EQUILIBRIUM STATES OF THE CHARNEY-DEVORE QUASI-GEOSTROPHIC EQUATION IN MID-LATITUDE ATMOSPHERE

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Abstract. On the predictability of atmospheric blocking in the mid-latitude atmosphere, a quasi-geostrophic beta-plane channel model equation was introduced in a pioneering work by Charney and DeVore in 1979 based on the understanding of quasi-stationary weather patterns. The equation is driven by a zonal thermal flow representing a mid-latitude westerly jet and a wave function representing an ocean land topography. They truncated the equation into a three spectral mode model, which gives rise to the existence of multiple equilibrium states showing a blocking mechanism profile. Moreover, they predicted numerically the absence of the multiple equilibrium states with respect to the flat topography situation. In the present paper, this absence observation is discussed from rigorous analysis and the coexistence of two stable equilibrium states, similar to the three mode model equilibrium states accounting for atmospheric blocking, is investigated numerically with the increment of topographic amplitude.

Keyword beta-plane channel model equation, Charney-DeVore equation, multiple equilibrium states, stability of equilibrium state, zonal flow, topography disturbance

1. Introduction

The persistence of planetary scale atmospheric anomalies and their influences on weather patterns have long been noted by scientists. Atmospheric blocking in mid-latitude atmosphere, disturbing eastward motion process, is a typical example of a persistent anomaly effecting atmospheric circulation. For example, in an omega shape blocking pattern, a high-pressure ridge covering the centre of the omega block experiences dry weather while low pressure troughs on either side of the block are dominated by rain and clouds.

The investigation into the maintenance of atmospheric blocking episodes are of great importance in practical relevance for weather prediction and climate change simulations. In the celebrated work of Charney and DeVore [3], a β-plane atmospheric flow in the troposphere between a topography h and a free surface is governed by the conversation of potential vorticity

$$\partial_t \left( \Delta \psi - \frac{\psi}{\lambda^2} \right) + J(\psi, \Delta \psi + h + \beta y) + \kappa \Delta \psi = \tau$$

(1)
on a synoptic scale channel domain $[0, 2\pi L] \times [0, \pi L]$. This equation was derived from a quasi-geostrophic approximation [2]. Here $\tau$ is a zonal thermal forcing, $J(\psi, \phi) = \partial_x \psi \partial_y \phi - \partial_y \psi \partial_x \phi$ is the Jacobian, $\lambda$ is a parameter defined by the Coriolis force and gravity effect on the atmosphere, and the coefficient $\kappa$ counts for the Ekman layers damping round the topography and the free surface.
The blocking events are explained as a quasi-stationary state. With nonlinear interactions of the zonal source and the Ekman damping, Charney and DeVore [3] suggested the coexistence of multiple equilibrium states of (1) due to topography disturbance around the zonal jet, which is a mean flow parallel to the circles of mid-latitude atmosphere.

In the study of Charney and DeVore [3], equation (1) is approximated by a three spectral mode truncation model as in the spectral mode truncation scheme of Lorenz [16]. Within certain parameter ranges, the three mode truncation model admits a high index, a low index and a medium index equilibrium states. The high index indicates the zonal flow component dominates stationary flow motion while the low index demonstrates the wave component related to $h$ control the stationary fluid motion. The high index and the low index equilibrium states are stable and the medium index equilibrium state is unstable. Blocking occurs when the flow is in the vicinity of the low index equilibrium circulation. This heuristic study has been widely confirmed by climatologists (see, for example, Crommelin et al. [10], Legras and Ghil [15], Eert [11], Ierley and Sheremet [13], Jiang et al. [14], Pedlosky [17]), Pierrehumbert and P. Malguzzi [18], Primeau [19], Reinhold and Pierrehumbert [20], Tung and Rosenthal [24], Holloway and Yoden [27, 28]).

Based on a finite difference scheme, numerical integrations of the barotropic atmospheric flow described by (1) were carried out on a $16 \times 16$ grid for the $\beta$-pane channel (see [3]). The existence of the multiple steady-state phenomenon was also confirmed by the grid-point scheme and the numerical integration flows are attracted by a low-index equilibrium states exhibiting atmospheric blocking (see [3, Figure 4]) when the non-dimensional topographic amplitude equals 0.2.

For the purely thermally driven flow without the topography involved, the multiple equilibria phenomenon was found in a six spectral mode truncation model but seems resultant from the truncation scheme as it disappears in the numerical integration via the grid-point model (see [3, Figure 5]). However, from view point of rigorous mathematical analysis, the understanding of equation (1) with respect to the equilibrium problem is missing, although [7, 8] considered rigorous bifurcation problems on some fluid motion equations different to (1). Especially, the existence of multiple equilibrium states was examined by Chen [5] for purely thermally driven flow governed by (1) over a wider channel domain. The growth of the channel width increases the freedom of the fluid motion and hence complexifies the nonlinear behaviours of the circulation flow. The nonlinear complexity of (1) with a different thermally driven source was also recently discussed by Chen [6].

