]$ is just a translation (by g) of $\tau(P)$, and therefore as above we obtain that $[\tau(\ell)], [\tau(\ell')]$ are equivalent.
For the quasiperiodity of $[\tau(\ell)]$ we fix a number $L > 0$ and we call M the maximum number of orbits of v which there are in a segment of length L, and L' is the number corresponds by quasiperiodicity for $L'' = L + 2Mlen(a)$. Now let P' be a l.s. of length L, then there is P which contains the same number of orbits of the folded turn and $[\tau(P)] = P'$ as above. Then P has length at most L'', some translation of it is contained in every l.s. of ℓ of length L'. Now let choose Q any l.s. of $[\tau(\ell)]$ of length L' then the preimage has length at least L', and therefore the preimage has the requested property. So Q contains a translation of P' as above. □

Definition 4.10. The stable lamination of $f : B \to B$ in the B-coordinates is the equivalence class $\Lambda^+_f(B)$ containing $\tau(\ell)$ for some (and by previous lemma any) leaf of $\Lambda^+_f(A)$.

Using again the property that τ is factored as the composition of a homeomorphism and a finite sequence of folds combined with the result for the $\Lambda^+_f(A)$, we have the following proposition.

Proposition 4.11. *Let a be a segment which is the period of the axis of a hyperbolic element in A, which is not N- periodic. Then the sequence $\{[\tau(f^i(a))]\}$ weakly converges to $\Lambda_f^+(B)$*

Lemma 4.12. *Suppose that $h : B \to B$ is any other train track map representing Φ. Then $\Lambda_f^+(B) = \Lambda_h^+(B)$*

Proof. Let a be a periodic segment as in 4.6 and 4.11. Then we have that the sequences $[\tau(f^i(a))], [h^i(\tau(a))]$ weakly converge to $\Lambda_h^+(B)$ and to $\Lambda_f^+(B)$, respectively by the previous propositions. But $\tau f^i, h^i\tau$ are $\mathcal{O}$-maps from A to $\phi(B)$, so their reduced images coincide in every path, after deleting some bounded segments near endpoints. Then there are leaves ℓ, ℓ' of $\Lambda_h^+(B)$ and $\Lambda_f^+(B)$ respectively with arbitrarily long common leaf segments. Since they are both quasiperiodic, it follows that they are equivalent. Indeed, let P be a l.s. of ℓ of length L then there exists L' s.t. for every l.s. of length L', P' there is some $g \in G$ s.t. $P \subseteq gP'$. But we can find a common segment Q of ℓ and ℓ' of length at least L', so by quasiperiodicity $P \subseteq gQ \subseteq \ell$ and since $Q \subseteq \ell'$ we have that $P \subseteq gQ \subseteq g\ell'$ and therefore $g^{-1}P \subseteq \ell'$.
We have proved that for every l.s. of ℓ there is an element of the group that map this l.s. to a l.s. of ℓ' and similarly we can prove the converse so ℓ and ℓ' are equivalent by definition. Therefore $\Lambda_h^+(B) = \Lambda_f^+(B)$ □

So we have proved that we can use any train track representative to define the set of laminations, in particular we give the following definition:

Definition 4.13. The **stable lamination** Λ_Φ^+ associated to some IWIP $\Phi \in Out(G, \mathcal{O})$ is the collection $\{\Lambda_f^+(B) | B \in \mathcal{O}\}$ where $f : A \to A$ is a train track representative of Φ. The **unstable lamination** Λ_Φ^- of Φ is the stable lamination of Φ^{-1}.

5 Action

Let ϕ be an IWIP and $f : T \to T$ be an optimal train track representative of ϕ.
We denote by $\mathcal{IL}$ the set of stable laminations Λ_ϕ^+, as ϕ ranges over all IWIP automophisms relative to $\mathcal{O}$. The group $Out(G, \mathcal{O})$ acts on $\mathcal{IL}$ via

$$\psi\Lambda_\phi^+ = \Lambda_{\psi\phi\psi^{-1}}^+ \tag{1}$$

More specifically, if ℓ is a leaf of Λ_ϕ^+ in the S-coordinates and $h : S \to S$ an $\mathcal{O}$ map representing ψ, then $[h(\ell)]$ represents a leaf of $\Lambda_{\psi\phi\psi^{-1}}^+$.
We are interested to study the stabiliser of the action for a fixed automorphism. Note that obviously the centraliser, which we denote by $C(\phi)$, of the IWIP ϕ in $Out(G, \mathcal{O})$ is

a subgroup of $Stab(\Lambda)$.
We will equip T with a specific train-track structure, the *minimal train-track structure*; more specifically we declare a turn legal, if it is crossed by some leaf of Λ^+_f. The properties of the lamination imply that a turn is legal iff there is a f-iterate of an edge of T that crosses the turn.

6 Subgroups carrying the lamination

From now and for the rest of the sections, we fix a group of finite Kurosh rank G with some (non-trivial) free product decomposition, the relative outer space $\mathcal{O}$ which corresponds to this decomposition, some expanding IWIP ϕ relative to $\mathcal{O}$ and the associated lamination $\Lambda^+_\phi = \Lambda$.
This section is devoted to prove that it is not possible for a proper subtree to contain every leaf of the lamination. Moreover, we will prove that every relative train track representative of some automorphism of the stabiliser, after passing to some power, induces the identity on the quotient restricted to any proper invariant subraph (which is union of strata).

Definition 6.1. Let A be a subgroup of G of finite Kurosh rank, and let's denote $T \in \mathcal{O}$ and T_A the minimal invariant A- subtree. We suppose also that for every $v \in V(T_A)$, $Stab_A(v) = Stab_G(v)$. Then we say that A *carries the lamination* Λ, if there exist some leaf ℓ of Λ which is contained in T_A.

Remark.
1. Every two leaves of the lamination project to the same bi-infinite path in Γ.
2. For every vertex v of T there exist $g \in G$ s.t. $gv \in T_A$ (in particular, T_A contains some orbit of any non-free vertex).

Proposition 6.2. *If a A is a subgroup of G, as in the previous definition, which carries Λ^+_ϕ then A has finite index in G.*

Proof. Let $f : T \to T$ be a train-track representative of ϕ , $\Gamma = G/T$ and let $H \to \Gamma$ be an isometric immersion corresponding to $A \leq G$. Then by our assumptions H is finite graph of groups and by the remarks contains every non-free vertex. Therefore (using also the assumption that the corresponding vertex groups are full), we can complete the immersion, by adding vertices (with trivial vertex group) and edges, to a connected finite-sheeted covering space $p : \Gamma' \to \Gamma$ and therefore we have that $T' = T$ (where T' is Bass-Serre tree of Γ').
Now we know that if A has infinite index, then we are really adding new edges in Γ' or equivalently we add new orbits of edges in T. But then using irreducibility we can reach

a contradiction.
More specifically, we choose e (edge of T) such that $f(e)$ starts with e. Then for every n the path $f^n(e)$ is a path of T_A. So if we choose any edge e_1 (lift of some edge in $\Gamma' - H$) there does not exist n and $g \in G$ such that $f^n(ge')$ passes through e_1 (since e_1 is in different orbit of edges in T_A), but this contradicts the fact that the transition matrix corresponding to f, denote it by $A(f)$, is irreducible. As a consequence, A must have finite index in G. □

Proposition 6.3. *Let $\psi \in Stab(\Lambda)$, and let $h : S \to S$ be a relative train-track representative of ψ. Then let's denote by S_0 some h-invariant G-subgraph of S (without free vertices of valence 1) that is a union of strata. Then there is a k s.t. if we restrict h^k to S_0 induces the identity in G/S_0.*

Proof. Let ℓ be a leaf in S-coordinates and let S_0 be a proper h-invariant subgraph. The quasiperiodicity implies that there is an upper bound to the length of both S_0 and $S - S_0$ segments, and hence only finitely many segments occur (since there are finitely many lengths corresponding to edge- paths of bounded length in the quotient and quasiperiodicity implies that there are finitely many orbits of leaf segments of a specific length). Using the same argument we have that it is not possible for ℓ to contain arbitarily long segments of a proper subgraph since then the quasiperiodicity implies that ℓ is contained in that subgraph which contradicts to the previous proposition. Therefore ℓ is a concatenation of non-degenerate segments in S_0 and in $S - S_0$ (otherwise would lift to a proper subgraph of H, which is impossible as we have noticed). Now we have that all S_0-segments are h- preperiodic (there exist M, N s.t. $h^M(L), h^N(L)$ are in the same orbit) or else h-iteration will produce arbitrarily long leaf segments contained in S_0 contradicting quasiperiodicity.
We can start with the disjoint union X of copies of the segments and the natural immersion $X \to S$ and we identify two endpoints of X if they are mapped to the same point of S. Then fold to convert the resulting map to an immersion $\pi : X' \to S$. But ℓ lifts to X' (by construction) and so by previous proposition we have again that $X' = S$ (it corresponds to a finite covering space of graph of groups). In particular, any simple periodic segment in S_0 lifts to X'. Consequently, this segment is a concatenation of paths in S_0 each of which is h-preperiodic, and therefore this segment is N-periodic (since it corresponds to an element of the group and so we have inverse). Thus every such segment a in S_0 is equivalent to some power $h^k(a)$ (note that there is a uniform bound for the powers) and hence for some k, h^k restricted to S_0 induces the identity on the quotient, since h is a relative train track.

□

7 Stretching map

In this section we will see that we can define a homomorphism from the stabiliser of the lamination to $\mathbb{R}$.

Lemma 7.1. *Suppose that $h : S \to S$ is an $\mathcal{O}$- map that represents $\psi \in Out(G, \mathcal{O})$. Then there exists a positive number $\lambda = \lambda(h, \Lambda)$ such that for every $\epsilon > 0$ there is $N > 0$ so that if L is a leaf segment of Λ of length $> N$, then $|\frac{length([h(L)])}{length(L)} - \lambda| < \epsilon$*

Proof. We note that since f is IWIP, we have that the transition matrix $M = A(f)$ is irreducible (as it is every power of M) and therefore we can apply the Perron - Frobenius theorem to M, as a consequence we have that long leaf segments of Λ cross orbits of edges of T with frequencies close to those determined by the components of the PF eigenvector.
Now fix large k and then large l.s. are concatenation of l.s. of the form $f^k(e)$, for some edges of T, each orbit of edges with definite frequency.(For $k = 1$ this is the statement above, for $k > 1$ apply $P.F$ theorem for f^k).
If M is large enough, then for any l.s. L with $length(L) > M$ we can think L as concatenation of l.s. of the form $f^k(e)$ (there are possible some shorts segments contained in the first and the final segment, which are not of this form but we can ignore them since their contribution in lengths is neligible).
Now let C be the bounded cancellation constant for $h : T \to T$, and let's denote $l_e = len(f^k(e))$, $l_e^h = len([h(f^k(e))])$, N_e be the number of occurrences of orbits of $f^k(e)$ in L and $N = \sum N_e$, then we have that $\frac{N_e}{N} \to r_e$, as $len(L) \to \infty$ (r_e is the PF component of the eigenvector that corresponds to e) by the PF theorem.
Note that the numbers N_e, l_e, l_e^h depends on k, so we define $a_k = \frac{\sum r_e l_e^h}{\sum r_e l_e}$. We have that $len(L) = \sum N_e l_e$ and by bounded cancellation lemma:

$$\frac{\sum N_e(l_e^h - 2C)}{\sum N_e l_e} \leq A_M = \frac{len([h(L)])}{len(L)} \leq \frac{\sum N_e l_e^h}{\sum N_e l_e} \tag{2}$$

and subdividing the sums by N we have that

$$\frac{\sum \frac{N_e}{N} l_e^h - 2C\frac{N_e}{N}}{\sum \frac{N_e}{N} l_e} \leq A_M = \frac{len([h(L)])}{len(L)} \leq \frac{\sum \frac{N_e}{N} l_e^h}{\sum \frac{N_e}{N} l_e} \tag{3}$$

where the term $2C\frac{N_e}{\sum N_e l_e}$ converges to 0 as $k \to \infty$ and as we noted above $\frac{N_e}{N} \to r_e$, as $len(L) \to \infty$. As a consequence, for every ϵ for large $k = k(\epsilon)$ and for large $M = M(\epsilon, k)$, $a_k - \epsilon \leq A_M \leq a_k + \epsilon$.
Firstly, we send $M \to \infty$ and then for every $\epsilon > 0$ for large k,

$$a_k - \epsilon \leq \liminf A_M \leq \limsup A_M \leq a_k + \epsilon \tag{4}$$

Therefore sending ϵ to 0, k to infinity, we have that, choosing a subsequence of a_k that converges to a,

$$a \leq \liminf A_M \leq \limsup A_M \leq a \tag{5}$$

and therefore $\lim A_M = \liminf A_M = \limsup A_M = a$.
As consequence we have the requested property that there exists a positive number λ s.t. $\frac{len([h(L)])}{len(L)} \to \lambda$, as $len(L)$ is going to infinity. $\square$

Lemma 7.2. *Using the notation as above and choosing any other representative h' of ψ, we have that $\lambda(h, \Lambda) = \lambda(h', \Lambda)$. In particular, the number doesn't depend on the representative but only on ψ.*

Proof. Let h, h' be $\mathcal{O}$-maps which represent ψ as in the previous lemma. Therefore by the proposition 3.1 for any L, $[h(L)] = [h'(L)]$, up to bounded error that doesn't depend on L. Therefore for every L, $len([h(L)]) \leq len([h'(L)]) + C$, where C is positive fixed and as a consequence

$$|\frac{len([h(L)] - len([h'(L)]))}{len(L)}| \leq \frac{C}{len(L)} \to 0$$

for large $len(L)$.
Therefore since $\frac{len([h(L)])}{len(L)} \to \lambda(h, \Lambda)$ and $\frac{len([h'(L)])}{len(L)} \to \lambda(h', \Lambda)$, we have as a consequence $\lambda(h, \Lambda) = \lambda(h', \Lambda)$. $\square$

Lemma 7.3. *Using the notation above we have that $\sigma : Stab(\Lambda) \to \mathbb{R}^+$, where $\sigma(\psi) = \lambda(h, \Lambda)$, is a well defined homomorphism.*

Proof. Since we have that $\psi \in Stab(\Lambda)$, this means that $[h(\ell)]$ is a leaf (for any leaf ℓ) and as a consequence σ is a well defined map.
We will prove that σ is homomorphism.
So we have to prove that for any $\psi_1, \psi_2 \in Stab(\Lambda)$ it holds that $\sigma(\psi_1)\sigma(\psi_2) = \sigma(\psi_1\psi_2)$.
We choose representatives h_1, h_2 of ψ_1, ψ_2 respectively, and by definitions $\frac{len([h_1(L)])}{len(L)} \to \sigma(\psi_1)$ and $\frac{len([h_2(L)])}{len(L)} \to \sigma(\psi_2)$. Moreover, h_1h_2 represents $\psi_1\psi_2$ (by previous lemma we can choose any representative).
Therefore since $\frac{len([h_1(h_2(L))])}{len(h_2(L)} \to \sigma(\psi_1\psi_2)$, for $len(L) \to \infty$ and actually we have the equality $\frac{len([h_1(h_2(L))])}{len[h_2(L)]} = \frac{len([h_1[h_2(L)]])}{len[h_2(L)]}\frac{len([h_2(L)])}{len[L]}$ up to bounded error. But now sending $len(L)$ to infinity, it holds $\frac{len([h_1[h_2(L)]])}{len[h_2(L)]} \to \sigma(\psi_1)$ (as $len[h_2(L)]$ converges to infinity when $len(L) \to \infty$ and the fact that $[h_1[h_2(L)]])$ and $[h_1(h_2(L))]$ are in bounded distance and the bound doesn't depend on L).
Therefore by uniqueness of the limit, we have that $\sigma(\psi_1\psi_2) = \sigma(\psi_1)\sigma(\psi_2)$.

$\square$

8 Kernel of the homomorphism

Now we investigate the properties of the kernel, We would like to prove that $ker(\sigma)$ contains as subgroup of finite index the intersection of the stabiliser with the kernel of the action. But firstly, we aim to prove that the subgroup $ker(\sigma)$ contains only non-exponentially growing automorphisms. We will prove it separately for irreducible and reducible automorphisms.

8.1 Reducible case

In the reducible case we will see that the automorphisms of the $Stab(\Lambda)$, have representatives of a very specific form. More specifically, every stratum except the top, is non-exponentially growing and moreover the representative restricted to each stratum is just a permutation of edges. Therefore we can calculate the value of σ, using only the top stratum if it is exponentially growing.

Proposition 8.1. *If $\psi \in Stab(\Lambda)$ is exponentially growing and there exists some k s.t. ψ^k reducible, then $\psi \notin Ker(\sigma)$*

Proof. Let $h : S \to S$ be a relative train track representative of ψ(we can change h with some power if it is necessary).
Firstly, we note that every stratum, except possibly the top one, is non-exponentially growing. This is true, since otherwise if some H_r is exponentially growing and $e \in H_r$ we have that the lengths of tightenings of h- iterates of e are arbitarily long (by the train track properties) and they are l.s. (by definition of the stabiliser of the lamination), but this means that we have arbitarily long segments contained in some proper subgraph (since $h(G_r) \subseteq G_r$), which is impossible as we have seen in 6.2.
Therefore if ψ is exponentially growing then we suppose, changing h with some iterate if it is necessary, that there exists H_0 which is union of strata, all of them are non-exponentially growing, h restricted to H_0 induces the identity in the quotient, and that the top stratum is exponentially growing, so if we have a leaf of the lamination and using the subgraph-overgraph decomposition of the leaf, it is implied that the lengths of long l.s. grow exponentially and in fact the actual value is the Perron-Frobenius eigenvalue that corresponds to the unique exponentially growing stratum. □

8.2 Irreducible case

Now let's suppose that ψ is an IWIP. We have two cases and we will prove the theorem independently for automorphisms that have the NGC and the rest automorphisms that have the GC (the dichotomy is the same as in the free case, but for the automorphisms with GC we need arguments of different nature). We will prove again that the value of σ corresponds to the Perron - Frobenious eigenvalue of ψ (or ψ^{-1}).

Lemma 8.2. *Let $h : S \to S$ be a train track map representing some irreducible $\psi \in Out(G, \mathcal{O})$.*

Then for every $C > 0$ there is a number $M > 0$ such that if L is any path, then one of the following holds:

1. *$[h^M(L)]$ contains a legal segment of length $> C$*
2. *$[h^M(L)]$ has fewer illegal turns that L*
3. *L is concatenation $x \cdot y \cdot z$, such that y is N-preperiodic and x, z have length $\leq 2C$ and at most one illegal turn.*

Proof. Choose M to be a natural number that exceeds the number of inequivalent legal edge paths of length $\leq 2C$.
Now assume that L is a path such that the second statement fails, so $[h^M(L)]$ has the same number of illegal turns with L (since h is train track map, sends edges to legal paths and legal turns to legal turns so it is not possible the image of a path to have more illegal turns than the path). So each h- iteration of L amounts to iterating maximal legal subsegments of L and cancelling portions of adjacent ones.
If, in addition, the first fail as well, then each maximal legal segment (which has length $\leq C$) of L, except possibly the ones that contain the endpoints must have two iterates that after cancellation yield equivalent segments (otherwise we will have M equivalent legal segments of length $\leq C$, but this contradicts to the choice of M).
Therefore, we have that each segment contains a preperiodic point so that these points subdivide L as $x \cdot y_1 \cdot \ldots \cdot y_m \cdot z$, and we have that this path satisfies the third statement. □

Firstly we will prove a useful lemma for IWIP automorphisms which satisfy the property NGC and then we see that how we can use it for GC automorphisms.

