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THE OPTIMAL CASH HOLDING MODELS FOR STOCHASTIC CASH MANAGEMENT OF CONTINUOUS TIME

Abstract. In business, enterprises need to maintain stable cash flows to meet the demands for payments in order to reduce the probability of possible bankruptcy. In this paper, we propose the optimal cash holding models in terms of continuous time and managers' risk preference in the framework of stochastic control theory in the setting of cash balance accounting with the interval of a safe area for cash holdings. Formulas for the optimal cash holdings are analytically derived with a widely used family of power utility functions. Our models can be seen as an extension of Miller-Orr model to solve the cash holding problem of continuous time from the accounting perspective. Numerical examples are also provided to illustrate the feasibility of the developed optimal cashing holding models of continuous time.

1. **Introduction.**

In business, in order to reduce the probability of possible bankruptcy, enterprises need to maintain stable cash flows to meet the demands for payments such as transaction and debt repayment. So far, most of the researches in the literature of cash management have focused on how to determine the optimal cash holdings. There are two main streams of optimal models for cash management based on objective functions. The first stream is pioneered by Baumol(1952) and Tobin(1956). In Baumol (1952), it was assumed that the cash income is sustainable while cash payments are quite on the opposite, where the optimal demand for cash is determined by the lowest total cost with the total cost including both opportunity and transaction costs. Based on Baumol's model, Tobin (1956) took the interest rate factor into account. Recent extensions along this stream can be found in Frenkel and Jovanovic(1980, 1981), Bar-Ilan(1990), Dixit(1991), Ben-Bassat and Gottlieb (1992), Chang(1999) and Perry and Stadje(2000), MAS Melo and Bilich(2013), to list a few.

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The second stream is pioneered by Miller and Orr(1966). Differently from the first stream, Miller and Orr (1966) considered that the cash flows are random with the optimal demand for cash determined by the lowest total cost. For recent extensions along this second stream, the reader is referred to Bar-Ilan, Perry and Stadje (2004), Bar-Ilan and Lederman (2007) and Moraes and Nagano(2013), among others.

The disadvantages associated with the above optimal decision models for cash management have been recognised in that they focused on how to reduce the total cost under the discrete time case and did not take managers' risk preference into account. Meanwhile, the influence of securities' risk on decision-making was completely ignored. In the Miller and Orr's model, it is moreover highlighted that the cash balance follows a normal distribution, which may be a very ideal state. Since then, a few scholars, including Giirgis (1968), Eppen and Fama (1969), Daellenbach (1971, 1974), Hausman and Sanchez-Bell (1975), Milbourne (1983), Vickson (1985) and Smith (1989), Baccarin (2002) and Bensoussan, Chutani and Sethi(2009), have tried to overcome the defects by using dynamic programming methods. See also Song, Ching, Siu and Yiu (2013) for a recent discussion on the issue.

In this paper, our main objective is to extend the Miller-Orr's model by supposing that cash balance dynamically satisfies a stochastic differential equation from the cash holding accounting perspective. Differently from the general investment portfolio problem in finance (c.f., Merton (1969 and 1971) and Karatzas et al (1991)), it is worth noting in accounting that a "safe area" of cash holdings is important for the companies. Firstly, all enterprises need some cash to keep liquidity, and they can usually set an upper bound H and a lower bound L of the cash holdings by the empirical data. Secondly, if the cash holding is not in the safe area $[L, H]$, there are two cases: One case is that the cash holding is more than H, which will result in the loss of opportunity cost. The excel of idle funds investing in risky assets could increase the profit of the enterprises. The other case is that the cash holding is less than L, in which the enterprises have the liquidity risk. Therefore the "safe area" is the warning lines for the enterprises. Thirdly, the difference between the cash holdings problem in accounting and the general investment portfolio problem in finance is that the former pursues the utility maximazation on the basis of the safe area while the latter usually does not take this safe area into account. The stochastic differential equations introduced for the risky assets in this paper have been popularly used in the literature in the setting of continuous-time investment financing by using dynamic programming methods; see, for example, Merton (1969 and 1971) and Karatzas et al (1991). However, the solution to the problem of general investment portfolio is not necessarily the solution to the problem of cash holdings owing to the safe area consideration. Those equations will contain the risk factor of securities, which will make up the Miller-Orr model which lacks of considering the securities' risk. In addition, in most of the literatures mentioned above, the optimal cash management either was determined by the lowest total cost including the opportunity and transaction costs, or did not consider a safe area for cash balance as in the setting of investment financing. In this paper, we will take the expected utility maximization as the ultimate goal of cash management in the setting of cash balance accounting with the interval of the safe area for cash holdings. This could be seen as an extension alternative to Song et al (2013). We notice that managers' risk preference is very important in the enterprises' decision-making. A risk-seeking manager tends to increase the investment in risky assets and reduce the cash holding, while a manager of risk aversion tends to reduce the investment in risky assets and increase the cash holding. In this regard, this paper will try to take the managers' preference into account, attempting to provide a decision-making base for managers. On the basis of this, the managers can decide how much of cash balance need to be held in the cash and how much need to be transferred to the risky assets.

The remaining of this paper is organized as follows. In Section 2 we will introduce the continuous-time models (CTMs) for stochastic cash management. Section 3 will first give the optimal cashing holdings with general utility function and then derive the analytical formulas for optimal cash holdings by using a popular family of specific power utility functions. In Section 4 some illustrative examples will be provided to demonstrate the feasibility of the CTMs. Section 5 finally concludes.

2. **Continuous-time cash management models.** Enterprises usually put their cash balance in commercial banks or purchase government bonds in order to seek greater security and liquidity but get lower risk-free return, which leads to higher opportunity cost. On the contrary the managers will pursue higher return by investing the remaining assets in risky long-term assets such as stocks, corporate bonds and so on, which brings transaction cost. So enterprises need to consider the trade-off between opportunity cost and benefits of cash holdings. In order to solve the problem, Baumol's model and Miller-Orr's model both aim at seeking the lowest total cost. Differently, we are supposing that managers' target is to get more profits on the basis of enterprises' stability. Therefore maximizing the enterprize's utility function of cash balance is the CTM's objective function.

