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A differential complex for CAT(0) cubical spaces

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ABSTRACT

In the 1980's Pierre Julg and Alain Valette, and also Tadeusz Pytlik and Ryszard Szwarc, constructed and studied a certain Fredholm operator associated to a simplicial tree. The operator can be defined in at least two ways: from a combinatorial flow on the tree, similar to the flows in Forman's discrete Morse theory, or from the theory of unitary operator-valued cocycles. There are applications of the theory surrounding the operator to C^*-algebra K-theory, to the theory of completely bounded representations of groups that act on trees, and to the Selberg principle in the representation theory of p-adic groups. The main aim of this paper is to extend the constructions of Julg and Valette, and Pytlik and Szwarc, to CAT(0) cubical spaces. A secondary aim is to illustrate the utility of the extended construction by developing an application to operator K-theory and giving a new proof of K-amenability for groups that act properly on finite dimensional CAT(0)-cubical spaces.

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1. Introduction

CAT(0) cubical spaces are geometric spaces that are assembled from cubes of various dimensions in much the same way that simplicial trees are assembled from edges. They play an important role in group theory that is more or less analogous to the roles that trees play in the theory of amalgamated free products and HNN extensions, and they have been especially prominent in recent work in group theory related to 3-manifold topology [1].

The goal of this paper and its sequel is to provide a geometric proof of the Baum-Connes conjecture (with coefficients) for groups acting on CAT(0)-cubical spaces. While the Baum-Connes isomorphism for these groups is a consequence of the Higson-Kasparov theorem [9], the approach taken here provides an explicit connection between the beautiful geometry of CAT(0) spaces and the KK-theoretic machinery needed for the proof of the conjecture. As a step towards this goal, we demonstrate how our constructions can be used to prove that groups acting on CAT(0)-spaces are K-amenable.

The first purpose of this paper is to associate to every base-pointed, finite-dimensional CAT(0) cubical space X a Fredholm complex of Hilbert spaces

$$\ell^2(X^0) \xrightarrow{d} \ell^2(X^1) \xrightarrow{d} \cdots \xrightarrow{d} \ell^2(X^{n-1}) \xrightarrow{d} \ell^2(X^n)$$

that is a combinatorial analogue of the Witten complex in Morse theory [22] associated to a symmetric space of noncompact type and the distance-squared function from a basepoint (this is a Morse function on the symmetric space, with a single critical point of index zero).

The second purpose is to construct a homotopy, in the topological sense, from the complex above to a base-point-independent Fredholm complex that is fully equivariant for any group acting isometrically on X. The Euler characteristic of the base-point-independent complex is the trivial representation.

The homotopy has an important interpretation in operator K-theory: it connects the γ-element to the identity in Kasparov's representation ring [14, Section 2], and so leads to a new, geometric proof of the Baum-Connes conjecture (with coefficients) for locally compact groups acting cocompactly and isometrically on a finite-dimensional CAT(0)-cubical space.

The main step in the Baum-Connes argument is the construction of the homotopy presented here. The remaining details involve arguments in operator K-theory rather than the geometry of cubical spaces, and will be presented elsewhere [3]. Here we shall focus on issues closely related to the geometry of CAT(0) cubical spaces. But even without going far into K-theoretic matters, we shall be able to use the homotopy alone to prove in Section 10 that any group that acts properly and isometrically on a CAT(0) cube complex is K-amenable [9–11].

Our complexes extend constructions of Julg and Valette [10,11], and of Pytlik and Szwarc [19], from simplicial tress to CAT(0) cubical spaces (a one-dimensional CAT(0)

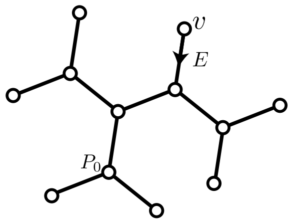

Fig. 1. Julg-Valette operator on a tree.

cubical space is the same thing as a simplicial tree). In the 1980's these authors constructed and studied a certain Fredholm operator associated to a simplicial tree, with applications to C^*-algebra K-theory [10,11], to the theory of completely bounded representations of groups that act on trees [19], and to the Selberg principle in the representation theory of p-adic groups [12,13].

Let us describe the construction of Julg and Valette, which is closer to the outlook of this paper than the construction of Pytlik and Szwarc. Fix a base vertex P_0 in a tree, and define an operator from the ℓ^2-space of vertices in the tree to the ℓ^2-space of edges by mapping each vertex v to the first edge E on the edge-path to the base vertex, except for the base vertex itself, which is mapped to zero. See Fig. 1. (The connection with Witten's complex may not be immediately apparent, but it can be made completely explicit with the help of Forman's Morse theory for cell complexes [6,7].)

The Julg-Valette operator is a bounded Fredholm operator of index one. It depends on the base vertex, but in a sense made precise by Kasparov's theory it is nearly equivariant. Moreover as Julg and Valette demonstrated the operator may be deformed through Fredholm operators, themselves nearly equivariant, to a Fredholm operator that is *exactly* equivariant. This is the key point in the applications above, as it is for us.

The analogous construction for general CAT(0) cubical spaces is as follows. Whereas for trees there is a canonical "gradient flow" towards a base vertex, in higher dimensions this is not so. For example a vertex is typically connected to the base vertex by a large number of edge-paths. We address this issue by having our Julg-Valette differential use all routes to the base vertex (the absence of a definite flow separates our work from actual Morse-theoretic constructions). See Fig. 2. In addition, in order to obtain the condition $d^2 = 0$, we need to assign orientations to cubes in a way that is unnecessary for trees. But after taking these considerations into account we obtain our complex, at least in its basic form. See Section 3 for details.

A greater challenge is the construction of the homotopy. Let us first describe the final, base-vertex-independent complex in the homotopy, which is built using the theory of *hyperplanes* in CAT(0) cubical spaces [17]. Hyperplanes are maximal, connected unions of midplanes of cubes that meet compatibly at faces (the precise definition is reviewed in Section 2). Hyperplanes determine a relation of *parallelism* on the cubes of X: two cubes are parallel if they are bisected by precisely the same set of hyperplanes, as illustrated in Fig. 3. See Section 4 for more details.

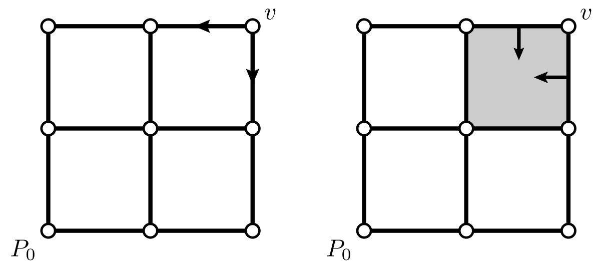

Fig. 2. The Julg-Valette differentials on the vertices (0-cubes) and edges (1-cubes) in a CAT(0) cubical space. Each cube is mapped to all adjacent cubes that are closer to the base vertex. If orientations are taken into account, then $d^2 = 0$.

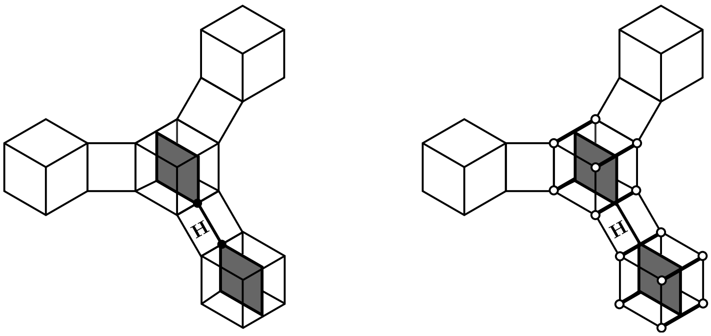

Fig. 3. A hyperplane (left) and an associated parallelism class of edges (right) in a CAT(0) cubical space.

More generally, we define a (p, q)-*cube pair* to be a $(p+q)$-cube together with a distinguished q-dimensional face, and we say that two (p, q)-cube pairs are *parallel* if the outer cubes are parallel, as are their distinguished q-dimensional faces. The q-cochain group in the final complex in the homotopy is the direct sum, over all p, of the ℓ^2-spaces of the sets of parallelism classes of (p, q)-cube pairs. The full construction is described in Section 5.

Let us examine the case of a tree. The hyperplanes are simply the midpoints of edges, and the only nontrivial parallelisms are among vertices, where two vertices are parallel if and only if they span an edge. There is one parallelism class of $(0, 0)$-cube pairs (because any two 0-cubes are parallel), and there is one parallelism class of $(0, 1)$-cube pairs for each edge. Meanwhile in degree 1 each edge constitutes its own parallelism class of $(1, 0)$-cube pairs. So in the case of a tree the final complex, which we shall call the *Pytlik-Szwarc complex*, has the form

$$\mathbb{C} \oplus \ell^2(E) \longrightarrow \ell^2(E)$$

where E is the set of edges. The differential is the obvious one. This is precisely the final complex obtained by Julg and Valette in their work on trees.

The construction of the Pytlik-Szwarc complex in higher dimensions is more intricate, largely because the structure of hyperplanes is more intricate (they are CAT(0)-cubical spaces in their own right). The same can be said of the homotopy that we construct, but again it reduces in the case of trees to the homotopy constructed by Julg and Valette, and uses the same essential idea, that if $d(x, y)$ is the edge-path distance function on vertices, then $\exp(-td(x, y))$ is a positive-definite kernel function for every $t > 0$, which may therefore be used to construct a family of inner products on the space of functions on the vertices. Roughly speaking, the associated family Hilbert spaces converges to the ℓ^2-space of the vertices as $t \to \infty$ and to the Pytlik-Szwarc space as $t \to 0$. This idea is made precise using the concept of continuous field of Hilbert space in Sections 6 and 7.

The differentials in the complexes along the homotopy are constructed in Section 8. Their construction essentially completes the proof that $\gamma = 1$ for groups acting properly and isometrically on CAT(0) cubical spaces. Some remaining functional-analytic details, and the application to K-amenability, are given in Sections 9 and 10.

We conclude by noting that our homotopy is interesting even in the case of a *finite* complex, where the cochain groups constitute the fibers of vector bundles over $[0, \infty]$. Examining the dimensions of the fibers in degree $q = 0$ we obtain the following result:

Theorem. *If X is finite* CAT(0) *cubical space, then the number of vertices of X is equal to the number of parallelism classes of cubes of all dimensions.*

This is the analogue for cubical spaces of the fact that the number of vertices in a finite tree is one plus the number of edges. There are similar fomulas involving parallelism classes of (p, q)-cubes for all $q > 0$, and they lead us to expect that aspects of our constructions will be of interest and value elsewhere in the theory of CAT(0) cube complexes.

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2. Cubes and hyperplanes

We shall begin by fixing some basic notation concerning the cubes and hyperplanes in a CAT(0) cubical space. We shall follow the exposition of Niblo and Reeves in [17], with some adaptations.

Throughout the paper X will denote a CAT(0) cubical space as in [17, Section 2.2]. Though not everywhere necessary, we shall assume throughout that X is finite-dimensional. Every q-cube in X contains exactly $2q$ codimension-one faces. Each such face is disjoint from precisely one other, which we shall call the *opposite* face.

We shall use the standard terms *vertex* and *edge* for 0-dimensional and 1-dimensional cubes.

The concept of a *midplane* of a cube is introduced in [17, Section 2.3]. If we identify a q-cube with the standard cube $[-\frac{1}{2}, \frac{1}{2}]^q$ in $\mathbb{R}^q$, then the midplanes are precisely the intersections of the cube with the coordinate hyperplanes in $\mathbb{R}^q$ (thus the midplanes of a cube C are in particular closed subsets of C). A q-cube contains precisely q midplanes (and in particular a vertex contains no midplanes).

Niblo and Reeves describe an equivalence relation on the set of all midplanes in a cubical space: two midplanes are (*hyperplane*) *equivalent* if they can be arranged as the first and last members of a finite sequence of midplanes for which the intersection of any two consecutive midplanes is again a midplane.

2.1 Definition. (See [17, Definition 2.5].) A *hyperplane* in X is the union of the set of all midplanes in an equivalence class of midplanes. A hyperplane *cuts* a cube if it contains a midplane of that cube. When a hyperplane cuts an edge, we say that the edge *crosses* the hyperplane. A hyperplane and a vertex are *adjacent* if the vertex is included in an edge that crosses the hyperplane.

2.2 Examples. If X is a tree, then the hyperplanes are precisely the midpoints of edges. If X is the plane, divided into cubes by the integer coordinate lines, then hyperplanes are the half-integer coordinate lines.

Hyperplanes are particularly relevant in the context of CAT(0) cubical spaces (such as the previous two examples) for the following reason:

2.3 Lemma. (See [20, Theorem 4.10] or [17, Lemma 2.7].) *If X is a* CAT(0) *cubical space, then every hyperplane is a totally geodesic subspace of X that separates X into two connected components.* □

The components of the complement of a hyperplane are the two *half-spaces* associated to the hyperplane. The half-spaces are open, totally geodesic subsets of X. Moreover the union of all cubes contained in a given half-space is a CAT(0) cubical space in its own right, and a totally geodesic subcomplex of X.

2.4 Lemma. *If two hyperplanes H and K in a* CAT(0) *cubical space X are disjoint, then one of the half-spaces of H is contained in one of the half-spaces of K.*

Proof. See [8, Lemma 2.10]. □

2.5 Lemma. *If k hyperplanes in a* CAT(0) *cubical space intersect pairwise, then all k intersect within some k-cube.*

Proof. See [20, Theorem 4.14]. □

2.6 Lemma. *Assume that k distinct hyperplanes in a* CAT(0) *cubical space have a non-empty intersection. If are they are all adjacent to a vertex, then they intersect in a k-cube that contains that vertex.*

Proof. See [17, Lemma 2.14 and Proposition 2.15]. $\square$

3. The Julg-Valette complex

Let X be a CAT(0) cubical space of dimension n. The aim of this section is to define a differential complex

$$\mathbb{C}[X^0] \xrightarrow{d} \mathbb{C}[X^1] \xrightarrow{d} \cdots \xrightarrow{d} \mathbb{C}[X^{n-1}] \xrightarrow{d} \mathbb{C}[X^n],$$

where X^q denotes the set of q-cubes of X. Our complex generalizes the one introduced by Julg and Valette in the case of a tree [10,11], and to motivate the subsequent discussion we recall their construction. Let T be a tree with vertex set T^0 and edge set T^1. Fix a base vertex P_0. The Julg-Valette differential

$$d : \mathbb{C}[T^0] \longrightarrow \mathbb{C}[T^1]$$

is defined by mapping a vertex $P \neq P_0$ to the first edge E on the unique geodesic path from P to P_0; P_0 itself is mapped to zero. There is an adjoint differential

$$\delta : \mathbb{C}[T^1] \longrightarrow \mathbb{C}[T^0]$$

that maps each edge to its furtherest vertex from P_0. The composite $d\delta$ is the identity on $\mathbb{C}[T^1]$, whereas $1 - \delta d$ is the natural rank-one projection onto the subspace of $\mathbb{C}[T^0]$ spanned by the base vertex. It follows easily that the cohomology of the Julg-Valette complex is $\mathbb{C}$ in degree zero and 0 otherwise.

For the higher-dimensional construction we shall need a concept of orientation for the cubes in X, and we begin there.

3.1 Definition. A *presentation* of a cube consists of a vertex in the cube, together with a linear ordering of the hyperplanes that cut the cube. Two presentations are *equivalent* if the edge-path distance between the two vertices has the same parity as the permutation between the two orderings. An *orientation* of a cube of positive dimension is a choice of equivalence class of presentations; an *orientation* of a vertex is a choice of sign $+$ or $-$.

3.2 Remark. Every cube has precisely two orientations, and if C is an oriented cube we shall write C^* for the same underlying unoriented cube equiped with the opposite orientation.

3.3 Definition. The space $\mathbb{C}[X^q]$ of *oriented q-cochains on X* is the vector space comprising the finitely-supported, *anti-symmetric*, complex-valued functions on the set of oriented q-cubes X^q. Here, a function f is anti-symmetric if $f(C) + f(C^*) = 0$ for every oriented cube C.

3.4 Remark. The space $\mathbb{C}[X^q]$ is a subspace of the vector space of *all* finitely supported functions on X^q, which we shall call the *full space of q-cochains*. The formula

$$f^*(C) = f(C^*)$$

defines an involution on the full space of q-cochains. We shall write C for both the Dirac function at the oriented q-cube C and for the cube itself; in this way C belongs to the full space of q-cochains. We shall write $\langle C\rangle$ for the oriented q-cochain

$$\langle C\rangle = C - C^* \in \mathbb{C}[X^q],$$

which is the difference of the Dirac functions at C and C^* (the two possible meanings of the symbol C^* agree).

Next, we introduce some geometric ideas that will allow us to define the Julg-Valette differential in higher dimensions. The first is the following generalization of the notion of adjacency introduced in Definition 2.1.

3.5 Definition. A q-cube C is *adjacent* to a hyperplane H if it is disjoint from H and if there exists a $(q+1)$-cube containing C as a codimension-one face that is cut by H.

3.6 Lemma. *A q-cube C is adjacent to a hyperplane H if and only if it is not cut by H and all of its vertices are adjacent to H.*

Proof. Clearly, if the cube C is adjacent to H then so are all of its vertices. For the converse, assume that all of the vertices of C are adjacent to H. By Lemma 2.5 it suffices to show that every hyperplane K that cuts C must also cross H. For this, let P and Q be vertices of C separated only by K, and denote by P^{op} and Q^{op} the vertices separated from P and Q only by H, respectively. These four vertices belong to the four distinct half-space intersections associated with the hyperplanes H and K, so that by Lemma 2.4 these hyperplanes intersect. □

We shall now fix a base vertex P_0 in the complex X.

