The bifurcation structure of a thin superconducting loop with small variations in its thickness

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Abstract: We study bifurcations between the normal and superconducting states, and between superconducting states with different winding numbers, in a thin loop of superconducting wire, of uniform thickness, to which a magnetic field is applied. We then consider the response of a loop with small thickness variations. We find that close to the transition between normal and superconducting states lies a region where the leading order problem has repeated eigenvalues. This leads to a rich structure of possible behaviours. A weakly nonlinear stability analysis is conducted to determine which of these behaviours occur in practice.

Keywords: Little-Parks experiment, superconductivity.

1 Introduction

Certain materials when cooled sufficiently undergo a second order phase transition which takes them from the *normal state*, in which they behave like conventional metals, to a state in which electric currents can be supported without resistance and in which unusual magnetic phenomena are exhibited. This state, termed the *superconducting state*, is one of the few examples where quantum mechanics manifests itself over a macroscopic scale. Indeed it is possible to attribute many of the peculiar properties of superconductors directly to the quantum mechanical nature of the phenomenon. In this context we note (I) the *Josephson effect* whereby two pieces of superconducting material separated by a non-superconductor interact via quantum mechanical tunelling, and (II) the quantisation of magnetic flux exemplified by a circulating current structure, termed a *vortex*, which is associated with one quantum of magnetic flux.

In this work we shall investigate, from a theoretical standpoint, the famous Little–Parks experiment [4] and other closely related set-ups. The inventors of this experiment used an extremely thin hollow cylinder to which an axial magnetic field is applied to show that the transition temperature at the onset of superconductivity has a periodic dependence on the

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flux within the ring. By noting that an integer plus-half-number of magnetic flux quanta within the ring penalises the superconducting state in comparison to an integer number of quanta, they argue that the period of variation in the transition temperature is exactly one quantum of magnetic flux. Recently there has also been considerable interest in a closely related set-up consisting of a thin closed loop of wire set in an applied magnetic field (see for example Fomin et al. [5] and Moshchalkov et. al. [6].)

Interest in the Little Parks experiment has not been solely restricted to experiment and much work has been done on the problem from a theoretical standpoint. In [3] the normal-superconducting transition is investigated by application of the Ginzburg-Landau model to a thin circularly-symmetric annulus. The work of Berger and Rubinstein [1] extends that performed in [3] with the discovery of a further set of transition lines between different superconducting solutions. It also investigates the effects that variations in the thickness of the ring have on the phase diagram, revealing novel symmetry breaking effects. Approximate one-dimensional Ginzburg-Landau models for a non-circularly-symmetric thin ring and a three dimensional thin loop of wire with varying thickness have been derived in [7] and [8] respectively and may be used to investigate the Little–Parks effect in non-circularly symmetric domains.

The present work is an extension of that performed by Berger and Rubinstein in [1]. We find that by formulating the problem in a different manner from these authors we are able to build up a detailed picture of the bifurcation structure of the problems, describing: (I) a thin ring or loop with uniform thickness, and (II) a thin ring or loop with small variations in the thickness. In the next section we formulate the problem for a thin wire of arbitrary geometry. In Section 3 we consider a loop of uniform thickness and describe the local behaviour of the system in the neighbourhood of the normal-superconducting bifurcation line, and in the neighbourhood of the transition lines between superconducting solutions with different topological winding numbers. In Section 4 we extend the analysis of the previous section to a wire with small variations in its thickness. In this section we also conduct a detailed analysis about the points in the phase diagram where the bifurcation lines intersect and where the symmetry breaking discovered in [1] is most marked. Finally, in Section 5, we draw our conclusions.

2 Problem Formulation

In the following we consider a loop of thin wire of length $2\pi l$ and with a typical cross sectional area η (where $\eta \ll 1$) lying along the curve $\boldsymbol{x} = \boldsymbol{q}(s)$.[1] Our starting point is the

[1] We also require that $\eta l \ll \xi$ where ξ is the coherence length. In effect this amounts to requiring that the temperature lie in some range centred on the critical temperature T_c since at this temperature ξ is infinite.

Ginzburg-Landau equations [2]

$$
\begin{aligned}
\frac{1}{m_s}\left(\hbar\nabla - ie_s\boldsymbol{A}\right)^2\Psi &= a(T)\Psi + b(T)|\Psi|^2\Psi,\\
\nabla\wedge\left(\frac{\boldsymbol{B}}{\mu}\right) &= -\frac{ie_s\hbar}{m_s}\left(\Psi^*\nabla\Psi - \Psi\nabla\Psi^*\right) - \frac{2e_s^2}{m_s}|\Psi|^2\boldsymbol{A},\\
\nabla\wedge\boldsymbol{A} &= \boldsymbol{B}.
\end{aligned}
$$

Here Ψ is a complex order parameter for the superconducting charge carriers, $\boldsymbol{B}$ is the magnetic field, T is the temperature and e_s and m_s are respectively the charge, and the effective mass of the superconducting charge carriers. We take the magnetic flux enclosed by the ring to be of the order of one flux quantum, and employ the nondimensionalisation

$$
\boldsymbol{x}\sim l \quad \Psi\sim\sqrt{\frac{|a(T)|}{b(T)}} \quad \boldsymbol{B}\sim\frac{\hbar}{e_s l^2} \quad \boldsymbol{A}\sim\frac{\hbar}{e_s l}.
$$

Following the analysis presented in [7] then leads to a one dimensional model for the averaged (over the thickness of the wire) order parameter ψ as a function of the dimensionless distance along the loop s

$$
\left(\frac{\partial}{\partial s} - iA\right)^2\psi + |\Gamma|(\mathrm{sgn}(\Gamma) - |\psi|^2)\psi + \frac{D'(s)}{D(s)}\left(\frac{\partial\psi}{\partial s} - iA\psi\right) = 0. \tag{1}
$$

In the above $D(s)$ is the cross-sectional area of the loop scaled with η, A is is the component of the vector potential $\boldsymbol{A}$ tangential to the curve $\boldsymbol{x} = \boldsymbol{q}(s)$, and Γ is a temperature-dependent parameter defined by

$$
\Gamma = \frac{1}{\xi^2} = \frac{-l^2 2ma}{\hbar^2}.
$$

Close to T_c, the critical temperature below which superconducting state is energetically favourable in the absence of magnetic field, Γ can be approximated by $\Gamma = k(T_c - T)$ where k is a positive constant. Remarkably this model is entirely independent of the geometry of the wire and depends on the magnetic field solely through the magnetic flux cutting the loop. Indeed, if we select a gauge for $\boldsymbol{A}$ such that its tangential component along the wire is constant we find A is related to $\mathcal{F}$ as follows:

$$
A = \frac{\mathcal{F}}{2\pi}.
$$

Although we are primarily interested in equilibrium states of the Little–Parks set-up, which can be obtained by solution of equation (1), we will also make use of an analogous time dependent model to investigate the stability of the equilibrium states. This model is derived in [7] and, where we employ the same nondimensionalisation as above, may be written as follows:

$$-|\Gamma|\left(\frac{\partial\psi}{\partial t}+i\Phi\psi\right)+\left(\frac{\partial}{\partial s}-iA\right)^2\psi+\frac{1}{D}\frac{\partial D}{\partial s}\left(\frac{\partial\psi}{\partial s}-iA\psi\right)-|\Gamma|\psi\left(|\psi|^2-\mathrm{sgn}(\Gamma)\right)=0, \quad (2)$$

$$\frac{\partial}{\partial s}\left(D\left(|\psi|^2A+\frac{i}{2}\left(\psi^*\frac{\partial\psi}{\partial s}-\psi\frac{\partial\psi^*}{\partial s}\right)\right)\right)+\sigma\frac{\partial}{\partial s}\left(D\left(\frac{\partial\Phi}{\partial s}+\frac{\partial A}{\partial t}\right)\right)=0, \quad (3)$$

where Φ is the dimensionless scalar electric potential. It is sometimes convenient to replace (3) by

$$\frac{|\Gamma|i}{2}\left(\psi\frac{\partial\psi^*}{\partial t}-\psi^*\frac{\partial\psi}{\partial t}\right)+|\Gamma||\psi|^2\Phi=\frac{\sigma}{D}\frac{\partial}{\partial s}\left(D\left(\frac{\partial\Phi}{\partial s}+\frac{\partial A}{\partial t}\right)\right). \quad (4)$$

which is easily derived from (2)-(3).

