COUNTING THE NUMBER OF HOMOTOPY ASSOCIATIVE MULTIPLICATIONS ON CERTAIN H-SPACES

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Abstract. We determine an upper bound for the number of homotopy associative multiplications on certain H-spaces. This is applied to $SU(3)$ and $Sp(2)$ at odd primes, and to give examples of p-local H-spaces with more than one multiplication but a unique homotopy associative multiplication.

1. Introduction

Throughout this paper, all spaces are pointed, connected and have the homotopy type of a CW-complex. A space X is an H-space if there is a continuous map $m\colon X \times X \longrightarrow X$ whose restriction to either factor is homotopic to the identity map on X. The map m is called a multiplication on X. Counting the number of distinct (non-homotopic) multiplications on an H-space is a problem with a long history. James [J2] showed that there are 12 distinct multiplications on S^3. Arkowitz and Curjel [AC] generalized this greatly by showing that if X is an H-space with a homotopy associative multiplication then there is a one-to-one correspondence between the number of distinct multiplications on X and the homotopy classes of maps $[X \wedge X, X]$. Zabrodsky [Z, Theorem 1.4.3] later showed that the same result holds for any H-space. These generalizations recovered James' result, since $[S^3 \wedge S^3, S^3] \cong \pi_6(S^3) \cong \mathbb{Z}/12\mathbb{Z}$. Mimura [M] used Arkowitz and Curjel's result to determine the number of distinct multiplications on $SU(3)$ and $Sp(2)$.

When X is both homotopy associative and homotopy commutative more counting results are known. Hubbuck [H] showed that if X is a finite H-space then there are no multiplications which are both homotopy associative and homotopy commutative unless X is a torus, and then the multiplication is unique. (In fact, the homotopy associativity hypothesis is not needed for this.) Localized at a prime p there can be non-torus finite H-spaces with multiplications which are both homotopy associative and homotopy commutative, but the third author [Th2] showed that if they are "retractile" then the homotopy associative and homotopy commutative multiplication is unique.

There remains the question of counting the number of homotopy associative multiplications. James [J2] showed that, of the 12 distinct multiplications on S^3, precisely 8 of them are homotopy associative. This does not obviously correspond to the number of classes in a homotopy set $[A, S^3]$ for

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any space A related to S^3, so the problem of counting the number of distinct homotopy associative multiplications is inherently more subtle than counting the number of multiplications.

In this paper we give an upper bound on the number of homotopy associative multiplications for retractile H-spaces. Let p be an odd prime, localize spaces and maps at p, and take homology with mod-p coefficients. An H-space B is retractile if there is a triple (A, i, B) such that: (i) $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$, where $\Lambda(\ \ )$ is the free exterior algebra functor, (ii) there is a map $i\colon A \longrightarrow B$ inducing the inclusion of the generating set in $H_*(B)$, and (iii) Σi has a left homotopy inverse. As will be discussed in Section 2 there are many examples of retractile H-spaces, and these include simply-connected, simple compact Lie groups when localized at sufficiently large primes. We determine an upper bound on the number of p-local homotopy associative multiplications on retractile H-spaces B.

Theorem 1.1. *Let B be a retractile H-space. If B has a homotopy associative multiplication then the number of homotopy associative multiplications on B is bounded above by the order of the group $[A \wedge B, B]$.*

Theorem 1.1 is in line with Arkowitz and Curjel's statement regarding the number of multiplications on B. Note that if $B = S^{2m+1}$ then $A = S^{2m+1}$ as well, so $[A \wedge B, B] \cong [B \wedge B, B]$ and no new information is gained. However, if B has more than one generator in homology then A has fewer cells than B so $[A \wedge B, B]$ may be a much smaller group than $[B \wedge B, B]$. Theorem 1.1 is applied to $SU(3)$ and $Sp(2)$ at primes $p \geq 3$ to show that many of the multiplications are not homotopy associative. Observe also that Theorem 1.1 implies that if $[A \wedge B, B] \cong 0$ then the homotopy associative multiplication on B is unique. We give examples where B has multiple distinct multiplications but a unique homotopy associative multiplication.

It is tempting to suspect that, as A "generates" B in some sense, the number of homotopy associative multiplications should be in terms of $[A \wedge A, B]$. But this would be like saying that different group structures G_1 and G_2 on a set T, which have the same generating set S and the same list of elements, are determined by products involving only two generators. However, this is not true. For example, $D_8 \times \mathbb{Z}/2\mathbb{Z} = \langle a, b, c \mid a^4 = b^2 = c^2 = e, ab = ba, bc = cb, cac = a^3 \rangle$ and $\mathrm{SmallGroup}(16, 3) = \langle a, b, c \mid a^4 = b^2 = c^2 = e, ab = ba, bc = cb, cac = ab \rangle$ are nonisomorphic groups of order 16, have the same generating set and the same list of elements, but the product structure on $\mathrm{SmallGroup}(16, 3)$ is not determined by products involving only two elements. So it is unlikely that, in general, the number of homotopy associative multiplications on a retractile H-space B is bounded above by $[A \wedge A, B]$.

2. Properties of loop suspensions and retractile H-spaces

This section discusses the background results needed for what is to come. We begin with some properties of loop suspensions. Let A be a path-connected, pointed space and let

$$E\colon A \longrightarrow \Omega\Sigma A$$

be the suspension map. For $k \geq 1$, let $A^{\times k}$ be the Cartesian product of A with itself k times and let e_k be the composite

$$e_k\colon A^{\times k} \xrightarrow{E^{\times k}} (\Omega\Sigma A)^{\times k} \xrightarrow{\mu} \Omega\Sigma A$$

where μ is the loop multiplication (which is homotopy associative). Note that $e_1 = E$.

It is well known that there is a natural homotopy equivalence $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$. Iterating, we obtain a retraction of $\Sigma A^{\wedge k}$ off $\Sigma A^{\times k}$, where $A^{\wedge k}$ is the k-fold smash product of A with itself. Using this we obtain a composite

$$\psi_k\colon \Sigma A^{\wedge k} \longrightarrow \Sigma A^{\times k} \xrightarrow{\Sigma e_k} \Sigma\Omega\Sigma A.$$

Taking the wedge sum of the maps ψ_k for $k \geq 1$ gives a map

$$\psi\colon \bigvee_{k=1}^{\infty} \Sigma A^{\wedge k} \longrightarrow \Sigma\Omega\Sigma A.$$

James [J1] proved the following.