3.7 Definition. Let H be a hyperplane in X. Define an operator

$$H \wedge \_ : \mathbb{C}[X^q] \longrightarrow \mathbb{C}[X^{q+1}]$$

as follows. Let C be an oriented q-cube in X.

(a) We put $H \wedge C = 0$ if C is not adjacent to H.
(b) In addition, we put $H \wedge C = 0$ if C is adjacent to H, but C lies in the same H-half-space as the base point P_0.
(c) If C is adjacent to H, and is separated by H from the base point, then we define $H \wedge C$ to be the unique cube containing C as a codimension-one face that is cut by H.

As for the orientations in (c), if C has positive dimension and is oriented by the vertex P, and by the listing on hyperplanes $H_1, \ldots, H_q$, then we orient $H \wedge C$ by the vertex that is separated from P by the hyperplane H alone, and by the listing of hyperplanes $H, H_1, \ldots, H_q$. If C is a vertex with orientation $+$ then $H \wedge C$ is oriented as above; if C has orientation $-$ then $H \wedge C$ receives the opposite orientation.

3.8 Remark. The linear operator $H \wedge \_$ of the previous definition is initially defined on the full space of q-cochains by specifying its values on the oriented q-cubes C, which form a basis of this space. We omit the elementary check that for an oriented q-cube C we have

$$H \wedge C^* = (H \wedge C)^*, \tag{3.1}$$

which allows us to restrict $H \wedge \_$ to an operator on the spaces of oriented q-cochains. We shall employ similar conventions consistently throughout, so that all linear operators will be defined initially on the full space of cochains and then restricted to the space of oriented cochains. Some formulas will hold only for the restricted operators and we shall point these few instances out.

3.9 Definition. The *Julg-Valette differential* is the linear map

$$d\colon \mathbb{C}[X^q] \longrightarrow \mathbb{C}[X^{q+1}]$$

given by the formula

$$d\,C = \sum_H H \wedge C,$$

where the sum is taken over all hyperplanes in X. Note that only finitely many terms in this sum are nonzero; even if X is not locally finite, only finitely many hyperplanes H are adjacent to C *and* separate C from the base point.

3.10 Example. In the case of a tree, if P is any vertex distinct from the base point P_0, then $H \wedge P$ is the first edge on the geodesic edge-path from P to P_0 and our operator d agrees the one defined by Julg and Valette. Once a base point is chosen every edge (in any CAT(0) cubical space) is canonically oriented by selecting the vertex nearest to the base point; vertices are canonically oriented by the orientation $+$. Thus, because the original construction of Julg and Valette involves only vertices and edges and assumes a base point, orientations do not appear explicitly.

3.11 Lemma. *If H_1 and H_2 are any two hyperplanes, and if C is any oriented cube, then*

(a) *$H_1 \wedge H_2 \wedge C$ is nonzero if and only if H_1 and H_2 are distinct, they are both adjacent to C, and they both separate C from P_0.*
(b) *$H_1 \wedge H_2 \wedge C = (H_2 \wedge H_1 \wedge C)^*$.*

3.12 Remark. Here, $H_1 \wedge H_2 \wedge C$ means $H_1 \wedge (H_2 \wedge C)$, and so on.

Proof. Item (a) follows from Lemmas 2.6 and 2.4. To prove (b), note first that as a result of (a) the left hand side is nonzero if and only if the right hand side is nonzero. In this case, both have the same underlying unoriented $(q+2)$-cube, namely the unique cube containing C as a codimension-two face and cut by H_1 and H_2. As for orientation, suppose C is presented by the ordering $K_1, \dots, K_q$ and the vertex P. The cube $H_1 \wedge H_2 \wedge C$ is then presented by the ordering $H_1, H_2, K_1, \dots, K_q$ and the vertex Q, the vertex immediately opposite both H_1 and H_2 from P; the cube $H_2 \wedge H_1 \wedge C$ is presented by the ordering $H_2, H_1, K_1, \dots, K_q$ and the same vertex. The same argument applies when C is a vertex with the orientation $+$, and the remaining case follows from this and the identity (3.1). $\square$

3.13 Lemma. *The Julg-Valette differential d, regarded as an operator on the space of oriented cochains, satisfies $d^2 = 0$.*

Proof. Let C be any q-cube, so that

$$d^2 \langle C \rangle = \sum_{H_1, H_2} H_1 \wedge H_2 \wedge \langle C \rangle.$$

As a consequence of Lemma 3.11 we have $H_1 \wedge H_2 \wedge \langle C \rangle + H_2 \wedge H_1 \wedge \langle C \rangle = 0$, and the sum vanishes. It is important here that we work on $\mathbb{C}[X^q]$ and not on the larger full space of q-cochains, where the result is not true. See Remarks 3.4 and 3.8. $\square$

3.14 Definition. Let H be a hyperplane and let $q \geq 1$. Define an operator

$$H \lrcorner \_ : \mathbb{C}[X^q] \longrightarrow \mathbb{C}[X^{q-1}]$$

as follows. Let C be an oriented q-cube in X.

(a) If H does not cut C, then $H \lrcorner C = 0$.
(b) If H does cut C then we define $H \lrcorner C$ to be the codimension-one face of C that lies entirely in the half-space of H that is separated from the base point by H.

As for orientations in (b), if C is presented by the ordered list $H, H_1, \ldots, H_{q-1}$ and the vertex P, and P is *not* separated from the base point by H, then $H \lrcorner C$ is presented by the ordered list $H_1, \ldots, H_{q-1}$ and the vertex separated from P by H alone. If C is an edge presented by the vertex P *not* separated from the base point by H then $H \lrcorner C = P^{op}$, the vertex of C opposite to P, with the orientation $+$; if C is presented by the vertex P and P *is* separated from the base point by H then $H \lrcorner C = P$ with the orientation $-$.

3.15 Remark. For convenience we shall define the operator $H \lrcorner \_$ to be zero on vertices.

3.16 Example. Let us again consider a tree T with a selected base vertex P_0. If E is any edge then $H \lrcorner E$ is zero unless H cuts E. In this case $H \lrcorner E = P$, where P is the vertex of E which is farthest away from P_0; we choose the orientation $-$ if E was oriented by the vertex P, and the orientation $+$ otherwise.

3.17 Definition. Let $q \geq 0$. Define an operator

$$\delta \colon \mathbb{C}[X^{q+1}] \longrightarrow \mathbb{C}[X^q]$$

by

$$\delta C = \sum_H H \lrcorner C,$$

again noting that only finitely many terms in this sum are nonzero.

3.18 Definition. The oriented q-cubes are a vector space basis for the full space of q-cochains. We equip this space with an inner product by declaring this to be an orthogonal basis and each oriented q-cube to have length $1/\sqrt{2}$. The subspace $\mathbb{C}[X^q]$ of oriented q-cochains inherits an inner product in which

$$\langle \langle C_1 \rangle, \langle C_2 \rangle \rangle = \begin{cases} 1, & \text{if } C_1 = C_2 \\ -1, & \text{if } C_1 = C_2^* \\ 0, & \text{otherwise.} \end{cases}$$

Thus, selecting for each unoriented q-cube one of its possible orientations gives a collection of oriented q-cubes for which the corresponding $\langle C \rangle$ form an orthonormal basis of the space of oriented q-cochains; this basis is canonical up to signs coming from the relations $-\langle C \rangle = \langle C^* \rangle$.

3.19 Proposition. *The operators d and δ of Definitions 3.9 and 3.17 are formally adjoint and bounded with respect to the inner products in Definition 3.18.*

Proof. The fact that the operators are bounded follows from our assumption that the complex X is finite dimensional. The fact that they are formally adjoint follows from the following assertion: for a hyperplane H, an oriented q-cube C and an oriented $(q+1)$-cube D we have that $H \wedge C = D$ if and only if $H \lrcorner D = C$. See Definitions 3.7 and 3.14. $\square$

To conclude the section, let us compute the cohomology of the Julg-Valette complex. We form the *Julg-Valette Laplacian*

$$\Delta = (d + \delta)^2 = d\delta + \delta d, \tag{3.2}$$

where all operators are defined on the space of oriented cochains (and not on the larger full space of cochains), where we have available the formula $d^2 = 0$ and hence also $\delta^2 = 0$.

3.20 Proposition. *If C is an oriented q-cube then*

$$\Delta\langle C\rangle = (q + p(C))\,\langle C\rangle,$$

where $p(C)$ is the number of hyperplanes that are adjacent to C and separate C from P_0. In particular, the $\langle C\rangle$ form an orthonormal basis of eigenvectors of Δ, which is invertible on the orthogonal complement of $\langle P_0\rangle$ (and so also on the space of oriented q-cochains for $q > 0$).

Proof. Each oriented vertex P is an eigenvector of $d\delta + \delta d$ (acting on the space of full space of cochains) with eigenvalue $p(P)$; indeed, $d\delta P = 0$ for dimension reasons while $\delta d P = p(P)P$, no matter the orientation. While the analogous statement is *not* true for an oriented q-cube C of dimension $q \geq 1$, the oriented cochain $\langle C\rangle$ is an eigenvector with eigenvalue as in the statement. To see this, let C be such a cube and observe that

$$\delta d C = \sum_{H_1, H_2} H_1 \lrcorner H_2 \wedge C,$$

and similarly

$$d\delta C = \sum_{H_1, H_2} H_1 \wedge H_2 \lrcorner C.$$

Adding these, and separating the sum into terms where $H_1 = H_2$ and terms where $H_1 \neq H_2$ we obtain

$$(d\delta + \delta d)C = \sum_{H} (H \lrcorner H \wedge C + H \wedge H \lrcorner C) + \sum_{H_1 \neq H_2} (H_1 \lrcorner H_2 \wedge C + H_2 \wedge H_1 \lrcorner C)\,. \tag{3.3}$$

According to Lemma 3.21 below, when $d\delta + \delta d$ is applied to an oriented cochain $\langle C\rangle$ the terms coming from the second sum in (3.3) cancel. To understand the first sum in (3.3), observe that if H is any hyperplane and C is any oriented cube, then

$$H \wedge (H \lrcorner C) = \begin{cases} C, & \text{if } H \text{ cuts } C \\ 0, & \text{otherwise,} \end{cases}$$

and also

$$H \lrcorner (H \wedge C) = \begin{cases} C, & \text{if } C \text{ is adjacent to } H \text{ and is separated by } H \text{ from } P_0 \\ 0, & \text{otherwise.} \end{cases}$$

Applying this to both C and its opposite cube, the proposition follows. □

3.21 Lemma. *If H_1 and H_2 are distinct hyperplanes, then*

$$H_1 \lrcorner H_2 \wedge C = H_2 \wedge H_1 \lrcorner C^*$$

for every oriented cube C. In particular, $H_1 \lrcorner H_2 \wedge \langle C \rangle = -H_2 \wedge H_1 \lrcorner \langle C \rangle$.

Proof. If C is a vertex then both sides of the formula are zero. More generally, if C is a q-cube and one of the following two conditions fails then both sides of the formula are zero:

(a) H_2 is adjacent to C, and separates it from the base point;
(b) H_1 cuts C and crosses H_2.

Assume both of these conditions, and suppose that C may be presented by the listing of hyperplanes $H_1, K_2, \ldots, K_q$ and vertex P, *and that H_1 separates P from the base point*; if C is not an edge this is always possible. We shall leave the exceptional case in which C is an edge oriented by its vertex closest to the base point to the reader.

Now, let Q be the vertex of C separated from P by H_1 alone, and let P^{op} and Q^{op} be the vertices directly opposite H_2 from P and Q, respectively. The cube $H_2 \wedge C$ is presented by the listing $H_2, H_1, K_2, \ldots, K_q$ together with the vertex P^{op}, hence also by the listing $H_1, H_2, K_2, \ldots, K_q$ and the vertex Q^{op}. It follows that $H_1 \lrcorner H_2 \wedge C$ is presented by the listing $H_2, K_2, \ldots, K_q$ and the vertex P^{op}. As for the right hand side, C^* is presented by the same listing as C but with the vertex Q, so that $H_1 \lrcorner C^*$ is presented by the listing $K_2, \ldots, K_q$ and the vertex P. It follows that $H_2 \wedge H_1 \lrcorner C^*$ is presented by the listing $H_2, K_2, \ldots, K_q$ and the vertex P^{op}, as required. □

3.22 Corollary. *The cohomology of the Julg-Valette complex is $\mathbb{C}$ in degree zero and 0 otherwise.*

Proof. In degree $q = 0$ the kernel of d is one dimensional and is spanned by $\langle P_0 \rangle$. In degrees $q \geq 1$ proceed as follows. From $d^2 = 0$ it follows that $d\Delta = d\delta d = \Delta d$, so that also $d\Delta^{-1} = \Delta^{-1} d$. Now the calculation

$$f = \Delta\Delta^{-1} f = (d\delta + \delta d)\Delta^{-1} f = d(\delta\Delta^{-1}) f$$

shows that an oriented q-cocycle f is also an oriented q-coboundary. □

We conclude the section with a slight generalization that will be needed later.

3.23 Definition. A *weight function* for X is a positive-real-valued function w on the set of hyperplanes in X. The *weighted Julg-Valette differential* is the linear map

$$d_w \colon \mathbb{C}[X^q] \longrightarrow \mathbb{C}[X^{q+1}]$$

given by the formula

$$d_w\, C = \sum_H w(H)\, H \wedge C.$$

In addition the adjoint operator

$$\delta_w \colon \mathbb{C}[X^{q+1}] \longrightarrow \mathbb{C}[X^q]$$

is defined by

$$\delta_w\, C = \sum_H w(H)\; H \lrcorner\, C.$$

3.24 Remark. We are mainly interested in the following examples, or small variations on them:

(a) $w(H) \equiv 1$.
(b) $w(H) =$ the minimal edge-path distance to the base point P_0 from a vertex adjacent to H.

The calculations in this section are easily repeated in the weighted context: the operators d_w and δ_w are formally adjoint, although *unbounded* in the case of an unbounded weight function as, for example, in (b); both are differentials when restricted to the spaces of oriented cochains; and the cohomology of either complex is $\mathbb{C}$ in degree zero and 0 otherwise. We record here the formula for the weighted Julg-Valette Laplacian. Compare Proposition 3.20.

3.25 Proposition. *If C is an oriented q-cube then*

$$\Delta_w \langle C\rangle = (q_w(C) + p_w(C))\langle C\rangle,$$

where $q_w(C)$ is the sum of the squares of the weights of the hyperplanes that cut C and $p_w(C)$ is the sum of the squares of the weights of the hyperplanes that are adjacent to C and separate C from the base vertex. □

4. Parallelism classes of cubes

The remaining aspects of our generalization of the Julg-Valette and Pytlik-Szwarc theory to CAT(0) cubical spaces all rest on the following geometric concept:

4.1 Definition. Two cubes D_1 and D_2 in a CAT(0) cubical space X are *parallel* if they have the same dimension, and if every hyperplane that cuts D_1 also cuts D_2.

Every parallelism class of q-cubes in X is determined by, and determines, a set of q pairwise intersecting hyperplanes, namely the hyperplanes that cut all the cubes in the parallelism class. Call these the *determining hyperplanes* for the parallelism class.

4.2 Proposition. *The intersection of the determining hyperplanes associated to a parallelism class of q-cubes carries the structure of a* CAT(0) *cube complex in which the p-cubes are the intersections of this space with the $(p+q)$-cubes in X that are cut by every determining hyperplane, and in which the hyperplanes are the nonempty intersections of this space with the non-determining hyperplanes in X.*

Proof. The case when $q = 0$ is the assertion that X itself is a CAT(0) cubical space. The case when $q = 1$ is the assertion that a hyperplane in a CAT(0) cubical space X is itself a CAT(0) cubical space in the manner described above, and this is proved by Sageev in [20, Thm. 4.11].

For the general result, we proceed inductively as follows. Suppose given k distinct hyperplanes $K_1, \ldots, K_k$ in X. The intersection $Z = K_2 \cap \cdots \cap K_k$ is then a CAT(0) cube complex as described in the statement, and the result will follow from another application of [20, Thm. 4.11] once we verify that $K_1 \cap Z$ is a hyperplane in Z. Now the cubes, and so also the midplanes of Z are exactly the non-empty intersections of the cubes and midplanes of X with Z. So, we must show that if two midplanes belonging to the hyperplane K_1 of X intersect Z non-trivially then their intersections are hyperplane equivalent *in* Z. But this follows from the fact that Z is a totally geodesic subspace of X. $\square$

4.3 Proposition. *Let X be a* CAT(0) *cubical space and let P be a vertex in X. In each parallelism class of q-cubes there is a unique cube that is closest to P, as measured by the distance from closest point in the cube to P in the edge-path metric.*

Before beginning the proof, we recall that the edge-path distance between two vertices is equal to the number of hyperplanes separating the vertices; see for example [20, Theorem 4.13]. In addition, let us make note of the following simple fact:

4.4 Lemma. *A hyperplane that separates two vertices of distinct cubes in the same parallelism class must intersect every determining hyperplane.*

Proof. This is obvious if the hyperplane is one of the determining hyperplanes. Otherwise, the hyperplane must in fact separate two cubes in the parallelism class, and so it must separate two midplanes from each determining hyperplane. Since hyperplanes are connected the result follows. □

Proof of Proposition 4.3. Choose a vertex R from among the cubes in the parallelism class such that

$$d(P,R) \leq d(P,S) \tag{4.1}$$

for every other such vertex S. We shall prove the *addition formula*

$$d(P,S) = d(P,R) + d(R,S), \tag{4.2}$$

and this will certainly prove the uniqueness of R and hence of the nearest cube in the parallelism class.