To address the importance of the bottom surface topography of the multiple equilibria technique given by Charney and DeVore [3], the purpose of the present study is two-fold. First, if the topographic wave is either flat or parallel to the zonal thermal flow, it is proved rigorously that the basic equilibrium state of (1) admits is globally stable with respect to the Fourier expansion perturbation of [3, 4] in a Sobolev space. This result is stated as Theorem 3.1 in Section 3 and rules out the existence of the atmospheric blocking phenomenon. Second, for the topographically driven flow of [3] with the photographic amplitude increasing from 0 to 0.2, a quasi-spectral code is developed from [9, 25, 26] in Section 4 on numerical simulation of the losing global stability and the occurrence of multiple equilibrium states as described in the three mode truncation model of [3].
2. Charney and DeVore’s atmospheric circulation model and the three-mode truncation model

To the understanding of the difference and alikeness of the present study and the original investigations of Charney and DeVore [3], it is beneficial to present their three-mode model results accordingly.

Charney and DeVore [3] considered a barotropic atmosphere between lower topography $h$ and a free surface of height $H + \eta$ and mean height $H$ confined between zonal walls a distance $\pi L$ apart. The stream function $\psi = q\eta/f_0$ defining the synoptic scale fluid motion driven by a thermal zonal forcing $\kappa \Delta \psi^*$ is governed by the equation (Charney and DeVore [3])

$$\partial_t (\Delta \psi - \frac{f_0^2}{gH} \psi) + J(\psi, \Delta \psi + f_0 \frac{h}{H} + \frac{2\Omega \cos \phi_0}{a} y) = -\frac{f_0}{2H} \sqrt{\frac{2\nu_e}{f_0}} \Delta (\psi - \psi^*),$$

where $\Omega$ is the angular speed of the earth’s rotation, $a$ is the radius of the earth, $\phi_0$ is a central latitude, $f_0 = 2\Omega \sin \phi_0$ is the Coriolis force at the central latitude $\phi_0 = 45^\circ$, $\nu_e$ is the bulk eddy viscosity in the Ekman frictional boundary layer and $\frac{f_0}{2H} \sqrt{\frac{2\nu_e}{f_0}} \Delta \psi^*$ denotes the vorticity source of the thermal wind driven by the radiation field.

The nondimensionlization of $t$ by $f_0^{-1}$, $x$ and $y$ by $L$, $\psi$ and $\psi^*$ by $L^2 f_0$, and $h$ by $H$ gives rise to the dimensionless form of (2) expressed as [3, equation (9) and Section 3]

$$\partial_t \left( \Delta \psi - \frac{\psi}{\lambda^2} \right) + J(\psi, \Delta \psi + h + \beta y) + \kappa \Delta (\psi - \psi^*) = 0$$

for the dimensionless quantities

$$\lambda^2 = \frac{gH}{f_0^2 L^2}, \quad \beta = \frac{L}{a} \cot \phi_0, \quad \kappa = \frac{1}{2H} \sqrt{\frac{2\nu_e}{f_0}}$$

and the topography

$$h = \frac{h_0}{2} 2 \cos 2x \sin y.$$  (4)

The $\beta$-plane channel reduces to the non-dimensional domain $[0, 2\pi] \times [0, \pi]$. The unknown stream function $\psi$ satisfies periodic boundary condition in the $x$ direction

$$\psi|_{x=0} = \psi|_{x=2\pi}$$

and the boundary condition on the zonal walls at $y = 0$ and $y = \pi$:

$$\partial_x \psi|_{y=0} = \partial_x \psi|_{y=\pi} = 0.$$  (6)

The driving source is defined by the stream function

$$\psi^* = \psi_1^* \sqrt{2} \cos y,$$

where the quantity $\psi_1^*$ is the driving Rossby number $U/(\sqrt{2} L f_0)$ with $U$ the dimensional amplitude of the driving zonal wind.

Charney and DeVore [3] examined the barotropic flow solution to (3)-(7) in the following Fourier expansion

$$\psi(t) = \sum_{m, n \geq 1} (a_{m,n}(t) \cos mx \sin ny + b_{m,n}(t) \sin mx \sin ny) + \sum_{n \geq 1} a_n(t) \cos ny.$$  (8)
This solution form was also considered by Charney and Straus [4] in the examination of baroclinic instabilities. By the spectral mode truncation scheme [16], the solution (8) is approximated by the three spectral mode expansion

$$
\psi = \psi_1 \sqrt{2} \cos y + \psi_2 \cos 2x \sin y + \psi_3 \sin 2x \sin y
$$

(9)

and thus (3) with the simplification $\lambda \to \infty$ reduces to the truncation model [7, equations (21), (27)-(29)].

