BIFURCATING STEADY-STATE SOLUTIONS OF THE DISSIPATIVE QUASI-GEOSTROPHIC EQUATION IN LAGRANGIAN FORMULATION

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ABSTRACT. It is shown that the non-homogeneous dissipative quasi-geostrophic equation

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = \sin x_2, \quad u = (-\partial_{x_2}, \partial_{x_1})(-\Delta)^{-\beta/2} \theta$$

with $\alpha = 0$ and $\beta > 1$ loses stability at a critical value $\kappa_c > 0$ and this instability gives rise to a circle of steady-state solutions.

Keywords: quasi-geostrophic equation, steady-state bifurcation, Lagrange formulation

Mathematics Subject Classification: 35B32, 35Q35, 86A10

1. INTRODUCTION

A surface quasi-geostrophic flow is the first order approximation of a geostrophic flow with respect to the quasi-geostrophic approximation under a small Rossby number [27]. Instability examination of quasi-geostrophic equation is significant important in the understanding of geophysical variabilities [16]. From the mathematical formulation of [10], the energy dissipation controlled by the fractional Laplacian $\kappa(-\Delta)^\alpha$ is applied to the quasi-geostrophic flow [11, 32] and the homogeneous flow is thus governed by the dissipative quasi-geostrophic equation

$$(1) \quad \frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa(-\Delta)^\alpha \theta = 0, \quad u = (-\partial_{x_2}, \partial_{x_1})(-\Delta)^{-1/2} \theta$$

for $0 \leq \alpha \leq 1$, the dissipative parameter $\kappa > 0$, the gradient operator ∇ and the Laplacian $\Delta = \nabla \cdot \nabla$.

Equation (1) provides an interesting simple model that exhibits a number of non-linear and dissipative characters of the three-dimensional Navier-Stokes equations, and hence has been studied extensively since the late 1990s. Chae and Lee [2], Chen et al. [4] and Wu [33] derived the existence of local regular solutions and global small solutions in the framework of Besov spaces for $\alpha > 0$. This well-posedness problem was also examined by Córdoba and Córdoba [13] in H^m spaces for $\alpha > 0$. For $\alpha > \frac{1}{2}$ (the subcritical case), it is well known (see [11, 18]) that a smooth initial function θ_0 with a compact support leads to a global regular solution. For the critical case $\alpha = \frac{1}{2}$, similar to the three-dimensional Navier-Stokes equations, the nonlinear term is difficult to be controlled by the dissipative term for large solutions. However, due to the validity of the maximal principle for (1), the global existence of regular solution was obtained by Kiselev et al. [21] (see also Caffarelli and Vasseur [1]). When $0 < \alpha < \frac{1}{2}$ (the supercritical case), the dissipation is too

weak to ensure the global existence of large regular solutions. Therefore for the efforts on finite blow up examinations, we refer to Kiselev [19, 20] and references therein. However, the global existence is obtainable if additional conditions are applied (see, for example, Constantin and Wu [12] and Dong and Pavlovic [15]). One may also refer to Chen and Xiong [8] for computational steady-state bifurcation results of Charney and DeVore quasi-geostrophic equation and to Kato and Ponce [22], Ju [18], Danchin [14] and Chen and Chen [3] for commutator estimates applicable to equation (1).

Note that the dissipative quasi-geostrophic equation (1) defines a flow without external forcing. As a result of the energy dissipation, the flow motion decays for $\alpha > 0$ (see Constantin and Wu [11] and Schonbek and Schonbek [28]). The also implies the nonlinear stability of the trivial steady-state solution $\theta = 0$ and then the nonlinear stability for the quasi-geostrophic equation driven by a small steady-state external forcing. However, when the external forcing is not small and the strength of nonlinear convection is overtaking the strength of the energy dissipation by decreasing the parameter κ , nonlinear instability arises and steady-state bifurcation may occur.

In the present study, the dissipative quasi-geostrophic flow is driven by the typical thermal forcing $f = \sin x_2$ (see Iserley and Sheremet [17], Pedlosky [27] and Veronis [30]) and thus is determined by the non-homogeneous dissipative quasi-geostrophic equations

$$(2) \quad \frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa (-\Delta)^\alpha \theta = \sin x_2, \quad u = (-\partial_{x_2}, \partial_{x_1}) (-\Delta)^{-1/2} \theta.$$

Here the solution θ is spatially periodic in the domain $[0, 2\pi b] \times [0, 2\pi]$ for $0 < b \leq 1$ and (2) admits a basic steady-state solution $\hat{\theta} = \frac{1}{\kappa} \sin x_2$.

It has been shown in [7] that $\hat{\theta}$ is globally nonlinear stable for $b = 1$ and $\hat{\theta}$ bifurcates into a circle of steady-state solutions as κ increasing across a critical value $\kappa_c > 0$ for $\frac{1}{2} \leq b < 1$. The bifurcating solutions are constructed by the amplitude expansion technique, which is only applicable to the case $\alpha \geq \frac{1}{2}$. It was expected in [7] to extend this bifurcation result to the lower limit situation $\alpha = 0$ of the strong critical case. Unfortunately, we have not obtained an available estimate to ensure the compactness required in the bifurcation analysis.

However, the steady-state bifurcation result can be obtained if we generalize the dissipative quasi-geostrophic equation (2) by modifying the vector field with $u = (-\partial_{x_2}, \partial_{x_1}) (-\Delta)^{-\beta/2} \theta$ as discussed by Constantin et al. [9], Kiselev [20] and Yamazaki [34].

More precisely, the main result of this paper reads as follows:

Theorem 1.1. *For $\frac{3}{4} \leq b < 1$, $\beta > 1$ and $2/(\beta - 1) < p < \infty$, the dissipative quasi-geostrophic equation*

$$(3) \quad \frac{\partial \theta}{\partial t} + u \cdot \nabla \theta + \kappa \theta = \sin x_2, \quad u = (-\partial_{x_2}, \partial_{x_1}) (-\Delta)^{-\beta/2} \theta$$

admits a positive critical value κ_c satisfying

$$(4) \quad \kappa_c^4 < \frac{\beta}{4} b^{4-\beta} (1 - b^\beta)$$

and a classical steady-state solution $\theta_{\kappa, \phi} \in H_p^{1+\beta}(\Omega_b)$ for $0 \leq \phi < 2\pi$ bifurcating from the basic steady-state solution $\hat{\theta}_0 = \frac{1}{\kappa_c} \sin x_2$ when κ varies across κ_c .

