A Two-Period Model with Portfolio Choice: Understanding Results from Different Solution Methods*

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Abstract

Using a stylized two-period model we compare portfolio solutions from two local solution approaches – the approach of Judd and Guu (2001) and the approach of Devereux and Sutherland (2010, 2011) – with the true nonlinear portfolio solution.

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1 Introduction

We present a stylized two-period model of portfolio choice and parameterize it to some key moments of returns on aggregate stock market indices. We use the model to compare the true nonlinear portfolio solution with the solutions from two approaches that belong to the class of local approximation methods, developed by Judd and Guu (2001, hereafter 'JG') and Devereux and Sutherland (2010, 2011, hereafter 'DS').

The DS solution approach has received considerable attention in solving portfolio problems in dynamic macroeconomic models in the recent past.[1] While the two-period setting of the present paper ignores the main advantages of the DS method, which lie in obtaining portfolio solutions in *dynamic* settings (possibly in environments with many states variables), it nevertheless is able to shed light on some of its properties.[2]

While DS and JG solution approaches are fundamentally similar, as they both are based on a Taylor-series approximation around the non-stochastic steady state, we find important differences between the results that they produce (as currently implemented). Devereux and Sutherland (2010, 2011) are mainly interested in incorporating the portfolio problem into dynamic macroeconomic models, and so they concentrate on approximating the solution in the direction of the model's state variables, at the same time neglecting the effect of the size of the shocks.[3] As a result, we find that in our two-period model, their approach delivers the constant portfolio solution independent of the size of the shocks.

At the same time, we show that the true solution generally depends on the size of uncertainty, with skewness, kurtosis and higher-order moments of the distribution of underlying shocks affecting the results. The JG bifurcation method is able to capture this dependency: its zero-order portfolio solution component coincides with DS, while its higher-order solutions components account for variations of the size of uncertainty. Even the second-order JG solution is able to account for the effects of skewness and kurtosis of equity returns on the solution.

We show that the resulting discrepancy between the DS and JG solutions can be nontrivial. This makes extending the DS approach to take into account the effect of the size of uncertainty a valuable exercise.[4]

2 Model

The world consists of two countries. In each there lives a representative investor for two periods, consuming a single consumption good in period 2. In period 1 investors decide on a portfolio over two assets: equity – a claim on the total world's output –, and a risk-free bond. The bond yields one unit of period-2-consumption and serves as numeraire, i.e., the period 1 bond price is normalized to 1. Each share has price p in period 1 and has a random period 2

[1] Together with the contributions of Tille and van Wincoop (2007) and Evans and Hnatkovska (2005, 2012).

[2] We perform a more extensive evaluation of the DS method, in a *dynamic* setting, in a companion paper, Rabitsch et al. (2014).

[3] This is for simplicity of exposition. It also squares with the intuition that in standard macro models the size of the shocks does not affect the solution up to the first-order of approximation (see Schmitt-Grohé and Uribe (2004) and Jin and Judd (2002)). However, the JG solutions shows that this intuition does not apply to a model with portfolio choice.

[4] Since both approaches are based on Taylor series approximations, the intuition suggests that this should be possible. We thank the referee for this point.

value, $Y = 1 + \varepsilon z$. We assume $E\{z\} = 0$ and $E\{z^2\} = 1$. In addition, we assume that the support for z is bounded from below, so that $Y > 0$ for all ε and z.

Each investor i starts with b_i^0 units of bonds and θ_i^0 shares of equity. Investors' utility is given by $u_i(C_i) = C_i^{1-\gamma_i}/(1-\gamma_i)$. C_i denotes investor i's period- 2-consumption which equals her final wealth. Without loss of generality, we assume $\theta_1^0 + \theta_2^0 = 1$; this implies that z denotes aggregate risk in the world endowment Y. Let θ_i be the shares of equity and b_i bonds held by investor i after trading in period 1. Investor i solves:

$$\begin{aligned}
&\max_{\theta_i, b_i} Eu_i(C_i) \\
&\text{s.t.:} \\
\theta_i^0 p + b_i^0 &= \theta_i p + b_i \quad \text{(budget constraint in period 1)} \\
C_i &= \theta_i Y + b_i, \forall Y \quad \text{(budget constraints in period 2)}
\end{aligned}$$

Market-clearing implies $\theta_1 + \theta_2 = 1$, $b_1 + b_2 = 0$. Define $\theta = \theta_1$; then $\theta_2 = 1 - \theta$. Also, denote $b_1 = b = -b_2$. Similarly, initial endowments $\theta^0 = \theta_1^0$, $\theta_2^0 = 1 - \theta^0$, and $b_1^0 = b^0 = -b_2^0$. The model's equilibrium conditions can be reduced to a system of two equations in θ and p:

$$\begin{aligned}
H(\theta(\varepsilon), p(\varepsilon), \varepsilon) = \qquad (1) \\
\begin{bmatrix} E\left[u_1'(\theta Y + b^0 + (\theta^0 - \theta)p)(Y - p)\right] \\ E\left[u_2'((1-\theta)Y - b^0 - (\theta^0 - \theta)p)(Y - p)\right] \end{bmatrix} = \mathbf{0}.
\end{aligned}$$

2.1 Portfolio solution methods

We comment only on the main points of the various portfolio solution approaches, and refer the interested reader to the appendix for further documentation. To obtain the *nonlinear (quadrature) portfolio solution* in this simple economy, called 'true solution' hereafter, we approximate the expectations operator using quadrature methods and solve system (1) using a nonlinear equations solver.

