Subgroups of almost finitely presented groups

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Abstract

We show that every countable group embeds in a group of type FP_2.

1 Introduction

In the late 1940's, Higman, Neumann and Neumann showed that every countable group embeds in a 2-generator group, in the same paper in which they introduced HNN-extensions [6]. Neumann had already shown that there are uncountably many 2-generator groups, from which it follows that they cannot all embed in finitely presented groups [9]. It was not until the early 1960's that Higman was able to characterize the finitely generated subgroups of finitely presented groups [5]. The Higman embedding theorem is a high-point of combinatorial group theory that makes precise the connection between group presentations and logic: it states that a finitely generated group G embeds in some finitely presented group if and only if G is recursively presented, i.e., there is an algorithm to write down the relations that hold in G [5].

A group G is almost finitely presented[1] or FP_2 if its augmentation ideal I_G is finitely presented as a module for its group algebra $\mathbb{Z}G$ (see [3, VIII.5] or [2] for more details). Every finitely presented group is FP_2, and every FP_2 group is finitely generated. Bestvina and Brady gave the first examples of FP_2 groups that are not finitely presented [1], although these examples arose as subgroups of finitely presented groups. In [7] the author constructed groups of type FP_2 that do not embed in any finitely presented group. Given these examples it becomes natural to look for an analogue of the Higman embedding theorem for FP_2 groups. Our main theorem answers this question.

Theorem 1.1. *Every countable group embeds in an FP_2 group.*

Although the statement is similar to the Higman-Neumann-Neumann embedding theorem, the proof is much closer to the Higman embedding theorem. In fact it is modelled on Valiev's proof of the Higman embedding theorem as described in [8, Sec. IV.7], which

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[1]This definition was used by Bieri and Strebel [2]; many authors use the phrase 'almost finitely presented' for the weaker condition that the augmentation ideal in the mod-2 group algebra is finitely presented.

is a simplification of Valiev's first proof [11]. Our proof is simpler than these antecedents because we are not obliged to consider recursively enumerable sets. We make the following definition, which is an analogue of Higman's notion of a benign subgroup.

Definition 1.2. *A subgroup H of a finitely generated group G is a homologically benign subgroup if the HNN-extension*

$$G_H = \langle G, t \colon t^{-1}ht = h \ h \in H\rangle$$

can be embedded in an FP_2 group.

Theorem 1.1 implies that all subgroups of finitely generated groups are homologically benign, however showing that various subgroups are homologically benign plays a major role in the proof of Theorem 1.1. The result below details what we need from [7]; after the statement we outline how to deduce it from results stated in [7].

Theorem 1.3. *For any fixed $l \geq 4$ and any set S of integers with $0 \in S$, there is an FP_2 group $J = J(l, S)$ and a sequence $j_1, \dots, j_l$ of elements of J such that $j_1^s j_2^s \cdots j_l^s = 1$ if and only if $s \in S$.*

Proof. The groups $G_L(S)$ that are constructed in [7] depend on a connected flag simplicial complex L and a set $S \subseteq \mathbb{Z}$. If L has perfect fundamental group and contains an edge loop of length l that is not homotopic to a constant map, then $J = G_L(S)$ has the claimed properties. See [7, section 2] for an explicit example of a suitable L in the case $l = 4$; examples for larger l can be obtained by taking subdivisions of this L.

We expand a little by giving the precise results within [7] that guarantee the various properties of the group $J = J(l, S)$. When $0 \in S \subseteq \mathbb{Z}$, [7, theorem 1.2] gives a presentation for $G_L(S)$ with generators the directed edges of L. By [7, theorem 1.3], the group $G_L(S)$ is FP_2 if and only if the fundamental group of L is perfect. If $j_1, \dots, j_l$ is a directed loop in L that does not bound a disk then by [7, Lemma 14.4], the word $j_1^s j_2^s \cdots j_l^s$ in the given generators for J is equal to the identity if and only if $s \in S$. □

Theorem 1.3 enables one to encode arbitrary subsets of the natural numbers $\mathbb{N}$ in presentations for FP_2 groups. This theorem replaces those parts of Valiev's proof that concern Diophantine equations or those parts of Higman's proof that concern recursive functions, each of which is used to encode recursively enumerable subsets of $\mathbb{N}$ in finite presentations.

