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CLASSIFICATION OF PHASE TRANSITIONS IN THIN STRUCTURES WITH SMALL GINZBURG–LANDAU PARAMETER*

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Abstract. Thin superconducting structures are considered. We compute the limit where the thickness *and* the Ginzburg–Landau parameter tend simultaneously to zero with a preferred scaling. The new equations enable us to divide the parameter space into regimes of first order or second order phase transition. The results are discussed in light of recent experiments.

Key words. superconductivity, phase transitions

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1. Introduction. Superconducting materials exhibit a phase transition from a normal state, in which they behave like conventional metals, to a superconducting state in which they can support electric currents without resistance and exhibit the so-called Meissner effect, namely a tendency to expel magnetic fields. This phase transition is commonly associated with a critical temperature T_c, below which the material enters the superconducting state. This, however, applies only in the absence of a magnetic field. Where a magnetic field H is applied to the sample the transition takes place across some (sample geometry dependent) curve in H-T space which we represent schematically in Figure 1.1. The purpose of this paper is to investigate this phase transition for thin structures made from a certain class of superconducting materials at low magnetic fields. In order to do this we make use of the Ginzburg–Landau (GL) model of superconductivity [7]. In this model the superconducting charge carriers (electron pairs) are represented by a complex order parameter $\psi(\boldsymbol{x})$, which is defined such that $|\psi(\boldsymbol{x})|^2$ is proportional to the number density of these charge carriers.

Phase transitions are classified into two types: first order (discontinuous) and second order (continuous). First order phase transitions are associated with a jump in some quantity (in this case the GL order parameter ψ) as a controlling parameter is varied (in this case temperature T). They are also associated with hysteresis in this quantity as the controlling parameter is swept up and down through the phase transition. In a second order phase transition the quantity of interest (here ψ) bifurcates continuously from its initial state (the normal state $\psi = 0$). Both types of transition are exhibited by superconductors in the change from the normal state $\psi = 0$ to the superconducting state $\psi \neq 0$. In bulk samples the order of this transition depends upon a material property, namely the GL parameter κ. It is known that the transition is of first order if $\kappa < 1/\sqrt{2}$ and second order if $\kappa > 1/\sqrt{2}$ [17]. This is one of the facts that has led to a distinction being drawn between type-I materials ($\kappa < 1/\sqrt{2}$) and type-II materials ($\kappa > 1/\sqrt{2}$). In contrast to bulk superconductors recent experiments [11, 19] on thin structures indicate a second order transition even in materials

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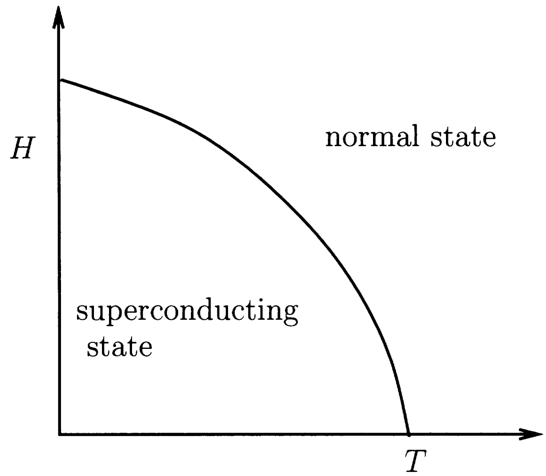

FIG. 1.1. *A schematic representation of the dependence of normal/superconducting phase transition on temperature T and magnetic field H.*

with rather small values of κ (for example, aluminum, where $\kappa = 0.28$). In order to explain this difference in behaviors we investigate the limit in which the thickness of the structure goes to zero at the same time as κ tends to zero. In particular we choose the distinguished limit which captures the crossover between type-II behavior, favored by thin geometries, and type-I behavior, favored by low values of κ.

From a mathematical viewpoint, a second order phase transition occurs whenever the bifurcation of the superconducting solution from the normal solution is supercritical, and a first order transition occurs whenever it is subcritical. A heuristic argument can be made for the magnetic field generated by superconducting currents favoring a subcritical bifurcation. Therefore one can say that a strong Meissner effect (low κ) is generally associated with a first order transition, whereas a weak Meissner effect (large κ) is typically associated with a second order phase transition. If, however, the sample has at least one thin dimension, then the sample's ability to change the magnetic field via the Meissner effect is diminished. Thus thin samples of low-κ materials may exhibit second order transitions. We shall derive models for superconductivity in thin domains in the distinguished limit, as the thickness of the domain and κ tend to zero, in which it is possible to find both first and second order phase transitions. We shall use each of these models to derive an eigenvalue problem whose solution determines the position of the normal/superconducting phase transition in H–T phase space; and to find a criterion which may be used to determine whether this phase transition is of first or second order. A similar eigenvalue problem and criterion have been derived from the full GL equations by Chapman [5] for a body of arbitrary shape. However, the practical application of this more general treatment is limited by the difficulty of solving the eigenvalue problem.

Limit models for thin superconducting domains have previously been derived by Chapman, Du, and Gunzburger [6] (thin films) and by Rubinstein and Schatzman in [14] (thin strips) and [15] (thin networks). In these works the authors consider the limit where the thickness of the domain goes to zero. In [13] we have derived a different model in which we considered the limit of vanishing thickness together with strong applied magnetic fields. All these models have proved useful to study a variety of problems. In particular they have been used to confirm experimental results such as the Little–Parks oscillation [10], [8], and even to predict new effects [3] (see also [12]). The new models we derive here complement these works. They enable us to derive precise criteria for classifying the phase transitions. They also serve as a useful tool for understanding such structures well into the nonlinear regime.

In section 2 we formulate the GL model for superconductivity. It turns out that the appropriate canonical scaling depends on the geometry and even on the topology of the sample. We therefore consider separately the cases of thin cylindrical shells (section 3), thin films (section 4), and thin wires (section 5). In each case we consider how the magnitude of the GL order parameter ψ varies with temperature for different constant magnetic fields (in practice this is mathematically convenient). However, we lose no generality in doing so as the surface representing $|\psi|$ as a function of magnetic field and temperature can be reconstructed from these slices. Finally we discuss our findings in section 6. One of our main conclusions is that the classification of materials into type-I or type-II is fairly meaningless in mesoscopic domains. A material can exhibit both types of behavior depending upon its geometry. In fact, even when the geometry is fixed, the type of phase transition can vary as the applied magnetic field changes.

2. Problem formulation. The GL equations are, appropriately nondimensionalized, as follows:

$$(\nabla - i\boldsymbol{A})^2 \psi = \Gamma\left(|\psi|^2 - 1\right)\psi, \tag{2.1}$$

$$\nabla \wedge (\nabla \wedge \boldsymbol{A}) = -\frac{\Gamma}{\kappa^2}\left(|\psi|^2 \boldsymbol{A} + \frac{i}{2}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)\right), \tag{2.2}$$

$$\boldsymbol{B} = \nabla \wedge \boldsymbol{A}, \tag{2.3}$$

$$[\boldsymbol{B}\cdot\boldsymbol{N}]_{\partial V} = 0, \qquad \left[\frac{1}{\mu}\boldsymbol{B}\wedge\boldsymbol{N}\right]_{\partial V} = 0, \tag{2.4}$$

$$\boldsymbol{N}\cdot(\nabla - i\boldsymbol{A})\,\psi|_{\partial V} = 0. \tag{2.5}$$

Here the dimensionless parameter κ is the GL parameter, and Γ is related to the coherence length $\xi(T)$ by

$$\Gamma = \frac{l^2}{\xi^2},$$

where l is the problem lengthscale. Close to T_c the critical temperature, below which superconducting state is energetically favorable in the absence of magnetic field, Γ can be approximated by $\Gamma = a(T_c - T)$ where a is a positive constant.

It is possible to reduce the number of dependent variables in (2.1)–(2.3) by the introduction of the gauge invariant variables

$$\psi = f e^{i\chi}, \qquad \boldsymbol{Q} = \boldsymbol{A} - \nabla\chi. \tag{2.6}$$

This leads to the following nondimensional system of equations in V:

$$\nabla^2 f = \Gamma\left(f^3 - f\right) + f|\boldsymbol{Q}|^2, \tag{2.7}$$

$$\boldsymbol{j} = \nabla \wedge \boldsymbol{B} = -\frac{\Gamma}{\kappa^2}\left(f^2 \boldsymbol{Q}\right), \tag{2.8}$$

$$\boldsymbol{B} = \nabla \wedge \boldsymbol{Q}, \tag{2.9}$$

which couple to Maxwell's equations

$$\left.\begin{array}{ll}\nabla \wedge \boldsymbol{B} = \mathbf{0}, & \nabla \cdot \boldsymbol{B} = 0, \\ \boldsymbol{B} \to B_{ext}\boldsymbol{e}_z & \text{as } |\boldsymbol{x}| \to \infty\end{array}\right\}, \tag{2.10}$$

in the exterior domain V^c via the jump conditions (2.4) and the boundary conditions

$$\boldsymbol{Q} \cdot \boldsymbol{N}|_{\partial V} = 0, \qquad \left.\frac{\partial f}{\partial N}\right|_{\partial V} = 0. \tag{2.11}$$

3. A thin cylindrical shell. Consider a thin superconducting cylindrical shell V, with axis in the z-direction, subject to an axial magnetic field. Then the center surface of the cylinder can in general be described by $\boldsymbol{r} = (x, y) = \mathbf{q}(s)$ and the inner and outer surfaces by

$$\boldsymbol{r} = \mathbf{q}(s) + \epsilon D(s)\boldsymbol{n}(s), \qquad \boldsymbol{r} = \mathbf{q}(s) - \epsilon D(s)\boldsymbol{n}(s),$$

respectively, where $\boldsymbol{n}(s)$ is the inward pointing normal to the surface $\boldsymbol{r} = \mathbf{q}(s)$ and $\epsilon = \alpha\kappa^2 \ll 1$ (note that $\boldsymbol{n}$ is not the same as $\boldsymbol{N}$, the normal to the surface of the shell). Points $\boldsymbol{x}$ within the cylinder may be described in terms of the orthogonal local coordinates X, s, and z such that

$$\boldsymbol{x} = \mathbf{q}(s) - \epsilon X\boldsymbol{n}(s) + z\boldsymbol{e}_z$$