3 A loop with uniform thickness $D \equiv 1$

Equation (1) has the solution $\psi = 0$ which corresponds to the normal state. We search for a bifurcation from this solution to a superconducting solution by linearising about $\psi = 0$ as follows:

$$\psi = \epsilon\psi_1 + \cdots \qquad \epsilon \ll 1.$$

and substituting into (1). At $O(\epsilon)$ we find that ψ_1 satisfies the linear equation

$$L_\Gamma\psi_1 = \frac{\partial^2\psi_1}{\partial s^2} - 2iA\frac{\partial\psi_1}{\partial s} - A^2\psi_1 + \Gamma\psi_1 = 0, \quad (5)$$

with periodic boundary conditions on $[0, 2\pi]$. This has the non-trivial eigensolution

$$\psi_1 = Ee^{ims} \qquad \text{for} \qquad \Gamma = (m-A)^2,$$

where m is an integer. Hence as Γ increases from zero the first solution to bifurcate is that with winding number $nint(A)$, where we define $nint(A)$ to be the integer closest to A. In fact we can extend the solution (6) to all values of $\Gamma > (m-A)^2$ and hence to $|\psi| = O(1)$ by noting that

$$\psi = \gamma e^{ims} \qquad |\gamma|^2 = \mathrm{sgn}(\Gamma) - \frac{(m-A)^2}{|\Gamma|}, \quad (6)$$

is a solution to the full problem (1) with $D \equiv 1$. A schematic diagram of this bifurcation, for constant A, is shown in figure 1a.

3.1 Linear stability of the normal solution

It is important to distinguish between branches of this bifurcation which correspond to stable solution, and are thus observable, and those which correspond to unstable solutions. We begin by investigating the linear stability of the normal solution by expanding about $\psi = 0$ in powers of ϵ ($\epsilon \ll 1$)

$$\begin{aligned}\psi &= \epsilon e^{\alpha t}\psi_1(s) + O(\epsilon^2),\\ \Phi &= O(\epsilon^2),\end{aligned}$$

and then substituting this expansion into the time dependent model (2)-(3). At $0(\epsilon)$ we find

$$-\alpha|\Gamma|\psi_1 + L_\Gamma\psi_1 = 0, \qquad \psi_1(0) = \psi_1(2\pi), \quad \psi_1'(0) = \psi_1'(2\pi).$$

This has solutions of the form

$$\psi_{1,m} = e^{ims} \qquad \alpha_m = \mathrm{sgn}(\Gamma) - \frac{(m-A)^2}{|\Gamma|},$$

where m is an integer. Hence one can see that small perturbations to the normal state will in general grow if we can find an integer m such that $(m-A)^2/\Gamma < \mathrm{sgn}(\Gamma)$.

3.2 Linear stability of the superconducting state with winding number m

The stability of the steady superconducting solution $\psi = \gamma e^{ims}$ can be investigated in a similar manner. We make the following expansion about this solution:

$$\begin{aligned}\psi &= \gamma e^{ims} + \epsilon e^{\alpha t}\psi_1(s) + \cdots,\\ \Phi &= \epsilon e^{\alpha t}\Phi_1(s) + \cdots.\end{aligned}$$

Making the assumption that γ is real (with no loss of generality) and substituting the above expansion into (2) and (3) we find, at $O(\epsilon)$,

$$\hat{L}_\Gamma\psi_1 = \frac{\partial^2\psi_1}{\partial s^2} - 2iA\frac{\partial\psi_1}{\partial s} - A^2\psi_1 + \Gamma(1-2|\gamma|^2)\psi_1 - \Gamma\gamma^2 e^{2ims}\psi_1^* = \Gamma\left(\alpha\psi_1 + i\gamma e^{ims}\Phi_1\right), \quad (7)$$

$$\left(A - \frac{m}{2}\right)\left(e^{ims}{\psi_1}^* + e^{-ims}\psi_1\right) + \frac{i}{2}\left(e^{-ims}\frac{\partial\psi_1}{\partial s} - e^{ims}\frac{\partial{\psi_1}^*}{\partial s}\right) + \frac{\sigma}{\gamma}\frac{\partial\Phi_1}{\partial s} = F, \quad (8)$$

where F is an arbitrary constant. We look for periodic solutions of the form

$$\begin{aligned}\psi_{1,j} &= e^{ims}\left(b_j e^{ijs} + b_{-j}e^{-ijs}\right),\\ \Phi_{1,j} &= i\left(d_j e^{ijs} + d_{-j}e^{-ijs}\right),\end{aligned}$$

where j is a non-zero integer. Substitution of the above into (7) and (8) leads to the following equations for (b_j, b_{-j}) and (d_j, d_{-j}):

$$\Gamma\gamma d_j = \Gamma\gamma^2 b^*_{-j} + P_j b_j = N_j b_j - N_{-j} b^*_{-j}, \tag{9}$$

where

$$N_j = \frac{\Gamma\gamma^2}{\sigma j}\left(A - m - \frac{j}{2}\right) \quad \text{and} \quad P_j = ((m+j) - A)^2 + \Gamma(2\gamma^2 - 1 + \alpha).$$

By making the transfromation $j \to -j$ in (9) we find a relation between b_{-j} and b^*_j. Then, on use the definition of γ found in (6), we can show that these relations are consistent only if α, the growth factor, is a solution to the quadratic

$$\Gamma^2\alpha^2 + \Gamma\alpha\left(\frac{\Gamma\gamma^2}{\sigma} + 2j^2 + 2\left(\Gamma - (m-A)^2\right)\right) + \left(j^2 + \frac{\Gamma\gamma^2}{\sigma}\right)\left(j^2 - 6(m-A)^2 + 2\Gamma\right) = 0.$$

Since $\Gamma \geq (m-A)^2$ the coefficient multiplying α in the equation above must be positive. Hence solutions α of this quadratic have positive real part iff $6(m-A)^2 - 2\Gamma - j^2 > 0$. The winding number m solution to (1) $\psi = \gamma e^{ims}$ is thus unstable for values of Γ and α for which

$$f_m(\Gamma, A) = 6(m-A)^2 - 2\Gamma - 1 > 0.$$

We must now investigate the case $j = 0$. Here we find a solution to (7)-(8), valid for all Γ and A_2,

$$\psi_1 = iEe^{ims} \quad \Phi_1 = 0 \quad \alpha = 0 \quad E \in \mathbb{R}. \tag{10}$$

It follows that the winding number m solution to equation (1) $\psi = \gamma e^{ims}$ is only neutrally stable for $\{(\Gamma, A) : f_m(\Gamma, A) < 0\}$. This should cause little surprise since $\psi = \gamma e^{ims+\nu}$ is a steady solution to (1) for all real ν. The reader should note that a change in phase of the solution makes no difference to the observable quantities of the system and does not alter the conclusions we draw in this work.