2 The proofs

Since this section is closely modelled on Lyndon and Schupp's account of the Higman embedding theorem [8, Sec. IV.7], we have tried to stay close to the notation that they use. We also omit arguments that are identical to those in [8].

Since we will be working with presentations, it is convenient to have a characterization of the FP_2 property in terms of presentations. Recall that the Cayley complex for a presentation of a group G is the universal cover of the presentation 2-complex. The group G acts freely on its Cayley complex, with one orbit of vertices and with orbits of

1- and 2-cells corresponding to the generators and relators respectively in the presentation. We define a partial Cayley complex to be a G-invariant subcomplex of the Cayley complex; partial Cayley complexes are in bijective correspondence with subcomplexes of the presentation complex.

Proposition 2.1. *Let H be given by a presentation with finitely many generators and a countable set of relators $r_1, r_2, \ldots$. The following are equivalent.*

(i) H is FP_2.

(ii) There exists m so that for each $i > m$, the loop defined by r_i represents zero in the homology of the partial Cayley complex corresponding to all the generators and the relators $r_1, \ldots, r_m$.

(iii) There is a connected free H-CW-complex with finitely many orbits of cells and perfect fundamental group.

Proof. Equivalence of (i) and (ii). Let X be the Cayley complex for H and let X_m be the partial Cayley complex containing all 1-cells and only the 2-cells that correspond to the relators $r_1, \ldots, r_m$. Let $C_*(X)$ and $C_*(X_m)$ denote the cellular chain complexes of X and X_m. The image of the map $d_1 : C_1(X) \to C_0(X)$ is isomorphic to the augmentation ideal I_H. Hence H is FP_2 if and only if the kernel of d_1 is finitely generated as a $\mathbb{Z}H$-module. Since $H_1(X)$ is trivial, this kernel is equal to the image $d_2(C_2(X))$. The stated condition on loops is equivalent to $d_2(C_2(X_m)) = d_2(C_2(X))$. If this holds then clearly $d_2(C_2(X))$ is finitely generated. Conversely, any finite subset of $d_2(C_2(X))$ is contained in some $d_2(C_2(X_m))$, so if $d_2(C_2(X))$ is finitely generated then there exists m with $d_2(C_2(X_m)) = d_2(C_2(X))$.

(ii) $\Longrightarrow$ (iii) and (iii) $\Longrightarrow$ (i). Each X_i is a connected H-CW-complex with finitely many orbits of cells, and if (ii) holds then $H_1(X_m) \cong H_1(X)$ is trivial. Given any H-CW-complex Y as in (iii), pick a maximal subtree T in Y/H, let $\widetilde{T}$ be the set of lifts of T in Y, and note that $\widetilde{T}$ is equivariantly isomorphic to $T \times H$. The cellular chain complex $C_*(Y, \widetilde{T})$ gives a finite presentation for the relative homology group $H_1(Y, \widetilde{T})$ as a $\mathbb{Z}H$-module. Since $H_1(Y) = 0$, $H_1(Y, \widetilde{T})$ is isomorphic to I_H. □

Next we give the homological version of the Higman Rope Trick [8, IV.7.6].