An equilibrium solution of this model is equivalent to a root of the function

$$
f(\psi_1) = \left( \frac{64 \sqrt{2} \psi_1}{15\pi} - \frac{2}{5} \beta \psi_3 - \kappa \psi_2 \right) \left( \psi_1 - \psi_1^* \right) + \frac{1}{5} \left( \frac{8 \sqrt{2} \psi_1}{3\pi} \right)^2 \psi_1,
$$

(10)

which is a function of the coefficient $\psi_1$ measuring the magnitude of zonal flow disturbance (see Figure 1). When $\psi_1$ is small and thus the wave disturbance measured by $\psi_2$ and $\psi_3$ dominates the atmospheric circulation, the corresponding equilibrium state is said to be low index. Therefore the zonal flow movement is disturbed and atmospheric blocking occurs (see Figure 2(a)). When $\psi_1$ is large, the corresponding equilibrium state is said to be high index. Hence as illustrated in Figure 2(b), the zonal flow jet dominates the atmospheric circulation and precludes the atmospheric blocking.

With the increment of the topography wave amplitude $h_0$, a saddle node bifurcation arises and the existence of a single equilibrium state at $h_0 = 0$ becomes the coexistence of the low-index state with $\psi_1 = 0.0294$, the medium index state with $\psi_1 = 0.121$ and the high index state with $\psi_1 = 0.154$ at $h_0 = 0.2$ (see Figure 2 and [3, Table 1]).
Figure 2. Flow patterns over ocean-land (blue-red) contrast topographic contours for $\psi_1^* = 0.2$, $\kappa = 0.01$, $\beta = 0.25$ and $h_0 = 0.2$: (a) the non-blocking circulation described by the high index equilibrium state ($\psi_1 = 0.154$) and (b) the $\Omega$ shape blocking circulation described by the low-index equilibrium state ($\psi_1 = 0.0294$) presented in [3, Figure 4].

3. Stability analysis for $h_0 = 0$

Let us begin with discussion of coexistence of equilibrium states of (3)-(7) in the case of the flat topography $h_0 = 0$. It should be noted that $\psi + c$ with a constant $c$ satisfies (3)-(7) whenever $\psi$ is a solution of the system. Thus it is convenient to adopt the following average condition

$$\int_0^{2\pi} \int_0^{\pi} \psi dx dy = 0$$

(11)

to avoid the occurrence of the trivial constant $c$.

However, for an arbitrary constant $c$, the function

$$\psi_c = \psi^* + c(\frac{\pi}{2} - y) = \psi_1^* \sqrt{2} \cos y + c(\frac{\pi}{2} - y)$$

(12)

is an equilibrium state satisfying (3)-(7) and (11) with $h_0 = 0$ and $\psi_c$ is approximated by the Fourier expansion

$$\psi_1^* \sqrt{2} \cos y + c \sum_{n=1}^{\infty} \frac{2[1 - (-1)^n]}{\pi n^2} \cos ny$$

(13)

in the sense of the $L_2$ norm

$$\|\phi\|_{L_2} = \left( \int_0^{2\pi} \int_0^{\pi} |\phi|^2 dx dy \right)^{\frac{1}{2}}.$$
This expansion is obtained by the even function extension to the domain \([-\pi, 0]\). This \(2\pi\)-periodic function is not continuously differentiable at \(y = 0\) and hence the expansion (13) does not converge in the Sobolev space

\[ \mathcal{H}^3 = \{ \psi \in L^2([0, 2\pi] \times [0, \pi]); \ \Delta \psi, \partial_x \Delta \psi, \partial_y \Delta \psi \in L^2([0, 2\pi] \times [0, \pi]) \} \]  

because the expansion

\[ \left\| \Delta \psi_e - \Delta \left( \psi^* - c \sum_{n=1}^{N} \frac{2[1 - (-1)^n]}{\pi n^2} \cos n y \right) \right\|_{L^2} \]  

(15)

\[ = \left\| c \sum_{n=1}^{N} \frac{2[1 - (-1)^n]}{\pi} \cos n y \right\|_{L^2} = 2|c| \left( \sum_{n=1}^{N} (1 - (-1)^n)^2 \right)^{\frac{1}{2}} \]  

(16)

diverges as \(N \to \infty\). Since (3) is a third-order partial differential equation, the classical solution of the system (3)-(7) is in the space \(\mathcal{H}^3\). It is necessary to assume the barotropic flow solution is approximately by the expansion (8) in the norm of \(\mathcal{H}^3\) to exclude the non-physical equilibrium state \(\psi; \psi \neq 0\) and to make sense for Fourier mode truncation schemes.