Lemma 8.3. *Let ψ, ψ^{-1} irreducible automorphisms (IWIP'S), $h : S \to S$ train track map representing ψ, $h' : S' \to S'$ representing ψ^{-1} and let's suppose that there is an integer m so that it is impossible to concatenate more than m N- periodic in S and in S'. Let $\tau : S \to S$, $\tau' : S' \to S'$, $\mathcal{O}$-maps.*
Then for any $C > 0$ there are constants $N_0 > 0$ and L_0 such that if j is line or a path of length $\geq L_0$ and if j' the isometric immersion obtained from $[\tau j]$, then one of the following holds:

(A) $[h^M(j))]$ contains a legal segment of length $> C$

(B) $[h'^M(j')]$ contains a legal segment of length $> C$

Proof. Without loss, we may assume that C is larger than the critical constants for h and for h'. Let M be the larger of the two integers guaranteed by previous lemma applied to h, C and h', C. We will fix a large integer $s = s(h, h', \tau, \tau', M)$. Suppose that (A) does not hold with $N_0 = sM$. We will apply the previous lemma only to h^M-admissible segments (a segment $L \subseteq j$ so that $h^M(\partial L) \subseteq [h^M(j)]$). By our assumption the first of the previous lemma doesn't hold. If we further restrict to segments L with $> m + 2$ illegal turns, then we can't have the third case either. So for such segments the second is always true. We can represent j as a concatenation of such segments of uniformly bounded length and the uniform bound does not depend on j, but only on h, h', τ, τ', M (since we will apply the same argument using $[\tau h(j)], h'$ instead of j, h respectively).
Say p is an upper bound to the number of illegal turns in each segment (there are finitely since they are of uniformly bounded length). Fix a with $\frac{p-1}{p} < a < 1$. For long enough segments L in j the ratio $\frac{\text{number of illegal turns in } [h^M(L)]}{\text{number of illegal turns in } L} < a$ (since the number of illegal turns in L than p and number of illegal turns in $[h^M(L)]$ is strictly less that the number of illegal turns in L).
By applying the same argument to $h^M(j)$ and then to $h^{2M}(j)$ etc, we see that for given $s > 0$ and long enough segments $L \subseteq j$ (the length depends on s as well), we have $\frac{\text{number of illegal turns in } [h^{sM}(L)]}{\text{number of illegal turns in } L} < a^s$,
or else (A) holds with $N_0 = sM$. Since legal segments have length above by C and below by the length of the shortest edge(with the exception of the two containing the endpoints), the length can be compared with two inequalities to the number of illegal turns. Therefore if (A) fails, there exists a constant $A = A(h, C)$ with the property $\frac{\text{length}[h^{sM}(L)]}{\text{length}(L)} < Aa^s$. Similarly, we can use the same argument using $[\tau h^{sM} j]$ in place of j and with h' in place of h. If (B) fails as well,(with $N_0 = sM$) we reach a similar conclusion that $\frac{\text{length}[h'^{sM}\tau h^{sM}(L)]}{\text{length}[\tau h^{sM}(L)}< Ba^s$ for some B depends only on h', C.
Firstly, we note that $h'^{sM}\tau h^{sM}, \tau$ are both $\mathcal{O}$-maps so they coincide to every path, except some bounded error near endpoints, in particular for long L, we have that the ratio of their lengths is bounded above by 2 and below by 1/2. Therefore multiplying the above inequalities and changing $h'^{sM}\tau h^{sM}$ by τ we have the inequality :

$$\frac{\text{length}[\tau(L)]}{\text{length}[\tau h^{sM}(L)]} \frac{\text{length}[h^{sM}(L)]}{\text{length}(L)} < 2ABa^{2s}. \tag{6}$$

On the other hand, $\frac{\text{length}[h^{sM}(L)]}{\text{length}(\tau h^{sM} L)} \frac{\text{length}[\tau(L)]}{\text{length}(L)} > \frac{1}{2Lip(\tau)Lip(\tau')}$ using again that $\tau'\tau$ and the identity are both $\mathcal{O}$- maps as above.
But sending s to infinity we have a contradiction, since $a < 1$. □

Geometric Case: In the proof of the previous lemma we have used the property that there is an integer m so that it is impossible to concatenate more than m N-periodic paths in j (and the iterates $[h^M(j)]$) and the same is true for j' (and the iterates $[h'^M(j')]$). The previous lemma is true for NGC automorphisms for every j.

But if we apply this when j is some leaf of the lamination and $h \in Stab(\Lambda)$, we can prove that this always the case.

Lemma 8.4. *If ℓ is some leaf of the lamination, then there is an integer m so that it is not possible for ℓ to contain a concatenation of m subpaths that each of them is N-periodic.*

Proof. Choose $f : T \to T$, stable train track representative (this is possible by [5], since N-periodic paths correspond to Nielsen periodic paths in the quotient or see [23] for a different approach), then there is exactly one path in $\Gamma = G/T$ in which every (indivisible) N-periodic path projects. We suppose that there is no bound in the number of concatenation of INP in ℓ. So by quasiperiodicity we have that every leaf segment is contained in some concatenation of equivalent paths of the form $P_1 P_2 ... P_n$ (where every P_i is a path that projects to the loop P). But then the subgroup that is constructed by the graph of groups corresponding to this loop (see the section 2.6. of the preliminaries), carries the lamination and therefore has finite index (by 6.2) in G, which is impossible. □

Therefore the lemma 8.3 is true, in this case, if we restrict to $h \in Stab(\Lambda)$ and ℓ some leaf of the lamination.

Definition 8.5. We say that a sequence $\{\Lambda_i\}$ of irreducible laminations in $\mathcal{IL}$ if for some (any) tree H every leaf segment of Λ in S - coordinates is a leaf segment of Λ_i in S -coordinates for all but finitely many i.

Proposition 8.6. *Let $\Lambda = \Lambda^+_\phi \in \mathcal{IL}$ and let $\psi \in Aut(G, \mathcal{O})$ which is an IWIP. Suppose that $\psi \in Stab(\Lambda)$, then $\Lambda = \Lambda^+_\psi$ or $\Lambda = \Lambda^-_\psi$.*

We note again that if a segment contains a legal segment with length larger than C_{crit} then the length of reduced iterates converge to infinity.

Proof. In the non-geometric case:
Using the notation of the previous lemmas. Let ℓ be a leaf of Λ in the S -coordinates. We apply the lemma to $[h^K \ell]$ with $K > 0$ and C larger the critical constants of h and h'. If for some $K > 0$ (A) holds, then if follows from quasiperiodicity that the forward iterates weakly converges to Λ^+_ψ, since we have that the length of reduced images converges to infinity and so we have arbitarily long legal segments and the quasiperiodicity implies that some translation of every leaf segment is finally contained in the reduced images.
The remaining possibility is that $[\tau h^K \ell]$ contains an S' legal segment of length $> C$ for all $K > 0$. But this means that $[\tau \ell]$ which equals to $[h'^K \tau h^K \ell]$ up to bounded error, contains an arbitarily high h'-iterate of a legal segment and quasiperiodicity now implies that $\Lambda = \Lambda^-_h$.

Now in the geometric case, we use the same argument but only for $h \in Stab(\Lambda)$ and we have the same result that $\Lambda = \Lambda_h^{\pm}$ □

Note that we have proved that for automorphisms with the property NGC, it is true for every IWIP ψ (relative to $\mathcal{O}$) either the forward ψ -iterates of Λ weakly converges to Λ_ψ^+ or $\Lambda = \Lambda_\psi^-$.

Corollary 8.7. *If $\psi \in Stab(\Lambda)$ is exponentially growing, then $\psi \notin Ker(\sigma)$*

Proof. For reducible automorphisms, we have already proved it in 8.1.
For irreducible ones, we have by the previous proposition that $\Lambda = \Lambda_\psi^+$ (changing ψ with ψ^{-1}, if it necessary) and so we can choose $f = h$, where h is the train track representative of ψ, in the proof of 7.1, and then $\sigma(\psi)$ is obviously equal to the Perron - Frobenious eigenvalue which is greater than 1, since ψ is exponentially growing(it is an IWIP). □

8.3 Discreteness of the Image

We will prove that the image of the homomorphism σ is discrete and therefore we can see σ as a homomorphism $\sigma : Stab(\Lambda) \to \mathbb{Z}$.

Lemma 8.8. *$\sigma(Stab(\Lambda))$ is a discrete set.*

Proof. This is true since by the proofs of the propositions (8.1, 8.7), every $\sigma(\psi)$ other than 1, occurs as the Perron- Frobenius eigenvalue for an irreducible integer matrix of uniformly bounded size. It is well known then that the set of such numbers form a discrete set and as a consequence $\sigma(Stab(\Lambda))$ is an infinite discrete subset of $\mathbb{R}$ and is hence isomorphic to $\mathbb{Z}$. □

9 Main Results

In this section, we will state and prove the main theorems. We use the same notation as in the sections above.

Lemma 9.1. *Let $h : S \to S$ be a relative train track representative of $\psi \in Ker(\sigma)$. Then there is some k such that h^k induces the identity on $G \backslash S$. Moreover, there are appropriate representatives of orbits of non -free vertices $v_1, ..., v_q$, such that $h(v_i) = v_i$. Finally, if $\psi \in Ker(\sigma) \cap Out(G, \{G_i\}^t)$ then ψ is an automorphism of finite order.*

Proof. Let $\psi \in Ker(\sigma)$ and $h' : S \to S$ be a RTT train track representative of ψ.

By 8.7, possibly after changing ψ with some iterate ψ^k, we can suppose that there is a relative train track representative, $h'^k = h : S \to S$ and a maximal proper h-invariant G-subgraph S_0 of S (we denote by H_0 the quotient S_0/G) s.t. the restriction of h on S_0 induces the identity in H_0. For the top stratum we can suppose that it contains a single edge e and that $h(e) = ea$, where a is some segment of S_0 (since it is non-exponentially growing). But then since h is a relative train track and $h \in Stab(\Lambda)$, we have that h-iterates of e produces arbitarily long segments of the lamination that are contained in S_0 which contradicts quasiperiodicity, except if the leaf of the lamination is of the form (in the quotient, so every a, e correspond to orbits):

$$...ea^{b_{-1}}e^{-1}x_{-1}ea^{b_0}e^{-1}x_0ea^{b_1}e^{-1}x_1ea^{b_2}e^{-1}...$$

for some integers b_i and x_i are contained in S_0 (or H_0 in the quotient). In this case, the lamination is carried by the subgroup which is the fundamental group of the graph of groups which consists of the disjoint union of two graph of groups corresponding to H_0 (which contains all the non-free vertices with full stabilisers) that are joined by an edge corresponding to e. But by 6.2, this leads to a contradiction since it is obvious that this subgroup is not of finite index (and by construction it contains the full stabilisers of vertices). Therefore we have that $h(e) = e$ and then h induces also the identity on $\Gamma - H_0$ ($\Gamma = G/T$) and so on Γ.
Now suppose $h(e_1) = e_1, h(e_2) = g_0e_2$, where $g_0 \in G_v$ as above, where e_1e_2 is a legal path. Then since h is a isometry we have that we will have as leaf segments of the form $e_1g_ne_2$ where $g_n = \psi^n(g_1)$. But since there are finitely many inequivalent paths of a specific length, we can get that after passing some power if needed, that there is some $g \in G_v$ such that $h(ge_2) = ge_2$ and after changing the fundamental domain (in particular, e_2 with ge_2), we have that h fixes pointwise the fundamental domain. Since this can be done for every vertex we have that we can suppose that every edge of the fundamental domain is fixed by h (after possibly passing to some power). Therefore h is an automorphism that sends a path of the form $g_1e_1, ..., g_me_m$ to the path $\psi(g_1)e_1, ..., \psi(g_m)e_m$ where $g_i, \psi(g_i) \in G_{\varnothing(e_i)}$, and as a consequence h depends only on the induced automorphisms on each G_i.
From the discussion above, if we assume also that $\psi \in' Out((G, \{G_i\}^t)$ the induced automorphism on each G_i is the identity and therefore h is the identity, which implies that there is some k such that ψ^k is represented by the identity and that means that ψ^k is the identity. □

As a consequence, let's consider the subgroup $A = Out((G, \{G_i\}^t) \cap Ker(\sigma)$ which, by the previous lemma, is periodic. Then $Stab(\Lambda)/A$ has a normal subgroup $B = Ker(\sigma)/A$, which is isomorphic to a subgroup $\bigoplus_{i=1}^{p} A_i$ of $\bigoplus_{i=1}^{p} Out(G_i)$ and $(Stab(\Lambda)/A)/B$ is an infinite cyclic group. If we further assume that $Out(G)$ is virtually torsion free (as for example in the free and the relative free case), then we have that $Stab(\Lambda)$ has a (torsion free) finite index subgroup A'. Then $A' \cap A = 1$ (since A is torsion free and A' periodic) and so that $A'/B \cap A'$ is isomorphic to $\mathbb{Z}$, where $A' \cap B$ is isomorphic to a subgroup of $\bigoplus_{i=1}^{p} Out(G_i)$. Finally, let's also assume that every $Out(G_i)$ is finite, then we get that A' is actually isomorphic to $\mathbb{Z}$, so we have exactly the same result as in the classical case of the free group, and more precisely $Stab(\Lambda)$ is virtually $\mathbb{Z}$. As conclusion of the discussion above, we get:

Theorem 9.2. *1. There is a normal periodic subgroup A of $Stab(\Lambda)$, such that the group $Stab(\Lambda)/A$ has a normal subgroup B isomorphic to subgroup of $\bigoplus_{i=1}^{q} Out(G_i)$ and $(Stab(\Lambda)/A)/B$ is isomorphic to $\mathbb{Z}$.*

2. *Let's also suppose that $Out(G)$ is virtually torsion free. Then $Stab(\Lambda)$ has a (torsion free) finite index subgroup K such that K/B' is isomorphic to $\mathbb{Z}$, where B' is a normal subgroup of K isomorphic to subgroup of $\bigoplus_{i=1}^{q} Out(G_i)$.*

3. *Finally, if we further suppose that every $Out(G_i)$ is finite, then $Stab(\Lambda)$ is virtually (infinite) cyclic.*

A direct corollary of the previous theorem is the following. Let's denote $C(\phi)$ the relative centraliser of ϕ in $Out(G, \mathcal{O})$. As we have seen, $C(\phi)$ is a subgroup of $Stab(\Lambda)$ and so we get:

Theorem 9.3. *1. There is a normal periodic subgroup A_1 of $C(\phi)$, such that the group $C(\phi)/A_1$ has a normal subgroup B_1 isomorphic to a subgroup of $\bigoplus_{i=1}^{q} Out(G_i)$ and $(C(\phi)/A_1)/B_1$ is isomorphic to $\mathbb{Z}$.*

2. *Let's also suppose that $Out(G)$ is virtually torsion free. Then $C(\phi)$ has a (torsion free) finite index subgroup K_1' such that K_1'/B_1' is isomorphic to $\mathbb{Z}$, where B_1' is a normal subgroup of K_1' isomorphic to subgroup of $\bigoplus_{i=1}^{q} Out(G_i)$.*

3. *Finally, if we further suppose that every $Out(G_i)$ is finite, then $C(\phi)$ is virtually (infinite) cyclic.*

Note that in the case in which $\mathcal{O}$ corresponds to the Grushko decomposition of G, we have that the previous theorem generalises the theorem in the classical case that the centraliser of an IWIP (for f.g. free groups with the absolute notion of irreducibility) is virtually cyclic since there are no G_i's and so the factor automorphisms are trivial in the free case. Additionally, we can take also relative results for the free and for the general case. This is possible since we can use the fact that every automorphism is irreducible relative to some appropriate space.

Moreover, note that if ϕ doesn't commute with the automorphisms of the free factors then $C(\phi)$ is virtually cyclic. But as we will see, in the general case there are examples that this is not true. In particular, we can find centralisers of IWIP automorphisms (relative to some space) which contain big subgroups and as a consequence they are not virtually cyclic.

In fact, we can get something stronger than the previous theorem. Remember that if G is a group and H is a subgroup of G, the commensurator (or virtual normalizer) of H in G is the subgroup $Comm_G(H) =: \{g \in G | [H : H \cap g^{-1}Hg] < \infty$, and $[g^{-1}Hg : H \cap g^{-1}Hg] < \infty\}$. Here we have that the commensurator $Comm_{Out(G,\mathcal{O})}(\phi)$ contains $C_{Out(G,\mathcal{O})}(\phi)$ for every automorphism ϕ. But for every IWIP ϕ the subgroup $Comm_{Out(G,\mathcal{O})}(\phi)$ stabilises the lamination, since for $\psi \in Comm_{Out(G,\mathcal{O})}(\phi)$ there are n, m such that $\psi\phi^m\psi^{-1} = \phi^n$, we get a similar statement as above for commensurators of IWIP automorphisms instead of centralisers.

Now let's give an example of a relative IWIP which has (relative) centraliser which fails to be virtually cyclic.

Example 9.4. As in the introduction, we fix the free product decomposition $G = G_1 * < a > * < b >$, where a, b are of infinite order and we denote by $F_2 =< a > * < b >$ the "free part". Then in the corresponding outer space $\mathcal{O}(F_4, G_1, F_2)$, which we denote by $\mathcal{O}$. In each tree $T \in \mathcal{O}$ there is exactly one non free vertex v_1 s.t $G_{v_1} = G_1$. Then we define the outer automorphism ϕ, which satisfies $\phi(g) = g$ for every $g \in G_1$, $\phi(a) = bg_1, \phi(b) = ab$ for some non-trivial $g_1 \in G_1$, then $\phi \in Out(G, \mathcal{O})$ is an IWIP relative to $\mathcal{O}$. But every factor automorphism of G_1 that fixes g_1 commutes with ϕ and therefore $C(\phi)$ contains the subgroup A of $Aut(G_1)Inn(G)$ that fixes g_1. So the centraliser is not virtually cyclic if A is sufficiently big.
We will prove that ϕ is an IWIP relative to $\mathcal{O}$. Firstly, note that there are no ϕ - invariant free factor systems of the form $\{[G_1], [< b >]\}$ or $\{< G_1, b >\}$ that contain the free factor system $\{[G_1]\}$. So the only possible case is the case where there is a ϕ-invariant free factor system of the form $\{[G_1], [< x, y >]\}$. Using the fact that we have two free factors, we can assume that the free factors G_1 and $< x, y >$ are actually ϕ-invariant. Therefore after possibly changing the basis we can suppose that the projection map from G to $G/ << G_1 >>=< a, b >$ sends x, y to a, b, respectively. Moreover, we can see that $x = am$, $y = bn$, where $m, n \in<< G_1 >>$. By the relations, $\phi(G_1) = G_1$ and $\phi(< x, y >) =< x, y >$, we have that ϕ induces the identity on G_1 (after possibly conjugacy with an element of G_1). In the first case, we can see that $\phi(x) = xy$ and $\phi(y) = x$. Then we get the identities $(am)(bn) = ab(\phi(m))$ and $am = ag_1\phi(n)$. By combining these together, we have that $mbg_1^{-1}\phi^{-1}(m) = b\phi(m)$ which easily leads to a contradiction to the fact that $m \in<< G_1 >>$. Similarly, we get a contradiction in the second case. Therefore there is no such a ϕ-invariant free factor system.
In the case that G_1 is isomorphic to F_3 and g_1 an element of its free basis, we have that $C(\phi)$ contains a subgroup which is isomorphic to $Aut(F_2)Inn(G)$.