Lemma 2.2. *If R is a homologically benign normal subgroup of a finitely generated group F, then F/R is embeddable in an FP_2 group.*

Proof. Fix R as in the statement, and let H be an FP_2 group containing the group $F_R = \langle F, t \colon t^{-1}rt = r,\ r \in R \rangle$. Let L be the subgroup of $F_R \leq H$ generated by F and $t^{-1}Ft$, so that $L \cong F *_R F$. As in [8, IV.7.6] there is a homomorphism $\phi : L \to F/R$ whose restriction to F is equal to the quotient map $F \to F/R$ and whose restriction to $t^{-1}Ft$ is the trivial homomorphism. Viewing L as a subgroup of H, the map $l \mapsto (l, \phi(l))$ defines a second copy of L inside $H \times F/R$. Let K be the HNN-extension in which the stable letter conjugates these two copies:

$$K = \langle H \times F/R, s \colon s^{-1}(l, 1)s = (l, \phi(l)),\ l \in L \rangle.$$

The group K is generated by the generators for H, the generators for F/R and the element s. As defining relators we may take the relators for F/R, the relators for H, finitely many relators stating that the generators for H and the generators for F/R commute, and finitely many relators of the form $s^{-1}(l,1)s(l,\phi(l))^{-1}$ for l in some generating set for L. As in [8, IV.7.6], the relators that hold between the generators for F/R can be eliminated from this presentation for K, leaving just the relators for H and finitely many other relators.

To see that K is FP_2, we use Proposition 2.1 applied to the presentation 2-complex with the generators and relators described above. The generators and relators for H are contained in those for K, so we may look at the partial Cayley complex for K corresponding to just these generators and relators. This 2-complex is isomorphic to a disjoint union of copies of the Cayley complex for H (one copy for each coset of H in K). Let $r_1, r_2 \ldots$ be an enumeration of the relators for H. Since H is FP_2, there exists m so that for $i > m$, the relator r_i represents zero in the homology of the partial Cayley complex for H with just the relators $r_1, \ldots, r_m$. It follows that these same loops represent zero in the homology of the partial Cayley complex for K discussed above.

Now consider the partial Cayley complex for K, taking all the generators, the commutation relators between generators for H and F/R, the finitely many relators involving s, and the relators $r_1, \ldots, r_m$. For $i > m$, the loops in this complex defined by r_i represent the zero element of homology, since they already represent 0 in the smaller partial Cayley complex consisting of a disjoint union of copies of the Cayley complex for H. Hence this presentation for K satisfies condition (ii) of Proposition 2.1, and so K is FP_2. □

Lemma 2.3. *Let G be a finitely generated group which is embeddable in an FP_2 group.*

- *Every finitely generated subgroup of G is homologically benign in G.*
- *If H and K are homologically benign subgroups of G, then so are their intersection and the subgroup that they generate.*

Proof. Almost identical to the proof of [8, Lemma IV.7.7], except that it relies on the fact that a free product with amalgamation $P = M *_G N$ is FP_2 provided that M and N are FP_2 and G is finitely generated rather than on a similar statement for finite presentability. This can be proved easily using Proposition 2.1. □

Lemma 2.4. *Fix $l \geq 4$, and for any integer s define $v_s := c_0^s c_1^s \cdots c_l^s d e^s$, an element of the free group $H = \langle c_0, \ldots, c_l, d, e\rangle$ of rank $l+3$. For any $S \subseteq \mathbb{Z}$ with $0 \in S$, the subgroup*

$$V_S := \langle v_s \colon s \in S\rangle \leq \langle c_0, \ldots, c_l, d, e\rangle$$

is homologically benign and is freely generated by the given elements.

Proof. If a reduced word in the elements v_s is written out in terms of the elements $c_0, \ldots, c_l, d, e$, the only cancellation that can take place involves c_0 and e. Thus the subwords $(c_1^s c_2^s \cdots c_l^s d)^{\pm 1}$ survive uncancelled, which implies that the elements v_s are free generators for the subgroup $V_{\mathbb{Z}}$ of H.

We claim that $V_{\mathbb{Z}}$ is benign, and hence homologically benign. To see this, define an ascending HNN-extension of the free group $H = \langle c_0, \ldots, c_l, d, e\rangle$ by

$$u^{-1} c_i u = c_0 c_1 \cdots c_{i-1} c_i c_{i-1}^{-1} c_{i-2}^{-1} \cdots c_0^{-1}, \quad u^{-1} d u = c_0 c_1 \cdots c_l d e, \quad u^{-1} e u = e.$$

Since $u^{-1}v_su = v_{s+1}$ for all $s \in \mathbb{Z}$ and $v_0 = d$ it follows that

$$\langle d, u\rangle \cap H = V_{\mathbb{Z}}.$$

Hence $V_{\mathbb{Z}}$ is benign in $\langle c_0, \dots, c_l, d, e, u\rangle$ and therefore also in the free group H.