3.3 Bifurcation between superconducting states with different winding numbers

In §3.2 above we saw that the winding number m solution to (1) $\psi = \gamma e^{ims}$ becomes unstable for $3(m-A)^2 - 1/2 > \Gamma$. One can associate the loss of stability with a bifurcation occuring on $\Gamma = 3(m-A)^2 - 1/2$. In order to investigate the bifurcation further we look for a steady solution to (1) of the form

$$\psi = \gamma e^{ims} + \epsilon\psi_1 + \cdots \qquad \epsilon \ll 1.$$

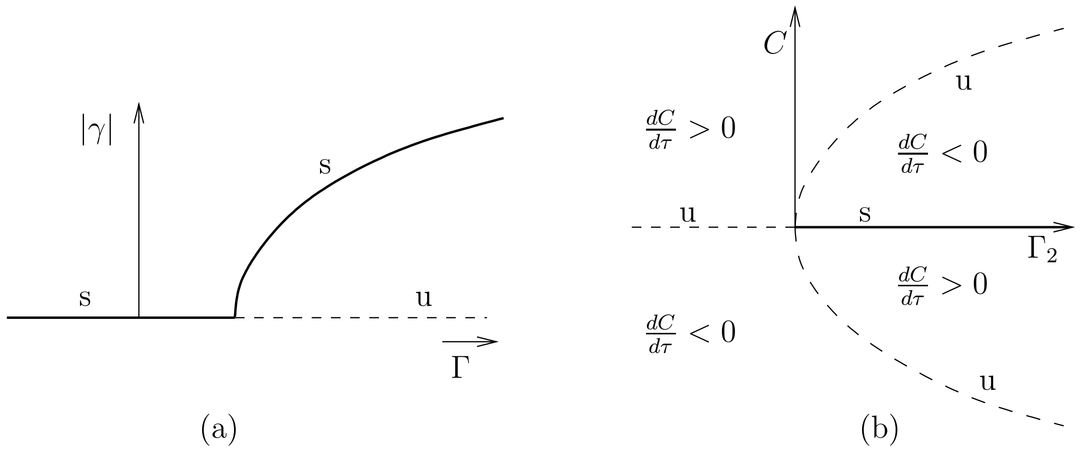

Figure 1: Bifurcation diagrams for (a) the normal-superconducting transition (b) the onset of instability of the superconducting solution

At first order we find the following equation for ψ_1:

$$\hat{L}_\Gamma \psi_1 = 0, \tag{11}$$

together with periodic boundary conditions on $[0, 2\pi]$. By comparison with §3.2 above we can immediately see that this problem has eigenvalue $\Gamma = 3(m-A)^2 - 1/2$. The corresponding eigenvector is

$$\psi_1 = Ce^{ims}\left(2i(m-A)\sin(s-\nu) - \cos(s-\nu)\right) \qquad C, \nu \in \mathbb{R}.$$

We now seek to determine the dependence of C, the amplitude of ψ_1, on small deviations in Γ away from the eigenvalue of (11) by proceeding to higher orders in the expansions of ψ and Γ

$$\begin{aligned} \psi &= \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 + \cdots, \\ \Gamma &= \Gamma_0 + \epsilon^2\Gamma_2 + \epsilon^3\Gamma_3 + \cdots. \end{aligned}$$

where

$$\psi_0 = \gamma_0 e^{ims}, \qquad \psi_1 = Ce^{ims}\left(2i(m-A)\sin(s-\nu) - \cos(s-\nu)\right),$$

and

$$g_m = m - A, \quad \Gamma_0 = 3g_m^2 - 1/2, \quad \gamma_0 = \left(1 - \frac{g_m^2}{(3g_m^2 - 1/2)}\right)^{1/2}. \tag{12}$$

At $O(\epsilon^2)$ we find the following equation for ψ_2

$$\hat{L}_{\Gamma_0}\psi_2 = H(s) = -\Gamma_2\psi_0(1 - |\psi_0|^2) + \Gamma_0\left(2\psi_0|\psi_1|^2 + {\psi_0}^*\psi_1\psi_1\right), \tag{13}$$

together with periodic boundary conditions on $[0, 2\pi]$. Since nontrivial solutions exist to the homogenous problem

$$\hat{L}_{\Gamma_0} w = 0 \qquad w \quad \text{periodic on} \quad [0, 2\pi], \tag{14}$$

we can use the Fredholm alternative to find a solvability condition on the right-hand side of (13); this is

$$\mathrm{Re}\left(\int_0^{2\pi} H(s) w^* ds\right) = 0 \quad \forall \quad w \quad \text{satisfying (14).} \tag{15}$$

The right-hand side of (13) satisfies (15) for all C and ν; we therefore solve for ψ_2 to find

$$\psi_2 = \gamma_0 e^{ims}\left(C^2(iM_2 \sin(2s) + M_1 \cos(2s)) + h_1 + h_2 C^2\right) + P\psi_1$$

where

$$M_1 = \frac{3\Gamma_0}{2(6g_m^2 - 2\Gamma_0 - 4)}, \quad M_2 = g_m\left(\Gamma_0/2 - M_1\right),$$
$$h_1 = \frac{\Gamma_2 g_m^2}{2\Gamma_0(\Gamma_0 - g_m^2)}, \qquad h_2 = \frac{\Gamma_0(3/2 + 2g_m^2)}{2(g_m^2 - \Gamma_0)},$$

and P is an arbitrary constant. Proceeding to $O(\epsilon^3)$ we find the following equation for ψ_3

$$\begin{aligned}\hat{L}_{\Gamma_0}\psi_3 = \big[&\Gamma_0(|\psi_1|^2\psi_1 + 2\psi_0\psi_1\psi_2{}^* + 2\psi_0\psi_2\psi_1{}^* + 2\psi_1\psi_2\psi_0{}^*) + \\ &\Gamma_2((2|\psi_0|^2 - 1)\psi_1 + \psi_0\psi_0\psi_1{}^*) - \Gamma_3(1 - |\psi_0|^2)\psi_0\big]\,.\end{aligned} \tag{16}$$

together with periodic boundary conditions on $[0, 2\pi]$. Applying the solvability condition (15) to equation (16) leads to the following relation between Γ_2 and C

$$C(C^2(18g_m^2 - 3)(4g_m^2 + 1) - 4\Gamma_2) = 0. \tag{17}$$

The coefficient multiplying C^3 is positive since $g_m^2 > 1/4$. The bifurcation diagram resulting from (17) is thus qualitatively similar to the picture in figure 1b.

3.4 Weakly nonlinear stability analysis of the bifurcation from the superconducting solution with winding number m

In the vicinity of this bifurcation, the linear stability analysis carried out in §3.2 reveals that all single wavenumber perturbations to the solution $\psi = \gamma e^{ims}$ decay exponentially in time t with the exception of the zero'th mode, which is always neutrally stable, and the first mode

$$\psi_{1,1} = Ce^{ims}\left(2i(m - A)\sin(s - \nu) - \cos(s - \nu)\right).$$

At leading order the latter has zero growth rate α. In order to investigate the stability of perturbations of this form in the neighbourhood of the bifurcation one needs to conduct a weakly nonlinear stability analysis. This we do by introducing the long time scale $\tau = \epsilon t$ and looking for a solution to the time-dependent model of the form

$$\begin{aligned}\psi &= \gamma e^{ims} + \epsilon C(\tau) e^{ims} \left(2i(m-A)\sin(s-\nu) - \cos(s-\nu)\right) + \epsilon^2 \psi_2 + \epsilon^3 \psi_3 \cdots ,\\ \Phi &= \epsilon^3 N(\tau) \phi_3(s) + \cdots .\end{aligned}$$

By adopting an approach similar to that employed in §3.3 we were able to find an expression for $dC/d\tau$. As expected the lines upon which $dC/d\tau = 0$ are those given by equation (17). They divide regions in which $dC/d\tau$ is positive from regions in which it is negative. The stability of a particular bifurcation branch depends on the sign of $dC/d\tau$ above and below it and the importance of the weakly nonlinear stability analysis is in identifying the sign of $dC/d\tau$ in the different regions. The results in this particular case are summarised figure 1b.