To apply the *Devereux and Sutherland solution approach*, we use DS' notation convention and express portfolio holdings in terms of assets in zero-net supply, $\alpha_t = [\alpha_e; \alpha_b] = \left[(\theta - \theta^0)p; b - b^0.\right]$. Following Schmitt-Grohé and Uribe (2004) and Jin and Judd (2002) we can think of the true policy function for α_t, in a recursive economy, as a function that depends on the model's state variables, x_t, and on a parameter that scales the variance-covariance matrix of the model's exogenous shock processes, ε; that is, $\alpha_t = \alpha(x_t, \varepsilon)$. In contrast to a standard Taylor series expansion to $\alpha_t = \alpha(x_t, \varepsilon)$, the DS approximate portfolio solution, as described in Devereux and Sutherland (2010, 2011), considers only how variations in the model's state variables, x_t, affect the optimal portfolio solution, but ignores the effect of variations in the size of uncertainty, ε. Because our model is static (we have $\hat{x} = 0$), the portfolio solution under DS is:

$$\overline{\alpha}_e = \frac{\gamma_2 - \gamma_1}{\gamma_1(1-\theta^0) + \gamma_2\theta^0}\theta^0(1-\theta^0). \qquad (2)$$

Or, for θ:

$$\theta = \theta^0 + \frac{\alpha_e}{p}, \text{ where } \alpha_e = \overline{\alpha}_e. \qquad (3)$$

The property of α_e which is key here, is that it is invariant to the size of the shock z, and as a result, of any other statistical properties (skewness, kurtosis etc.).

To obtain the *Judd-Guu portfolio approach*, using bifurcation methods, we closely follow the steps outlined in Judd and Guu (2001). Unlike the DS approach, the JG solution depends on the size of uncertainty, and, as result, on higher-order moments of assets' returns. Namely, the first-order terms of JG's approximate solution depend on the returns' skewness, while the second-order terms depend on their kurtosis.

3 Results

Consider a setup of countries with identical initial endowments, $b_i^0 = 0$ and $\theta_i^0 = 0.5$ for country $i = 1, 2$, but assume country 2 is twice as risk averse, reflected by $\gamma_1 = \gamma_2/2$. In our numerical examples, we take the robust empirical stylized fact of positive and non-normally distributed equity premia seriously. We model world output endowment, $Y = 1+\varepsilon z$, through a Normal-inverse Gaussian (N.I.G.) distribution.[5] This gives us enough flexibility to target mean, standard deviation, skewness and kurtosis of equity (excess) returns in our model, to the observed moments of excess returns of aggregate stock market indices reported in Guidolin and Timmermann (2008), for Pacific-ex-Japan, United Kingdom, United States, Japan, Europe-ex-UK, and World, based on monthly MSCI indices – repeated in columns 1-4 of Table 1.[6]

Figure 1 plots the portfolio solution for country 1's equity share, θ, as a function of the size of uncertainty ε, for two illustrative examples: 'United Kingdom' (panel A) and 'Pacific-ex-Japan' (panel B). The first region's MSCI displays positive, the latter's negative skewness; both display substantial kurtosis.

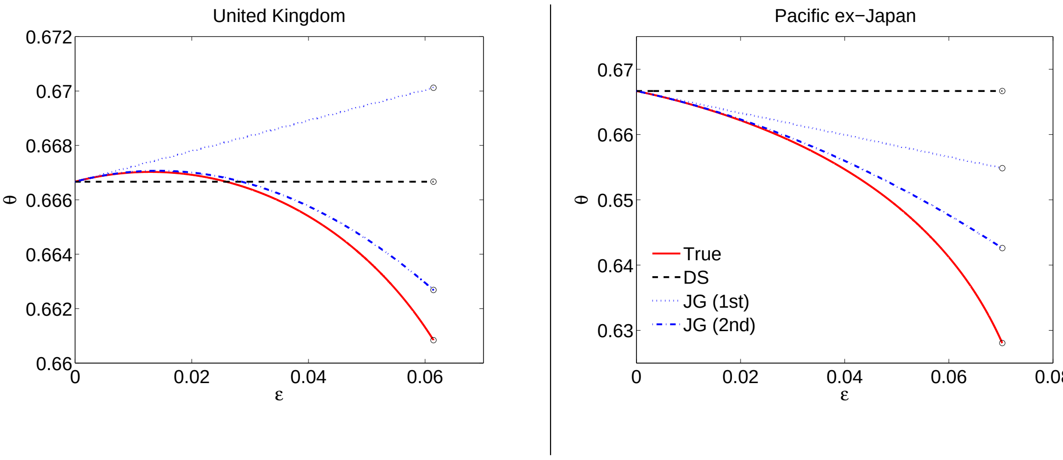
The solid red line displays the true portfolio solution: as country 1 is less risk averse, it chooses to hold a higher share of equity than initially endowed with ($\theta > \theta^e = 0.5$), which it finances by going short in debt. Also, the solution for θ depends on the size of uncertainty: for the UK case we observe that country 1's optimal share in equity initially increases, and then decreases, as ε increases. For Pacific-ex-Japan θ continuously decreases.