Theorem 2.1. *The map ψ is a homotopy equivalence.* $\square$

The map ψ will be used to prove a criterion for when two maps out of a product $\Omega\Sigma A \times B$ are homotopic. For a space X let 1_X be the identity map on X.

Lemma 2.2. *Let Y be an H-space and suppose that there are maps $f, g\colon \Omega\Sigma A \times B \longrightarrow Y$. If the composites $A^{\times k} \times B \xrightarrow{e_k \times 1_B} \Omega\Sigma A \times B \xrightarrow{f} Y$ and $A^{\times k} \times B \xrightarrow{e_k \times 1_B} \Omega\Sigma A \times B \xrightarrow{g} Y$ are homotopic for each $k \geq 1$, then f is homotopic to g.*

Proof. Since Y is an H-space, it retracts off $\Omega\Sigma Y$, so to show that $f \simeq g$ it suffices to show that $\Sigma f \simeq \Sigma g$. Let f_1 and f_2 be the restrictions of f to $\Omega\Sigma A$ and B respectively. After suspending, we have $\Sigma(\Omega\Sigma A \times B) \simeq \Sigma\Omega\Sigma A \vee \Sigma B \vee \Sigma(\Omega\Sigma A \wedge B)$, and Σf is homotopic to the wedge sum of Σf_1, Σf_2 and the map

$$f_3\colon \Sigma(\Omega\Sigma A \wedge B) \xrightarrow{q} \Sigma(\Omega\Sigma A \times B) \xrightarrow{\Sigma f} Y$$

where q has been chosen so that it has a left homotopy inverse. Similarly, Σg is the wedge sum of maps Σg_1, Σg_2 and $g_3 = \Sigma g \circ q$. To show that $\Sigma f \simeq \Sigma g$ we will show that $\Sigma f_1 \simeq \Sigma g_1$, $\Sigma f_2 \simeq \Sigma g_2$, and $f_3 \simeq g_3$.

First consider f_1 and g_1. The hypotheses imply that the composites $A^{\times k} \xrightarrow{e_k} \Omega\Sigma A \xrightarrow{f_1} Y$ and $A^{\times k} \xrightarrow{e_k} \Omega\Sigma A \xrightarrow{g_1} Y$ are homotopic for all $k \geq 1$. Therefore, by the definition of ψ_k, the composites

$\Sigma A^{\wedge k} \xrightarrow{\psi_k} \Sigma\Omega\Sigma A \xrightarrow{\Sigma f_1} \Sigma Y$ and $\Sigma A^{\wedge k} \xrightarrow{\psi_k} \Sigma\Omega\Sigma A \xrightarrow{\Sigma g_1} \Sigma Y$ are homotopic for all $k \geq 1$. Taking the wedge sum of the maps ψ_k then implies that the composites $\bigvee_{k=1}^{\infty} \Sigma A^{\wedge k} \xrightarrow{\psi} \Sigma\Omega\Sigma A \xrightarrow{\Sigma f_1} \Sigma Y$ and $\bigvee_{k=1}^{\infty} \Sigma A^{\wedge k} \xrightarrow{\psi} \Sigma\Omega\Sigma A \xrightarrow{\Sigma g_1} \Sigma Y$ are homotopic. But ψ is a homotopy equivalence by Theorem 2.1, implying that $\Sigma f_1 \simeq \Sigma g_1$.

A similar argument using $(\bigvee_{k=1}^{\infty} \Sigma A^{\wedge k}) \wedge B \xrightarrow{\psi \wedge 1_B} (\Sigma\Omega\Sigma A) \wedge B$ shows that $f_3 \simeq g_3$.

Finally, observe that the restrictions of $f \circ (e_1 \times 1_B)$ and $g \circ (e_1 \times 1_B)$ to B are f_2 and g_2 respectively. By hypothesis, $f \circ (e_1 \times 1_B) \simeq g \circ (e_1 \times 1_B)$, so $f_2 \simeq g_2$. $\square$

One more property about loop suspensions proved by James [J1] that we will need is the following.

Theorem 2.3. *If Y is a homotopy associative H-space then any map $f \colon A \longrightarrow Y$ can be extended to an H-map $\overline{f} \colon \Omega\Sigma A \longrightarrow Y$.* $\square$

Now we turn to retractile H-spaces. Let p be an odd prime, localize spaces and maps at p, and take homology with mod-p coefficients. Recall from the Introduction that an H-space B is retractile if there is a triple (A, i, B) such that: (i) $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$, where $\Lambda(\ \ )$ is the free exterior algebra functor, (ii) there is a map $i \colon A \xrightarrow{i} B$ inducing the inclusion of the generating set in $H_*(B)$, and (iii) Σi has a left homotopy inverse. A large family of examples was constructed in different ways by Cooke, Harper and Zabrodsky [CHZ] and Cohen and Neisendorfer [CN]. They showed that if A is any CW-complex consisting of ℓ odd dimensional cells, where $\ell < p-1$, then after localization at p there is a retractile triple (A, i, B). In [Th1] it was shown that if $\ell < p-2$ then B has a homotopy associative, homotopy commutative multiplication. Sometimes for other reasons it is known that there are retractile H-spaces when $\ell = p-1$. For instance, $SU(p)$ at p is retractile, with the space A being $\Sigma\mathbb{C}P^{p-1}$, and $Sp((p+1/2))$ at p is also retractile. Note that in these cases the loop structure on the Lie group is homotopy associative.

