Plane Wave Identification With Circular Arrays By Means Of A Finite Rate Of Innovation Approach

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Abstract

Many problems in the field of acoustic measurements depend on the direction of incoming wave fronts w.r.t. a measurement device or aperture. This knowledge can be useful for signal processing purposes such as noise reduction, source separation, de-aliasing and super-resolution strategies among others.

This paper presents a signal processing technique for the identification of the directions of travel for the principal plane wave components in a sound field measured with a circular microphone array. The technique is derived from a finite rate of innovation data model and the performance is evaluated by means of a simulation study for different numbers of plane waves in the sound field.

1 Introduction

There are many methods of decomposing a microphone array measurement of a sound field into an assessable number of parameters. Ignoring the differences between two-dimensional and three-dimensional technologies, a convenient approach to perform this analysis/decomposition is to decompose a sound field into a set of plane waves, e.g. as proposed in [1, 2, 3, 4]. Plane waves have the advantage that they are elementary entities defined by only two parameters: the direction of travel (DOT) and the complex amplitude. If the number of plane waves in a sound field is limited then it is also perfectly described by a small number of parameters. In that case, these parameters can be identified from the measurement made with a suitable microphone array.

One typically distinguishes between non-parametric (i.e. no underlying signal model) and parametric (e.g. based on a signal model) source detection or DOT estimation techniques. Examples of non-parametric techniques are adaptive beamforming based on maximum likelihood estimators or the MVDR beamformer [5, 6, 7]. Prominent examples of parametric techniques are the MUSIC algorithm [8, 6], the EB-ESPRIT and the EB-DETECT algorithm [2], and Matching Pursuit variants such as CLEAN [9]. Compressed sensing techniques could also be utilised to estimate parameters, yet are they typically used improve the performance of sensor arrays directly [10, 11]. Some of these techniques require a priori knowledge (e.g. the number of sources) or deploy statistical models to obtain estimates. This may potentially lead to usable, yet slightly inaccurate results. Many applications and techniques, however, require the exact knowledge of the DOT to ensure the best possible performance. Examples are source separation, de-noising and de-aliasing techniques based on beamforming for acoustical and sonar applications, as well as super-resolution techniques [12, 13, 14, 15].

This work presents a novel application of a signal processing approach that theoretically allows to analytically recover the set of DOT of a finite number of principal plane waves from the measured data of a circular microphone array. Once recovered, this information can be used to further process the measured data, e.g. with simple beamforming techniques to suppress sources outside the main listening direction or even de-aliasing strategies [15]. This work is to be understood as an initial study based on idealised simulations and aims to introduce the proposed method. Therefore the impact of measurement noise, model mismatch and transducer imperfections is not within the scope of this work and will be addressed in future publications.

The next section introduces the sound field model, which is suitable to be combined with circular microphone arrays. The third section briefly describes how the data defining the sound field can be gained from the array observation. In the fourth section, the proposed method to identify the plane waves in the sound field based on a finite rate of innovation data model is presented. In the penultimate section, the proposed method is analysed with respect to its accuracy for different idealised sound field scenarios and the last section concludes the findings and gives an outlook on future work.

2 Sound Field Model

The general sound field model for this work is based on the Herglotz Wave Function (HWF) in the frequency domain [16, 17] given by

$$p(\mathbf{x},\omega) = \int_{\Omega} H(\mathbf{x},\hat{\mathbf{y}},\omega) q(\hat{\mathbf{y}}) d\Omega(\hat{\mathbf{y}}),\ \mathbf{x} \in \Lambda, \tag{1}$$

where Λ denotes a sphere of a given radius around the coordinate origin within which the homogeneous wave equation is satisfied at all points $\mathbf{x}$. The integration domain $\Omega = \{\hat{\mathbf{y}} : \sqrt{y_1^2 + y_2^2 + y_3^2} = 1\}$ encompasses all the points on the surface of the unit sphere. Equation (1) basically describes the sound field as the superposition of plane waves travelling in the directions $\hat{\mathbf{y}}$. The magnitude of each individual plane wave is determined by the *Herglotz Density* (HD) $q(\hat{\mathbf{y}})$ and its propagation characteristics are given through the *Herglotz Kernel* $H(\mathbf{x}, \hat{\mathbf{y}}, \omega) = e^{ik\mathbf{x}\cdot\hat{\mathbf{y}}}$.