Remark It is nearly always possible to infer the sign of the growth rate of the most unstable mode (in this case $dC/d\tau$) in some region from a linear stability analysis conducted along any two of the bifurcation branches. It follows that we can also infer the stability of the other bifurcation branches.

3.5 Description of the behaviour of the uniform ring

In §3 we examined the bifurcation structure of a thin ring with uniform thickness. It was found, that as Γ increases from zero, the normal solution first becomes unstable when $\Gamma = (nint(A) - A)^2$ at which point a stable superconducting solution, of the form $\psi = \gamma e^{ims}$ ($m = nint(A)$) bifurcates from the normal solution (see figure 1a). We were also able to analyse the stability of the superconducting solution, with winding number m, far from the normal-superconducting transition. We found that this solution becomes unstable as the function $f_m(\Gamma, A) = 6(m-A)^2 - 2\Gamma - 1$, increases through zero at which point a subcritical bifurcation occurs (see figure 1b). We expect that after the winding number m solution loses stability the system will evolve to a solution with winding number $m-1$ or $m+1$ depending on whether $f_{m-1}(\Gamma, A)$ or $f_{m+1}(\Gamma, A)$ is negative. Physically this corresponds to a vortex moving across the ring. These results are summarised in in the phase diagram plotted in figure 2. Here the solid line represents the normal-superconducting transition; the middle two dashed lines bound the region of stability of the solution $\psi = \gamma e^{ims}$ and the dotted lines bound the regions of stability of the solutions $\psi = \gamma e^{i(m+1)s}$ and $\psi = \gamma e^{i(m-1)s}$. It is noteable that the transitions between different winding numbers (for instance m and $m-1$) occur at different places in the phase diagram depending on whether A is increasing as a transition from winding number $m-1$ to m is made or decreasing as the reverse transition is made (the arrows in figure 2 show the direction of transition). This phenomenon, commonly termed superheating or supercooling, is associated with the system adopting a metastable state

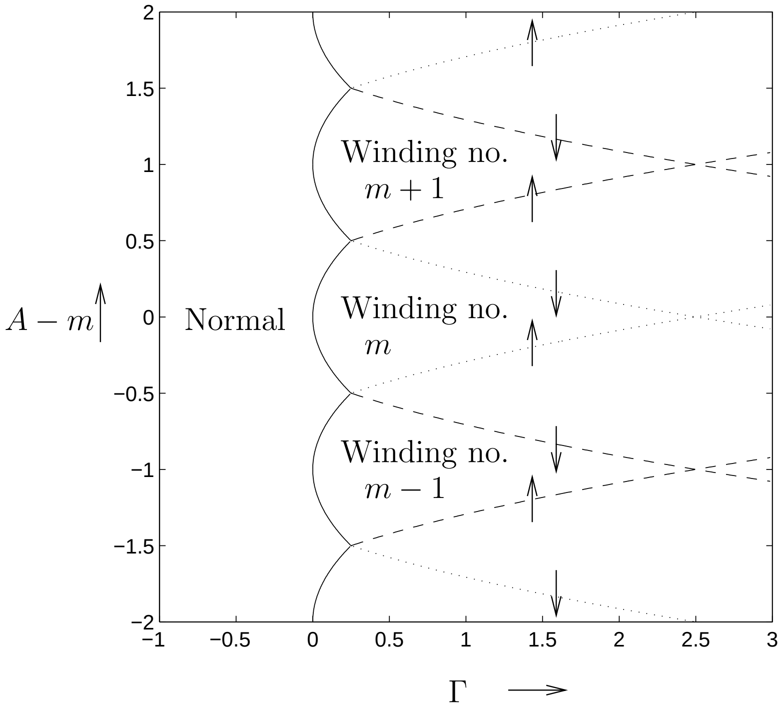

Figure 2: The phase diagram for a ring of uniform thickness

with higher energy than the global minimum. As the parameters of state move through some critical value there is a catastrophic loss of stability which sends the system into an equilibrium far from its original state. In common with other systems which exhibit superheating/supercooling we expect that where the system lies in the metastable state a sufficiently large perturbation may drive it out of this equilibrium into an equilibrium with lower energy.

In this context we also note the work of Zhang and Price [9] in which experimental evidence of this hysteresis is presented.

4 A loop with small variations in thickness

In this section we examine the effect that a small variation in the thickness of the ring has on the bifurcation structure of equation (1). We write D, the measure of the thickness, as follows:

$$D = 1 + \delta D_1 \qquad \delta \ll 1 \tag{18}$$

and expand D_1 as a Fourier series in s, choosing the origin of the s coordinate such that the coefficient of $\sin(s)$ vanishes

$$D_1(s) = \sum_{n=1}^{\infty} \beta_n \cos(ns) + \sum_{n=2}^{\infty} \alpha_n \sin(ns),$$

We substitute (18) into equation (1) and look for solutions. Clearly $\psi \equiv 0$ (the normal state) is still a solution. We can also find a superconducting solution by looking for a regular perturbation of the exact solution for a uniform ring (see equation (6)) of the form

$$\psi = \gamma e^{ims} + \delta\psi_1 + \cdots , \tag{19}$$

where

$$\gamma = \left(1 - g_m^2/\Gamma\right)^{1/2}. \tag{20}$$

At $O(\delta)$ substitution of the above into (1) gives rise to the following equation for ψ_1:

$$\hat{L}_\Gamma \psi_1 = -ig_m D_1' \gamma e^{ims}.$$

together with periodic boundary conditions on $[0, 2\pi]$. This problem has solution

$$\begin{aligned}\psi_1 = -\gamma g_m e^{ims} \Bigg(&\sum_{n=1}^{\infty} \frac{\beta_n}{6g_m^2 - 2\Gamma - n^2} \left(2g_m \cos(ns) + \frac{i(2g_m^2 - 2\Gamma - n^2)}{n} \sin(ns) \right) \\ &+ \sum_{n=2}^{\infty} \frac{\alpha_n}{6g_m^2 - 2\Gamma - n^2} \left(2g_m \sin(ns) - \frac{i(2g_m^2 - 2\Gamma - n^2)}{n} \cos(ns) \right) \Bigg).\end{aligned} \tag{21}$$

The regular expansion (19) breaks down close to the normal-superconducting transition as $\gamma \to 0$. For non-zero β_1 this expansion also breaks down along the transition lines between winding numbers. This is because the denominator of the first term in (21), $6g_m^2 - 2\Gamma - 1$, vanishes. In §4.1, §4.2 and §4.3 we examine the effect that variations in the thickness of the wire have upon these bifurcations.