(here s is the dimensionless distance around the two-dimensional curve $\boldsymbol{r} = \mathbf{q}(s)$ which increases by 2π as a complete circuit is made). Since this coordinate system is orthogonal we can use standard results to write the vector operators grad, div, curl, and Laplacian in terms of the derivatives of X, s, and z and the basis vectors $\boldsymbol{e}_x = \boldsymbol{x}_X/|\boldsymbol{x}_X|$, $\boldsymbol{e}_s = \boldsymbol{x}_s/|\boldsymbol{x}_s|$, and $\boldsymbol{e}_z$. In terms of these coordinates we find

$$\nabla\Omega = \frac{\boldsymbol{e}_x}{\epsilon}\frac{\partial\Omega}{\partial X} + \frac{\boldsymbol{e}_s}{(1+\epsilon kX)}\frac{\partial\Omega}{\partial s} + \boldsymbol{e}_z\frac{\partial\Omega}{\partial z}, \tag{3.1}$$

$$\nabla \cdot \boldsymbol{A} = \frac{1}{\epsilon}\frac{\partial A_1}{\partial X} + \frac{kA_1}{(1+\epsilon kX)} + \frac{1}{(1+\epsilon kX)}\frac{\partial A_2}{\partial s} + \frac{\partial A_3}{\partial z}, \tag{3.2}$$

$$\nabla^2\Omega = \frac{1}{\epsilon^2}\frac{\partial^2\Omega}{\partial X^2} + \frac{k}{\epsilon(1+\epsilon kX)}\frac{\partial\Omega}{\partial X} + \frac{1}{(1+\epsilon kX)}\frac{\partial}{\partial s}\left(\frac{1}{(1+\epsilon kX)}\frac{\partial\Omega}{\partial s}\right) + \frac{\partial^2\Omega}{\partial z^2}, \tag{3.3}$$

$$\begin{aligned}\nabla \wedge \boldsymbol{A} = {}& \boldsymbol{e}_x\left(\frac{1}{(1+\epsilon kX)}\frac{\partial A_3}{\partial s} - \frac{\partial A_2}{\partial z}\right) + \boldsymbol{e}_s\left(\frac{\partial A_1}{\partial z} - \frac{1}{\epsilon}\frac{\partial A_3}{\partial X}\right) \\ &+ \boldsymbol{e}_z\left(\frac{1}{\epsilon}\frac{\partial A_2}{\partial X} + \frac{kA_2}{(1+\epsilon kX)} - \frac{1}{(1+\epsilon kX)}\frac{\partial A_1}{\partial s}\right),\end{aligned} \tag{3.4}$$

where we write $\boldsymbol{A} = A_1\boldsymbol{e}_x + A_2\boldsymbol{e}_s + A_3\boldsymbol{e}_z$ and k is the curvature of $\boldsymbol{r} = \boldsymbol{q}(s)$. We look for a solution to (2.7)–(2.9) of the form

$$\boldsymbol{B} = B(X,s)\boldsymbol{e}_z, \qquad \boldsymbol{Q} = Q_1(X,s)\boldsymbol{e}_x + Q_2(X,s)\boldsymbol{e}_s, \qquad f = f(X,s),$$

use the vector operators found in (3.1)–(3.4), and expand in powers of ϵ to find

$$\frac{1}{\epsilon^2}f_{XX} + \frac{k}{\epsilon}f_X - k^2Xf_X + f_{ss} = \Gamma(f^3 - f) + f(Q_1^2 + Q_2^2) + O(\epsilon f), \tag{3.5}$$

$$B_s\boldsymbol{e}_x - \frac{1}{\epsilon}B_X\boldsymbol{e}_s = -\frac{\alpha\Gamma}{\epsilon}f^2\left(Q_1\boldsymbol{e}_x + Q_2\boldsymbol{e}_s\right) + O(\epsilon B), \tag{3.6}$$

$$B = \frac{1}{\epsilon}Q_{2,X} + kQ_2 - Q_{1,s} + O(\epsilon|\boldsymbol{Q}|). \tag{3.7}$$

We take the divergence of (2.8) and expand this in powers of ϵ to find

$$\frac{1}{\epsilon}\frac{\partial}{\partial X}\left(f^2 Q_1\right) + \frac{\partial}{\partial s}\left(f^2 Q_2\right) + k f^2 Q_1 = O(\epsilon f^2 |\boldsymbol{Q}|), \tag{3.8}$$

and finally we write down the boundary conditions (2.11) in terms of the new variables and in powers of ϵ:

$$\left(\frac{1}{\epsilon} f_X - \epsilon D'(s) f_s\right)\bigg|_{X=D(s)} = O(\epsilon^2 f), \quad \left(\frac{1}{\epsilon} f_X + \epsilon D'(s) f_s\right)\bigg|_{X=-D(s)} = O(\epsilon^2 f), \tag{3.9}$$

$$(Q_1 - \epsilon D'(s) Q_2)|_{X=D(s)} = O(\epsilon^2 |\boldsymbol{Q}|), \quad (Q_1 + \epsilon D'(s) Q_2)|_{X=-D(s)} = O(\epsilon^2 |\boldsymbol{Q}|). \tag{3.10}$$

We now seek an asymptotic solution to the system (3.5)–(3.10) of the form

$$\left.\begin{aligned} f &= f^{(0)}(X,s) + \epsilon f^{(1)}(X,s) + \epsilon^2 f^{(2)}(X,s) + \cdots, \\ \boldsymbol{Q} &= \boldsymbol{Q}^{(0)}(X,s) + \epsilon \boldsymbol{Q}^{(1)}(X,s) + \cdots, \\ \boldsymbol{B} &= B^{(0)}(X,s)\boldsymbol{e}_z + \epsilon B^{(1)}(X,s)\boldsymbol{e}_z + \cdots \end{aligned}\right\}. \tag{3.11}$$

Substituting this expansion into (3.5) and expanding to $O(1/\epsilon)$ yields

$$f^{(0)} = f^{(0)}(s), \qquad f^{(1)} = f^{(1)}(s), \tag{3.12}$$

while proceeding to $O(1)$ gives an equation for $f^{(2)}$:

$$f^{(2)}_{XX} + f^{(0)}_{ss} = \Gamma({f^{(0)}}^3 - f^{(0)}) + f^{(0)}({Q_1^{(0)}}^2 + {Q_2^{(0)}}^2). \tag{3.13}$$

Boundary conditions on this problem are then obtained by substituting (3.11) into (3.9) and expanding to $O(\epsilon)$; they are

$$f^{(2)}_X - D'(s) {f^{(0)}}'(s)\Big|_{X=D(s)} = 0, \qquad f^{(2)}_X + D'(s) {f^{(0)}}'(s)\Big|_{X=-D(s)} = 0. \tag{3.14}$$

Equations and boundary conditions for $\boldsymbol{Q}^{(0)}$ are obtained from the leading order expansions of (3.7), (3.8), (3.9), and (3.10) and have the solution

$$\boldsymbol{Q}^{(0)} = Q_2^{(0)}(s)\boldsymbol{e}_s. \tag{3.15}$$

Proceeding to $O(1)$ in (3.8) we find the equation for $Q_1^{(1)}$

$${f^{(0)}}^2 Q^{(1)}_{1,X} + \frac{\partial}{\partial s}\left({f^{(0)}}^2 Q_2^{(0)}\right) = 0 \tag{3.16}$$

with boundary conditions arising from the $O(\epsilon)$ term of (3.10):

$$Q_1^{(1)} - D' Q_2^{(0)}\Big|_{x=D} = 0, \qquad Q_1^{(1)} + D' Q_2^{(0)}\Big|_{x=-D} = 0. \tag{3.17}$$

Integrating (3.13) and (3.16) between $X = -D$ and D and applying the boundary conditions (3.14) and (3.17), respectively, we find the governing equations for $f^{(0)}(s)$ and $Q_2^{(0)}(s)$; these are

$$\frac{1}{D}\frac{d}{ds}\left(D\frac{df^{(0)}}{ds}\right) = \Gamma({f^{(0)}}^3 - f^{(0)}) + f^{(0)}{Q_2^{(0)}}^2, \tag{3.18}$$

$$\frac{d}{ds}\left(D{f^{(0)}}^2 Q_2^{(0)}\right) = 0. \tag{3.19}$$

Consider now the behavior of the magnetic field. Substituting expansion (3.11) into (2.10) which holds in V_{ext}, the unbounded part of V^c, and the nonsuperconducting region enclosed by the cylinder V_{int}, we see that

$$B^{(0)} = B_{ext} \quad \text{in } V_{ext}, \qquad B^{(0)} = B_{int}^{(0)} = \text{const.} \quad \text{in } V_{int}.$$

In the superconducting domain V we find

$$B_X^{(0)} = \alpha\Gamma {f^{(0)}}^2 Q_2^{(0)}.$$

Integrating this between $X = -D$ and $X = D$ (recall that the inner surface is given by $X = -D$ and the outer by $X = D$) then gives the following expression for the magnetic field in the interior of the cylinder:

$$B_{int}^{(0)} = B_{ext} - 2\alpha D\Gamma {f^{(0)}}^2 Q_2^{(0)}. \tag{3.20}$$

Consider now the magnetic vector potential $\boldsymbol{A}$. To leading order this is solely determined by the magnetic fields in V_{ext} and V_{int} such that where we expand $\boldsymbol{A}$ as $\boldsymbol{A} = \boldsymbol{A}^{(0)} + \epsilon \boldsymbol{A}^{(1)} + \cdots$ and substitute this into (2.3), we find

$$\nabla \wedge \boldsymbol{A}^{(0)} = B_{ext} \quad \text{in } V_{ext}, \qquad \nabla \wedge \boldsymbol{A}^{(0)} = B_{int} \quad \text{in } V_{int}.$$

In order to solve for $\boldsymbol{A}^{(0)}$ an equation giving its gauge must also be specified (typically $\nabla \cdot \boldsymbol{A}^{(0)} = 0$). In order to relate (3.18) and (3.19) to the magnetic vector potential it is helpful to transform back to complex variables $(\psi, \boldsymbol{A})$ where

$$\psi^{(0)} = f^{(0)}(s)\exp(i\chi^{(0)}(s)), \qquad Q_2^{(0)} = A_2^{(0)}(s) - {\chi^{(0)}}'(s),$$
$$A_2^{(0)}(s) = \boldsymbol{A}^{(0)} \cdot \boldsymbol{q}'(s)\Big|_{\boldsymbol{x}=\boldsymbol{q}(s)}.$$

Equations (3.18) and (3.19) can easily be shown to be equivalent to

$$\left.\begin{array}{c} \left(\dfrac{d}{ds} - iA_2^{(0)}\right)^2 \psi^{(0)} + \dfrac{1}{D}\dfrac{dD}{ds}\left(\dfrac{d\psi^{(0)}}{ds} - iA_2^{(0)}\psi^{(0)}\right) = \Gamma\psi^{(0)}\left(|\psi^{(0)}|^2 - 1\right), \\ \psi^{(0)} \quad \text{periodic on} \quad (0, 2\pi) \end{array}\right\}.$$