As pointed out in Section 1, differently from the extension by Song *et al.* (2013), in this paper, we consider the setting where we assume the manager, as a decision maker in the enterprises, determines the split of the given level of cash balance M_t between cash holdings X_t and risky assets R_t at time t with a safe area taken into account. In order to simplify the problem, we assume the risky assets are stocks. Miller and Orr pointed out that enterprises can get an upper bound H and a lower bound L of cash holdings from the empirical data. And under normal operating conditions, the company should have $H \leq M_t$. There are two principles. One is when $X_t > H$, the cash holdings is excessive and managers will decide to transfer a sizeable quantity of funds to risky assets for temporary investment. The other one is when $X_t < L$, enterprises may be in the face of a prolonged net drain and managers will liquidate some risky assets or will borrow to restore the cash holdings to an "safe level", which is, in Miller-Orr's model, the adequate working level. We will need the same upper bound H, lower bound L and the above-stated two principles.

The stochastic dynamics of the prices of the stock S_t and the money market account B_t at time t are supposed to evolve according to, respectively, the Brownian motion based stochastic differential equation and the ordinary differential equation as usually assumed in the literature of finance (c.f., Merton (1969 and 1971) and Karatzas et al (1991), etc.). We suppose S_t and B_t are modelled by the popular geometric Brownian motion and the ordinary differential equation, respectively, as follows:

$$\begin{cases} \frac{\mathrm{d}S_t}{S_t} = r_1 \mathrm{d}t + \sigma_1 \mathrm{d}W_t \\ S_0 = a \end{cases} \tag{1}$$

$$\frac{\mathrm{d}B_t}{B_t} = r_0 \mathrm{d}t, \tag{2}$$

where r_1 represents the instantaneous expected rate of return, r_0 is the interest rate of the cash. Here r_0, r_1, σ_1 are constants with $r_1 > r_0$, and σ_1 represents the instantaneous volatility, W_t is a one-dimensional Brownian motion on a given probability space $(\Omega, \mathscr{F}, P, \mathscr{F}_t)$, where $\mathscr{F}_t$ is the natural filtration, i.e. $\mathscr{F}_t = \sigma\{W_t : 0 < s < t\}$.

The interval $[L, H]$ is the "safe area". If cash holdings is in $[L, H]$, that is $L \leq X_t \leq H$, the manager does not need to make adjustment about X_t. However, if the cash holdings is beyond the safe area $[L, H]$, we need to make adjustment to maximize the enterprize's utility function of the cash balance. Note that, in general, from the *investment financing* perspective, an investor could borrow as much cash as possible to invest into the stock. However, in the setting of this paper, from the *cashing holding accounting* perspective, the first principle is to meet the daily cash needs for the remaining cash purposes, and then is to achieve the value growth. So in our paper, differently from the financing case as considered in the literature such as Merton (1969, 1971) and Karatzas et al (1991), etc., we are rather considering the accounting case with the "safe area" of interval $[L, H]$. Here, when a company wants to borrow cash, we then take it to be contained in the cash flow X_t at time t.

Now we discuss the problem in the following two cases. In the sequel, we suppose that the cash balance is M_t at time t, and the proportion of cash balance in cash holdings is β_t, then we have $X_t = \beta_t M_t$, $R_t = (1-\beta_t)M_t$.

2.1. **Case 1:** $X_t > H$**.** If $X_t > H$ at time t, we need to transfer part of cash holdings to risky assets. We suppose $0 \leqslant \lambda_t \leqslant 1$ denotes the fraction of X_t invested in stocks and then $1 - \lambda_t$ denotes the cash holdings percentage at time t. On the other hand, cash holdings should always be in the safe interval $[L, H]$, so we require $L \leq (1 - \lambda_t)\beta_t M_t \leq H$, that is λ_t must be in$[1 - \frac{H}{\beta_t M_t}, 1 - \frac{L}{\beta_t M_t}]$. Because $X_t = \beta_t M_t \geq H \geq L$, $0 \leq 1 - \frac{H}{\beta_t M_t} \leq \lambda_t \leq 1 - \frac{L}{\beta_t M_t} \leq 1$.

Now we denote by Λ the set of all admissible transfer strategies λ_t.

The dynamics of cash process M_t satisfies the following equation

$$dM_t = \frac{(1-\lambda_t)\beta_t M_t}{B_t} dB_t + \frac{\lambda_t \beta_t M_t + (1-\beta_t)M_t}{S_t} dS_t \tag{3}$$

Substituting (1) and (2) into (3) leads to

$$\begin{aligned} dM_t &= \frac{(1-\lambda_t)\beta_t M_t}{B_t} dB_t + \frac{\lambda_t \beta_t M_t + (1-\beta_t)M_t}{S_t} dS_t \\ &= \frac{(1-\lambda_t)\beta_t M_t r_0 B_t}{B_t} dt + \frac{\lambda_t \beta_t M_t + (1-\beta_t)M_t S_t}{S_t} (r_1 dt + \sigma_1 dW_t) \\ &= [r_0 \beta_t (1-\lambda_t) M_t + r_1 (1-\beta_t) M_t + r_1 \lambda_t \beta_t M_t] dt + [(1-\beta_t)M_t + \lambda_t \beta_t M_t] \sigma_1 dW_t \end{aligned}$$

The aim of managers is to find an optimal cash holdings process. Now we suppose that there is a utility function $U(x)$, i.e., a strictly concave, increasing and differentiable function $U(x)$ satisfying

$$\begin{cases} U^{'}(x) > 0, \\ U^{''}(x) < 0. \end{cases}$$

Then the problem can be described as

$$\text{[M-1]:}\begin{cases} \max\limits_{\lambda_t} E[U(M_T)] \\ s.t. \begin{cases} dM_t = [r_0\beta_t(1-\lambda_t)M_t + r_1(1-\beta_t)M_t + r_1\lambda_t\beta_t M_t]dt + [(1-\beta_t)M_t + \lambda_t\beta_t M_t]\sigma_1 dW_t \\ \lambda_t \in \Lambda \end{cases} \end{cases}$$

where $U(M_T)$ is the manager's preferences over the terminal wealth of the cash balance.

2.2. **Case 2:** $X_t < L$**.** If $X_t < L$ at time t, we need to transfer part of risky assets to cash holdings. Suppose that $R_t = (1-\beta_t)M_t$ is the risky assets holdings' value at time t, and in order to increase the cash holdings X_t, we need to transfer part of R_t to X_t. Let $0 \leqslant \mu_t \leqslant 1$ is the conversion ratio. Similar to case 1, $\beta_t M_t + \mu_t(1-\beta_t)M_t$ should be in the safe area $[L, H]$, then $\mu_t \in [\frac{L-\beta_t M_t}{(1-\beta t)M_t}, \frac{H-\beta_t M_t}{(1-\beta t)M_t}]$. Because $X_t = \beta_t M_t < L < H < M_t$, $0 \le \frac{L-\beta_t M_t}{(1-\beta t)M_t} \le \mu_t \le \frac{H-\beta_t M_t}{(1-\beta t)M_t} \le 1$. We also can denote by Π the set of all admissible transfer strategies μ_t .