While equation (1) is valid within $\Lambda \in \mathbb{R}^3$, this work considers only two-dimensional scenarios to reconcile the sound field model with the intended signal processing approach. Hence, the sound field is assumed as height-invariant, i.e. the pressure is constant along the x_3-axis, so that

$$p(x_1, x_2, x_3) = p(x_1, x_2).$$

It has been shown in [17] that, using the Jacobi-Anger expansion [18], the model of the pressure in equation (1) can be rewritten in polar coordinates as a function of the radius r and the polar angle ϕ

$$p(r, \phi) = 2\pi \sum_{n=-\infty}^{\infty} i^n R_n(kr) \frac{e^{in\phi}}{\sqrt{2\pi}} \int_0^{2\pi} \frac{e^{-in\phi'}}{\sqrt{2\pi}} q(\phi') d\phi'. \tag{2}$$

Note that the angular frequency ω has been dropped from the notation for the sake of brevity. The radial function

$$R_n(kr) = \begin{cases} J_n(kr) & ,\mathrm{FF} \\ J_n(kr) - \frac{J_n'(kr_s)}{H_n^{(1)\prime}(kr_s)} H_n^{(1)}(kr) & ,\mathrm{CS} \end{cases} \tag{3}$$

allows for the modelling of free field conditions (FF) or the presence of a cylindrical scatterer at the origin (CS) with radius r_s [19, 2]. $J_n(\cdot)$ and $H_n^{(1)}(\cdot)$ denotes the Bessel-function and the Hankel-function of the first kind, respectively, where $J_n'(\cdot)$ and $H_n^{(1)\prime}(\cdot)$ denote their respective derivatives. The integration variable ϕ' in eq. (2) replaces the DOT $\hat{\mathbf{y}}$, so that it represents the angle between the positive x_1-axis and the vector $\hat{\mathbf{y}}$. The integration domain has become the unit circle $\Omega_c = \{\hat{\mathbf{y}} : \sqrt{x_1^2 + x_2^2} = 1\}$.

It is assumed here that the Herglotz density holds the basic information that defines the plane waves in the sound field, and that this needs to be recovered from a finite number of pressure observations made by a circular microphone array before the planned signal processing approach can be applied. This is typically referred to as the inverse problem, which is solved in the next section.

3 Recovering the Herglotz Density from Circular Array Measurements

The procedure for solving the inverse problem such as the one posed by equation (2) has been discussed in a number of works [19, 20, 2, 21]. For the sake of brevity, this derivation has been omitted. The analytical solution to the inverse problem to equation (2) is given by

$$q(\phi) = \sum_{n=-\infty}^{\infty} q_n \frac{e^{in\phi}}{\sqrt{2\pi}}, \tag{4}$$

with

$$q_n = \frac{1}{2\pi i^n R_n(kr_V)} \int_0^{2\pi} \frac{e^{-in\phi'}}{\sqrt{2\pi}} p(\phi') d\phi'. \tag{5}$$

For a circular microphone array with $L = 2N + 1$ uniformly distributed pressure sensors, the solution for the coefficients q_n in (5) can be approximated by

$$\tilde{q}_n = \frac{1}{i^n R_n(kr_V) L} \sum_{n=l}^{L} \frac{e^{-in\Delta\phi l}}{\sqrt{2\pi}} p(\Delta\phi l), \tag{6}$$

with $\Delta\phi = \frac{2\pi}{L}$. As a consequence of the discretisation, the HD can only be reconstructed up to the Nth order [22, 17], so that

$$\tilde{q}(\phi) = \sum_{n=-N}^{N} \tilde{q}_n \frac{e^{in\phi}}{\sqrt{2\pi}}. \tag{7}$$

Furthermore, discretisation is also bound to lead to aliasing effects since the spatial complexity of sound fields can typically not be expressed through an order-limited HD, especially at high frequencies. However, the problem of aliasing is not within the scope of this work and considered a separate problem, but it has been described for circular arrays with pressure sensors in previous work by Poletti [22] and Alon and Rafaely [15]. In the following, it shall be assumed that the recovered coefficients $\tilde{q}_n$ are not corrupted by either aliasing, measurement noise or misalignment of the sensors.