By making the one-dimensional gauge transformation

$$\psi^{(0)} = \tilde{\psi}(s)\exp\left(i\int_0^s A_2^{(0)}(u) - \frac{\mathcal{F}^{(0)}}{2\pi}du\right), \qquad \mathcal{F}^{(0)} = \int_0^{2\pi} A_2^{(0)}(u)du,$$

this in turn may be shown to be equivalent to

$$\left.\begin{array}{c} \left(\dfrac{d}{ds} - i\dfrac{\mathcal{F}^{(0)}}{2\pi}\right)^2 \tilde{\psi} + \dfrac{1}{D}\dfrac{dD}{ds}\left(\dfrac{d\tilde{\psi}}{ds} - i\dfrac{\mathcal{F}^{(0)}}{2\pi}\tilde{\psi}\right) = \Gamma\tilde{\psi}\left(|\tilde{\psi}|^2 - 1\right), \\ \tilde{\psi} \quad \text{periodic on} \quad (0, 2\pi) \end{array}\right\}. \tag{3.21}$$

Here $\mathcal{F}^{(0)}$ is the leading order magnetic flux threading the cylinder. It is related to the superconducting current flowing in the ring through (3.20) which, when we introduce $\mathcal{S}$ (the area enclosed by the cylinder), we can write in the following form:

$$\frac{\mathcal{F}^{(0)}}{\mathcal{S}} = B_{ext} - 2\alpha D\Gamma\left(|\tilde{\psi}|^2\frac{\mathcal{F}^{(0)}}{2\pi} + \frac{i}{2}\left(\tilde{\psi}^*\frac{d\tilde{\psi}}{ds} - \tilde{\psi}\frac{d\tilde{\psi}^*}{ds}\right)\right). \tag{3.22}$$

By taking $\tilde{\psi}$ multiplied by the complex conjugate of (3.21) and subtracting $\tilde{\psi}^*$ multiplied by (3.21), it may be verified that the right-hand side of (3.22) is independent of s.

Summary. We have derived a model, comprised of (3.21) and (3.22), which describes superconductivity in a thin hollow cylinder made from a low-κ material. The equations (3.21) and (3.22) form a closed system. Given the applied magnetic field B_{ext}, we can solve it to find the superconducting order parameter $\tilde{\psi}$ in the cylinder and the magnetic flux $\mathcal{F}$ enclosed by it.

3.1. Transition to the superconducting state: A first or second order phase transition? In this section we shall (I) show how the model (3.21)–(3.22) can be used to calculate the value of Γ at which a superconducting solution bifurcates from the normal solution $\tilde{\psi} = 0$ and (II) derive a criterion to work out whether this bifurcation is subcritical (first order) or supercritical (second order). Dropping the superscript on $\mathcal{F}$ we look for an asymptotic solution lying close to the normal solution by making the following expansion:

$$\tilde{\psi} = \delta\tilde{\psi}_0 + \cdots ,$$
$$\Gamma = \Gamma_0 + \cdots ,$$
$$\mathcal{F} = \mathcal{F}_0 + \cdots ,$$

where $\delta \ll 1$. Substituting the above into (3.21)–(3.22) and expanding to leading order we find the following eigenvalue problem for $\tilde{\psi}_0$:

$$\left.\begin{aligned} L\tilde{\psi}_0 = \left(\frac{d}{ds} - i\frac{\mathcal{F}_0}{2\pi}\right)^2 \tilde{\psi}_0 + \frac{1}{D}\frac{dD}{ds}\left(\frac{d\tilde{\psi}_0}{ds} - i\frac{\mathcal{F}_0}{2\pi}\tilde{\psi}_0\right) + \Gamma_0\tilde{\psi}_0 = 0, \\ \tilde{\psi}_0 \quad \text{periodic on} \quad (0, 2\pi) \end{aligned}\right\}, \tag{3.23}$$

$$\mathcal{F}_0 = \mathcal{S}B_{ext}. \tag{3.24}$$

Thus, given the applied magnetic field B_{ext} it is possible to calculate $\mathcal{F}_0$ and then, by looking for solutions to (3.23), calculate the value of $\Gamma(B_{ext})$ at which a superconducting solution first bifurcates from the normal solution as Γ is increased (and temperature decreases); this is the first eigenvalue Γ_0 of (3.23). In order to calculate whether this bifurcation is subcritical or supercritical we need to proceed to higher orders in the asymptotic expansion of $\tilde{\psi}$ to investigate how the amplitude of $\tilde{\psi}_0$ depends upon small deviations of Γ away from Γ_0. The expansion proceeds as follows:

$$\tilde{\psi} = \delta\tilde{\psi}_0 + \delta^3\tilde{\psi}_1 + \cdots ,$$
$$\Gamma = \Gamma_0 + \delta^2\Gamma_1 + \cdots ,$$
$$\mathcal{F} = \mathcal{F}_0 + \delta^2\mathcal{F}_1 + \cdots .$$

Substituting this into (3.21)–(3.22) and expanding to first order we find the following inhomogeneous problem for $\tilde{\psi}_1$:

$$\left.\begin{aligned} L\tilde{\psi}_1 = H(s) = \frac{\mathcal{F}_1}{2\pi}\left(\frac{\mathcal{F}_0}{\pi}\tilde{\psi}_0 + i\left(2\frac{d\tilde{\psi}_0}{ds} + \frac{1}{D}\frac{dD}{ds}\tilde{\psi}_0\right)\right) + \Gamma_0|\tilde{\psi}_0|^2\tilde{\psi}_0 - \Gamma_1\tilde{\psi}_0, \\ \tilde{\psi}_1 \quad \text{periodic on} \quad (0, 2\pi) \end{aligned}\right\} \tag{3.25}$$

together with an equation for the constant $\mathcal{F}_1$

$$\mathcal{F}_1 = -2\alpha\mathcal{S}D\Gamma_0\left(|\tilde{\psi}_0|^2\frac{\mathcal{F}_0}{2\pi} + \frac{i}{2}\left(\tilde{\psi}_0^*\frac{d\tilde{\psi}_0}{ds} - \tilde{\psi}_0\frac{d\tilde{\psi}_0^*}{ds}\right)\right). \tag{3.26}$$

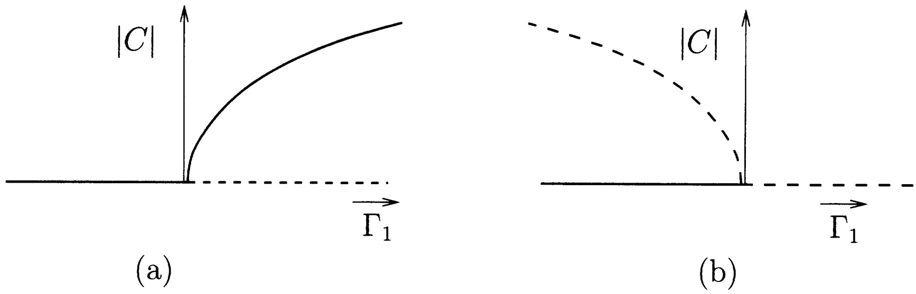

FIG. 3.1. *The dependence of* $|C|$ *on* Γ_1. *In* (a) *there is a supercritical bifurcation while in* (b) *there is a subcritical bifurcation.*

Using the solution $\tilde{\psi}_0$ to the homogenous version of (3.25), we can write a solvability condition for (3.25):

$$\text{A solution exists to (3.25) iff } \int_0^{2\pi} D\tilde{\psi}_0^* H(s)ds \equiv 0. \tag{3.27}$$

We now seek to relate the amplitude of $\tilde{\psi}_0$ to Γ_1 via this solvability condition. In order to do this it proves helpful to write

$$\tilde{\psi}_0 = Cw, \qquad \text{where } C \in \mathbb{C}$$

and w is a solution of (3.23) normalized such that $\int_0^{2\pi} D|w|^2 ds = 1$. Making the above substitution in (3.27) and evaluating the integral using the definitions of $H(s)$ and $\mathcal{F}_1$ found in (3.25) and (3.26), we find

$$\Gamma_1 |C|^2 = \Gamma_0 |C|^4 \left(\int_0^{2\pi} D|w|^4 ds - 4\alpha S K_w^2 \right),$$
$$K_w = D\left(|w|^2 \frac{SB_{ext}}{2\pi} + \frac{i}{2}\left(w^* \frac{dw}{ds} - w\frac{dw^*}{ds} \right) \right) = \text{const.}$$

The bifurcation is supercritical (second order) if a superconducting solution exists for $\Gamma_1 > 0$ (see Figure 3.1(a)) and subcritical if no superconducting solution exists for $\Gamma_1 > 0$ (see Figure 3.1(b)). It is straightforward to show that $\Gamma_0 > 0$, and it thus follows that

$$\left.\begin{aligned} &\int_0^{2\pi} D|w|^4 ds - 4\alpha S K_w^2 > 0 \Longrightarrow \text{supercritical bifurcation,} \\ &\int_0^{2\pi} D|w|^4 ds - 4\alpha S K_w^2 < 0 \Longrightarrow \text{subcritical bifurcation} \end{aligned}\right\}. \tag{3.28}$$

3.2. An example: The uniform ring. We consider a shell of uniform dimensionless thickness D and set $\alpha = 1$. A superconducting solution, of the form $\tilde{\psi} = \delta C e^{ims}$, where C is a complex constant, bifurcates from the normal solution at

$$\Gamma = \Gamma_0 = \left(m - \frac{SB_{ext}}{2\pi} \right)^2, \quad \text{where} \quad m = \text{nint}\left(\frac{SB_{ext}}{2\pi} \right),$$

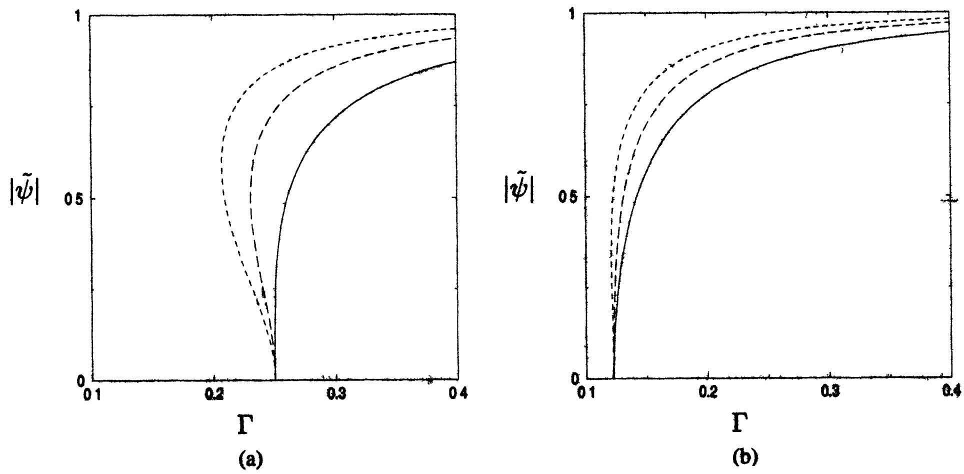

FIG. 3.2. *The dependence of* $|\tilde{\psi}_0|$ *on* Γ *for a uniform ring with* $\alpha = 1$. *In graph* (a) $(B_{ext}/2.0 - m)^2 = 0.25$; *in case* (b) $(B_{ext}/2.0 - m)^2 = 0.125$. *In each case the three different plots represent different values of* D. *For the dotted line* $D = 5.0$, *for the dashed line* $D = 3.5$, *and for the solid line* $D = 2.0$.