The addition formula (4.2) is a consequence of the following *hyperplane property* of any R satisfying (4.1): *every hyperplane that separates P from R is parallel to* (*that is, it does not intersect*) *at least one determining hyperplane.* Indeed, it follows from Lemma 4.4 and the hyperplane property that no hyperplane can separate R from both P and S, so that (4.2) follows from the characterization of the edge path distance given above.

It remains to prove the hyperplane property for any R satisfying (4.1). For this we shall use the notion of *normal cube path* from [17, Section 3]. There exists a normal cube path from R to P with vertices

$$R = R_1, \ldots, R_l = P.$$

This means that every pair of consecutive R_i are diagonally opposite a cube, called a *normal cube*, all of whose hyperplanes separate R from P, and every such separating hyperplane cuts exactly one normal cube. It also means that every hyperplane K separating R_i from R_{i+1} is parallel to at least one of the hyperplanes H separating R_{i-1} from R_i (so each normal cube is, in turn, as large as possible). Note that the hyperplane K is contained completely in the half-space of H that contains P.

No hyperplane H separating $R = R_1$ from R_2 can intersect every determining hyperplane, for if it did, then it would follow from Lemma 2.6 that H and the determining hyperplanes would intersect in a $(q+1)$-cube having R as a vertex. The vertex S separated from R by H alone would then belong to a cube in the parallelism class, and would be strictly closer to P than R.

Consider the second normal cube, with opposite vertices R_3 and R_2. Any hyperplane K separating R_3 from R_2 is parallel to some hyperplane H separating R_2 from R_1, and this is in turn parallel to some determining hyperplane. But K is contained completely in the half-space of H that contains P, while the determining hyperplane is contained completely in the half-space of H that contains R. So K does not meet this determining hyperplane.

Continuing in this fashion with successive normal cubes, we find that every hyperplane that separates P from R is indeed parallel to some determining hyperplane, as required. $\square$

We can now verify the combinatorial formula mentioned at the end of the introduction. Fix, as usual, a base-vertex P in X. Associate to each vertex Q in X the first cube C in the normal cube path (see [17, Section 3] again) connecting Q to P.

4.5 Proposition. *The above correspondence induces a bijection from the set of vertices in X to the set of parallelism classes of cubes in X.*

Proof. Let Q be a vertex in X and let C be the first cube in the normal cube path from Q to P, as above. We claim that C is the nearest cube in its parallelism class to P; this fact will prove injectivity of our map. It follows from the addition formula in the proof of Proposition 4.3 that if C was *not* nearest to P within its parallelism class, then any hyperplane that separated Q from the nearest cube would also separate Q from P. Choosing a hyperplane that is adjacent to Q but does not cut C, we would find that C is not the first cube in the normal cube path from Q to P.

Conversely, if C is the nearest cube to P in any given parallelism class, and if Q is the vertex of C that is most remote from P, then all of the hyperplanes determining the parallelism class of P separate Q from P. Moreover no other hyperplane that is adjacent to Q can separate Q from P; this is a consequence of the hyperplane property proved in Proposition 4.3. So C is the first cube in the normal cube path from Q to P. This proves surjectivity. $\square$

4.6 Proposition. *Let X be a* $\mathrm{CAT}(0)$ *cubical space and let P and Q be vertices in X that are separated by a single hyperplane H. The nearest q-cubes to P and Q within a parallelism class are either the same, or are opposite faces, separated by H, of a $(q+1)$-cube that is cut by H.*

Proof. Denote by R and S the nearest vertices to P and Q, respectively, among the vertices of cubes in the equivalence class, and suppose that a hyperplane K separates R from S. Then it must separate P from S by the addition formula (4.2) applied to the nearest point R, and also separate Q from R, by the addition formula applied to the nearest point S. So it must separate P from Q, and hence must be H. So either there is no hyperplane separating R from S, in which case of course $R = S$ and the nearest cubes to P and Q are the same, or R is opposite S across H. If H is a determining hyperplane, then R and S are vertices of the same q-cube in the parallelism class; if H is not a determining hyperplane, then R and S belong to q-cubes that are opposite to one another across H, as required. $\square$

5. The Pytlik-Szwarc complex

As described in the introduction, our ultimate goal involves deforming the Julg-Valette complex into what we call the *Pytlik-Szwarc complex*, a complex with the same cohomology but which is equivariant in the case of a group acting on the CAT(0) cube complex. In this short section we describe the (algebraic) Pytlik-Szwarc complex.

As motivation for what follows we consider how to compare orientations on parallel cubes. The key observation is that a vertex in a q-cube is uniquely determined by its position relative to the cutting hyperplanes $K_1, \dots, K_q$. Thus, there is a natural isometry between (the vertex sets of) any two parallel q-cubes. We shall say that parallel q-cubes of positive dimension are *compatibly oriented* if their orientations are presented by vertices P_1 and P_2 which correspond under this isometry and a common listing of the cutting hyperplanes $K_1, \dots, K_q$; vertices are *compatibly oriented* if they are oriented by the same choice of sign.

We shall now generalize these considerations to pairs comprising a cube and one of its faces.

5.1 Definition. A *cube pair* is a pair (C, D) in which C is a cube containing D as a face. Two cube pairs (C_1, D_1) and (C_2, D_2) are *parallel* if the cubes C_1 and C_2 are parallel, and the cubes D_1 and D_2 are parallel too. When D is a q-cube, and C is a $(p+q)$-cube, we shall call (C, D) a (p, q)-cube pair, always keeping in mind that in this notation p is the *codimension* of D in C.

We may describe the parallelism class of a (p, q)-cube pair (C, D) by grouping the determining hyperplanes of the parallelism class of C into a symbol

$$\{\, H_1, \dots, H_p \,|\, K_1, \dots, K_q \,\}, \tag{5.1}$$

in which the $K_1, \dots, K_q$ determine the parallelism class of D. The hyperplanes $H_1, \dots, H_p$ which cut C but not D are the *complementary hyperplanes* of the cube pair, or of the parallelism class.

An *orientation* of a cube pair (C, D) is an orientation of the face D. In order to compare orientations of parallel cube pairs (C_i, D_i) we can compare the orientations on the faces D_i, which are themselves parallel cubes, but must also take into account the position of the faces within the ambient cubes C_i. For this we introduce the following notion.

5.2 Definition. Two parallel cube pairs (C_1, D_1) and (C_2, D_2) have the *same parity* if the number of complementary hyperplanes that separate D_1 from D_2, is *even.* Otherwise they have the *opposite parity.*

5.3 Definition. Let (C_1, D_1) and (C_2, D_2) be parallel cube pairs, each with an orientation. The orientations are *aligned* if one of the following conditions holds:

(a) (C_1, D_1) and (C_2, D_2) have the same parity, and D_1 and D_2 are compatibly oriented; or
(b) (C_1, D_1) and (C_2, D_2) have the opposite parity, and D_1 and D_2 are not compatibly oriented.

In the symbol (5.1) describing the parallelism class of a cube pair (C, D), the hyperplanes are not ordered; the only relevant data is which are to the left, and which to the right of the vertical bar. If the cube pair (C, D) is oriented, then the symbol receives additional structure coming from the orientation of D. We group the determining hyperplanes as before, and include a vertex R of D into a new symbol

$$\{ H_1, \dots, H_p \,|\, K_1, \dots, K_q \,|\, R \} . \tag{5.2}$$

Here, in the case $q > 0$, the hyperplanes $K_1, \dots, K_q$ form an *ordered list* which, together with the vertex R are a presentation of the oriented cube D. In the case $q = 0$ this list is empty and we replace it by the sign representing the orientation of the vertex $D = R$, obtaining a symbol of the form

$$\{ H_1, \dots, H_p \,|\, + \,|\, R \} \quad \text{or} \quad \{ H_1, \dots, H_p \,|\, - \,|\, R \} . \tag{5.3}$$

In either case the hyperplanes $H_1, \dots, H_p$ remain an *unordered set*. Conversely, a formal expression as in (5.2) or (5.3) is the symbol of some oriented (p, q)-cube pair precisely when the hyperplanes $H_1, \dots, K_q$ are distinct and have nonempty (pairwise) intersection, and the vertex R is adjacent to all of them.

The following definition captures the notion of alignment of orientations in terms of the associated symbols.

5.4 Definition. Symbols

$$\{ H_1, \dots, H_p \,|\, K_1, \dots, K_q \,|\, R \} \quad \text{and} \quad \{ H'_1, \dots, H'_p \,|\, K'_1, \dots, K'_q \,|\, R' \}$$

of the form (5.2) are *equivalent* if

(a) the sets $\{ H_1, \dots, H_p \}$ and $\{ H'_1, \dots, H'_p \}$ are equal;
(b) the $K_1, \dots, K_q$ are a permutation of the $K'_1, \dots, K'_q$; and
(c) the number of hyperplanes among the $H_1, \dots, K_q$ separating R and R' has the same parity as the permutation in (b).

In the case of symbols of the form (5.3) we omit (b) and replace (c) by

(c′) the number of hyperplanes among the $H_1, \dots, H_p$ separating R and R' is even if the orientation signs agree, and odd otherwise.

An *oriented* (p,q)*-symbol* is an equivalence class of symbols. We shall denote the equivalence class of the symbol (5.2) by

$$[\,H_1,\dots,H_p\,|\,K_1,\dots,K_q\,|\,R\,],$$

or simply by $[\,H\,|\,K\,|\,R\,]$ when no confusion can arise, and we use similar notation in the case of symbols of the form (5.3). We shall denote the set of oriented (p,q)-symbols by $\mathcal{H}_q^p$, and the (disjoint) union $\mathcal{H}_q^0 \cup \cdots \cup \mathcal{H}_q^{n-q}$ by $\mathcal{H}_q$.

5.5 Proposition. *The oriented symbols associated to oriented* (p,q)*-cube pairs agree precisely when the orientations of the cube pairs are aligned.* $\square$

Our generalization of the Pytlik-Szwarc complex will be a differential complex designed to capture the combinatorics of oriented, aligned cube pairs:

$$\mathbb{C}[\mathcal{H}_0] \xrightarrow{d} \mathbb{C}[\mathcal{H}_1] \xrightarrow{d} \cdots \xrightarrow{d} \mathbb{C}[\mathcal{H}_{n-1}] \xrightarrow{d} \mathbb{C}[\mathcal{H}_n]. \tag{5.4}$$

5.6 Definition. The space of oriented q-cochains of type p in the Pytlik-Szwarc complex is the space of finitely supported, *anti-symmetric*, complex-valued functions on $\mathcal{H}_q^p$. Here, a function is anti-symmetric if

$$f([\,H\,|\,K\,|\,R\,]) + f([\,H\,|\,K\,|\,R\,]^*) = 0,$$

where we have used the involution on $\mathcal{H}_q^p$ defined by reversing the orientation of the symbol. We shall denote this space by $\mathbb{C}[\mathcal{H}_q^p]$. The space of oriented q-cochains is defined similarly using the oriented symbols of type (p,q) for all $0 \le p \le n-q$. It splits as the direct sum

$$\mathbb{C}[\mathcal{H}_q] = \mathbb{C}[\mathcal{H}_q^0] \oplus \cdots \oplus \mathbb{C}[\mathcal{H}_q^{n-q}].$$

5.7 Remark. As with the Julg-Valette cochains, the space of oriented Pytlik-Szwarc q-cochains of type p is a subspace of the *full space of Pytlik-Szwarc q-cochains of type p*, which is the vector space of *all* finitely supported functions on the set $\mathcal{H}_q^p$. We shall follow conventions similar to those in Section 3: we write

$$[\,H_1,\dots,H_p\,|\,K_1,\dots,K_q\,|\,R\,] \quad \text{or} \quad [\,H\,|\,K\,|\,R\,]$$

for both the Dirac function at an oriented symbol and the symbol itself, and

$$\langle\,H\,|\,K\,|\,R\,\rangle = [\,H\,|\,K\,|\,R\,] - [\,H\,|\,K\,|\,R\,]^* \in \mathbb{C}[\mathcal{H}_q^p]$$

for the difference of the Dirac functions. Further, linear operators will be defined on the full space of cochains by specifying their values on the basis of Dirac functions at the oriented symbols. We shall typically omit the elementary check that an operator commutes with the involution and so restricts to an operator on the spaces of oriented cochains.

We now define the differential in the Pytlik-Szwarc complex (5.4).

5.8 Definition. The Pytlik-Szwarc differential is the linear map $d : \mathbb{C}[\mathcal{H}_q] \to \mathbb{C}[\mathcal{H}_{q+1}]$ which is 0 on oriented symbols of type $(0, q)$ and which satisfies

$$d\,[\,H_1, \dots, H_p \,|\, K_1, \dots, K_q \,|\, R\,] = \sum_{i=1}^{p} \Big[\,H_1, \dots, \widehat{H_i}, \dots, H_p \,|\, H_i, K_1, \dots, K_q \,|\, R_i\,\Big]$$

for oriented (p, q)-symbols with $p, q \geq 1$. Here, R_i is the vertex separated from R by H_i alone and, as usual, a 'hat' means that an entry is removed. When $q = 0$ the same formula is used for symbols of the form $[H \,|\, + \,|\, R]$ which, together with the requirement that d commute with the involution, determines d on symbols of the form $[H \,|{-}|\, R]$. Since d maps an oriented symbol of type (p, q) to a linear combination of oriented symbols of type $(p - 1, q + 1)$ in all cases, it splits as the direct sum of linear maps

$$d : \mathbb{C}[\mathcal{H}^p_q] \longrightarrow \mathbb{C}[\mathcal{H}^{p-1}_{q+1}]$$

for $0 < p \leq n - q$, and is 0 on the $\mathbb{C}[\mathcal{H}^0_q]$.

5.9 Lemma. *The Pytlik-Szwarc differential d, regarded as an operator on the space of oriented cochains, satisfies $d^2 = 0$.* □

5.10 Example. Let T be a tree. The Pytlik-Szwarc complex has the form

$$d : \mathbb{C} \oplus \mathbb{C}[\mathcal{H}^1_0] \longrightarrow \mathbb{C}[\mathcal{H}^0_1],$$

where d is 0 on $\mathbb{C}$ and, after identifying each of $\mathbb{C}[\mathcal{H}^1_0]$ and $\mathbb{C}[\mathcal{H}^0_1]$ with the space of finitely supported functions on the set of edges of T, the identity $\mathbb{C}[\mathcal{H}^1_0] \to \mathbb{C}[\mathcal{H}^0_1]$. For the identifications, note that both $\mathcal{H}^0_1$ and $\mathcal{H}^1_0$ are identified with the set of oriented edges in T and that the involution acts by reversing the orientation. So the space of anti-symmetric functions on each identifies with the space of finitely supported functions on the set of edges.

Our goal for the remainder of this section is to analyze the Pytlik-Szwarc complex. Emphasizing the similarities with the Julg-Valette complex we begin by providing a formula for the formal adjoint of the Pytlik-Szwarc differential.

5.11 Definition. Let $\delta : \mathbb{C}[\mathcal{H}_q] \to \mathbb{C}[\mathcal{H}_{q-1}]$ be the linear map which is 0 on oriented symbols of type $(p, 0)$ and which satisfies

$$\delta\,[\,H_1,\ldots,H_p\,|\,K_1,\ldots,K_q\,|\,R\,] = \sum_{j=1}^{q}(-1)^j\left[\,H_1,\ldots,H_p,K_j\,|\,K_1,\ldots,\widehat{K_j},\ldots,K_q\,|\,R\,\right],$$

for oriented symbols of type (p,q) with $q \geq 1$. Again a 'hat' means that an entry is removed. Since δ maps an oriented symbol of type (p,q) to a linear combination of oriented symbols of type $(p+1,q-1)$ it splits as a direct sum of linear maps

$$\delta : \mathbb{C}[\mathcal{H}^p_q] \to \mathbb{C}[\mathcal{H}^{p+1}_{q-1}]$$

for $0 < q \leq n-p$, and is 0 on the $\mathbb{C}[\mathcal{H}^p_0]$.

5.12 Definition. We define an inner product on the full space of Pytlik-Szwarc q-cochains by declaring that the elements of $\mathcal{H}_q$ are orthogonal, and that each has length $1/\sqrt{2}$. The subspace $\mathbb{C}[\mathcal{H}_q]$ of oriented Pytlik-Szwarc q-cochains inherits an inner product in which

$$\langle\langle\, H\,|\,K\,|\,R\,\rangle\,,\langle\, H'\,|\,K'\,|\,R'\,\rangle\rangle = \begin{cases} 1, & [\,H\,|\,K\,|\,R\,] = [\,H'\,|\,K'\,|\,R'\,] \\ -1, & [\,H\,|\,K\,|\,R\,] = [\,H'\,|\,K'\,|\,R'\,]^* \\ 0, & \text{otherwise} \end{cases}$$

5.13 Lemma. *The operators d and δ of Definitions 5.8 and 5.11 are formally adjoint and bounded with respect to the inner products in Definition 5.12.* □

5.14 Proposition. *The Pytlik-Szwarc Laplacian*

$$\Delta = (d+\delta)^2 = d\delta + \delta d : \mathbb{C}[\mathcal{H}_q] \longrightarrow \mathbb{C}[\mathcal{H}_q]$$

acts on the summand $\mathbb{C}[\mathcal{H}^p_q]$ as scalar multiplication by $p+q$.