Fix some $S \subseteq \mathbb{Z}$, and claim that V_S is homologically benign in H. To see this, let $J = J(l, S)$ and $j_1, \dots, j_l \in J$ be as in the statement of Theorem 1.3, and let $K = K(S)$ be

$$K = \langle c_0, d, e\rangle * (\langle c_1, c_2, \dots, c_l\rangle \times J) = H *_{\langle c_1,\dots,c_l\rangle} (\langle c_1, c_2, \dots, c_l\rangle \times J).$$

The group K is FP_2, since it has a presentation in which the only relators are the relators of J and finitely many commutation relators between $c_1, \dots, c_l$ and the generators of J.

Define an HNN-extension $M = M(S)$ of K, with base group H and stable letter t via

$$t^{-1}c_0t = c_0, \quad t^{-1}c_it = c_ij_i \text{ for } i > 0, \quad t^{-1}dt = d, \quad t^{-1}et = e.$$

The group M is FP_2 and its subgroups $V_{\mathbb{Z}}$, $t^{-1}V_{\mathbb{Z}}t$ and H are all homologically benign. The elements $t^{-1}v_st$ freely generate the free group $t^{-1}V_{\mathbb{Z}}t$. In terms of the generators for K, $t^{-1}v_st = c_0^sc_1^s \cdots c_l^sj_1^s \cdots j_l^sde^s$. When a reduced word in the elements $t^{-1}v_st$ is written in these terms, the only cancellation that can take place involves c_0 and e, thus the subwords $(c_1^s \cdots c_l^sj_1^s \cdots j_l^sd)^{\pm 1}$ survive uncancelled. It follows that such a reduced word is in H if and only if each subword $j_1^s \cdots j_l^s$ is equal to 1, or equivalently each s that occurs lies in S. Hence V_S is equal to $t^{-1}V_{\mathbb{Z}}t \cap H$ and is homologically benign in M and in H. □

As in [8, IV.7], let L be the free group $L = \langle a, b\rangle$, and let F be the free group of rank $l + 6$ with $F = \langle a, b, c_0, \dots, c_l, d, e, h\rangle$. Define a Gödel numbering γ of *all* words on the alphabet $\{a, b, a^{-1}, b^{-1}\}$ by the formula

$$\gamma(\emptyset) = 0, \quad \gamma(a) = 1, \quad \gamma(b) = 2, \quad \gamma(a^{-1}) = 3, \quad \gamma(b^{-1}) = 4,$$

and extending to longer words by concatenation, viewing a concatenation of digits as a number. Thus γ is a bijection between the words and the subset of $\mathbb{N}$ consisting of zero and all integers whose decimal digits lie in the set $\{1, 2, 3, 4\}$.

To any word w on $\{a, b, a^{-1}, b^{-1}\}$, associate a codeword $g_w \in L$ defined by

$$g_w := whc_0^{\gamma(w)}c_1^{\gamma(w)} \cdots c_l^{\gamma(w)}de^{\gamma(w)}.$$

The subgroup G of F generated by all the elements g_w is freely generated by them.