where $\mathrm{nint}(A)$ is defined to be the integer closest to A. We can use the criterion (3.28) to show that this bifurcation is subcritical (first order) iff

$$\frac{2D\mathcal{S}}{\pi}\left(m - \frac{\mathcal{S}B_{ext}}{2\pi}\right)^2 > 1.$$

Since the solution of the full problem (3.21)–(3.22) has the form

$$\tilde{\psi} = Ee^{ims}, \qquad E \in \mathbb{C},$$

we can pursue the solution which bifurcates at $\Gamma = \Gamma_0$ into the fully nonlinear regime. Substituting for $\tilde{\psi}$ in (3.21)–(3.22) we find the following problem for E and $\mathcal{F}$:

$$\text{either} \qquad E = 0,$$

$$\text{or} \quad \left\{ \begin{array}{c} |E|^2 = 1 - \left(m - \dfrac{\mathcal{F}}{2\pi}\right)^2 / \Gamma, \\ -D\left(m - \dfrac{\mathcal{F}}{2\pi}\right)^3 + \left(D\Gamma + \dfrac{\pi}{\mathcal{S}}\right)\left(m - \dfrac{\mathcal{F}}{2\pi}\right) + \left(\dfrac{B_{ext}}{2} - \dfrac{\pi m}{\mathcal{S}}\right) = 0. \end{array} \right.$$

Numerical solutions to this problem, which show the dependence of $|\tilde{\psi}| = |E|$ upon Γ, are plotted in Figure 3.2 for the special case of $\mathcal{S} = \pi, \alpha = 1$, and for different values of D and $(B_{ext}/2 - \pi m/\mathcal{S})$.

4. The thin film problem. In this section we consider a thin superconducting film occupying the domain

$$V_\epsilon = \left\{(x, y, z) \in \mathbb{R}^3 : (x, y) \in V_0, \quad -\epsilon d_1(x, y) < z < \epsilon d_2(x, y)\right\}$$

and subject to a perpendicular magnetic field $B_{ext}\mathbf{e}_z$. Here we take

$$\epsilon = \alpha\kappa^2, \qquad \kappa \ll 1,$$

where α, d_1, and d_2 are all $O(1)$. In order to determine the effect of the applied magnetic field on the superconductor we need to solve the GL equations in V_ϵ and couple these to Maxwell's equations in the exterior. Since the film is thin we consider a method for approximating the GL equations in V_ϵ. We rescale z with the thickness of the thin film by introducing the scaled variable Z such that

$$z = \epsilon Z,$$

and rewrite (2.7)–(2.8), which hold in V_ϵ, accordingly:

$$\frac{1}{\epsilon^2} f_{ZZ} + (f_{xx} + f_{yy}) = \Gamma\left(f^3 - f\right) + f|\boldsymbol{Q}|^2, \tag{4.1}$$

$$\epsilon B_{3,y} - B_{2,Z} = -\alpha\Gamma f^2 Q_1, \tag{4.2}$$

$$B_{1,Z} - \epsilon B_{3,x} = -\alpha\Gamma f^2 Q_2, \tag{4.3}$$

$$\epsilon\left(B_{2,x} - B_{1,y}\right) = -\alpha\Gamma f^2 Q_3, \tag{4.4}$$

$$Q_{1,Z} = O(\epsilon|\boldsymbol{B}|), \qquad Q_{2,Z} = O(\epsilon|\boldsymbol{B}|). \tag{4.5}$$

Taking the divergence of (2.8) gives an additional equation that proves useful when carrying out the asymptotic analysis

$$\frac{1}{\epsilon}\frac{\partial}{\partial Z}\left(f^2 Q_3\right) + \left(\frac{\partial}{\partial x}\left(f^2 Q_1\right) + \frac{\partial}{\partial y}\left(f^2 Q_2\right)\right) = 0. \tag{4.6}$$

On the upper and lower boundaries of the thin film the conditions (2.11) yield

(4.7)
$$f_Z - \epsilon^2\left(d_{1,x} f_x + d_{1,y} f_y\right)\big|_{Z=d_1} = O(\epsilon^2 f), \quad f_Z + \epsilon^2\left(d_{2,x} f_x + d_{2,y} f_y\right)\big|_{Z=-d_2} = O(\epsilon^2 f),$$
$$Q_3 - \epsilon\left(d_{1,x} Q_1 + d_{1,y} Q_2\right)\big|_{Z=d_1} = 0, \qquad Q_3 + \epsilon\left(d_{2,x} Q_1 + d_{2,y} Q_2\right)\big|_{Z=-d_2} = 0.$$

We now seek an asymptotic solution to (4.1)–(4.6) of the form

$$\boldsymbol{B} = \boldsymbol{B}^{(0)} + O(\epsilon), \qquad \boldsymbol{Q} = \boldsymbol{Q}^{(0)} + \epsilon\boldsymbol{Q}^{(1)} + \cdots,$$
$$f = f^{(0)} + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots.$$

Substituting this expansion into (4.1) (at leading and first order) and (4.5) (at leading order) we see that

$$f^{(0)} = f^{(0)}(x,y), \quad f^{(1)} = f^{(1)}(x,y),$$
$$Q_1^{(0)} = Q_1^{(0)}(x,y), \quad Q_2^{(0)} = Q_2^{(0)}(x,y).$$

Since $f^{(0)}$ is independent of Z it follows from (4.6) that $Q_3^{(0)}$ is also independent of Z; applying the boundary condition (4.8) leads to the conclusion that $Q_3^{(0)} \equiv 0$ and hence that

$$\boldsymbol{Q}^{(0)} = Q_1^{(0)}(x,y)\boldsymbol{e}_x + Q_2^{(0)}(x,y)\boldsymbol{e}_y. \tag{4.8}$$

Proceeding to $O(\epsilon)$ in (4.6) we find

$${f^{(0)}}^2 Q_{3,Z}^{(1)} + \frac{\partial}{\partial x}\left({f^{(0)}}^2 Q_1^{(0)}\right) + \frac{\partial}{\partial y}\left({f^{(0)}}^2 Q_2^{(0)}\right) = 0.$$

Integrating this equation with respect to Z between $Z=-d_2$ and $Z=d_1$ and applying the boundary condition (4.8) at first order ($O(\epsilon)$) gives rise to the following relation between $f^{(0)}$ and $\boldsymbol{Q}^{(0)}$:

$$\frac{\partial}{\partial x}\left(D{f^{(0)}}^2 Q_1^{(0)}\right)+\frac{\partial}{\partial y}\left(D{f^{(0)}}^2 Q_2^{(0)}\right)=0, \tag{4.9}$$

where $D(x,y)=d_1(x,y)+d_2(x,y)$ is the dimensionless thickness of the film. We can find a further relation between $f^{(0)}$ and $\boldsymbol{Q}^{(0)}$ by taking the $O(1)$ expansion of (4.1), integrating this with respect to Z between $Z=-d_2$ and $Z=d_1$, and applying the boundary condition (4.8) at $O(\epsilon^2)$; this relation is

$$\frac{1}{D}\left(\frac{\partial}{\partial x}\left(D\frac{\partial f^{(0)}}{\partial x}\right)+\frac{\partial}{\partial y}\left(D\frac{\partial f^{(0)}}{\partial y}\right)\right)=\Gamma\left({f^{(0)}}^3-f^{(0)}\right)+f^{(0)}|\boldsymbol{Q}^{(0)}|^2. \tag{4.10}$$

In order to couple (4.9) and (4.10) to Maxwell's equations in the region exterior to V_ϵ (and thus relate $\boldsymbol{Q}^{(0)}$ and $f^{(0)}$ to the applied magnetic field) it is advantageous to reintroduce the original variables ψ and $\boldsymbol{A}$, and the phase of the order parameter χ (as defined in (2.6)). These have the following expansions:

$$\begin{aligned}\boldsymbol{A}&=\boldsymbol{A}^{(0)}+\epsilon\boldsymbol{A}^{(1)}+\cdots,\\ \chi&=\chi^{(0)}+\epsilon\chi^{(1)}+\cdots,\\ \psi&=\psi^{(0)}+\epsilon\psi^{(1)}+\cdots.\end{aligned}$$

Comparing these variables to the leading order behavior of $\boldsymbol{Q}$, through (2.6) and (4.8), we see that

$$\begin{aligned}\chi^{(0)}&=\chi^{(0)}(x,y),\\ \psi^{(0)}&=f^{(0)}(x,y)\exp(i\chi^{(0)}(x,y)).\end{aligned}$$

It follows that the behavior of the superconducting thin film is determined, to leading order, by the components of the leading order vector potential $\boldsymbol{A}^{(0)}$ tangential to the film. In fact we can rewrite (4.9) and (4.10), in terms of $\boldsymbol{a}^{(0)}=A_1^{(0)}(x,y,0)\boldsymbol{e}_x+A_2^{(0)}(x,y,0)\boldsymbol{e}_y$ and $\psi^{(0)}$, as follows:

$$(\nabla-i\boldsymbol{a}^{(0)})^2\psi^{(0)}+\frac{1}{D}\nabla D\cdot\left(\nabla\psi^{(0)}-i\boldsymbol{a}^{(0)}\psi^{(0)}\right)=\Gamma^{(0)}\left(|\psi^{(0)}|^2-1\right)\psi^{(0)}. \tag{4.11}$$