The next section introduces a method of estimating the DOTs of up to N plane waves from the set of recovered coefficients $\{\tilde{q}_n : n \in [-N \dots N]\}$ on the basis of a finite rate of innovation signal model.

4 Finite Rate of Innovation Approach

This section presents how the parameters defining the plane waves in a sound field can be recovered based on a Finite Rate of Innovation (FRI) signal model. A comprehensive introduction to the theory behind signals with finite rate of innovation is beyond the scope of this work and the interested reader is referred to the pertinent literature [23, 24, 25].

First, the new signal model for the Herglotz density is introduced. This model is based on a form for the HD specified by equation (7) but makes another assumption regarding the nature of the sound field. Then, a set of annihilation filters is calculated that is applied to the new signal model. These filters are specified by the DOT of the incoming plane waves. Finally, it is shown how these parameters can be extracted from the identified annihilation filters.

4.1 HD of a Single Plane Wave

From the definition of the HWF in equations (1) and (2), it can be seen that the HD of a single plane wave is theoretically a weighted Dirac delta shifted to the direction of travel θ and is given by

$$q(\phi) = b\delta(\phi - \theta), \tag{8}$$

where b is the complex amplitude of the plane wave. For a function with a period of 2π, the Dirac delta can be approximated by the Fourier Series

$$q(\phi) = \sum_{n=-\infty}^{\infty} b\frac{e^{-in\theta}}{\sqrt{2\pi}}\frac{e^{in\phi}}{\sqrt{2\pi}}. \tag{9}$$

The above form of the HD can now be utilised in the FRI signal model, which is introduced in the following subsection.

4.2 New Model for the Herglotz Density

In the following, only Herglotz densities that define sound fields with up to $M = N$ plane waves are considered. Accordingly, these can be represented by the sum of the Herglotz densities of the M individual plane waves

$$q(\phi) = \sum_{m=1}^{M} b_m \sum_{n=-\infty}^{\infty} \frac{e^{in(\phi-\theta_m)}}{2\pi}. \tag{10}$$

A Herglotz Density of the above form has exactly $2M$ degrees of freedom, i.e. M directions of travel and M complex amplitudes. Alternatively, since the HD is periodic, one can say that it has a 'finite rate of innovation', which is an alternative way of expressing a 'finite number of degrees of freedom'.

Rearranging equation (10) yields

$$q(\phi) = \sum_{n=-\infty}^{\infty} \sum_{m=1}^{M} b_m \frac{e^{-in\theta_m}}{\sqrt{2\pi}} \frac{e^{in\phi}}{\sqrt{2\pi}} = \sum_{n=-\infty}^{\infty} q_n \frac{e^{in\phi}}{\sqrt{2\pi}} \tag{11}$$

with the Fourier coefficients

$$q_n = \sum_{m=1}^{M} b_m \frac{e^{-in\theta_m}}{\sqrt{2\pi}}. \tag{12}$$

The coefficients q_n can be seen as an infinite sequence. Alternatively, the coefficients q_n in (12) can be seen as the superposition of M infinite sequences defined by

$$q_n^{(m)} = b_m \frac{e^{-in\theta_m}}{\sqrt{2\pi}} \tag{13}$$

for $m \in [1 \dots M]$. Note that each of these sequences is uniquely related to *one* of the plane waves only. The following subsection describes how the individual parameters b_m and θ_m can be extracted from the q_n.