The portfolio solution obtained by the Judd-Guu approach can help understand the mechanisms that drive these results in more detail. The positive skewness of the UK's MSCI return index (0.75) leads to a positive slope of the first-order (linear) Judd-Guu solution: positive skewness means shifting more weight to 'good' outcomes, such that an investor would demand more of the risky asset. Positive skewness therefore works to increase country 1's optimal equity holdings, θ, as ε increases. While this logic applies to both investors JG show that the

[5] The N.I.G. distribution has experienced recent interest in the finance literature because of its flexibility in capturing non-normal properties of asset pricing data (see e.g. Colacito et al. (2012)).

[6] In particular, for each MSCI index we consider, we choose 4 parameters of the N.I.G. distribution to make sure that $E\left\{z^3\right\}$ and $E\left\{z^4\right\}$ match the observed skewness and kurtosis of that MSCI index' returns from the data, and that $E\left\{z\right\} = 0$ and $E\left\{z^2\right\} = 1$ (the normalization assumed by Judd and Guu (2001), which we follow here). Since $E\left\{z^2\right\} = 1$, we control the volatility of the return process through the choice of ε. In our model the variance of gross equity return, R_e, is given by $var(R_e) = var\left(\frac{1+\varepsilon z}{p}\right) = \frac{\varepsilon^2}{p^2} = \varepsilon^2\left[E\left(R_e\right)\right]^2$, because $E[z^2] = 1$ and $[E\left(R_e\right)] = 1/p$. Using this result, we set $\varepsilon = \frac{std(r_e^{data})}{\left[E\left(r_e^{data}\right)+1\right]}$, where r_e^{data} is the net return in the data. Finally, we pick our final free parameter, γ_2, to match the observed mean excess equity return.

Figure 1: Equity shares held by country 1 investor. Panel A and B refer to the parameterizations for the UK and Pacific stock market facts respectively. Circles correspond to the value of ε used in the calibration.



strength with which equity demand increases in such case depends on investors' relative 'skew-tolerance'. For the CRRA preference specification we use, skew-tolerance is always larger for the less risk-averse country, implying that country 1's appetite for taking risk increases more strongly and its chosen equity position goes up under positive skewness as ε increases.[7] Panel B, 'Pacific-ex-Japan', provides a different example: returns display negative skewness (-2.3). This implies that the return distribution is more heavily shifted towards 'bad' outcomes, so investors demand less of the risky asset. Since the skew-tolerance coefficient continues to be higher for country 1, but now, because of negative skewness, multiplies a negative number $E\left(z^3\right)$, the slope from the first-order part of the JG solution is negative: the less risk averse country 1 decreases its holdings of risky assets as ε increases. The second-order JG solution is able to the capture effects of kurtosis on the portfolio solution. MSCI return-indices of both regions are characterized by substantial kurtosis (10.3 for UK, 22.3 for Pacific-ex-Japan). Kurtosis means putting more weight to tail events, so as ε increases, this leads an investor to reduce demand for the risky asset. Again, this logic applies to both investors, the relative strength of this effect depends on investors' relative 'kurtosis-tolerance'. For CRRA preferences kurtosis-tolerance is lower for the less risk-averse country[8], so that the reduction in the demand for risky assets due to (excess) kurtosis is more pronounced for the less risk-averse country: as ε increases, country 1's equity share further decreases.

Finally, the black dashed line in Figure 1 shows results from applying the DS solution approach. The DS solution coincides with the constant (zero-order) component of the Judd-Guu solution. As explained in section 2 the portfolio solution under DS is a function of state

[7] Judd and Guu (2001) define 'skew-tolerance' as $\rho\left(C_i\right) = \frac{1}{2}\frac{u'(C_i)}{u''(C_i)}\frac{u'''(C_i)}{u''(C_i)}$, for country $i = 1, 2$. For CRRA preferences this is given by $\rho\left(C_i\right) = \frac{1}{2}\frac{\gamma_i+1}{\gamma_i}$. Note that in this case $\frac{\partial\rho}{\partial\gamma_i} = -\frac{1}{2\gamma_i^2} < 0$. Therefore, with $\gamma_1 < \gamma_2$ we have that $\rho\left(C_1\right) > \rho\left(C_2\right)$.