4.1 Normal-Superconducting transition

As $(1 - g_m^2/\Gamma)^{1/2}$ decreases towards zero so does the amplitude of ψ. For sufficiently small ψ small deviations in the loop thickness directly affect the leading order behaviour of the order parameter. Scaling ψ appropriately will thus allow us to investigate the effect of the small thickness variations on the normal-superconducting bifurcation. As we shall demonstrate it is appropriate to scale ψ with δ; this motivates the following expansion:

$$\begin{aligned}\psi &= \delta\psi_0 + \delta^2\psi_1 + \delta^3\psi_2 + \cdots , \\ \Gamma &= \Gamma_0 + \delta^2\Gamma_2 + \cdots ,\end{aligned}$$

where $\Gamma_0 = g_m^2$ (g_m is defined in (12)). Substituting into (1) we find the leading order solution satisfies

$$L_{\Gamma_0}\psi_0 = 0, \tag{22}$$

together with periodic boundary conditions on $[0, 2\pi]$. Since $\Gamma_0 = g_m^2$ is an eigenvalue of (22) there is a nontrivial periodic solution, namely

$$\psi_0 = Ce^{ims} \qquad C \in \mathbb{C},$$

At $O(\delta^2)$ we find an equation for ψ_1

$$L_{\Gamma_0}\psi_1 = -D_1' \left(\psi_0' - iA\psi_0\right), \tag{23}$$

together with periodic boundary conditions on $[0, 2\pi]$. Although the problem for ψ_1 is an inhomogenous version of the eigenvalue problem for ψ_0, the inhomogeneity is such that we can find a periodic solution for ψ_1; this is

$$\psi_1 = Ce^{ims}\left(\sum_{n=1}^{\infty}\frac{\beta_n}{n^2-4g_m^2}\left(2g_m^2\cos(ns) - ing_m\sin(ns)\right)\right.$$
$$\left. + \sum_{n=2}^{\infty}\frac{\alpha_n}{n^2-4g_m^2}\left(2g_m^2\sin(ns) + ing_m\cos(ns)\right)\right) + R\psi_0,$$

where R is an arbitrary constant. Proceeding to $O(\delta^3)$ we find the following equation for ψ_2:

$$L_{\Gamma_0}\psi_2 = H(s) = D_1 D_1' \left(\psi_0' - iA\psi_0\right) + \Gamma_0|\psi_0|^2\psi_0 - D_1'\left(\psi_1' - iA\psi_1\right) - \Gamma_2\psi_0, \tag{24}$$

together with periodic boundary conditions on $[0, 2\pi]$. This is, once again, an inhomogeneous version of the eigenvalue problem for ψ_0 and, in order for it to have a periodic solution, we require that the right-hand side satisfy the solvability condition

$$\int_0^{2\pi} H(s){\psi_0}^* ds = 0.$$

Applying this condition to (24) gives rise to the following relation between C and Γ_2:

$$C = 0 \quad \text{or} \quad \Gamma_2 = g_m^2\left(|C|^2 - \frac{\beta_1^2}{2(1-4g_m^2)} - \sum_{n=2}^{\infty}\frac{n^2(\alpha_n^2+\beta_n^2)}{2(n^2-4g_m^2)}\right). \tag{25}$$

Providing $g_m^2 \neq 1/4$ the qualitative features of the bifurcation remain unchanged from those of the uniform ring (see figure 1a). Quantitatively one can see that the effect of the non-uniformity is to slightly decrease the value of Γ at which the bifurcation occurs (i.e at which $C = 0$), although notably this critical value of Γ still has minimum value zero attained when $g_m = m - A = 0$. Perhaps the most interesting feature of equation (25) is that where $\beta_1 \neq 0$ there is a singularity of C at the points $g_m = m - A = \pm 1/2$. This is associated with the breakdown of the expansion around these points.

4.2 Transition between superconducting states with different winding numbers

We now procced to examine the effect of small variations in the thickness of the wire upon the transition between superconducting states with different winding numbers (these transitions are represented by dotted and dashed lines in the phase diagram plotted in figure 2). As in §4.1 above we need to identify the size of the perturbation to the uniform solution $\psi = \gamma e^{ims}$ at which effects resulting from variations in the thickness of the wire are of the same order as those resulting from the proximity of the bifurcation. It turns out that the appropriate scaling for the perturbation of ψ is $\delta^{1/3}$; this motivates the following expansion:

$$\begin{aligned} \psi &= \psi_0 + \delta^{1/3}\psi_1 + \delta^{2/3}\psi_2 + \delta\psi_3 + \cdots , \\ \Gamma &= \Gamma_0 + \delta^{2/3}\Gamma_2 + \delta\Gamma_3 + \cdots . \end{aligned}$$

where

$$\psi_0 = \gamma_0 e^{ims}, \quad \gamma_0 = \left(1 - \frac{2g_m^2}{6g_m^2 - 1}\right)^{1/2}, \quad \Gamma_0 = 3g_m^2 - \frac{1}{2}. \tag{26}$$

Substituting the above into equation (1) we retrieve the same equations at first and second order (i.e $O(\delta^{1/3})$ and $O(\delta^{2/3})$) for ψ_1 and ψ_2 as we did in §3.3, namely (11) and (13). Upto the constants C and ν the solutions of these equations are the same as those in §3.3. However, since variation in the thickness of the wire is associated with a loss of symmetry, we can no longer expect the phase ν of ψ_1 to remain completely arbitrary. It is thus more convenient for our present purposes to write ψ_1 and ψ_2 in the form

$$\begin{aligned} \psi_1 &= Ce^{ims}\left(2ig_m \sin(s) - \cos(s)\right) + Ee^{ims}\left(2ig_m \cos(s) + \sin(s)\right), \\ \psi_2 &= \gamma_0 e^{ims}\big((C^2 - E^2)(iM_2 \sin(2s) + M_1 \cos(2s)) + 2CE(iM_2 \cos(2s) - M_1 \sin(2s)) \\ &\quad + h_1 + h_2(C^2 + E^2)\big), \end{aligned}$$

where C and E are real constants still to be determined and M_1, M_2, h_1 and h_2 are as defined in §3.3. Proceeding to $O(\delta)$ we find the following inhomogenous equation for ψ_3:

$$\begin{gathered} \hat{L}_{\Gamma_0}\psi_3 = \big[\Gamma_0(|\psi_1|^2\psi_1 + 2\psi_0\psi_1{\psi_2}^* + 2\psi_0\psi_2{\psi_1}^* + 2\psi_1\psi_2{\psi_0}^*) + \\ \Gamma_2((2|\psi_0|^2 - 1)\psi_1 + \psi_0\psi_0{\psi_1}^*) - \Gamma_3(1 - |\psi_0|^2)\psi_0 - D_1'(\psi_0' - iA\psi_0)\big] . \end{gathered} \tag{27}$$

together with periodic boundary conditions on $[0, 2\pi]$. In order to find a solution for ψ_3 we require the right hand side of (27) to satisfy the solvability condition (15). Applying this condition results in two coupled relations for C and E which are independent of α_n, β_n for $n \geq 2$,

$$\begin{aligned} (C^3 + E^2C)(72g_m^4 + 6g_m^2 - 3) &= 4\beta_1\sqrt{\frac{4g_m^2 - 1}{6g_m^2 - 1}} + 4\Gamma_2 C, \\ (E^3 + C^2E)(72g_m^4 + 6g_m^2 - 3) &= 4\Gamma_2 E. \end{aligned}$$

It is straightforward to show that $E = 0$ and thus that

$$C^3(18g_m^2 - 3)(4g_m^2 + 1) - 4\Gamma_2 C = 4\beta_1\sqrt{\frac{4g_m^2 - 1}{6g_m^2 - 1}}.$$

The presence of the term on the right hand side of this equation, for $\beta_1 \neq 0$, is enough to cause a qualitative change in the dependence of C on Γ_2 from that found in the uniform ring. Without loss of generality where β_1 is non-zero we can set it equal to 1. Having made this simplification we plot C versus Γ_2, for two values of g_m, in figure 3. In accordance with the remark made in §3.4 we can infer the stability of the bifurcation branches from our knowledge of the stability of the solution for large $|\Gamma_2|$ (see §3.1).