Here $H_p^\tau(\Omega_b)$ denotes the fractional Sobolev space on the two-dimensional periodic domain $\Omega_b = \mathbb{R}^2/(2\pi bZ \times 2\pi Z)$:

$$(5) \quad H_p^\tau(\Omega_b) = \left\{ f \in L_p(\Omega_b); \int_0^{2\pi b} \int_0^{2\pi} f dx_1 dx_2 = 0, \|f\|_{H_p^\tau} < \infty \right\}$$

with the norm

$$\|f\|_{H_p^\tau} = \|f\|_{L_p} + \|(-\Delta)^{\frac{\tau}{2}} f\|_{L_p}$$

and the L_p norm

$$\|f\|_{L_p} = \left(\int_0^{2\pi b} \int_0^{2\pi} |f|^p dx_1 dx_2 \right)^{1/p}.$$

The fractional Laplacian $(-\Delta)^\tau$ is in the following sense:

$$(-\Delta)^\tau e^{-i(mb x_1 + n x_2)} = ((mb)^2 + n^2)^\tau e^{-i(mb x_1 + n x_2)}.$$

The instability property of (3) exhibits the coexistence of a circle of steady-state solutions $\theta_{\kappa, \phi}$ for $0 \leq \phi < 2\pi$ and its profile is displayed in Figure 1. For a circle bifurcation theory and the stability of bifurcating steady-state flows, one may refer to Ma and Wang [24]. The present steady-state bifurcation analysis is developed from the author [5] by using a Lagrange formulation technique and the traditional steady-state bifurcation theorem given by Krasnoselskii [23].

This paper is organized as follows. In Section 2, a Lagrangian formulation approach is developed from [5] so that the dissipative quasi-geostrophic flow in a neighborhood of the basic flow $\hat{\theta}$ in fractional Sobolev spaces is estimated and the compactness required by the bifurcation analysis is obtained. Section 3 exhibits the Lagrangian formulation for the stability derivation in Krasnos

FIGURE 1. The nonlinear instability profile for a circle of steady-state flows branching off the basic steady-state flow as κ varying across the critical value κ_c .

2. LAGRANGIAN FORMULATION FOR THE STATIONARY EQUATION OF (3)

For convenience, we employ the notation $\theta' = \kappa\theta$ and $u' = \kappa u$ to rewrite the stationary equation of (3) as

$$(6) \quad u \cdot \nabla \theta + \kappa^2 \theta = \kappa^2 \sin x_2, \quad u = (-\partial_{x_2}, \partial_{x_1}) (-\Delta)^{-\beta/2} \theta,$$

where the prime symbol have been omitted. The corresponding steady-state solution $\hat{\theta}$ of (3) becomes

$$\theta_0 = \sin x_2,$$

the basic solution of (6).

The divergence free vector field u in (6) is in the following Banach space

$$H_\infty^1(\Omega_b) = \{v \in L_\infty(\Omega_b); \|v\|_{H_\infty^1} = \|v\|_{L_\infty} + \|\nabla v\|_{L_\infty} < \infty\}.$$

Since the bifurcation in Theorem 1.1 is local, we consider the flow motion when u is close to the steady-state vector field

$$u_0 = (-\cos x_2, 0) = (-\partial_{x_2}, \partial_{x_1}) (-\Delta)^{-\frac{\beta}{2}} \theta_0.$$

For a trajectory $y = (y_1, y_2)$ initiating from a particle $x = (x_1, x_2)$ in the fluid domain, we have the Lagrangian formulation

$$(7) \quad \begin{cases} -\frac{\partial}{\partial s} y(x, s) &= u(y(x, s)), \quad s > 0, \\ y(x, 0) &= x \in \Omega_b. \end{cases}$$

Applying the gradient operator $\nabla = (\partial_{x_1}, \partial_{x_2})$ to (7), we have

$$(8) \quad -\frac{\partial}{\partial s} \nabla y(x, s) = \nabla y \cdot \nabla_y u(y(x, s)), \quad s > 0,$$

$$(9) \quad \nabla y(x, 0) = I.$$

for the 2×2 identity matrix I and the gradient $\nabla_y = (\partial_{y_1}, \partial_{y_2})$.

The flow motion around the basic steady-state vector field u_0 is described by the result.

Lemma 2.1. *Let $u \in H_\infty^1(\Omega_b)$ so that $\|\nabla u - \nabla u_0\|_{L_\infty} \leq \frac{1}{2}$. Then the estimate*

$$(10) \quad \|\nabla y(\cdot, s)\|_{L_\infty} \leq \sqrt{2}(1+s)e^{2s\sqrt{\|\nabla u - \nabla u_0\|_{L_\infty}}}$$

holds true for $s \geq 0$.

To show the validity of (10) with $u = (u_1, u_2)$, we see that

$$\|\partial_{y_2} u_1(y) - \sin y_2\|_{L_\infty} \leq \|\nabla u - \nabla u_0\|_{L_\infty}.$$

and for $(i, j) \neq (2, 1)$

$$\|\partial_{y_i} u_j(y)\|_{L_\infty} \leq \|\nabla u - \nabla u_0\|_{L_\infty}.$$