We need to establish some properties of homotopy assocative retractile H-spaces. Let B be a retractile H-space with an associated triple (A, i, B), and suppose that B has a homotopy associative multiplication m. By Theorem 2.3 there is an extension

$$(1)\qquad \begin{tikzcd} A \arrow[r, "i"] \arrow[d, "E"'] & B \\ \Omega\Sigma A \arrow[ur, "r(m)"'] & \end{tikzcd}$$

where $r(m)$ is an H-map with respect to the multiplication m on B. Since B is retractile, there is a map

$$t \colon \Sigma B \longrightarrow \Sigma A$$

which is a left homotopy inverse of Σi. Let $\overline{s}(m)$ be the composite

$$\overline{s}(m) \colon B \xrightarrow{E} \Omega\Sigma B \xrightarrow{\Omega t} \Omega\Sigma A.$$

Lemma 2.4. *The following hold:*

(a) *there is a homotopy commutative diagram*

$$\begin{tikzcd} A \arrow[d, "i"'] \arrow[dr, "E"] & \\ B \arrow[r, "\overline{s}(m)"] & \Omega\Sigma A; \end{tikzcd}$$

(b) *the composite* $B \xrightarrow{\overline{s}(m)} \Omega\Sigma A \xrightarrow{r(m)} B$ *is a homotopy equivalence.*

Proof. By the naturality of E and the definition of $\overline{s}(m)$ there is a homotopy commutative diagram

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "E"] & B \arrow[d, "E"] \arrow[dr, "\overline{s}(m)"] & \\ \Omega\Sigma A \arrow[r, "\Omega\Sigma i"] & \Omega\Sigma B \arrow[r, "\Omega t"] & \Omega\Sigma A. \end{tikzcd}$$

Since the composite along the bottom row is the identity map, we obtain $E \simeq \overline{s}(m) \circ i$ as asserted in part (a).

For part (b), first observe that $\overline{s}(m) \circ i \simeq E$ by part (a) and $i \simeq r(m) \circ E$ by (1). Therefore $r(m) \circ \overline{s}(m) \circ i \simeq r(m) \circ E \simeq i$. Since i_* is the inclusion of the generating set in $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$, we see that $r(m) \circ \overline{s}(m)$ is a self-map of B with the property that $r(m)_* \circ \overline{s}(m)_*$ is the identity map when restricted to $\widetilde{H}_*(A)$. An inductive argument using the reduced diagonal (or dualizing to cohomology) then shows that $r(m)_* \circ \overline{s}(m)_*$ is an isomorphism, and so $r(m) \circ \overline{s}(m)$ is a homotopy equivalence. □

We wish to replace the map $\overline{s}(m)$ in Lemma 2.4 with one that gives the identity map on B in part (b) but preserves property (a). To do this, let $e = r(m) \circ \overline{s}(m)$ be the homotopy equivalence from Lemma 2.4 (b). Define $\overline{i}$ by the composite

$$\overline{i} \colon A \xrightarrow{i} B \xrightarrow{e} B.$$

Consider the diagram

$$\begin{tikzcd} & A \arrow[dl, "\overline{i}"'] \arrow[d, "i"] \arrow[dr, "E"] & & \\ B \arrow[r, "e^{-1}"] & B \arrow[r, "\overline{s}(m)"] & \Omega\Sigma A \arrow[r, "r(m)"] & B. \end{tikzcd} \tag{2}$$

The left triangle commutes by the definition of $\overline{i}$, and the right triangle homotopy commutes by Lemma 2.4 (a). Since $r(m) \circ \overline{s}(m) = e$, the bottom row is the identity map on B. Thus the homotopy commutativity of (2) implies that $\overline{i} \simeq r(m) \circ E$, but $r(m) \circ E \simeq i$ by (1). Therefore $\overline{i} \simeq i$. So if we define $s(m)$ by the composite

$$s(m) \colon B \xrightarrow{e^{-1}} B \xrightarrow{\overline{s}(m)} \Omega\Sigma A$$

then we obtain $s(m) \circ i \simeq s(m) \circ \bar{i} = \bar{s}(m) \circ e^{-1} \circ \bar{i} \simeq E$, where the last homotopy is from the homotopy commutativity of the two triangles in (2). Further, by definition of $s(m)$, the composite $r(m) \circ s(m)$ is homotopic to the identity map on B. We record these properties for later use.

Lemma 2.5. *Let B be a retractile H-space with an associated triple (A, i, B) and let m be a homotopy associative multiplication on B. Then there is a map $B \xrightarrow{s(m)} \Omega\Sigma A$ such that:*

(a) *there is a homotopy commutative diagram*

$$\begin{tikzcd} A \arrow[d, "i"] \arrow[dr, "E"] & \\ B \arrow[r, "s(m)"] & \Omega\Sigma A; \end{tikzcd}$$

(b) *the composite $B \xrightarrow{s(m)} \Omega\Sigma A \xrightarrow{r(m)} B$ is homotopic to the identity map on B.*

□

3. An upper bound on the number of homotopy associative multiplications

Let B be an H-group, that is, B is a homotopy associative H-space with a homotopy inverse. (Note that the homotopy inverse hypothesis is redundant since a homotopy associative H-space having the homotopy type of a CW-complex is equivalent to an H-group by using the shearing map.) Fix a map

$$m \colon B \times B \longrightarrow B$$

inducing an H-group structure on B. Then for any space X, the multiplication m induces a group structure on the homotopy classes of maps $[X, B]$. In what follows, we will compare the fixed multiplication m with other homotopy associative multiplications m' on B, and all additions and differences calculated in $[X, B]$ will be made with respect to the structure induced by the fixed multiplication m.

Let m' be a homotopy associative multiplication on B. By Theorem 2.3 there is an extension

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "E"] & B \\ \Omega\Sigma A \arrow[ur, "r(m')"'] & \end{tikzcd} \tag{3}$$

where $r(m')$ is an H-map with respect to the multiplication m'. Observe that there is also an H-map $r(m)$ corresponding to the fixed multiplication m, and $r(m')$ need not be homotopic to $r(m)$. Let

$$s(m') \colon B \longrightarrow \Omega\Sigma A$$

be the right homotopy inverse of $r(m')$ given in Lemma 2.5.

Define $\theta(m')$ by the composite

$$\theta(m')\colon A\times B\xrightarrow{E\times s(m')}\Omega\Sigma A\times\Omega\Sigma A\xrightarrow{\mu}\Omega\Sigma A\xrightarrow{r(m')}B. \tag{4}$$

In the next several lemmas we will determine some properties of $\theta(m')$.

Lemma 3.1. *The map $A\times B\xrightarrow{\theta(m')}B$ has the following properties:*

(a) *the restriction of $\theta(m')$ to A is homotopic to i;*

(b) *the restriction of $\theta(m')$ to B is homotopic to 1_B;*

(c) *there is a homotopy commutative diagram*

$$\begin{tikzcd} A\times B \arrow[r, "\theta(m')"] \arrow[d, "i\times 1_B"] & B \arrow[d, equal] \\ B\times B \arrow[r, "m'"] & B. \end{tikzcd}$$

Proof. The restriction of $\theta(m')$ to A is the composite $A\xrightarrow{E}\Omega\Sigma A\xrightarrow{r(m')}B$. By (3) this composite is homotopic to i.