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Stabiliser of an Attractive Fixed Point of an IWIP Automorphism of a free product

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Abstract

For a group G of finite Kurosh rank and for some arbiratily free product decomposition of G, $G = H_1 * H_2 * ... * H_r * F_q$, where F_q is a finitely generated free group, we can associate some (relative) outer space $\mathcal{O}(G, \{H_1, ..., H_r\})$. We define the relative boundary $\partial(G, \{H_1, ..., H_r\}) = \partial(G, \mathcal{O})$ corresponding to this free product decomposition, as the set of infinite reduced words (with respect to free product length). By denoting $Out(G, \{H_1, ..., H_r\})$ the subgroup of $Out(G)$ which is consisted of the outer automorphisms which preserve the set of conjugacy classes of H_i's, we prove that for the stabiliser $Stab(X)$ of an attractive fixed point in $X \in \partial(G, \{H_1, ..., H_r\})$ of an irreducible with irreducible powers automorphism relative to $\mathcal{O}$, it holds that it has a (normal) subgroup B isomorphic to subgroup of $\bigoplus_{i=1}^{r} Out(H_i)$ such that $Stab(X)/B$ is isomorphic to $\mathbb{Z}$. The proof relies heavily on the machinery of the attractive lamination of an IWIP automorphism relative to $\mathcal{O}$.

Contents

1 Introduction

The outer automorphism group $Out(F_n)$ of a finitely generated free group F_n has been extensively studied. In particular, $Out(F_n)$ has been studied via its action on the outer space CV_n, which has been introduced by Culler and Vogtmann in [6]. Let G be a group of finite Kurosh rank, i.e. G can be written as a finite free product of the form $G = G_1 * G_2 * ... * G_m * F_n$, where all the G_i's are freely indecomposable and F_n is a finitely generated free group. The concept of the outer space can be generalised for such a group G and there is a contractible space of G-trees, $\mathcal{O}(G, \{G_1, G_2, ..., G_m\}, F_n)$ on which $Out(G)$ acts. This space was introduced by Guirardel and Levitt in [9]. In fact, for a group G as above and any non-trivial free product decomposition $G = H_1 * H_2 * ... * H_r * F_q$ (here H_i may be freely decomposable or isomorphic to $\mathbb{Z}$), they constructed a relative outer $\mathcal{O}(G, \{H_1, H_2, ..., H_r\}, F_q)$ on which the subgroup $Out(G, \{H_1, ..., H_r\})$ of $Out(G)$ acts, where $Out(G, \{H_1, ..., H_r\}) = \{\Phi \in Out(G) |$ for every $i = 1, .., r$, there is some j s.t. $\phi(H_i) = g_i H_j g^{-1}\}$.
For a finitely generated free group F_n, it is well known that we can define the boundary ∂F_n which is a Cantor set and it can be viewed as the set of infinite reduced words (for some fixed basis of F_n). Moreover, an automorphism $\phi \in Aut(F_n)$, can be seen as a quasi-isometry of F_n and therefore it induces a homeomorphism of the boundary ∂F_n, which we denote by $\partial\phi$. As a consequence, we can study the subgroup $Stab(X)$ of automorphisms that fix the infinite word $X \in \partial F_n$, i.e. $\phi \in Stab(X)$ iff $\partial\phi(X) = X$. In this paper, we would like to study the corresponding notions for a group G (of finite Kurosh rank) relative to some fixed non-trivial free product decomposition and especially the $Aut(G, \{H_1, H_2, ..., H_r\})$- stabiliser of infinite reduced words. Firstly, we fix the outer space corresponding to some free product decomposition of G, as above. In this case, we can define a (relative) boundary $\partial(G, \{H_1, ..., H_r\})$ as the set of infinite reduced words for the free product length (for some fixed basis of the free group). Similarly, every $\phi \in Aut(G, \{H_1, ..., H_r\})$ induces a homeomorphism $\partial\phi$ of the relative boundary $\partial(G, \{H_1, ..., H_r\})$. It is natural to ask if we can compute the subgroups $Stab(X)$ for any $X \in \partial(G, \{H_1, ..., H_r\})$. However, there is no full answer even in the free case. There is a result of Hilion in this direction (see [11]). An automorphism of $Aut(F_n)$ is said IWIP (i.e. irreducible, with irreducible powers), if no non-trivial free factor of F_n is mapped by some power $k > 1$ to a conjugate of itself. Therefore, using this terminology, Hilion's result can be stated as:

Theorem. (Hilion, [11]) If $X \in \partial F_n$ is an attractive fixed point of an IWIP automorphism, then $Stab(X)$ is infinite cyclic.

In the general case, we say that an automorphism ϕ is IWIP relative to $\mathcal{O}$, if there is non-trivial free factor B of G that strictly contain some conjugate of some $H_i, i = 1, ..., r$, it is mapped by some power ϕ^k, $k > 1$ to a conjugate of itself. Then the main result of the present paper is a generalisation of the previous theorem.

However, here there is a difference that arises from the factor automorphisms of the H_i's. More precisely:

Main Theorem. If $X \in \partial(G, \{H_1, ..., H_r\})$ is an attractive fixed point of an IWIP automorphism ϕ, then $Stab(X)$ has a subgroup B isomorphic to a subgroup of $\bigoplus_{i=1}^{p} Out(H_i)$ and $Stab(X)/B$ is infinite cyclic.

In our case, there are examples of X, as above, where $Stab(X)$ is not infinite cyclic. We describe such an example in the last section, see 4.6. The main idea is that there are attractive fixed words of IWIP automorphisms that contain even finitely many elements of the elliptic free factors. Moreover, if a factor automorphism (an automorphism of some H_i) fixes these words, then it stabilises the attractive fixed point. So if H_i's are sufficiently big, there are non-trivial automorphisms of H_i that fix these words. As a consequence, we can find arbitrarily large subgroups of $Stab(X)$. On the other hand, if we suppose that every $Out(H_i)$ is finite, then we have a similar result as in the free case. In particular:

Corollary 1.1. *If $X \in \partial(G, \{H_1, ..., H_r\})$ is an attractive fixed point of an IWIP automorphism ϕ and every $Out(H_i)$ is finite, then $Stab(X)$ is virtually cyclic.*

Our proof is similar to that of [11], but we have to adjust the notions and to use the generalisations of the results used by Hilion, for the general case of free products. In particular, we use the work of Francaviglia and Martino [7] for train track representatives of IWIP automorphisms of a free product instead of the classical notion of train track representatives of automorphisms of free groups [2]. In fact, here an IWIP automorphism relative to $\mathcal{O}$ can be represented by a train-track map which is a G-equivariant, Lipschitz map $f : T \to T$, where $T \in \mathcal{O}$ and $f(gx) = \phi(g)f(x)$, with the property that no backtracking subpath occurs if one iterates the train-track map on any edge of T. As a consequence of the general notion of train-track representatives, the author in [19] generalised the work of Bestvina, Handel and Mosher in the free case [1], and in particular we have the notion of the attractive lamination of an IWIP automorphism.
Now let us describe the basic steps of the proof. Fistly, we construct a nice splitting of a given attractive fixed point X of an IWIP automorphism, using train track representatives, which matches the language of the attractive lamination. Then we relate the subgroup $Stab(X)$ to the stabiliser of the attractive lamination, and so using the fact proved in [19] about the stabiliser of the attractive lamination and a technical lemma, and more specifically the fact that $Stab(X) \cap Out(G, \{H_i\}^t)$ is torsion free, we get the main result.
As we have seen, there are a lot of facts that they are shard by CV_n and the general space $\mathcal{O}$. As we have already mentioned that the train track representatives can be generalised in the general case. In the same paper, there is the construction and the properties of the Lipschitz metric for $\mathcal{O}$ which is a metric that the same authors previously studied for

CV_n (see [8]). Recently, there are more papers that they indicate that we can find more similarities between CV_n and $\mathcal{O}$. For example, the construction of hyperbolic spaces on which $Out(G, \{H_1, ..., H_r\})$ acts ([10]), [13]), the boundary of outer space ([12]), the Tits alternative for subgroups of $Out(G)$ ([14]), the study of the asymmetry of the Lipschitz metric([20]) and the study of the centralisers of IWIP automorphisms([19]).
OUTLINE: In Section 2, we recall some useful definitions and facts, in additional we generalise some well known notions for free groups to the free product case and we prove some basic preliminary results that we need for the main theorem. In Section 3, we describe the construction of the attractive lamination for an IWIP automorphism and we list some properties. The last section is devoted to the proof of the main theorem.

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2 Preliminaries

2.1 $\mathbb{R}$ -trees, Kurosh rank

Let G be a group of *finite Kurosh rank* i.e G splits as a free product $G = H_1 * ... * H_s * F_r$, where every H_i is non-trivial, not isomorphic to $\mathbb{Z}$ and freely indecomposable. Here the Kurosh rank of G is just the number $s + r$. This decomposition is called the *Gruskho decomposition.* It is the "minimal" decomposition of G and it is unique, in the sense that the free rank r is well defined and the H_i's are unique up to conjugation. The class of groups of finite Kurosh rank contain strictly the class of finitely generated groups (by the Grushko theorem). We are interested only for groups which have finite Kurosh rank. In particular, for such a group G we fix an *arbitary* (non-trivial) free product decomposition $G = H_1 * ... * H_m * F_n$ (i.e. we don't assume that every H_i is not infinite cyclic or even freely indecomposable). However, we usually assume that $m + n > 2$. These groups admit co-compact actions on $\mathbb{R}$-trees (and vice-versa). It is useful that we can also apply the theory in the case that G is free, and the H_i's are certain free factors of G (relative free case).

We consider isometric actions of the group G on $\mathbb{R}$-trees induced by the free product decomposition and, more specifically, we say that T is a *G-tree,* if it is a simplicial metric tree (T, d_T), where G acts simplicially on T (sending vertices to vertices and edges to edges) and for all $g \in G, e \in E(T)$ we have that e and ge are isometric. Moreover, we suppose that every G-action is *minimal*, which means that there is no G-invariant proper subtree.
Now let's fix a G-tree T. An element $g \in G$ is called *hyperbolic*, if it doesn't fix any points of T. Any hyperbolic element g of G acts by translations on a subtree of T

homeomorphic to the real line, which is called the axis of g and is denoted by $axis_T(g)$. The *translation length* of g is the distance that g translates its axis. The action of G on T defines a length function denoted by

$$\ell_T : G \to R, \ell_T(g) := \inf_{x \in T} d_T(x, gx).$$

In this context, the infimum is always minimum and we say that $g \in G$ is hyperbolic if and only if $\ell_T(g) > 0$. Otherwise, g is called *elliptic* and it fixes a (unique) point of T. For more details about group actions on $\mathbb{R}$-trees, see [5].

2.2 Relative Outer Space

In this subsection we recall some basic definitions and properties. More details about the relative outer space can be found in [7].
We consider G-trees as in the previous subsection. We will define an outer space $\mathcal{O} = \mathcal{O}(G, (H_i)_{i=1}^m, F_n)$ relative to some fixed free product decomposition of G. More specifically, the elements of the outer space can be thought as simplicial metric G-trees, up to G-equivariant homothety. Moreover, we require that these G-trees also satisfy the following conditions:

- The action of G on T is minimal.
- The edge stabilisers are trivial.
- There are finitely many orbits of vertices with non-trivial stabiliser, more precisely for every H_i, $i = 1, ..., m$ (as above) there is exactly one vertex v_i with stabiliser H_i (all the vertices in the orbits of v_i's are called *non-free vertices*).
- All other vertices have trivial stabiliser (and we call them *free vertices*).
- The quotient G/T is a finite graph of groups

Note that the last condition follows from the others, but we mention it in order to emphasise the importance of the co- compactness of the action.

Action: Let $Aut(G, \mathcal{O})$ be the subgroup of $Aut(G)$ that preserve the set of conjugacy classes of the H_i 's. Equivalently, $\phi \in Aut(G)$ belongs to $Aut(G, \mathcal{O})$ iff $\phi(H_i)$ is conjugate to one of the H_j 's (in general, i may be different to j). The group $Aut(G, \mathcal{O})$ admits a natural action on a simplicial tree by "changing the action", i.e. for $\phi \in Aut(G, \mathcal{O})$ and $T \in \mathcal{O}$, we define $\phi(T)$ to be the metric tree with T, but the action is given by $g * x = \phi(g)x$ (where the action in the right hand side is the action of the G-tree T). As $Inn(G)$ acts on $\mathcal{O}$ trivially, $Out(G, \mathcal{O}) = Aut(G, \mathcal{O})/Inn(G)$ acts on $\mathcal{O}$. Note also that in the case of the Grushko decomposition, we have $Out(G) = Out(G, \mathcal{O})$.

Remark. Note that for a $g \in G$ and $T, S \in \mathcal{O}$, it holds that g is hyperbolic relative to T iff g is hyperbolic relative to S. Therefore it makes sense to say that g is hyperbolic relative to $\mathcal{O}$, as we will do. We denote by $Hyp(\mathcal{O})$ the set of hyperbolic elements of $\mathcal{O}$.

2.3 Topological Representatives, $\mathcal{O}$ - Maps

Edge paths: Firstly, we would like to define the notion of an edge path for some tree $T \in \mathcal{O}$. More specifically, since T is an $\mathbb{R}$-tree we have that any edge is isometric to the interval $[0, \ell(e)]$. We say that an edge path is a reduced path of the form $e_1 e_2 ... e_n$ (without backtracking). We can also define an infinite edge path, as an infinite reduced path of the form $e_1 e_2 ... e_n e_{n+1} ...$. Similarly, we can define a bi-infinite edge path. We usually call paths lines these paths.
Tightening: Every path p is homotopic (relative endpoints) to a unique edge path $[p]$ in T. Actually, we can obtain from p the path $[p]$, after removing the backtracking, and we say that $[p]$ is obtained by tightening p.

$\mathcal{O}$ - maps:

Definition 2.1. We say that a map between trees $A, B \in \mathcal{O}$, $f : A \to B$ is an **$\mathcal{O}$- map**, if it is a G-equivariant, Lipschitz continuous, surjective function.

It is very useful to know that there are such maps between any two trees. This is true and, additionally, by their construction they coincide on the non - free vertices. More specifically, by [7] we get:

Lemma 2.2. *For every pair $A, B \in \mathcal{O}$; there exists a $\mathcal{O}$-map $f : A \to B$. Moreover, any two $\mathcal{O}$-maps from A to B coincide on the non-free vertices.*

Now we will prove that every $\mathcal{O}$-map is a quasi-isometry. But firstly, we need a technical lemma:

Lemma 2.3. *Let $T \in \mathcal{O}$ and v be a vertex of T. Then the inclusion map ι from the G-orbit of v, $A = G \cdot b$ to T is a quasi-isometry. As a consequence, any projection p from T to A is again a quasi-isometry.*

Proof. It is obvious that the inclusion map is $1-1$ and satisfies that $d_A(x, y) = d_T(x, y)$. So it remains to show that ι is quasi- onto, which means that there is some M s.t. for every $x \in T$ there is some $g \in G$ with $d_T(x, gv) \leq M$. This follows from the fact that the quotient $\Gamma = G/T$ is compact and therefore we can choose M to be the maximum distance in Γ between the projection of v and the other vertices. Therefore the result follows. □

Lemma 2.4. *Let $T, S \in \mathcal{O}$ and $f : T \to S$ be an $\mathcal{O}$-map. Then f is a quasi-isometry.*

Proof. Let choose some vertex $v \in T$, then f induces a Lipschitz map from $A = Gv$ to $B = Gf(v)$. Note also that by the construction of $\mathcal{O}$-maps there is an $\mathcal{O}$ - map h from S to T which is the inverse function of $f_{|A}$ restricted to B and it is again Lipschitz. Therefore $f_{|A}$ is an isomorphism between A and B (and in particular quasi-isometry). Using now the lemma 2.3 and the fact that the inverse of a quasi-isometry is a quasi-isometry we get that: $T \xrightarrow{p} A \xrightarrow{f} B \xrightarrow{q} S$, where p is the projection of T to A and ι is the inclusion map, where the maps $p, q, f_{|A}$ are quasi-isometries, and it follows that f is a quasi- isometry. □

Using the lemma 2.4 and the existence of $\mathcal{O}$-maps between every two elements of $\mathcal{O}$ (see 2.2), we get that:

Proposition 2.5. *Let $T, S \in \mathcal{O}$, then we have that the metric trees T and S are quasi -isometric.*

Topological representatives: It is very useful to see an outer automorphism as a map between a tree $T \in \mathcal{O}$. More specifically:

Definition 2.6. Let $\Phi \in Out(G, \mathcal{O})$ and $T \in \mathcal{O}$, then we say that a Lipschitz surjective map $f : T \to T$ **represents** Φ if for any $g \in G$ and $t \in T$ we have $f(gt) = \Phi(g)(f(t))$. In other words, f is an $\mathcal{O}$-map from T to $\Phi(T)$.

Applying again 2.2 (the existence of $\mathcal{O}$-maps), we get:

Lemma 2.7. *Let $\Phi \in Out(G, \mathcal{O})$ and $T \in \mathcal{O}$. Then there is a (simplicial) topological representative of Φ in T.*

The topological representatives many times produce paths which are not reduced, and then we have cancellation in their images. Therefore we have to define a map (induced by f) from the reduced edge-paths of T to itself, and we denote it by $f_{\#}$, by the rule $f_{\#}(w) = [f(w)]$ for every edge-path w of T.

As these maps represent an outer automorphism Φ, if we change the tree T with $\iota_h(T)$ where $\iota_h \in Inn(G)$ is just the conjugation by some $h \in G$, we get an other $\mathcal{O}$-map that still represents Φ. Therefore each regular automorphism $\phi \in \Phi$ corresponds to some topological representative. In particular, for a topological representative $f : T \to T$ of an automorphism Φ, and for every $\phi \in \Phi$, changing appropriately the tree T with some $\iota_h(T)$, we can choose f to satisfy $\phi(g)f = fg$, for every $g \in G$. We say that f, ϕ are **mated**. Note that in this case, for $g \in G$ (as we can see g as an isometry of T):

Remark. $g \in Fix(\phi)$ if and only g and f commute.