By an elementary calculation and the use of (8), we have

$$\begin{aligned} & \frac{\partial}{\partial s} \begin{pmatrix} \|\partial_{x_1} y_1\|_{L_\infty} & \|\partial_{x_2} y_1\|_{L_\infty} \\ \|\partial_{x_1} y_2\|_{L_\infty} & \|\partial_{x_2} y_2\|_{L_\infty} \end{pmatrix} \\ & \leq \begin{pmatrix} \|\partial_{y_1} u_1\|_{L_\infty} & \|\partial_{y_2} u_1\|_{L_\infty} \\ \|\partial_{y_1} u_2\|_{L_\infty} & \|\partial_{y_2} u_2\|_{L_\infty} \end{pmatrix} \begin{pmatrix} \|\partial_{x_1} y_1\|_{L_\infty} & \|\partial_{x_2} y_1\|_{L_\infty} \\ \|\partial_{x_1} y_2\|_{L_\infty} & \|\partial_{x_2} y_2\|_{L_\infty} \end{pmatrix} \\ & \leq \begin{pmatrix} \sigma & 1+\sigma \\ \sigma & \sigma \end{pmatrix} \begin{pmatrix} \|\partial_{x_1} y_1\|_{L_\infty} & \|\partial_{x_2} y_1\|_{L_\infty} \\ \|\partial_{x_1} y_2\|_{L_\infty} & \|\partial_{x_2} y_2\|_{L_\infty} \end{pmatrix} \end{aligned}$$

for $\sigma = \|\nabla u - \nabla u_0\|_{L_\infty}$. Without loss of generality, we may assume $\sigma > 0$. The combination of the previous equation and the initial identity condition (9) implies

$$\begin{aligned} & \begin{pmatrix} \|\partial_{x_1} y_1\|_{L_\infty} & \|\partial_{x_2} y_1\|_{L_\infty} \\ \|\partial_{x_1} y_2\|_{L_\infty} & \|\partial_{x_2} y_2\|_{L_\infty} \end{pmatrix} \\ & \leq \exp \left(s \begin{pmatrix} \sigma & 1+\sigma \\ \sigma & \sigma \end{pmatrix} \right) \\ & = \begin{pmatrix} \frac{\sqrt{\sigma^2+\sigma}}{2\sigma} & \frac{\sqrt{\sigma^2+\sigma}}{2\sigma} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} e^{s(\sigma+\sqrt{\sigma^2+\sigma})} & 0 \\ 0 & e^{s(\sigma-\sqrt{\sigma^2+\sigma})} \end{pmatrix} \begin{pmatrix} \frac{\sigma}{\sqrt{\sigma^2+\sigma}} & 1 \\ \frac{\sigma}{\sqrt{\sigma^2+\sigma}} & -1 \end{pmatrix}, \end{aligned}$$

and hence, for the eigenvalues $\lambda_1 = \sigma + \sqrt{\sigma^2 + \sigma}$ and $\lambda_2 = \sigma - \sqrt{\sigma^2 + \sigma}$,

$$\begin{aligned} \|\nabla y(\cdot, s)\|_{L_\infty}^2 & \leq \frac{2(e^{\lambda_1 s} + e^{\lambda_2 s})^2 + \left(\frac{\sigma+1}{\sigma} + \frac{\sigma}{\sigma+1}\right)(e^{\lambda_1 s} - e^{\lambda_2 s})^2}{4} \\ & \leq \frac{8e^{2\lambda_1 s} + \left(\frac{\sigma+1}{\sigma} + \frac{\sigma}{\sigma+1}\right)(\lambda_1 - \lambda_2)^2 s^2 e^{2\lambda_1 s}}{4} \\ & \leq (2 + s^2)e^{2s(\sigma+\sqrt{\sigma^2+\sigma})}. \end{aligned}$$

This implies (10) due to the condition $\sigma \leq \frac{1}{2}$.

With the use of the Lagrangian formulation, the equation in Euler formulation can be written in the equation in Lagrangian formulation.

Lemma 2.2. *Assume that $1 \leq p < \infty$, $\kappa > 0$, $f \in H_p^1(\Omega_b)$ and $u \in H_\infty^1(\Omega_b)$. Then the operator $\kappa^2 + u \cdot \nabla$ has a bounded inverse represented as*

$$(11) \quad (\kappa^2 + u \cdot \nabla)^{-1} f = \int_0^\infty e^{-s\kappa^2} f(y(x, s)) ds$$

in the following sense

$$(12) \quad \|(\kappa^2 + u \cdot \nabla)^{-1} f\|_{L_p} \leq \frac{1}{\kappa^2} \|f\|_{L_p}.$$

Additionally, if u is subject to the condition

$$(13) \quad \|\nabla u - \nabla u_0\|_{L_p} \leq \frac{\kappa^4}{5},$$

then there holds the estimate

$$(14) \quad \|\nabla(\kappa^2 + u \cdot \nabla)^{-1} f\|_{L_p} \leq \frac{16\sqrt{2}(\kappa^2 + 1)}{\kappa^4} \|\nabla f\|_{L_p}.$$

Proof. For $f \in H_p^1(\Omega_b)$, let φ satisfy the equation

$$(15) \quad (\kappa^2 + u(x) \cdot \nabla) \varphi(x) = f(x).$$

By the Lagrange formulation (7), we rewrite the previous equation as

$$(16) \quad (\kappa^2 + u(y(x, s)) \cdot \nabla_y) \varphi(y(x, s)) = f(y(x, s))$$

for $s \geq 0$, and hence

$$(17) \quad \left(\kappa^2 - \frac{\partial y(x, s)}{\partial s} \cdot \nabla_y \right) \varphi(y(x, s)) = f(y(x, s)),$$

or, after the multiplication of the function $-e^{-\kappa^2 s}$ and the use of (7),

$$(18) \quad \frac{\partial [e^{-\kappa^2 s} \varphi(y(x, s))]}{\partial s} = -e^{-\kappa^2 s} f(y(x, s)).$$

This gives the expression

$$(19) \quad \varphi(x) = \int_0^\infty e^{-s\kappa^2} f(y(x, s)) ds.$$

It is readily seen that the kernel of the operator $\kappa^2 + u \cdot \nabla$ is the set $\{0\}$. Thus (11) is valid.