The restriction of $\theta(m')$ to B is the composite $B\xrightarrow{s(m')}\Omega\Sigma A\xrightarrow{r(m')}B$. Since $s(m')$ is a right homotopy inverse of $r(m')$, this composite is homotopic to 1_B.

Finally, by parts (a) and (b) and the fact that $r(m')$ is an H-map, there is a homotopy commutative diagram

$$\begin{tikzcd} A\times B \arrow[r, "E\times s(m')"] \arrow[dr, "i\times 1_B"'] & \Omega\Sigma A\times\Omega\Sigma A \arrow[r, "\mu"] \arrow[d, "r(m')\times r(m')"] & \Omega\Sigma A \arrow[d, "r(m')"] \\ & B\times B \arrow[r, "m'"] & B. \end{tikzcd}$$

The upper direction around the diagram is the definition of $\theta(m')$. Thus $\theta(m')\simeq m'\circ(i\times 1_B)$. $\square$

The next lemma combines with Lemma 3.1 to show that the map $A\times B\xrightarrow{\theta(m')}B$ is a kind of action of A on B. Recall the map $A^{\times k}\xrightarrow{e_k}\Omega\Sigma A$ defined at the beginning of Section 2.

Lemma 3.2. *There is a homotopy commutative diagram*

$$\begin{tikzcd} A\times A \arrow[r, "e_2"] \arrow[d, "1_A\times i"] & \Omega\Sigma A \arrow[d, "r(m')"] \\ A\times B \arrow[r, "\theta(m')"] & B. \end{tikzcd}$$

Proof. Consider the diagram

$$\begin{tikzcd} A\times A \arrow[r, equal] \arrow[d, "1_A\times i"] & A\times A \arrow[d, "E\times E"] \arrow[dr, "e_2"] & & \\ A\times B \arrow[r, "E\times s(m')"] & \Omega\Sigma A\times\Omega\Sigma A \arrow[r, "\mu"] & \Omega\Sigma A \arrow[r, "r(m')"] & B. \end{tikzcd}$$

The square homotopy commutes by Lemma 2.5 (a) and the triangle commutes by the definition of e_2. Therefore the diagram as a whole homotopy commutes. Observing that the bottom row is the definition of $\theta(m')$, the lemma is proved. $\square$

We next show that the "action" of A on B has an associativity property which is compatible with the multiplication m' on B. Define the map $\varphi(m')$ by the composite

$$\varphi(m')\colon \Omega\Sigma A \times B \xrightarrow{1_{\Omega\Sigma A}\times s(m')} \Omega\Sigma A \times \Omega\Sigma A \xrightarrow{\mu} \Omega\Sigma A \xrightarrow{r(m')} B.$$

Note that $\varphi(m')\circ(E\times 1_B) = \theta(m')$.

Lemma 3.3. *There are homotopy commutative diagrams*

$$\begin{tikzcd} A\times A\times B \arrow[r, "1_A\times\theta(m')"] \arrow[d, "e_2\times 1_B"] & A\times B \arrow[d, "\theta(m')"] \\ \Omega\Sigma A\times B \arrow[r, "\varphi(m')"] & B \end{tikzcd} \qquad \begin{tikzcd} A\times A\times B \arrow[rr, "1_A\times\theta(m')"] \arrow[d, "i\times i\times 1_B"] & & A\times B \arrow[d, "\theta(m')"] \\ B\times B\times B \arrow[r, "1\times m'"] & B\times B \arrow[r, "m'"] & B. \end{tikzcd}$$

Proof. We begin with the right diagram. Consider the string of homotopies

$$\begin{aligned} \theta(m')\circ(1_A\times\theta(m')) &\simeq m'\circ(i\times 1_B)\circ(1_A\times(m'\circ(i\times 1_B))) \\ &\simeq m'\circ(i\times(m'\circ(i\times 1_B))) \\ &\simeq m'\circ(1_B\times m')\circ(i\times i\times 1_B). \end{aligned}$$

The first homotopy holds by Lemma 3.1 (c), the second just compresses the data, and the third reorganizes it. Thus the right diagram in the statement of the lemma homotopy commutes.

Next, consider the diagram

$$\begin{tikzcd} A\times A\times B \arrow[r, "e_2\times 1_B"] \arrow[d, "i\times i\times 1_B"] & \Omega\Sigma A\times B \arrow[r, "1_{\Omega\Sigma A}\times s(m')"] \arrow[d, "r(m')\times 1_B"] & \Omega\Sigma A\times\Omega\Sigma A \arrow[r, "\mu"] \arrow[d, "r(m')\times r(m')"] & \Omega\Sigma A \arrow[d, "r(m')"] \\ B\times B\times B \arrow[r, "m'\times 1_B"] & B\times B \arrow[r, equal] & B\times B \arrow[r, "m'"] & B. \end{tikzcd}$$

The left square homotopy commutes since $e_2 = \mu\circ(E\times E)$, $r(m')$ is an H-map, and $r(m')\circ E\simeq i$; the middle square homotopy commutes since $s(m')$ is a right homotopy inverse for $r(m')$; and the right square homotopy commutes since $r(m')$ is an H-map. Therefore the diagram as a whole homotopy commutes. By definition, $\varphi(m') = r(m')\circ\mu\circ(1_{\Omega\Sigma A}\times s(m'))$, so the upper direction around the diagram is $\varphi(m')\circ(e_2\times 1_B)$. Thus the homotopy commutativity of the diagram gives $\varphi(m')\circ(e_2\times 1_B)\simeq m'\circ(m'\times 1_B)\circ(i\times i\times 1_B)$. Refining, as m' is homotopy associative, $m'\circ(m'\times 1_B)\simeq m'\circ(1_B\times m')$, and by the first part of the lemma, $m'\circ(1_B\times m')\circ(i\times i\times 1_B)\simeq\theta(m')\circ(1_A\times\theta(m'))$. Thus we obtain $\varphi(m')\circ(e_2\times 1_B)\simeq\theta(m')\circ(1_A\times\theta(m'))$, proving that the left diagram in the statement of the lemma homotopy commutes. $\square$

The associativity properties in Lemma 3.3 allow us to iterate with repeated actions of A on B. Let $\Theta_1(m') = \theta(m')$ and for $k \geq 2$, define $\Theta_k(m')$ recursively by the composite

$$\Theta_k(m')\colon A^{\times k} \times B = A \times (A^{(k-1)} \times B) \xrightarrow{1_A \times \Theta_{k-1}(m')} A \times B \xrightarrow{\theta(m')} B.$$

For $k \geq 2$, let

$$M'_k\colon B^{\times k} \longrightarrow B$$

be the map obtained by iteratively using m' to multiply the identity map on B with itself. Note that as m' is homotopy associative, the order in which this multiplication occurs does not matter. Then arguing exactly as in Lemma 3.3 proves the following.

