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The Mixed Boundary Condition for the Ginzburg Landau Model in Thin Films

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Abstract—The mixed boundary condition for the Ginzburg Landau model of superconductivity is considered in thin films. A simplified model is derived in the asymptotic limit of very small thickness. We also show that under certain conditions there is no nucleation of superconductivity at all.

Keywords—Superconductivity, Ginzburg Landau model, Thin films, Mixed boundary conditions.

1. INTRODUCTION

The Ginzburg Landau (GL) model for superconductivity concerns two unknown fields, see [1]. A complex order parameter u and the magnetic vector potential A. We consider the GL model in cylindrical domains of the form

$$\Omega = \{x, y, z \mid (x, y) \in \Omega_0,\ -l < z < l\}, \tag{1}$$

where Ω_o is a bounded two-dimensional domain. The lengthscale of the problem is normalized by $R = (\Omega_0/2\pi)^{1/2}$.

The energy functional is then written as [2]

$$G(u, A) = \int_\Omega \frac{\lambda}{2}\left(|u|^2 - 1\right)^2 + |(\nabla - iA)\,u|^2 + \kappa^2\lambda^{-1}\int_{\mathbf{R}^3} |\nabla \times A - H_e|^2. \tag{2}$$

Here H_e is the external magnetic field, the parameter λ is related to the temperature T through

$$\lambda = \frac{R^2}{\xi_0^2}\frac{T_c - T}{T_c}, \tag{3}$$

κ is a temperature independent material parameter, T_c is the critical temperature in the absence of magnetic fields, and ξ_0 is a material parameter. Equating the first variation to zero gives rise

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to the associated Euler Lagrange equations

$$-\Delta u + i\nabla \cdot Au + iA\, \nabla u + |A|^2 u + \lambda u \left(|u|^2 - 1\right) = 0, \qquad x \in \Omega, \tag{4}$$

$$\nabla \times (\nabla \times A - H_e) = \kappa^{-2}\lambda \Im(\bar{u}(\nabla - iA)u)1_\Omega, \qquad x \in \mathbf{R}^3, \tag{5}$$

$$(\nabla - iA)u \cdot \nu = 0, \qquad x \in \partial\Omega. \tag{6}$$

In [3], de Gennes has proposed to modify (6) when the superconducting region is adjacent to a metal. His new condition takes the form

$$(\nabla - iA)u \cdot \nu = -\beta u, \tag{7}$$

where β is a positive parameter and ν is the normal to the boundary. Equivalently, one can modify the GL functional by adding a surface energy term $\int_{\partial\Omega} \beta|u|^2$ to $G(u, A)$. We shall refer to the new functional as the *Modified* Ginzburg Landau (MGL) functional, and to the parameter β as the de Gennes coefficient. The mutual interaction between the superconducting material and its surrounding normal material is termed the proximity effect.

Considerable experimental activity in superconductivity concerns materials with a small lateral dimension. It is, therefore, useful to compute the asymptotics of the MGL functional as the thickness tends to zero. Such limits provide simpler equations for the equilibrium states and prevent the need to define a very fine mesh in the process of seeking a numerical solution. Limits of this kind have been derived [4,5] for GL functionals of type (2). We shall derive the associated asymptotic limit for the MGL model. For this purpose, we set $l = \epsilon$, where ϵ is a small positive parameter, and denote the superconducting domain by Ω_ϵ.

We expect the energy in the thin film to be of $O(\epsilon)$. We thus rewrite a scaled version of the MGL in the form

$$G_\epsilon(u, A) = \frac{1}{2\epsilon}\int_{\Omega_\epsilon} \frac{\lambda}{2}\left(|u|^2 - 1\right)^2 + |(\nabla - iA)u|^2 + \frac{\kappa^2}{2\epsilon\lambda}\int_{\mathbf{R}^3} |\nabla \times A - H_e|^2 + \frac{1}{2}\int_{\partial\Omega_\epsilon} b|u|^2. \tag{8}$$

Notice that we implicitly assume the canonical scaling $\beta = \epsilon b$. It is convenient to split the boundary of Ω_ϵ into

$$\partial\Omega_\epsilon = \bigcup_{i=1}^{3} D_i, \qquad D_1 = \Omega_0 \times [-\epsilon], \qquad D_2 = \Omega_0 \times [\epsilon], \qquad D_3 = \partial\Omega_0 \times [-\epsilon, \epsilon]. \tag{9}$$

To characterize the minimizers (u_ϵ, A_ϵ) of G_ϵ, we introduce the functional

$$F(v) = \int_{\Omega_0} \frac{\lambda}{2}\left(|v|^2 - 1\right)^2 + b|v|^2 + |\left(\nabla_2 - iA_e\right)v|^2, \tag{10}$$

where A_e is the vector potential associated with the applied field, i.e., $\nabla \times A_e = H_e$, and ∇_2 is the two-dimensional gradient operator. The natural setting to formulate a convergence result is to associate with a function u, defined over Ω_ϵ, its lateral average $\tilde{u} = (1/2\epsilon)\int_{-\epsilon}^{\epsilon} u(x, y, z)\, dz$. The natural function space for minimizing F is the Sobolev space $H^1(\Omega_0)$.