Moreover, by the incompressible flow transformation property

$$\det(\nabla y(x, s)) \equiv 1$$

due to the divergence free condition $\nabla \cdot u = 0$, we have the desired bounds

$$(20) \quad \|(\kappa^2 + u \cdot \nabla)^{-1} f(x)\|_{L_p} \leq \int_0^\infty e^{-s\kappa^2} \|f(y(x, s))\|_{L_p} ds = \frac{1}{\kappa^2} \|f\|_{L_p},$$

and, by (10) and (13),

$$\begin{aligned} \|\nabla(\kappa^2 + u \cdot \nabla)^{-1} f(x)\|_{L_p} &\leq \int_0^\infty e^{-s\kappa^2} \|\nabla f(y(x, s))\|_{L_p} ds \\ &\leq \int_0^\infty e^{-s\kappa^2} \|\nabla y(x, s)\|_{L_\infty} \|\nabla_y f(y(x, s))\|_{L_p} ds \\ &\leq \int_0^\infty e^{-s\kappa^2} \sqrt{2}(1+s) e^{s\kappa^2 2/\sqrt{5}} ds \|\nabla f\|_{L_p} \\ &\leq \frac{\sqrt{2}}{\left(1 - \frac{2}{\sqrt{5}}\right)^4} \left(\frac{1}{\kappa^2} + \frac{1}{\kappa^4}\right) \|\nabla f\|_{L_p}. \end{aligned}$$

This gives the desired estimate and the proof is complete.

As a consequence of Lemma 2.2, equation (6) becomes

$$(21) \quad \theta(x) - \kappa^2(\kappa^2 + u \cdot \nabla)^{-1} \theta_0 = 0, \quad u = (-\partial_{x_2}, \partial_{x_1})(-\Delta)^{-\beta/2} \theta,$$

or

$$(22) \quad \psi(x) - \kappa^2(-\Delta)^{-\beta/2}(\kappa^2 + u \cdot \nabla)^{-1} \psi_0 = 0, \quad u = (-\partial_{x_2}, \partial_{x_1})\psi.$$

for the stream function $\psi = (-\Delta)^{-\beta/2} \theta$ and $\psi_0 = \theta_0 = \sin x_2$.

3. LINEAR INSTABILITY

Linearizing the operator $-\kappa^2(-\Delta)^{-\beta/2}(\kappa^2 + u \cdot \nabla)^{-1}\psi_0$ with respect to ψ around ψ_0 , we have the linear operator

$$\begin{aligned} M_\kappa \psi &= -(-\Delta)^{-\beta/2}(\kappa^2 + u_0 \cdot \nabla)^{-1}u_0 \cdot \nabla \psi \\ (23) \quad &= (-\Delta)^{-\beta/2}(\kappa^2 - \cos x_2 \partial_{x_1})^{-1} \cos x_2 \partial_{x_1} \psi. \end{aligned}$$

Therefore, we have the expression

$$(24) \quad \psi(x) - \kappa^2(\kappa^2 + u \cdot \nabla)^{-1}\psi_0 = \psi - \psi_0 + M_\kappa(\psi - \psi_0) + N(\psi, \kappa)$$

for a nonlinear operator N .

Thus to show the bifurcation behaviour of (22), it is necessary to obtain the existence of a critical value $\kappa = \kappa_c$ and an eigenfunction ψ so that

$$(25) \quad \psi + M_\kappa \psi = 0,$$

or

$$(26) \quad \kappa^2(-\Delta)^{\beta/2}\psi - \cos x_2 \partial_{x_1} [(-\Delta)^{\beta/2} - 1]\psi = 0.$$

Lemma 3.1. *Let $\frac{3}{4} \leq b < 1$, $0 \leq \phi < 2\pi$, $1 \leq \beta$ and $1 < p < \infty$. Assume that $m \neq 1$ is an integer. Then for any $\kappa > 0$*

$$(27) \quad \left\{ \sum_{n=-\infty}^{\infty} a_n \cos(bx_1 + \phi + nx_2 + \frac{n\pi}{2}) \in H_p^{1+\beta}(\Omega_b); \psi + M_\kappa \psi = 0 \right\} = \{0\},$$

and there exists a critical value $\kappa = \kappa_c$ so that

$$(28) \quad \bigcup_{k=1}^{\infty} \left\{ \psi = \sum_{n=-\infty}^{\infty} \frac{a_n \cos(bx_1 + \phi + nx_2 + \frac{n\pi}{2})}{b[(b^2 + n^2)^{\frac{\beta}{2}} - 1]} \in H_p^{1+\beta}(\Omega_b); (1 + M_\kappa)^k \psi = 0 \right\} = 1.$$

This result is obtained by a continued fraction technique developed from Chen et al. [6], Chen and Price [7] and Chen [5] and originated from Mishalkin and Sinai [25] and Yudovich [35].

Proof. Multiplying (26) by $\kappa^{-2}[(-\Delta)^{\frac{\beta}{2}} - 1]\psi$ and integrating the resultant equation over the domain Ω_b , we have the integral equation

$$(29) \quad 0 = \int_0^{2\pi/b} \int_0^{2\pi} (-\Delta)^{\frac{\beta}{2}} \psi [(-\Delta)^{\frac{\beta}{2}} \psi - \psi] dx_1 dx_2,$$

after taking integration by parts. The substitution of the function

$$\psi = \sum_{n=-\infty}^{\infty} a_n \cos(bmx_1 + \phi + nx_2 + \frac{n\pi}{2}), \quad m \neq 1.$$

into (29) implies $a_n \equiv 0$ and hence (27) is verified. Here we have used the average condition

$$(30) \quad \int_{\Omega_b} \psi dx_1 dx_2 = 0$$

implied in the space $H_p^1(\Omega_b)$ defined in (5) to confirm $a_0 = 0$ whenever $m = 0$.

For the derivation of the critical number κ_c , we use the eigenfunction

$$(31) \quad \psi = \sum_{n=-\infty}^{\infty} \frac{a_n}{b[(b^2 + n^2)^{\frac{\beta}{2}} - 1]} \cos(bx_1 + \phi + nx_2 + \frac{n\pi}{2})$$

to rewrite (26) as

$$\begin{aligned}
0 &= \sum_{n=-\infty}^{\infty} \frac{2\kappa^2(b^2 + n^2)^{\frac{\beta}{2}} a_n}{b[(b^2 + n^2)^{\frac{\beta}{2}} - 1]} \cos(bx_1 + \phi + nx_2 + \frac{n\pi}{2}) \\
&\quad + \sum_{n=-\infty}^{\infty} 2a_n \cos x_2 \sin(bx_1 + \phi + nx_2 + \frac{n\pi}{2}) \\
&= \sum_{n=-\infty}^{\infty} \left(\frac{2\kappa^2(b^2 + n^2)^{\frac{\beta}{2}} a_n}{b[(b^2 + n^2)^{\frac{\beta}{2}} - 1]} - a_{n-1} + a_{n+1} \right) \cos(bx_1 + \phi + nx_2 + \frac{n\pi}{2}),
\end{aligned}$$

and so

$$(32) \quad 2\kappa^2 d_n a_n + a_{n+1} - a_{n-1} = 0, \quad n \in Z$$

for

$$(33) \quad d_n = \frac{(b^2 + n^2)^{\frac{\beta}{2}}}{b[(b^2 + n^2)^{\frac{\beta}{2}} - 1]}.$$