2.4 N-periodic Paths

Here we will define the notion of an N - path. See more about the properties of N-periodic paths in [19]. A difference between the free and our case is that it is not always true that there are finitely many orbits of paths of a specific length (if there are non-free vertices with infinite stabiliser), but it is true that there are finitely many paths that have different projections in the quotient G/T. Therefore the notion of an N- path (we define it below) plays the role of a Nielsen path. Note that here if $h : S \to S$, we say that a point $x \in S$ is h-periodic, if there are $g \in G$ and some natural k s.t. $h^k(x) = gx$.

Definition 2.8.

1. Two paths p, q in $S \in \mathcal{O}$ are called *equivalent*, if they project to the same path in the quotient G/S. In particular, their endpoints $o(p), o(q)$ and $t(p), t(q)$ are in the same orbits, respectively.
2. Let $h : S \to S$ be a representative of some outer automorphism Ψ, let p be a path in S and let's suppose that the endpoints of p and $h(p)$ are in the same orbits (respectively), then we say that a path p in S is *N-path* (relative to h), if the paths $[h(p)], p$ are equivalent.

In the free case, we need representatives of outer automorphisms for which we can control the number of Nielsen paths. For example the notion of stable and appropriate train track representatives. As we will see, we can define the corresponding notions in our case but using N-paths.

2.5 Relative Boundary

Let's fix some relative outer space $\mathcal{O}$ with respect to a some fixed free product decomposition of G. We will give the definition of the relative boundary relative to $\mathcal{O}$.

For every $T \in \mathcal{O}$, we can use the Gromov hyperbolic boundary ∂T, as T is a 0-hyperbolic space (or a tree), by defining it as the set of equivalence classes of sequences of points in T that converge to infinity with respect to the Gromov product (with respect to some fixed base point p). However, it is more convenient for our purposes to define it as the set of lines passing through a base point $x \in T$. The two definitions coincide in the case of a proper (i.e. the closed balls are compact) hyperbolic metric space. But in the case of trees, we don't need the properness. For more details about the Gromov Boundary, see the very interesting survey for boundaries of hyperbolic spaces [15].

More specifically, for any two lines ℓ, ℓ' starting from $x \in T$, we define the equivalence relation by $\ell \equiv \ell'$ iff ℓ, ℓ' have an infinite common subline. Now we denote the boundary by $\partial_x T = \{[\ell] | \ell : [0, \infty) \to T \text{ is a geodesic ray with } \ell(0) = x\}$. It is not difficult to see that this definition does not depend on the base point and so we will usually omit the base point from the notation.

Let $p, q \in V(T) \cup \partial T$, we define the operation $\wedge$ as follows: $p \wedge q$ is the common initial

subpath (starting from x) of the unique edge paths $[x,p],[x,q]$ that connect p,q with the base point x. We can also define the r neighbourhood of a point r in the boundary, as $V(p,r)=\{q\in\partial_x T|$ for any geodesic rays ℓ_1,ℓ_2 starting at x and with $[\ell_1]=p,[\ell_2]=q$ we have $\liminf_{n\to\infty}|\ell_1(n)\wedge\ell_2(n)|\geq r\}$. Now we topologise ∂T by setting the basis of neighborhoods for any $p\in\partial T$ to be the collection $\{V(p,r)|r\geq 0\}$. Moreover, this topology is metrisable and in particular, the metric on ∂T is given by $d(p,q)=e^{-|[x,p]\wedge[x,q]|}$ for $p,q\in\partial T$ (where $e^{-\infty}=0$).
It is not difficult to see that any quasi-isometry $f:T\to S$, induces a homeomorphism between the boundaries $\partial T,\partial S$, as constructed. In particular, since any $\mathcal{O}$-map $f:T\to S$ is a quasi-isometry, it can be extended to the boundary and it induces a well defined homeomorphism, which we denote by $\partial f:\partial T\to\partial S$. Therefore we get that:

Lemma 2.9. *Let $T,S\in\mathcal{O}$. Then ∂T is homeomorpic to ∂S.*

Note that in our case, if there is some infinite H_i it is easy to see that ∂T is not compact in the metric topology. For example, if we have a point of infinite valence we can produce a sequence of lines that they have constant distance between each other. Therefore we have a sequence in ∂T, which has not converging subsequence. However, it is possible to find other interesting topologies for which $T\cup\partial T$ is compact. For example, see [4] for the observers' topology.

We can also define the set $\partial(G,\mathcal{O})$ of infinite reduced words with respect to the free product length which is induced by our fixed free product decomposition. For any $A,B\in G\cup\partial(G,\{H_1,...,H_r\})$, we define the operation $\wedge$ as follows: $A\wedge B$ is the longest common initial subword of A,B. It is easy to see that the map $d(A,B)=e^{-|A\wedge B|}$, for $A\neq B$ and $d(A,A)=0$ is a metric on the space $G\cup\partial G$. Finally, since any $\phi\in Aut(G,\mathcal{O})$ can be seen as a quasi-isometry of G, we have that it induces a homeomorphism of $\partial(G,\{H_1,...,H_r\})$ which we denote by $\partial\phi$. Note that the two notions of the boundary can be identified, in particular:

Lemma 2.10. *Let $T\in\mathcal{O}$. Then ∂T is homeomorphic to $\partial(G,\{H_1,...,H_r\})$.*

Proof. Consider the universal cover S of the rose of m cycles with n edges attached, corresponding to the free product decomposition $G=H_1*...*H_m*F_n$ with length of edges corresponding to the n simple loops to be 1 and of the rest of edges to have length 1/2. It is easy to see now that of an edge path starting from a base point v (it can be chosen to be the lift of the unique free vertex of the quotient) correspond to a word in G (and vice versa) and the length of the edge path is exactly the free product length of the word. Moreover, the lines starting from the base point correspond to infinite reduced words of G with respect to the free product length. Therefore there is a bijection from the set ∂S of lines of S starting from v to $\partial\partial(G,\{H_1,...,H_r\})$. Since the metrics are the similar, it is easy to see that this map is a actually a homeomorphism. But now for every $T\in\mathcal{O}$, we have that ∂T is homeomorphic to ∂S and the lemma follows. □

Note also that since $\partial T, \partial(G, \{H_1, ..., H_r\})$ are homeomorphic, we can identify ∂f and $\partial\phi$.

2.6 Rational and non-Rational Points

For every hyperbolic element g of $\mathcal{O}$, the sequence of elements g^k have arbitrarily large (free product) length and so it has a limit in the relative boundary $\partial(G, \{H_1, ..., H_r\})$, which we denote it by g^∞. We can also define $g^{-\infty}$ as $(g^{-1})^\infty$.
If $g \in G$, we denote by ι_u the inner automorphism of G given by $\iota_u(g) = ugu^{-1}$, for every $g \in G$. If $u \in Hyp(\mathcal{O})$, it is easy to see that then $\partial\iota_u$ fixes exactly two points of the relative boundary $\partial(G, \{H_1, ..., H_r\})$ and more specifically the points $u^\infty, u^{-\infty}$. Note that since edge stabilisers of elements of $\mathcal{O}$ are trivial, for an elliptic element u then the inner automorphism ι_u cannot fix a point of the boundary.
We say that infinite words of the form $u^\infty, u^{-\infty}$ for a hyperbolic element u, are **rational points** of the boundary. Alternatively, we could define the rational points as the fixed points of inner automorphisms corresponding to hyperbolic elements.

Proposition 2.11. *If $X \in \partial(G, \mathcal{O})$ is not a rational point, then the restriction of the quotient map $Aut(G, \mathcal{O}) \to Out(G, \mathcal{O})$ to $Stab(X)$ is injective.*

Proof. Let $T \in \mathcal{O}$ and let's assume that $\phi \in Aut(G, \mathcal{O})$. Suppose also that X is a fixed point of $\partial\phi$.
Let u be a non-trivial of $\mathcal{O}$ and suppose that X is fixed by $\partial(i_u \circ \phi)$, which implies that it is a fixed point of ∂i_u. As a consequence, u is a hyperbolic element and in particular X it is rational point or equivalently the axis of the hyperbolic element u, and so $X = u^\infty$ or $X = u^{-\infty}$, which leads us to a contradiction. □

2.7 Regular and Singular Fixed Points

For an automorphism $\phi \in Aut(G, \mathcal{O})$, we denote by $Fix\phi$ the fixed subgroup of ϕ: $Fix\phi = \{g \in G | \phi(g) = g\}$. Since by [3], we have that $Fix\phi$ has finite Kurosh rank, its (relative) boundary $\partial Fix\phi$ embeds into $\partial(G, \{H_1, ..., H_r\})$ and it is actually a subgroup of $Fix\partial\phi$ of fixed infinite words by $\partial\phi$. We will use the same terminology as in the free case and we distinguish two cases for infinite fixed words of an automorphism $\phi \in Aut(G, \mathcal{O})$ i.e. the elements of $Fix\partial\phi$: either it belongs to $\partial Fix\phi$ and then it is called **singular**, or otherwise it is called **regular**.
For the singular fixed points, the are two subcases. We use the notion topologically attractive (and repulsive) fixed points as in [16], but there is also the a metric notion. These definitions are different in the free product case, while they coincide in the free case. For more details, see [16].
We say that a fixed point X of $\partial\phi$ **attractive**, if there is an integer N s.t. if $|Y \wedge X| \geq N$,

then $\lim_{n\to\infty} \phi^n(Y) = X$. A fixed point X is said to be **repulsive**, if it is attractive for $\partial\phi^{-1}$. A classification of fixed points of $\partial\phi$ has been proved in the proposition 5.1.14. of [16] and more specifically:

Proposition 2.12. *Let $\phi \in Aut(G, \mathcal{O})$. A fixed point of $\partial\phi$ is :*

- *either singular*
- *or attractive*
- *or repulsive*

2.8 Bounded Cancellation Lemma

Let $T, T' \in \mathcal{O}$ and $f : T \to T'$ be an $\mathcal{O}$- map. If we have a concatenation of paths ab, ever if $f(a) = f_\#(a)$ and $f(b) = f_\#(b)$, it is possible to have cancellation in $f(a)f(b)$. However, the cancellation is bounded above by some some M which depends only on f and not on a, b . In particular, we can define the bounded cancellation constant of f (let's denote it $BCC(f)$) to be the supremum of all real numbers N with the property that there exist A, B, C some points of T with B in the (unique) reduced path between A and C such that $d_{T'}(f(B), [f(A), f(C)]) = N$ (the distance of $f(B)$ from the reduced path connecting $f(A)$ and $f(C)$), or equivalently is the lowest upper bound of the cancellation for a fixed $\mathcal{O}$-map.
The existence of such number is well known, for example a bound has given in [13]:

Lemma 2.13. *Let $T \in \mathcal{O}$, let $T' \in \mathcal{O}$, and let $f : T \to T'$ be a Lipschitz map. Then $BCC(f) \le Lip(f)qvol(T)$, where $qvol(T)$ the quotient volume of T, defined as the infimal volume of a finite subtree of T whose G-translates cover T.*

Therefore we can define a new map, in particular:

Definition 2.14. Let $f : T \to T$ be a topological representative of $\Phi \in Out(G, \mathcal{O})$ and let's denote by C the bounded cancellation lemma of f. Then for every edge path w of T, we can define the map $f_{\#,C}(w)$ as the path obtained by removing both extremities of length C from the reduced image of $f_\#$.

2.9 Train Track Representatives

In this section we will define the notion of a "good" representative of an outer automorphism $\Phi \in Out(G, \mathcal{O})$. As we have seen there are representatives of every outer automorphism (i.e. $\mathcal{O}$-maps from T to $\phi(T)$), but sometimes we can find representatives with better properties. In particular, we want a topological representative f, where $f^k(e) = f^k_\#(e)$ for every k and for every edge e. These maps, which are called *train track*

maps, are very useful and every irreducible automorphism has such a representative (we can choose it to be simplicial, as well).
We give below a more general definition of a train track map representing an outer automorphism. We are interested for these maps because we can control the cancellation (as we have seen, it is not possible to avoid it). Firstly, we need the notions of a legal path relative to some fixed train track structure.

Definition 2.15.

1. A **pre-train track structure** on a G-tree T is a G-invariant equivalence relation on the set of germs of edges at each vertex of T. Equivalence classes of germs are called **gates**.
2. A **train track structure** on a G-tree T is a pre-train track structure with at least two gates at every vertex.
3. A **turn** is a pair of germs of edges emanating from the same vertex. A **legal turn** is called a turn for which the two germs belong to different equivalent classes. A **legal path**, is a path that contains only legal turns.

Now we can define the train track maps.

Definition 2.16. An $\mathcal{O}$-map $f : T \to T$, which is representing Φ is called a train track map, if there is a train track structure on T so that

1. f maps edges to legal paths (in particular, f does not collapse edges).
2. If $f(v)$ is a vertex, then f maps inequivalent germs at v to inequivalent germs at $f(v)$.

However, we can not have such representatives for any outer automorphism. But it can be proved that for an interesting class of outer automorphisms can be represented by such a map. We will describe this class for regular automorphisms, but it can easily be defined for outer automorphisms as well.
In the free case, an automorphism $\phi \in Aut(G, \mathcal{O})$ is called *irreducible*, if it there is no ϕ-invariant free factor up to conjugation. In our case we know that the H_i's are invariant free factors, but we don't want to have "more invariant free factors". More precisely, we will define the irreducibility of some automorphism *relative* to the space $\mathcal{O}$ or to the free product decomposition. Similarly, we can define irreducibility for outer automorphisms of G.

Firstly, we will give the algebraic definition, but we need the notion of a free factor system. Suppose that G can be written as a free product, $G = G_1 * G_2 * ... G_k * F_n$. Then we say that the set $\mathcal{A} = \{[G_i] : 1 \leq i \leq k\}$ is a **free factor system for** G, where $[A]$ $= \{gAg^{-1} : g \in G\}$ is the set of conjugates of A.
Now we define an order which we denote by $\sqsupseteq$ on the set of free factor systems of G.

More specifically, given two free factor systems $\mathcal{G} = \{[G_i] : 1 \leq i \leq k\}$ and $\mathcal{H} = \{[H_j] : 1 \leq j \leq m\}$, we write $\mathcal{G} \sqsubseteq \mathcal{H}$ if for each i there exists a j such that $G_i \leq gH_jg^{-1}$ for some $g \in G$. The inclusion is strict, and we write $\mathcal{G} \sqsubset \mathcal{H}$, if some G_i is contained strictly in some conjugate of H_j. We can see $\{[G]\}$ as a free factor system and in fact, it is the maximal (under $\sqsubseteq$) free factor system. Any free factor system that is contained strictly to $\mathcal{G}$ is called **proper**. Note also that the Grushko decomposition induces a free factor system, which is actually the minimal free factor system (relative to $\sqsubseteq$).
We say that $\mathcal{G} = \{[G_i] : 1 \leq i \leq k\}$ is ϕ - **invariant** for some $\phi \in Aut(G)$, if ϕ preserves the conjugacy classes of G_i's. In each free factor system $\mathcal{G} = \{[Gi] : 1 \leq i \leq p\}$, we associate the outer space $\mathcal{O} = \mathcal{O}(G, (G_i)_{i=1}^p, F_k)$ and any $\phi \in Out(G)$ leaving $\mathcal{G}$ invariant, will act on $\mathcal{O}$ in the same way as we have described earlier.

Definition 2.17. Let $\mathcal{G}$ be a free factor system of G which is Φ- invariant for some $\Phi \in Out(G)$. Then Φ is called *irreducible relative to* $\mathcal{G}$, if $\mathcal{G}$ is a maximal (under $\sqsubseteq$) proper, Φ-invariant free factor system.

We could alternatively define the notion of irreducibility as:

Definition 2.18. We say $\phi \in Aut(G, \mathcal{O})$ is $\mathcal{O}$*-irreducible* if for any $T \in \mathcal{O}$ and choose some $f : T \to T$ representing Φ, where $\phi \in \Phi$ and f mated with ϕ, if $W \subseteq T$ is a proper f-invariant G-subgraph then W does not contain the axis of a hyperbolic element.

The next lemma confirms that the two definitions of irreducibility are related.

Lemma 2.19. *Suppose $\mathcal{G}$ is a free factor system of G with associated space of trees $\mathcal{O}$, and further suppose that $\mathcal{G}$ is ϕ-invariant. Then ϕ is irreducible relative to $\mathcal{G}$ if and only if ϕ is $\mathcal{O}$-irreducible.*

Now let's give the definition of an irreducible automorphism with irreducible powers relative to $\mathcal{O}$, which are the automorphisms that we will study.

Definition 2.20. An outer automorphism $\phi \in Out(G, \mathcal{O})$ is called **IWIP** (*irreducible with irreducible powers* or *fully irreducible*), if every ϕ^k is irreducible relative to $\mathcal{O}$.

Now as we have said above, every irreducible outer automorphism has a train track representative. This fact it generalises the well known theorem of Bestvina and Handel (see [2]) . In particular, we can apply it on every power of some IWIP automorphism.

Theorem 2.21 (Francaviglia- Martino)**.** *Let $\Phi \in Out(G, \mathcal{O})$ be irreducible. Then there exists a simplicial train track map representing Φ.*

An interesting remark is the following:

Remark. Every outer automorphism $\phi \in Aut(G)$ is irreducible relative to some appropriate space (or relative to some free product decomposition). Moreover, there are two cases: either ϕ is IWIP relative to $\mathcal{O}$ or it fixes a point of $\mathcal{O}$ (i.e. there is $T \in \mathcal{O}$ s.t. ϕ can be seen as an isometry of T).

In particular, using the remark above, in the relative free case we have some results for automorphisms of $Out(F_n)$ that they are not IWIP relative to CV_n, but they are IWIP relative to some appropriate space $\mathcal{O}$.

Splittings and Appropriate train track maps:

In this section, we will define the notion of an appropriate train track representative which is similar to the definition of the free case. As we have discussed the notion of a Nielsen path in our case it has to be replaced by the notion of a N-path. Using N-paths, we can define the stable train track representatives.

Definition 2.22. We say that a train track representative f of an outer automorphism Φ is stable, if it supports at most one equivalence class of N-paths.

It is well known that every outer automorphism can be represented by a stable train track representative (for example see [3] or [17]). Let's denote by p some representative of the unique class of the N-path that f supports, if it exists. Here we need some even better notion of train track representatives, but firstly we need the notion of a splitting. More specifically, let f as above, and let w be a path in T. We say that $w = ...w_m w_{m+1}...$, where w_i's are non-trivial subpaths of w, is a splitting for f if for all $k \geq 1$, $f^k_{\#}(w) = ...f^k_{\#}(w_m)f^k_{\#}(w_{m+1})....$. Then we use the notation: $w = ... \cdot w_m \cdot w_{m+1} \cdot ...$ and the w_i's are called the bricks of w.