Lemma 3.4. *For $k \geq 1$ there are homotopy commutative diagrams*

$$\begin{tikzcd} A^{\times k} \times B \arrow[d, "e_k \times 1_B"] \arrow[drr, "\Theta_k(m')"] & & \\ \Omega\Sigma A \times B \arrow[rr, "\varphi(m')"] & & B \end{tikzcd} \qquad \begin{tikzcd} A^{\times k} \times B \arrow[d, "i^{\times k} \times 1_B"] \arrow[drr, "\Theta_k(m')"] & & \\ B^{\times k} \times B \arrow[rr, "M'_{k+1}"] & & B. \end{tikzcd}$$

□

Notice that Lemmas 3.1, 3.3 and 3.4 all hold for any homotopy associative multiplication m'. In particular, the analogous diagrams hold for the fixed multiplication m. This is now used to prove a reduction: to show that m' and m are homotopic it suffices to prove the weaker statement that $\theta(m')$ and $\theta(m)$ are homotopic.

Proposition 3.5. *Let m' be a homotopy associative multiplication on B. Then $\theta(m') \simeq \theta(m)$ if and only if $m' \simeq m$.*

Proof. Suppose that $m' \simeq m$. Then by Lemma 3.1 (c), $\theta(m') \simeq m' \circ (i \times 1_B) \simeq m \circ (i \times 1_B) \simeq \theta(m)$.

Conversely, suppose that $\theta(m') \simeq \theta(m)$. The recursive definition sof $\Theta_k(m')$ and $\Theta_k(m)$ in terms of $\theta(m')$ and $\theta(m)$ respectively imply that for each $k \geq 1$ we have $\Theta_k(m') \simeq \Theta_k(m)$. Therefore, by Lemma 3.4 the composites

$$A^{\times k} \times B \xrightarrow{e_k \times 1_B} \Omega\Sigma A \times B \xrightarrow{\varphi(m')} B$$

$$A^{\times k} \times B \xrightarrow{e_k \times 1_B} \Omega\Sigma A \times B \xrightarrow{\varphi(m)} B$$

are homotopic for all $k \geq 1$. By Lemma 2.2, this implies that $\varphi(m') \simeq \varphi(m)$.

We now relate $\varphi(m')$ and $\varphi(m)$ to the multiplications m' and m. Consider the diagram

$$\begin{tikzcd} B \times B \arrow[r, "s(m') \times 1_B"] \arrow[d, equal] & \Omega\Sigma A \times B \arrow[r, "1_{\Omega\Sigma A} \times s(m')"] & \Omega\Sigma A \times \Omega\Sigma A \arrow[r, "\mu"] \arrow[d, "r(m') \times r(m')"] & \Omega\Sigma A \arrow[d, "r(m')"] \\ B \times B \arrow[rr, equal] & & B \times B \arrow[r, "m'"] & B. \end{tikzcd}$$

The left rectangle homotopy commutes since $s(m')$ is a right homotopy inverse for $r(m')$ and the right square homotopy commutes since $r(m')$ is an H-map. Therefore the entire diagram homotopy commutes. By definition, $\varphi(m') = r(m') \circ \mu \circ (1_{\Omega\Sigma A} \times s(m'))$, so the upper direction around the diagram is homotopic to $\varphi(m') \circ (s(m') \times 1_B)$. The homotopy commutativity of the diagram therefore implies that $\varphi(m') \circ (s(m') \times 1_B) \simeq m'$. On the other hand, consider the diagram

$$\begin{tikzcd}
B \times B \arrow[r, "s(m')\times 1_B"] \arrow[d, "(r(m)\circ s(m'))\times 1_B"] & \Omega\Sigma A \times B \arrow[r, "1_{\Omega\Sigma A}\times s(m)"] & \Omega\Sigma A \times \Omega\Sigma A \arrow[r, "\mu"] \arrow[d, "r(m)\times r(m)"] & \Omega\Sigma A \arrow[d, "r(m)"] \\
B \times B \arrow[rr, equal] & & B \times B \arrow[r, "m"] & B.
\end{tikzcd}$$

Arguing as before, the diagram homotopy commutes. This time we only obtain $\varphi(m) \circ (s(m') \times 1_B) \simeq m \circ ((r(m) \circ s(m')) \times 1_B)$. We claim that $r(m) \circ s(m') \simeq 1_B$. If so then $\varphi(m) \circ (s(m') \times 1_B) \simeq m$, and therefore as $\varphi(m') \simeq \varphi(m)$ we obtain

$$m' \simeq \varphi(m') \circ (s(m') \times 1_B) \simeq \varphi(m) \circ (s(m') \times 1_B) \simeq m,$$

completing the proof.

It remains to show that $r(m) \circ s(m')$ is homotopic to the identity map on B. Since $\varphi(m') \simeq \varphi(m)$, their restrictions to $\Omega\Sigma A$ are homotopic. By the definition of $\varphi(m')$, its restriction to $\Omega\Sigma A$ is the map $\Omega\Sigma A \xrightarrow{r(m')} B$, and similarly the restriction of $\varphi(m)$ to $\Omega\Sigma A$ is the map $\Omega\Sigma A \xrightarrow{r(m)} B$. So we have $r(m') \simeq r(m)$. Therefore, as $s(m')$ is a right homotopy inverse for $r(m')$, we obtain $r(m) \circ s(m') \simeq r(m') \circ s(m') \simeq 1_B$, as required. $\square$

Recall that the fixed multiplication m on B is used to induce the group structure in $[X, B]$ for any space X. Let m' be any homotopy associative multiplication on B. By Lemma 3.1 (a) and (b), the restrictions of $\theta(m)$ and $\theta(m')$ to $A \vee B$ are both homotopic to $i \vee 1_B$. Therefore in $[A \times B, B]$ the difference $D(m') = \theta(m') - \theta(m)$ is null homotopic when restricted to $A \vee B$, and so it extends to a map $d(m')\colon A \wedge B \longrightarrow B$. Note that the choice of extension is uniquely determined since the connecting map δ in the homotopy cofibration sequence $A \vee B \longrightarrow A \times B \longrightarrow A \wedge B \xrightarrow{\delta} \Sigma(A \vee B)$ is null homotopic.