This yields

$$(34) \quad 2\kappa^2 d_n \pm \frac{a_{\pm(n+1)}}{a_{\pm n}} \mp \frac{a_{\pm(n-1)}}{a_{\pm n}} = 0 \quad \text{for } n \geq 0.$$

After an iteration for $n \geq 0$, equation (34) reduces to

$$(35) \quad \gamma_{\pm n} \equiv \frac{a_{\pm n}}{a_{\pm(n-1)}} = \frac{\pm 1}{2\kappa^2 d_n \pm \frac{a_{\pm(n+1)}}{a_{\pm n}}} = \frac{\pm 1}{2\kappa^2 d_n + \frac{1}{2\kappa^2 d_{n+1} + \frac{1}{\ddots}}} \quad \text{for } n \geq 1.$$

Combining (35) for $n = 1$ and (33)-(34) for $n = 0$, we have equivalent formulation of the iteration equation (32):

$$(36) \quad -d_0 = \frac{b^{\beta-1}}{1 - b^{\beta}} = \frac{1}{2\kappa^4 d_1 + \frac{1}{d_2 + \frac{1}{2\kappa^4 d_3 + \frac{1}{d_4 + \frac{1}{\ddots}}}}}.$$

The function $P(\kappa)$, representing the right-hand side term of (36), is the Stieltjes continued fraction. It follows from [29] or [31, Theorem 28.1] that $P(\kappa)$ uniformly convergent to a positive value and is an analytic function of $\kappa > 0$. Upon the observation of P being strictly monotone function of κ such that

$$\lim_{\kappa \rightarrow \infty} P(\kappa) = 0, \quad \lim_{\kappa \rightarrow 0} P(\kappa) = \infty,$$

there exists a unique critical value $\kappa = \kappa_c > 0$ satisfying (36). Thus for such a critical value $\kappa = \kappa_c$, the coefficients a_n of the associated eigenfunction ψ in the form of (31) are subject to the expression

$$(37) \quad a_n = \begin{cases} \gamma_1 \cdots \gamma_n, & n \geq 1, \\ 1, & n = 0, \\ (-1)^n a_{-n}, & n \leq -1. \end{cases}$$

It is readily seen that eigenfunction ψ obtained is smooth. Thus we have

$$(38) \text{Im} \left\{ \psi = \sum_{n=-\infty}^{\infty} \frac{a_n \cos(bx_1 + \phi + nx_2 + \frac{n\pi}{2})}{b[(b^2 + n^2)^{\frac{\beta}{2}} - 1]} \in H_p^{1+\beta}(\Omega_b); (1+M_\kappa)\psi = 0 \right\} = 1.$$

For the spectral simplicity given in (28), it is sufficient to verify the property

$$(39) \text{Im} \left\{ \psi = \sum_{n=-\infty}^{\infty} \frac{a_n \cos(bx_1 + \phi + nx_2 + \frac{n\pi}{2})}{b[(b^2 + n^2)^{\frac{\beta}{2}} - 1]} \in H_p^{1+\beta}(\Omega_b); (1+M_\kappa)^2\psi = 0 \right\} = 1.$$

We see that the equation $(1+M_\kappa)^2\psi = 0$ can be written in the form

$$(40) \quad \psi' + M_\kappa \psi' = 0 \quad \text{and} \quad \psi' = \psi + M_\kappa \psi$$

or, equivalently,

$$(41) \quad \kappa^2(-\Delta)^{\frac{\beta}{2}}\psi' - \cos x_2((-\Delta)^{\frac{\beta}{2}} - 1)\partial_{x_1}\psi' = 0,$$

$$(42) \quad \kappa^2(-\Delta)^{\frac{\beta}{2}}\psi - \cos x_2((-\Delta)^{\frac{\beta}{2}} - 1)\partial_{x_1}\psi = (\kappa^2 - \cos x_2\partial_{x_1})(-\Delta)^{\frac{\beta}{2}}\psi'.$$

for

$$\psi' = \sum_{n=-\infty}^{\infty} \frac{a'_n \cos(bx_1 + \phi + nx_2 + \frac{n\pi}{2})}{b[(b^2 + n^2)^{\frac{\beta}{2}} - 1]}.$$

Thus (41)-(42) reduce respectively to the iteration equations

$$(43) \quad 2\kappa^2 d_n a'_n + a'_{n+1} - a'_{n-1} = 0,$$

and

$$(44) \quad 2\kappa^2 d_n a_n + a_{n+1} - a_{n-1} = 2\kappa^2 d_n a'_n + b d_{n+1} a'_{n+1} - b d_{n-1} a'_{n-1}$$

for any arbitrary integer n . Therefore from the demonstration of (38), it remains to prove that $\psi' = 0$ or $a'_n \equiv 0$.

On the contrary, we assume that $a'_n \neq 0$ and hence satisfies (37). We show a contradiction from this assumption.