It now remains to find one more set of equations relating $\boldsymbol{a}^{(0)}$ to $\psi^{(0)}$ and the applied magnetic field $\boldsymbol{B}=B_{ext}\boldsymbol{e}_z$. Returning to (4.2)–(4.4), we see that there is an $O(1)$ jump in the magnetic field across the film. Integrating these equations between $Z=-d_2$ and $Z=d_1$ we find

$$[B_1]_{z=-\epsilon d_2}^{z=\epsilon d_1}=j_2^{(0)}(x,y)+O(\epsilon),\quad [B_2]_{z=-\epsilon d_2}^{z=\epsilon d_1}=-j_1^{(0)}(x,y)+O(\epsilon),\quad [B_3]_{z=-\epsilon d_2}^{z=\epsilon d_1}=O(\epsilon), \tag{4.12}$$

where

$$\boldsymbol{j}^{(0)}=-\alpha\Gamma^{(0)}D\left(|\psi^{(0)}|^2\boldsymbol{a}^{(0)}+\frac{i}{2}\left({\psi^{(0)}}^*\nabla\psi^{(0)}-\psi^{(0)}\nabla{\psi^{(0)}}^*\right)\right).$$

By taking account of the continuity of magnetic field across the boundaries $z = \epsilon d_1$ and $z = -\epsilon d_2$, linearizing (4.12) onto $z = 0$, and applying Maxwell's equations (2.10), we can write the leading order problem for the magnetic vector potential outside the film as follows:

$$\begin{aligned} \nabla^2 \boldsymbol{A}^{(0)} &= -\boldsymbol{j}^{(0)}(x,y)\delta(z), \\ \nabla \cdot \boldsymbol{A}^{(0)} &= 0, \\ \nabla \wedge \boldsymbol{A}^{(0)} &\to B_{ext}\boldsymbol{e}_z \qquad \text{as } |\boldsymbol{x}| \to \infty. \end{aligned}$$

Solving for $\boldsymbol{A}^{(0)}$ gives

$$\boldsymbol{A}^{(0)}(x,y,z) = B_{ext}\hat{\boldsymbol{A}}(x,y) + \frac{1}{4\pi}\int\int_{V_0} \frac{\boldsymbol{j}^{(0)}(\zeta,\eta)}{\left((x-\zeta)^2+(y-\eta)^2+z^2\right)^{1/2}} d\zeta d\eta, \tag{4.13}$$

where $\hat{\boldsymbol{A}}$ is such that

$$\nabla \wedge \hat{\boldsymbol{A}} = \boldsymbol{e}_z, \qquad \nabla \cdot \hat{\boldsymbol{A}} = 0. \tag{4.14}$$

Setting $z = 0$ in (4.13) thus gives rise to a further relation between $\boldsymbol{a}^{(0)}$ and $\psi^{(0)}$, namely,

$$\begin{aligned} \alpha\Gamma_0 \int\int_{V_0} &\frac{\left[D\left(|\psi^{(0)}|^2\boldsymbol{a}^{(0)} + \frac{i}{2}\left(\psi^{(0)*}\nabla\psi^{(0)} - \psi^{(0)}\nabla\psi^{(0)*}\right)\right)\right](\zeta,\eta)}{\left((x-\zeta)^2+(y-\eta)^2\right)^{1/2}} d\zeta d\eta \\ &= 4\pi\left(B_{ext}\hat{\boldsymbol{A}}(x,y) - \boldsymbol{a}^{(0)}(x,y)\right). \end{aligned} \tag{4.15}$$

In order to close the system comprised of (4.11) and (4.15) we need to specify the boundary conditions on ∂V_0; it is clear that this is

$$\boldsymbol{N} \cdot \left(\nabla - i\boldsymbol{a}^{(0)}\right)\psi^{(0)}\Big|_{\partial V_0} = 0. \tag{4.16}$$

The system (4.11) and (4.15) together with the boundary condition (4.16) bear some similarity to a system of integral equations proposed in [16] for the particular case of superconducting disks.

4.1. Transition to the superconducting state: A first or second order phase transition? In this section we use the model (4.11), (4.15), and (4.16) to find an eigenvalue problem whose solution gives the values of Γ, as a function of the applied magnetic field B_{ext}, at which a superconducting solution bifurcates from the normal solution $\psi = 0$ $\boldsymbol{a} = B_{ext}\hat{\boldsymbol{A}}$. By taking account of the nonlinear terms in the model we then find a criterion which can be used to work out whether the bifurcation of the superconducting solution is supercritical (second order) or subcritical (first order).

Henceforth the superscripts on ψ and $\boldsymbol{a}$, in (4.11), (4.15), and (4.16) are dropped. We search for superconducting solutions of these equations bifurcating from the normal solution by looking for an asymptotic solution of the form

$$\begin{aligned} \psi &= \delta\psi_0 + \cdots, \\ \boldsymbol{a} &= \boldsymbol{a}_0 + \cdots, \\ \Gamma &= \Gamma_0 + \cdots, \end{aligned}$$

where $\delta \ll 1$. Substituting this expansion into (4.11), (4.15), and (4.16) and taking the leading order terms in δ leads to the following eigenvalue problem for ψ_0:

$$\left.\begin{aligned} \tilde{L}\psi_0 = (\nabla - i\boldsymbol{a}_0)^2\psi_0 + \frac{1}{D}\nabla D\cdot(\nabla\psi_0 - i\boldsymbol{a}_0\psi_0) + \Gamma_0\psi_0 = 0, \\ \boldsymbol{N}\cdot(\nabla - i\boldsymbol{a}_0)^2\psi_0\Big|_{\partial V_0} = 0, \end{aligned}\right\} \tag{4.17}$$

$$\boldsymbol{a}_0 = B_{ext}\hat{\boldsymbol{A}}(x,y). \tag{4.18}$$

For any given applied field B_{ext} the value of Γ at which a superconducting solution first bifurcates from the normal solution, as Γ is increased, is the first eigenvalue of Γ_0 of (4.17). In order to investigate whether this bifurcation is subcritical or supercritical it is necessary to proceed to higher orders in the expansion of ψ, $\boldsymbol{a}$, and Γ:

$$\begin{aligned} \psi &= \delta\psi_0 + \delta^3\psi_1 + \cdots, \\ \boldsymbol{a} &= \boldsymbol{a}_0 + \delta^2\boldsymbol{a}_1 + \cdots, \\ \Gamma &= \Gamma_0 + \delta^2\Gamma_1 + \cdots. \end{aligned}$$

Substituting the above into (4.11), (4.15), and (4.16) and expanding to first order gives rise to the following inhomogeneous problem for ψ_1:

$$\begin{aligned} \tilde{L}\psi_1 = H(x,y) = \Bigg(&\Gamma_0|\psi_0|^2\psi_0 - \Gamma_1\psi_0 + 2\boldsymbol{a}_0\cdot\boldsymbol{a}_1\psi_0 \\ &+ i\left(\boldsymbol{a}_1\cdot\nabla\psi_0 + \nabla\cdot(\boldsymbol{a}_1\psi_0) + \frac{1}{D}\nabla D\cdot\boldsymbol{a}_1\psi_0\right)\Bigg), \end{aligned} \tag{4.19}$$

$$\boldsymbol{N}\cdot(\nabla - i\boldsymbol{a}_0)\psi_1|_{\partial V_0} = i\boldsymbol{N}\cdot\boldsymbol{a}_1\psi_0|_{\partial V_0}, \tag{4.20}$$

$$\boldsymbol{a}_1(x,y) = -\frac{\alpha\Gamma_0}{4\pi}\int\!\!\int_{V_0}\frac{\left[D\left(|\psi_0|^2\boldsymbol{a}_0 + \frac{i}{2}\left(\psi_0{}^*\nabla\psi_0 - \psi_0\nabla\psi_0{}^*\right)\right)\right](\zeta,\eta)}{\left((x-\zeta)^2 + (y-\eta)^2\right)^{1/2}}d\zeta d\eta. \tag{4.21}$$

Since ψ_0 is a solution to the homogeneous version of (4.19)–(4.20), $H(x,y)$ will need to satisfy a solvability condition in order to show that a solution to these equations exists. It is possible to derive this condition starting from the relationship

$$\int\!\!\int_{V_0} D\left(\psi_0^*\tilde{L}\psi_1 - \psi_1\left(\tilde{L}\psi_0\right)^*\right)dV = \int_{\partial V_0} D\left[\psi_0^*(\nabla - i\boldsymbol{a}_0)\psi_1 - \psi_1(\nabla + i\boldsymbol{a}_0)\psi_0^*\right]\cdot\boldsymbol{N}dS.$$

We can substitute for various terms in the above using the complex conjugate of (4.17), (4.19), and (4.20) to find the following condition:

$$\int\!\!\int_{V_0} D\psi_0^*H(x,y)dV = \int_{\partial V_0} iD|\psi_0|^2\boldsymbol{a}_1\cdot\boldsymbol{N}dS. \tag{4.22}$$

Finally, by manipulating $D\psi_0^*H(x,y)$ we can find a surface term to cancel with that on the right-hand side of (4.22) and this leaves the following solvability condition:

$$\int\!\!\int_{V_0} D\left(\Gamma_0|\psi_0|^4 - \Gamma_1|\psi_0|^2\right) + 2\boldsymbol{a}_1\cdot\left(D\left(\boldsymbol{a}_0|\psi_0|^2 + \frac{i}{2}\left(\psi_0^*\nabla\psi_0 - \psi_0\nabla\psi_0^*\right)\right)\right)dV = 0. \tag{4.23}$$

Our goal now is to use this condition to relate some norm of ψ_0 to Γ_1 and in particular to find out whether there is a nontrivial solution of ψ_0 for positive values of Γ_1 (a supercritical bifurcation) or for negative values of Γ_1 (a subcritical bifurcation). In order to do this it is helpful to write ψ_0 in terms of w, a normalized solution of the eigenvalue problem (4.17):

$$\psi_0 = Cw, \quad \text{where} \quad C \in \mathbb{C} \quad \text{and} \quad \int\!\!\int_{V_0} D|w|^2 dV = 1.$$

Substituting for ψ_0 and $\boldsymbol{a}_1$ in (4.23) we find the following relationship between $|C|$ and Γ_1:

$$\Gamma_1 |C|^2 = \Upsilon |C|^4, \qquad \text{where} \tag{4.24}$$

$$\left\{\begin{aligned} &\Upsilon = \Gamma_0 \left[\int\!\!\int_{V_0} D|w|^4 dV - \frac{\alpha}{2\pi}\int\!\!\int_{V_0}\left(\int\!\!\int_{V_0} \frac{\boldsymbol{K}_w(x,y)\cdot\boldsymbol{K}_w(\zeta,\eta)}{((x-\zeta)^2+(y-\eta)^2)^{1/2}} d\zeta d\eta\right) dx\, dy\right], \\ &\boldsymbol{K}_w = D\left(B_{ext}\hat{\boldsymbol{A}}|w|^2 + \frac{i}{2}\left(w^*\nabla w - w\nabla w^*\right)\right). \end{aligned}\right. \tag{4.25}$$