[8] JG's definition of 'kurtosis-tolerance' is given by $\kappa\left(C_i\right) = -\frac{1}{3}\frac{u''''(C_i)}{u''(C_i)}\frac{u'(C_i)}{u''(C_i)}\frac{u'(C_i)}{u''(C_i)}$. For CRRA preferences, $\kappa\left(C_i\right) = -\frac{1}{3}\frac{(\gamma_i+1)(\gamma_i+2)}{\gamma_i^2}$. Note that in this case $\frac{\partial\kappa}{\partial\gamma_i} = \frac{\gamma_i+2}{\gamma_i^3} > 0$. Therefore, with $\gamma_1 < \gamma_2$ we have $\kappa\left(C_1\right) < \kappa\left(C_2\right)$.

Asset	Data				$\gamma_2/\gamma_1 = 2$			$\gamma_2/\gamma_1 = 3$		
	Mean,%	SD,%	Skew	Kurt	θ^{DS}	θ^{JG}	θ^{true}	θ^{DS}	θ^{JD}	θ^{true}
UK	0.7503	6.1898	0.7587	10.316	0.6667	0.6626	0.6608	0.7500	0.7449	0.7417
Pacific-ex-Japan	0.3892	7.0538	-2.2723	22.297	0.6667	0.6427	0.6282	0.7500	0.7195	0.6973
World	0.4560	5.174	-0.8711	6.9133	0.6667	0.6607	0.6588	0.7500	0.7424	0.7396
US	0.5415	4.4825	-0.7084	5.9138	0.6667	0.6623	0.6607	0.7500	0.7445	0.7421
Japan	0.3733	6.4830	0.0700	3.5044	0.6667	0.6653	0.6642	0.7500	0.7483	0.7466
Europe-ex-UK	0.4158	5.0578	-0.5672	4.6124	0.6667	0.6631	0.6620	0.7500	0.7454	0.7439

Table 1: Optimal equity holdings obtained by different portfolio solution methods; model calibrated to (various regions') return data on MSCI aggregate stock market indices by Guidolin and Timmerman (2008).

variables only, and not a direct function of the size of uncertainty, ε. Since, in this simple static model there is no variation in states, the obtained constant solution is not only the zero-order solution, but actually corresponds to the DS solution up to any order.

Table 1 reports the optimal portfolio solutions for all other regions, calibrated to the respective MSCI return indices. Columns 5-8 (9-12) report the true portfolio solutions, the (second-order) JG solution, and the DS solution, for the scenario in which country 2 is twice (three times) as risk averse as country 1. The largest discrepancies emerge for MSCI Pacific-ex-Japan: the difference to the true solution of the equity share obtained by the (second-order) JG solution is -2.31% (-3.14%), the difference of the DS solution -6.13% (-7.56%).

4 Conclusions

In a two-period model, calibrated to match key moments of returns on aggregate stock market indices, we find that DS and JG solutions coincide in the limit where uncertainty vanishes, but else differ. As currently implemented, the DS approach does not account for variations in the size of uncertainty (and its interactions with other statistical properties of returns, such as skewness and kurtosis), unlike JG. We show that the resulting discrepancy between the DS and JG solutions can be non-trivial. This makes extending the DS solution to take into account the effect of the size of uncertainty an interesting direction for future research.

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A Appendix

A.1 Model Equilibrium Conditions

The optimization problem of investor i, for $i = 1, 2$, and market-clearing gives rise to the following system of equilibrium conditions:

$$\begin{array}{ll} \text{(E1): } \lambda_1 = E\left[u_1'(C_1)\right], & \text{(E2): } \lambda_2 = E\left[u_2'(C_2)\right], \\ \text{(E3): } p\lambda_1 = E\left[u_1'(C_1)Y\right], & \text{(E4): } p\lambda_2 = E\left[u_2'(C_2)Y\right], \\ \text{(E5): } C_1 = \theta Y + b, \forall Y, & \text{(E6): } C_2 = (1-\theta)Y - b, \forall Y, \\ \text{(E7): } \theta^0 p + b^0 = \theta p + b, & \end{array}$$

with unknowns: $C_1, C_2, \theta, b, p, \lambda_1, \lambda_2$; λ_i denotes the Lagrange multiplier on investor i's period 1 budget constraint. In addition, denote the return on equity by $R_e = Y/p$, bond return $R_b = 1$, and excess return, $R_x = R_e - R_b$. The above equilibrium conditions can be further reduced to a system of two equations in variables θ and p, which correspond to equation (1) in the main text, which is restated below for convenience.