Proof. The proof is a direct calculation. The result of applying δd to an oriented symbol $[\,H_1,\ldots,H_p\,|\,K_1,\ldots,K_q\,|\,R\,]$ of type (p,q) is the sum

$$(-1)\sum_{i=1}^{p}[\,H_1,\ldots,H_p\,|\,K_1,\ldots,K_q\,|\,R_i\,] +$$
$$+\sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{j+1}[\,H_1,\ldots,\widehat{H}_i,\ldots,H_p,K_j\,|\,H_i,K_1,\ldots,\widehat{K}_j,\ldots,K_q\,|\,R_i\,],$$

where R_i is the vertex separated from R only by H_i. And the result of applying $d\delta$ is

$$\sum_{j=1}^{q}(-1)^j[\,H_1,\ldots,H_p\,|\,K_j,K_1,\ldots,\widehat{K}_j,\ldots,K_q\,|\,Q_j\,] +$$
$$+\sum_{j=1}^{q}(-1)^j\sum_{i=1}^{p}[\,H_1,\ldots,\widehat{H}_i,\ldots,H_p,K_j\,|\,H_i,K_1,\ldots,\widehat{K}_j,\ldots,K_q\,|\,R_i\,],$$

where now Q_j is separated from R only by K_j, and R_i is as before. When these are added, the second summands cancel (even at the level of full cochains). The first summands combine to give

$$-p[\,H|K|R\,]^* + \left\lfloor \tfrac{q}{2} \right\rfloor [\,H|K|R\,] - \left\lceil \tfrac{q}{2} \right\rceil [\,H|K|R\,]^*$$

where we have used that exchanging R for R_i, or for Q_j reverses the orientation of a symbol. Considering oriented cochains, it follows that $\langle H|K|R\rangle$ is an eigenvector of $d\delta + \delta d$ with eigenvalue $p+q$, as required. □

5.15 Corollary. *The cohomology of the Pytlik-Szwarc complex is $\mathbb{C}$ in dimension zero and 0 otherwise.* □

As a prelude to the next section, let us conclude by making note of a combinatorial result that generalizes the combinatorial theorem mentioned at the end of the introduction and makes evident a close connection between the Pytlik-Szwarc and Julg-Valette cochain spaces. The proof is the same as the proof of Proposition 4.5.

Fix a base-vertex P in X and associate to each q-cube D in X the (p,q)-cube pair (C,D) where C is the first cube in the normal cube path connecting the furthest vertex of D from P back to P.

5.16 Proposition. *For every fixed $q \geq 0$ the above correspondence induces a bijection from the set of q-cubes in X to the set of parallelism classes of all (p,q)-cube pairs (as p ranges through all possible values).* □

6. Continuous fields of Hilbert spaces

Our objective over the next several sections is to construct a family of complexes that continuously interpolates between the Julg-Valette complex and the Pytlik-Szwarc complex. We shall construct the interpolation within the Hilbert space context, using the concept of a *continuous field* of Hilbert spaces.

We refer the reader to [5, Chapter 10] for a comprehensive treatment of continuous fields of Hilbert spaces. In brief, a continuous field of Hilbert spaces over a topological space T consists of a family of Hilbert spaces parametrized by the points of T, together with a distinguished family Σ of sections that satisfies several axioms, of which the most important is that the pointwise inner product of any two sections in Σ is a *continuous* function on T. See [5, Definition 10.1.2]. The following theorem gives a convenient means of constructing continuous fields.

6.1 Theorem. *Let T be a topological space, let $\{\mathfrak{H}_t\}$ be a family of Hilbert spaces parametrized by the points of T, and let Σ_0 be a family of sections that satisfies the following conditions:*

(a) *The pointwise inner product of any two sections in* Σ_0 *is a continuous function on* T*.*
(b) *For every* $t \in T$ *the linear span of* $\{\sigma(t) : \sigma \in \Sigma\}$ *is dense in* $\mathfrak{H}_t$*.*

There is a unique enlargement of Σ_0 *that gives* $\{\mathfrak{H}_t\}_{t\in T}$ *the structure of a continuous field of Hilbert spaces.*

Proof. The enlargement Σ consists of all sections σ such that for every $t_0 \in T$ and every $\varepsilon > 0$ there is a section σ_0 in the linear span of Σ_0 such that

$$\|\sigma_0(t) - \sigma(t)\|_t < \varepsilon$$

for all t in some neighborhood of t_0. See [5, Proposition 10.2.3]. □

6.2 Definition. We shall call a family Σ_0, as in the statement of Theorem 6.1, a *generating family* of sections for the associated continuous field of Hilbert spaces.

Ultimately we shall use the parameter space $T = [0, \infty]$, but in this section we shall concentrate on the open subspace $(0, \infty]$, and then extend to $[0, \infty]$ in the next section. In both this section and the next we shall deal only with the construction of continuous fields of Hilbert spaces; we shall construct the differentials acting between these fields in Section 8.

We begin by completing the various cochain spaces from Section 3 in the natural way so as to obtain Hilbert spaces.

6.3 Definition. Denote by $\ell^2(X^q)$ the Hilbert space completion of the Julg-Valette oriented cochain space $\mathbb{C}[X^q]$ in the inner product of Definition 3.18 in which the basis comprised of the oriented cochains $\langle C \rangle$ is orthonormal.

6.4 Remark. As was the case in Section 3, we shall also consider the full cochain space comprised of the square-summable functions on the set of oriented q-cubes. This is the completion of the full space of Julg-Valette q-cochains in the inner product of Definition 3.18, and contains the space $\ell^2(X^q)$ of the previous definition as the subspace of anti-symmetric functions.

We shall now construct, for every $q \geq 0$, families of Hilbert spaces parametrized by the topological space $(0, \infty]$. These will be completions of the spaces of Julg-Valette q-cochains, both full and oriented, but with respect to a family of pairwise distinct inner products. Considering the oriented cochains, we obtain a family of Hilbert spaces $\ell^2_t(X^q)$ each of which is a completion of the corresponding $\mathbb{C}[X^q]$. The Hilbert space $\ell^2_\infty(X^q)$ will be the space $\ell^2(X^q)$ just defined.

6.5 Definition. If D_1 and D_2 are q-cubes in X, and if D_1 and D_2 are parallel and have compatible orientations, then denote by $d(D_1, D_2)$ the number of hyperplanes in X that are disjoint from D_1 and D_2 and that separate D_1 from D_2. If D_1 and D_2 are q-cubes in X, but are not parallel, or have incompatible orientations, then set $d(D_1, D_2) = \infty$.

If D_1 and D_2 are (compatibly oriented) vertices, then $d(D_1, D_2)$ is the edge-path distance from D_1 to D_2. In higher dimensions, if D_1 and D_2 are parallel then they may be identified with vertices in the CAT(0) cubical space which is the intersection of the determining hyperplanes for the parallelism class. If in addition they are compatibly oriented, then $d(D_1, D_2)$ is the edge-path distance in this complex. Compare Theorem 4.2.

6.6 Definition. Let $t > 0$ and $q \geq 0$. For every two oriented q-cubes D_1 and D_2 define

$$\langle D_1, D_2 \rangle_t = \tfrac{1}{2} \exp\big(-\tfrac{1}{2} t^2 d(D_1, D_2)\big),$$

where of course we set $\exp(-\frac{1}{2}t^2 d(D_1, D_2)) = 0$ if $d(D_1, D_2) = \infty$, and then extend by linearity to a sesqui-linear form on the full space of Julg-Valette q-cochains.

Note that the formula in the definition makes sense when $t = \infty$, where

$$\tfrac{1}{2} \exp\big(-\tfrac{1}{2} t^2 d(D_1, D_2)\big) = \begin{cases} \frac{1}{2}, & D_1 = D_2 \\ 0, & D_1 \neq D_2. \end{cases}$$

In particular, the form $\langle \ , \ \rangle_\infty$ is the one underlying Definition 3.18 that we used to define $\ell^2(X^q)$.

6.7 Theorem. *The sesqui-linear form* $\langle \ , \ \rangle_t$ *is positive semi-definite.*

Proof. Consideration of oriented, as opposed to unoriented, cubes merely gives two (orthogonal copies) of each space of functions. Aside from this, the result is proved in [17, Technical Lemma, p.6] in the case $q = 0$. See also [8, Prop. 3.6]. The case $q > 0$ reduces to the case $q = 0$ using Theorem 4.2. □

6.8 Definition. For $t \in (0, \infty]$ denote by $\ell^2_t(X^q)$ the Hilbert space completion of the Julg-Valette oriented cochain space $\mathbb{C}[X^q]$ in the inner product $\langle \ , \ \rangle_t$.

6.9 Remark. The Hilbert spaces of the previous definition are completions of the *quotient of* $\mathbb{C}[X^q]$ *by the elements of zero norm.* We shall soon see that every nonzero linear combination of oriented q-cubes has nonzero ℓ^2_t-norm for every t, so the natural maps from $\mathbb{C}[X^q]$ into the $\ell^2_t(X^q)$ are injective.

Next, we define a generating family of sections, using either one of the following lemmas; on the basis of Theorem 6.1, it is easy to check that the continuous fields arising from the lemmas are one and the same.

6.10 Lemma. *Let $t \in (0, \infty]$. The set of all sections of the form*

$$t \mapsto f \in \mathbb{C}[X^q] \subseteq \ell^2_t(X^q),$$

indexed by all $f \in \mathbb{C}[X^q]$, is a generating family of sections for a continuous field. $\square$

6.11 Lemma. *The set of all sections of the form*

$$t \mapsto f(t) \langle C \rangle \in \ell^2_t(X^q),$$

where f is a continuous scalar function on $(0, \infty]$ and C is an oriented q-cube, is a generating family of sections for a continuous field. $\square$

The continuous fields that we have constructed are not particularly interesting as continuous fields. In fact they are isomorphic to constant fields (they become much more interesting when further structure is taken into account, as we shall do later in the paper). For the sequel it will be important to fix a particular isomorphism, and we conclude this section by doing this.

The required unitary isomorphism will be defined using certain cocycle operators $W_t(C_1, C_2)$, which are analogues of those studied by Valette in [21] in the case of trees. In the case $q = 0$ the cocycle operators for general CAT(0) cubical spaces were constructed in [8]. The case where $q > 0$ involves only a minor elaboration of the $q = 0$ case, and so we shall refer to [8] for details in what follows.

6.12 Definition. If D is a q-cube that is adjacent to a hyperplane H, then define D^{op} to be the opposite face to D in the unique $(q + 1)$-cube that is cut by H and contains D as a q-face (such a cube exists by Lemma 3.6). In the case D is oriented, we orient D^{op} compatibly. In either case, we shall refer to a pair such as D and D^{op} as being adjacent across H.

6.13 Definition. Let C and C^{op} be unoriented q-cubes, adjacent across a hyperplane H as in the previous definition. If D is any oriented q-cube that is adjacent to H, then for $t \in (0, \infty]$ we define

$$W_t(C^{op}, C)D = \begin{cases} (1 - e^{-t^2})^{1/2} D - e^{-\frac{1}{2}t^2} D^{op}, & \text{if } D \text{ is separated from } C \text{ by } H \\ e^{-\frac{1}{2}t^2} D^{op} + (1 - e^{-t^2})^{1/2} D, & \text{if } D \text{ is not separated from } C \text{ by } H. \end{cases}$$

So $W_t(C^{op}, C)$ really only depends on the half-space decomposition determined by H, not on C and C^{op}. In addition we define

$$W_t(C^{op}, C)D = D \quad \text{if } D \text{ is not adjacent to } H.$$

We extend $W_t(C^{op}, C)$ by linearity to a linear operator on the spaces of (full and oriented) Julg-Valette q-cochains.

For example

$$W_0(C^{op}, C)C = C^{op} \quad \text{and} \quad W_0(C^{op}, C)C^{op} = -C,$$

while

$$W_\infty(C^{op}, C)C = C \quad \text{and} \quad W_\infty(C^{op}, C)C^{op} = C^{op},$$

and indeed $W_\infty(C^{op}, C)$ is the identity operator. More generally, when restricted to the two-dimensional space spanned by the ordered basis (D, D^{op}) with D adjacent to H but *not* separated from C by H, the operator $W_t(C^{op}, C)$ acts as the unitary matrix

$$\begin{bmatrix} (1 - e^{t^2})^{1/2} & -e^{-\frac{1}{2}t^2} \\ e^{-\frac{1}{2}t^2} & (1 - e^{t^2})^{1/2} \end{bmatrix}.$$

In particular, $W_t(C^{op}, C)$ extends to a unitary operator on the completed cochain spaces of Definition 6.3 and subsequent remark.

Let us now assume that two unoriented q-cubes C_1 and C_2 are parallel, but not necessarily adjacent across a hyperplane. It follows from Theorem 4.2 that there exists a path of q-cubes $E_1, E_2, \ldots, E_n$, with $E_1 = C_1$ and $E_n = C_2$, where each consecutive pair E_i, E_{i+1} consists of parallel and adjacent q-cubes. For all $t \geq 0$ let us define

$$W_t(C_1, C_2) = W_t(E_1, E_2)W_t(E_2, E_3) \ldots W_t(E_{n-1}, E_n). \tag{6.1}$$

This notation, which omits mention of the path, is justified by the following result:

6.14 Proposition. *The unitary operator $W_t(C_1, C_2)$ is independent of the path from C_1 to C_2.*

Proof. Let γ and γ' be two cube paths connecting cubes C_1 and C_2. As the cubes C_1 and C_2 are parallel, by Theorem 4.2 they can be thought of as vertices in the CAT(0) cubical space created from their parallelism class. The paths γ and γ' then give rise to vertex paths in this CAT(0) cubical space with common beginning and end vertices. In this way we reduce the general case of the proposition to the zero dimensional case, which has been proved in [8, Lemma 3.3]. $\square$

6.15 Definition. Fix a base vertex P_0 in X, and let $t \in (0, \infty]$. For every oriented q-cube D let

$$U_t D = W_t(D_0, D)D,$$

where D_0 is the cube nearest to the base vertex P_0 in the parallelism class of D (see Proposition 4.3). Extend U_t by linearity to a linear operator on the spaces of full and oriented Julg-Valette q-cochains; in particular, on oriented cochains we have

$$U_t : \mathbb{C}[X^q] \longrightarrow \mathbb{C}[X^q].$$

6.16 Lemma. *The linear operator U_t is a vector space isomorphism.*

Proof. Consider the increasing filtration of the cochain space, indexed by the natural numbers, in which the nth space is spanned by those cubes whose nearest vertex to P_0 in the edge-path metric is of distance n or less from P_0. The operator U_t preserves this filtration. In fact, a simple direct calculation (see [8, Lemma 4.7]) shows that

$$\begin{aligned} U_t D &= W_t(D_0, D)D \\ &= \text{constant} \cdot D + \text{linear combination of cubes closer to } P_0 \text{ than } D. \end{aligned}$$

This formula shows that the induced map on associated graded spaces is an isomorphism. So U_t is an isomorphism. □

6.17 Lemma. *If D_1 and D_2 are any two oriented q-cubes in X, then*

$$\langle U_t D_1, U_t D_2 \rangle = \langle D_1, D_2 \rangle_t,$$

where the inner product on the left hand side is that of $\ell^2(X^q)$.

6.18 Remark. The lemma implies that the sesqui-linear form $\langle\ ,\ \rangle_t$ is positive definite for each $t > 0$, since $\langle\ ,\ \rangle$ is positive-definite and U_t is an isomorphism.

Proof of the lemma. We can assume that the q-cubes D_1 and D_2 are parallel and compatibly oriented since otherwise both sides of the formula are zero. Let D_0 denote the q-cube in the parallelism class that is nearest to the base vertex P_0. Then the unitarity of W_t and Proposition 6.14 give

$$\begin{aligned} \langle U_t D_1, U_t D_2 \rangle &= \langle W_t(D_0, D_1)D_1, W_t(D_0, D_2)D_2 \rangle \\ &= \langle W_t(D_0, D_2)^* W_t(D_0, D_1)D_1, D_2 \rangle \\ &= \langle W_t(D_2, D_0) W_t(D_0, D_1)D_1, D_2 \rangle \\ &= \langle W_t(D_2, D_1)D_1, D_2 \rangle. \end{aligned}$$

But, by an elaboration of [8, Proposition 3.6] we have

$$W_t(D_2, D_1)D_1 = e^{-\frac{1}{2}t^2 d(D_2, D_1)} D_2 + \text{ multiples of oriented cubes other than } D_2. \quad (6.2)$$

Hence we conclude that

$$\langle W_t(D_2, D_1)D_1, D_2 \rangle = \tfrac{1}{2} e^{-\frac{1}{2}t^2 d(D_2, D_1)} = \langle D_1, D_2 \rangle_t,$$

as required. □

The following results are immediate consequences of the above:

6.19 Theorem. *For all $t \in (0, \infty]$ the map*

$$U_t\colon \mathbb{C}[X^q] \longrightarrow \mathbb{C}[X^q]$$

extends to a unitary isomorphism

$$U_t : \ell^2_t(X^q) \longrightarrow \ell^2_\infty(X^q). \quad \square$$

6.20 Theorem. *The unitary operators U_t determine a unitary isomorphism from the continuous field $\{\ell^2_t(X^q)\}_{t\in(0,\infty]}$ generated by sections in Lemmas 6.10 and 6.11 to the constant field with fiber $\ell^2(X^q)$.* $\square$

7. Extension of the continuous field

In this section we shall extend the continuous fields over $(0, \infty]$ defined in Section 6 by adding the following fibers at $t = 0$.

7.1 Definition. We shall denote by $\ell^2_0(X^q)$ the completion of the space of oriented Pytlik-Szwarc q-cochains in the inner product of Definition 5.12. It is the subspace of anti-symmetric functions in the Hilbert space of all square-summable functions on the set of oriented symbols $\mathcal{H}_q$.

The following two definitions focus on the particular continuous sections that we shall extend.

7.2 Definition. Let $p, q \geq 0$ and let (C, D) be an oriented (p, q)-cube pair. The associated *basic q-cochain of type p* is the linear combination

$$f_{C,D} = \sum_{E\|_C D} (-1)^{d(D,E)} E$$

in the full cochain space. Here, the sum is over those q-cubes E in C that are parallel to D, each of which is given the orientation compatible with the orientation of D. The associated basic *oriented* cochain is

$$f_{\langle C,D\rangle} = f_{C,D} - f_{C,D^*} = \sum_{E\|_C D} (-1)^{d(D,E)} \langle\, E \,\rangle,$$

belonging to the space $\mathbb{C}[X^q]$ of oriented q-cochains.