Now the topographic wave is assumed to be parallel with the zonal flow \(\psi^*\) such as \(h = \rho \psi^*\) with \(\rho \geq 0\). The quasi-geostrophic atmosphere is over this topographic wave and so the equation (3) is written as

\[ \partial_t \left( \Delta \psi - \frac{\psi}{\lambda^2} \right) + J(\psi, \Delta \psi + h + \beta y) + \kappa \Delta (\psi - \psi^*) = 0 \ (h = \rho \psi^* \sqrt{2} \cos y). \]  

(17)

When \(\rho = 0\), (17) reduces to the purely thermally driven flow problem.

The stability result reads

**Theorem 3.1.** For \(\kappa > 0\), \(\beta > 0\), \(\rho \geq 0\), \(\psi^* \neq 0\), \(t \geq 0\) and \(\lambda > 0\), let \(\psi(t)(x, y) = \psi(t, x, y)\) solve equations (5)-(6) and (17) and \(\psi\) be expanded in the form of (8) in the Sobolev space \(\mathcal{H}^3\). Then \(\psi(t)\) is convergent exponentially to the basic equilibrium state \(\psi^*\) as, for \(\rho > 0\),

\[ \| \Delta (\psi(t) - \psi^*) \|^2_{L^2} \leq \frac{(1 + \rho)}{\min\{1, \rho\}} \left( 1 + \frac{1}{\lambda^2} \right) e^{-\frac{2\pi^2}{\lambda^2} \kappa} \| \Delta (\psi(0) - \psi^*) \|^2_{L^2}. \]  

(18)

and, for \(\rho = 0\),

\[ \| \Delta (\psi(t) - \psi^*) \|_{L^2} \leq 2 \sqrt{1 + \frac{1}{\lambda^2}} e^{-\frac{\pi^2}{\lambda^2} \kappa} \| \Delta (\psi(0) - \psi^*) \|_{L^2} \] 

\[ + \frac{2}{\kappa^2} \left( 1 + \frac{1}{\lambda^2} \right) e^{-\frac{\pi^2}{\lambda^2} \kappa} \| \Delta (\psi(0) - \psi^*) \|^2_{L^2}. \]  

(19)

Since \(\psi^*\) is an equilibrium state of equation (17), this result shows the universal stability of \(\psi^*\) with respect to the barotropic flow solution (8) given by Charney and DeVore [3]. Especially, for the barotropic flow driven purely by the zonal thermal forcing \((\rho = 0)\) is stable and attracted by the equilibrium state \(\psi^*\). This confirms the numerical results derived by the grid-point scheme (see [3, Section 4]) and shows the absence of the atmospheric blocking phenomenon predicted by the three mode truncation model.
Thus multiplying (17) or (21) by (∆ + 1 − 1, ∂ + 1 − 1) and slipping boundary condition (6), we have

\[ \sum_{m,n \geq 1} (a_{m,n}(t) \cos mx \sin ny + b_{m,n}(t) \sin mx \cos ny) \]

\[ + \sum_{n \geq 1} c_n(t) \cos ny \]

(20)

for \( c_1 = a_1 - \psi_1^* \sqrt{2} \) and \( c_n = a_n \) with \( n \geq 2 \).

Since \( \psi^* \) is an equilibrium state, equation (17) can be rewritten as

\[ -i \frac{\partial}{\partial t} [\Delta(\psi - \psi^*) - \frac{\psi - \psi^*}{\lambda^2}] + J(\psi - \psi^*, \Delta(\psi - \psi^*)) + \kappa \Delta(\psi - \psi^*) \]

\[ = J(\psi^*, \Delta(\psi - \psi^*)) + J(\psi - \psi^*, \Delta \psi^*) + \partial_y \partial_x(\psi - \psi^*) + \beta \partial_x(\psi - \psi^*) \]

\[ = -\partial_y \partial^* \partial_x(\Delta + 1)(\psi - \psi^*) + \partial_y \partial_x(\psi - \psi^*) + \beta \partial_x(\psi - \psi^*) \]

\[ = \psi_1^* \sqrt{2} \sin \gamma \partial_x(\Delta + 1 - \rho)(\psi - \psi^*) + \beta \partial_x(\psi - \psi^*). \]