Definition 2.23. A stable train track representative $f : T \to T$ of an IWIP outer automorphism $\Phi \in Out(G, \mathcal{O})$ is called appropriate, if for any path w of T, there exists some positive integer K s.t. for all $k \geq K$, $f^k_{\#}(w)$ has a splitting where the bricks are either edger or they are N-paths equivalent to p.

The proof of the next lemma is the same as in the free case, but the main difference is that we don't have finitely many paths of a given length but finitely many inequivalent paths of a given length. Therefore in the conclusion we have N-paths (and no Nielsen paths).

Lemma 2.24. *Let $\Phi \in Out(G, \mathcal{O})$ be an IWIP automorphism. Then there exists some positive power of Φ, which can be represented by an appropriate train track representative.*

3 The Attractive lamination of an IWIP Automorphism

In this section we recall the notion of an algebraic lamination. In particular, we describe the construction and the properties of the attractive lamination of an IWIP automorphism which have been proved by the author in [19]. Note that this construction is a direct generalisation of the corresponding well known notion due to Bestvina, Feighn and Handel in the free case, see [1].

3.1 Laminations

We denote by $\partial^2(G, \mathcal{O})$ the pairs of the boundary which don't belong in the diagonal, i.e. the set $\{(X, Y)|X, Y \in \partial(G, \mathcal{O}), X \neq Y\}$. Note that the topology of $\partial(G, \mathcal{O})$ induces a topology on $\partial^2(G, \mathcal{O})$. Moreover, we have a natural action of G on $\partial(G, \mathcal{O})$ which induces a diagonal action on $\partial^2(G, \mathcal{O})$.

Definition 3.1. An **algebraic lamination** L of G is a subset of $\partial^2(G, \mathcal{O})$, which is closed, G-invariant and flip invariant (i.e. if $(X, Y) \in \partial^2(G, \mathcal{O})$, then $(Y, X) \in \partial^2(G, \mathcal{O})$).

The identification of $\partial(G, \{H_1, ..., H_r\})$ with ∂T where $T \in \mathcal{O}$, implies that the lamination L induces a set of lines $L(T)$ in T, which we call the **symbolic lamination** in T-coordinates associated to L. A line of the lamination is called **leaf**. Now we can define the **laminary language** $\mathcal{L}(L(T))$ in T-coordinates as the G-set of all (orbits of) finite edge paths which occur in some leaf of $L(T)$.

3.2 Action of $Out(G, \mathcal{O})$ on the set of Laminations

Now we can define an action of $Out(G, \mathcal{O})$ on the set of algebraic laminations of G, as follows: for $\phi \in Aut(G, \mathcal{O}), (X, Y) \in \partial^2(G, \mathcal{O})$ we define the map $\partial^2\phi$ by $\partial^2\phi(X, Y) = (\partial\phi(X), \partial\phi(Y))$ which is a well defined homeomorphism of $\partial^2(G, \mathcal{O})$ (since $\partial\phi$ is a homeomorphism). This implies that this map sends an algebraic lamination to an algebraic lamination. Note that from the G-invariance of the lamination follows that the image of this map depends only of the outer automorphism Φ, where $\phi \in \Phi$. As a consequence we have a well defined action of the group of outer automorphisms $Out(G, \mathcal{O})$ on the set of algebraic laminations .

Definition 3.2. Let $\Phi \in Out(G, \mathcal{O})$ and L be an algebraic lamination. Moreover, assume $f : T \to T$ for some $T \in \mathcal{O}$ is a topological representative of Φ and let denote by C the bounded cancellation constant corresponding to f.

- We say that Φ **stabilises the algebraic lamination** L, if $\Phi(L) = L$.

- We say that f **stabilises the laminary language** $\mathcal{L}(L(T))$, if for all $w \in \mathcal{L}(L(T))$, $f_{\#,C}(w) \in \mathcal{L}(L(T))$.

Actually, these definitions are closely related. More specifically, by [19] we have:

Proposition 3.3. *Let $\Phi \in Out(G, \mathcal{O})$ be an IWIP outer automorphism and L_Φ^+ be its attractive lamination. Then Ψ stabilises L_Φ^+ iff there is some representative $h : T \to T$ of Ψ, where $T \in \mathcal{O}$, which stabilises the laminary language $\mathcal{L}(L_\Phi^+)$ of Φ.*

3.3 The Attractive Lamination of an IWIP Outer Automorphism

Here we describe the construction of the attractive lamination relative to an IWIP outer automorphism $\Phi \in Out(G, \mathcal{O})$ and we list some interesting properties.
Firstly, we recall the notion of a quasi-periodic line ℓ of $T \in \mathcal{O}$.

Definition 3.4. A line ℓ of T is called **quasi-periodic** (or q.p.) if for every $L > 0$ there exists some L' (sufficiently large) s.t. for every subpath of length L of ℓ occurs as subpath of some orbit of every subpath of ℓ of length L'.

Note that the notion of quasi-periodicity for ℓ implies that (the orbits of) every path of ℓ occurs infinite many times in both ends of ℓ and moreover the distance between any two occurrences is bounded.
Now for an IWIP automorphism Φ, let's choose some train track representative $f : T \to T$ of Φ. We can define the laminary language of our attractive lamination as the G- set $\mathcal{L}_f^+$ of the orbits of finite edge paths in T s.t. an edge path $w \in \mathcal{L}_f^+$ iff there exists an edge e of T and an integer $k \geq 1$ s.t. w is a subpath of $f^k(e)$.
It can be proved that we have the following properties:

Proposition 3.5. *[19]*

1. *For any edge e of T and for all $w \in \mathcal{L}_f^+$, there is some k such that w is a subpath of some orbit of $f^k(e)$.*
2. *There exists an algebraic lamination L_Φ^+ whose laminary language in T-coordinates is $\mathcal{L}_f^+$.*
3. *This algebraic lamination does not depend on the choice of the train track map f representing Φ and of the tree T.*
4. *Every leaf of the L_Φ^+ is quasiperiodic.*

Moreover, we have a very interesting result about the stabiliser of the lamination which has been proved by the author in [19]:

Theorem 3.6. *Let's denote by $Stab(L^+_\Phi)$ the stabiliser of the lamination. Then there is a normal periodic subgroup A of $Stab(L^+_\Phi) \cap Out(G, \{H_i\}^t)$, such that the group $Stab(\Lambda)/A$ has a normal subgroup B isomorphic to a subgroup of $\bigoplus_{i=1}^{p} Out(H_i)$ and $(Stab(\Lambda)/A)/B$ is isomorphic to $\mathbb{Z}$.*

4 Attractive Fixed points of an IWIP automorphism

In this section, we will prove the main theorem of this paper. Our result and the method is a direct generalisation of the main result of [11].

4.1 Structure of an Attractive Fixed point of an IWIP Automorphism

Proposition 4.1. *Let $\Phi \in Out(G, \mathcal{O})$ be an IWIP automorphism which can be represented by an appropriate train-track map $f : T \to T$. Let $\phi \in \Phi$, and suppose that f is mated with ϕ. Let's denote by $X \in \partial(G, \{H_1, ..., H_r\})$ an attractive fixed point of ϕ. Then there is some vertex $v \in T$ such that:*

1. $[v, f^2(v)] = [v, f(v)] \cdot [f(v), f^2(v)]$
2. *if we denote by $R_v = [v, f(v)] \cdot [f(v), f^2(v)] \cdot \ldots \cdot [f^k(v), f^{k+1}(v)] \cdot \ldots$, we have that R_v represents the point X*
3. *the segment $[v, f(v)]$ has a splitting whose bricks are either an edge or belongs to the unique equivalence class of the N- path p of T (if it exists) and, in addition, the first brick of this splitting is an edge.*

Proof. Firstly, we will prove that we can find a point $v_0 \in T$ which satisfies the properties (i) and (ii).
By the definition of an attractive fixed point of $\partial\phi$ and since $f : T \to T$ is a train track map, there exists a vertex v_0 of T s.t. the limit of iterates $f^k(v_0)$ converges to X.
We denote by R_{v_0} the line that is constructed as in item (ii), corresponding to v_0, and by our assumption we get that R_{v_0} represents X. For every k we can define inductively the points v_k, obtaining $v_{k+1} \in R_{v_0}$ as the projection of the reduced image of $f(v_k)$ in R_{v_0}. Then it is clear to see that by construction that:

$$[v_{k+1}, v_{k+2}] \subseteq [f(v_k), f(v_{k+1})] = f_\#([v_k, v + k + 1]) \subseteq f([v_k, v_{k+1}])$$

For every $k = 0, 1, 2, ...,$ we define the set $V_k = \{x \in T | f^i(v) \in [v_i, v_{i+1}],$ for every $0 \leq i \leq k\}$, and since for $y \in V_k$ we get that $f^k(y) \in [v_k, v_{k+1}]$, we have that $f^k(V_k) \subseteq [v_k, v_{k+1}]$. We will prove that this is actually an equality, i.e. $f^k(V_k) = [v_k, v_{k+1}]$.
We will prove it by induction on $n = k$:

For $k = 0$, it is obvious since $V_0 = [v_0, v_1] = f^0([v_0, v_1]) = f^0(V_0)$. Suppose now that our induction hypothesis is true for $n = k$, i.e. $f^k(V_k) = [v_k, v_{k+1}]$ and we will prove it for $n = k + 1$.
Since $[v_{k+1}, v_{k+2}] \subseteq f([v_k, v_{k+1}])$, by the induction hypothesis we get that $[v_{k+1}, v_{k+2}] \subseteq f(f^k(V_k)) = f^{k+1}(V_k)$.
Now by definition of V_k, we have that:

$$x \in V_{k+1} \iff x \in V_k \text{ and } f^{k+1}(x) \in [v_{k+1}, v_{k+2}]$$

But for some $y \in [v_{k+1}, v_{k+2}]$, as have seen, there is some $x \in V_k$ s.t. $f^{k+1}(x) = y \in [v_{k+1}, v_{k+2}]$. Using the equivalence above, we get that $x \in V_{k+1}$ and so $y \in f^{k+1}(V_{k+1})$ and our claim has been proved.
Then the equality $f^k(V_k) = [v_k, v_{k+1}]$, implies that every V_k is non-empty for every k. Since the V_k's form a decreasing sequence of non-empty closed subsets of $[v_0, v_1]$ (thus compact), which implies that the intersection of all V_k's is non-empty. In particular, there is some point $v \in V_k$ for every k. By the construction of V_k, we get that $f^k(v) \in [v_k, v_{k+1}]$ for every k and so v satisfies the properties (i) and (ii).
Now we would like to prove that we can choose v to be a vertex which additionally satisfies (iii). Let's suppose that v is not a vertex. Then after passing to some power, we can choose an appropriate train track representative, and then the path $u = [v_0, f^2(v_0)]$ has a splitting for which the corresponding bricks are either edges or lifts of the unique (up to equivalence) N-path of f (if it exists). Then we consider the initial vertex v' of the brick of u that contains $f(v_0)$. By choice of v', we have that $f^k_{\#}(v') \in f^k_{\#}(u) = [f^k(v_0), f^{k+2}(v_0)]$ for every k. Moreover, as $v' \in [v, f(v)]$, we get that $f^k(v') \in [f^k(v), f^{k+1}(v)]$. Therefore $f^k(v') \in R_{v_0}$ for all k. Therefore v' can be chosen to be a vertex.
Finally, since X is an attractive fixed point of $\partial\phi$, we have that the distance between $f^k(v')$ and $f^{k+1}(v')$ is going to infinity, as k is going to infinity. In particular, there is a brick b of $[v', f(v')]$ s.t. the length of the reduced image of $f^k(b)$ is going to infinity, which implies that b must be an edge (since the lengths of $f^k_{\#}(p)$ are bounded). Changing v' by the initial vertex of b, we find a point that satisfies (i),(i), and (iii). □

Now assuming the splitting of $[v, f(v)] = b_0 \cdot b_1 \cdot \ldots \cdot b_q$, as in the previous proposition, we can group together the successive bricks which are N-paths and they are equivalent with p (we call this new splitting, the adapted splitting). Therefore we can assume that b_0 is a single edge and every other b_i is either a single edge (and then we say that b_i is a **regular** brick) or it is equivalent to a power of p (and then we say that b_i is **singular**). Note that, by construction, between 2 singular bricks there is at least one regular brick. Moreover, the adapted splitting of $[v, f(v)]$ induces a splitting of $[f^k(v), f^{k+1}(v)] = b_{0,k} \cdot b_{1,k} \cdot \ldots \cdot b_{q,k}$ for every k, where $b_{i,k} = f^k_{\#}(b_i)$. Similarly, we extend the notions of regularity and singularity, using the corresponding notions as in the adapted splitting of $[v, f(v)]$.

Finally, note that by construction and since every $[f^k(v), f^{k+1}(v)]$ starts with a regular brick, we have that the adapted splitting of R_v still satisfies the property: between 2 singular bricks there is at least one regular brick. Note also that the lengths of the singular bricks of R_v are bounded uniformly (for instance this follows by the quasiperiocity of the any leaf of the lamination).

4.2 The Stabiliser of an Attractive Fixed point of an IWIP Automorphism

Theorem 4.2. *If $X \in \partial(G, \{H_1, ..., H_r\})$ is an attractive fixed point of an IWIP automorphism $\phi \in Aut(G, \mathcal{O})$. Assume that $\psi \in Aut(G, \mathcal{O})$ fixes X, then if we denote by Φ, Ψ the outer automorphisms corresponding to ϕ, ψ respectively, it is true that Ψ stabilises the attractive lamination L^+_Φ.*

Proof. Firstly, we note that since X is an attractive fixed point of ϕ, then X is an attractive fixed point of ϕ^k for every $k \geq 0$. Moreover, by the construction of the (attractive) lamination (see [19]) we get again that $L^+_\Phi = L^+_{\Phi^k}$ for every positive integer k.
Therefore, after possibly changing Φ with Φ^k, we can assume that Φ is represented (by applying 2.24) by an appropriate train track representative $f : T \to T$. Here we fix our notation, more specifically let $h : T \to T$ be the $\mathcal{O}$-map which represents Ψ and let denote by C the bounded cancellation constant (2.13) corresponding to h. Finally, we denote by ℓ_0 the maximal length of a singular brick in R_v, using the notation of the proposition 4.1.
Now let u be an edge path of the laminary language of the symbolic attractive lamination. We need to prove that there is an occurrence (orbit) of the reduced image (after deleting some extremal paths of length C), $h_{\#,C}(u)$ in R_v which is completely contained in a regular brick of the adapted splitting. This implies that $h_{\#,C}(u)$ is contained in $\mathcal{L}(L(T))$ and then by applying 3.3, the theorem follows.
Now since every leaf of the lamination is quasiperiodic (see 3.5 for the properties of the lamination), we can find an edge path U in the laminary language corresponding to T, which has the type $U = uu_0u$ (where by u we mean that they are of the same orbit) and u_0 can be chosen arbitrarily long.
It is convenient for us to assume that the length of $h_\#(u_0)$ is longer than the number $\ell + 2C$. This can be done since every $\mathcal{O}$ - map, and in particular h, is a quasi-isometry. (Indeed if h is a (μ, ν) quasi- isometry, it is enough to consider the starting path u_0 to be longer than $\mu(\ell + 2C + \nu)$).
Using again the quasiperiodicity and the fact that the regular bricks have unbounded lengths, we have that there is some K, s.t. eventually for every $k \geq K$, we can find an occurrence of U in every regular brick $b_{i,k}$. In particular, we can find infinitely many occurrences of U in R_v and so, by the definition of the action of an automorphism on

the set of laminations, infinitely many occurrences of $h_{\#,C}(U)$ in $h_{\#}(R_v)$.
In order to prove it, firstly we note that since $\psi(X) = X$ and so $h_{\#}(R_v) \cap R_v$ is a subray of R_v, there are infinitely many occurrences of $h_{\#,C}(U)$ in R_v. Let's denote by w_j a sequence of these distinct occurrences.
As we have seen above, the regular bricks of R_v become arbitrarily long after some steps and therefore for every path fixed path m there is a finite number of occurrences of m that fully contain a regular brick.
If there is some w_i that is fully contained in a regular brick, then there is some occurrence of $h_{\#,C}(u) \subseteq h_{\#}(U)$ which is fully contained in this brick and our claim has been proved.
Otherwise, using the remark above, after passing to a subsequence, we can see that every w_j meet at most two regular and a singular bricks of R_v. In particular, we can suppose that are three possibly cases and we will prove that our claim is always true:

1. All w_j's meet two regular bricks and one singular brick which is joining the regular ones.
   In this case, $h_{\#,C}(U) = u_1 \cdot b \cdot u_2$, where each u_i is contained in some regular brick. By the choice of ℓ_0, we can suppose that at least one of u_i's satisfies the inequality:
   $$|u_i| \geq \frac{|h_{\#,C}(U)| - \ell_0}{2}.$$
   Moreover, since by the definition of the map $h_{\#,C}$ and using the bounded cancellation lemma, since $U = uu_0u$ we have that:
   $$|h_{\#,C}(U)| \geq 2 \cdot |h_{\#,C}(u)| + |h_{\#,C}(u_0)|.$$
   Combining these inequalities with the choice of u_0, we have that $|u_i| \geq |h_{\#,C}(u)|$, which means that $h_{\#,C}(u)$ is a subpath of u_i and so it is fully contained in a regular brick.

2. All w_j's meet two consecutive regular bricks.
   In thiss case, as above, $h_{\#,C}(U) = u_1 \cdot u_2$, where both u_i's are contained in some regular bricks. As previously, there is some u_i s.t.:
   $$|u_i| \geq \frac{|h_{\#,C}(U)|}{2} \geq \frac{2 \cdot |h_{\#,C}(u)| + |h_{\#,C}(u)|}{2} \geq |h_{\#,C}(u)|$$
   Therefore we have the same conclusion: that $h_{\#,C}(u)$ is contained in some u_i and therefore in some regular brick.

3. All w_j's meet two consecutive bricks: a regular and a singular one.
   In this case, without loss, we assume that $h_{\#,C}(U) = u_1 \cdot b$, where u_1 is contained in a regular brick and b' in a singular brick. Then by the choice of ℓ_0, we get that:
   $$|u_1| \geq |h_{\#,C}(U)| - \ell_0 \geq |h_{\#,C}(u)|$$

As in the previous cases, we get that $h_{\#,C}(u)$ is a subpath of a regular brick.