We can use the results obtained above to describe the transition from one winding number to an adjacent winding number in a slightly non-uniform ring. One can see that as Γ approaches the critical value $3(m - A)^2 - 1/2$ from above the perturbation to the solution $\psi = \gamma e^{ims}$, caused by the non-uniformity, grows from size $O(\delta)$ to $O(\delta^{1/3})$ until a certain value of Γ is reached, of $O(\delta^{2/3})$ greater than $3(m - A)^2 - 1/2$, at which there is a fold in the bifurcation diagram (see figure 3). Once this point has been reached the system jumps to a solution with winding number $m - 1$ or $m + 1$. Thus, despite the qualitative difference in the relation between C and Γ_2, the physical behaviour of the system is little altered from that of the symmetric ring.

It is also of note that as $g_m^2 \to 1/4$ so $\gamma \to 0$ and the expansion breaks down.

4.3 Transitions close to the critical point

In §4.1 and §4.2 above we examined the behaviour of the slightly non-uniform wire in the vicinity of the normal-superconducting transition line and the transition lines between states with different winding numbers. However we were not able to resolve about the *critical points* $g_m = (m - A)^2 = 1/4$, $\Gamma = 1/4$ where the two sorts of transition line intersect.[2] In the following we examine the behaviour of solutions to (1) around the general critical point $(\Gamma, A) = (1/4, m + 1/2)$. We investigate a region about this point of a size such that the effect of a typical deviation of (Γ, A) from $(1/4, m + 1/2)$ on ψ is comparable with the effect of an $O(\delta)$ perturbation in the thickness of the wire where, as above, we choose δ such that $\beta_1 = 1$. The appropriate scale for the deviation in (Γ, A) about the critical point is $O(\delta)$ and we thus expand as follows:

$$\begin{aligned} \psi &= \delta^{1/2}\psi_0 + \delta^{3/2}\psi_2 + \cdots , \\ A &= (m + 1/2) + \delta A_2 + \cdots , \\ \Gamma &= \frac{1}{4} + \delta\Gamma_2 + \cdots , \end{aligned}$$

[2] In figure 2 the critical points are represented by the intersection of dotted and dashed lines with the solid curve.

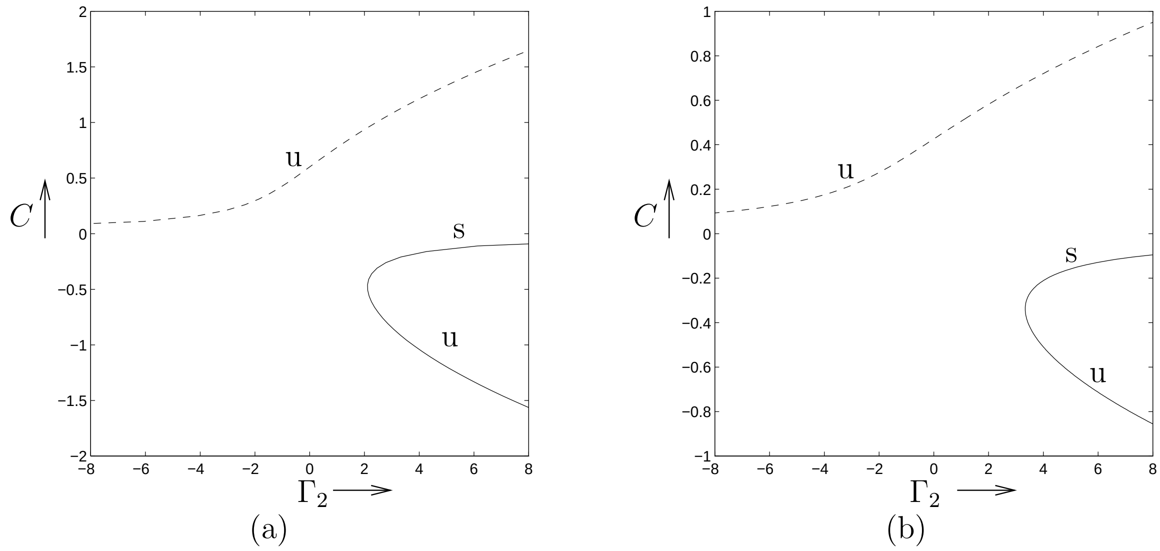

Figure 3: Bifurcation from a solution with winding number m for a ring with small variations in thickness; the plots are of C versus Γ_2 with A held constant. In (a) $g_m = m - A = 0.65$ and in (b) $g_m = m - A = 0.85$.

Substituting into (1) we find an equation for ψ_0 at $O(\delta)$

$$\tilde{L}_{1/4}\psi_0 = \psi_0{}'' - 2i(m + 1/2)\psi_0{}' - (m + 1/2)^2\psi_0 + \frac{1}{4}\psi_0 = 0, \tag{28}$$

together with periodic boundary conditions on $[0, 2\pi]$. Here the operator $\tilde{L}_\Gamma$ is equivalent to L_Γ (see equation (5)) where we set $A = m + 1/2$.

The reason for the singular behaviour of the system about the critical point becomes apparent when we search for solutions to (28); we find that $\Gamma = 1/4$ is a repeated eigenvalue of the problem associated with the doubly degenerate eigensolution

$$\psi_0 = Re^{ims} + Ue^{i(m+1)s} \qquad U, R \in \mathbb{C}.$$

Procceding to $O(\delta^{3/2})$ we find the following equation for ψ_2:

$$\begin{aligned} \tilde{L}_{1/4}\psi_2 &= H(s), \\ H(s) &= (2iA_2 - D_1')\,(\psi_0{}' - i(m + 1/2)\psi_0) + \frac{1}{4}|\psi_0|^2\psi_0 - \Gamma_2\psi_0. \end{aligned}$$

together with periodic boundary conditions on $[0, 2\pi]$. In order that this have solution $H(s)$ must satisfy the solvability conditions

$$\int_0^{2\pi} e^{-ims}H(s)ds = 0 \qquad \int_0^{2\pi} e^{-i(m+1)s}H(s)ds = 0.$$

These conditions gives rise to the following relations:

$$R\left(4(A_2-\Gamma_2)+(|R|^2+2|U|^2)\right) = U, \tag{29}$$
$$U\left(-4(A_2+\Gamma_2)+(|U|^2+2|R|^2)\right) = R. \tag{30}$$

It is clear that R and U must have the same argument. Matching to the other regions, in which we assume that the coefficients multiplying e^{ims} are real, one can see that R and U must also be real. Even after making this simplification it is still tedious to search for solutions to (29)-(30). In order to transform these equations into a more tracteable form we introduce new variables η and ρ such that

$$R=\rho\cos\eta \qquad U=\rho\sin\eta.$$

Substituting the above into equations (29) and (30) and performing some manipulations leads to the following decoupled relations for η and ρ:

$$\text{either} \qquad \rho=0 \tag{31}$$

$$\text{or} \qquad \left\{\begin{array}{l} \cos 2\eta = \dfrac{6A_2\sin 2\eta}{2\Gamma_2\sin 2\eta-1} \\ \rho^2=\dfrac{2}{3}\left(4\Gamma_2+\dfrac{1}{\sin 2\eta}\right). \end{array}\right. \tag{32}$$

These equations are relatively straightforward to solve numerically and in figure 4 we show solutions to (29) and (30), obtained in this manner, for various values of Γ_2. The solutions plotted are not complete. In particular, since equations (29)-(30) are invariant under the transformation $(R,U)\to(-R,-U)$, solutions occur in pairs, and we plot only one of every pair. We have also ommited the zero solution $(R,U)=(0,0)$ from some of the plots.