Then the dynamics of cash process M_t satisfies the following equation:

$$dM_t = \frac{\beta_t M_t + \mu_t(1-\beta_t)M_t}{B_t} dB_t + \frac{(1-\mu_t)(1-\beta_t)M_t}{S_t} dS_t. \tag{4}$$

Substituting (1) and (2) into (4), one arrives at

$$\begin{aligned} dM_t &= \frac{\beta_t M_t + \mu_t(1-\beta_t)M_t}{B_t} dB_t + \frac{(1-\mu_t)(1-\beta_t)M_t}{S_t} dS_t \\ &= \frac{\beta_t M_t + \mu_t(1-\beta_t)M_t}{B_t} r_0 B_t dt + \frac{(1-\mu_t)(1-\beta_t)M_t}{S_t} S_t(r_1 dt + \sigma_1 dW_t) \\ &= [r_0\beta_t M_t + r_0\mu_t(1-\beta_t)M_t + r_1(1-\mu_t)(1-\beta_t)M_t]dt + (1-\mu_t)(1-\beta_t)M_t\sigma_1 dW_t. \end{aligned}$$

Then the problem similar to [M-1] can be given as

$$\text{[M-2]:}\begin{cases} \max\limits_{\mu_t} E[U(M_T)] \\ s.t. \begin{cases} dM_t = [r_0\beta_t M_t + r_0\mu_t(1-\beta_t)M_t + r_1(1-\mu_t)(1-\beta_t)M_t]dt + (1-\mu_t)(1-\beta_t)M_t\sigma_1 dW_t \\ \mu_t \in \Pi. \end{cases} \end{cases}$$

[M-1] and [M-2] belong to stochastic control problems. Both of them are the CTM and can be regarded as a natural generalization from the Miller-Orr's discrete time case to the continuous time case.

3. **Optimal Solutions to [M-1] and [M-2].** We first present the general optimal solutions to [M-1] and [M-2] in Subsections 3.1 and 3.2, respectively, and then discuss the optimal solutions with specific utility functions in Subsection 3.3.

3.1. **The general optimal solution to [M-1].** In order to solve the model [M-1], we need to define a value function as follows:

$$I(t,x) = \max_{\lambda_t} E[U(M_T)|M_t = x], \ 0 < t < T. \tag{5}$$

By the dynamic programming maximal principle, we can get the following Hamilton-Jacobi-Bellman equation (for short, HJB-equation)

$$I_t + \sup_{\lambda_t} \{[r_0\beta_t(1-\lambda_t)x + r_1(1-\beta_t)x + r_1\lambda_t\beta_t x]I_x + \frac{1}{2}x^2(1-\beta_t+\lambda_t\beta_t)^2\sigma_1^2 I_{xx}\} = 0. \quad (6)$$

Taking differentiation of the part inside the brackets of (6) with respect to λ_t,

$$x(r_1-r_0)I_x + x^2(1-\beta_t+\lambda_t\beta_t)\sigma_1^2 I_{xx} = 0,$$

therefore,

$$\lambda_t^* = 1 - \frac{1}{\beta t} + \frac{r_0-r_1}{\beta_t x\sigma_1^2}\frac{I_x}{I_{xx}}. \quad (7)$$

Then we can get the equation about value function I by taking λ_t^* into (6)

$$I_t + [x(r_0 - \frac{(r_1-r_0)^2}{x\sigma_1^2}\frac{I_x}{I_{xx}})]I_x + \frac{1}{2}x^2(\frac{r_1-r_0}{x\sigma_1^2}\frac{I_x}{I_{xx}})^2\sigma_1^2 I_{xx} = 0,$$

that is,

$$I_t + r_0 x I_x - \frac{(r_1-r_0)^2}{2\sigma_1^2}\frac{I_x^2}{I_{xx}} = 0$$

with

$$I(T,x) = U(x). \quad (8)$$

The following verification theorem, following from Fleming and Soner (2006, Chapter III, Theorem 8.1), shows that the classical solution to the HJB equation yields to the solution to the optimization problem [M-1].

Theorem 3.1. *Assume that $V(t,x)$ is the solution to (6) with boundary condition (8). Then the optimal value function I of (6) and V coincide. Furthermore, let λ_t^* satisfy*

$$I_t + [r_0\beta_t(1-\lambda_t^*)x + r_1(1-\beta_t)x + r_1\lambda_t^*\beta_t x]I_x + \frac{1}{2}x^2(1-\beta_t+\lambda_t^*\beta_t)^2\sigma_1^2 I_{xx} = 0$$

for all $(t,x) \in [0,T)\times R$. Then the strategy λ_t^* is optimal with $E[U(M_T^*)|M_t^* = x] = V(t,x)$, where M_t^* denotes the cash balance process under the strategy λ_t^*.

3.2. **The general optimal solution to [M-2].** In order to solve the model [M-2], we also need to define a value function

$$J(t,y) = \max_{\mu_t} E[U(M_T)|M_t = y],\ 0 < t < T, \quad (9)$$

and the HJB-equation is

$$J_t + \sup_{\mu_t} \{[r_0\beta_t y + r_0\mu_t(1-\beta_t)y + r_1(1-\mu_t)(1-\beta_t)y]J_y + \frac{1}{2}y^2(1-\mu_t)^2(1-\beta_t)^2\sigma_1^2 J_{yy}\} = 0. \quad (10)$$

Taking the differential of (10) with respect to μ_t,

$$-y(r_1-r_0)J_y - y^2(1-\beta_t)(1-\mu_t)\sigma_1^2 J_{yy} = 0,$$

therefore,

$$\mu_t^* = 1 + \frac{r_1-r_0}{y(1-\beta_t)\sigma_1^2}\frac{J_y}{J_{yy}}. \quad (11)$$

Then we can get the equation about value function J by taking μ_t^* into (10),

$$J_t + [r_0 y - \frac{(r_1-r_0)^2}{\sigma_1^2}\frac{J_y}{J_{yy}}]J_y + \frac{1}{2}y^2(\frac{r_1-r_0}{y\sigma_1^2}\frac{J_y}{J_{yy}})^2\sigma_1^2 J_{yy} = 0,$$

that is,

$$J_t + r_0 y J_y - \frac{1}{2}\frac{(r_1 - r_0)^2}{\sigma_1^2}\frac{J_y^2}{J_{yy}} = 0,$$

where

$$J(T, y) = U(y). \tag{12}$$

The following verification theorem, again following from Fleming and Soner (2006, Chapter III, Theorem 8.1), shows that the classical solution to the HJB equation yields to the solution to the optimization problem [M-2].