4.3 Definition of the Annihilation Filter

For each sequence $q_n^{(m)}$, there is a corresponding filter

$$a_n^{(m)} = \delta_n - e^{-i\theta_m} \delta_{n-1}$$

that satisfies the equation

$$a_n^{(m)} * q_n^{(m)} = 0,$$

where the asterisk represents the discrete signal convolution and δ_n denotes the Kronecker delta [19, 26], which is defined by

$$\delta_n = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$

These filters are therefore also sometimes referred to as *annihilation filters* [23]. The z-transform of the filters $a_n^{(m)}$ is given by

$$A^{(m)}(z) = 1 - e^{-i\theta_m} z^{-1},$$

which is zero for $z = e^{i\theta_m}$. Hence each annihilation filter suppresses the component $e^{in\theta_m}$ in the sequence of the Fourier coefficients q_n.

If the individual progressions $q_n^{(m)}$ could be observed directly from the recovered coefficients q_n, it would be simple to identify the corresponding annihilation filters. Unfortunately this is not the case. From equations (12) and (13) it follows however that

$$q_n = \sum_{m=1}^{M} q_n^{(m)}, \forall n \in \mathbb{Z}.$$

It can thus be assumed that one non-trivial solution for a_n that satisfies

$$q_n * a_n = 0 \tag{14}$$

is the combination of all annihilation filters $a_n^{(m)}$ to one comprehensive annihilation filter a_n of the form

$$a_n = a_n^{(1)} * a_n^{(2)} * \ldots * a_n^{(M)} = \delta_n + \sum_{m=1}^{M} \alpha_m \delta_{m-n} \tag{15}$$

with its z-transform given by

$$A(z) = \prod_{m=1}^{M} A^{(m)}(z) = \prod_{m=1}^{M} (1 - u_m z^{-1}),\ u_m = e^{-i\theta_m}. \tag{16}$$

The z-transform $A(z)$ can also be written using the coefficients α_m defined in equation (15):

$$A(z) = 1 + \alpha_1 z^{-1} + \alpha_2 z^{-2} + \ldots + \alpha_M z^{-M}. \tag{17}$$

From equations (16) and (17), it can be seen that the coefficients u_m are in fact the roots of the characteristic polynomial

$$\eta(z) = z^M + \alpha_1 z^{M-1} + \alpha_2 z^{M-2} + \ldots + \alpha_M. \tag{18}$$

In conclusion, once the annihilation filter a_n has been identified and the roots of its characteristic polynomial have been calculated, then the directions of travel $\{\theta_m : m \in [1 \ldots M]\}$ of the M plane waves are successfully recovered. The next step is therefore to find the annihilation filter a_n.

4.4 Calculation of the Annihilation Filter

The convolution of discrete-time signals can be realised in a convolution matrix. With a limited length of the filter a_n, the linear equation system (LEQS) equivalent to (14) is given by

$$\begin{bmatrix} q_1 & q_0 & q_{-1} & \cdots & q_{-(M-1)} \\ q_2 & q_1 & q_0 & \cdots & q_{-(M-2)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ q_M & q_{M-1} & q_{M-2} & \cdots & q_0 \end{bmatrix} \cdot \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_M \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

It follows from equation (15) that $a_0 = 1$. This can be exploited to avoid the trivial solution for the filter a_n. Rearranging the above LEQS yields

$$\begin{bmatrix} q_0 & \cdots & q_{-(M-1)} \\ \vdots & \ddots & \vdots \\ q_{M-1} & \cdots & q_0 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ \vdots \\ a_M \end{bmatrix} = - \begin{bmatrix} q_1 \\ \vdots \\ q_M \end{bmatrix}, \tag{19}$$

which is widely known in the field of auto-regressive filtering as a Yule-Walker equation system. This can ideally be solved by plain matrix inversion, since it can be expected that the matrix is full rank. However, this may not be the case in practice if the M plane waves are not distinct [23]. Furthermore, equation (19) implicitly gives a condition for the set of modal coefficients $\{q_n : n \in [-N_{\min} \ldots N_{\min}]\}$ that is needed to perform this type of analysis. From the system matrix and the right hand side of equation (19) it follows that

$$2N_{\min} + 1 = 2M \iff N_{\min} = M - \frac{1}{2}. \tag{20}$$

The last step is then to find the M distinct roots u_m of the characteristic polynomial $\eta(z)$ in (18). These can be obtained through different algorithms. For this work, the built-in function *roots* of MATLAB has been utilised.