7.3 Example. A basic q-cochain of type $p = 0$ is just a single oriented q-cube. A basic q-cochain of type p=1 is a difference of q-cubes that are opposite faces of a $(q$+1$)$-cube;

a basic q-cochain of type $p = 2$ is a difference of basic q-cochains of type p=1; and so on. Conversely, if (C_1, D_1) and (C_2, D_2) are parallel oriented (p, q)-cube pairs, then $f_{\langle C_1,D_1\rangle}$ is equal to $f_{\langle C_2,D_2\rangle}$ plus a linear combination of basic cochains of type $(p+1, q)$.

7.4 Definition. A *basic section of type* p of the continuous field $\{\, \ell^2_t(X^q)\,\}_{t\in(0,\infty]}$ is a continuous section $\sigma_{\langle C,D\rangle}$ of the form

$$(0, \infty] \ni t \overset{\sigma_{\langle C,D\rangle}}{\longmapsto} t^{-p} f_{\langle C,D\rangle} \in \ell^2_t(X^q),$$

where (C, D) is an oriented (p, q)-cube pair.

We shall extend the basic sections to sections over $[0, \infty]$ by assigning to each of them a value at $t = 0$ in the Hilbert space $\ell^2_0(X^q)$, namely the Pytlik-Szwarc symbol associated to the cube pair (C, D), as in Section 5. We shall write the symbol as

$$\langle\, C, D\,\rangle = [C, D] - [C, D]^* \in \ell^2_0(X^q).$$

Compare Definition 5.4 and Remark 5.7. We shall prove the following result.

7.5 Theorem. *Let* $q \geq 0$*.*

(a) *The pointwise inner product*

$$\big\langle \sigma_{\langle C_1,D_1\rangle}, \sigma_{\langle C_2,D_2\rangle} \big\rangle_t$$

of any two basic sections (*of possibly different types*) *extends to a continuous function on* $[0, \infty]$*.*

(b) *The value of this continuous function at* $0 \in [0, \infty]$ *is equal to the inner product*

$$\big\langle \langle C_1, D_1\rangle, \langle C_2, D_2\rangle \big\rangle_0.$$

7.6 Example. Suppose that X is a tree. When $q = 1$, the only basic sections are those of type $p = 0$, and Theorem 7.5 is easily checked in this case. When $q = 0$ there are basic sections of type $p = 0$, which are again easily handled, but also basic sections of type $p = 1$. These have the form

$$t \mapsto t^{-1}(\langle P\rangle - \langle Q\rangle),$$

where P and Q are adjacent vertices in the tree. One calculates that

$$\big\langle t^{-1}(\langle P\rangle - \langle Q\rangle), t^{-1}(\langle P\rangle - \langle Q\rangle)\big\rangle_t = 2t^{-2}(1 - e^{-\frac{1}{2}t^2}),$$

which converges to 1 as $t \to 0$, in agreement with Theorem 7.5. In addition if $t^{-1}(\langle R\rangle - \langle S\rangle)$ is a second, distinct basic cochain, and if the vertices P, Q, R, S are arranged in sequence along a path in the tree, then a short calculation reveals that if d is the distance between Q and R, then

$$\left\langle t^{-1}(\langle P\rangle - \langle Q\rangle), t^{-1}(\langle R\rangle - \langle S\rangle)\right\rangle_t = -t^{-2}e^{-\frac{d}{2}t^2}(1-e^{-\frac{1}{2}t^2})^2 = O(t^2).$$

In particular the inner product converges to 0 as $t \searrow 0$, again in agreement with Theorem 7.5.

Theorem 7.5 allows us to extend our continuous field to $[0,\infty]$. It is an artifact of our orientation conventions that we *cannot* perform this extension at the level of full cochains; cube pairs (C_1, D_1) and (C_2, D_2) which are parallel, not compatibly oriented and of opposite parity determine the same oriented symbol at $t = 0$ but their associated basic cochains are orthogonal for all $t > 0$.

7.7 Definition. An *extended basic section of type* p of the continuous field of Hilbert spaces $\{\, \ell^2_t(X^q)\,\}_{t\in[0,\infty]}$ is a section of the form

$$t \longmapsto \begin{cases} \langle C, D\rangle & t = 0 \\ \sigma_{\langle C,D\rangle} & t > 0, \end{cases}$$

where (C, D) is an oriented (p, q)-cube pair.

The basic sections form a generating family of sections for the continuous field $\{\ell^2_t(X^q)\}_{t\in(0,\infty]}$, and of course the symbols $\langle C, D\rangle$ span $\ell^2_0(X^q)$. So it follows from the theorem that the extended basic sections form a generating family of sections for a continuous field over $[0,\infty]$ with fibers $\ell^2_t(X^q)$, whose restriction to $(0,\infty]$ is the continuous field of the previous section.

We shall prove Theorem 7.5 by carrying out a sequence of smaller calculations. The following formula is common to all of them, and it will also be of use in Section 8. Here, and subsequently, we shall use the expression $O(t^p)$ not only in its usual sense, but also for any finite sum of oriented q-cubes times coefficient functions, each of which is bounded by a constant times t^p as $t \searrow 0$.

7.8 Lemma. *If* (C, D) *is an oriented* (p, q)*-cube pair then*

$$\sum_{E\|_C D} (-1)^{d(D,E)} W_t(D, E)E = (-t)^p D^{\mathrm{op}} + O(t^{p+1}), \tag{7.1}$$

where D^{op} *is the* q*-face of* C *separated from* D *by the complementary hyperplanes of the pair* (C, D)*, with compatible orientation.*

Proof. We shall prove the lemma by induction on p. The case $p = 0$ is clear. As for the case $p > 0$, let H be a hyperplane that cuts C but not D. Our aim is to apply the induction hypothesis to the codimension-one faces of C separated by H. Denote these faces by $C_\pm$ with C_+ being the face containing D; denote $D_+ = D$ and D_- the face of C_- directly across H from D; and finally denote by $D^{\mathrm{op}}_\pm$ the face in $C_\pm$ separated from $D_\pm$ by all the complementary hyperplanes of the pair (C, D) *except* H. We have, in particular, $D^{\mathrm{op}} = D^{\mathrm{op}}_-$.

Now, the expression on the left hand side of (7.1) depends on the cube pair (C, D) and for the course of the proof we shall denote it by $g_{C,D}$. We compute the summand of $g_{C,D}$ corresponding to a face E that belongs to C_- using the path from D_+ to D_- and on to E. Doing so, we see that

$$\begin{aligned} g_{C,D} &= g_{C_+,D_+} - W_t(D_+, D_-) g_{C_-,D_-} \\ &= (1 - e^{-\frac{1}{2}t^2})\, g_{C_+,D_+} - (1 - e^{-t^2})^{\frac{1}{2}}\, g_{C_-,D_-}. \end{aligned}$$

Here, we have used that the coefficient of g_{C_+,D_+} at a face E of C_+ equals the coefficient of g_{C_-,D_-} at the face of C_- which is directly across H from E. By the induction hypothesis, $g_{C_+,D_+} = (-t)^{p-1} D^{\mathrm{op}}_+ + O(t^p)$, which is $O(t^{p-1})$. Since $1 - e^{-\frac{1}{2}t^2}$ is $O(t^2)$ the first term in the display is $O(t^{p+1})$. As for the second term, again by induction we have $g_{C_-,D_-} = (-t)^{p-1} D^{\mathrm{op}}_- + O(t^p)$, which is $O(t^{p-1})$. It follows that

$$\begin{aligned} -(1 - e^{-t^2})^{\frac{1}{2}}\, g_{C_-,D_-} &= -t\, g_{C_-,D_-} + (t - (1 - e^{-t^2})^{\frac{1}{2}})\, g_{C_-,D_-} \\ &= (-t)^p D^{\mathrm{op}}_- + O(t^{p+1}) + (t - (1 - e^{-t^2})^{\frac{1}{2}}) O(t^{p-1}) \\ &= (-t)^p D^{\mathrm{op}} + O(t^{p+1}), \end{aligned}$$

where we have used that $t - (1 - e^{-t^2})^{\frac{1}{2}}$ is $O(t^3)$. Putting things together, the lemma is proved. □

In the previous section we defined unitary isomorphisms $U_t : \ell^2_t(X^q) \to \ell^2(X^q)$. While these were defined using a specific choice of base point within each parallelism class of q-cubes, the choice is not important as far as the unitarity of U_t is concerned. We shall exploit this by making judicious choices of base point to calculate the inner products in Theorem 7.5.

7.9 Lemma. *Let (C, D) be an oriented (p, q)-cube pair, and let $f_{C,D}$ be the associated basic q-cochain of type p. The pointwise inner product*

$$\left\langle t^{-p} f_{C,D}, t^{-p} f_{C,D} \right\rangle_t$$

converges to $\frac{1}{2}$ as $t \searrow 0$.

Proof. Choose D as the base point for defining the unitary isomorphisms U_t. Then $U_t f_{C,D}$ is exactly the expression (7.1) in the previous lemma. It follows from the lemma that

$$\begin{aligned}\left\langle t^{-p} f_{C,D}, t^{-p} f_{C,D}\right\rangle_t &= \left\langle t^{-p} U_t f_{C,D}, t^{-p} U_t f_{C,D}\right\rangle_\infty \\ &= \left\langle (-1)^p D^{\mathrm{op}} + O(t), (-1)^p D^{\mathrm{op}} + O(t)\right\rangle_\infty \\ &= \tfrac{1}{2} + O(t),\end{aligned}$$

and the result follows. □

7.10 Lemma. *Let (C_1, D_1) and (C_2, D_2) be parallel (p, q)-cube pairs of the same parity, in which the q-dimensional faces are compatibly oriented. The pointwise inner product*

$$\left\langle t^{-p} f_{C_1,D_1}, t^{-p} f_{C_2,D_2}\right\rangle_t$$

converges to $\frac{1}{2}$ as $t \searrow 0$.

Proof. We may assume that D_2 lies on the same side of each of the complementary hyperplanes of the parallelism class as D_1; indeed replacing D_2 by this face, if necessary, does not change the corresponding basic cochain. Choose D_1 as the base point for defining the unitary isomorphisms U_t, so that by Lemma 7.8 we have

$$U_t f_{C_1,D_1} = (-t)^p D_1^{\mathrm{op}} + O(t^{p+1})$$

and also, using the identity $W_t(D_1, E) = W_t(D_1, D_2) W_t(D_2, E)$ for the q-dimensional faces E of C_2,

$$U_t f_{C_2,D_2} = (-t)^p W_t(D_1, D_2) D_2^{\mathrm{op}} + O(t^{p+1}).$$

But, the hyperplanes separating D_1 and D_2 are precisely those separating D_1^{op} and D_2^{op}, so that by (6.2) we have

$$\begin{aligned}W_t(D_1, D_2) D_2^{\mathrm{op}} &= W_t(D_1^{\mathrm{op}}, D_2^{\mathrm{op}}) D_2^{\mathrm{op}} \\ &= e^{-\frac{1}{2} t^2 d(D_1, D_2)} D_1^{\mathrm{op}} + O(1),\end{aligned}$$

in which the terms included under the $O(1)$ are orthogonal to D_1^{op}. Putting everything together we get

$$\begin{aligned}\left\langle t^{-p} f_{C_1,D_1}, t^{-p} f_{C_2,D_2}\right\rangle_t &= \left\langle t^{-p} U_t f_{C_1,D_1}, t^{-p} U_t f_{C_2,D_2}\right\rangle_\infty \\ &= e^{-\frac{1}{2} t^2 d(D_1, D_2)} \left\langle (-1)^p D_1^{\mathrm{op}}, (-1)^p D_1^{\mathrm{op}}\right\rangle_\infty + O(t)\end{aligned}$$

and the result follows from this. □

7.11 Lemma. *Let (C_1, D_1) and (C_2, D_2) be oriented cube pairs of types (p_1, q) and (p_2, q), respectively, and let f_{C_1,D_1} and f_{C_2,D_2} be the associated basic q-cochains. If (C_1, D_1) and*

(C_2, D_2) are not parallel, or if D_1 and D_2 are not compatibly oriented, then the pointwise inner product

$$\left\langle t^{-p_1} f_{C_1,D_1}, t^{-p_2} f_{C_2,D_2} \right\rangle_t$$

converges to 0 *as* $t \searrow 0$*. In particular, this is the case if* $p_1 \neq p_2$*.*

Proof. If D_1 and D_2 fail to be parallel or have incompatible orientations, then f_{C_1,D_1} and f_{C_2,D_2} are orthogonal in the full cochain space for all $t > 0$, and the lemma is proved. So we can assume that D_1 and D_2 *are* parallel and compatibly oriented, and therefore that C_1 and C_2 are not parallel. There is then, after reindexing if necessary, a hyperplane H that passes through C_2 but not C_1, and through neither D_1 nor D_2. Choose as a base point for the unitary U_t a q-dimensional face D of C_2 which is parallel to the D_i, compatibly oriented, and on the same side of H as the cube C_1. So $f_{C_2,D_2} = \pm f_{C_2,D}$ and also

$$\begin{aligned} \left\langle t^{-p_1} f_{C_1,D_1}, t^{-p_2} f_{C_2,D} \right\rangle_t &= \left\langle t^{-p_1} U_t f_{C_1,D_1}, t^{-p_2} U_t f_{C_2,D} \right\rangle_\infty \\ &= \left\langle t^{-p_1} U_t f_{C_1,D_1}, (-1)^{p_2} D^{\mathrm{op}} + O(t) \right\rangle_\infty, \end{aligned}$$

where D^{op} is the face of C_2 separated from D by all the complementary hyperplanes of the pair (C_2, D). In particular, D and D^{op} are on opposite sides of H. Now, it follows from the definition of U_t and basic properties of the cocycle W_t that all cubes appearing in the support of $U_t f_{C_1,D_1}$ are on the same side of H as D. Further, arguing as in the proof of the previous lemma, it follows from Lemma 7.8 that $U_t f_{C_1,D_1}$ is $O(t^{p_1})$, so that the inner product above is $O(t)$. □

Proof of Theorem 7.5. The possible values of the inner product in (b) are 0 and ± 1: the positive value occurs when the oriented cube pairs (C_1, D_1) and (C_2, D_2) are parallel and aligned; the negative value occurs when they are parallel and not aligned; and 0 occurs when they are not parallel. The result now follows from Lemmas 7.10 and 7.11. Indeed, the result is clear when the cube pairs are not parallel, and in the case of parallel cube pairs the inner product in (a) is the linear combination

$$\begin{aligned} &\langle t^{-p} f_{C_1,D_1}, t^{-p} f_{C_2,D_2} \rangle_t + \langle t^{-p} f_{C_1,D_1^*}, t^{-p} f_{C_2,D_2^*} \rangle_t \\ &\qquad - \langle t^{-p} f_{C_1,D_1^*}, t^{-p} f_{C_2,D_2} \rangle_t - \langle t^{-p} f_{C_1,D_1}, t^{-p} f_{C_2,D_2^*} \rangle_t. \end{aligned}$$

Assume the cube pairs are aligned. If (C_i, D_i) are compatibly oriented and have the same parity then the same is true of the cube pairs appearing in the first row, whereas the cube pairs appearing in the second row are not compatibly oriented; the inner products in the first row converge to $\frac{1}{2}$ and those in the second row converge to 0. Similarly, if (C_i, D_i) are not compatibly oriented and have the opposite parity then the inner products in the first row converge to 0, whereas those in the second row converge to $-\frac{1}{2}$. The case of non-aligned cube pairs is similar. □

In the sequel it will be convenient to work with the following generating family of continuous bounded sections. All sections we have encountered so far are *geometrically bounded* in the following sense.

7.12 Definition. A (not necessarily continuous) section σ of the continuous field $\{\ell^2_t(X^q)\}_{t\in[0,\infty]}$ is *geometrically bounded* if there is a finite set $A \subseteq X^q$ such that $\sigma(t)$ is supported in A for all $t \in (0, \infty]$.

7.13 Proposition. *The space of geometrically bounded, continuous sections of the continuous field $\{\ell^2_t(X^q)\}_{t\in[0,\infty]}$ is spanned over $C[0,\infty]$ by the extended basic continuous sections.*

Proof. Fix $q \geq 0$ and select one representative cube pair (C, D) from each parallelism class of (p, q)-cubes. The associated basic cochains $\pm f_{\langle C,D\rangle}$ span $\mathbb{C}[X^q]$ (the signs indicate that we need to choose orientations to define $f_{\langle C,D\rangle}$). That is, each q-cube is a complex linear combination of basic cochains of type (p, q) for possibly varying p. It follows that the basic sections $\pm\sigma_{\langle C,D\rangle}$ span the geometrically bounded sections over $(0, \infty]$, as a module over $C(0, \infty]$, since the individual cubes certainly do. Now suppose that σ is a geometrically bounded continuous section over $[0, \infty]$. From the above, for $t > 0$ we can write

$$\sigma(t) = \sum_{j=1}^{N} h_j(t)\sigma_{\langle C_j,D_j\rangle}$$

where $h_1, \ldots, h_N$ are continuous scalar functions on $(0, \infty]$. Using Theorem 7.5 and the boundedness of $\|\sigma(t)\|^2$ as $t \to 0$, we find that the functions h_j are bounded. Taking inner products with each $\sigma_{\langle C_i,D_i\rangle}$ and applying Theorem 7.5 again we find that each h_j extends to a continuous function on $[0, \infty]$. This proves the proposition. □

8. Differentials on the continuous field

The purpose of this section is to construct differentials

$$\ell^2_t(X^0) \xrightarrow{d_t} \ell^2_t(X^1) \xrightarrow{d_t} \cdots \xrightarrow{d_t} \ell^2_t(X^{n-1}) \xrightarrow{d_t} \ell^2_t(X^n)$$

that continuously interpolate between the Julg-Valette differentials at $t = \infty$ and the Pytlik-Szwarc differentials at $t = 0$. For later purposes it will be important to use *weighted* versions of the Julg-Valette differentials, as in Definition 3.23. But first we shall proceed without the weights, and then indicate at the end of this section how the weights are incorporated.