Multiplying the n th equation of (43) by $(-1)^n a_n$ and the n th equation of (44) by $(-1)^n a'_n$ and then summing the resultant equations respectively, we have

$$(45) \quad 0 = \sum_{n=-\infty}^{\infty} (-1)^n a_n [2\kappa^2 d_n a'_n + a'_{n+1} - a'_{n-1}],$$

and

$$(46) \quad \begin{aligned} & \sum_{n=-\infty}^{\infty} (-1)^n a'_n [2\kappa^2 d_n a_n + a_{n+1} - a_{n-1}] \\ &= \sum_{n=-\infty}^{\infty} (-1)^n a'_n [2\kappa^2 d_n a'_n + b d_{n+1} a'_{n+1} - b d_{n-1} a'_{n-1}]. \end{aligned}$$

Rearranging terms in the summations, we see that the right-hand side term of (45) is identical to the left-hand side term of (46). Thus (46) becomes

$$0 = \sum_{n=-\infty}^{\infty} (-1)^n 2\kappa^2 d_n a_n'^2 + \sum_{n=-\infty}^{\infty} (-1)^n b a'_n [d_{n+1} a'_{n+1} - d_{n-1} a'_{n-1}],$$

or

$$\begin{aligned}
 0 &= \sum_{n=1}^{\infty} (-1)^n 4\kappa^2 d_n a_n'^2 + 2\kappa^2 d_0 a_0'^2 \\
 (47) \quad &+ \sum_{n=0}^{\infty} 2(-1)^n b d_{n+1} a_n' a_{n+1}' - \sum_{n=1}^{\infty} 2(-1)^n b d_{n-1} a_n' a_{n-1}',
 \end{aligned}$$

after the use of the identities $a'_{-n} = (-1)^n a'_n$ and $d_n = d_{-n}$ given by (33) and (37). By the identities

$$\frac{a_n}{a_{n-1}} = \gamma_n, \quad \frac{1}{\gamma_n} - \gamma_{n+1} = 2\kappa^2 d_n, \quad \gamma_1 = -d_0 \kappa^2$$

implied from (32) and (35), the term of (47) including the last two summations is further manipulated as

$$\begin{aligned}
 &\sum_{n=0}^{\infty} 2(-1)^n b d_{n+1} a_n' a_{n+1}' - \sum_{n=1}^{\infty} 2(-1)^n b d_{n-1} a_n' a_{n-1}' \\
 &= \sum_{n=0}^{\infty} 2(-1)^n b \frac{d_{n+1} a_{n+1}'^2}{\gamma_{n+1}} - \sum_{n=1}^{\infty} 2(-1)^n b d_{n-1} a_{n-1}'^2 \gamma_n \\
 &= - \sum_{n=1}^{\infty} 2(-1)^n b d_n a_n'^2 \frac{1}{\gamma_n} + \sum_{n=0}^{\infty} 2(-1)^n b d_n a_n'^2 \gamma_{n+1} \\
 &= - \sum_{n=1}^{\infty} 4(-1)^n \kappa^2 b d_n^2 a_n'^2 + 2b d_0 a_0'^2 \gamma_1, \\
 (48) \quad &= - \sum_{n=1}^{\infty} 4(-1)^n \kappa^2 b d_n^2 a_n'^2 - 2b d_0^2 a_0'^2 \kappa^2.
 \end{aligned}$$

On the substitution of (48) into (47), we have

$$0 = \sum_{n=1}^{\infty} (-1)^n 4\kappa^2 (1 - b d_n) d_n a_n'^2 + 2\kappa^2 (1 - b d_0) d_0 a_0'^2.$$

That is, by employing (33) and removing the factor $4\kappa^2$,

$$(49) \quad 0 = \sum_{n=1}^{\infty} \frac{(-1)^n d_n a_n'^2}{(b^2 + n^2)^{\frac{\beta}{2}} - 1} - \frac{d_0 a_0'^2}{2(1 - b^\beta)}.$$

On the other hand, multiplying the n th equation of (43) by $a_n'/[4\kappa^2(1 - b^\beta)]$ and summing the resultant equations yield

$$(50) \quad 0 = \sum_{n=-\infty}^{\infty} \frac{d_n a_n'^2}{2(1 - b^\beta)} = \sum_{n=1}^{\infty} \frac{d_n a_n'^2}{1 - b^\beta} + \frac{d_0 a_0'^2}{2(1 - b^\beta)}.$$

Equation (49) plus equation (50) gives that

$$\begin{aligned}
0 &= \sum_{n=1}^{\infty} \frac{(-1)^n d_n a_n'^2}{(b^2 + n^2)^{\frac{\beta}{2}} - 1} + \sum_{n=1}^{\infty} \frac{d_n a_n'^2}{1 - b^\beta} \\
&\geq \left(\frac{1}{1 - b^\beta} - \frac{1}{(b^2 + 1)^{\frac{\beta}{2}} - 1} \right) d_1 a_1'^2 + \sum_{n=2}^{\infty} \left(\frac{1}{1 - b^\beta} - \frac{1}{(b^2 + n^2)^{\frac{\beta}{2}} - 1} \right) d_n a_n'^2 \\
&> 4 \sum_{n=2}^{\infty} \frac{(b^2 + 4)^{\beta/2} - 2}{(1 - b^\beta)[(b^2 + n^2)^{\frac{\beta}{2}} - 1]} d_n a_n'^2,
\end{aligned}$$

since $\frac{3}{4} \leq b < 1$ and $\beta > 1$. This leads to a contradiction and hence implies the desired assertion $a_n' \equiv 0$.

The proof of Lemma 3.1 is completed.

4. KRASNOSELSKII BIFURCATION THEOREM

Krasnoselskii bifurcation theorem is fundamental to the proof of Theorem 1.1. For convenience, it is stated in the following.

Theorem 4.1. (*Krasnoselskii [23] and Nirenberg [26]*) *For a Banach space X , a constant value $\kappa_{\text{crit}} > 0$ and an open neighborhood $\mathcal{D}$ of the point $(0, \kappa_{\text{crit}})$ in the metric space $X \times [0, \infty)$, let F , M_κ and N be the operators expressed as*

$$F(\psi, \kappa) = \psi + M_\kappa \psi + N(\psi, \kappa), \quad (\psi, \kappa) \in \mathcal{D},$$

and subject to the following conditions:

- (i): $F : \mathcal{D} \mapsto X$ is continuous,
- (ii): $M_\kappa : X \mapsto X$ is linear and $M_\kappa : \mathcal{D} \mapsto \mathcal{D}$ is compact and continuous,
- (iii): $N : \mathcal{D} \mapsto X$ is nonlinear and compact, $N(0, \kappa) \equiv 0$ and $N(\psi, \kappa) = o(\|\psi\|_X)$ uniformly for $(\psi, \kappa) \in \mathcal{D}$,
- (iv): the spectral simplicity condition

$$\dim \bigcup_{n=1}^{\infty} \{\psi \in X; (1 + M_{\kappa_{\text{crit}}})^n \psi = 0\} = 1$$

holds true at the critical value κ_{crit} .