Integrating by parts and using the periodic boundary condition (5) and the non-slip boundary condition (6), we have

\[ \int_0^{2\pi} \int_0^\pi J(\psi - \psi^*, \Delta(\psi - \psi^*))\Delta(\psi - \psi^*)dxdy = \int_0^{2\pi} \int_0^\pi J(\psi - \psi^*, \Delta(\psi - \psi^*))\Delta(\psi - \psi^*)dxdy \]

\[ = \int_0^{2\pi} \int_0^\pi J(\psi - \psi^*, \Delta(\psi - \psi^*))\Delta(\psi - \psi^*)dxdy(1 - \rho) \]

\[ = -\int_0^{2\pi} \int_0^\pi J(\psi - \psi^*, \Delta(\psi - \psi^*))\Delta(\psi - \psi^*)dxdy \]

\[ + \int_0^{2\pi} \int_0^\pi J(\psi - \psi^*, \Delta(\psi - \psi^*))\Delta(\psi - \psi^*)dxdy(1 - \rho) \]

\[ = -\int_0^{2\pi} \int_0^\pi J(\psi - \psi^*, \Delta(\psi - \psi^*))(\Delta + 1 - \rho)(\psi - \psi^*)dxdy = 0, \]

\[ \int_0^{2\pi} \int_0^\pi \partial_y \partial^* \partial_x(\Delta + 1 - \rho)(\psi - \psi^*)(\Delta + 1 - \rho)(\psi - \psi^*)dxdy = 0 \]

and

\[ \int_0^{2\pi} \int_0^\pi \beta \partial_x(\psi - \psi^*)(\Delta + 1 - \rho)(\psi - \psi^*)dxdy = 0. \]

Thus multiplying (17) or (21) by \((\Delta + 1 - \rho)(\psi - \psi^*)\) and then integrating over the domain \([0, 2\pi] \times [0, \pi]\), we have

\[ 0 = \int_0^{2\pi} \int_0^\pi \left[ \frac{\partial}{\partial t} \left( \Delta(\psi - \psi^*) - \frac{\psi - \psi^*}{\lambda^2} \right) + \kappa \Delta(\psi - \psi^*) \right] (\Delta + 1 - \rho)(\psi - \psi^*). \]
This implies, by (20),
\[
0 = 2\kappa \sum_{m,n \geq 1} (m^2 + n^2)(m^2 + n^2 + \rho - 1)(a_{m,n}^2 + b_{m,n}^2) + 2\kappa \sum_{n \geq 1} 2n^2(n^2 + \rho - 1)c_n^2
+ \frac{\partial}{\partial t} \sum_{m,n \geq 1} (m^2 + n^2 + \frac{1}{\lambda^2})(m^2 + n^2 + \rho - 1)(a_{m,n}^2 + b_{m,n}^2)
+ \frac{\partial}{\partial t} \sum_{n \geq 1} 2n^2(n^2 + \rho - 1)c_n^2. \tag{22}
\]

Employing the following notation
\[
A(t) = \sum_{m,n \geq 1} (m^2 + n^2)(m^2 + n^2 + \rho - 1)(a_{m,n}^2 + b_{m,n}^2) + \sum_{n \geq 1} 2n^2(n^2 + \rho - 1)c_n^2, \tag{23}
\]
\[
B(t) = \sum_{m,n \geq 1} (m^2 + n^2 + \rho - 1)(a_{m,n}^2 + b_{m,n}^2) + \sum_{n \geq 1} 2n^2(n^2 + \rho - 1)c_n^2, \tag{24}
\]
and integrating equation (22) with respect to \(t\), we have
\[
A(t) + \frac{1}{\lambda^2} B(t) + 2\kappa \int_s^t A(r)dr = A(s) + \frac{1}{\lambda^2} B(s)
\leq \left( 1 + \frac{1}{\lambda^2} \right) A(s) \tag{25}
\]
for \(0 \leq s < t < \infty\). Therefore equation (25) implies that
\[
\frac{\lambda^2}{\lambda^2 + 1} A(t) + 2\kappa' \int_s^t A(r)dr \leq A(s) \quad \text{for} \quad \kappa' = \frac{\kappa}{1 + \frac{1}{\lambda^2}}. \tag{26}
\]
Multiplying equation (26) by \(2\kappa'\) and then integrating it with respect to \(s\), we obtain
\[
\frac{\lambda^2}{\lambda^2 + 1} 2\kappa'(t-s)A(t) + 2\kappa' \int_s^t 2\kappa'(r-s)A(r)dr \leq 2\kappa' \int_s^t A(s)dr.
\]

Repeating this argument \(n\) more times yields
\[
\frac{\lambda^2}{\lambda^2 + 1} \frac{[2\kappa'(t-s)]^{n+1}}{(n+1)!} A(t) + 2\kappa' \int_s^t \frac{[2\kappa'(r-s)]^{n+1}}{(n+1)!} A(r)dr
\leq 2\kappa' \int_s^t \frac{[2\kappa'(r-s)]^n}{n!} A(s)dr.
\]