Once the roots have been found, the estimated directions of travel $\tilde{\theta}_m$ can be calculated by the formula

$$\tilde{\theta}_m = i \ln(u_m), \forall m \in [1 \dots M]$$

where $\ln(x)$ denotes the principal value of the natural logarithm.

4.5 Recovering the Complex Amplitudes of the Individual Plane Waves

With the DOT identified, equation (12) can be used to generate a linear equation system with the complex amplitudes $\{b_m : m \in [1 \dots M]\}$ as the unknown parameters and the Fourier coefficients q_n form the vector of constants [23].

$$\begin{bmatrix} q_0 \\ \vdots \\ q_{M-1} \end{bmatrix} = \frac{1}{2\pi} \begin{bmatrix} 1 & \dots & 1 \\ e^{-i\theta_1} & \dots & e^{-i\theta_M} \\ \vdots & \ddots & \vdots \\ e^{-i(M-1)\theta_1} & \dots & e^{-i(M-1)\theta_M} \end{bmatrix} \begin{bmatrix} b_1 \\ \vdots \\ b_M \end{bmatrix} \tag{21}$$

This so called Vandermonde equation system [23] can be solved by matrix inversion to identify the complex amplitudes b_m.

5 Performance Evaluation

This section reports the results of numerical simulations to evaluate the performance of the proposed technique by means of one specific example and the results of a small simulation study.

5.1 An Example

Figure 1 gives an indication of the accuracy of the estimated parameters $\{\theta_m : m \in [1 \dots M]\}$ when using the FRI based method with an artificially generated set of HD coefficients $\{q_n : n \in [-7 \dots 7]\}$, which are given by equation (11) for a set of $M = 4$ randomly generated DOT θ_m and complex amplitudes b_m. The top graph shows the real- and imaginary part of the original HD $q(\phi)$. The graph in the middle shows the real- and imaginary part of the HD $q_R(\phi)$ that was reconstructed on the basis of the identified parameters $\tilde{\theta}_m$ and $\tilde{b}_m$. The bottom graph shows the absolute error between the original HD and the reconstructed HD $|q - q_R|$. It can be seen that the FRI method approximates the set of parameters $\{\theta_m : m \in [1 \dots M]\}$ with very high accuracy. The nearly negligible error is

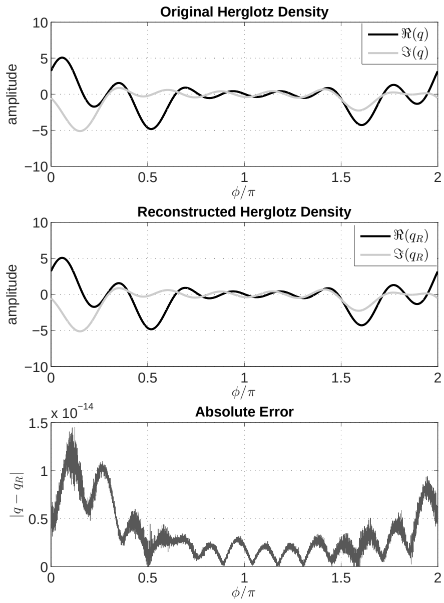

Figure 1: Example of the results achieved with the proposed method for $M = 4$ and $N = 7$, (Top) real- and imaginary part of the original HD, (Middle) real- and imaginary part of the reconstructed HD, (Bottom) absolute reconstruction error as an indicator for the parameter estimation performance.

accredited to the limited numerical accuracy during the matrix inversion.