The bifurcation is supercritical (second order) if a superconducting solution exists for $\Gamma_1 > 0$ (see Figure 3.1(a)) and subcritical if no superconducting solution exists for $\Gamma_1 > 0$ (see Figure 3.1(b)). It thus follows that

$$\left.\begin{array}{c} \Upsilon > 0 \Longrightarrow \text{supercritical bifurcation,} \\ \Upsilon < 0 \Longrightarrow \text{subcritical bifurcation} \end{array}\right\}. \tag{4.26}$$

4.2. An example: Calculation of the normal/superconducting transition in an annulus. We consider the annular domain $\{V_0 : r \in [\gamma, \beta]\}$ where (r, θ) are radial coordinates with origin at the center of the annulus (we shall also make use of the corresponding cartesians $x = r\cos\theta$ and $y = r\sin\theta$). In order to determine how a thin annular superconducting film of uniform dimensionless thickness D (and with $\alpha = 1$) makes the transition from the normal to superconducting states, we look for solutions to the eigenvalue problem (4.17) of the form

$$\begin{aligned} \psi_0 &= Cw(r,\theta), & \boldsymbol{a}_0 &= B_{ext}\hat{\boldsymbol{A}}, \\ w(r,\theta) &= f(r)e^{im\theta}, \quad 2\pi\int_\gamma^\beta Df^2(r)r\,dr = 1, & \hat{\boldsymbol{A}} &= \frac{r}{2}\boldsymbol{e}_\theta. \end{aligned}$$

Substitution of the above into (4.17) yields the following:

$$f'' + \frac{f'}{r} + f\left(\Gamma_0 - \left(\frac{B_0 r}{2} - \frac{m}{r}\right)^2\right) = 0, \tag{4.27}$$

$$f'(\gamma) = f'(\beta) = 0. \tag{4.28}$$

We compute the normal/superconducting transition curve by solving this eigenvalue problem numerically with different integer values of m. The value of Γ at which the transition occurs is given by $\Gamma = \Gamma_{crit}(B_{ext}) = \min_{m\in\mathbb{Z}}(\Gamma_0(B_{ext}, m))$. In Figures 4.1 and 4.2 plots of Γ_{crit} versus B_{ext} are made for annulus of inner radius $a = 0.5$ and outer radius $b = 1.0$.

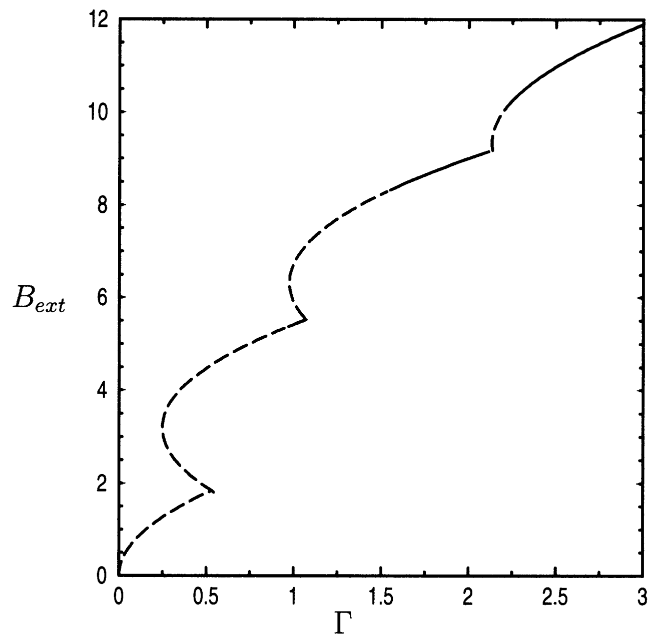

FIG. 4.1. *The normal/superconducting transition curve for an annulus of inner radius* $a = 0.5$ *and outer radius* $b = 1.0$ *and dimensionless thickness* $D = 5.0$. *Dashed lines represent a second order phase transition and solid lines a first order one.*

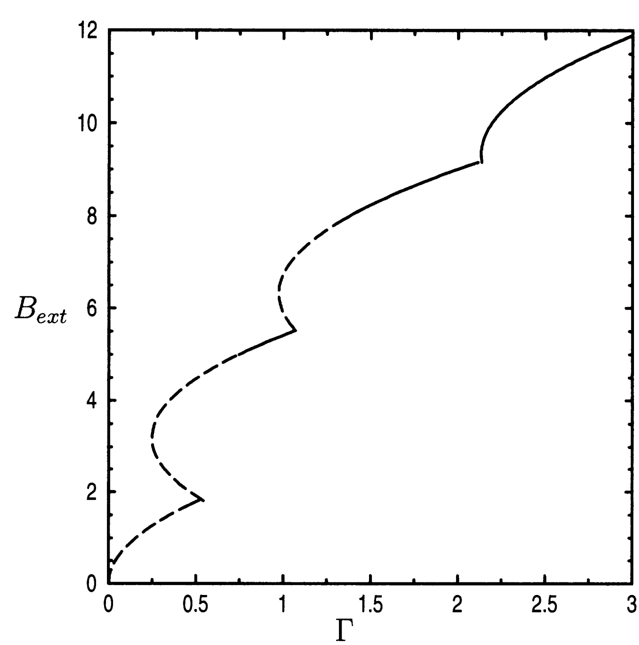

FIG. 4.2. *The normal/superconducting transition curve for an annulus of inner radius* $a = 0.5$ *and outer radius* $b = 1.0$ *and dimensionless thickness* $D = 7.0$.

In order to calculate whether this transition is subcritical (first order) or supercritical (second order) we need to evaluate Υ in (4.25). We note first that

$$\boldsymbol{K}_w = Df^2(r)\left(\frac{B_{ext}r}{2} - \frac{m}{r}\right)\boldsymbol{e}_\theta,$$

and then that

$$\int\int_{V_0} \frac{\boldsymbol{K}_w(\zeta,\eta)}{((x-\zeta)^2+(y-\eta)^2)^{1/2}} d\zeta d\eta$$

$$(4.29) = \int_{R=\gamma}^{\beta}\int_{\phi=0}^{2\pi} \frac{Df^2(R)\,(B_{ext}R/2 - m/R)}{(r^2+R^2-2rR\cos(\phi-\theta))^{1/2}} (\boldsymbol{e}_y\cos\phi - \boldsymbol{e}_x\sin\phi)\, R\, dR d\phi,$$

where $\zeta = R\cos\phi$, $\eta = R\sin\phi$, and $x = r\cos\theta$, $y = r\sin\theta$. With the aid of some simple substitutions and formulas 3.674 (part 3) and 8.126 (parts 2 and 4) of Gradshteyn and Ryzhik [9] we can perform the integral in ϕ contained in (4.29). After some algebra we obtain the following expression:

(4.30)

$$\int\int_{V_0} \frac{\boldsymbol{K}_w(\zeta,\eta)}{((x-\zeta)^2+(y-\eta)^2)^{1/2}} d\zeta d\eta = 4\boldsymbol{e}_\theta \int_\gamma^\beta Df^2(R)\left(\frac{B_{ext}R}{2} - \frac{m}{R}\right) G\left(\frac{R}{r}\right) dR,$$

where

$$G(p) = \frac{1}{2}\left[\frac{1+p^2}{1+p} K\left(\frac{2\sqrt{p}}{1+p}\right) - -(1+p)E\left(\frac{2\sqrt{p}}{1+p}\right)\right],$$

and where $K(.)$ and $E(.)$ are the complete elliptic integrals of the first and second kind, respectively.

By substituting (4.30) into (4.25) we find that

$$\Upsilon = 2\pi D\Gamma_0 \left[\int_\gamma^\beta rf^4(r)dr - \frac{2D}{\pi}\int_\gamma^\beta\int_\gamma^\beta rG\left(\frac{R}{r}\right)\left(f^2(R)\left(\frac{B_{ext}R}{2} - \frac{m}{R}\right)\right)\right.$$

$$(4.31) \qquad \left. \times \left(f^2(r)\left(\frac{B_{ext}r}{2} - \frac{m}{r}\right)\right) dRdr\right].$$

Where $\Upsilon > 0$ the normal/superconducting transition is supercritical, and where $\Upsilon < 0$ it is subcritical.

This criterion is used in Figures 4.1 and 4.2 to distinguish between sub- and supercritical sections of the curve $\Gamma = \Gamma_{crit}(B_{ext})$. We observe alternating sections of supercritical and subcritical transitions. This peculiar result means that the phase transition depends not only on the geometry but also on the applied magnetic field.

5. The thin wire. We consider a thin wire with typical variations in its cross-section occuring on a lengthscale comparable with its length (i.e., an $O(1)$ lengthscale), and of typical nondimensional width ϵ, where $\epsilon \ll 1$. Following a previous paper by the authors [13] we represent points $\boldsymbol{x}$ inside, or sufficiently close to, the wire in terms of local coordinates (X, Y, s) defined about the centerline of the $\boldsymbol{x} = \boldsymbol{r}(s)$ (here s is arclength). These are such that a point $\boldsymbol{x}$ with coordinates (X, Y, s) is at position

$$(5.1) \qquad \boldsymbol{x} = \boldsymbol{r}(s) + \epsilon X\boldsymbol{n}(s) + \epsilon Y\boldsymbol{b}(s),$$

where $\boldsymbol{n}$ and $\boldsymbol{b}$ are, respectively, the normal and the binormal to the curve $\boldsymbol{x} = \boldsymbol{r}(s)$ (again note that $\boldsymbol{n}$ is not the same as $\boldsymbol{N}$ the normal to the surface to the wire). This coordinate system is nonorthogonal since the basis vectors

$$\boldsymbol{e}_x = \frac{\partial \boldsymbol{x}/\partial X}{|\partial \boldsymbol{x}/\partial X|}, \qquad \boldsymbol{e}_y = \frac{\partial \boldsymbol{x}/\partial Y}{|\partial \boldsymbol{x}/\partial Y|}, \qquad \boldsymbol{e}_s = \frac{\partial \boldsymbol{x}/\partial s}{|\partial \boldsymbol{x}/\partial s|}$$

are not all mutually orthogonal off the curve $\boldsymbol{x} = \boldsymbol{r}(s)$. However, for $O(1)$ values of X and Y they are close to being orthogonal. This property is used in [13] to find asymptotic expansions, in powers of ϵ, for the vector operators gradient, divergence, and curl in terms of (X, Y, s) and the curvature k and torsion τ of the centerline of the wire. These expressions are as follows:

$$\nabla F = \frac{1}{\epsilon}\left(\boldsymbol{e}_x \frac{\partial F}{\partial X} + \boldsymbol{e}_y \frac{\partial F}{\partial Y}\right) + \boldsymbol{e}_s \frac{\partial F}{\partial s}(1 + \epsilon k X + O(\epsilon^2)), \tag{5.2}$$