Theorem 3.2. *Assume that $W(t, y)$ is the solution to (10) with boundary condition (12). Then the optimal value function J of (10) and W coincide. Furthermore, let μ_t^* satisfy*

$$J_t + [r_0\beta_t y + r_0\mu_t(1-\beta_t)y + r_1(1-\mu_t)(1-\beta_t)y]J_y + \frac{1}{2}y^2(1-\mu_t)^2(1-\beta_t)^2\sigma_1^2 J_{yy} = 0$$

for all $(t, y) \in [0, T)\times R$. Then the strategy μ_t^* is optimal with $E[U(M_T^*)|M_t^* = y] = W(t, y)$, where M_t^* denotes the cash process under the strategy μ_t^*.

As indicated above, solving the stochastic control problems [M-1] and [M-2] becomes solving equations (6) and (10), respectively. It is worth noticing that these partial differential equations are second order and non-linear, which are usually difficult to obtain explicit solutions. However, for [M-1], we see from Theorem 1 that if a classical solution V to (6) with the boundary condition (8) can be found, then we have the unique optimal value function I and the corresponding optimal strategy λ_t^*. In other words, for the problem of maximizing expected utility function, we need to solve the nonlinear partial differential equation (6) and seek λ_t^* which maximizes the function

$$I_t + [r_0\beta_t(1-\lambda_t)x + r_1(1-\beta_t)x + r_1\lambda_t\beta_t x]I_x + \tfrac{1}{2}x^2(1-\beta_t+\lambda_t\beta_t)^2\sigma_1^2 I_{xx}$$

for all $(t, x) \in [0, T) \times R$. Similarly, for [M-2], we also see from Theorem 2 that if a classical solution W to (10) with the boundary condition (12) can be found,then we have the unique optimal value function J and the corresponding optimal strategy μ_t^*. That is, for the problem of maximizing expected utility function, we need to solve the nonlinear partial differential equation (10) and seek μ_t^* which maximizes the function

$$J_t + [r_0\beta_t y + r_0\mu_t(1-\beta_t)y + r_1(1-\mu_t)(1-\beta_t)y]J_y + \tfrac{1}{2}y^2(1-\mu_t)^2(1-\beta_t)^2\sigma_1^2 J_{yy}$$

for all $(t, y) \in [0, T) \times R$. We can not find their explicit solutions for general utility functions, which may need numerical calculations. Fortunately, however, for a widely-used special class of HARA utility functions, their explicit solutions can be found. In following subsections we construct a solution V of (6) and a solution W of (10) with the boundary conditions in the case of power utility. Meanwhile, we will consider the influence of a safe area on the model solutions.

3.3. **Optimal analytical solutions with specific utility functions.** In this subsection we will solve [M-1] and [M-2] under HARA power utility function

$$U(x) = \tfrac{x^p}{p},\ p < 1 \text{ and } p \neq 0,$$

with the restriction of $\lambda_t \in \Lambda$ and $\mu_t \in \Pi$, respectively.

3.3.1. *Case for [M-1].* Firstly, for [M-1] we need to solve

$$I_t + \sup_{\lambda_t\in\Lambda}\{[r_0\beta_t(1-\lambda_t)x + r_1(1-\beta_t)x + r_1\lambda_t\beta_t x]I_x + \frac{1}{2}x^2(1-\beta_t+\lambda_t\beta_t)^2\sigma_1^2 I_{xx}\} = 0 \quad (13)$$

with the boundary condition

$$I(T,x) = \frac{x^p}{p}. \quad (14)$$

Next, we summarise the conclusions in the form of theorem.

Theorem 3.3. *There exists a classical solution V to (13) with the boundary condition (14). The solution $V(t,x)$ and the corresponding λ_t^* are as follows:*
(i) If $1-\frac{1}{\beta_t}+\frac{r_1-r_0}{\beta_t(1-p)\sigma_1^2} \le 1-\frac{H}{\beta_t M_t}$, then

$$V(t,x) = m_1(t)\tfrac{x^p}{p},\ m_1(t) = e^{-pd_1(t-T)},\ d_1 = r_1 + \tfrac{r_0-r_1}{M_t}H + \tfrac{(M_t-H)^2}{2M_t^2}\sigma_1^2(p-1)$$

and

$$\lambda_t^* = 1-\tfrac{H}{\beta_t M_t}.$$

(ii) If $1-\frac{H}{\beta_t M_t} < 1-\frac{1}{\beta_t}+\frac{r_1-r_0}{\beta_t(1-p)\sigma_1^2} < 1-\frac{L}{\beta_t M_t}$, then

$$V(t,x) = m_2(t)\tfrac{x^p}{p},\ m_2(t) = e^{-d_2(t-T)},\ d_2 = r_0 p - \tfrac{p(r_1-r_0)^2}{2(p-1)\sigma_1^2}$$

and

$$\lambda_t^* = 1-\tfrac{1}{\beta_t}+\tfrac{r_1-r_0}{\beta_t(1-p)\sigma_1^2}.$$

(iii) If $1-\frac{L}{\beta_t M_t} \le 1-\frac{1}{\beta_t}+\frac{r_1-r_0}{\beta_t(1-p)\sigma_1^2}$, then

$$V(t,x) = m_3(t)\tfrac{x^p}{p},\ m_3(t) = e^{-pd_3(t-T)},\ d_3 = r_1 + \tfrac{r_0-r_1}{M_t}L + \tfrac{(M_t-L)^2}{2M_t^2}\sigma_1^2(p-1)$$

and

$$\lambda_t^* = 1-\tfrac{L}{\beta_t M_t}.$$

Proof. First we suppose that (13)-(14) has a solution V, which has the following form:

$$V(t,x) = m(t)\frac{x^p}{p}, \quad (15)$$

where $m(t)$ is a suitable function which we can guess is an exponential function, and the boundary condition implies that $m(T) = 1$. Inserting this trivial solution (15) into (13) results in

$$m'(t)\frac{1}{p}+m(t)\sup_{\lambda_t\in\Lambda}\{\frac{1}{2}(p-1)\sigma_1^2\beta_t^2\lambda_t^2+[(r_1-r_0)\beta_t+(p-1)\beta_t\sigma_1^2-(p-1)\beta_t^2\sigma_1^2]\lambda_t+l\} = 0, \quad (16)$$

where $l = r_0\beta_t + r_1 - r_1\beta_t + \frac{1}{2}(p-1)\sigma_1^2 + \frac{1}{2}(p-1)\sigma_1^2\beta_t^2 - (p-1)\sigma_1^2\beta_t$.
For λ_t^* without restriction, and $p<1$, so the supremum of (16) is attained at

$$\lambda_t^* = 1-\frac{1}{\beta_t}+\frac{r_1-r_0}{\beta_t(1-p)\sigma_1^2}. \quad (17)$$