$$H\left(\theta\left(\varepsilon\right), p\left(\varepsilon\right), \varepsilon\right) = \tag{4}$$
$$\begin{bmatrix} E\left[u_1'(\theta Y + b^0 + (\theta^0 - \theta)p)(Y - p)\right] \\ E\left[u_2'((1-\theta)Y - b^0 - (\theta^0 - \theta)p)(Y - p)\right] \end{bmatrix} = \mathbf{0}.$$

Asset	NIG parameters						
	μ	α	β	δ	ε	γ_2^{*}	γ_2^{**}
Pacific-ex-Japan	0.1439	0.4163	−0.1745	0.3114	0.0703	0.886	1.120
UK	−0.1138	0.6932	0.1171	0.6638	0.0614	2.969	3.920
World	0.2903	1.0839	−0.3176	0.9473	0.0515	2.344	3.096
US	0.3104	1.2331	−0.3352	1.0990	0.0446	3.750	4.950
Japan	−0.1406	2.4628	0.1411	2.4507	0.0646	2.294	3.051
Europe-ex-UK	0.4776	1.7463	−0.5250	1.5150	0.0504	2.320	3.079

*– for the case $\frac{\gamma_2}{\gamma_1} = 2$, **– for the case $\frac{\gamma_2}{\gamma_1} = 3$

Table 2: Calibrated parameter values

A.2 Details of the Nonlinear (Quadrature) Solution

To obtain the nonlinear (quadrature) portfolio solution in this simple economy, we approximate the expectations operator using quadrature methods and simply solve the system given in (4) using a nonlinear equations solver.[9]

The key step in obtaining the quadrature solution is to replace the integrals in (4) with finite sums. We do so by using the Gauss-Chebyshev quadrature. We assume that z follows a truncated normal inverse Gaussian distribution (NIG). The NIG distribution is completely characterized by 4 parameters (μ_{nig}, α_{nig}, β_{nig} and δ_{nig}). This allows us to match the first 4 moments of the returns from the data. In addition, we assume that the support of z is bounded from below, $z >= \underline{Z}$, so that $Y > 0$ for all values of ε that we consider. In practice, we assume that $\underline{Z} = -10$ in all cases, except for when we consider MSCI Pacific-ex-Japan with $\gamma_2 = 3\gamma_1$, where we assume that $\underline{Z} = -9$. This ensures that $C_1 > 0$ and $C_2 > 0$ for all values of ε that we consider. After fixing $\underline{Z}$ and some large upper bound $\bar{Z}$[10], we set the values for μ_{nig}, α_{nig}, β_{nig} and δ_{nig}, apply the Gauss-Chebyshev quadrature with 1000 nodes[11] to compute the resulting first 4 moments, and change values of μ_{nig}, α_{nig}, β_{nig} and δ_{nig} until we obtain $E[z] = 0$, $E[z^2] = 1$, and $E[z^3]$ and $E[z^4]$ that match the skewness and kurtosis of assets' returns in the data.

After this, we solve the system in (4) with a non-linear solver on a fine grid over $[\underline{\varepsilon}, \bar{\varepsilon}_i]$, where $\bar{\varepsilon}_i$ corresponds to the standard deviation of the asset i's returns in the data.

A.3 Details of the Devereux-Sutherland Solution

The contributions by Devereux and Sutherland (2011, 2010) provide easy-to-apply methods to obtain approximate portfolio solutions in a *dynamic* stochastic GE model. While we apply their method in a model that is essentially static in the sense that there is no variation in state variables, it is indicative to reflect first on how their method works in the general case of a dynamic setting. In particular, denote with α_t the true (unknown) function of optimal holdings of any asset that is zero-net supply.[12] In the above contributions, DS show that a

[9]The nonlinear solution in this static economy is simple to obtain. In more general, dynamic settings nonlinear methods providing a globally valid approximation for portfolios is substantially more complex. Such global portfolio solution methods have been proposed by Kubler and Schmedders (2003).

[10]In practice, we set $\bar{Z} = 30$, and check that the results are not sensitive to changing this value.

[11]We check that the results are not sensitive the the number of quadrature nodes selected as well.

[12]DS' exposition of their method is in terms of assets in zero-net supply. This is not in any way restrictive. For assets in positive net supply, such as equities, this can be easily achieved by defining portfolio positions in

zero-order (first-order) approximation to the true portfolio solution can be obtained from a second (third) order Taylor series expansion to the model's portfolio optimality conditions, in conjunction with a first (second) order Taylor series expansion to the model's other optimality and equilibrium conditions. Applying these steps one obtains an approximate portfolio solution of the format:

$$\alpha_t = \bar{\alpha} + \alpha' \widehat{x}_t. \tag{5}$$

where $\bar{\alpha}$ is the zero-order (constant) part of the solution, α' is a vector of the first-order coefficients, x_t is the vector of the model's state variables, and $\widehat{x}_t$ refers to the state variables expressed as (log-)deviations from their steady state values.