Summing the previous equation with respect to \(n \geq 0\) and employing equation (26), we have
\[
\frac{\lambda^2}{\lambda^2 + 1} e^{2\kappa'(t-s)} A(t) + 2\kappa' \int_s^t e^{2\kappa'(r-s)} A(r)dr
\leq \frac{\lambda^2}{\lambda^2 + 1} \left( A(t) + 2\kappa' \int_s^t A(r)dr \right) + 2\kappa' \int_s^t e^{2\kappa'(r-s)} A(s)dr
\leq A(s) + 2\kappa' \int_s^t e^{2\kappa'(r-s)} A(s)dr,
\]
or
\[
A(t) \leq e^{-2\kappa' t} \left( 1 + \frac{1}{\lambda^2} \right) A(0). \tag{27}
\]
This together with (23) yields, for $\rho > 0$,

$$\min\{1, \rho\} \left( \sum_{m,n \geq 1} (m^2 + n^2)^2(a_{m,n}^2(t) + b_{m,n}^2(t)) + \sum_{n \geq 1} 2n^4c_n^2(t) \right)$$

$$\leq (1 + \rho)e^{-2\kappa' t} \left( 1 + \frac{1}{\lambda^2} \right) \left( \sum_{m,n \geq 1} (m^2 + n^2)^2(a_{m,n}^2(0) + b_{m,n}^2(0)) + \sum_{n \geq 1} 2n^4c_n^2(0) \right),$$

and hence the validity of (18).

When $\rho = 0$, upon the observation of equation (23), equation (27) is rewritten as

$$\sum_{m,n \geq 1} (m^2 + n^2)^2(a_{m,n}^2(t) + b_{m,n}^2(t)) + \sum_{n \geq 2} 2n^4c_n^2(t)$$

$$\leq 2e^{-2\kappa' t} \left( 1 + \frac{1}{\lambda^2} \right) \left( \sum_{m,n \geq 1} (m^2 + n^2)^2(a_{m,n}^2(0) + b_{m,n}^2(0)) + \sum_{n \geq 2} 2n^4c_n^2(0) \right).$$

That is, by equation (20),

$$\|\Delta(\psi(t) - \varphi(t))\|_{L_2}^2 \leq 2e^{-2\kappa' t} \left( 1 + \frac{1}{\lambda^2} \right) \|\Delta(\psi(0) - \varphi(0))\|_{L_2}^2$$

for

$$\varphi(t) = c_1(t) \cos y + \psi^*.$$  

(31)

On the other hand, to estimate the remaining part with respect to $\varphi - \psi^*$, we employ integration by parts to produce the equations

$$\int_0^{2\pi} \int_0^{\pi} |\psi|^2 \sqrt{2} \sin y \partial_x \Delta + 1 - \rho)(\psi - \psi^*) + \beta \partial_x (\psi - \psi^*)(\varphi - \psi^*) dxdy = 0$$

and

$$\int_0^{2\pi} \int_0^{\pi} J(\psi - \psi^*, \Delta(\psi - \psi^*))(\varphi - \psi^*) dxdy$$

$$= - \int_0^{2\pi} \int_0^{\pi} \partial_x (\psi - \psi^*) \Delta(\psi - \psi^*) \partial_y (\varphi - \psi^*) dxdy$$

$$= - \int_0^{2\pi} \int_0^{\pi} \partial_x (\psi - \varphi) \Delta(\psi - \psi^*) \partial_y (\varphi - \psi^*) dxdy$$

$$= - \int_0^{2\pi} \int_0^{\pi} \partial_x (\psi - \varphi) \Delta(\psi - \varphi) \partial_y (\varphi - \psi^*) dxdy.$$
Therefore, multiplying equation (21) by $\phi - \psi^*$ and then integrating over the periodic channel domain yield

$$
0 = \int_0^{2\pi} \int_0^{2\pi} \left[ \frac{\partial}{\partial t} [\Delta(\psi - \psi^*) - \psi - \psi^*] + J(\psi - \psi^*, \Delta(\psi - \psi^*) + \kappa \Delta(\psi - \psi^*)) \right] (\phi - \psi^*) dx dy
$$

Thus it follows that, after multiplication by $e^{\kappa t}$ and then integrating over the periodic channel domain yield

$$
0 = \int_0^{2\pi} \int_0^{2\pi} \left[ \frac{1}{2} \left( 1 + \frac{1}{\lambda^2} \right) \frac{\partial}{\partial t} (\phi - \psi^*)^2 - \partial_x (\phi - \psi^*) \Delta(\psi - \phi) \partial_y (\phi - \psi^*) - \kappa (\phi - \psi^*)^2 \right] dx dy
$$

This gives the estimate (19) and thus the proof of Theorem 3.1 is complete.