Recall that the operators

$$U_t : \mathbb{C}[X^q] \longrightarrow \mathbb{C}[X^q]$$

from Definition 6.15 were proved to be isomorphisms in Lemma 6.16.

8.1 Definition. For $t \in (0, \infty]$ we define

$$d_t = U_t^{-1} d U_t \colon \mathbb{C}[X^q] \longrightarrow \mathbb{C}[X^{q+1}],$$

where d is the Julg-Valette differential from Definition 3.9. In addition, we define

$$d_0 \colon \mathbb{C}[\mathcal{H}_q] \longrightarrow \mathbb{C}[\mathcal{H}_{q+1}]$$

to be the Pytlik-Szwarc differential from Definition 5.8.

We aim to prove the following continuity statement concerning these operators:

8.2 Theorem. *If $\{\sigma(t)\}$ is any continuous and geometrically bounded section of the continuous field $\{\ell^2_t(X^q)\}_{t\in[0,\infty]}$, then the pointwise differential $\{d_t\sigma(t)\}$ is a continuous and geometrically bounded section of $\{\ell^2_t(X^{q+1})\}_{t\in[0,\infty]}$. In fact if (C, D) is an oriented (p, q)-cube pair, and if $\sigma_{\langle C,D\rangle}$ is the associated extended basic cochain, then*

$$d_t\sigma_{\langle C,D\rangle}(t) = \sum_H \sigma_{\langle C,H\wedge D\rangle}(t) + O(t)$$

for $t > 0$, where the sum is over the hyperplanes that cut C but not D.

It follows formula above and the definition of d_0 that $d\sigma_{\langle C,D\rangle}$ is a continuous section, and according to Proposition 7.13 the space of continuous and geometrically bounded sections is generated as a module over $C[0, \infty]$ by the extended basic sections. So the formula implies the first statement in the theorem.

8.3 Lemma. *Let (C, D) be an oriented (p, q)-cube pair and assume that all the complementary hyperplanes of the pair (C, D) separate D from the base point P_0. The associated basic q-cochain of type p satisfies*

$$t^{-p} U_t f_{C,D} = W_t(D_0, D_1)D + O(t),$$

where D_0 is the q-cube in X that is closest to the base point P_0 among cubes parallel to D, and D_1 is the face of C that is parallel to D and separated from D by all the complementary hyperplanes.

Proof. According to our definitions, $f_{C,D} = (-1)^p f_{C,D_1}$ and

$$\begin{aligned} U_t f_{C,D_1} &= \sum_{E\|_C D} (-1)^{d(D_1,E)} W_t(D_0, E)E \\ &= W_t(D_0, D_1) \sum_{E\|_C D} (-1)^{d(D_1,E)} W_t(D_1, E)E \\ &= (-t)^p W_t(D_0, D_1)D + O(t^{p+1}), \end{aligned}$$

where we have applied Lemma 7.8. The result follows. □

8.4 Lemma. *Let (C, D) and D_1 be as in the previous lemma. Let C_0 be the nearest cube to P_0 in the parallelism class of C, let F be the face of C_0 which is parallel to D and separated from the base point P_0 by the complementary hyperplanes, and let F_1 be the face of C_0 that is parallel to D and separated from F by the complementary hyperplanes. Then*

(a) *$H \wedge F$ is nonzero if and only if H is a complementary hyperplane, in which case $H \wedge F \subseteq C_0$;*
(b) *$d(C_0, C) = d(F, D) = d(F_1, D_1)$;*
(c) *$W_t(D_0, D_1)D = F + O(t)$.*

Proof. Consider first the case $q = 0$. In this case, $D_0 = P_0$ and the vertex F_1 is characterized by the following *hyperplane property* from the proof of Proposition 4.3: every hyperplane separating P_0 and F_1 is parallel to at least one determining hyperplane of parallelism class of C (and C_0).

For (a), $H \wedge F$ is nonzero exactly when H is adjacent to F and separates it from P_0. The hyperplanes cutting C_0 certainly satisfy this condition. Conversely, a hyperplane satisfying this condition must intersect all determining hyperplanes by Lemma 2.4, so cannot separate F_1 from P_0 and so must cut C_0.

For (b), no determining hyperplane (of the parallelism class of C) separates F and D. It follows easily that a hyperplane separates C and C_0 if and only if it separates F and D. The same argument applies to F_1 and D_1.

For (c), from the cocycle property we have

$$W_t(D_0, D_1)D = W_t(D_0, F_1)W_t(F_1, D_1)D.$$

To evaluate this, observe that a hyperplane appearing along (a geodesic) path from F_1 to D_1 must cross every determining hyperplane. It follows that $W_t(F, F_1) = W(D, D_1)$ commutes with $W_t(F_1, D_1)$ and we have

$$\begin{aligned} W_t(F_1, D_1)D &= W_t(F, F_1)W(D_1, D)W_t(F_1, D_1)D \\ &= W_t(F, D)D \\ &= e^{\frac{1}{2}dt^2}F + O(t), \end{aligned}$$

where $d = d(F, D)$, and the last equality follows from an elaboration of [8, Proposition 3.6]. Finally, no hyperplane separating D_0 and F_1 is adjacent to F so that $W_t(D_0, F_1)F = F$. Putting things together, the result follows.

We reduce the general case to the case $q = 0$ using Proposition 4.2, according to which the set of q-cubes parallel to D is the vertex set of a CAT(0) cube complex in such a way that the $(p + q)$-cubes in X correspond to the p-cubes in this complex. The key observation is that the p-cube in this complex corresponding to the $(p + q)$ cube C_0 in the statement of the lemma is the p-cube closest to the vertex corresponding to D_0. $\square$

Proof of Theorem 8.2. We need to show that

$$d_t\sigma_{\langle C,D\rangle}(t) = \sum_H \sigma_{\langle C,H\wedge D\rangle}(t) + O(t),$$

where the sum is over the hyperplanes that cut C but not D. After possibly changing a sign, we can assume that D is the furthest from the base point among the q-dimensional faces of C parallel to D. In other words, we can assume that the complementary hyperplanes $H_1,\dots,H_p$ of the pair (C,D) separate D from the base point. Each $H_i\wedge D$ is therefore a $(q+1)$-dimensional face of C, and we shall show that

$$d_t(\sigma_{\langle C,D\rangle}(t)) = \sum_{i=1}^p \sigma_{\langle C,H_i\wedge D\rangle}(t) + O(t).$$

We have equality when $t=0$, so it suffices to treat the case for $t>0$ where we need to show that

$$d_t(t^{-p}f_{C,D}) = t^{-(p-1)}\sum_{i=1}^p f_{C,H_i\wedge D} + O(t).$$

This is equivalent to

$$d\,U_t(t^{-p}f_{C,D}) = \sum_{i=1}^p U_t(t^{-(p-1)}f_{C,H_i\wedge D}) + O(t). \tag{8.1}$$

As for the left hand side of (8.1), applying Lemmas 8.3 and 8.4 we have

$$d\,U_t(t^{-p}f_{C,D}) = d\,F + O(t) = \sum_{i=1}^p H_i\wedge F + O(t),$$

where F is as in the statement of Lemma 8.4. So, to complete the verification of (8.1) it suffices to check that

$$U_t(t^{-(p-1)}f_{C,H_i\wedge D}) = H_i\wedge F + O(t).$$

But this follows from Lemmas 8.3 and 8.4, applied to the $(p-1,q+1)$-cube pair $(C,H_i\wedge D)$ (although a little care must be taken here since the base cube D_0 that is nearest to P_0 within the parallelism class of D should be replaced by an analogous base cube for the parallelism class of $H_i\wedge D$). $\square$

Consider now the adjoint operators

$$\delta_t = U_t^{-1}\delta U_t\colon \mathbb{C}[X^q]\longrightarrow \mathbb{C}[X^{q+1}] \tag{8.2}$$

for $t > 0$, together with the adjoint Pytlik-Szwarc differential

$$\delta_0 \colon \mathbb{C}[\mathcal{H}^q] \longrightarrow \mathbb{C}[\mathcal{H}^{q+1}]$$

8.5 Theorem. *If $\{\sigma(t)\}$ is any continuous and geometrically bounded section of the continuous field $\{\ell^2_t(X^{q+1})\}_{t\in[0,\infty]}$, then $\{\delta_t\sigma(t)\}$ is a continuous and geometrically bounded section of the continuous field $\{\ell^2_t(X^q)\}_{t\in[0,\infty]}$. In fact if $\Sigma_{\langle C,D\rangle}$ is any basic extended section, then*

$$\delta_t\sigma_{\langle C,D\rangle}(t) = \sum \sigma_{\langle C,H_i \lrcorner D\rangle}(t) + O(t),$$

where the sum is over the hyperplanes that cut D.

Proof. While this could be approached through computations similar to those above, there is a shortcut. We can write X as an increasing union of finite, combinatorially convex subcomplexes K, each of which contains the base point; see [16, Theorem B4]. Each $\mathbb{C}[K^q]$ is invariant under d and δ, and under all the operators U_t. The latter restrict to isomorphisms by the argument of Lemma 6.16. It follows that for each (p, q)-cube pair (C, D) there is some finite subcomplex K within which $\delta_t\sigma_{\langle C,D\rangle}$ is supported for all $t > 0$. So for $t > 0$ there is a *finite* expansion

$$\delta_t\sigma_{\langle C,D\rangle} = \sum_{j=1}^{N} h_j(t)\sigma_{\langle C_j,D_j\rangle},$$

as in the proof of Proposition 7.13, where $h_1, \ldots, h_N$ are continuous on $(0, \infty]$. Using the adjoint relation

$$\langle\sigma_{\langle C_i,D_i\rangle}, \delta_t\sigma_{\langle C,D\rangle}\rangle = \langle d_t\sigma_{\langle C_i,D_i\rangle}, \sigma_{\langle C,D\rangle}\rangle$$

and Theorem 8.2 we see that the inner products on the left-hand side are continuous on $[0, \infty]$, and indeed

$$\langle\sigma_{\langle C_i,D_i\rangle}, \delta_t\sigma_{\langle C,D\rangle}\rangle = \begin{cases} 1 + O(t) & \text{if } (C_i, D_i) \text{ is some } (C, H \lrcorner D) \\ O(t) & \text{otherwise.} \end{cases}$$

The theorem follows from this. □

Finally, we return to the issue of weights, which will be important in the next section when we work in the context of Kasparov theory. Let w_t the function on hyperplanes defined by the formula

$$w_t(H) = \begin{cases} 1 + t\,\mathrm{dist}(H, P_0), & 0 < t \le 1 \\ 1 + \mathrm{dist}(H, P_0) & 1 \le t \le \infty. \end{cases} \tag{8.3}$$

In the next section we shall work with the weighted operators

$$d_t = U_t^{-1} d_{w_t} U_t \colon \mathbb{C}[X^q] \longrightarrow \mathbb{C}[X^{q+1}], \tag{8.4}$$

for $t > 0$, where as before U_t is the isomorphism from Definition 6.15, and where d_{w_t} is the weighted Julg-Valette differential described in Definition 3.23.

If $t > 0$, then operator in (8.4) does *not* extend from $\mathbb{C}[X^q]$ to a bounded operator between ℓ^2_t-spaces. But since the pointwise values of a geometrically bounded section lie in $\mathbb{C}[X^q]$, Theorem 8.2 makes sense in the weighted case without extending the domains of the operators d_t in (8.4) beyond $\mathbb{C}[X^q]$. Moreover the theorem remains true for the weighted family of operators. The proof reduces immediately to the unweighted case because the weighted and unweighted differentials, applied to a continuous and geometrically bounded section, differ by an $O(t)$ term. The same applies to Theorem 8.5.

9. Equivariant Fredholm complexes

We shall assume from now on that a second countable, locally compact Hausdorff topological group[4] G acts on a locally finite and finite-dimensional CAT(0) cubical space X (preserving the cubical structure). We shall *not* assume that G fixes any base point in X.

Our goal in this section to place the Julg-Valette and Pytlik-Szwarc complexes within the context of equivariant Fredholm complexes, and we need to begin with some definitions.

9.1 Definition. A *Fredholm complex* of Hilbert spaces is a bounded complex of Hilbert spaces and bounded operators for which the identity morphism on the complex is chain homotopic, through a chain homotopy consisting of bounded operators, to a morphism consisting of compact Hilbert space operators.

In other words, a Fredholm complex of Hilbert spaces is a complex of the form

$$\mathfrak{H}^0 \xrightarrow{d} \mathfrak{H}^1 \xrightarrow{d} \cdots \xrightarrow{d} \mathfrak{H}^n,$$

with each $\mathfrak{H}^p$ a Hilbert space and each differential a bounded operator. Moreover there exist bounded operators

$$h : \mathfrak{H}^p \longrightarrow \mathfrak{H}^{p-1} \qquad (p = 1, \ldots, n)$$

such that each operator

[4] The topological restrictions on the group G are not really necessary, but they will allow us to easily fit the concept of equivariant Fredholm complex into the context of Kasparov's KK-theory in the next section.

$$dh + hd : \mathfrak{H}^p \longrightarrow \mathfrak{H}^p \qquad (p = 0, \dots, n)$$

is a compact perturbation of the identity operator.

The Fredholm condition implies that the cohomology groups of a Fredholm complex are all finite-dimensional, which is the main reason for the definition. But we are interested in the following concept of *equivariant* Fredholm complex, for which the cohomology groups are not so relevant.

9.2 Definition. Let G be a second countable Hausdorff locally compact topological group. A *G-equivariant Fredholm complex* of Hilbert spaces is a bounded complex of separable Hilbert spaces and bounded operators satisfying the following conditions:

(a) Each Hilbert space carries a continuous unitary representation of G.
(b) The differentials d are not necessarily equivariant, but the differences $d - gdg^{-1}$ are compact operator-valued and norm-continuous functions of $g \in G$.
(c) The identity morphism on the complex is chain homotopic, through a chain homotopy consisting of bounded operators, to a morphism consisting of compact Hilbert space operators.
(d) The operators h in the chain homotopy are not necessarily equivariant, but the differences $h - ghg^{-1}$ are compact operator-valued and norm-continuous functions of $g \in G$.

9.3 Remark. Because the differentials are not necessarily equivariant, the cohomology groups of an equivariant Fredholm complex of Hilbert spaces do not necessarily carry actions of G, and so are not of direct interest themselves as far as G is concerned. Nevertheless the above definition, which is due to Kasparov (in a minor variant form), has played an important role in a number of mathematical areas, most notably the study of the Novikov conjecture in manifold topology [14] (see [2] for a survey of other topics).

We are going to manufacture equivariant Fredholm complexes from the Julg-Valette and Pytlik-Szwarc complexes. The Julg-Valette complex is the more difficult of the two to understand. Disregarding the group action, the Julg-Valette differentials from Definition 3.9 extend to bounded operators on the Hilbert space completions of the cochain spaces associated to the inner products in (3.18), and the resulting complex of Hilbert spaces and bounded operators is Fredholm, as in Definition 9.1. Moreover the group G certainly acts unitarily. But the Julg-Valette differentials typically fail to be G-equivariant, since they are defined using a choice of base point in the complex X which need not be fixed by G. This means that the technical items (b) and (d) in Definition 9.2 need to be considered carefully.

In fact to handle these technical items it will be necessary to finally make use of the weight functions $w(H)$ that we introduced in Definition 3.23. The following computation will be our starting point. Assemble together all the Julg-Valette cochain spaces to form the single space

$$\mathbb{C}[X^\bullet] = \bigoplus_{q=0}^{\dim(X)} \mathbb{C}[X^q],$$

and then form the Hilbert space completion

$$\ell^2(X^\bullet) = \bigoplus_{q=0}^{\dim(X)} \ell^2(X^q).$$

9.4 Lemma. *For any weight function $w(H)$ the Julg-Valette operator*

$$D = d + \delta \,:\, \ell^2(X^\bullet) \longrightarrow \ell^2(X^\bullet),$$

viewed as a densely-defined operator with domain $\mathbb{C}[X^\bullet]$, is essentially self-adjoint.

Proof. The operator D is formally self-adjoint in the sense that

$$\langle Df_1, f_2\rangle = \langle f_1, Df_2\rangle$$

for all $f_1, f_2 \in \mathbb{C}[X^\bullet]$. The essential self-adjointness of D is a consequence of the fact that the range of the operator

$$I + \Delta = I + D^2$$

is dense in $\ell^2(X^\bullet)$, and this in turn is a consequence of the fact that the Julg-Valette Laplacian is a diagonal operator, as indicated in Proposition 3.25. □

Since D is an essentially self-adjoint operator, we can study the resolvent operators $(D \pm iI)^{-1}$, which extend from their initial domains of definition (namely the ranges of $(D \pm iI)$ on $\mathbb{C}[X^\bullet]$) to bounded operators on $\ell^2(X^\bullet)$.

9.5 Lemma. *If w is a weight function that is* proper *in the sense that for every $d > 0$ the set $\{ H : w(H) < d \}$ is finite, then the resolvent operators*

$$(D \pm iI)^{-1} :\, \ell^2(X^\bullet) \longrightarrow \ell^2(X^\bullet)$$

are compact Hilbert space operators.