Then there exists a continuous family $(\psi_\kappa, \kappa) \in \mathcal{D}$, different to the trivial one $(0, \kappa)$, such that

$$(51) \quad F(\psi_\kappa, \kappa) = 0.$$

5. PROOF OF THEOREM 1.1

To verify the conditions of Krasnoselskii Theorem, we define the operator

$$F(\psi, \kappa) = \psi - \kappa^2 (-\Delta)^{-\frac{\beta}{2}} (\kappa^2 + u \cdot \nabla)^{-1} \psi_0.$$

Therefore, $F = 0$ represents the steady-state equation of Theorem 1.1. Thus we have the desired decomposition form

$$F(\psi, \kappa) = \psi - \psi_0 + M_\kappa(\psi - \psi_0) + N(\psi, \kappa), \quad \psi_0 = \sin x_2$$

for the linear operator M_κ and the nonlinear operator N given in (23) and (24).

Firstly, for a constant τ so that $1 + 2/p < \tau < \beta$, we introduce the space $X = X_\phi$ based on the spectral analysis in Section 3 as

$$X_\phi = \left\{ \psi \in H_p^{1+\tau}(\Omega_b); \psi = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} a_{m,n} \cos(mbx_1 + m\phi + nx_2 + \frac{n\pi}{2}) \right\}.$$

Upon the observation of the flow invariance property that

$$u' \cdot \nabla \psi = (-\partial_{x_2} \psi', \partial_{x_1} \psi') \cdot \nabla \psi = \sum_{m=0}^{\infty} \sum_{n=-\infty}^{\infty} \tilde{a}_{m,n} \cos(mbx_1 + m\phi + nx_2 + \frac{n\pi}{2})$$

for some coefficients $\tilde{a}_{m,n}$ whenever $\psi', \psi \in X_\phi$ and $u = (-\partial_{x_2} \psi, \partial_{x_1} \psi)$, we have the flow invariance property

$$(-\Delta)^{-\frac{\beta}{2}} [\kappa^2 + u \cdot \nabla]^{-1} : X_\phi \mapsto X_\phi \quad \text{for } \psi \in X_\phi$$

due to Lemma 2.2. By Lemma 2.2, the choice of the space X also ensures the existence of the critical value $\kappa_c > 0$ and the spectral simplicity condition (iv)

$$\dim \bigcup_{n=1}^{\infty} \{\psi \in X; (1 + M_{\kappa_c})^n \psi = 0\} = 1.$$

Moreover, we define the metric space

$$\mathcal{D} = \left\{ \psi \in X_\phi; \|\nabla^2 \psi - \nabla^2 \psi_0\|_{L_\infty} \leq \frac{(\kappa_c - \epsilon)^4}{5} \right\} \times [\kappa_c - \epsilon, \kappa_c + \epsilon].$$

Secondly, we verify the assumptions (i) of Theorem 4.1. By Lemma 2.2, we have the bound

$$(52) \quad \|F(\psi, \kappa) - \psi\|_{H_p^{1+\beta}} = \kappa^2 \|(\kappa^2 + u \cdot \nabla)^{-1} \psi_0\|_{H_p^1} \leq c \|\psi_0\|_{H_p^1},$$

where and in what follows c denotes a generic constant independent of $(\psi, \kappa) \in \mathcal{D}$.

For the continuity of F with respect to (ψ, κ) , $(\psi', \kappa') \in \mathcal{D}$ and $u = (-\partial_{x_2} \psi, \partial_{x_1} \psi)$ and $u' = (-\partial_{x_2} \psi', \partial_{x_1} \psi')$, we further use Lemma 2.2 to obtain

$$\begin{aligned} & \| [F(\psi', \kappa') - F(\psi, \kappa) - (\psi' - \psi)] \|_{H_p^\beta} \\ & \leq |\kappa'^2 - \kappa^2| \|(\kappa'^2 + u' \cdot \nabla)^{-1} \psi_0\|_{L_p} + \kappa^2 \|(\kappa^2 + u \cdot \nabla)^{-1} \psi_0 - (\kappa'^2 + u' \cdot \nabla)^{-1} \psi_0\|_{L_p} \\ & \leq c |\kappa^2 - \kappa'^2| \|\psi_0\|_{L_p} \\ & \quad + \kappa^2 \|(\kappa'^2 + u' \cdot \nabla)^{-1} [\kappa^2 - \kappa'^2 + (u - u') \cdot \nabla] (\kappa^2 + u \cdot \nabla)^{-1} \psi_0\|_{L_p} \\ & \leq c |\kappa^2 - \kappa'^2| + c \|[\kappa^2 - \kappa'^2 + (u - u') \cdot \nabla] (\kappa^2 + u \cdot \nabla)^{-1} \psi_0\|_{L_p} \\ & \leq c |\kappa^2 - \kappa'^2| + c (|\kappa^2 - \kappa'^2| + \|u - u'\|_{L_\infty}) \|(\kappa^2 + u \cdot \nabla)^{-1} \psi_0\|_{H_p^1} \\ & \leq c |\kappa^2 - \kappa'^2| + c (|\kappa^2 - \kappa'^2| + \|\nabla \psi - \nabla \psi'\|_{L_\infty}) \|\psi_0\|_{H_p^1}. \end{aligned}$$

Therefore, combining (52) and the interpolation inequality

$$(53) \quad \|\varphi\|_{H_p^{1+\tau}} \leq c \|\varphi\|_{H_p^\beta}^{\beta-\tau} \|\varphi\|_{H_p^{1+\beta}}^{1-(\beta-\tau)},$$

we have $F : \mathcal{D} \mapsto \mathcal{D}$ and the continuity

$$(54) \quad \|F(\psi', \kappa') - F(\psi, \kappa) - (\psi' - \psi)\|_{H_p^{1+\tau}} \leq c (|\kappa^2 - \kappa'^2| + \|\psi - \psi'\|_{H_\infty^2})^{\beta-\tau}.$$