□

Proposition 4.3. *Let $\phi \in Aut(G, \mathcal{O})$ be an IWIP automorphism. If X is an attractive fixed point of ϕ, and $\psi \in Aut(G, \{H_i\}^t)$ with $\psi(X) = X$ and ψ has finite order, then ψ is the identity.*

Proof. Firstly, note that since ψ has finite order, as in the free case we have that any fixed point of $\partial\psi$ is singular, i.e. $X \in \partial Fix(\psi)$.
From the Kurosh subgroup theorem we have that $Fix(\psi) = A_1 * ... * A_q * F$, where each H_i is contained in some conjugate of some G_j and F is a free group. Since $\psi \in Aut(G, \{H_i\}^t)$, it's easy to see that every A_i is exactly a conjugate of the corresponding H_j.
Combining the facts that the Kurosh rank of a the fixed subgroup is less than the Kurosh rank of G and the Schreier formula for Kurosh ranks (see [3] and [18]) if ψ is not the identity, then $Fix(\psi)$ is not of finite index and therefore applying the Proposition 6.2 of [19], does not carry the lamination. In other words, if we consider some (optimal) topological representative $h : T \to T$ of ψ and we denote by T' the subtree of T corresponding to $Fix(\psi)$, we have that X is fixed by ψ iff some line representing X is contained in T'. As we have seen, it is not possible for T' to contain some leaf ℓ of $L^+_\Phi(T)$.

So there is a finite subpath of some $\ell \in L^+_\Phi(T)$ which cannot be lifted in T'. Using the notation of the previous propositions, we denote by R_v to be the line that represents X. By the definition of the lamination, w appears in all sufficiently long regular bricks of R_v and in particular infinitely many times in R_v. Also, it appears infinitely many times in any ray R in T representing X. As a consequence, no ray representing X can be lifted to T'. Therefore X cannot be fixed by $\partial\psi$, with only exception the case where ψ is the identity.

□

Now Theorem 1 is just a corollary of the previous statements. In particular,

Corollary 4.4. *Let $\Phi \in Out(G, \mathcal{O})$ be an IWIP outer automorphism. If $X \in \partial(G, \{H_1, ..., H_r\})$ is an attractive fixed point of an IWIP automorphism $\phi \in \Phi$, then $Stab(X)$ injects into $Stab(L^+_\Phi)$ via the quotient map $Aut(G, \mathcal{O}) \to Out(G, \mathcal{O})$. Moreover, there is a normal subgroup B of $Stab(X)$ isomorphic to a subgroup of $\bigoplus_{i=1}^{p} Out(H_i)$ and $Stab(X)/B$ is isomorphic to $\mathbb{Z}$.*

Proof. Since ϕ is an IWIP, and in particular it is not an inner automorphism, we have that X is not a rational point, and therefore we can apply the proposition 2.11 and the theorem 4.2, we get that $Stab(X)$ can be seen as a subgroup of $Stab(\Lambda^+_\Phi)$. Using the basic result of [19] (Theorem 3.6), we get that there is a normal periodic subgroup A' of $Stab(X)$, such that the group $Stab(X)/A'$ has a normal subgroup B' isomorphic to a subgroup of $\bigoplus_{i=1}^{p} Out(H_i)$ and $(Stab(X)/A)/B$ is isomorphic to $\mathbb{Z}$. But then by applying 4.3, we get that $Stab(X) \cap Out(G, \{H_i\}^t)$ is torsion free, therefore A' (which is a subgroup of $Stab(X) \cap Out(G, \{H_i\}^t)$ so torsion free and periodic) is the trivial group and we can conclude that there is a normal subgroup B of $Stab(X)$ isomorphic to a subgroup of $\bigoplus_{i=1}^{p} Out(H_i)$ such that $Stab(X)/B$ is isomorphic to $\mathbb{Z}$. □

An obvious corollary is the following:

Corollary 4.5. *If $X \in \partial(G, \{H_1, ..., H_r\})$ is an attractive fixed point of an IWIP automorphism ϕ and suppose that every $Out(H_i)$ is finite, then $Stab(X)$ is virtually infinite cyclic.*

Example of an automorphism ϕ and an attractive fixed point of X of ϕ , such that $Stab(X)$ is not cyclic.

Example 4.6. Let's suppose that our free product decomposition is of the form $G = G_1 * < b_1 > * < b_2 >$, where b_i are of infinite order. Here G_1 is an elliptic subgroup, we denote by $F_2 =< b_1 > * < b_2 >$ the "free part" and by $\mathcal{O}$ the corresponding outer space $\mathcal{O}(G, G_1, F_2)$. Then we define the automorphism ϕ, which satisfies $\phi(a) = a$ for every $a \in G_1$, $\phi(b_1) = b_2 g_1, \phi(b_2) = b_1 b_2$ where $g_1 \in G_1$, and it is easy to see that $\phi \in Aut(G, \mathcal{O})$ is an IWIP automorphism relative to $\mathcal{O}$. Actually, the automorphism induces a train track representative. Since $Aut(G_1)$ can be seen as a subgroup of $Aut(G, \mathcal{O})$, it follows that there is an attractive fixed point X of ϕ which contains just the letters b_1, b_2, g_1 of ϕ and we have that for every $\psi \in Aut(G_1)$ fixes g_1, i.e. $\psi(g_1) = g_1$, we have that $\psi(X) = X$. Therefore $Stab(X)$ contains the subgroup A of $Aut(G_1)$ of automorphisms of G_1 that fix g_1. Therefore since we can choose G_1 with arbitarily big $Aut(G_1)$ and in particular A to not be infinite cyclic, $Stab(X)$ isn't always infinite cyclic. For example, if G_1 is isomorphic to F_3 and g_1 an element of its free basis, we have that $Stab(X)$ contains a subgroup which is isomorphic to $Aut(F_2)$.

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Asymmetry of Outer Space of a Free Product

Dionysios Syrigos

Abstract

For every free product decomposition $G = G_1 * ... * G_q * F_r$ of a group of finite Kurosh rank G, where F_r is a finitely generated free group, we can associate some (relative) outer space $\mathcal{O}$. We study the asymmetry of the Lipschitz metric d_R on the (relative) Outer space $\mathcal{O}$. More specifically, we generalise the construction of Algom-Kfir and Bestvina, introducing an (asymmetric) Finsler norm $\| \cdot \|^L$ that induces d_R. Let's denote by $Out(G, \mathcal{O})$ the outer automorphisms of G that preserve the set of conjugacy classes of G_i's. Then there is an $Out(G, \mathcal{O})$-invariant function $\Psi : \mathcal{O} \to \mathbb{R}$ such that when $\| \cdot \|^L$ is corrected by $d\Psi$, the resulting norm is quasisymmetric. As an application, we prove that if we restrict d_R to the ϵ-thick part of the relative Outer space for some $\epsilon > 0$, is quasi-symmetric . Finally, we generalise for IWIP automorphisms of a free product a theorem of Handel and Mosher, which states that there is a uniform bound which depends only on the group, on the ratio of the relative expansion factors of any IWIP $\phi \in Out(F_n)$ and its inverse.

Contents

1 Introduction

Outer Space is a very well studied space, which can be used to study the group of outer automorphisms $Out(F_n)$ of a finitely generated free group F_n. There are a lot of combinatorial and topological methods to study the space. However, Francaviglia and Martino in [8] introduced a natural asymmetric Lipschitz metric d_R on CV_n. We could define also a symmetric version of this metric, but the non-symmetric one is geodesic and seems natural in terms of studying the dynamics of free group automorphisms. Recently, this metric theory and the resulting geometric point of view have been used extensively to study the Outer Space. As a consequence, we can get many new results, as well as more elegant new proofs of older results, for example see: [1], [2], [3], [7] and [11].
On the other hand, Guirardel and Levitt in [10] constructed an outer space relative to any free product decomposition of a group $G = G_1 * ... * G_q * F_r$ of finite Kurosh rank. There are a lot of analogies between the classical and the general Outer Space.

Firstly, Francaviglia and Martino in [9] introduced and studied the Lipschitz metric for the general case. In the same paper, they proved as an application, the existence of train track representatives for (relative) IWIP automorphisms. Moreover, many well known constructions and theorems of the free case can be generalised in the general case (for example, see [5], [12], [14], [15], [16], [17] and [18]). This is a motivation to study further analogies, and in particular here we study the asymmetric metric d_R.
In this paper, we generalise the construction of Algom-Kfir and Bestvina in [2] following closely their approach, as we introduce an asymmetric Finsler norm on the tangent space of the relative Outer space that induces the asymmetric Lipschitz metric. We also show how to correct this norm to make it quasi-symmetric. Our main result explains the lack of quasi-symmetry in terms of a certain function and more specifically:

Theorem 1.1. *There is an $Out(G, \mathcal{O})$-invariant continuous, piecewise smooth function $\Psi : \mathcal{O} \to \mathbb{R}$ and constants $A, B > 0$ (depending only on the numbers r, q) such that for every $T, S \in \mathcal{O}$ we have $d(T, S) \leq A \cdot d(S, T) + B \cdot [\Psi(T) - \Psi(S)]$.*

As an application, we prove that if we restrict the asymmetric metric d_R to the ϵ-thick part of the relative Outer space for $\epsilon > 0$, which is the subspace of $\mathcal{O}$ of the points for which all hyperbolic elements have length bounded below by ϵ, is quasi-symmetric (actually, we just need the multiplicative constant). Finally, we generalise a theorem of Handel and Mosher (see [13]), that there is a uniform bound, which depends only on the numbers r and q, on the ratio of the relative expansion factors of any IWIP $\phi \in Out(G, \mathcal{O})$ and its inverse. Since any automorphism $\phi \in Out(G)$ is irreducible relative to some appropriate space $\mathcal{O}$, we can apply the general theorem to get a result for the expansion factors of any automorphism $\phi \in Out(F_n)$ and its inverse, as in the general theorem of [13].

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2 Preliminaries

2.1 Kurosh rank and $\mathbb{R}$-trees

Let's suppose that G is a group which splits as a finite free product $G = H_1 * ... * H_r * F_n$, where every H_i is non-trivial, not isomorphic to $\mathbb{Z}$ and freely indecomposable. We say that such a group has *finite Kurosh rank* and such a decomposition is called *Gruskho decomposition*. Note that the G_i's are unique, up to conjugacy and the ranks n, r are well defined. The number $r + n$ is called the Kurosh rank of G. Finally, every f.g. group admits a splitting as above (by the theorem of Grushko). We are interested only for

groups which have finite Kurosh rank.
Now for a group G of finite Kurosh rank, we fix an arbitary (non-trivial) free product decomposition $G = H_1 * ... * H_r * F_n$, i.e without assuming that each H_i is not isomorphic to $\mathbb{Z}$ or freely indecomposable. Note that these groups admit co-compact actions on $\mathbb{R}$-trees (and vice-versa).
More specifically, for a simplicial tree T (not necessarily locally compact), we denote by $V(T)$ and $E(T)$ the set of vertices and edges of T, respectively. We put also a metric on the tree T, by assigning a positive length to each edge and we can think T as a $\mathbb{R}$-tree. Now, for $x, y \in T$, we denote by $[x, y]$ the unique path from x to y, and for any reduced path p in T we denote by $\ell_T(p)$ the length of p in T which is defined by summing the lengths of the edges that p crosses.
We consider only isometric actions of the group G on $\mathbb{R}$-trees and, more specifically, we say that T is a *G-tree,* if it is a simplicial metric tree (T, d_T), where G acts simplicially on T (sending vertices to vertices and edges to edges) and for all $g \in G, e \in E(T)$ we have that e and ge are isometric. Moreover, we suppose that every G-action is *minimal*, which means that there is no G-invariant proper subtree.
Now let's fix a G-tree T. An element $g \in G$ is called *hyperbolic*, if it doesn't fix any points of T. Any hyperbolic element g of G acts by translation on a subtree of T homeomorphic to the real line, which is called the axis of g and denoted by $axis_T(g)$. The *translation length* of g is the distance that g translates its axis. The action of G on T defines a length function denoted by

$$\ell_T : G \to R, \ell_T(g) := \inf_{x \in T} d_T(x, gx).$$

In this context, the infimum is always minimum and we say that $g \in G$ is hyperbolic if and only if $\ell_T(g) > 0$. Otherwise, g is called *elliptic* and it fixes a point of T. Finally, if g is hyperbolic, we can find some $v \in axis_T(g)$ s.t. the unique reduced path from v to gv has length exactly $\ell_T(g)$. Sometimes, the segment $[v, gv]$ (or even the loop α on which $[v, gv]$ projects to $\Gamma = G/T$) is called the *period of the axis.* For more details about $\mathbb{R}$-trees, see [6].

2.2 Outer Space and The simplex of Metrics

Let's fix an arbitrarily free product decomposition $G = G_1 * ... * G_r * F_n$ of a group G of finite Kurosh rank. Note that it is useful that we can also apply the theory in the case that G is free, and the G_i's are certain free factors of G (relative free case).
Following [10], we define an outer space $\mathcal{O} = \mathcal{O}(G, (G_i)_{i=1}^r, F_n)$ relative to this free product decomposition (relative outer space).

Definition 2.1. An element T of the outer space $\mathcal{O}$ can be thought as simplicial metric G-tree, up to G-equivariant homothety. Moreover, we require that:

- The edge and the tripod stabilisers are trivial.
- There are finitely many orbits of vertices with non-trivial stabiliser and more precisely for every G_i, $i = 1, ..., r$ there is exactly one vertex v_i with stabiliser G_i (all the vertices in the orbits of v_i's are called *non-free vertices*).
- All other vertices have trivial stabiliser (and we call them *free vertices*).

The mimimality implies that we have finitely many orbits of edges for every tree T and we denote by $E_1(T)$ the finite set which contains exactly one edge of each orbit. Also, for convenience we normalise the length of edges and we suppose that the sum of the lengths of edges in $E_1(T)$ is 1.
Note that by a remark of [9], the hyperbolic elements of $T \in \mathcal{O}$ depends only on the space $\mathcal{O}$ and we denote them by $Hyp(\mathcal{O})$.
On the other hand, for a G-tree T as above, we can consider a lot of different metrics ℓ s.t. $(T, \ell) \in \mathcal{O}$. More specifically, we say that a G-invariant function $\ell : E(T) \to [0, 1]$ is a *metric* (relative to $\mathcal{O}$) on T, if there is no hyperbolic element $g \in G$ in $\mathcal{O}$ s.t. $\ell_T(g) = 0$. We denote by Σ_T the set of all metrics in T. The space Σ_T of all metrics ℓ on T is a "simplex with missing faces", where the missing faces correspond to metrics that vanish on a G-subgraph that contains the axes of hyperbolic elements. Therefore in that case (T, ℓ) is not an element of $\mathcal{O}$.
Alternatively, we could define $\mathcal{O}$ as the disjoint union of the simplices Σ_T, where T varies over all the G-trees T which satisfy the assumptions of the Definition 2.1.
We would like to define a natural action of $Out(G)$ on $\mathcal{O}$, but this is not possible since it not always the case that the automorphisms of G preserve the structure of the trees, as they may not preserve the conjugacy classes of the G_i's. However, we can describe here the action of a specific subgroup of $Out(G)$ (namely, the automorphisms that preserve the decomposition or, equivalently, the structure of the trees) on $\mathcal{O}$.
Let $Aut(G, \mathcal{O})$ be the subgroup of $Aut(G)$ that preserve the set of conjugacy classes of the G_i 's. Equivalently, $\phi \in Aut(G)$ belongs to $Aut(G, \mathcal{O})$ iff $\phi(G_i)$ is conjugate to one of the G_j 's (in general, i is different to j). The group $Aut(G, \mathcal{O})$ admits a natural action on a simplicial tree by "changing the action", i.e. for $\phi \in Aut(G, \mathcal{O})$ and $T \in \mathcal{O}$, we define $\phi(T)$ to be the element with the same underlying tree with T, the same metric but the action is given by $g * x = \phi(g)x$ (where the action in the right hand side is the action of the G-tree T). As $Inn(G)$ acts on $\mathcal{O}$ trivially, there is a induced action of $Out(G, \mathcal{O}) = Aut(G, \mathcal{O})/Inn(G)$ on $\mathcal{O}$. Note also that in the case of the Grushko decomposition we have $Out(G) = Out(G, \mathcal{O})$.

2.3 Tangent spaces

For every $\ell \in \Sigma_T$, we define the tangent space

$$T_\ell(\Sigma_T) = \Big\{\tau : E(T) \to \mathbb{R} | \sum_{e \in E(T)} \tau(e) = 0\Big\}.$$

Since the tangent space does not depend on the metric, for every two metrics ℓ, ℓ' the natural identification between $T_\ell(\Sigma_T)$ and $T_{\ell'}(\Sigma_T)$, implies that the total tangent space can be written as $T(\Sigma_T) \cong \Sigma_T \times \mathbb{R}^{N-1}$ where N is the number of edges of Σ_T.

Definition 2.2. A tangent vector $\tau \in T_\ell(\Sigma_T)$ is *integrable* (relative to ℓ), if $\tau(e) < 0$ implies that $\ell(e) > 0$ for all $e \in E(\Sigma_T)$, i.e. it is not possible to find an edge e with $\tau(e) < 0$ and $\ell(e) = 0$.

Note that if τ is integrable, then for all sufficiently small $t \geq 0$ we have that $\ell + t\tau \in \Sigma_T$. As a consequence, we can define $\tau(p)$ for any reduced path p in T, as $\sum_e \tau(e)$ where e varies all over the edges that p crosses, counted with multiplicity. Therefore if $g \in Hyp(\mathcal{O})$ and L_g is the period of the axis of g, we can define $\tau(g) := \tau(L_g)$.

2.4 Lipschitz metric and Optimal maps

In this section, we follow [9]. Let $A, B \in \mathcal{O}$ be two elements of the outer space and let's denote by ℓ_A, ℓ_B the corresponding translation functions of A and B, respectively. Here we define the (right) stretching factor as:

$$\Lambda_R(A, B) := \sup_{g \in Hyp(\mathcal{O})} \frac{\ell_B(g)}{\ell_A(g)}$$

and the (right) asymmetric pseudo-distance as:

$$d_R(A, B) = d(A, B) := \log \Lambda_R(A, B)$$

In the case where $r = 2$ and $n = 0$, we have just one tree with exactly one orbit of edges. Therefore the metric vanishes. However, in any other case the metric is not symmetric and in fact is not even quasi-symmetric. If $n \geq 2$, we can adjust the counter- examples of the free case in order to work in the general case as well. We will give an examples for the case where $r = n = 1$.

Example 2.3. Suppose that $r = n = 1$ and so is of the form $G = G_1 * \mathbb{Z}$, where G_1 is any group of finite Kurosh rank. Then we have two simplices (marked trees, if we forget the metric) of G-trees and let's denote them by T, S s.t. G/T is a loop with a non free vertex and G/S a loop with one edge attached connecting the loop and the non-free

vertex which has valence 1. Now the unique representative of edges of T has length 1 while we give length ϵ in the edge corresponding to the loop of G/S and $1-\epsilon$ to the other and let's denote this metric tree by $S_\epsilon \in \mathcal{O}$.
Now all the hyperbolic elements in T have length 1, while in S_ϵ there are hyperbolic elements of length ϵ and some others with length $2-\epsilon$. Therefore choosing ϵ sufficiently small, we can see that $d_R(T, S_\epsilon) = 2-\epsilon$, while $d_R(S_\epsilon, T) = \frac{1}{\epsilon} \to \infty$.