The reader should observe how the dependence of (R,U) on A_2 changes as Γ_2 increases. For sufficiently small Γ_2 (figure 4(a)) there is a region centred on $A_2=0$ where the only solution is the zero solution. Beyond $\Gamma_2=-1/4$ this region dissapears such that for all values of A_2 a non-zero solution pair exists. As Γ_2 is increased still further the curves representing the original solution pairs $R=R(A_2)$ and $U=U(A_2)$ develop a fold and, for some values of A_2, another solution pair appears. It is possible to find formulae which give the positions, in the (Γ_2,A_2) plane, of (I) the boundary of the *normal region* in which only the zero solution exists and (II) the fold in the solution surface. We calculate the former by linearising (29) and (30) about $(R,U)=(0,0)$ and looking for bifurcations from the zero solution. These occur along the lines

$$\Gamma_2=-\left(A_2^2+\frac{1}{16}\right)^{1/2} \quad \text{and} \quad \Gamma_2=\left(A_2^2+\frac{1}{16}\right)^{1/2}.$$

The normal region is bounded by the first of these two lines and lying in

$$\Gamma_2<-\left(A_2^2+\frac{1}{16}\right)^{1/2}.$$

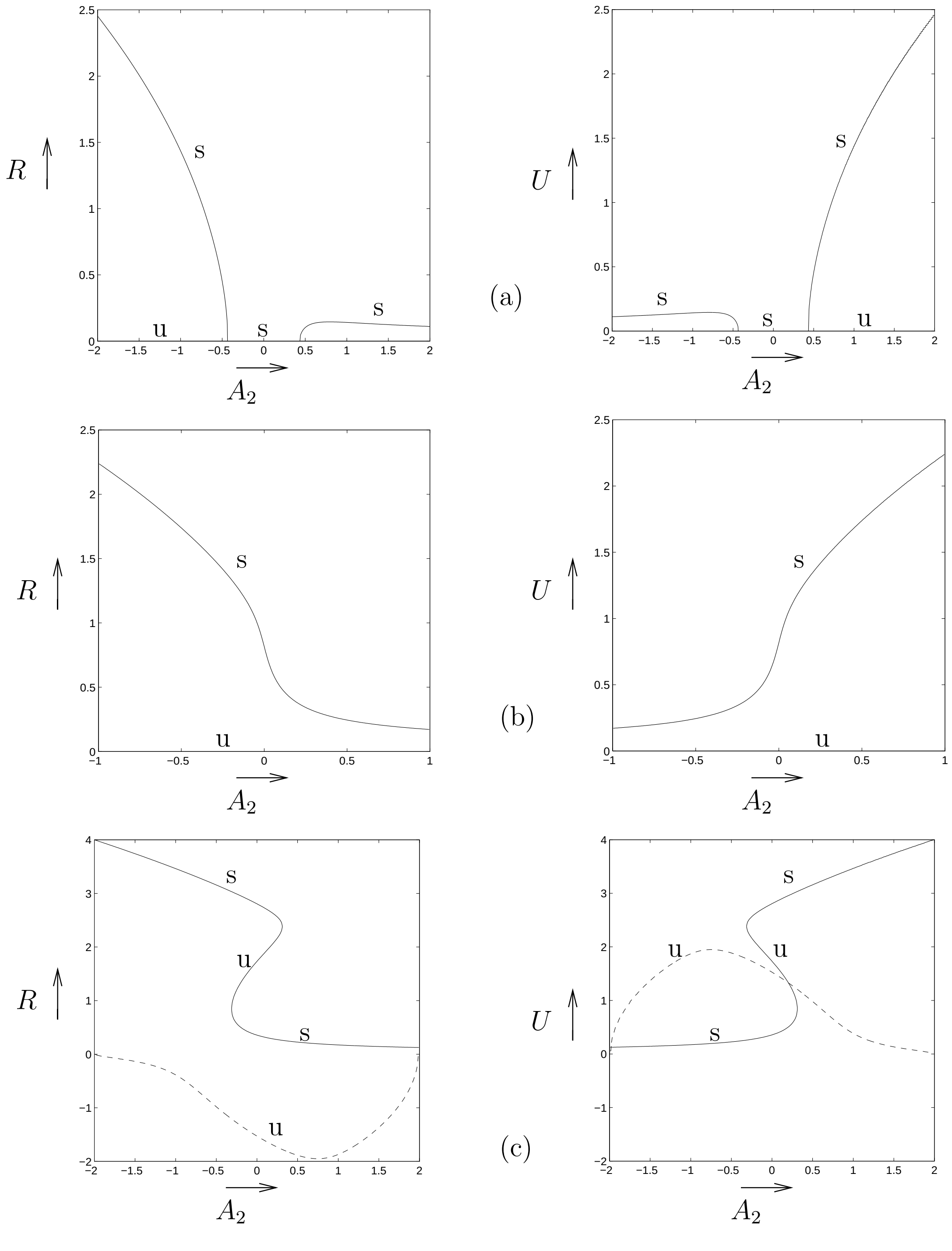

Figure 4: The figure shows plots of R and U versus A_2 for constant values of Γ_2. In (a) $\Gamma_2 = -0.5$, in (b) $\Gamma_2 = 0.25$ and in (c) $\Gamma_2 = 2.0$

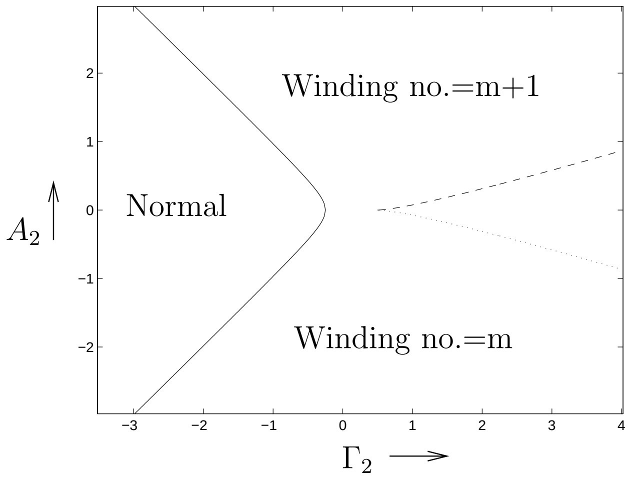

Figure 5: The phase diagram for a ring with small variations in thickness in the vicinity of a critical point.

We look for the fold in the solution surface by searching for infinities of $\partial\eta/\partial A_2$. These occur for values of η satisfying

$$\sin 2\eta = \left(\frac{1}{2\Gamma_2}\right)^{1/3} \quad \text{and} \quad \cos 2\eta = \left(\frac{3A_2}{\Gamma_2}\right)^{1/3}. \tag{33}$$

The fold thus lies along a curve in the (Γ_2, A_2) plane satisfying

$$\left(\frac{1}{2\Gamma_2}\right)^{2/3} + \left(\frac{3A_2}{\Gamma_2}\right)^{2/3} = 1. \tag{34}$$

This curve is plotted, along with the boundary of the normal region, in figure 5. The dashed and dotted lines represent the fold and the solid line the boundary of the normal region.