Thirdly, the validity of the condition (ii) of Theorem 4.1 with respect to the linear operator M_κ is obtained in the similar manner. It follows from Lemma 2.2 that for $(\kappa, \psi), (\kappa', \psi') \in \mathcal{D}$,

$$\begin{aligned} & \|M_\kappa \psi - M_{\kappa'} \psi'\|_{H_p^{1+\beta}} \\ & \leq \|(\kappa^2 + u_0 \cdot \nabla)^{-1} (u \cdot \nabla \psi_0 - u' \cdot \nabla \psi_0)\|_{H_p^1} + \|[(\kappa^2 + u_0 \cdot \nabla)^{-1} - (\kappa'^2 + u_0 \cdot \nabla)^{-1}] u \cdot \nabla \psi_0\|_{H_p^1} \\ & \leq c \|u \cdot \nabla \psi_0 - u' \cdot \nabla \psi_0\|_{H_p^1} + |\kappa^2 - \kappa'^2| \|(\kappa^2 + u_0 \cdot \nabla)^{-1} (\kappa'^2 + u_0 \cdot \nabla)^{-1} u' \cdot \nabla \psi_0\|_{H_p^1} \\ & \leq c \|\psi - \psi'\|_{H_p^2} + c |\kappa^2 - \kappa'^2| \|\psi'\|_{H_p^2}. \end{aligned}$$

This gives the required compactness and continuity property

$$(55) \quad \|M_\kappa \psi - M_{\kappa'} \psi'\|_{H_p^{1+\beta}} \leq c |\kappa - \kappa'| + c \|\psi - \psi'\|_{H_p^{1+\tau}},$$

since the imbedding $H_p^{1+\beta} \hookrightarrow H_p^{1+\tau}$ is compact.

Finally, to verify the assumptions (iii), we notice that the compactness and continuity of the operator

$$N(\psi, \kappa) = F(\psi, \kappa) - \psi + \psi_0 - \kappa M_\kappa(\psi - \psi_0)$$

has been given by (54) and (55) since the imbedding $H_p^{1+\tau}(\Omega_b) \hookrightarrow H_\infty^2(\Omega_b)$ is compact due to the condition $1 + 2/p < \tau$.

It now remains to prove the nonlinear property. To do so, we apply the identities

$$(-\Delta)^{-\frac{\beta}{2}} \kappa^2 (\kappa^2 + u_0 \cdot \nabla)^{-1} \psi_0 = (-\Delta)^{-\frac{\beta}{2}} \psi_0 = \psi_0,$$

and

$$u_0 \cdot \nabla (\psi - \psi_0) = -(u - u_0) \cdot \nabla \psi_0$$

to the operator

$$N = -\kappa^2 (-\Delta)^{-\frac{\beta}{2}} (\kappa^2 + u \cdot \nabla)^{-1} \psi_0 + \psi_0 + (-\Delta)^{-\frac{\beta}{2}} (\kappa^2 + u_0 \cdot \nabla)^{-1} u_0 \cdot \nabla (\psi - \psi_0)$$

to obtain that

$$\begin{aligned} (-\Delta)^{\frac{\beta}{2}} N(\psi, \kappa) &= \kappa^2 (\kappa^2 + u \cdot \nabla)^{-1} (u - u_0) \cdot \nabla (\kappa^2 + u_0 \cdot \nabla)^{-1} \psi_0 \\ &\quad - (\kappa^2 + u_0 \cdot \nabla)^{-1} (u - u_0) \cdot \nabla \psi_0 \\ &= [(\kappa^2 + u \cdot \nabla)^{-1} - (\kappa^2 + u_0 \cdot \nabla)^{-1}] (u - u_0) \cdot \nabla \psi_0 \\ &= (\kappa + u \cdot \nabla)^{-1} (u - u_0) \cdot \nabla (\kappa + u_0 \cdot \nabla)^{-1} (u - u_0) \cdot \nabla \psi_0. \end{aligned}$$

This together with Lemma 2.2 implies

$$\begin{aligned} \|N(\psi, \kappa)\|_{H_p^\beta} &\leq c \|(u - u_0) \cdot \nabla (\kappa^2 + u_0 \cdot \nabla)^{-1} (u - u_0) \cdot \nabla \psi_0\|_{L_p} \\ &\leq c \|\nabla \psi - \nabla \psi_0\|_{L_\infty} \|\nabla (\kappa^2 + u_0 \cdot \nabla)^{-1} (u - u_0) \cdot \nabla \psi_0\|_{L_p} \\ (56) \quad &\leq c \|\nabla \psi - \nabla \psi_0\|_{L_\infty} \|\psi - \psi_0\|_{H_p^2} \leq c \|\psi - \psi_0\|_{H_p^2}^2 \end{aligned}$$

and

$$\begin{aligned} \|N(\psi, \kappa)\|_{H_p^{1+\beta}} &= \|[(\kappa^2 + u \cdot \nabla)^{-1} - (\kappa^2 + u_0 \cdot \nabla)^{-1}] (u - u_0) \cdot \nabla \psi_0\|_{H_p^1} \\ &\leq c \|(u - u_0) \cdot \nabla \psi_0\|_{H_p^1} \\ (57) \quad &\leq c \|\psi - \psi_0\|_{H_p^2}. \end{aligned}$$

Applying the interpolation inequality (53) to the estimates (56) and (57), we have the desired inequality

$$(58) \quad \|N(\psi, \kappa)\|_{H_p^{1+\tau}} \leq c \|\psi - \psi_0\|_{H_p^2}^{2(\beta-\tau)+1-(\beta-\tau)} \leq c \|\psi - \psi_0\|_{H_p^{1+\tau}}^{1+(\beta-\tau)}.$$

The bifurcation assertion of Theorem 1.1 is thus follows from Theorem 4.1. The upper bound (4) of the critical value κ_c is from the inequality

$$\frac{b^{\beta-1}}{1-b^\beta} \leq \frac{1}{2\kappa_c^4 \frac{(1+b^2)^{\frac{\beta}{2}}}{b[(1+b^2)^{\frac{\beta}{2}}-1]}} \leq \frac{\beta b^3}{4\kappa_c^4},$$

which is derived from (36). The proof of Theorem 1.1 is complete.

Acknowledgment. This research is partially supported by NSFC of China (Grant No. 11571240).

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