DS also state that their solution principle, which builds up on earlier work by Samuelson (1970), could be successively applied to higher orders: to obtain an n-th order accurate portfolio solution, one needs to approximate the portfolio optimality conditions up to order $n+2$, in conjunction with an approximation to the model's other optimality and equilibrium conditions of order $n+1$. E.g., going one order higher, one would obtain the approximate portfolio solution as $\alpha_t = \bar{\alpha} + \alpha' \widehat{x}_t + \frac{1}{2} \widehat{x}_t' \alpha'' \widehat{x}_t$.

It is important to realize that the expression in equation (5) is, however, not the same as what would result from a Taylor series expansion of the true policy function α_t. Following Schmitt-Grohé and Uribe (2004) and Jin and Judd (2002) we can think of the true policy function in a recursive economy as a function that depends on the model's state variables, x_t, and on a parameter that scales the variance-covariance matrix of the model's exogenous shock processes, ε; that is, $\alpha_t = \alpha(x_t, \varepsilon)$. A Taylor series to policy function α_t, evaluated at approximation points $x_t = \overline{x}$ and $\varepsilon = 0$, would then result in:

$$\alpha_t = \alpha(\overline{x}, 0) + \alpha_x(\overline{x}, 0)\widehat{x}_t + \alpha_\varepsilon(\overline{x}, 0)\varepsilon + \frac{1}{2}\widehat{x}_t' \alpha_{xx}(\overline{x}, 0)\widehat{x}_t + \alpha_{x\varepsilon}(\overline{x}, 0)\widehat{x}_t \varepsilon + \frac{1}{2}\alpha_{\varepsilon\varepsilon}(\overline{x}, 0)\varepsilon^2 + ... \tag{6}$$

That is, in contrast to the Taylor series expansion in equation (6) the DS approximate portfolio solution does only consider how variations in the model's state variables affect the optimal portfolio solution, but ignores the effect of variations in the size of uncertainty.[13,14]

Let us return to finding the DS portfolio solution in our two-period model. To apply their method, it is convenient to reformulate the portfolio positions in zero-sum value terms. In our model, this means defining portfolio positions as:

$$\alpha_e = (\theta - \theta^0)p, \quad \alpha_b = b - b^0.$$

terms of deviations from some initial portfolio endowments, and then multiplying them by their price.

[13]The comparison of the DS solution with equation (6) is simply for reasons of exposition. We are of course not suggesting that an approximate solution to the true unknown portfolio function actually can be obtained by taking a simple Taylor series expansion around the non-stochastic steady state. This is not feasible using standard local approximation methods (using the standard implicit function theorem) – the portfolio is indeterminate both at the non-stochastic steady state and in a first-order approximation of the stochastic setting. This is exactly the problem that the DS method and the JG method have addressed and proposed (different) ways of solving for.

[14]In the general case of a *dynamic* model, this still does not imply that the size of uncertainty cannot have an effect on optimal portfolios. In principle there could be an effect of the size of uncertainty, ε, on the portfolio through the effect of ε on the states themselves. This, however, would only be happening at higher orders, as the (state) variables are not affected by ε at first-order (certainty equivalence) and only through a constant at second-order (see Schmitt-Grohé and Uribe (2004)).

We can then re-write home investor's budget constraints as:

$$
\begin{aligned}
0 &= \underbrace{(\theta-\theta^0)p}_{\alpha_e} + \underbrace{\left(b-b^0\right)}_{\alpha_b} = \alpha_e + \alpha_b = W \\
C_1 &= (\theta-\theta^0)p\underbrace{\frac{Y}{p}}_{=R_e} + \left(b-b^0\right)\underbrace{1}_{=R_b} + b^0 + \theta^0 Y = \alpha_e R_e + \alpha_b R_b + b^0 + \theta^0 Y \\
&= W R_b + \alpha_e \underbrace{(R_e - R_b)}_{=R_x} + b^0 + \theta^0 Y
\end{aligned}
$$

Since the first equation implies that $W = 0$, the equilibrium system can be written as:

$$
\begin{array}{ll}
\text{(E1'): } \lambda_1 = E\left[u_1'(C_1)R_b\right], & \text{(E2'): } \lambda_2 = E\left[u_2'(C_2 R_b)\right], \\
\text{(E3'): } \lambda_1 = E\left[u_1'(C_1)R_e\right], & \text{(E4'): } \lambda_2 = E\left[u_2'(C_2)R_e\right], \\
\text{(E5'): } C_1 = \alpha_e R_x + b^0 + \theta^0 Y, \forall Y, & \text{(E6'): } C_1 + C_2 = Y, \forall Y, \\
\text{(E7'): } \theta^0 p + b^0 = \theta p + b, &
\end{array}
$$