4.3.1 Weakly nonlinear stability analysis.

We need now to investigate the stability of the large number of solutions to equation (1) that exist in the vicinity of the critical point. A linear stability analysis about the normal state reveals that all single wavenumber perturbations to the solution $\psi = 0$ decay exponentially in time with the exception of the $m'th$ and $(m+1)'th$ modes which have zero growth rate at leading order. This motivates us to examine the growth rate of these modes by looking for solutions to the time dependent model (2) and (4) of the form

$$\begin{aligned}\psi &= \delta^{1/2}\left(R(\tau)e^{ims} + U(\tau)e^{i(m+1)s}\right) + \delta^{3/2}\psi_2 + \cdots,\\ A &= (m+1/2) + \delta A_2 + \cdots,\\ \Gamma &= \frac{1}{4} + \delta\Gamma_2 + \cdots,\\ \Phi &= O(\delta^2),\end{aligned}$$

where τ is the long time variable given by $\tau = \delta t$. Adopting an approach similar to that in §4.3 above we find the following equations for $R'(\tau)$ and $U'(\tau)$:

$$\frac{dR}{d\tau} + R\left(4(A_2 - \Gamma_2) + (|R|^2 + 2|U|^2)\right) = U, \tag{35}$$

$$\frac{dU}{d\tau} + U\left(-4(A_2 + \Gamma_2) + (|U|^2 + 2|R|^2)\right) = R. \tag{36}$$

We can ascertain the stability of solutions to the steady problem (29) and (30) for any given (Γ_2, A_2) by plotting a phase diagram. Alternatively we can attempt to make a more general statement regarding the stability of the solution manifold by linearising equations (35)-(36) about the steady solution $(R, U) = (\rho\sin\eta, \rho\cos\eta)$

$$\begin{aligned}R &= \rho\sin\eta + \epsilon e^{\lambda\tau} r_1,\\ U &= \rho\cos\eta + \epsilon e^{\lambda\tau} u_1,\end{aligned}$$

where $\epsilon \ll 1$. Substituting the above into (35)-(36) and proceeding to $O(\epsilon)$ one finds that $(r_1, u_1)^T$ is an eigenvector of the matrix

$$\begin{pmatrix} -(\rho^2(3\sin^2\eta + 2\cos^2\eta) + 4(A_2 - \Gamma_2)) & 1 - 2\rho^2\sin 2\eta \\ 1 - 2\rho^2\sin 2\eta & 4(A_2 + \Gamma_2) - \rho^2(3\cos^2\eta + 2\sin^2\eta) \end{pmatrix}$$

with corresponding eigenvalue λ. This eigenvalue satisfies the quadratic equation

$$a\lambda^2 + b\lambda + c = 0, \tag{37}$$

whose coefficients are determined in the usual manner. After considerable manipulation, in which we eliminate ρ and $\cos 2\eta$ by reference to equation (32), we find that these may be written in the form

$$a = 3\sin^2 2\eta \qquad b = 2\sin^2 2\eta\left(8\Gamma_2 + \frac{5}{\sin 2\eta}\right)$$

$$c = 8\sin 2\eta\left(\Gamma_2 + \frac{1}{\sin 2\eta}\right)\left(1 - 2\Gamma_2\sin^3 2\eta\right).$$

In order for a solution to be stable we require that both solutions to equation (37) have negative real part. Since $a > 0$ this amounts to requiring both b and c be positive. Referring to equation (32b) we see that the physically relevant solutions are those for which

$$4\Gamma_2 + \frac{1}{\sin 2\eta} > 0. \tag{38}$$

For $\Gamma_2 > 0$ such solutions are stable (i.e $b > 0$ and $c > 0$) iff

$$0 < \sin 2\eta < \left(\frac{1}{2\Gamma_2}\right)^{1/3}.$$

For $\Gamma_2 < 0$ all solutions which satisfy the condition (38) are stable.

Remark It is clear form equation (32a) that $\sin 2\eta = 0$ is never a solution. It follow that a sheet of the solution manifold may only change from unstable to stable or vice-versa along lines on which $\sin 2\eta = (1/(2\Gamma_2))^{1/3}$ and $\Gamma_2 > 0$. It is straightforward to show that solutions of (32a) can only adopt this value of $\sin 2\eta$ at positions in the (Γ_2, A_2) plane satisfying (34) and furthermore that $\cos 2\eta = (3A_2/\Gamma_2)^{1/3}$. Referring back to equation (33) we can see that the switch in stability occurs along a fold in the sheet. Hence the sheet that bifurcates from the normal solution along the line $\Gamma_2 = -(A_2^2 + 1/16)^{1/2}$ is always stable except where it has folded over on itself (this sheet is represented by the solid curve in figure 4). The other non-zero sheet to the solution manifold bifurcates from the zero solution along $\Gamma_2 = (A_2^2 + 1/16)^{1/2}$ and is always unstable R and U take opposite signs along this sheet, corresponding to negative vales of $\sin 2\eta$ (this sheet is represented by the dashed line in figure 4c).

We can investigate the stability of the normal solution $(R, U) = (0, 0)$ by adopting a similar approach to that taken above. We linearise equations (35)-(36) about the zero solution by substitution of

$$R = \epsilon e^{\lambda\tau} r_1, \quad U = \epsilon e^{\lambda\tau} u_1, \quad \epsilon \ll 1.$$

At $O(\epsilon)$ we find an eigenvalue problem for $(r_1, u_1)^T$ with corresponding eigenvector λ. Determining λ in the usual manner we find

$$\lambda = 4\left(\Gamma_2 \pm \left(A_2^2 + \frac{1}{16}\right)^{1/2}\right).$$

The normal solution is therefore stable only for

$$\Gamma_2 < -\left(A_2^2 + \frac{1}{16}\right)^{1/2}.$$

Physical interpretation of the solutions. We have investigated the solution structure, and stability, of a loop with slightly non-uniform thickness in the vicinity of a critical point. We can interpret the results physically with the aid of figure 5. To the left of the solid line the only solution is $\psi = 0$ corresponding to the normal state. As this line is crossed the normal solution loses stability and a stable superconducting solution bifurcates. This solution has winding number $m+1$ for $A_2 > 0$ and winding number m for $A_2 < 0$. A completely smooth transition between winding numbers occurs if the system traverses the phase plane, from top to bottom or vice-versa, without crossing the dotted or dashed lines (see figure 4(b)). If however a transversal of the phase plane is made from bottom to top which crosses the dashed line the transition between winding number occurs suddenly; as the dashed line is crossed the winding number m solution loses stability and the system evolves to the winding number $m+1$ solution. The reverse transition, from winding number $m+1$ to m, occurs along the dotted line.

5 Conclusion

In this work we have investigated the bifurcations of a loop of superconducting wire between different states. For a wire of uniform thickness we found, in agreement of the results of Berger and Rubinstein [1] and Gorff and Parks [3], a supercritical bifurcation between the normal state and a superconducting state whose topological winding number is determined by the magnetic field. We also found a series of subcritical bifurcations occuring between superconducting states with different winding numbers. Investigation of the stability of these states led us to infer that the system exhibits hysteresis as a transition is made from one winding number to an adjacent one and back again. The lines representing these two different types of transition in state space (see figure 2) intersect at a series of points (the critical points).

In the latter portion of this paper we investigated the effect that small nonuniformities in the thickness of the wire loop have upon the bifurcation structure of the system. We found that a supercritical bifurcation still occurs between the normal and superconducting states; the effect of the nonuniformity is only to increase the temperature (lower Γ) at which it occurs. The transition between superconducting states of different winding numbers no longer occurs at a subcritical bifurcation, but on a fold (see figure 3). The most marked change, though, is in the vicinity of a critical point. Here variations in the thickness cause the folded region of the solution surface to separate from the normal-superconducting bifurcation. Thus in a narrow region close to this bifurcation the transition between superconducting states with different winding numbers is completely smooth.

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