Let's recall the definition of [9] and some useful properties. We say that a map $f : A \to B$, where $A, B \in \mathcal{O}$, is an $\mathcal{O}$- map, if it is a G-equivariant, Lipschitz continuous, surjective function. One interesting property is the following:

Lemma 2.4. *For every pair $A, B \in \mathcal{O}$; there exists an $\mathcal{O}$-map $f : A \to B$. Moreover, any two $\mathcal{O}$-maps from A to B coincide on the non-free vertices.*

In addition, it can be proved that for every A, B there is an $\mathcal{O}$-map f which realises the distance between them, which means that the Lipschitz constant of $Lip(f)$ is exactly $d(A, B)$. These maps are called *optimal.* In particular, for every IWIP automorphism $\phi \in Out(G, \mathcal{O})$ relative to $\mathcal{O}$, there is an optimal train track representative $f : T \to \phi(T)$ of ϕ that stretches every edge by a specific number, which is called the *expansion factor* of ϕ (relative to $\mathcal{O}$).
Finally, we list some useful properties of the metric:

Proposition 2.5. *(Francaviglia and Martino, [9])*

1. *For every $A, B \in \mathcal{O}$ there is an optimal map $f : A \to B$ with* $Lip(f) = \inf\{Lip(h) | h \text{ is an } \mathcal{O} \text{ -map from } A \text{ to } B\}$

2. *$d(A, B) \geq 0$ with equality only if $A = B$.*

3. *$d(A, C) \leq d(A, B) + d(B, C)$ for all $A, B, C \in \mathcal{O}$.*

4. *d is a geodesic metric. Moreover, a path that realises the distance $d(A, B)$ for every $A, B \in \mathcal{O}$ can be chosen to be piecewise linear, and even linear in each simplex.*

5. *$Out(G, \mathcal{O})$ acts on $\mathcal{O}$ by isometries.*

2.5 Candidates

Definition 2.6. An element $g \in G$ is a candidate in T, if it is hyperbolic in T and, denoting by $axis_T(g)$ its axis in T, there exists $v \in axis_T(g)$ such that the segment $[v, gv]$ projects to a loop α in the quotient graph $\Gamma := G/T$ which is either (see also Figure 1)

1. an embedded loop, or

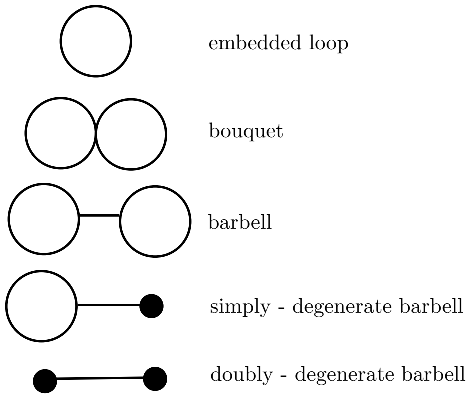

FIGURE 1: Projections of candidates

2. a bouquet of two circles in Γ, i.e. $\alpha = \alpha_1\alpha_2$, where α_1 and α_2 are embedded circles in Γ which meet in a single point, or

3. a barbell graph, i.e. $\alpha = \alpha_1\beta\alpha_2\bar{\beta}$, where α_1 and α_2 are embedded circles in Γ that do not meet, and β is an embedded path in Γ that meets α_1 and α_2 only at their origin (and we denote by $\bar{\beta}$ the path β crossed in the opposite direction), or

4. a simply-degenerate barbell, i.e. α is of the form $\alpha_1\beta\bar{\beta}$, where α_1 is an embedded loop in Γ and β is an embedded path in Γ, with two distinct endpoints, which meets α_1 only at its origin, and whose terminal endpoint is a non-free vertex in Γ, or

5. a doubly-degenerate barbell, i.e. α is of the form $\beta\bar{\beta}$, where β is an embedded path in Γ whose two distinct endpoints are different non-free vertices.

We denote by C_T the set of candidates in T.

Definition 2.7. Let $g, g' \in G$ be hyperbolic elements in $\mathcal{O}$, for which $pr(axis_T(g)) = pr(axis_T(g'))$, or in other words they project to the same path to the quotient $\Gamma = G/T$, then we say that they are projectively equivalent in T (or just projectively equivalent).

Remark. Note that if g, g' are projectively equivalent in T, then for any $\ell \in \Sigma_T$: $\tau(g) = \tau(g')$, $\ell(g) = \ell(g')$. Moreover, there are finitely many projectively inequivalent hyperbolic elements of bounded length. In particular, there are finitely many projectively inequivalent candidates.

The next proposition shows that the distance is realised on a candidates and it is essential for our arguments. In particular:

Proposition 2.8. *For any $A, B \in \mathcal{O}$,*

$$d(A, B) = \max_{g \in C_A} \frac{\ell_B(g)}{\ell_A(g)}$$

3 Basic Lemma

Let's assume that $A, B \in \mathcal{O}$. One main question is that if we change slightly B, can we compute the distance $d(A, B)$ using the same candidate of A? We will prove that this is possible under some conditions.

Definition 3.1. A closed convex cone, in a finite dimensional real vector space V, is a closed subset C of V such that $v, w \in C$ implies that $tv + sw \in C$ for all $t, s \in [0, \infty)$.

One main example of a closed convex cone is the set of integrable vectors in $T_\ell(\Sigma_T)$.

Notational Convention: When we restrict our attention to a specific simplex Σ_T for a specific G-tree T in Outer space, we may identify the point $(T, *, \ell)$, where we denote the G-action on T by $*$, by only specifying the metric ℓ.

Firstly, we prove a very useful proposition which states that in a specific case we can use the same candidate which realises the distance.

Proposition 3.2. *We follow Proposition 6 in [2].*

1. *Let $\tau \in T_\ell(\Sigma_T)$ be an integrable vector. Then there is a candidate α in Σ_T such that*
$$d(\ell, \ell + t\tau) = log\frac{(\ell + t\tau)(\alpha)}{\ell(\alpha)}$$
for all sufficiently small $t \geq 0$, i.e. the same candidate a realises the distance $d(\ell, \ell + t\tau)$ for small t. Moreover, α has the property that for any other hyperbolic element g, $\frac{\tau(\alpha)}{\ell(\alpha)} \geq \frac{\tau(g)}{\ell(g)}$.
2. $\lim_{t \to 0^+} \frac{d(\ell, \ell+t\tau)}{t} = \frac{\tau(\alpha)}{\ell(\alpha)}$, *where α is the candidate of item (i).*
3. *The set of integrable vectors in $T_\ell(\Sigma_T)$ can be written as a finite union of closed convex cones $B_1, B_2, ..., B_M$ such that for any B_i, there is a (projective equivalence class of a) candidate α_i that realises the distance $d(\ell, \ell + t\tau)$ for any $\tau \in B_i$ and for all sufficiently small $t \geq 0$.*

Proof. Let α be a candidate in T that realises $d(\ell, \ell + t\tau)$. This is equivalent to the inequalities :
$$\frac{(\ell + t\tau)(\alpha)}{\ell(\alpha)} \geq \frac{(\ell + t\tau)(g)}{\ell(g)}$$
for all hyperbolic elements g in $\mathcal{O}$. But since the distance can be realised by a candidate, it is enough to consider these inequalities only for the candidates. Moreover, as we have seen we have finitely many classes of projectively inequivalent candidates and so we need finitely many of these inequalities. Let choose a representative of each class and let's

denote them by $\alpha_i \in C_T$, $i = 1, ..., M$.
On the other hand, we can simplify these inequalities to $\frac{\tau(\alpha)}{\ell(\alpha)} \geq \frac{\tau(g)}{\ell(g)}$ when $t > 0$.
This is a finite system of linear inequalities which determines a closed convex cone B_i associated to each α_i as in (iii) and more specifically

$$\tau \in B_i \Longleftrightarrow \frac{\tau(\alpha_i)}{\ell(\alpha_i)} \geq \frac{\tau(a_j)}{\ell(a_j)}, \text{ for every } j = 1, ..., M.$$

The inequalities do not depend on t and so we have (i), since we can choose the same candidate to realise the distance for all small t and the second part of the statement is evident by the discussion above.
Finally, using the item (i), we can divide by t in order to calculate the limit which is straightforward and then the item (ii) follows. □

4 Norm

As in the section above, we fix a tree T. We can now define a function in $\Sigma_T \times T_\ell(\Sigma_T)$, which is a norm. Therefore, we will have a norm in the tangent space which induces the Lipschitz metric. We fix a metric $\ell \in \Sigma_T$ and we give the next definition:

Definition 4.1. Let $\tau \in T_\ell(\Sigma_T)$. Then we define:

$$\|(\ell, \tau)\|^L = \sup\left\{\frac{\tau(g)}{\ell(g)}\Big| g \in Hyp(\mathcal{O})\right\}$$

We will prove that we have an (asymmetric) Finsler norm for the Lipschitz metric.

Proposition 4.2. *1. If τ is integrable, then $\|(\ell,\tau)\|^L = \lim\limits_{t\to 0^+} \frac{d(\ell,\ell+t\tau)}{t}$*

2. The supremum in the definition is achieved on a candidate of Σ_T.

3. $\|(\ell,\tau)\|^L$ is continuous on $T(\Sigma_T)$.

4. $\|(\ell,\tau)\|^L \geq 0$ with equality iff $\tau = 0$.

5. $\|(\ell,\tau_1+\tau_2)\|^L \leq \|(\ell,\tau_1)\|^L + \|(\ell,\tau_2)\|^L$

6. If $c > 0$, then $\|(\ell,c\tau)\|^L = c\|(\ell,\tau)\|^L$

Proof. We follow Proposition 8 in [2].

Since τ is integrable, we can use 3.2 (iii) and we have that there is some i s.t. $\tau \in B_i$, but then using the item (ii) of the same proposition we get that there exists some candidate α_i with the property

$$\lim_{t\to 0^+} \frac{d(\ell,\ell+t\tau)}{t} = \frac{\tau(\alpha_i)}{\ell(\alpha_i)}$$

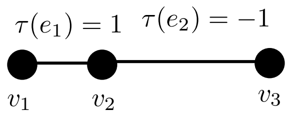

FIGURE 2: The quotient G/T

Therefore, (i) of Proposition 3.2 establishes that $\frac{\tau(\alpha_i)}{\ell(\alpha_i)} \geq \frac{\tau(g)}{\ell(g)}$ for any other hyperbolic element g in $\mathcal{O}$, which means that the supremum in the definition can be achieved on the candidate α_i.

2. If τ is integrable, then as we have seen above there is a candidate α with the property $\|(\ell,\tau)\|^L = \frac{\tau(\alpha)}{\ell(\alpha)}$, and so the supremum is realised on some candidate of T.
   Now let τ be not integrable, which means that there is some edge e s.t. $\ell(e) = 0$ and $\tau(e) < 0$. But we can always find some ℓ' which is as close as we want to ℓ (which means that $\ell'(e) = \epsilon$ for small ϵ) so that τ becomes integrable (relative to ℓ'). Therefore if the perturbation is sufficiently small, the candidate that works for the pair (ℓ,τ), works for (ℓ',τ) as well.

3. This follows from (ii), since we can replace the sup of the definition by a maximum over a finite set (of projectively inequivalent classes of candidates for graphs in the simplex Σ_T, i.e. the projections of candidates to $\Gamma = G/T$).

4. If $\tau \neq 0$, then we can produce some $g \in Hyp(\mathcal{O})$ so that $\tau(g) > 0$. We can do this without dependence on ℓ, so we may assume that (as above, changing ℓ) τ is integrable, and then by 3.2(i) and the item (i) above, we have that for sufficiently small t: $0 < d(\ell, \ell + t\tau) = log(1 + t\|(\ell,\tau)\|^L)$, which implies that there exists such element g.

(v) and (vi) are straightforward, using the definition and the properties of supremum.

□

However, as in the free case the norm is not quasi-symmetric. For $n > 1$, we can essentially use the examples (adjusted appropriately) of the free case. We give an example if there is no free part.

Example 4.3. For $n = 0$ and $r = 3$ and so $G = G_1 * G_2 * G_3$. Let T be the tree with three orbits of non-free vertices which we denote by v_1, v_2, v_3 with vertex groups G_1, G_2, G_3, respectively. Let's also suppose that there are exactly two orbits of edges and namely we denote by e_1 the edge connecting v_1 with v_2 and by e_2 the edge connecting v_2 with v_3. Moreover, if we assume that $\ell(e_1) = \epsilon, \ell(e_2) = 1 - \epsilon$ and $\tau(e_1) = 1, \tau(e_2) = -1$, it's easy to see that there are three types of candidates (see Figure 2). More specifically, the candidates C_1, C_2, C_3 that correspond to $[v_1, v_2]$, $[v_2, v_3]$, $[v_1, v_3]$, respectively. Their lengths are 2ϵ for C_1, $2(1-\epsilon)$ for C_2 and 2 for C_3 and their τ-values are 2 for C_1, -2 for C_2 and 0 for C_3. Therefore we can see that $\|(\ell, \tau)\|^L = \frac{1}{\epsilon}$, while $\|(\ell, -\tau)\|^L = \frac{1}{1-\epsilon}$. Now sending ϵ to 0, we get $\|(\ell, \tau)\|^L \to \infty$, while $\|(\ell, -\tau)\|^L \to 1$.

5 Corrected norm

We would like to define a new norm on $\Sigma_T \times T_\ell(\Sigma_T)$ which is quasi- symmetric. As in the free case, we have to correct $\|\cdot\|^L$ by adding the directional derivative of a function which is the sum of - log's of the lengths of candidates. The first problem is that the candidates are not the same, if we change the marking. Therefore, we need to consider the double covers but not of the graph of groups, as we don't get all the candidates. Also, we have to face one other problem which is the existence of G_i's and which makes the homology of the graph of groups insufficient. However, we can change the graph of groups with an actual graph by introducing a loop of length 0 corresponding to some non-free vertex and we will see that this is enough as every degenerate barbell corresponds to an actual barbell. This idea is due to the anonymous referee.
More precisely, for every $T \in \mathcal{O}$ let's denote by Γ the graph that corresponds to the graph of groups T/G, as we have described above, by changing each non-free vertex v_i with an embedded loop b_i of length 0 (we call the b_i's special loops). The metric and the tangent vectors of T induce a metric and tangent vectors on Γ. Also, we set $\ell(b_i) = \tau(b_i) = 0$. In particular, the fundamental group of Γ is isomorphic to F_{r+n}. Now we will use the homology with $\mathbb{Z}_2$ - coefficients, $H_1(\Gamma, \mathbb{Z}_2)$, of Γ which makes sense as Γ is an actual graph. Every group element can be naturally written as a loop in Γ (but not uniquely, even if all the natural choices have the same length) where every element of the G_i's corresponds to the same loop b_i. We will describe a canonical choice of a loop in Γ, if the group element correspond is a candidate.

Definition 5.1. Let g be a candidate in T. We will associate to g a unique loop of Γ, which we will denote by α_Γ.

1. If g belongs to one the first three cases of 2.6 (i.e. its projection α is either an embedded loop, or an actual barbell or a figure eight), then we denote by α_Γ the natural projection of α in Γ.

2. If g corresponds to a simply degenerate barbell, where its projection α in G/T is of the form $\alpha_1\beta\bar{\beta}$, where α_1 is an embedded loop in G/T and β is an embedded path in G/T, where the terminal point of β is the non-free vertex v_i, we denote by α_Γ the actual barbell of the form $\alpha_1'\beta' b_i \bar{\beta}'$, where we denote by α_1', β' the natural projections of α_1, β in Γ, respectively.

3. Finally, if g corresponds to a doubly degenerate barbell, where its projection α in G/T is of the form $\beta\bar{\beta}$, where β is an embedded path in G/T, where the terminal point of β is the non-free vertex v_i and the origin is the non-free vertex v_j, we denote by α_Γ the actual barbell of the form $b_j\beta' b_i\bar{\beta}'$, where we denote by β' the natural projection of β in Γ.

By the construction, we get that $\ell(\alpha_\Gamma) = \ell(\alpha) = \ell(g)$ and $\tau(\alpha_\Gamma) = \tau(\alpha) = \tau(g)$. Let $y = [y] \in H_1(\Gamma, \mathbb{Z}_2)$ be non-trivial. Then we define $\ell(y)$ to be the infimum of $\ell(x)$, where x ranges over all loops in the class of y. We will prove that this infimum is actually minimum. Now as Γ is an actual graph, we get that Proposition 9 of [2] immediately implies the following lemma:

Lemma 5.2. *Let $y \in H_1(\Gamma, \mathbb{Z}_2)$, then there are finitely many loops $h_1, \ldots, h_k$ so that $\ell(y)$ is realized by some h_i for all $\ell \in \Sigma_\Gamma$. Moreover, if b is an embedded loop then for all $\ell \in \Sigma_\Gamma$, b is the shortest loop representing $b \in H_1(\Gamma, \mathbb{Z}_2)$.*

We can see that Σ_T can be naturally identified with Σ_Γ (as the special loops have length 0). The set of linear inequalities $\ell(h_i) \leq \ell(h_j)$ for the set of h_i's in the previous proposition divides the simplex Σ_T into closed convex subsets $C_1, ..., C_k$ s.t. for each C_i there is an h_i with the property $\ell(h_i) \leq \ell(h_j)$ for all j. In fact, we define the C_i's by:

$$\ell \in C_i \Longleftrightarrow \ell(h_i) \leq \ell(h_j), \text{ for every } j = 1, ..., k.$$

As a consequence we get the following corollary:

Corollary 5.3. *Each simplex Σ_T is covered by closed convex subsets $C_1, ..., C_k$ s.t. for each $x \in H_1(\Gamma, \mathbb{Z}_2)$, there is a j such that $\ell([x]) = \ell(h_j)$ for all $\ell \in C_j$.*

Moreover, we can get as a corollary:

Corollary 5.4. *Let choose $[x] \in H_1(\Gamma, \mathbb{Z}_2)$. For every integrable $\tau \in T_\ell(\Sigma_T)$, there is a j s.t. $\ell, \ell + t\tau \in C_j$ (for all small $t > 0$) and the derivative from the right at 0 of $t \to (\ell + t\tau)([x])$ is $\tau(h_j)$.*
Moreover, it equals to $\min\{\tau(h_i) | [h_i] = [x], h_i \text{ realises } \ell([x])\}$.