Following Devereux and Sutherland , we obtain the zero-order or constant portfolio solution, $\bar{\alpha}_e$, from the second-order approximation of both countries' first order optimality conditions with respect to portfolio allocations[15], which, once combined, result in an expression that contains only first-order terms of the model's macro variables. The second order approximation to the Euler equations w.r.t. to equity and w.r.t. to the bond, gives:

$$
\begin{aligned}
\bar{C}_1^{-\gamma_1}\bar{R}_b \cdot E\left[\hat{r}_x - \gamma_1\hat{c}_1\hat{r}_x + \frac{1}{2}(\hat{r}_e^2 - \hat{r}_b^2)\right] &= 0 \\
\bar{C}_2^{-\gamma_2}\bar{R}_b \cdot E\left[\hat{r}_x - \gamma_2\hat{c}_2\hat{r}_x + \frac{1}{2}(\hat{r}_e^2 - \hat{r}_b^2)\right] &= 0
\end{aligned}
$$

Combining, we get:

$$
E\left[(\gamma_1\hat{c}_1 - \gamma_2\hat{c}_2)\,\hat{r}_x\right] = 0 \tag{7}
$$

That is, we need first order expressions for consumptions of country 1 and 2, and of excess returns. Those are found by log-linearizing (E5'), (E6') and the definition of excess returns, $R_e = Y/p$, and substituting the $\bar{\alpha}\hat{r}_x$ term with a mean-zero shock ξ in (E5'):

$$
\begin{aligned}
\bar{C}_1\hat{c}_1 &= \xi_1 + \theta_0\bar{Y}\hat{y}_1, \\
\bar{C}_2\hat{c}_2 &= \bar{Y}\hat{y} - \bar{C}_1\hat{c}_1, \\
\hat{r}_x &= \bar{Y}\hat{y}
\end{aligned}
$$

Plugging the above expressions for $\hat{c}_1$, $\hat{c}_1$ and $\hat{r}_x$ into equation (7), using the fact that $\bar{C}_1 = \theta^0\overline{Y}$ and $\bar{C}_2 = \left(1-\theta^0\right)\overline{Y}$ and plugging back $\alpha_e\hat{r}_x$ for ξ, we get:

$$
\left(\frac{\gamma_1(\alpha_e+\theta^0)}{\theta^0} - \frac{\gamma_2(1-\theta^0-\alpha_e)}{1-\theta^0}\right)E\hat{y}_1^2 = \left(\frac{\gamma_1(\alpha_e+\theta^0)}{\theta^0} - \frac{\gamma_2(1-\theta^0-\alpha_e)}{1-\theta^0}\right)\varepsilon^2 = 0
$$

[15] That is, both countries' Euler equations with respect to the risky and with respect to the safe asset.

where we used $\hat{y} = \varepsilon z$ and $Ez^2 = 1$.

Solving the last equation for α_e, we get:

$$\overline{\alpha}_e = \frac{\gamma_2 - \gamma_1}{\gamma_1(1-\theta^0) + \gamma_2\theta^0}\theta^0(1-\theta^0) \tag{8}$$

Because our model is static and we have $\hat{x} = 0$, and because the size of uncertainty, ε, does not in any other way affect the portfolio solution under the DS method, there is a strong implication: it turns out that in our two-period model also higher-order approximations, up to any order, are identical to the constant zero-order part of the solution, $\bar{\alpha}_e$. The DS portfolio solution for θ is then obtained as:

$$\theta = \theta^0 + \frac{\alpha_e}{p}, \text{ where } \alpha_e = \overline{\alpha}_e. \tag{9}$$

The property of α_e which is key here, is that it is invariant to the size, or any other statistical properties (i.e. skewness, kurtosis etc.), of the shock z in the model. It should be clear that this is true in our model from inspecting equation (2) – α_e only depends on the difference between the two investors' risk aversion parameters and the initial equity endowments.[16].

Once the optimal α_e is found, the solution to θ can be found from $\theta = \theta^0 + \frac{\alpha_e}{p}$. While from equation (8) it is clear that α_e does not depend on the size of shocks, ε, this is not generally true for θ, as p in general will depend on ε in higher-order approximations. To see, how the portfolio solutions from the DS method would perform if one accounted for this, we use the solution for p from the true portfolio solution method. The idea is, that at best, an infinite-order Taylor approximation would converge to the true function $p(\varepsilon)$. As the first row of Figure 2 shows, $p(\varepsilon)$ is, however, a decreasing function of ε (the return on the risky asset increases as the size of shocks increases, so its price falls). This implies, that allowing p to vary with ε would actually worsen the portfolio solution results from the DS method, which is confirmed in the second row of Figure 2; θ^{DS} increases as ε increases.