We also can take $V_x = m(t)\frac{x^{p-1}}{p}$ and $V_{xx} = m(t)(p-1)\frac{x^{p-2}}{p}$ into (7), then (17) can also be obtained. But if the right hand side of (17) is less than $1-\frac{H}{\beta_t M_t}$ or larger than $1-\frac{L}{\beta_t M_t}$, then we have to truncate it by $1-\frac{H}{\beta_t M_t}$ and $1-\frac{L}{\beta_t M_t}$, respectively. Thus we need to handle the following three cases:

(1) The case of $1-\frac{1}{\beta_t}+\frac{r_1-r_0}{\beta_t(1-p)\sigma_1^2}\leq 1-\frac{H}{\beta_t M_t}$.
In this case, the symmetry axis $\lambda_t=1-\frac{1}{\beta_t}+\frac{r_1-r_0}{\beta_t(1-p)\sigma_1^2}$ of the downward sparabola

$$\frac{1}{2}(p-1)\sigma_1^2\beta_t^2\lambda_t^2+[(r_1-r_0)\beta_t+(p-1)\beta_t\sigma_1^2-(p-1)\beta_t^2\sigma_1^2]\lambda_t+l=h \tag{18}$$

is on the left side of the safe area $[1-\frac{H}{\beta_t M_t},1-\frac{L}{\beta_t M_t}]$, and (18) is monotone decreasing for λ_t in $[1-\frac{H}{\beta_t M_t},1-\frac{L}{\beta_t M_t}]$, so (16) attains its maximum at

$$\lambda_t^*=1-\frac{H}{\beta_t M_t}.$$

This allows us to replace (16) by

$$m_1'(t)+pd_1m_1(t)=0$$

subject to the condition $m_1(T)=1$. Then we can derive the following solution to (16)

$$m_1(t)=e^{-pd_1(t-T)}.$$

Thus, (i) is proved.
(2) The case of $1-\frac{H}{\beta_t M_t}<1-\frac{1}{\beta_t}+\frac{r_1-r_0}{\beta_t(1-p)\sigma_1^2}<1-\frac{L}{\beta_t M_t}$.
In this case, the supremum of (16) is attained at

$$\lambda_t^*=1-\frac{1}{\beta_t}+\frac{r_1-r_0}{\beta_t(1-p)\sigma_1^2}$$

and it is $\in\Lambda$. Substituting it into (16), we obtain

$$m_2'(t)+pd_2m_2(t)=0$$

subject to the condition $m_2(T)=1$. Then we can derive the following solution to (16)

$$m_2(t)=e^{-pd_2(t-T)}.$$

Thus, (ii) is proved.
(3) The case of $1-\frac{L}{\beta_t M_t}\leq 1-\frac{1}{\beta_t}+\frac{r_1-r_0}{\beta_t(1-p)\sigma_1^2}$.
In this case, the symmetry axis $\lambda_t=1-\frac{1}{\beta_t}+\frac{r_1-r_0}{\beta_t(1-p)\sigma_1^2}$ of the downward sparabola (18) is on the right side of the safe area $[1-\frac{H}{\beta_t M_t},1-\frac{L}{\beta_t M_t}]$, and (18) is monotone increasing for λ_t in $[1-\frac{H}{\beta_t M_t},1-\frac{L}{\beta_t M_t}]$, so (16) attains its maximum at

$$\lambda_t^*=1-\frac{L}{\beta_t M_t}.$$

This allows us to replace (16) by

$$m_3'(t)+pd_3m_3(t)=0$$

subject to the condition $m_3(T)=1$. Then we can derive the following solution to (16)

$$m_3(t)=e^{-pd_3(t-T)}.$$

Thus, (iii) is proved. □

Combining Theorem 1 with Theorem 3, we obtain

Corollary 1. *The optimal transfer strategy λ_t^* is*

$$\lambda_t^* = \begin{cases} 1-\frac{H}{\beta_t M_t} & if\ 1-\frac{1}{\beta_t}+\frac{r_1-r_0}{\beta_t(1-p)\sigma_1^2} \le 1-\frac{H}{\beta_t M_t}; \\ 1-\frac{1}{\beta_t}+\frac{r_1-r_0}{\beta_t(1-p)\sigma_1^2} & if\ 1-\frac{H}{\beta_t M_t} < 1-\frac{1}{\beta_t}+\frac{r_1-r_0}{\beta_t(1-p)\sigma_1^2} < 1-\frac{L}{\beta_t M_t}; \\ 1-\frac{L}{\beta_t M_t} & if\ 1-\frac{L}{\beta_t M_t} \le 1-\frac{1}{\beta_t}+\frac{r_1-r_0}{\beta_t(1-p)\sigma_1^2}. \end{cases}$$

3.3.2. *Case for [M-2].* Now, for [M-2], we need to solve

$$J_t+\sup_{\mu_t\in\Pi}\{[r_0\beta_t y + r_0\mu_t(1-\beta_t)y + r_1(1-\mu_t)(1-\beta_t)y]J_y + \frac{1}{2}y^2(1-\mu_t)^2(1-\beta_t)^2\sigma_1^2 J_{yy}\} = 0 \tag{19}$$

with the boundary condition

$$J(T,y) = \frac{y^p}{p}. \tag{20}$$

Theorem 3.4. *There exists a classical solution W to (19) with the boundary condition (20). The solution $W(t,y)$ and the corresponding μ_t^* are as follows:*
(i) If $1-\frac{r_1-r_0}{(1-\beta_t)(1-p)\sigma_1^2} \le \frac{L-\beta_t M_t}{(1-\beta_t)M_t}$, then

$$W(t,y) = n_1(t)\frac{y^p}{p},\ n_1(t) = e^{-pq_1(t-T)},\ q_1 = r_1 + \frac{(r_0-r_1)L}{M_t} + \frac{(M_t-L)^2}{2M_t^2}\sigma_1^2(p-1)$$