THEOREM. *Let (u_ϵ, A_ϵ) be a sequence of minimizers of G_ϵ. Then it is possible to extract from it subsequences $(\tilde{u}_\epsilon, A_\epsilon)$ such that $\tilde{u}_\epsilon$ converges in $H^1(\Omega_0)$ to a minimizer of $F(v)$, and A_ϵ converges in $H^1(\mathbf{R}^3)$ to A_e.*

PROOF. Consider the functional

$$G_\epsilon^1(u, A) = \frac{1}{2\epsilon}\int_{\Omega_\epsilon} \frac{\lambda}{2}\left(|u|^2 - 1\right)^2 + b|u|^2 + |(\nabla - iA)u|^2 + \frac{\kappa^2}{2\epsilon\lambda}\int_{\mathbf{R}^3} |\nabla \times A - H_e|^2. \tag{11}$$

Clearly $G_\epsilon^1 - G_\epsilon = bR_\epsilon$, where $R_\epsilon = (1/2\epsilon)\int_{\Omega_\epsilon} |u|^2 - \sum_1^3 (1/2)\int_{D_i} |u|^2$.

The statement of the theorem holds for the minimizers of $G^1_\epsilon(u,A)$ by the same arguments as in [4,5]. Therefore, it remains to show that G^1_ϵ is a small perturbation of G, i.e., that $\lim_{\epsilon\to 0} R_\epsilon(u_\epsilon) = 0$.

A minimizer u_ϵ of G_ϵ satisfies the maximum principle $|u_\epsilon| \le 1$. Therefore, $\int_{D_3} |u_\epsilon|^2 \le C\epsilon$. Define the restriction of u_ϵ to D_i, $i = 1,2$ by $u_{i,\epsilon}$. We can write

$$R_\epsilon(u_\epsilon) = \sum_1^2 \frac{1}{4\epsilon}\int_{\Omega_\epsilon}\left(|u_\epsilon|^2 - |u_{i,\epsilon}|^2\right) + O(\epsilon). \tag{12}$$

Since all terms in G_ϵ are positive, the inequality

$$G_\epsilon(u_\epsilon, A_\epsilon) \le G_\epsilon(1, A_e) \le C$$

implies

$$\int_{\Omega_\epsilon} |(\nabla - iA_\epsilon)\, u_\epsilon|^2 \le 2C\epsilon.$$

In particular,

$$\int_{\Omega_\epsilon} \left|\frac{\partial |u_\epsilon|}{\partial z}\right|^2 \le 2C\epsilon. \tag{13}$$

Combining (12) and (13), we obtain the desired estimate for R_ϵ and complete the proof.

REMARK. It is easy to generalize the theorem to include the case in which the de Gennes coefficient b and the domain thickness l vary with x and y.

When the temperature (or the magnetic field) is sufficiently high, the global minimizer is given by the pair $(u, A) = (0, A_e)$, where $\nabla \times A_e = H_e$. This solution is called the *normal state.* A phase transition occurs when this state bifurcates into a stable nontrivial solution for the order parameter. Returning to our original formulation (4)–(6), the bifurcation equation for the MGL is given by the spectral problem for the first eigenvalue λ_p for the operator $-(\nabla - iA_e)^2$, i.e.,

$$(\nabla - iA_e)^2\psi = -\lambda_p\psi, \qquad x \in \Omega, \tag{14}$$

together with the boundary condition

$$(\nabla - iA_e)\psi\cdot\nu = -\beta\psi, \qquad x \in \partial\Omega. \tag{15}$$

The eigenvalue λ_p is determined by the applied field H_e through the potential A_e. The transition temperature T is then determined by (3). It is clear that the effect of the de Gennes coefficient is to increase λ_p, i.e., to reduce T. One might think that if β is above a certain critical value β_0, the shift in λ_p will push the transition temperature below absolute zero, and therefore, there will be no phase transition at all.

We note, however, that $\lambda_p(\beta) \le \lambda_0$, where λ_0 is the first eigenvalue for (14) with a homogeneous Dirichlet boundary condition. In addition, ξ_0^2/R^2 is typically very small. Thus, the effect of shifting $\lambda_p(\beta)$ is not significant. The situation is different in a thin film. Assume that the nondimensional thickness ϵ is of order ξ_0/R. Then if $\beta = O(1)$, we get $\lambda_p = O(\epsilon^{-2})$. Thus, if β is larger then some $O(1)$ critical value, the transition temperature will be too small, and therefore, the normal state will not lose stability and no phase transition will occur.

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