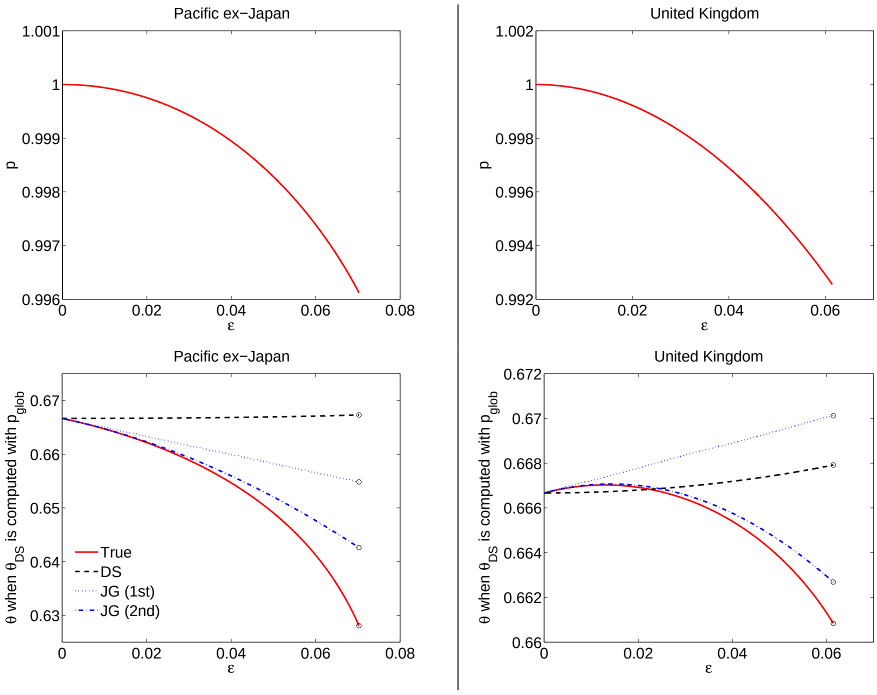
A.4 Details of the Judd-Guu Solution

The system in (1) implicitly defines $\theta(\varepsilon)$ and $p(\varepsilon)$. Denote this system $H(\theta(\varepsilon), p(\varepsilon), \varepsilon) = 0$. However, the implicit function theorem cannot be applied to analyze (1) around $\varepsilon = 0$, since assets are perfect substitutes in such case and must trade at the same price; that is, we must have $p(0) = 1$. However, $\theta(0)$ is indeterminate because $H(\theta, p, 0) = 0$ for all θ. The indeterminacy of θ implies that $H_\theta(\theta, 1, 0) = 0$, ruling out application of the implicit function theorem.

Judd and Guu (2001) show how one can use the bifurcation theorem to solve the above problem. The bifurcation approach requires that the Jabobian matrix $H_{(\theta,p)}$ is a zero matrix. While at $\varepsilon = 0$, $\theta(0)$ is indeterminate, there is only a single possible value for $p(0)$ and $p'(0)$; so the Jacobian $H_{(\theta,p)}$ would in fact not be a zero matrix. We follow Judd and Guu (2001) in solving this problem by reformulating the problem in terms of the price of risk, π, instead of the price of equity, p. That is, we parameterize the equity price as $p\varepsilon = 1 - \varepsilon^2\pi(\varepsilon)$, where

[16] Strictly speaking, the finding that α_e is invariant to changes in the size of uncertainty does not imply that the same is true for θ, as the equity price, p, generally does depend on ε. In appendix A.3, we show that taking into the account the effect of the size of shocks on p would, in fact, worsen the performance of the DS solution for θ.

Figure 2: Equity price from true solution, p^{true}, and solutions for equity shares held by country 1 investor.



$\pi(\varepsilon)$ is the risk premium in the ε-economy. Since $\sigma_z^2 = 1$, ε^2 is the variance of risk and $\pi(\varepsilon)$ is the risk premium per unit variance. This way, the system in (1) can be rewritten as

$$\mathbf{H}\left(\theta\left(\varepsilon\right),\pi\left(\varepsilon\right),\varepsilon\right) = \tag{10}$$
$$\left[\begin{array}{c} E\left[u_1'(\theta\left(1+\varepsilon z\right)+b^e+\left(\theta_0-\theta\right)\left[1-\varepsilon^2\pi\right])(z+\varepsilon\pi)\right] \\ E\left[u_2'((1-\theta)\left(1+\varepsilon z\right)-b^e-\left(\theta_0-\theta\right)\left[1-\varepsilon^2\pi\right])(z+\varepsilon\pi)\right] \end{array}\right] = \mathbf{0}.$$

Obtaining the coefficients of the Taylor series expansion of $\theta(\varepsilon)$, given by

$$\theta\left(\varepsilon\right) = \theta_0 + \theta'\left(0\right)\varepsilon + \theta''\left(0\right)\frac{\varepsilon^2}{2} + \theta'''\left(0\right)\frac{\varepsilon^3}{6} + ..., \tag{11}$$

is then conceptually straightforward. To find θ_0, one needs to differentiate function $\mathbf{H}$ with respect to ε, to find $\theta'(0)$ one needs to differentiate function $\mathbf{H}$ w.r.t. ε the second time, to find $\theta''(0)$ the third time, etc., and needs to evaluate those derivatives at $\varepsilon = 0$.