Let $HA(B)$ be the set of homotopy associative multiplications on B. Define a map

$$\psi\colon HA(B) \longrightarrow [A \wedge B, B]$$

by $\psi(m') = d(m')$. We now prove Theorem 1.1, restated as follows.

Theorem 3.6. *The map $HA(B) \xrightarrow{\psi} [A \wedge B, B]$ is an injection.*

Proof. Let m' be in the kernel of ψ. Then $\psi(m') = d(m')$ is null homotopic. This implies that $D(m') = \theta(m') - \theta(m)$ is null homotopic. Therefore $\theta(m') \simeq \theta(m)$ and so by Proposition 3.5, $m' \simeq m$. Hence the kernel of ψ consists only of the element m, implying that ψ is an injection. $\square$

One immediate consequence of Theorem 3.6 identifies cases when the homotopy associative multiplication on B is unique.

Corollary 3.7. *If $[A\wedge B, B] \cong 0$ then there is a unique homotopy associative multiplication on B.* $\square$

Before moving on to some examples in the next section, we make two further observations. First, recall that the set of all multiplications on B is in one-to-one correspondence with $[B \wedge B, B]$. The correspondence is constructed in [AC] by means of a difference map analogous to ψ. Observe that $[A \wedge B, B]$ injects into $[B \wedge B, B]$, for if m is a fixed multiplication on B, then the composite

$$[A\wedge B, B] \xrightarrow{E_*} [A\wedge B, \Omega\Sigma B] \cong [\Sigma A\wedge B, \Sigma B] \xrightarrow{(t\wedge 1)^*} [\Sigma B\wedge B, \Sigma B] \cong [B\wedge B, \Omega\Sigma B] \xrightarrow{r(m)_*} [B\wedge B, B]$$

is a right inverse of the map $(i \wedge 1)^* : [B \wedge B, B] \to [A \wedge B, B]$. Thus the composite $HA(B) \xrightarrow{\psi} [A\wedge B, B] \longrightarrow [B\wedge B, B]$ represents the inclusion of the homotopy associative multiplications on B into the full set of multiplications on B.

Second, we relate $HA(B)$ to a different set which is of some interest in its own right. An H-map $r : \Omega\Sigma A \to B$ is called an *H-retraction* if there exists a map (not necessarily an H-map) $s : B \to \Omega\Sigma A$ such that $r \circ s$ is homotopic to the identity map on B. Let $HRet[\Omega\Sigma A, B]$ be the set of homotopy classes of H-retractions. Observe that there is a map

$$\tau\colon HRet[\Omega\Sigma A, B] \longrightarrow [A \times B, B]$$

defined by sending $r \in HRet[\Omega\Sigma A, B]$ (with its associated right homotopy inverse s) to the composite $A \times B \xrightarrow{E\times s} \Omega\Sigma A \times \Omega\Sigma A \xrightarrow{\mu} \Omega\Sigma A \xrightarrow{r} B$.

Lemma 3.8. *Let B be a retractile H-space with an associated triple (A, i, B). Then there is an injection $HA(B) \longrightarrow HRet[\Omega\Sigma A, B]$ with the property that the composite $HA(B) \longrightarrow HRet[\Omega\Sigma A, B] \xrightarrow{\tau} [A \times B, B]$ send the multiplication $m \in HA(B)$ to the map $\theta(m)$ defined in (4).*

Proof. Let $m \in HA(B)$. Using the homotopy associative H-structure m on B, Lemma 2.3 implies that there is an extension

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, "E"'] & B \\ \Omega\Sigma A \arrow[ur, "r(m)"'] & \end{tikzcd}$$

where $r(m)$ is an H-map. By Lemma 2.5, there is a map $s(m)\colon B \longrightarrow \Omega\Sigma A$ which is a right homotopy inverse of $r(m)$. Thus $r(m) \in HRet[\Omega\Sigma A, B]$. Therefore, we obtain a map $\sigma\colon HA(B) \longrightarrow HRet[\Omega\Sigma A, B]$ given by $\sigma(m) = r(m)$.

Further, observe that $\tau(r(m))$ is the composite $A\times B \xrightarrow{E\times s(m)} \Omega\Sigma A\times\Omega\Sigma A \xrightarrow{\mu} \Omega\Sigma A \xrightarrow{r(m)} B$, which is precisely the definition of $\theta(m)$. Thus the composite $HA(B) \xrightarrow{\sigma} HRet[\Omega\Sigma A, B] \xrightarrow{\tau} [A \times B, B]$ send the multiplication $m \in HA(B)$ to the map $\theta(m)$. In particular, Proposition 3.5 implies that $\tau \circ \sigma$ is an injection. Therefore σ is also an injection. $\square$

4. Examples

4.1. **Multiplications on $SU(3)$ and $Sp(2)$.** Mimura [M] determined the number of distinct multiplications on $SU(3)$ and $Sp(2)$ by calculating $[SU(3) \wedge SU(3), SU(3)]$ and $[Sp(2) \wedge Sp(2), Sp(2)]$.

Theorem 4.1 (Mimura). *There are $2^{15} \cdot 3^9 \cdot 5 \cdot 7$ distinct multiplication on $SU(3)$, and $2^{20} \cdot 3 \cdot 5^5 \cdot 7$ distinct multiplications on $Sp(2)$.* □

Theorem 4.1 is an integral statement; localized at a prime p, the number of distinct multiplications is the p-component of the total number of integral multiplications. We give upper bounds on the number of odd primary homotopy associative multiplications on $SU(3)$ and $Sp(2)$. In the case of $SU(3)$ the space corresponding to A is $\Sigma\mathbb{C}P^2$ - the 5-skeleton of $SU(3)$; and in the case of $Sp(2)$ the space corresponding to A is the 7-skeleton of $Sp(2)$, which we will simply call A. Write $|[A \wedge B, B]|$ for the cardinality of the group $[A \wedge B, B]$.