$$\nabla \cdot \boldsymbol{P} = \frac{1}{\epsilon}\left(\frac{\partial P_1}{\partial X} + \frac{\partial P_2}{\partial Y}\right) + \frac{\partial P_3}{\partial s} - kP_1 + O(\epsilon|\boldsymbol{P}|), \tag{5.3}$$

$$\begin{aligned} \nabla \wedge \boldsymbol{P} = \frac{1}{\epsilon}&\left(\left(\frac{\partial P_2}{\partial X} - \frac{\partial P_1}{\partial Y}\right)\boldsymbol{e}_s + \frac{\partial P_3}{\partial Y}\boldsymbol{e}_x - \frac{\partial P_3}{\partial X}\boldsymbol{e}_y\right) \\ &+ \left(\tau Y\left(\frac{\partial P_2}{\partial X} - \frac{\partial P_1}{\partial Y}\right) - \frac{\partial P_2}{\partial s}\right)\boldsymbol{e}_x \\ &+ \left(\frac{\partial P_1}{\partial s} + kP_3 + \tau X\left(\frac{\partial P_1}{\partial Y} - \frac{\partial P_2}{\partial X}\right)\right)\boldsymbol{e}_y \\ &+ \tau\left(X\frac{\partial P_3}{\partial X} + Y\frac{\partial P_3}{\partial Y}\right)\boldsymbol{e}_s + O(\epsilon|\boldsymbol{P}|), \end{aligned} \tag{5.4}$$

where we write the vector $\boldsymbol{P}$ in the form $\boldsymbol{P} = P_1\boldsymbol{e}_x + P_2\boldsymbol{e}_y + P_3\boldsymbol{e}_s$. It proves convenient to define a function $G(X, Y, s)$ which takes negative values inside the wire V, is zero on the boundary ∂V, and is positive outside the wire in V^c. The cross section of the wire $\Omega(s_0)$ about the point $\boldsymbol{x} = \boldsymbol{r}(s_0)$ is then given by

$$\Omega(s_0) = \{(X, Y) : \quad G(X, Y, s_0) < 0\}.$$

We write the vector operators in (2.7), the divergence of (2.8), and the boundary conditions (2.11) in terms of the local coordinates. We obtain

$$\begin{aligned} \frac{1}{\epsilon^2}(f_{XX} + f_{YY}) - \frac{k}{\epsilon}f_X = &\left[\Gamma\left(f^3 - f\right) + f|\boldsymbol{Q}|^2 - f_{ss}\right] \\ &+ O\left(f_X, f_Y, \epsilon f_s\right), \end{aligned} \tag{5.5}$$

$$G_X f_X + G_Y f_Y + \epsilon^2 G_s f_s|_{\partial\Omega(s)} = O\left(\epsilon^2 f_X, \epsilon^2 f_Y, \epsilon^3 f_s\right), \tag{5.6}$$

$$\frac{1}{\epsilon}\left(\frac{\partial}{\partial X}\left(f^2 Q_1\right) + \frac{\partial}{\partial Y}\left(f^2 Q_2\right)\right) + \frac{\partial}{\partial s}\left(f^2 Q_3\right) - kf^2 Q_1 = O\left(\epsilon f, \epsilon|\boldsymbol{Q}|\right), \tag{5.7}$$

$$G_X Q_1 + G_Y Q_2 + \epsilon G_s Q_3|_{\partial\Omega(s)} = O\left(\epsilon^2|\boldsymbol{Q}|\right). \tag{5.8}$$

We assume that f and $\boldsymbol{Q}$ have expansions in ϵ which have an $O(1)$ term at leading order; this assumption is justified a posteriori. Solving for f in $\Omega(s)$ using (5.5) and (5.6) gives

$$f = F(s) + O(\epsilon^2), \tag{5.9}$$

while solving for $\boldsymbol{Q}$ on $\Omega(s)$ using (5.7) and (5.8) gives

$$\boldsymbol{Q} = Q_3\boldsymbol{e}_s + O(\epsilon). \tag{5.10}$$

Integrating (5.5) over $\Omega(s)$ then yields

(5.11)
$$\frac{1}{\epsilon^2}\int_{\partial\Omega(s)} \frac{f_X G_X + f_Y G_Y}{(G_X^2 + G_Y^2)^{1/2}} dl = D\left[\Gamma(F^3 - F) - F''\right] + F\int\int_{\Omega(s)} Q_3^2 dX dY + O(\epsilon),$$

where we define $D(s)$, the dimensionless thickness of the wire, by

$$D(s) = \int\int_{\Omega(s)} dX dY.$$

Substituting for $(f_X G_X + f_Y G_Y)|_{\partial\Omega(s)}$ in (5.11), using (5.6), and then making use of the relation

$$\frac{d}{ds}\int\int_{\Omega(s)} \theta(X,Y,s) dX\, dY = \int\int_{\Omega(s)} \frac{\partial\theta}{\partial s} dX\, dY - \int_{\partial\Omega(s)} \frac{G_s\theta}{\sqrt{G_X^2 + G_Y^2}} dl, \tag{5.12}$$

which holds for any smooth function $\theta(X,Y,s)$, we find the following equation for $F(s)$:

$$\frac{d}{ds}(DF') = D\Gamma(F^3 - F) + F\int\int_{\Omega(s)} Q_3^2 dX dY + O(\epsilon), \tag{5.13}$$

$$F(s) \quad \text{periodic on } [0, 2\pi]. \tag{5.14}$$

Performing similar manipulations on (5.7) and (5.8) as we did for (5.5) and (5.6) above, we find an equation for the conservation of electric current in the wire:

$$\frac{d}{ds}\left[\int\int_{\Omega(s)} F^2(s) Q_3(X,Y,s) dX dY\right] = O(\epsilon). \tag{5.15}$$

We can now use (2.8) to calculate the total current I flowing across any cross section $\Omega(s)$,

$$I = \frac{\epsilon^2}{\kappa^2}\mathcal{I} + O(\epsilon), \quad \text{where} \quad \mathcal{I} = -\Gamma\int\int_{\Omega(s)} F^2 Q_3 dX dY = \text{const.} + O(\epsilon). \tag{5.16}$$

We obtain an equation for $Q_3(X,Y,s)$ in a cross section $\Omega(s)$, valid to $O(\epsilon)$, by substituting (2.9) into (2.8) and using the expansion of curl found in (5.4),

$$Q_{3,XX} + Q_{3,YY} = \frac{\Gamma\epsilon^2}{\kappa^2} F^2 Q_3 + O(\epsilon) \qquad \text{in } \Omega(s). \tag{5.17}$$

This couples to Maxwell's equations (2.10a) (with zero current density) outside the wire in $\overline{\Omega}^c$. We choose to write these in the form $\nabla \wedge (\nabla \wedge \boldsymbol{Q}) = \boldsymbol{0}$, $\nabla \wedge \boldsymbol{Q} = \boldsymbol{B}$, where $\boldsymbol{Q}$ and its first derivatives are continuous across the boundary $\partial\Omega(s)$. Taking the $\boldsymbol{e}_s$ component of the former of these two equations we find

$$Q_{3,XX} + Q_{3,Y,Y} = O(\epsilon) \qquad \text{in } \overline{\Omega}^c(s). \tag{5.18}$$

As $R = (X^2 + Y^2)^{1/2}$ tends to infinity, the right-hand side of (5.17) appears as a point source for the potential Q_3. Therefore Q_3 has the following asymptotic behavior:

$$Q_3 \sim -\frac{\epsilon^2 \mathcal{I}}{2\pi\kappa^2} \log R + \beta(s) + O(\epsilon \log R) \qquad \text{as } R \to \infty, \tag{5.19}$$

where $\mathcal{I}$ is given by (5.16). Notice that this behavior represents the effect of a current size $\epsilon^2 \mathcal{I}/\kappa^2$ flowing along the wire. Notice also that the logarithmic growth uniquely determines the far field behavior, and that the additive $O(1)$ term $\beta(s)$ is determined by the shape of the cross section $\Omega(s)$.

Hence, if we can find $\mathcal{I}$, we shall have sufficient boundary data to solve for $F(s)$ and $Q_3(X, Y, s)$ up to $O(\epsilon)$. In what follows we shall consider how, by matching the preceding inner analysis for $Q_3(X, Y, s)$ to the vector potential $\boldsymbol{A}$ in a far field region we can find a relation between $\mathcal{I}$ and the average of $\beta(s)$ over the ring. In order to accomplish the matching it is helpful to write down the following relation between Q_3 and A_3 (the tangential component of the vector potential in the inner region) which is based on (2.6):

$$Q_3 = A_3 - n - h'(s) + O(\epsilon). \tag{5.20}$$

Here n is an integer, termed the winding number, defined by

$$n = \int_0^{2\pi} \left. \frac{\partial \chi}{\partial s} \right|_{X=Y=0} ds,$$

and $h(s)$ is a 2π periodic function of s.

Far field region and matching. We consider a far field with characteristic lengthscale comparable to the length of the wire (i.e., $O(1)$). On this lengthscale the thickness of the wire is small $O(\epsilon)$ and we thus write $\boldsymbol{B}$ and $\boldsymbol{A}$ as follows:

$$\boldsymbol{B}_o = \frac{\epsilon^2}{\kappa^2} \boldsymbol{b}_o + B_{ext} \boldsymbol{e}_z + O\left(\frac{\epsilon^3}{\kappa^2}\right), \qquad \boldsymbol{A}_o = \frac{\epsilon^2}{\kappa^2} \boldsymbol{a}_o + B_{ext} \hat{\boldsymbol{A}} + O\left(\frac{\epsilon^3}{\kappa^2}\right).$$

Here the subscript o denotes a far-field term and $\hat{\boldsymbol{A}}$ is defined by (4.14). Substituting the above into (2.10) we find the following system for $\boldsymbol{b}_o$:

$$\nabla \wedge \boldsymbol{b}_o = \mathcal{I} \int_\gamma \delta\left(\boldsymbol{x} - \boldsymbol{q}(s)\right) \boldsymbol{t}(s) ds, \quad \nabla \cdot \boldsymbol{b}_o = 0, \tag{5.21}$$

$$\boldsymbol{b}_o \to 0 \qquad \text{as } |\boldsymbol{x}| \to \infty, \tag{5.22}$$

where γ is the curve $\boldsymbol{x} = \boldsymbol{q}(s)$ and $\boldsymbol{t}(s)$ is the tangent to this curve. Here the singular term on the right-hand side of (5.21) arises because a current size $\epsilon^2 \mathcal{I}/\kappa^2$ flows along the wire. We solve this system in terms of $\boldsymbol{a}_o$ for which we choose the gauge

$$\nabla \cdot \boldsymbol{a}_o = 0.$$

Substituting for $\boldsymbol{b}_o = \nabla \wedge \boldsymbol{a}_o$ in (5.21) yields an equation for $\boldsymbol{a}_o$,

$$\nabla^2 \boldsymbol{a}_o = -\mathcal{I} \int_\gamma \delta\left(\boldsymbol{x} - \boldsymbol{q}(s)\right) \boldsymbol{t}(s) ds,$$

with solution

$$\boldsymbol{a}_o = \frac{\mathcal{I}}{4\pi} \int_\gamma \frac{\boldsymbol{t}(\tau)}{|\boldsymbol{x} - \boldsymbol{q}(\tau)|} d\tau. \tag{5.23}$$

