Social Welfare in One-Sided Matching Mechanisms

(Extended Abstract)

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ABSTRACT

We study the Price of Anarchy of mechanisms for the well-known problem of one-sided matching, or house allocation, with respect to the social welfare objective. We consider both ordinal mechanisms, where agents submit preference lists over the items, and cardinal mechanisms, where agents may submit numerical values for the items being allocated. We present a general lower bound of $\Omega(\sqrt{n})$ on the Price of Anarchy, which applies to *all* mechanisms and we show that a very well-known mechanisms, Probabilistic Serial achieves a matching upper bound. We extend our lower bound to the Price of Stability of a large class of mechanisms that satisfy a common proportionality property.

Keywords

One-sided matching, probabilistic serial, truthfulness, price of anarchy, Nash equilibrium

1. INTRODUCTION

One-sided matching (also called the house allocation problem) is the fundamental problem of assigning items to agents, such that each agent receives exactly one item, which has numerous applications. In this setting, agents are often asked to provide *ordinal preferences*, i.e. preference lists, or rankings of the items. We assume that underlying these ordinal preferences, agents have numerical values specifying how much they value each item [7]. In game-theoretic terms, these are the agents' von Neumann-Morgenstern utility functions [12] and the associated preferences are often referred to as *cardinal preferences*.

A *mechanism* is a function that maps agents' valuations to matchings. However, agents are rational strategic entities that might not always report their valuations truthfully; they may misreport their values if that results in a better matching (from their own perspective). Assuming the agents report their valuations strategically to maximize their utilities, it is of interest to study the *Nash equilibria* of the induced game, i.e. strategy profiles from which no agent wishes to unilaterally deviate.

A natural objective for the designer is to choose the matching that maximizes the *social welfare*, i.e. the sum of agents' valuations for the items they are matched with, which is the most prominent measure of aggregate utility in the literature. Given the strategic nature of the agents, we are interested in mechanisms that maximize the social welfare *in the equilibrium*. We use the standard measure of equilibrium inefficiency, the *Price of Anarchy* [8], that compares the maximum social welfare attainable in any matching with the *worst-case* social welfare that can be achieved in an equilibrium.

We evaluate the efficiency of a mechanism with respect to the Price of Anarchy of the induced game. We study both deterministic and randomized mechanisms: in the latter case the output is a probability mixture over matchings, instead of a single matching. We are interested in the class of *cardinal* mechanisms, which use cardinal preferences, and generalize the ordinal mechanisms.

Note that our setting involves no monetary transfers and generally falls under the umbrella of *approximate mechanism design without money* [10]. In general settings without money, one has to fix a canonical representation of the valuations. A common approach in the literature is to consider the *unit-sum* normalization, i.e. each agent has a total value of 1 for all the items.

Our main contributions

Our main contribution can be summarized by the following theorem.

THEOREM 1. *The Price of Anarchy of Probabilistic Serial is $O(\sqrt{n})$ and the Price of Anarchy of any mechanism is $\Omega(\sqrt{n})$. This implies that Probabilistic Serial is (asymptotically) optimal among all mechanisms for the problem.*

The upper bound of the theorem holds with respect to the very general equilibrium notion, the *coarse correlated equilibrium*, as well as the *Bayes-Nash equilibrium*, when agents have incomplete information about the preference orderings of other agents.

We extend the lower bound of Theorem 1 to the the *Price of Stability*, a more optimistic measure of efficiency, for all

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mechanisms that satisfy a simple *proportionality* property, which requires that any agent can report some preference list that guarantees her an allocation at least as good as an equal split (in terms of probability) of each item, regardless of the reports of the other agents. This property is satisfied by most mechanisms in literature, including Probabilistic Serial.

Finally, for deterministic mechanisms, we prove that no mechanism can achieve a Price of Anarchy smaller than $\Omega(n)$, proving that randomization is essential for non-trivial guarantees to be achievable. The proofs to our results can be found in the full version of the paper [5].

Discussion

The one-sided matching problem was introduced in [7] and has been studied extensively ever since, with various desirable objectives related to truthfulness, fairness and economic efficiency. The objective of social welfare maximization for one-sided matching problems has been studied before in the literature, for mechanisms that are truthful [6]. The authors in [6] prove that Random Priority, a well-known truthful mechanism, achieves an approximation ratio of $O(\sqrt{n})$, which is asymptotically optimal among all *truthful* mechanisms. Our lower bounds are more general, since they apply to *all* mechanisms, not just truthful ones. In particular, our lower bound on the Price of Anarchy of all mechanisms generalizes the corresponding bound for truthful mechanisms in [6].

We establish the tightness of the bounds in Theorem 1, by bounding the inefficiency of perhaps the most popular one-sided matching mechanism, Probabilistic Serial. The mechanism was introduced in [3] and since then, it has been in the center of attention of the matching literature, with related work on characterizations, strategic aspects and hardness of manipulation. Somewhat surprisingly, the Nash equilibria of the mechanism were only recently studied [1].

A somewhat different recent branch of study considers ordinal measures of efficiency instead of social welfare maximization, under the assumption that agents' preferences are only expressed through preference orderings over items [2, 9, 4]. While interesting, these measures of efficiency do not accurately encapsulate the socially desired outcome the way that social welfare does, especially since an underlying cardinal utility structure is part of the setting [3, 7, 12]. Our results actually suggest that in order to achieve the optimal welfare guarantees, one does not even need to elicit this utility structure; agents can only be asked to report preference orderings, which is arguably more appealing.

Finally, we point out that our work is in a sense analogous to the literature that studies the Price of Anarchy in item-bidding auctions for settings without money. Furthermore, the extension of our results to very general solution concepts (coarse correlated equilibria) and settings of incomplete information (Bayes-Nash equilibria) is somehow reminiscent of the *smoothness* framework [11] for games. While our results are not proven using the smoothness condition, our extension technique is similar in spirit.

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