Proof. Let $[x] \in H_1(\Gamma, \mathbb{Z}_2)$ and (without loss, after reordering) assume that $h_1, ..., h_m$ are the loops in Γ in the homothety class of $[x]$ which realise $\ell([x])$, i.e. $\ell([x]) = \ell(h_i), i = 1, ..., m$. Assuming that $\tau(h_1) \leq \tau(h_i), i = 1, ..., m$, we have that, for all sufficiently small $t > 0$,

$$(\ell + t\tau)(h_1) = \ell(h_1) + t\tau(h_1) \leq \ell(h_i) + t\tau(h_i) = (\ell + t\tau)(h_i), i = 1, ..., m.$$

Therefore $(\ell+t\tau)(h_1)$ realises $(\ell+t\tau)([x])$. As a consequence, using the previous corollary, $\ell, \ell + t\tau \in C_1$. In this case, it is straightforward that the derivative is exactly $\tau(h_1)$, since actually $(\ell + t\tau)([x]) = (\ell + t\tau)(h_1)$ for all small $t > 0$. ☐

Here we have to consider all the non-trivial double covers of Γ, $\Gamma_i \to \Gamma, i = 1, 2, ..., 2^{n+r} - 1$. Note that the only loops of length 0 in some Γ_i is a lift of some b_i, as there are no two special loops with a common point in Γ. We can naturally get lifts of ℓ and τ to each Γ_i, and we denote them by ℓ_i and τ_i, respectively. Similarly, we can define the space of metrics Σ_{Γ_i} and the corresponding tangent space $T_{\ell_i}(\Sigma_{\Gamma_i})$. A very important lemma which is the reason that we consider the double covers is the following:

Lemma 5.5. *If g is a candidate in T and let α_Γ the loop in Γ as in the Definition 5.1, then there exists a double cover $\Gamma_i \to \Gamma$ and a lift $\tilde{\alpha_\Gamma}$ of α_Γ in Γ_i, so that $\tilde{\alpha_\Gamma}$ is the unique loop that realises the length of* $[\tilde{\alpha_\Gamma}]$.

Proof. We will apply Lemma 5.2 on the appropriate Γ_i. The most important remark is that every candidate can be identified in Γ with either an embedded loop or a figure eight or an actual barbell, which we denote by α_Γ see the Definition 5.1. It is enough to prove that every embedded loop, every figure eight and every barbell in Γ lift to an embedded loop in some appropriate double lift. This fact follows immediately from the Lemma 12 of [2], as we have actual graphs and the proof is exactly the same. □

We denote by $H(\Gamma_i)$ the set of non-trivial classes $[a]$ such that a is not represented by (the lift of) some special loop b_i, $i = 1, ..r$. Note that every elliptic element can be represented by some special loop in Z_2 coefficients, as it is conjugate to some $g_i \in G_i$.

Definition 5.6. Fixing some $\ell \in \Sigma_T$ and $\tau \in T_\ell(\Sigma_T)$, we define the number

$$N(\ell, \tau) = -\sum_{\Gamma_i} \sum_{[\delta] \in H(\Gamma_i)} \frac{\min \tau_i([\delta])}{\ell_i([\delta])} \tag{1}$$

where minimum is taken over the loops h in the class of $[\delta]$ which realise $\ell_i([\delta])$, where Γ_i are all the double covers of Γ.

We are now in position to define the new norm by :

$$\|(\ell, \tau)\|^N = \|(\ell, \tau)\|^L + \frac{1}{K+1} N(\ell, \tau) \tag{2}$$

where K is the number of summands in 1 (and it depends on r, n).
We write $\|\tau\|^{\cdot}$ instead of $\|(\ell, \tau)\|^{\cdot}$, for simplicity.

Define the map $\Psi : \Sigma_T \to \mathbb{R}$ by

$$\Psi(\ell) = -\frac{1}{K+1} \sum_{\Gamma_i} \sum_{[\delta] \in H(\Gamma_i)} log \ell_i([\delta]) \tag{3}$$

where ℓ_i is the lift of ℓ to Γ_i. Again, note that Ψ is smooth on each convex set C_j of the Corollary 5.3.

For both cases it is true that:

Lemma 5.7.

$$\frac{1}{K+1} max\{\|\tau\|^L, \|-\tau\|^L\} \leq \|\tau\|^N \leq 2\|\tau\|^L + \|-\tau\|^L$$

Proof. Firstly, we choose some candidates α, β which realise $\|\tau\|^L$, $\|-\tau\|^L$, respectively. Then we have that by 3.2 for any hyperbolic element g in $\mathcal{O}$, $\frac{\tau(g)}{\ell(g)} \leq \frac{\tau(\alpha)}{\ell(\alpha)} = \|\tau\|^L$, $\|$ and $\frac{-\tau(g)}{\ell(g)} \leq \frac{-\tau(\beta)}{\ell(\beta)} = \|-\tau\|^L$. But since the minimum in 1 , varies over the loops h in Γ that realise $\ell(\delta)$, we have that for each h in the sum we get: $\frac{\tau(h)}{\ell(\delta)} \leq \frac{\tau(\alpha)}{\ell(\alpha)} = \|\tau\|^L$ and similarly $\frac{-\tau(h)}{\ell(\delta)} \leq \frac{-\tau(\beta)}{\ell(\beta)} = \|-\tau\|^L$. Therefore we have that the positive summands in $N(\ell,\tau)$ are dominated by $\|-\tau\|^L$ and similarly the absolute value of negative summands are dominated by $\|\tau\|^L$ and the right hand side follows.
Now the inequality $\frac{1}{K+1}\|\tau\|^L \leq \|\tau\|^N$ is equivalent to $-N(\ell,\tau) \leq K\|\tau\|^L$, which is true using again the same argument as above.
Also, we have to prove that $\frac{1}{K+1}\|-\tau\|^L \leq \|\tau\|^N$, which is equivalent to the inequality

$$\|-\tau\|^L - N(\ell,\tau) \leq (K+1)\|\tau\|^N$$

Let α be a candidate that realises $\|-\tau\|^L$, then by applying the Lemma 5.5, we get that there is a term in $-N(\ell,\tau)$ of the form $\frac{\tau(\alpha)}{\ell(\alpha)}$ which is cancelled out with $\|-\tau\|^L = \frac{-\tau(\alpha)}{\ell(\alpha)}$, because $\ell(\alpha) = \ell(\alpha_\Gamma) > 0, \tau(\alpha) = \tau(\alpha_\Gamma)$ (see also Definition 5.1) and there is some appropriate double lift Γ_i of Γ where the lift of α_Γ is the unique loop that realises its homothety class. Finally, we apply again that each positive term of $-N(\ell,\tau)$ is dominated by $\|\tau\|^L$ and the left hand side inequality follows. □

Therefore $\|\cdot\|^N$ is a (non-symmetric) norm, just like $\|\cdot\|^L$ (positivity follows from the previous lemma, while subadditivity and multiplicativity with positive scalars are evident from the definition and the properties of $\|\cdot\|^L$). As an immediate corollary, we can get a nice relation between $\|\tau\|^N$ and $\|-\tau\|^N$, which implies that the new norm is quasi-symmetric.

Corollary 5.8. *There is a constant $A = 3(K+1)$ so that*

$$\|\tau\|^N \leq A\|-\tau\|^N$$

Proposition 5.9. *If $\ell \in \Sigma_T$ and $\tau \in T_\ell(\Sigma_T)$ is integrable then*

$$\|\tau\|^N = \|\tau\|^L + d_\tau\Psi$$

where the third term is the derivative of Ψ in the direction of τ, i.e. the derivative from the right at 0 of $t \to \Psi(\ell + t\tau)$.

Proof. We just need to prove that $d_\tau \Psi = \frac{1}{K+1} N(\ell, \tau)$. We use the chain rule and the Corollary 5.4.
Applying the Corollary 5.4 on Γ_i, ℓ_i and τ_i, we get that for any $[\delta] \in H(\Gamma_i)$:

$$d_{\tau_i}\ell_i([\delta]) = \tau_i(\delta_i)$$

where a_i is the loop in Γ_i that realises $\ell_i(\alpha)$ and on which τ_i is minimal. Therefore using the chain rule:

$$d_{\tau_i} log \ell_i([\delta]) = \frac{\tau_i(\delta_i)}{\ell_i(\delta_i)}$$

Therefore the resulting equations follows immediately, if we take the double sum. □

As in the free case, it is not difficult to see that we can extend the map (and its properties) to the whole Outer space $\mathcal{O}$. Firstly, we note that $\|\cdot\|^L, \|\cdot\|^N$ and Ψ commute with the inclusions of simplices corresponding to collapsing forests without non-free vertices (since for the edges e in that forest we have that $\ell(e) = \tau(e) = 0$).
Let's denote by $R_{r,n}$ the 'natural rose' corresponding to the free product with the identity as the marking. More precisely, it has exactly r non-free vertices $v_1, ..., v_r$ such that $G_{v_i} = G_i$ and exactly one free vertex v. It is consisted of n loops corresponding to F_n and r additional edges which we denote by E_i connecting v_i with v. We denote by $R^*_{r,q}$ the graph that is induced by changing the non-free vertices with special loops (of length 0). Moreover, as the marking permutes the conjugacy classes of the G_i's, every special loop is sent to some special loop in Z_2-coefficients, and so there is a permutation between the sets $H(R^*_{r,n})$ and $H(\Gamma)$ by construction and so we can identify their homothety classes. Similarly we can identify the double covers of $R^*_{r,n}$ with the double covers of Γ, and the isomophism between their fundamental groups lifts to isomophisms between the fundamental groups of their double covers.
Therefore we can define Ψ globally. Moreover, let $\phi \in Out(G, \mathcal{O})$ be an automorphism and $T \in \mathcal{O}$ be an element of the outer space, then $\phi(T)$ is again an element with the same underlying tree as T with the same metric, but with a different G-action. But the change of the action, only induces a permutation of the summands in the definition of Ψ (as we described previously, Φ preserves the homothety classes of special loops). As a consequence, Ψ is $Out(G, \mathcal{O})$-invariant.

6 Lengths of Paths

In the following sections all the ideas and the proofs are essentially the same as in [2], however we include the most of the proofs for the convenience of the reader and for completeness.

Let $\gamma : [0,1] \to \mathcal{O}$ be a piecewise linear path. This means that γ can be subdivided into finitely many subpaths so that each one is contained in some C_j as in Corollary 5.3 on which Ψ is smooth.
On the other hand, the Lipschitz length of γ is

$$len_L(\gamma) = \sup\Big\{\sum_{i=1}^{p} d(\gamma(t_{i-1}), \gamma(t_i)) : 0 = t_0 < t_1 < ... < t_p = 1\}$$

Suppose that $\Delta t_i = t_i - t_{i-1}$ is small. Since $\gamma(t_i)$ is ℓ and $\gamma'(t_i)$ is the vector of the tangent space τ, we get:

$$d(\gamma(t_{i-1}, \gamma(t_i)) = \frac{d(\gamma(t_{i-1}), \gamma(t_{i-1} + \Delta t_i))}{\Delta t_i} \Delta t_i \sim \|(\gamma(t_{i-1}), \gamma'(t_{i-1}))\|^L \Delta t_i$$

Thus

$$len_L(\gamma) = \int_0^1 \|(\gamma(t), \gamma'(t))\|^L dt$$

Similarly, we can also define the length corresponding to the new norm.

$$len_N(\gamma) = \int_0^1 \|(\gamma(t), \gamma'(t))\|^N dt$$

Proposition 6.1. *Let $T, S \in \mathcal{O}$ and $\gamma : [0,1] \to \mathcal{O}$ be a path from T to S in $\mathcal{O}$. Then $len_N(\gamma) = len_L(\gamma) + \Psi(S) - \Psi(T)$.*

Proof. We can use the Fundamental Theorem of Calculus to $\Psi \circ \gamma$, since Ψ and γ are piecewise differentiable. In order to simplify the notation, we write $\|\gamma'(t)\|^{\cdot}$ instead of $\|(\gamma(t), \gamma'(t))\|^{\cdot}$
Therefore:

$$len_N(\gamma) = \int_0^1 \|\gamma'(t)\|^N dt$$

On the other hand, combining it also with 5.9, we get that:

$$\int_0^1 \|\gamma'(t)\|^N dt = \int_0^1 [\|\gamma'(t)\|^L + d_{\gamma'(t)}\Psi] dt = len_L(\gamma) + \Psi(S) - \Psi(T)$$

□

Proposition 6.2. *Let $T, S \in \mathcal{O}$ and $\gamma : [0,1] \to \mathcal{O}$ be a path from T to S in $\mathcal{O}$. Let $-\gamma : [0,1] \to \mathcal{O}$ be the reverse path $-\gamma(t) = \gamma(1-t)$. Then*

$$len_N(-\gamma) \leq A len_N(\gamma)$$

where A is the constant from Corollary 5.8.

Proof. Since γ is piecewise C^1, for all but finitely many points $[-\gamma'](s) = -\gamma'(1-s)$. Thus using the simplification of the notation as in the previous proof and changing the variable ($s = 1-t$), we get :

$$len_N(-\gamma) = \int_0^1 \|[-\gamma'](s)\|^N ds = \int_0^1 \| - \gamma'(t)\|^N dt$$

But now we apply the Corollary 5.8 and we have that:

$$\int_0^1 \| - \gamma'(t)\|^N dt \leq \int_0^1 A\|\gamma'(t)\|^N dt = Alen_N(\gamma)$$

□

7 Applications

Now we are in position to prove the Main theorem and different applications. Let A be the constant from Corollary 5.8.

Corollary 7.1. *For any $T \in \mathcal{O}$, for any $\phi \in Out(G, \mathcal{O})$ and any piecewise linear path γ from T to $\phi(T)$,*

$$len_L(\gamma) = len_N(\gamma)$$

Therefore

$$len_L(\gamma) \leq Alen_L(-\gamma)$$

Proof. By Proposition 6.1, we get that:

$$len_N(\gamma) = len_L(\gamma) + \Psi(\phi(T)) - \Psi(T).$$

But since as we have seen Ψ is $Out(G, \mathcal{O})$ -invariant, which means $\Psi(\phi(T)) = \Psi(T)$, and therefore $len_N(\gamma) = len_L(\gamma)$. So using the Corollary 5.8, the result follows. □

Theorem 7.2. *For any $T, S \in \mathcal{O}$ and for any piecewise linear path γ from T to S, fo*

$$len_N(\gamma) \leq Alen_N(-\gamma) + (A+1)[\Psi(T) - \Psi(S)]$$

where A is the constant of 5.8.

Proof. Combining the Propositions 6.1 and 6.2:

$$len_L(\gamma) + \Psi(S) - \Psi(T) = len_N(\gamma)$$

$$\leq Alen_N(-\gamma) = Alen_L(-\gamma) + A[\Psi(T) - \Psi(S)].$$

Therefore we get the requested result for $len_L(\gamma)$. □

Theorem 7.3. *For any $T, S \in \mathcal{O}$,*

$$d(T,S) \leq Ad(S,T) + (A+1)[\Psi(T) - \Psi(S)]$$

Proof. Let $T, S \in \mathcal{O}$ and let's choose a piecewise linear geodesic path from S to T which we denote by $-\gamma$. We apply the previous theorem to γ, which is a path from T to S. □

Remark. Note here that since ϕ is $Out(G, \mathcal{O})$-invariant, if $T, S \in \mathcal{O}$ are in the same orbit, then $d(T,S) \leq Ad(S,T)$

Now we prove a theorem about the relation between the expansion factors of an IWIP relative to $\mathcal{O}$ and its inverse. This is a generalisation of the theorem of Handel and Mosher in [13], about the relation of the expansions factors of an IWIP automorphism of a free group and its inverse.

Theorem 7.4. *For any IWIP automorphism $\phi \in Out(G, \mathcal{O})$ relative to $\mathcal{O}$, let λ be the expansion factor of ϕ and μ be the expansion factor of the IWIP ϕ^{-1}. Then $\mu \leq \lambda^A$, where A as above.*

Proof. Let $f : T \to T$ be an optimal train track representative of ϕ and $h : S \to S$ be an optimal train track representative of ϕ^{-1}, which means that $d(\phi^k(T), T) = k \log \lambda$ and $d(\phi^{-k}(S), S) = k \log \mu$, for every natural number k .
Let choose a number $D \geq \max\{d(T,S), d(S,T)\}$ and then, by the triangle inequality, we get that for any natural number k, $d(\phi^k(T), T) \geq d(\phi^k(S), S) - d(\phi^k(S), \phi^k(T)) - d(T,S) \geq klog\mu - 2D$. On the other hand, using the Main Theorem, $d(\phi^k(T), T) \leq A \cdot d(T, \phi^k(T)) = A \cdot klog\lambda$. Therefore combining the inequalities we get

$$A \cdot k \cdot log\lambda \geq k \cdot log\mu - 2D$$

As consequence, for every k we have that $log\mu \leq A \cdot log\lambda + \frac{2D}{k}$ and sending k to infinity, we get $\frac{log\mu}{log\lambda} \geq A$ or equivalently $\mu \leq \lambda^A$ □

However, Handel and Mosher proved also a more general theorem for automorphisms of free groups, and more specifically they proved a relation between the sets of expansions factors of any automorphism and its inverse, using the notion of strata of relative train tracks representatives and the powerful machinery of laminations of Bestvina, Feighn and Handel (see for example [4]). Using the theorem above for the general case, we can get as a corollary a special case of this theorem.
As an application of the method above and [8]. If $\phi \in Out(F_n)$ and $f : T \to T$ is a relative train track representative of ϕ, denoting by λ the expansion factor of the top stratum, then by a remark of [9] there is a relative outer space $\mathcal{O}$ on which $\phi \in Out(G, \mathcal{O})$ and ϕ is irreducible relative to $\mathcal{O}$ (equivalently, a maximal free factor system). Moreover, the same is true for ϕ^{-1} using the same space (equivalently the same free factor system).

We distinguish two cases. If $\lambda = 1$, then ϕ and ϕ^{-1} fix some point of $\mathcal{O}$. Which means that there is a relative train track representative of ϕ^{-1}, for which the expansion factor of the top stratum is 1. If $\lambda > 1$, then ϕ, ϕ^{-1} are IWIP relative to $\mathcal{O}$, and let's denote by $\mu > 1$ the expansion factor of ϕ^{-1} relative to $\mathcal{O}$, which means that there is a relative train track representative h of ϕ^{-1} with the expansion factor of the top stratum to be μ. Using the theorem above, we get that $\log \mu$ and $\log \lambda$ are comparable and the constant depends on the group. Note that, in general, we don't have the uniqueness of the maximal free factor system, however using this approach we can get a correspondence between the maximal free factor systems of ϕ and ϕ^{-1}, and in particular their relative expansion factors.

Let ϵ be a positive number. Let denote by $\mathcal{O}_{\geq\epsilon}$ the thick part of $\mathcal{O}$, i.e. the set of all trees of $\mathcal{O}$, which don't contain hyperbolic elements shorter than ϵ, then it is co-compact for every ϵ.

Theorem 7.5. *For every $\epsilon > 0$ there is a constant B so that for every $T, S \in \mathcal{O}_{\geq\epsilon}$ and any piecewise linear path γ from T to S:*

$$\frac{1}{A} len(\gamma) - B \leq len(-\gamma) \leq A len(\gamma) + B$$

Moreover, there is a constant D such that for all $T, S \in \mathcal{O}_{\geq\epsilon}$

$$d(S, T) \leq D d(T, S)$$

The proof is exactly the same in the free case using the fact the ϵ-thick part of $\mathcal{O}$ is co-compact.

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