and

$$\mu_t^* = \frac{L-\beta_t M_t}{(1-\beta_t)M_t}.$$

(ii) If $\frac{L-\beta_t M_t}{(1-\beta_t)M_t} < 1-\frac{r_1-r_0}{(1-\beta_t)(1-p)\sigma_1^2} < \frac{H-\beta_t M_t}{(1-\beta_t)M_t}$, then

$$W(t,y) = n_2(t)\frac{y^p}{p},\ n_2(t) = e^{-pq_2(t-T)},\ q_2 = r_0 p - \frac{p(r_1-r_0)^2}{2(p-1)\sigma_1^2}$$

and

$$\mu_t^* = 1-\frac{r_1-r_0}{(1-\beta_t)(1-p)\sigma_1^2}.$$

(iii) If $\frac{H-\beta_t M_t}{(1-\beta_t)M_t} \le 1-\frac{r_1-r_0}{(1-\beta_t)(1-p)\sigma_1^2}$, then

$$W(t,y) = m_3(t)\frac{y^p}{p},\ n_3(t) = e^{-pq_3(t-T)},\ q_3 = r_1 + \frac{r_0-r_1}{M_t}H + \frac{(M_t-H)^2}{2M_t^2}\sigma_1^2(p-1)$$

and

$$\mu_t^* = \frac{H-\beta_t M_t}{(1-\beta_t)M_t}.$$

Proof. First we suppose that (19)-(20) has a solution W, which has the following form:

$$W(t,x) = n(t)\frac{y^p}{p}, \tag{21}$$

where $n(t)$ is a suitable function which we can guess is an exponential function, and the boundary condition implies that $n(T) = 1$. Inserting this trivial solution

(21) into (19) results in

$$n^{'}(t)\frac{1}{p}+n(t)\sup_{\mu_t\in\Pi}\{\frac{1}{2}(p-1)\sigma_1^2(1-\beta_t)^2\mu_t^2+[(r_0-r_1)(1-\beta_t)-(p-1)(1-\beta_t)^2\sigma_1^2-(p-1)\beta_t^2\sigma_1^2]\mu_t+k\}=0, \tag{22}$$

where $k = r_0\beta_t + r_1 - r_1\beta_t + \frac{1}{2}(p-1)\sigma_1^2(1-\beta_t)^2$.
For μ_t^* without restriction and $p < 1$, the supremum of (22) is attained at

$$\mu_t^* = 1 - \frac{r_1 - r_0}{(1-\beta_t)(1-p)\sigma_1^2}; \tag{23}$$

we also can take $W_y = n(t)\frac{y^{p-1}}{p}$ and $W_{xx} = n(t)(p-1)\frac{y^{p-2}}{p}$ into (11), then (23) can also be obtained. But if the right hand side of (23) is less than $\frac{L-\beta_t M_t}{(1-\beta_t)M_t}$ or larger than $\frac{H-\beta_t M_t}{(1-\beta_t)M_t}$, then we have to truncate it by $\frac{L-\beta_t M_t}{(1-\beta_t)M_t}$ and $\frac{H-\beta_t M_t}{(1-\beta_t)M_t}$, respectively. Thus we need to handle the following three cases:
(1) The case of $1 - \frac{r_1-r_0}{(1-\beta_t)(1-p)\sigma_1^2} \le \frac{L-\beta_t M_t}{(1-\beta_t)M_t}$.
In this case, the symmetry axis $\mu_t = 1 - \frac{r_1-r_0}{(1-\beta_t)(1-p)\sigma_1^2}$ of the downward sparabola

$$\frac{1}{2}(p-1)\sigma_1^2(1-\beta_t)^2\mu_t^2+[(r_0-r_1)(1-\beta_t)-(p-1)(1-\beta_t)^2\sigma_1^2-(p-1)\beta_t^2\sigma_1^2]\mu_t+k = z \tag{24}$$

is on the left side of the safe area $[\frac{L-\beta_t M_t}{(1-\beta_t)M_t}, \frac{H-\beta_t M_t}{(1-\beta_t)M_t}]$, and (24) is monotone decreasing for μ_t in $[\frac{L-\beta_t M_t}{(1-\beta_t)M_t}, \frac{H-\beta_t M_t}{(1-\beta_t)M_t}]$, so (22) attains its maximum at

$$\mu_t^* = \frac{L-\beta_t M_t}{(1-\beta_t)M_t}.$$

This allows us to replace (22) by

$$n_1^{'}(t) + pq_1 n_1(t) = 0$$

subject to the condition $n_1(T) = 1$. Then we can derive the following solution to (22)

$$n_1(t) = e^{-pq_1(t-T)}.$$

Thus, (i) is proved.
(2) The case of $\frac{L-\beta_t M_t}{(1-\beta_t)M_t} < 1 - \frac{r_1-r_0}{(1-\beta_t)(1-p)\sigma_1^2} < \frac{H-\beta_t M_t}{(1-\beta_t)M_t}$.
In this case, the supremum of (22) is attained at

$$\mu_t^* = 1 - \frac{r_1 - r_0}{(1-\beta_t)(1-p)\sigma_1^2}$$

and it is $\in \Pi$. Substituting it into (22), we obtain

$$n_2^{'}(t) + pq_2 n_2(t) = 0$$

subject to the condition $n_2(T) = 1$. Then we can derive the following solution to (22)

$$n_2(t) = e^{-pq_2(t-T)}.$$

Thus, (ii) is proved.
(3) The case of $\frac{H-\beta_t M_t}{(1-\beta_t)M_t} \le 1 - \frac{r_1-r_0}{(1-\beta_t)(1-p)\sigma_1^2}$.
In this case, the symmetry axis $\mu_t = 1 - \frac{r_1-r_0}{(1-\beta_t)(1-p)\sigma_1^2}$ of the downward sparabola (24)

is on the right side of the safe area $[\frac{L-\beta_t M_t}{(1-\beta_t)M_t}, \frac{H-\beta_t M_t}{(1-\beta_t)M_t}]$, and (24) is monotone increasing for μ_t in $[\frac{L-\beta_t M_t}{(1-\beta_t)M_t}, \frac{H-\beta_t M_t}{(1-\beta_t)M_t}]$, so (22) attains its maximum at

$$\mu_t^* = \frac{H-\beta_t M_t}{(1-\beta_t)M_t}.$$

This allows us to replace (22) by

$$n_3^{'}(t) + pq_3 n_3(t) = 0$$

subject to the condition $n_3(T) = 1$. Then we can derive the following solution to (22)

$$n_3(t) = e^{-pq_3(t-T)}.$$

Thus, (iii) is proved.