Proof. The two resolvent operators are adjoint to one another, and so it suffices to show that the product

$$(I+\Delta)^{-1} = (D+iI)^{-1}(D-iI)^{-1}$$

is compact. But the compactness of $(I+\Delta)^{-1}$ is clear from Proposition 3.25. □

Let us now examine the dependence of the Julg-Valette operator D on the initial choice of base point in X.

9.6 Lemma. *If w is a weight function that is G-*bounded *in the sense that*

$$\sup_H \left| w(H) - w(gH) \right| < \infty$$

for every $g \in G$, then

$$\left\| D - g(D) \right\| < \infty.$$

That is, the difference $D - g(D)$, which is a linear operator on $\mathbb{C}[X^\bullet]$, extends to a bounded linear operator on $\ell^2(X^\bullet)$.

Proof. It suffices to prove the estimate for d in place of $D = d + \delta$, since d and δ are adjoint to one another. Now

$$dC - g(d)C = \sum_H w(H) H \wedge_{P_0} C - \sum_H w(g(H)) H \wedge_{g(P_0)} C,$$

where $\wedge_{P_0}$ and $\wedge_{g(P_0)}$ denote the operators of Definition 3.7 associated to the two indicated choices of base points. Since $w(H) - w(gH)$ is uniformly bounded we can replace $w(g(H))$ by $w(H)$ in the second sum, and change the overall expression only by a term that defines a bounded operator. So it suffices to show that for any pair of base points P_0 and P_1 the expression

$$\sum_H w(H)\big(H \wedge_{P_0} C - H \wedge_{P_1} C\big)$$

defines a bounded operator. But the expression in parentheses is only non-zero when H separates P_0 from P_1, and there are only finitely many such hyperplanes. So the lemma follows from the fact that for any hyperplane H the formula

$$H \wedge_{P_0} C - H \wedge_{P_1} C$$

defines a bounded operator. □

From now on we shall assume that the Julg-Valette complex is weighted using a proper and G-bounded weight function. Since the weighted Julg-Valette differential is not bounded, we shall need to make an adjustment to fit the weighted complex into the framework of Fredholm complexes of Hilbert spaces and bounded operators. We do this by forming the *normalized* differentials

$$d' = d(I+\Delta)^{-\frac{1}{2}} \colon \ell^2(X^q) \longrightarrow \ell^2(X^{q+1})$$

(where, strictly speaking, by d in the above formula we mean the closure of d in the sense of unbounded operator theory). The *normalized Julg-Valette complex* is the complex

$$\ell^2(X^0) \xrightarrow{d'} \ell^2(X^1) \xrightarrow{d'} \cdots \xrightarrow{d'} \ell^2(X^n). \tag{9.1}$$

It is indeed a complex because d and $(I+\Delta)^{-\frac{1}{2}}$ commute with one another, and it is a Fredholm complex because the adjoints d'^* constitute a chain homotopy between the identity and a compact operator-valued cochain map. In fact

$$d'd'^* + d'^*d' = D^2(I+D^2)^{-1} = I - (I+D^2)^{-1},$$

and $(I+D^2)^{-1}$ is compact by Lemma 9.5.

We shall use the following computation from the functional calculus to show that the normalized complex is an equivariant Fredholm complex of Hilbert spaces.

9.7 Lemma *(Compare [4]). If T is a positive, self-adjoint Hilbert space operator that is bounded below by some positive constant, then*

$$T^{-\frac{1}{2}} = \frac{2}{\pi}\int_0^\infty (\lambda^2 + T)^{-1}\, d\lambda.$$

(*The integral is defined as a limit of Riemann sums and is convergent in the norm topology.*) □

9.8 Theorem. *The normalized Julg-Valette complex*

$$\ell^2(X^0) \xrightarrow{d'} \ell^2(X^1) \xrightarrow{d'} \cdots \xrightarrow{d'} \ell^2(X^n)$$

defined using a proper and G-bounded weight function is an equivariant Fredholm complex.

Proof. In light of the discussion above it suffices to show that the normalized operator

$$D' = D(I+D^2)^{-1/2} = d' + d'^*$$

has the property that $g(D') - D'$ is a compact operator-valued and norm-continuous function of $g \in G$. For this we use Lemma 9.7 and the formula

$$D(\lambda^2+1+D^2)^{-1} = \frac{1}{2}\big((D+i\mu)^{-1} + (D-i\mu)^{-1}\big),$$

where $\mu = (\lambda^2+1)^{1/2}$, to write the difference $g(D') - D'$ as a linear combination of two integrals

$$\int_0^\infty \Big((g(D)\pm i\mu)^{-1} - (D\pm i\mu)^{-1}\Big)\, d\lambda.$$

The integrand is

$$(g(D)\pm i\mu)^{-1}\big(D - g(D)\big)(D\pm i\mu)^{-1}, \tag{9.2}$$

which, by Lemmas 9.5 and 9.6, is a compact operator-valued and norm-continuous function of $\lambda\in[0,\infty)$ whose norm is $O(\lambda^{-2})$ as $\lambda \nearrow \infty$. So the integrals converge to compact operators, as required. □

Having dealt with the Julg-Valette complex, let us now examine the Pytlik-Szwarc complex. The inner products on the Pytlik-Szwarc cochain spaces given in Definition 5.12 are G-invariant, and the Pytlik-Szwarc differentials given in Definition 5.8 are bounded and G-equivariant, so the story here is much simpler.

9.9 Theorem. *The Pytlik-Szwarc complex*

$$\ell^2_0(X^0) \longrightarrow \ell^2_0(X^1) \longrightarrow \cdots \longrightarrow \ell^2_0(X^n)$$

is an equivariant Fredholm complex.

Proof. It follows from Proposition 5.14 that the formula

$$h = \frac{1}{p+q}\,\delta : \mathbb{C}[\mathcal{H}^p_q] \longrightarrow \mathbb{C}[\mathcal{H}^{p+1}_{q-1}]$$

(we set $h=0$ when $p=q=0$) defines an exactly G-equivariant and bounded chain homotopy between the identity and a compact operator-valued cochain map, namely the orthogonal projection onto $\mathbb{C}[\mathcal{H}^0_0]\cong\mathbb{C}$ in degree zero, and the zero operator in higher degrees. □

To conclude this section we introduce the following notion of (topological, as opposed to chain) homotopy between two equivariant Fredholm complexes. In the next section we shall construct a homotopy between the Julg-Valette and Pytlik-Szwarc equivariant Fredholm complexes we constructed above using the continuous field of complexes constructed in Section 8.

9.10 Definition. Two equivariant complexes of Hilbert spaces $(\mathfrak{H}_0^\bullet, d_0)$ and $(\mathfrak{H}_1^\bullet, d_1)$ are *homotopic* if there is a bounded complex of continuous fields of Hilbert spaces over $[0,1]$ and adjointable families of bounded differentials satisfying the following conditions:

(a) Each continuous field carries a continuous unitary representation of G.
(b) The differentials $d = \{d_t\}$ are not necessarily equivariant, but the differences $d - gdg^{-1}$ are compact families and norm-continuous functions of $g \in G$.
(c) The identity morphism on the complex is chain homotopic, through a chain homotopy consisting of adjointable families of bounded operators, to a morphism consisting of compact operators between continuous fields.
(d) The operators $h = \{h_t\}$ in the homotopy above are again not necessarily equivariant, but the differences $h - ghg^{-1}$ are compact families and norm-continuous functions of $g \in G$.
(e) The restrictions of the complex to the points $0, 1 \in [0,1]$ are the complexes $(\mathfrak{H}_0^\bullet, d_0)$ and $(\mathfrak{H}_1^\bullet, d_1)$.

We need to supply definitions for the operator-theoretic concepts mentioned above. These are usually formulated in the language of Hilbert modules, as for example in [15], but for consistency with the rest of this paper we shall continue to use the language of continuous fields of Hilbert spaces.

9.11 Definition. An *adjointable family of operators* (soon we shall contract this to *adjointable operator*) between continuous fields $\{\mathfrak{H}_t\}$ and $\{\mathfrak{H}'_t\}$ over the same compact space T is a family of bounded operators

$$A_t \colon \mathfrak{H}_t \longrightarrow \mathfrak{H}'_t$$

that carries continuous sections to continuous sections, whose adjoint family

$$A_t^* \colon \mathfrak{H}'_t \longrightarrow \mathfrak{H}_t$$

also carries continuous sections to continuous sections. An adjointable operator is *unitary* if each A_t is unitary.

9.12 Definition. A representation of G as unitary adjointable operators on a continuous field $\{\mathfrak{H}_t\}$ is *continuous* if the action map

$$G \times \{\,\text{continuous sections}\,\} \longrightarrow \{\,\text{continuous sections}\,\}$$

is continuous. We place on the space of continuous sections the topology associated to the norm $\|\sigma\| = \max \|\sigma(t)\|$.

9.13 Definition. An adjointable operator $A = \{A_t\}$ between continuous fields of Hilbert spaces over the same compact base space T is *compact* if it is the norm limit, as a Banach space operator

$$A : \{\,\text{continuous sections}\,\} \longrightarrow \{\,\text{continuous sections}\,\},$$

of a sequence of linear combinations of operators of the form

$$\sigma \longmapsto \langle \sigma_1, \sigma \rangle \sigma_2,$$

where σ_1 and σ_2 are continuous sections (of the domain and range continuous fields, respectively). The compact operators form a closed, two-sided ideal in the C^*-algebra of all adjointable operators.

Here, then, is the theorem that we shall prove in the next section:

9.14 Theorem. *The equivariant Fredholm complexes obtained from the Julg-Valette and Pytlik-Szwarc complexes in Theorems 9.8 and 9.9 are homotopic (in the sense of Definition 9.10).*

10. K-amenability

The purpose of this section is to prove Theorem 9.14. Before giving the proof, we shall explain the K-theoretic relevance of the theorem. To do so we shall need to use the language of Kasparov's equivariant KK-theory [14] and so shall assume familiarity with this theory. We emphasize, however, that the proof of Theorem 9.14 will involve only the definitions from the last section and our work earlier in the paper.

A G-equivariant complex of Hilbert spaces, as in Definition 9.2, determines a class in Kasparov's equivariant representation ring

$$R(G) = KK_G(\mathbb{C}, \mathbb{C}),$$

in such a way that the following conditions are satisfied:

(a) Homotopic complexes, as in Definition 9.10, determine the same element.
(b) A complex whose differentials are exactly G-equivariant determines the same class as the complex of cohomology groups (these are finite-dimensional unitary representations of G) with zero differentials.
(c) A complex with the one-dimensional trivial representation in degree zero, and no higher-dimensional cochain spaces, determines the multiplicative identity element $1 \in R(G)$.

10.1 Definition. See [11, Definition 1.2]. A second countable and locally compact Hausdorff topological group G is *K-amenable* if the multiplicative identity element $1 \in R(G)$ is representable by an equivariant Fredholm complex of Hilbert spaces

$$\mathfrak{H}^0 \longrightarrow \mathfrak{H}^1 \longrightarrow \cdots \longrightarrow \mathfrak{H}^n$$

in which each cochain space $\mathfrak{H}^p$, viewed as a unitary representation of G, is weakly contained in the regular representation of G.

10.2 Theorem *(See [11, Corollary 3.6].). If G is K-amenable, then the natural homomorphism of C^*-algebras*

$$C^*_{\max}(G) \longrightarrow C^*_{\mathrm{red}}(G)$$

induces an isomorphism of K-theory groups

$$K_*(C^*_{\max}(G)) \longrightarrow K_*(C^*_{\mathrm{red}}(G)). \quad \square$$

10.3 Remarks. The C^*-algebra homomorphism in the theorem is itself an isomorphism if and only if the group G is amenable; this explains the term *K-amenable.* Not every group is K-amenable; for example an infinite group with Kazhdan's property T is certainly not K-amenable, because the K-theory homomorphism is certainly not an isomorphism.

After having quickly surveyed this background information, we can state the main result of this section. The following theorem is not new; it was proved by Higson and Kasparov in [9, Theorem 9.4] using a different argument that is more general (it applies to a broader class of groups) but less geometric than the argument we shall present here.

10.4 Theorem. *If a second countable and locally compact group G admits a proper action on a locally finite and finite dimensional* $\mathrm{CAT}(0)$ *cubical space, then G is K-amenable.*

Julg and Valette proved this theorem in [11] in the case where the cubical space is a tree. They used the Julg-Valette complex, as we have called it, for a tree, and showed that the continuous field of complexes that we have constructed in this paper is a homotopy connecting the Julg-Valette and Pytlik-Szwarc complexes. We shall do the same in the general case. The construction of this homotopy proves the theorem in view of the following simple result, whose proof we shall omit.

10.5 Lemma. *Assume that a second countable and locally compact group G acts properly on a* $\mathrm{CAT}(0)$ *cubical space. The Hilbert spaces in the Julg-Valette complex are weakly contained in the regular representation of G.* $\square$

In light of these remarks, Theorem 10.4 follows from Theorem 9.14, and we now turn to the proof of that theorem. We shall construct the required homotopy by modifying the constructions in Section 8 in more or less the same way that we modified the Julg-Valette complex to construct the complex (9.1). We shall therefore be applying the functional calculus to the family of operators

$$D_t = U_t^*(d_{w_t} + \delta_{w_t})U_t \colon \ell_t^2(X^\bullet) \longrightarrow \ell_t^2(X^\bullet), \tag{10.1}$$

where d_{w_t} is a weighted Julg-Valette differential, and of course δ_{w_t} is the adjoint differential. Henceforth, we shall work exclusively with the weight functions w_t defined in (8.3), which are G-bounded and proper when the underlying CAT(0) cubical space is locally finite. To apply the functional calculus we shall need to know that the family of resolvent operators

$$(D_t + i\lambda)^{-1} \colon \ell_t^2(X^\bullet) \longrightarrow \ell_t^2(X^\bullet)$$

carries continuous sections to continuous sections. Actually we shall need a small variation on this, involving the following operators.

10.6 Definition. Denote by $P = \{P_t\}$ the operator that is in each fiber the orthogonal projection onto the span of the single basic 0-cochain $f_{\langle P_0,P_0\rangle}$ of type $p = 0$ for $t > 0$ (of course this basic cochain is just $\langle P_0\rangle$).

It follows from the formula for the Julg-Valette Laplacian in Proposition 3.25 that the operators $P_t + \Delta_t$ are essentially self-adoint and bounded below by 1. So we can form the resolvent operators $(D_t + P_t + i\lambda)^{-1}$ for any $\lambda \in \mathbb{R}$, including $\lambda = 0$.

10.7 Proposition. *Let λ be a nonzero real number. The families of operators*

$$\big\{\, (D_t + i\lambda)^{-1} \colon \ell_t^2(X^\bullet) \to \ell_t^2(X^\bullet) \,\big\}_{t\in[0,\infty]} \quad \textit{and}$$
$$\big\{\, (D_t + P_t + i\lambda)^{-1} \colon \ell_t^2(X^\bullet) \to \ell_t^2(X^\bullet) \,\big\}_{t\in[0,\infty]}$$

map the space of continuous sections to itself.

We shall prove this by examining action of the Laplacians

$$\Delta_t = D_t^2 = U_t^*(d_{w_t} + \delta_{w_t})^2 U_t \tag{10.2}$$

on continuous and geometrically bounded sections of the field $\{\ell_t^2(X^\bullet)\}_{t\in[0,\infty]}$. The essential point is that these Laplacians are diagonalized (up to geometrically bounded $O(t)$ terms) by the extended basic sections.

Proof of Proposition 10.7. Since the continuous sections are generated by the extended basic sections, it suffices to prove that each such section is mapped to a continuous section. Since it is only continuity at $t = 0$ that is in question, it suffices to show that a basic section is mapped to a continuous section, plus a section that vanishes at $t = 0$. For this, it suffices to show that the ranges of the families $\{D_t + i\lambda\}$ and $\{D_t + P_t + i\lambda\}$ as they act on all continuous and geometrically bounded sections include all basic sections, modulo sections that vanish at $t = 0$. Finally, using

$$\Delta_t + \lambda^2 I = (D_t + i\lambda)(D_t - i\lambda)$$

and

$$\Delta_t + P_t + \lambda^2 I = (D_t + P_t + i\lambda)(D_t + P_t - i\lambda),$$

it suffices to show that the ranges of the families $\{\Delta_t + \lambda^2 I\}$ and $\{\Delta_t + P_t + \lambda^2 I\}$ have this property.

Let $\sigma_{\langle C,D\rangle}$ be an extended basic cochain of type (p, q). Lemmas 8.3 and 8.4 tell us that

$$\sigma_{\langle C,D\rangle}(t) = t^{-p} f_{C,D},$$

so according to our formula for the Julg-Valette Laplacian in Proposition 3.25,

$$(d_{w_t} + \delta_{w_t})^2 U_t : t^{-p}\sigma_{\langle C,D\rangle} \longmapsto (p_t + q_t) \cdot F + O(t),$$

where p_t is the sum of the squares of the weights of the hyperplanes adjacent to F that separate it from P_0, and q_t is the sum of the squares of the weights of the hyperplanes cutting F. Applying U_t^* to both sides we get

$$\big(\Delta_t^2 + \lambda^2 I\big) : t^{-p}\sigma_{\langle C,D\rangle} \longmapsto \big((p_t + q_t) + \lambda^2\big) \cdot t^{-p}\sigma_{\langle C,D\rangle} + O(t).$$

Similarly

$$\big(\Delta_t + P_t + \lambda^2 I\big) : t^{-p}\sigma_{\langle C,D\rangle} \longmapsto \big(\max\{1, (p_t + q_t)\} + \lambda^2\big) \cdot t^{-p}\sigma_{\langle C,D\rangle} + O(t).$$

So the ranges of the families $\{\Delta_t + \lambda^2 I\}$ and $\{\Delta_t + P_t + \lambda^2 I\}$ contain $O(t)$ perturbations of every basic section, as required. $\square$

Now form the bounded self-adjoint operators

$$F_t = D_t(P_t + D_t^2)^{-\frac{1}{2}}.$$

By the above and Lemma 9.7 the family $\{F_t\}_{t\in[0,\infty]}$ maps continuous sections to continuous sections. So we can consider the bounded complex of continuous fields of Hilbert spaces over $[0, 1]$ and bounded adjointable operators

$$\{\ell^2_t(X^0)\}_{t\in[0,\infty]} \xrightarrow{\{d'_t\}_{t\in[0,\infty]}} \{\ell^2_t(X^1)\}_{t\in[0,\infty]} \xrightarrow{\{d'_t\}_{t\in[0,\infty]}} \cdots \xrightarrow{\{d'_t\}_{t\in[0,\infty]}} \{\ell^2_t(X^n)\}_{t\in[0,\infty]} \tag{10.3}$$

in which each differential $\{d'_t\}$ is the component of $\{F_t\}$ mapping between the indicated continuous fields.