Lemma 2.5. *The subgroup G is benign in F.*

Proof. Almost identical to the argument in [8, IV.7]. Make a group F^* defined as the fundamental group of a graph of groups with one vertex group F, and four edges corresponding to stable letters u_λ for $\lambda \in \{a, b, a^{-1}, b^{-1}\}$, each of which defines an ascending HNN-extension of F with relations

$$u_\lambda^{-1}au_\lambda = a, \quad u_\lambda^{-1}bu_\lambda = b, \quad u_\lambda^{-1}c_iu_\lambda = c_0^{\gamma(\lambda)}c_1^{\gamma(\lambda)} \cdots c_{i-1}^{\gamma(\lambda)}c_i^{10}c_{i-1}^{-\gamma(\lambda)} \cdots c_0^{-\gamma(\lambda)},$$

$$u_\lambda^{-1}du_\lambda = c_0^{\gamma(\lambda)}c_1^{\gamma(\lambda)} \cdots c_l^{\gamma(\lambda)}de^{\gamma(\lambda)}, \quad u_\lambda^{-1}eu_\lambda = e^{10}, \quad u_\lambda^{-1}hu_\lambda = \lambda h.$$

In F^*, we have that for any word $w = \lambda_1 \cdots \lambda_n$,

$$u_{\lambda_1}^{-1} \cdots u_{\lambda_n}^{-1} g_\emptyset u_{\lambda_n} \cdots u_{\lambda_1} = u_{\lambda_1}^{-1} \cdots u_{\lambda_n}^{-1} h d u_{\lambda_n} \cdots u_{\lambda_1} = g_w,$$

and if $w = u\lambda$ then $u_\lambda g_w u_\lambda^{-1} = g_u$.

To show that G is benign in F, it suffices to show that in F^*,

$$G = F \cap \langle g_\emptyset, u_a, u_b, u_{a^{-1}}, u_{b^{-1}} \rangle.$$

From the equations given above, it is clear that the left-hand side is contained in the right-hand side. As in [8, IV.7], to prove the converse it suffices to show that whenever $z \in G$ and $\lambda \in \{a, b, a^{-1}, b^{-1}\}$ are such that $u_\lambda z u_\lambda^{-1} \in F$, then in fact $u_\lambda z u_\lambda^{-1} \in G$, or equivalently $z \in u_\lambda^{-1} G u_\lambda$. For this, write $z = g_{w_1}^{\epsilon_1} \cdots g_{w_n}^{\epsilon_n}$ as a reduced word in the elements g_w, with $\epsilon_i = \pm 1$. When this expression for z is rewritten in terms of the generators for F and reduced, each subword of the form $(c_1^{\gamma(w_i)} c_2^{\gamma(w_i)} \cdots c_l^{\gamma(w_i)} d)^{\epsilon_i}$ survives uncancelled, and any two such subwords are separated by a non-trivial word in the other generators a, b, c_0, e, h. Each of the natural free generators for $u_\lambda^{-1} F u_\lambda$ except $u_\lambda^{-1} d u_\lambda = c_0^{\gamma(\lambda)} c_1^{\gamma(\lambda)} \cdots c_l^{\gamma(\lambda)} d e^{\gamma(\lambda)}$ has total exponent of each c_i divisible by 10. From this it follows that each $\gamma(w_i)$ is congruent to $\gamma(\lambda)$ modulo 10, and hence that $w_i = x_i \lambda$ for some shorter word x_i, so that $z \in u_\lambda^{-1} G u_\lambda$ as required. □

Corollary 2.6. *Every subgroup of the free group $L = \langle a, b \rangle$ is homologically benign.*

Proof. Let N be a subgroup of L, and define a subset $S = S(N) \subseteq \mathbb{N}$ as the set of Gödel codes for words w on $\{a, b, a^{-1}, b^{-1}\}$ that are equal (as elements of L) to an element of N:

$$S = \{\gamma(w) \colon w \in_L N\}.$$

Now let Y_S be the free product $\langle a, b, h \rangle * V_S \leq F$, where V_S is as defined in the statement of Lemma 2.4. By that lemma, V_S is homologically benign, and hence Y_S is homologically benign in F. Since Y_S is freely generated by $\{a, b, h, v_s \colon s \in S\}$, it is easy to see that $G \cap Y_S$ is freely generated by $\{g_w \colon w \in N\}$. (Recall that $v_s = c_0^s c_1^s \cdots c_l^s d e^s$.) Hence $G \cap Y_S$ is homologically benign. The subgroup generated by $G \cap Y_S$ and the finite set $\{c_0, \ldots, c_l, d, e, h\}$, which is equal to $N * \langle c_0, \ldots, c_l, d, e, h \rangle$, is therefore also homologically benign and the intersection of this group with L is equal to N. □