In order to use Van Dyke's matching principle [18] we need to calculate the expansion of $\boldsymbol{a}_o(\boldsymbol{x}(X,Y,s))$ in terms of the inner variables defined in (5.1). We write $R = (X^2+Y^2)^{1/2}$ and expand the integral in (5.23) as follows:

$$\text{(5.24)} \quad \int_\gamma \frac{\boldsymbol{t}(\tau)}{|\boldsymbol{x}(X,Y,s)-\boldsymbol{q}(\tau)|}d\tau = \left[\int_0^{S-L} + \int_{S+L}^{2\pi} + \int_{S-L}^{S+L}\right] \frac{\boldsymbol{t}(\tau)}{|\boldsymbol{x}(X,Y,s)-\boldsymbol{q}(\tau)|}d\tau,$$

where $\epsilon \ll L \ll 1$. Following Batchelor [1] we write $\epsilon u = (\tau - s)/R$ in the last of these integrals and expand $\boldsymbol{t}(\tau)$ and $|\boldsymbol{x}(X,Y,s)-\boldsymbol{q}(\tau)|$ in powers of ϵ about $\tau = s$ to obtain an asymptotic expansion for the term

$$\int_{S-L}^{S+L} \frac{\boldsymbol{t}(\tau)}{|\boldsymbol{x}(X,Y,s)-\boldsymbol{q}(\tau)|}d\tau$$
$$= \boldsymbol{t}(s)\left[\int_{-L/\epsilon R}^{L/\epsilon R} \frac{1}{(1+u^2)^{1/2}}du + \epsilon\frac{kX}{2}\int_{-L/\epsilon R}^{L/\epsilon R}\frac{u^2}{(1+u^2)^{3/2}}du\right] + O(\epsilon L).$$

Evaluating the integrals in the above expansion and substituting the result into (5.24) and (5.23) yields the leading terms of the expansion in inner variables of $\boldsymbol{a}_o$:

$$\text{(5.25)} \quad \boldsymbol{a}_o(\boldsymbol{x}(X,Y,s)) = \mathcal{I}\left(\frac{1}{2\pi}\log\left(\frac{2}{\epsilon R}\right) + M(s)\right)\boldsymbol{t}(s) + O(\epsilon\log\epsilon),$$

$$\text{(5.26)} \quad M(s) = \frac{1}{4\pi}\lim_{L\to 0}\left(\left[\int_0^{S-L} + \int_{S+L}^{2\pi}\right]\frac{\boldsymbol{t}(\tau)}{|\boldsymbol{q}(s)-\boldsymbol{q}(\tau)|}d\tau + 2\log L\boldsymbol{t}(s)\right).$$

From this it follows that the expansion of the tangential component of the far-field vector potential (in inner coordinates) is

$$\boldsymbol{t}(s)\cdot\boldsymbol{A}_o(\boldsymbol{x}(X,Y,s)) = -\frac{\epsilon^2\mathcal{I}}{2\pi\kappa^2}\log R + \frac{\epsilon^2\mathcal{I}}{\kappa^2}\left(\frac{1}{2\pi}\log\left(\frac{2}{\epsilon}\right) + M(s)\right) + B_{ext}\hat{\boldsymbol{A}}\cdot\boldsymbol{t}(s) + \cdots.$$

Matching $\int_0^{2\pi} Q_3 ds$ to $\int_0^{2\pi}\boldsymbol{t}(s)\cdot\boldsymbol{A}_o ds$, using (5.20), we find the following relation between $\mathcal{I}$ and the averaged value of $\beta(s)$:

$$\text{(5.27)} \quad \int_0^{2\pi}\beta(s)ds = \frac{\epsilon^2\mathcal{I}}{\kappa^2}\left(\log\left(\frac{2}{\epsilon}\right) + \int_0^{2\pi}M(s)ds\right) + B_{ext}\int_0^{2\pi}\hat{\boldsymbol{A}}\cdot\boldsymbol{t}(s)ds - 2\pi n.$$

Given the winding number n, the equation above is sufficient to close the inner problem, consisting of equations (5.13), (5.14), (5.16), (5.17), (5.18), and (5.19), for $F(s)$ and $Q_3(X,Y,s)$.

5.1. Asymptotic solution of the model given by equations (5.13), (5.14), (5.16), (5.17), (5.18), (5.19), and (5.27). Since F and Q_3 are expected to have expansions in ϵ with $O(1)$ leading terms $\mathcal{I}$ will likewise have an $O(1)$ leading term. The solution of the system under discussion will only differ at leading order from the model discussed previously in [12, 14] if the first term on the right-hand side of (5.27) is of comparable size to the other term on the right-hand side of this equation. The canonical scaling for ϵ is thus

$$\epsilon^2\log\left(\frac{1}{\epsilon}\right) = \kappa^2.$$

Having made this assumption we seek an asymptotic solution to (5.13), (5.14), (5.16), (5.17), (5.18), (5.19), and (5.27) in powers of $\log(1/\epsilon)$ of the form

$$\mathcal{I} = \mathcal{I}^{(0)} + \frac{\mathcal{I}^{(1)}}{\log(1/\epsilon)} + \cdots, \qquad \beta = \beta^{(0)} + \frac{\beta^{(1)}}{\log(1/\epsilon)} + \cdots,$$
$$Q_3 = Q_3^{(0)} + \frac{Q_3^{(1)}}{\log(1/\epsilon)} + \cdots, \qquad F = F^{(0)} + \frac{F^{(1)}}{\log(1/\epsilon)} + \cdots.$$

Substituting this expansion into (5.13), (5.14), (5.16), (5.17), (5.18), (5.19), and (5.27), taking the leading order term, and solving for $Q_3^{(0)}$ we find the following:

$$Q_3^{(0)} = \beta^{(0)}(s), \tag{5.28}$$
$$\frac{1}{D}\frac{d}{ds}\left(DF^{(0)\prime}\right) = \Gamma({F^{(0)}}^3 - F^{(0)}) + F^{(0)}{Q_3^{(0)}}^2, \tag{5.29}$$
$$\mathcal{I}^{(0)} = -\Gamma D {F^{(0)}}^2 Q_3^{(0)} = \text{const.}, \tag{5.30}$$
$$\int_0^{2\pi} \beta^{(0)} ds + 2\pi n = B_{ext} \int_0^{2\pi} \hat{\boldsymbol{A}} \cdot \boldsymbol{t}(s) ds + \mathcal{I}^{(0)}. \tag{5.31}$$

We adopt the same methods as those used in section 3 to formulate this model in terms of the complex variable $\tilde{\psi}^{(0)}$ and the leading order magnetic flux cutting the ring $\mathcal{F}^{(0)}$ such that

$$\tilde{\psi}^{(0)} = F^{(0)}(s)\exp(i\chi^{(0)}(s)), \qquad Q_3^{(0)} = \frac{\mathcal{F}^{(0)}}{2\pi} - \chi^{(0)\prime}(s).$$

In terms of these new variables (5.28)–(5.31) are

$$\left.\begin{array}{c} \left(\dfrac{d}{ds} - i\dfrac{\mathcal{F}^{(0)}}{2\pi}\right)^2 \tilde{\psi}^{(0)} + \dfrac{1}{D}\dfrac{dD}{ds}\left(\dfrac{d\tilde{\psi}^{(0)}}{ds} - i\dfrac{\mathcal{F}^{(0)}}{2\pi}\tilde{\psi}^{(0)}\right) = \Gamma\tilde{\psi}^{(0)}\left(|\tilde{\psi}^{(0)}|^2 - 1\right), \\ \tilde{\psi}^{(0)} \quad \text{periodic on} \quad (0, 2\pi) \end{array}\right\}, \tag{5.32}$$

(5.33)
$$\mathcal{F}^{(0)} = B_{ext}\int_0^{2\pi} \hat{\boldsymbol{A}} \cdot \boldsymbol{t}(s) ds - D\Gamma\left(|\tilde{\psi}^{(0)}|^2 \frac{\mathcal{F}^{(0)}}{2\pi} + \frac{i}{2}\left((\tilde{\psi}^{(0)})^* \frac{d\tilde{\psi}^{(0)}}{ds} - \tilde{\psi}^{(0)}\frac{d{\tilde{\psi}^{(0)}}^*}{ds}\right)\right).$$

To leading order we have just the same problem that we derived for the thin cylindrical shell, namely, (3.21) and (3.22). At this order the analysis of the phase transition therefore proceeds as in section 3.1. A higher degree of accuracy can be obtained by taking further terms in the expansions of $\mathcal{I}$, β, Q_3, and $\mathcal{F}$ in inverse powers of $\log(1/\epsilon)$.

6. Conclusion. We have derived canonical models for thin superconducting geometries in certain distinguished limits as the thickness of the superconducting domain and the GL parameter κ tend to zero. These models capture the competing effects of small κ, which favors type-I behavior (a first order normal/superconducting transition), and the small aspect ratio of the domain, which favors type-II behavior (a second order normal/superconducting transition). We found that the preferred scaling in which the various effects are of the same order depends crucially on the geometry. For example, the scaling for thin shells and thin films was $\kappa^2 \sim \epsilon$, while the scaling for thin wires was $\kappa^2 \sim \epsilon^2 \log \epsilon^{-1}$, where ϵ is the domain aspect ratio. Our main results are found in (3.21) and (3.22) for thin cylindrical shells, (4.11), (4.15), and (4.16) for thin films, and (5.13), (5.14), (5.16), (5.17), (5.18), (5.19), and (5.27) for thin wires.

We used our new equations to compute the boundaries in the parameter space that separate regimes corresponding to different kind of phase transition. Of particular interest are Figures 3.2 and 4.1 for the case of annular thin films. While the normal-superconducting transition curves exhibit the expected Little–Parks oscillations, a large set of superconducting solutions, bifurcating from the normal state, are subcritical (first order). This effect becomes more pronounced as the applied magnetic field is increased. This is in agreement with the heuristic argument we presented in the introduction, according to which the Meissner effect favors a subcritical transition.

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