□

Combining Theorem 2 with Theorem 4, we obtain

Corollary 2. *The optimal transfer strategy μ_t^* is*

$$\mu_t^* = \begin{cases} \frac{L-\beta_t M_t}{(1-\beta_t)M_t} & if\ 1-\frac{r_1-r_0}{(1-\beta_t)(1-p)\sigma_1^2} \leq \frac{L-\beta_t M_t}{(1-\beta_t)M_t}; \\ 1-\frac{r_1-r_0}{(1-\beta_t)(1-p)\sigma_1^2} & if\ \frac{L-\beta_t M_t}{(1-\beta_t)M_t} < 1-\frac{r_1-r_0}{(1-\beta_t)(1-p)\sigma_1^2} < \frac{H-\beta_t M_t}{(1-\beta_t)M_t}; \\ \frac{H-\beta_t M_t}{(1-\beta_t)M_t} & if\ \frac{H-\beta_t M_t}{(1-\beta_t)M_t} \leq 1-\frac{r_1-r_0}{(1-\beta_t)(1-p)\sigma_1^2}. \end{cases}$$

As can be seen from Corollary 1 and Corollary 2, when (17) and (23) are not in Λ and Π, then the transfer ratio is not related with risk preference p. This also fully discloses the importance of the safe area for companies. Here, managers' risk preference has no influence on the companies' decision.

4. Illustrative examples.

4.1. Illustration of solutions to [M-1] and [M-2].

We first give an example to demonstrate the feasibility of the CTM. In order to compute the model, we need to know the daily average returns and variance of the risky securities, and also the parameter p. We randomly select four profitable stocks from the Shanghai and Shenzhen stock markets, China Vanke(000002), *ST Tianwei (600550), Pret (002324) and Tianxing Instrument and Meter (000710), denoted by A, B, C and D, respectively, and then get their daily average yield and variance as shown in Table 1 using the daily close data from 1 January 2012 to 30 August 2014.

TABLE 1. The daily average yield and variance

stock number	daily average yield	variance
A	0.000399	0.000437
B	0.000433	0.000687
C	0.000443	0.0019
D	0.000463	0.00055

As managers usually are cautious about the cash balance, one can take $p = 0.1$. We suppose the enterprise' cash flows range is $[1000, 7000] = [L, H]$, which can be obtained by the mangers' estimation.

Table 2. The optimal cash holdings in case 1: $X_t > H$

stock number	invest ratio λ_t^* in (17)	$(1-\lambda_t^*)X_t$	optimal cash holdings
A	0.756891	1944.871	1944.871
B	0.542302	3661.586	3661.586
C	0.04379	7649.684	7000
D	0.815414	1476.685	1476.685

If the cash balance $M_t = 10000$ and $\beta_t = 0.8$ at time t and $r_0 = 0.03/365$, then $X_t = 8000 > 7000$, so the managers will buy some stocks to achieve the wealth utility maximization. The cash holding policy for different stocks can be seen in Table 2.

In particular, Table 2 shows that if the managers decide to buy stock C , then the cash holdings is 7649.684 . It is still higher than the upper bound H. According to Theorem 3, we can get the optimal cash holdings is 7000. Alternatively, if the managers will buy stock B, then the whole process can be expressed by Figure 1. From Figure 1, when the cash balance exceeds the upper bound H to point E, and the managers can spend 4338.414 fund to buy stock B, then point F is the optimal cash holding point.

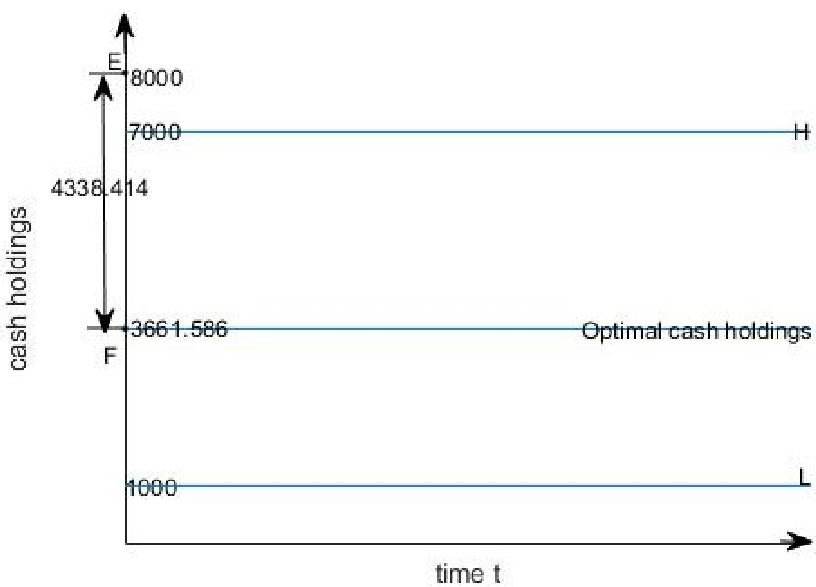

Figure 1. optimal cash holding model in case 1

Alternatively, if the cash balance $M_t = 10000$ and $\beta_t = 0.08$ at time t, then $X_t = 800 < 1000$, and the managers will sell some stocks to restore the cash holdings to the "safe level". Then the conversion policy and the cash holding policy for different stocks can be seen in Table 3.

In particular, Table 3 shows that if the managers decide to sell stock C, then the cash holdings is 7890.0104, which is higher than the upper bound H, where, according to Theorem 4, we can get the optimal cash holdings is 7000. Now if the managers will sell stock B, then the whole process can be expressed by Figure 2.

TABLE 3. The optimal cash holdings in case 2: $X_t < L$

stock number	conversion ratio μ_t^* in (23)	$\mu_t^* R_t + X_t$	optimal cash holdings
A	0.124443	1944.8711	1944.8711
B	0.383288	4326.2458	4326.2458
C	0.770653	7890.0104	7000
D	0.163794	2306.9047	2306.9047

From Figure 2, when the cash holdings is 800, which is down to the lower bound L to point M, and the managers will sell 3526.246 fund of stock B and point N is the optimal cash holding point.