We are now ready to complete the proof of Theorem 1.1. By the Higman-Neumann-Neumann embedding theorem [6, 8], any countable group can be embedded in a 2-generator group. This 2-generator group is isomorphic to L/N for some normal subgroup N. By Corollary 2.6, N is homologically benign, and so by Lemma 2.2, L/N can be embedded in an FP_2 group.

3 Closing remarks

An opinion attributed to Gromov [4, Ch. 1] is that any statement that is valid for every countable group should be trivial. With this in mind, is there an easier, more direct proof of Theorem 1.1? Is there one that is not modelled on a proof of the Higman embedding theorem and that does not does not rely on Theorem 1.3, or other results from [1, 7]?

To prove Theorem 1.1, we only need the groups $J(l, S)$ for some fixed $l \geq 4$. Our motivation for allowing l to vary comes from the above question. For any $l \geq 4$ and any S with $0 \in S \subseteq \mathbb{Z}$, define a group $J'(l, S)$ by the presentation

$$J'(l,S) = \langle j_1, \ldots, j_l \colon j_1^s j_2^s \cdots j_l^s = 1 \;\; s \in S \rangle.$$

If one could show that $J'(l, S)$ embeds in a group of type FP_2 and that $j_1^s j_2^s \cdots j_l^s \neq 1$ if $s \notin S$ without invoking [1, 7], one would obtain a different proof of Theorem 1.1. If $l \geq 13$, the given presentation for $J'(l, S)$ satisfies the $C'(1/6)$ small cancellation condition [8, Ch. 5]. This can be used to give a different proof that $j_1^s j_2^s \cdots j_l^s \neq 1$ for $s \notin S$.

The proof of the Higman-Neumann-Neumann embedding theorem in [8, IV.3] implies that any FP_2 group embeds in a 2-generator FP_2 group. It follows that every countable group embeds in a 2-generator FP_2 group.

The groups $J = J(l, S)$ in Theorem 1.3 may be chosen to have cohomological dimension $\operatorname{cd} J = 2$ in addition to the stated properties. By keeping track of the cohomological dimension at each stage of the argument one obtains the following strengthened version of Corollary 2.6, and hence a strengthened version of Theorem 1.1:

Corollary 3.1. *For every subgroup N of the free group $L = \langle a, b \rangle$, the HNN-extension $\langle L, t \colon t^{-1}nt = n \;\; n \in N \rangle$ embeds in an FP_2 group of cohomological dimension five.*

Theorem 3.2. *Every countable group G embeds in a 2-generator FP_2 group G^*, with* $\operatorname{cd} G^* \leq \operatorname{cd} G + 5$. *Every torsion element in G^* is conjugate to an element of G.*

The proof of the Higman embedding theorem in [8, IV.7] shows that every recursively presented group G of finite cohomological dimension embeds in a finitely presented group G^* of finite cohomological dimension. However, $\operatorname{cd} G^*$ increases with the complexity of the Diophantine equation used to encode the relators in G. Applying Sapir's aspherical version of the Higman embedding theorem [10] gives the following.

Theorem 3.3. *For every recursive subgroup N of the free group $L = \langle a, b \rangle$, the HNN-extension $\langle L, t \colon t^{-1}nt = n \;\; n \in N \rangle$ embeds in a finitely presented group of cohomological dimension two.*

Combining this with the Higman rope trick [8, IV.7.6] gives a version of the Higman embedding theorem which is an analogue of Theorem 3.2, but with a better bound on $\operatorname{cd} G^*$.

Theorem 3.4. *Every recursively presented group G embeds into a finitely presented 2-generator group G^* with* $\operatorname{cd} G^* \leq \operatorname{cd} G + 2$. *Every torsion element in G^* is conjugate to an element of G.*

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