10.8 Proposition. *Disregarding the G-action, the complex (10.3) is a homotopy of Fredholm complexes.*

Proof. If we set $h_t = d'^{*}_t$, then

$$h_t d'_t + d'_t h_t = \Delta_t(P_t + \Delta_t)^{-1} = I - P_t(P_t + \Delta_t)^{-1},$$

and $\{P_t(P_t+\Delta_t)^{-1}\}$ is a compact operator on the continuous field $\{\ell^2_t(X^\bullet)\}_{t\in[0,\infty]}$. □

It remains show that (10.3) is an *equivariant* homotopy. If the resolvent families $\{(D_t + P_t + i\lambda)^{-1}\}$ were compact, then we would be able to follow the route taken in the previous section to prove equivariance of the Fredholm complex associated to the Julg-Valette complex. But compactness fails at $t = 0$, and so we need to be a bit more careful. The following two propositions will substitute for the Lemmas 9.5 and 9.6 that were used to handle the Julg-Valette complex in the previous section.

10.9 Proposition. *For every $\varepsilon > 0$ and for every $\lambda \in \mathbb{R}$ the restricted family of operators*

$$\{(D_t + P_t \pm i\lambda)^{-1}\}_{t\in[\varepsilon,\infty]}$$

is compact.

10.10 Proposition. *For every $g \in G$ the operators $D_t - g(D_t)$ are uniformly bounded in t:*

$$\sup_{t\in[0,\infty]} \|D_t - g(D_t)\| < \infty.$$

Moreover

$$\|D_t - g(D_t)\| = O(t),$$

as $t \searrow 0$.

Taking these for granted for a moment, here is the result of the calculation:

10.11 Theorem. *The complex (10.3) is a homotopy of equivariant Fredholm complexes in the sense of Definition 9.10.*

Proof. We need to check that the families of differentials $\{d'_t\}$ in the complex (10.3) are G-equivariant modulo compact operators, and also that $\{d'_t - g(d'_t)\}$ varies norm-continuously with $g \in G$.

Let us discuss norm-continuity first. If g is sufficiently close to the identity in G, then g fixes the base point P_0, and for such g we have $g(d'_t) = d'_t$ for all t. So $\{g(d'_t)\}$ is actually locally constant as a function of g.

The proof of equivariance modulo compact operators is a small variation of the proof of Theorem 9.8. We need to show that the family of operators $\{g(F_t) - F_t\}$ is compact. Since

$$\begin{aligned} F_t &= D_t(P_t + \Delta_t)^{-\frac{1}{2}} \\ &= (P_t + D_t)(P_t + \Delta_t)^{-\frac{1}{2}} + \text{compact operator}, \end{aligned}$$

it suffices to prove that the operator

$$E_t = (P_t + D_t)(P_t + \Delta_t)^{-\frac{1}{2}}$$

is equivariant modulo compact operators. Applying Lemma 9.7 we find that

$$\begin{aligned} E_t &= \frac{2}{\pi} \int_0^\infty (P_t + D_t)(\lambda^2 I + P_t + \Delta_t)^{-1}\, d\lambda \\ &= \frac{1}{\pi} \int_0^\infty \big((D_t + P_t - i\lambda)^{-1} + (D_t + P_t + i\lambda)^{-1} \big)\, d\lambda \end{aligned}$$

(the integral converges in the strong topology). Therefore the difference $g(E_t) - E_t$ is the sum of the two integrals

$$\frac{1}{\pi} \int_0^\infty \Big((g(D_t) + g(P_t) \pm i\lambda)^{-1} - (D_t + P_t \pm i\lambda)^{-1} \Big)\, d\lambda. \tag{10.4}$$

(we shall see that these integrals are norm-convergent). The integrands in (10.4) can be written as

$$\begin{aligned} &(g(P_t) + g(D_t) \pm i\lambda)^{-1} \big(D_t - g(D_t) \big) (P_t + D_t \pm i\lambda)^{-1} \\ &\quad + (g(P_t) + g(D_t) \pm i\lambda)^{-1} \big(P_t - g(P_t) \big) (P_t + D_t \pm i\lambda)^{-1}. \end{aligned} \tag{10.5}$$

Both terms in (10.5) are norm-continuous, compact operator-valued functions of $\lambda \in [0, \infty)$, the first by virtue of Proposition 10.10 and the second because P_t is compact. Moreover the norms of both are $O(\lambda^{-2})$ as $\lambda \nearrow \infty$. So the integrals associated to the integrands (10.5) converge in the norm to compact operators, as required. □

It remains to prove Propositions 10.9 and 10.10.

Proof of Proposition 10.9. We want to show that the family of operators

$$\{K_t\}_{t\in[\varepsilon,\infty]} = \{(D_t + P_t \pm i\lambda)^{-1}\}_{t\in[\varepsilon,\infty]}$$

is compact. Since the compact operators form a closed, two-sided ideal in the C^*-algebra of all adjointable families of operators it suffices to show that the family

$$\{K_t^* K_t\}_{t\in[\varepsilon,\infty]} = \{(\Delta_t + P_t + \lambda^2)^{-1}\}_{t\in[\varepsilon,\infty]}$$

is compact; compare [18, Proposition 1.4.5]. Conjugating by the unitaries U_t it suffices to prove that the family

$$\left\{(d_{w_t}\delta_{w_t} + \delta_{w_t}d_{w_t} + P_t + \lambda^2)^{-1}\right\}_{t\in[\varepsilon,\infty]}$$

on the constant field of Hilbert spaces with fiber $\ell^2(X^\bullet)$ is compact; this is one of the things that restricting to $t \in [\varepsilon, \infty]$ makes possible. But this final assertion is a simple consequence of the explicit formula for the Julg-Valette Laplacian in Proposition 3.25, together with the fact that the weight functions w_t are *uniformly* proper in $t \in [\varepsilon, \infty]$ in the sense that for every d, all but finitely many hyperplanes H satisfy $w_t(H) \geq d$ for all $t \in [\varepsilon, \infty]$. Compare to the proof of Lemma 9.5. □

We turn now to Proposition 10.10, the proof of which shall occupy us for the remainder of the section. We aim to analyze the difference $D_t - g(D_t)$, and a complicating factor is that G not only fails to preserve the Julg-Valette differential, but also fails to preserve the unitary operators U_t that appear in the definition of D_t. In fact Proposition 10.10 is only correct because the two failures cancel out one another out.

10.12 Definition. Let P and Q be vertices in X. Define a unitary operator

$$\widehat{W}_t(Q,P)\colon \ell^2(X^q) \longrightarrow \ell^2(X^q)$$

as follows. When $q = 0$, we define $\widehat{W}_t(Q,P)$ to be the cocycle operator $W_t(Q,P)$ of Definition 6.13. On higher cubes, $\widehat{W}_t(Q,P)$ respects the decomposition of $\ell^2(X^q)$ according to parallelism classes, and on a summand determined by an equivalence class we set $\widehat{W}_t(Q,P) = W_t(C_Q, C_P)$, where C_Q and C_P are the cubes in the class nearest to Q and P.

Recall that P_0 is our fixed base point. We write $Q_0 = g(P_0)$, and introduce the abbreviation $\widehat{W}_t := \widehat{W}_t(Q_0, P_0)$. With this notation, it is immediate from the definition of the unitary operator U_t in Definition 6.15 that

$$g(U_t) = \widehat{W}_t U_t : \ell^2_t(X^\bullet) \to \ell^2(X^\bullet).$$

Combining this with the definition of D_t we obtain

$$g(D_t) = U_t^* \widehat{W}_t^* \big(g(d_{w_t}) + g(\delta_{w_t})\big) \widehat{W}_t U_t,$$

so that difference we wish to analyze is given by

$$D_t - g(D_t) = U_t^* \Big((d_{w_t} + \delta_{w_t}) - \widehat{W}_t^* \big(g(d_{w_t}) + g(\delta_{w_t})\big) \widehat{W}_t \Big) U_t.$$

The right-hand side of this expression can be rearranged as

$$U_t^* \widehat{W}_t^* \Big(\widehat{W}_t d_{w_t} - g(d_{w_t}) \widehat{W}_t \Big) U_t + U_t^* \Big(\delta_{w_t} \widehat{W}_t^* - \widehat{W}_t^* g(\delta_{w_t}) \Big) \widehat{W}_t U_t,$$

the norm of which is bounded by

$$\big\| \widehat{W}_t d_{w_t} - g(d_{w_t}) \widehat{W}_t \big\| + \big\| \delta_{w_t} \widehat{W}_t^* - \widehat{W}_t^* g(\delta_{w_t}) \big\|.$$

So, to prove Proposition 10.10 it suffices to show that the operators

$$\widehat{W}_t d_{w_t} - g(d_{w_t}) \widehat{W}_t \quad \text{and} \quad \delta_{w_t} \widehat{W}_t^* - \widehat{W}_t^* g(\delta_{w_t})$$

satisfy the conclusions of that proposition. In fact, since the second operator is the adjoint of the first it suffices to prove these conclusions for the first operator alone, and this is what we shall do.

Before we proceed, let us adjust our notation a bit, as follows. Given a vertex P in X, we shall denote by d_{P,w_t} the Julg-Valette differential that is defined using the base vertex P and the weight function (8.3), for whose definition we also use the base vertex P rather than P_0. With this new notation we can drop further mention of the group G. Proposition 10.10 is now a consequence of the following assertion:

10.13 Proposition. *The operator*

$$\widehat{W}_t(Q,P) d_{P,w_t} - d_{Q,w_t} \widehat{W}_t(Q,P) \colon \mathbb{C}[X^q] \longrightarrow \mathbb{C}[X^{q+1}]$$

is bounded for all $t > 0$, and moreover

$$\lim_{t\to 0} \big\| \widehat{W}_t(Q,P) d_{P,w_t} - d_{Q,w_t} \widehat{W}_t(Q,P) \big\| = 0.$$

Recall now that the Julg-Valette differential is defined using the operation $H \wedge C$ between hyperplanes and cubes. Since the operation depends on a choice of base vertex, we shall from now on write $H \wedge_P C$ to indicate that choice, as we did earlier.

To prove Proposition 10.13 it suffices to consider the case where P and Q are at distance 1 from one another. We shall make this assumption from now on, and shall denote by K the (unique) hyperplane that separates P and Q.

10.14 Lemma. *If a hyperplane H fails to separate P from Q, then*

$$H \wedge_P \widehat{W}_t(P,Q)D = \widehat{W}_t(P,Q)(H \wedge_Q D)$$

for all oriented q-cubes D.

Proof. First, since we assume that H fails to separate P from Q the operators $H\wedge_P$ and $H\wedge_Q$ are equal, and we shall drop the subscripts for the rest of the proof.

Next, if H cuts D, then it cuts all the cubes parallel to D, and therefore cuts all the cubes that make up $\widehat{W}_t(P,Q)D$. In this case both sides of the equation in the lemma are zero. So we assume from now on that H is disjoint from D.

Recall that K is the hyperplane that separates Q from P. According to Proposition 4.6 the cubes C_P and C_Q nearest to P and Q in the parallelism class of D are either equal or are opposite faces across K of a $(q+1)$-cube that is cut by K. So $\widehat{W}_t(P,Q)D$ is either just D or is a combination

$$\widehat{W}_t(P,Q)D = aD + bE \tag{10.6}$$

of D and another cube E that is an opposite face from D in a $(q+1)$-cube that is cut by K.

If H fails to separate D from P, or equivalently, if it fails to separate D from Q, then it also fails to separate any of the terms in $\widehat{W}_t(P,Q)D$ from P or Q, and accordingly both sides of the equation in the lemma are zero. So we can assume from now on that H does separate D from P and Q.

Suppose now that K fails to be adjacent to D, either because it cuts D or because some vertex of D is not adjacent to K. The left-hand side of the equation is then $H \wedge D$. This is either zero, in which case the equation obviously holds, or it is a $(q+1)$-cube to which K also fails to be adjacent, in which case the right-hand side of the equation is simply $H \wedge D$. So we can assume that K is adjacent to D.

Let E be the q-cube that is separated from D by K alone, as in (10.6). Since H fails to separate D from E, or P from Q, but separates D and E from P and Q, we see from Lemma 2.4 that H and K intersect. By Lemma 2.6, if H is adjacent to either of D or E, then there is a $(q+2)$-cube that is cut by H and K and contains both D and E as faces. In this case both sides of the equation in the lemma are

$$a\, H \wedge D \,+\, b\, H \wedge E$$

with a and b as in (10.6). Finally, if H is adjacent to neither D nor E, then both sides of the equation are zero. □

10.15 Lemma. *Recall that K separates P from Q. We have:*

$$K \wedge_P \widehat{W}_t(P,Q)D - \widehat{W}_t(P,Q)(K \wedge_Q D) = f(t)K \wedge_Q D - g(t)K \wedge_P D,$$

where f and g are smooth, bounded functions on $[0,\infty)$ that vanish at $t = 0$.

Proof. If D fails to be adjacent to K, then both sides in the displayed formula are zero. So suppose D is adjacent to K. In this case

$$\widehat{W}_t(P,Q)(K \wedge_Q D) = K \wedge_Q D.$$

Now according to the definitions

$$\widehat{W}_t(P,Q)D = \pm e^{-\frac{1}{2}t^2} E + (1 - e^{-t^2})^{\frac{1}{2}} D,$$

where E is the q-cube opposite D across K, and where the sign is $+1$ if D is separated from P by K, and -1 if it is not. We find then that

$$K \wedge_P \widehat{W}_t(P,Q)D = \pm e^{-\frac{1}{2}t^2} K \wedge_P E + (1 - e^{-t^2})^{\frac{1}{2}} K \wedge_P D.$$

But $K \wedge_P E = 0$ if E is not separated from P by K, which is to say if D *is* separated from P by K. So we can write

$$K \wedge_P \widehat{W}_t(P,Q)D = -e^{-\frac{1}{2}t^2} K \wedge_P E + (1 - e^{-t^2})^{\frac{1}{2}} K \wedge_P D.$$

In addition

$$K \wedge_P E = -K \wedge_Q D$$

so that

$$K \wedge_P \widehat{W}_t(P,Q)D = e^{-\frac{1}{2}t^2} K \wedge_Q D + (1 - e^{-t^2})^{\frac{1}{2}} K \wedge_P D.$$

Finally we obtain

$$\widehat{W}_t(P,Q)(K \wedge_Q D) - K \wedge_P \widehat{W}_t(P,Q)D = (e^{\frac{1}{2}t^2} - 1) K \wedge_Q D - (1 - e^{-t^2})^{\frac{1}{2}} K \wedge_P D,$$

as required. □

Proof of Proposition 10.13. We shall use the previous lemmas and the formula

$$d_{P,w_t} D = \sum_H w_{P,t}(H)\, H \wedge_P D$$

for the Julg-Valette differential. We get

$$\begin{aligned}
&\widehat{W}_t(Q,P) d_{P,w_t} - d_{Q,w_t} \widehat{W}_t(Q,P) \\
&\quad = \sum_H \Big(w_{P,t}(H) \widehat{W}_t(Q,P)\,(H \wedge_P D) - w_{Q,t}(H) H \wedge_Q \widehat{W}_t(Q,P)D \Big).
\end{aligned} \tag{10.7}$$

Let us separate the sum into a part indexed by hyperplanes that do *not* separate P from Q, followed by the single term indexed by the hyperplane K that does separate P from Q. According to Lemma 10.14 the first part is

$$\sum_{H \neq K} \big(w_{P,t}(H) - w_{Q,t}(H)\big)\, \widehat{W}_t(Q,P)\,(H \wedge_P D)\,.$$

Inserting the definition of the weight function, we obtain

$$t \sum_{H \neq K} \big(\mathrm{dist}(H,P) - \mathrm{dist}(H,Q)\big)\, \widehat{W}_t(Q,P)\,(H \wedge_P D)\,, \tag{10.8}$$

and moreover

$$\big|\mathrm{dist}(H,P) - \mathrm{dist}(H,Q)\big| \leq 1.$$

As for the part of (10.7) indexed by K, keeping in mind that

$$\mathrm{dist}(K,P) = \tfrac{1}{2} = \mathrm{dist}(K,Q),$$

we obtain from Lemma 10.15 the following formula for it:

$$(1 + \tfrac{1}{2}t) f(t) K \wedge_Q D - (1 + \tfrac{1}{2}t) g(t) K \wedge_P D, \tag{10.9}$$

where f and g are bounded and vanish at 0. The required estimates follow, because the terms in (10.8) and (10.9) are uniformly bounded in number, are supported uniformly close to D, are uniformly bounded in size, and vanish at $t = 0$. $\square$

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