Lemma 4.2. *Localized away from 2, $|[\Sigma\mathbb{C}P^2 \wedge SU(3), SU(3)]| = 3^5 \cdot 5$ and $|[A \wedge Sp(2), Sp(2)]| = 3 \cdot 5^4 \cdot 7$.*

Proof. First consider the $SU(3)$ case. By [MT], localized away from 2 there are homotopy equivalences $SU(3) \simeq S^3 \times S^5$ and $\Sigma\mathbb{C}P^2 \simeq S^3 \vee S^5$. Using the fact that $(S^3 \vee S^5) \wedge (S^3 \times S^5) \simeq S^6 \vee S^8 \vee S^8 \vee S^{10} \vee S^{11} \vee S^{13}$ and Toda's [To] calculations of the odd primary homotopy groups of spheres, we obtain

$$\begin{aligned}
[\Sigma\mathbb{C}P^2 \wedge SU(3), SU(3)] &\cong [(S^3 \vee S^5) \wedge (S^3 \times S^5), S^3 \times S^5] \\
&\cong \pi_6(S^3 \times S^5) \oplus 2 \cdot \pi_8(S^3 \times S^5) \oplus \pi_{10}(S^3 \times S^5) \oplus \pi_{11}(S^3 \times S^5) \oplus \pi_{13}(S^3 \times S^5) \\
&\cong \mathbb{Z}/3\mathbb{Z} \oplus 2 \cdot \mathbb{Z}/3\mathbb{Z} \oplus (\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}) \oplus 0 \oplus \mathbb{Z}/3\mathbb{Z}.
\end{aligned}$$

Consequently, $|[\Sigma\mathbb{C}P^2 \wedge SU(3), SU(3)]| = 3^5 \cdot 5$.

Next, consider $Sp(2)$. The space $Sp(2)$ does not split as a product of spheres at the prime 3 - in this case a close look at the calculation of $[Sp(2) \wedge Sp(2), Sp(2)]$ in [M] shows that, localized at 3, there are isomorphisms $[A \wedge Sp(2), Sp(2)] \cong [Sp(2) \wedge Sp(2), Sp(2)] \cong \mathbb{Z}/3\mathbb{Z}$. Localized away from 2 and 3, by [MT] there are homotopy equivalences $Sp(2) \simeq S^3 \times S^7$ and $A \simeq S^3 \vee S^7$. Using the fact that $(S^3 \vee S^7) \wedge (S^3 \times S^7) \simeq S^6 \vee S^{10} \vee S^{10} \vee S^{13} \vee S^{14} \vee S^{17}$ and Toda's calculations, we obtain

$$\begin{aligned}
[A \wedge Sp(2), Sp(2)] &\cong [(S^3 \vee S^7) \wedge (S^3 \times S^7), S^3 \times S^7] \\
&\cong \pi_6(S^3 \times S^7) \oplus 2 \cdot \pi_{10}(S^3 \times S^7) \oplus \pi_{13}(S^3 \times S^7) \oplus \pi_{14}(S^3 \times S^7) \oplus \pi_{17}(S^3 \times S^7) \\
&\cong 0 \oplus 2 \cdot \mathbb{Z}/5\mathbb{Z} \oplus 0 \oplus (\mathbb{Z}/5\mathbb{Z} \oplus \mathbb{Z}/7\mathbb{Z}) \oplus \mathbb{Z}/5\mathbb{Z}.
\end{aligned}$$

Consequently, $|[A \wedge Sp(2), Sp(2)]| = 3 \cdot 5^4 \cdot 7$. □

Theorem 3.6 and Lemma 4.2 immediately imply the following.

Proposition 4.3. *Localized away from 2 there are at most $3^5 \cdot 5$ distinct homotopy associative multiplications on $SU(3)$ and $3 \cdot 5^4 \cdot 7$ distinct homotopy associative multiplications on $Sp(2)$.* □

In particular, notice that when $SU(3)$ is localized at 7 it has 7 distinct multiplications but Lemmas 3.7 and 4.2 imply that there is a unique homotopy associative multiplication, which must be the one induced by the standard multiplication on $SU(3)$.

4.2. **Unique homotopy associative multiplications.** We begin with a family of examples. Let $B = \prod_{i=1}^k S^3$, $A = \bigvee_{i=1}^k S^3$, and $i\colon A \longrightarrow B$ be the inclusion of the wedge into the product. Then (A, i, B) is retractile. Iterating the suspension splitting $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$ shows that $A \wedge B$ is homotopy equivalent to a wedge of spheres of the form S^{3j} for $j \in \{2, \cdots, k+1\}$.

Lemma 4.4. *Let p be an odd prime satisfying $p \equiv -1 \bmod 3$. Let $k = (2p-1)/3$. Then localized at p, $B = \prod_{i=1}^k S^3$ has more than one multiplication but a unique homotopy associative multiplication.*

Proof. First notice that $p \equiv -1 \bmod 3$ implies that $2p \equiv -2 \equiv 1 \bmod 3$ so $(2p-1)/3$ is an integer. Also, S^3 is a loop space, implying that $B = \prod_{i=1}^k S^3$ has a homotopy associative multiplication.

We have $A \wedge B \simeq \bigvee_{\mathcal{I}} S^{3j}$ where $2 \leq j \leq k+1$. Thus $A \wedge B$ is 5-connected and has dimension $3(k+1) = 2p+2$. The dimension result implies that there are no sphere summands of $A \wedge B$ in dimension $2p$. But in dimensions $\leq 2p+2$, the only nontrivial homotopy homotopy group of $\pi_m(S^3)$ for $6 \leq m \leq 2p+2$ occurs when $m = 2p$. Thus $[A \wedge B] \cong 0$. Therefore, by Corollary 3.7, B has a unique homotopy associative multiplication.