Multiplying the previous equation by $(1 + 1/\lambda^2)^{-1/2} \| \phi - \psi^* \|_{L_2}$ and employing equation (30), we have

$$
\frac{\partial}{\partial t} \| \phi(t) - \psi^* \|_{L_2} + \kappa \| \phi(t) - \psi^* \|_{L_2} \leq \frac{1}{\pi} \frac{1}{1 + \frac{1}{\lambda^2}} \| \Delta(\phi(t) - \phi(t)) \|_{L_2}
$$

Thus it follows that, after multiplication by $e^{\kappa t}$ and integration with respect to $t$ in equation (32),

$$
\| \phi(t) - \psi^* \|_{L_2} \leq e^{-\kappa t} \| \phi(0) - \psi^* \|_{L_2} + \frac{2}{\pi \kappa^2} e^{-\kappa t} \| \Delta(\phi(0) - \phi(0)) \|_{L_2}
$$

or

$$
\| \Delta(\phi(t) - \psi^*) \|_{L_2} \leq e^{-\kappa t} \| \Delta(\phi(0) - \psi^*) \|_{L_2} + \frac{2}{\pi \kappa^2} e^{-\kappa t} \| \Delta(\phi(0) - \phi(0)) \|_{L_2}.
$$

Combining this estimate with (30), we have

$$
\| \Delta(\phi(t) - \psi^*) \|_{L_2} \leq 2 e^{-\kappa t} \sqrt{1 + \frac{1}{\lambda^2} \| \Delta(\phi(0) - \psi^*) \|_{L_2}}
$$

This gives the estimate (19) and thus the proof of Theorem 3.1 is complete.

4. Numerical simulation on multiple equilibrium states of the topographically driven flow

In the previous section, under the purely thermal driving source assumption $h = 0$, it is proved that the basic equilibrium state

$$
\psi_{h=0} = \psi^* = \psi^*_1 \sqrt{2 \cos y}
$$

is stable with respect to the perturbation in the form of (8). This basic equilibrium state depending only on the variable $y$ is uni-directional. If the barotropic flow is also driven by the topographic wave $h = h_0 \cos 2x \sin y$ with $h_0 \neq 0$, the corresponding basic equilibrium state is not uni-directional. What is more, there is no equilibrium state, which can be expressed in an analytic form. Therefore the stability analysis in the previous section is not applicable to the present situation and hence numerical simulation is employed to understand equilibrium state behaviours.
As in [3] for simplicity, we assume that \( \lambda \to \infty \) in (3) to consider the equation

\[
\partial_t \Delta \psi + J(\psi, \Delta \psi + \psi h_0 \cos 2x \sin y + \beta y) = -\kappa \Delta (\psi - \psi^1 \sqrt{2} \cos y)
\]

(34)

with the solution \( \psi \) subject to boundary conditions (5) and (6). In order to compare with the results of [3], the parameter values

\[
\beta = 0.25, \psi^1 = 0.2, \kappa = 0.01
\]

(35)

are fixed throughout the present section, while the topographic amplitude \( h_0 \) varies in the range [0, 0.2].

Instead of using the vorticity formulation (34), we consider the velocity formulation expressed as

\[
\nabla \cdot u = 0,
\]

(36)

\[
\partial_t u + u \cdot \nabla u - (h_0 \cos 2x \sin y + \beta y)u^\bot + \nabla p + \kappa u = \kappa u^*
\]

(37)

for

\[
u = (-\partial_y \psi, \partial_x \psi), \quad u^\bot = (\partial_x \psi, \partial_y \psi), \quad u^* = (-\partial_y \psi^*, 0) = (\psi^1 \sqrt{2} \sin y, 0).
\]

The velocity field is subject to the boundary condition

\[
\partial_x \psi|_{y=0} = \partial_x \psi|_{y=\pi} = 0, \quad u|x=0 = u|x=2\pi.
\]

(38)

Moreover, the pressure is assumed to satisfy the periodic condition

\[
p|x=0 = p|x=2\pi
\]

(39)

which implies the net pressure gradient to be zero and rules out the occurrence of the non-physical equilibrium states

\[
\psi^* + e(\frac{\pi}{2} - y), \quad e \neq 0
\]

(40)

under the purely thermally driven situation \( h_0 = 0 \).