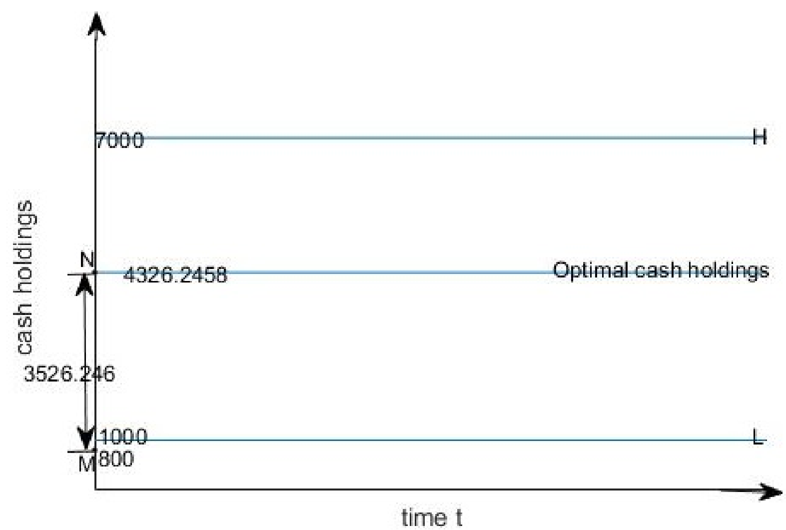

FIGURE 2. optimal cash holding model in case 2

4.2. **Illustration of the optimal cash holdings with different p values with the power utility.** Taking stock C as an example, we consider the effect of different p values in the HARA power utility on the cash holding decision-making. We still suppose the safe cash flows range is [1000,7000] and the cash balance is 10000 and β_t is 0.8 at time t. Then the cash holding policy with different p values can be seen in Table 4, which is also plotted in Figure 3.

From Figure 3 we can see clearly that the larger the parameter p, the lower the optimal allocation to cash until $p \geq 0.9$. In particular, from Table 4, when p is 0.9, λ_t^* is greater than 1, where the manager is a serious risk seeker, even willing to borrow money to invest in risky asset. However, in order to keep adequate working capital, the optimal cash holdings should be 1000 according to Theorem 3.

4.3. **Illustration of the comparison between the CTM model and the Miller-Orr model.** By the Miller-Orr (1966) model, we get the optimal cash holding using $M^* = \frac{4}{3}(3\gamma\sigma^2/4\upsilon)^{\frac{1}{3}} + L$, where M^* is the firm's optimal average yield, γ is the transaction cost, σ^2 is the variance of daily cash flows, υ is the marginal and

TABLE 4. The optimal cash holdings with different p values

p	invest ratio λ_t^* in (17)	$(1-\lambda_t^*)X_t$	optimal cash holdings
0.1	0.04379	7649.684	7000
0.2	0.04672	7626.262	7000
0.3	0.08911	7287.156	7000
0.4	0.14562	6835.016	6835
0.5	0.22475	6202.019	6202
0.6	0.34344	5252.523	5252
0.7	0.54125	3670.031	3670
0.8	0.93687	505.047	1000
0.9	2.12374	-8989.91	1000

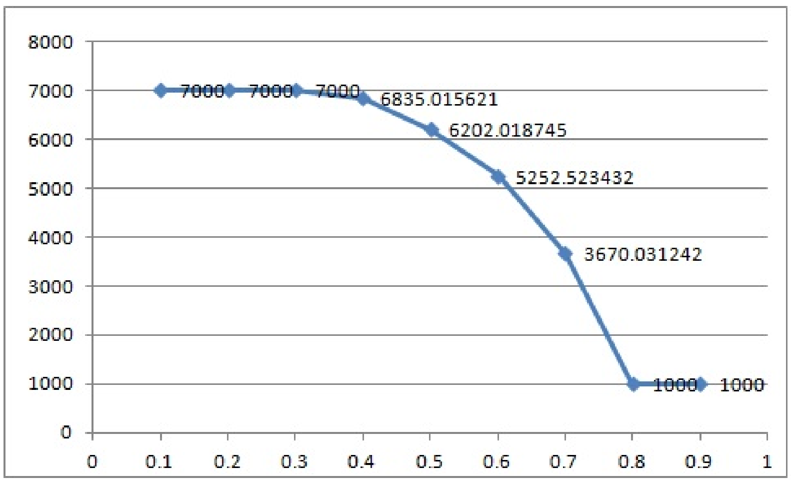

FIGURE 3. optimal allocation to cash for different p values

average yield. Now suppose for one company that $\gamma = 50$ and $\sigma = 800$. Then, for stocks A, B, C and D, we can get $M_A^* = 6224.181$, $M_B^* = 6083.670$, $M_C^* = 6045.156$, $M_D^* = 5971.439$. Obviously, the Miller-Orr model ignores the risk of risky assets and the managers' preferences, so the results are close to each other. However, in Subsection 4.1 above, our CTM model considers these factors. The CTM model decides the cash holding policy relying on the managers' preferences and the return and variance of the risky assets. So the CTM model solves the cash holding problem from a different perspective of the Miller-Orr model. In the future, we can consider to combine the two models to solve the problem.

5. **Conclusion.** We summarise the main contributions of this paper as follows.

Firstly, we have obtained the optimal cash holdings formulas based on [M-1] and [M-2] in a general form using the HJB equation derived by the dynamic programming principle. For the cases of specific power utility functions, we can deduce the optimal decisions in the explicit forms in Corollary 1 and Corollary 2. That is, we arrive at the optimal cash holdings $(1-\lambda_t^*)X_t$ or $(X_t + \mu_t^* R_t)$ at time t. This result can be regarded as an extension from a different perspective of the work of Miller and Orr (1966).

Secondly, as can be seen from Corollary 1 and Corollary 2, the conversion ratio λ_t^* and μ_t^* are both related with the parameter p, the coefficient of the relative risk aversion. The larger the parameter p, the lower degree of the relative risk aversion, which means the managers more likely to pursue risks and thus the higher the ratio investing on risky assets.

Thirdly, the CTM avoids to characterize the distribution of cash flows. The result shows that the managers making decisions only need to consider the return and risk of the investment objective. This conclusion makes up the Miller-Orr model's lack of considering the securities' risk.

Finally, we demonstrate that the CTM is feasible, which the managers can use to make decisions.

There are several topics worthy further research. In this paper, we take specific power utility functions to get the analytical solutions, so in future we may try to obtain general numerical solutions based on general utility functions. Furthermore, the CTM model does not consider the variance of the daily cash flows, and we may consider the cash holding policy by combining the CTM model with the Miller-Orr model.

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