On the other hand, observe that $B \wedge B$ has dimension $6k$. As $k = (2p-1)/3$ we have $6k = 4p-2$. By [To], there is an isomorphism $\pi_{4p-2}(S^3) \cong \mathbb{Z}/p\mathbb{Z}$. As $S^3 \simeq \Omega BS^3$, we obtain isomorphisms

$$[B \wedge B, B] = [B \wedge B, \prod_{i=1}^k \Omega BS^3] \cong [B \wedge B, \Omega(\prod_{i=1}^k BS^3)] \cong [\Sigma(B \wedge B), \prod_{i=1}^k BS^3].$$

Since B is a product of spheres, $\Sigma(B \wedge B)$ is homotopy equivalent to a wedge of spheres. The dimension of $\Sigma(B \wedge B)$ implies that one of these spheres occurs in dimension $4p-1$. Thus $[B \wedge B, B]$ contains a summand of the form $\oplus_{i=1}^k \mathbb{Z}/p\mathbb{Z}$. Therefore B has more than one multiplication. □

Next, we give an example of a 5-local sphere bundle over a sphere which has more than one multiplication but only one homotopy associative multiplication. This requires a general lemma.

Lemma 4.5. *Let X be a finite, connected CW-complex with cells in dimensions $m_0 = 0$ and $m_1, \ldots, m_k$ where $m_i > 0$ for $i > 0$. Let Y be a connected space with the property that $\pi_n(Y) \cong 0$ for every $n \in \{m_0, m_1, \ldots, m_k\}$. Then $[X, Y] \cong 0$.* □

Proof. For a positive integer t, let X_t be the t-skeleton of X. Note that X_{m_0} is the basepoint. Relabelling if necessary, assume that $m_1 \leq \cdots \leq m_k$. Since X has cells in the positive dimensions

$\{m_1, \ldots, m_k\}$, for $1 \leq i \leq k$ there are homotopy cofibration sequences

$$\bigvee S^{m_i-1} \xrightarrow{f_i} X_{m_{i-1}} \longrightarrow X_{m_i} \xrightarrow{q_i} \bigvee S^{m_i}$$

for attaching maps f_i and pinch maps q_i.

Clearly the restriction of $X \xrightarrow{f} Y$ to X_{m_0} is null homotopic. Suppose inductively that the restriction of f to $X_{m_{i-1}}$ is null homotopic. Then from the homotopy cofibration $X_{m_{i-1}} \longrightarrow X_{m_i} \xrightarrow{q_i} \bigvee S^{m_i}$ we see that the restriction of f to X_{m_i} factors through q_i to a map $e_i\colon \bigvee S^{m_i} \longrightarrow Y$. But as $\pi_{m_i}(Y) = 0$, the map e_i is null homotopic. Therefore the restriction of f to X_{m_i} is null homotopic. By induction, the restriction of f to $X_{m_k} = X$ - that is, f itself - is null homotopic. $\square$

Localize at $p = 5$. Let α_1 be the generator of the stable stem $\pi_{12}(S^5) \cong \mathbb{Z}/5\mathbb{Z}$. Define the space A by the homotopy cofibration

$$S^{12} \xrightarrow{\alpha_1} S^5 \longrightarrow A.$$

Since A has two cells and $p = 5$, by [CN], there is a retractile H-space B such that $H_*(B) \cong \Lambda(\widetilde{H}_*(A))$, and there is a homotopy fibration

$$S^5 \longrightarrow B \longrightarrow S^{13}. \tag{5}$$

Since A has 2-cells and $p = 5$, by [Th1] the H-space B has a multiplication which is both homotopy associative and homotopy commutative.

Lemma 4.6. *The H-space B has more than one distinct multiplication at* 5 *but precisely one of these is homotopy associative.*

Proof. We will calculate $[A \wedge B, B]$ and $[B \wedge B, B]$. First, observe that $A \wedge B$ has cells in dimensions $\{10, 18, 23, 26, 31\}$, with 2 cells occurring in dimension 18 and one cell in each of the other four dimensions. By [To], there are 5-local isomorphisms $\pi_n(S^5) \cong 0$ and $\pi_n(S^{13}) \cong 0$ for every $n \in \{10, 18, 23, 26, 31\}$. Therefore, the homotopy fibration (5) implies that $\pi_n(B) \cong 0$ for every $n \in \{10, 18, 23, 26, 31\}$. By Lemma 4.5, this implies that $[A \wedge B, B] \cong 0$. Hence, by Corollary 3.7, B has a unique homotopy associative multiplication.

Next, observe that the top cell of $B \wedge B$ occurs in dimension 36 and is not a cell of $A \wedge B$. Consider the homotopy cofibration sequence

$$S^{35} \xrightarrow{h} (B \wedge B)_{31} \longrightarrow B \wedge B \xrightarrow{q} S^{36} \xrightarrow{\Sigma h} \Sigma(B \wedge B)_{31}$$

where $(B \wedge B)_{31}$ is the 31-skeleton of $B \wedge B$, h attaches the top cell to $B \wedge B$, and q is the pinch map to the top cell. Since B is retractile, ΣA retracts off ΣB, so as B only has one more cell than A, there is a homotopy equivalence $\Sigma B \simeq \Sigma A \vee \Sigma S^{18}$. Iterating this implies that the top cell splits off $\Sigma B \wedge B$. Therefore Σh is null homotopic. The homotopy cofibration $B \wedge B \longrightarrow S^{35} \xrightarrow{\Sigma h} \Sigma(B \wedge B)_{31}$ induces an exact sequence $[(\Sigma(B \wedge B)_{31}, B] \xrightarrow{(\Sigma h)^*} [S^{35}, B] \xrightarrow{q^*} [B \wedge B, B]$. Since Σh is null homotopic, $(\Sigma h)^*$ is the zero map, and so q^* is an injection.

On the other hand, by [To], there are 5-local isomorphisms $\pi_{36}(S^5) \cong \mathbb{Z}/5\mathbb{Z}$ and $\pi_{36}(\Omega S^{13}) \cong \pi_{37}(S^{13}) \cong 0$. So the homotopy fibration sequence $\Omega S^{13} \longrightarrow S^5 \xrightarrow{i} B \longrightarrow S^{13}$ induces an exact sequence $\pi_{36}(\Omega S^{13}) \longrightarrow \pi_{36}(S^5) \xrightarrow{i_*} \pi_{36}(B)$, which implies that i_* is an injection. Therefore $\pi_{36}(B) \ncong 0$. Hence $[B \wedge B, B] \ncong 0$, implying that B has more than one distinct multiplication at 5. $\square$

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