This atmospheric circulation problem (36)-(39) is solved numerically by a quasi-spectral code, which is developed from Chevalier et al. [9], Xiong [25] and Xiong et al. [26]. In the zonal \( x \)-direction, the velocity field \( u \) is expanded in \( m \) Fourier spectral modes, while in the meridional \( y \)-direction \( u \) is discretized by the Chebyshev-Gauss-Lobatto collocation method with \( n \) collocation nodes. Therefore the system (36)-(39) is approximated by a system of ordinary differential equations. This system is discretized by a combined scheme of Crank-Nicolson method and Runge-Kutta method. The discretized system is numerically convergent and stable with respect to a suitable choice of small time step sizes. It is obtained from numerical experiments that very little difference amongst numerical results for the choices \( m \geq 12 \) and \( n \geq 65 \) can be spotted out. Therefore numerical results are selected when \( m = 12 \) and \( n = 65 \).

According to the numerical computation, there exits a branch of equilibrium states \( \psi_{h_0} \) for \( 0 < h_0 \leq 0.2 \) extended from the pure zonal one \( \psi_{h_0=0} \).

The stability property of \( \psi_{h_0} \) for \( 0 \leq 0.2 \) was confirmed numerically. For small \( h_0 \), no equilibrium state different to \( \psi_{h_0}^* \) was observed. For example, the streamlines of \( \psi_{h_0=0.01} \) over the topography with the small amplitude \( h_0 = 0.01 \) are displayed in Figure 3, which shows that \( \psi_{h_0=0.01} \), an almost pure zonal flow, is similar to \( \psi_{h_0=0} \) without movement in the meridional direction.

When \( h_0 \) rises and is close to 0.03, the attraction basin of the equilibrium state \( \psi_{h_0} \) reduces and a locally stable equilibrium state \( \hat{\psi}_{h_0} \) away from \( \psi_{h_0} \) arises. This nonlinear bifurcation phenomenon is similar to the saddle node bifurcation of the
three mode truncation model described in Figure 1. That is, the local stable equilibrium state $\psi_{h_0}$ coupled with an unstable equilibrium state $\psi_{h_0}'$ away from $\psi_{h_0}$ emerges, but $\psi_{h_0}'$ eludes from the present numerical experiments. As an example, the two stable equilibrium states $\psi_{h_0}$ and $\hat{\psi}_{h_0}$ with $h_0 = 0.04$ are illustrated in Figure 4, which shows that $\psi_{h_0}$ remains a high index equilibrium state while the new bifurcating equilibrium state $\hat{\psi}_{h_0}$ is a low index equilibrium state displaying an omega block circulation as illustrated by the three mode model in Section 2.

For $0.04 < h_0 \leq 0.2$, the local stable behaviours of $\psi_{h_0}$ and $\hat{\psi}_{h_0}$ remain unchanged. Further selected numerical results on the two local stable equilibrium
states are displayed in Figures 5 and 6. The flow pattern of the high index equilibrium state $\psi_{h_0}$ exhibits the movement in the zonal direction prevailing the circulation, although the increase of $h_0$ value results in the growth of movement disturbance in the meridional direction. In contrast, the flow pattern described by the low index equilibrium state $\hat{\psi}_{h_0}$ endures a block in an omega shape (see Figures 5 and 6).

5. Discussion

Charney and DeVore [3] introduced a simplified atmospheric circulation equation and used numerical computations and a three mode truncation scheme to predict the existence of multiple equilibrium states resultant from nonlinear interaction of a zonal forcing, a topography and Ekman damping and the absence of multiple equilibrium states when the topography is ignored. Their work offers fundamental understanding of the mechanisms of atmospheric blocking persistence in the atmospheric science from the coexistence of multiple equilibrium states. However, rigorous mathematical theory on the equilibrium state problem of the Charney-DeVore quasi-geostrophic equation is missing. The present investigation provides the first effect on this problem from viewpoint of partial differential equations and proves the uniqueness of the equilibrium states in the sense of barotropic flow solutions of [3] when the topography $h = 0$. Furthermore, with the increment of topographic amplitude from $h_0 = 0$, numerical simulations of the Charney-DeVore equation are presented to confirm the equilibrium state behaviours predicted by the three mode truncation model.
The present investigation shows the influence of the topography in the multiple equilibria technique of Charney and DeVore [3], where the flow is driven by the force in the case of a cos(y) profile. If the thermally driven force is replaced by a cos(2y) profile, equilibrium state bifurcation arises in (2) with the flat topography (h₀ = 0). This instability can be derived from the rigorous analysis of [5]. If h₀ ≠ 0 and the forcing is of a cos(2y) profile, the instability of the corresponding flow is much more complicated as discussed in [6].

The barotropic instability technique of [3] was developed by Charney and Straus [4] into the examination of instabilities of baroclinic quasi-geostrophic equations modeling a thermally driven atmosphere in the presence of topography. The problem was further examined by Cehelsky and Tung [1] and Rambaldi and Salustri [22]. For baroclinic instability problems without the presence of topography, one may refer to Holm and Wingate [12] and Ripa [23].

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References

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