5.2 Simulation Study

To assess both the average performance and the competitiveness of the proposed method, it was tested in a simulation study against the matching pursuit variant CLEAN [9]. The latter is an iterative algorithm that can be described as follows:

1. Fit the signature of a single plane wave (see eq. (9)) in the Herglotz Density domain to the recovered data $\tilde{q}(\phi)$ so that their global maxima are aligned and even. The parameter that best aligns the two maxima is the DOT estimate $\tilde{\theta}_m$ for the current iteration's strongest plane wave. [1]

2. Subtract the identified signature from $\tilde{q}(\phi)$.

3. Repeated the first two steps until the designated number of M plane waves have been identified.

In this simulation study, 5001 uniformly spaced samples were used for the required reconstruction in the HD domain.

[1] It can be shown that $\tilde{\theta}_m$ also maximises the cross-correlation between the single plane wave signature and the current HD data $\tilde{q}(\phi)$.

For the study, both methods were evaluated w.r.t. their average Parameter Identification Error (PIE) for the estimation of the DOTs. For an estimated parameter $\tilde{\theta}_m$, the PIE in percent is given by

$$\mathrm{PIE} = \left(|\theta_m - \tilde{\theta}_m| \ \mathbf{mod}\ \pi \right) 100\%.$$

The above measure was calculated for all identified DOTs in 5000 different Herglotz Densities and the results were averaged afterwards. This was repeated for different numbers of plane waves M in the sound field. Every HD was of the form given in (11) with randomly chosen θ_m and b_m, respectively. That corresponds to data perfectly acquired through a circular array with $L = 15$ microphones (i.e. no spatial aliasing), allowing to reconstruct the sound field up to the order $N = 7$.

Table 1 shows the study's results for $M = 4, 5, 6, 7$, where $M = 7$ is the largest number of plane waves that can be identified with the proposed method from the simulated sound field data (compare eq. (20)). It can be seen that, even for $M = 7$, the proposed method outperforms the CLEAN method

Number of Plane Waves	Average PIE with FRI	Average PIE with CLEAN
$M = 4$	$4.0574 \cdot 10^{-9}$ %	6.3412 %
$M = 5$	$9.8317 \cdot 10^{-8}$ %	8.4108 %
$M = 6$	$5.1642 \cdot 10^{-6}$ %	9.7623%
$M = 7$	$6.3900 \cdot 10^{-3}$ %	10.4769%

Table 1: PIE in percent for $M = 4, 5, 6, 7$, calculated for each value M from 5000 different Herglotz Densities $q(\phi)$, each specified by a set of randomly chosen directions of travel θ_m and complex amplitudes b_m.

significantly, providing sufficient accuracy for most applications.

Note that this simulation study has been conducted based on simulated, measurement noise-free and thus ideal conditions, where the number of plane waves identified always matched the number of plane waves in the field.

6 Conclusion

A novel method to estimate the direction of travel of a given number of plane waves from the measurement of a circular microphone array has been presented. The method is based on FRI signal theory and has been evaluated by means of an initial simulation study w.r.t. the achieved average parameter identification error. It has been shown that the latter does not exceed 0.01% for up to $M = 7$ plane waves with the proposed method. This assumes ideal measurement data acquired with 15 microphones in the absence of noise. The remaining inaccuracy can most presumably be blamed on the limited numerical precision of the computer system. The performance comparison of the proposed method and the CLEAN algorithm indicates that, for the conditions simulated, the FRI method clearly surpasses the CLEAN algorithm in terms of accuracy.

Future work is going to investigate the performance of the proposed method when applied to non-ideal data, covering the evaluation of its robustness against measurement noise, non-ideal plane waves (e.g. point sources in the near field of the measurement aperture) and transducer/aperture imperfections. Especially potential ill-conditioning of the matrices involved in the FRI method may have a crucial impact on its performance. Furthermore, the problem of when the number of plane waves to be identified does not match the number of plane waves in the sound field, i.e. it is greater or smaller, will be considered, as well as a comparison of the proposed method to further alternatives (e.g. MUSIC, EB-ESPRIT, etc.).

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