LINE DISCLINATION DYNAMICS IN UNIAXIAL NEMATIC LIQUID CRYSTALS

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Summary

We analyse the motion of a line disclination in a nematic liquid crystal using the methods of formal asymptotics in the limit as the thickness of its core tends to zero. The disclination appears as a singularity in the field equations for the director and the fluid velocity. Its motion is governed by a law obtained from a local analysis about its core.

1. Introduction

So-called *nematic liquid crystals* are comprised of rigid rod-like molecules with rotational symmetry about their long axis and reflectional symmetry about the plane made by their two short axes. At high temperatures nematics are isotropic and behave in every way much like a conventional liquid. However as the temperature of the substance is lowered it undergoes a phase transition that takes it from the *isotropic state*, in which there is no ordering between molecules, to the *nematic state* in which the individual molecules show a tendency to align along a common direction. The alignment of the molecules in the nematic state results in optical anisotropy, which is readily detected, and may also result in markedly anisotropic flow properties.

Using the concept of a director $\boldsymbol{n}$, which is a vector that gives, in some sense, the average direction of the nematic molecules, Frank and Oseen (**1**, **2**) modelled the static properties of the nematic state. The resulting theory does not distinguish between $-\boldsymbol{n}$ and $\boldsymbol{n}$ since the ends of a particular molecule are indistinguishable. The Frank–Oseen theory was subsequently generalized, to include time-dependence, by Ericksen and Leslie (**3**, **4**, **5**, **6**) resulting in a model which predicts the evolution of the director field $\boldsymbol{n}$ and the fluid flow field $\boldsymbol{v}$.

While both these theories have been rather successful at modelling many of the phenomena exhibited by nematics they have not been able to satisfactorily account for singularities in the director field, termed *disclinations*, which are commonly observed in such materials (see de Gennes (**7**)). Such disclinations occur either at a point, from which the director radiates, or along a curve. In the latter case, as a circuit is made about the disclination curve the director field rotates through an integer multiple of π, and thereby preserves, in the sense that $-\boldsymbol{n}$ and $\boldsymbol{n}$ are indistinguishable, the single-valuedness of the director off the curve but leads to a singularity on the curve. Viewed in terms of the Frank–Oseen and the Ericksen–Leslie models the disclination line has not only a singular director field but also infinite energy. In order to resolve this paradox Ericksen (**8**) has noted that the degree of alignment between the rod-like nematic molecules may vary spatially (even at constant temperature) and that the molecules should be totally unaligned (in the isotropic state) on the disclination curve. Modelling this variable alignment requires the introduction of an order

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parameter S which gives a statistical measure of the orientation of the molecules and is such that if $S = 1$ the molecules are all perfectly aligned parallel to the director $\boldsymbol{n}$; if $S = 0$ there is no ordering of the molecules and the material is in the *isotropic state*; and if $S = -\frac{1}{2}$ all the molecules are aligned in a plane perpendicular to $\boldsymbol{n}$. Another model along similar lines has been proposed by de Gennes (**7**, **9**) but requires the introduction of a second-order parameter. Given that no compelling evidence has been presented to favour either model over the other we opt for the simpler Ericksen model.

The aim of this paper is to use the methods of formal asymptotics to calculate the velocity of a rectilinear disclination line in a physically relevant regime in which the ratio of the width of the disclination core to all other lengthscales in the problem, ϵ, is vanishingly small. We base our calculation on the very general model given in (**8**) making some additional assumptions about the precise nature of the liquid crystal. The calculation is thus not entirely general and we would expect the result we obtain to be slightly modified for materials with properties other than those specified. Our result bears some similarity to a particle-field model proposed by Denniston (**10**) to describe nematic disclination dynamics. We start in section 2 by writing down the Ericksen model (**8**) and then use this to repeat calculations made by Pismen and Rubinstein (**11**) and Mottram and Hogan (**12**) of the core structure of a static disclination. In section 3 we proceed to derive the law of motion for a disclination, a calculation that bears some similarity to the derivation of the law of motion for a superconducting vortex (see Peres and Rubinstein (**13**) and Chapman and Richardson (**14**)) and to the law for defects in a complex scalar field subject to the nonlinear heat equation (see Neu (**15**)), but which also has some novel features brought about by the interaction of the disclination with the fluid velocity field $\boldsymbol{v}$. In this context we note the work of Pismen and Rubinstein (**11**) who also derive the disclination law of motion from (**8**) but neglect the fluid velocity field and so arrive at a result substantively different from ours. In section 4 we illustrate the law of motion with some examples and finally in section 5 we draw our conclusions.

2. The model

The Ericksen model (**8**) describes the evolution of the director $\boldsymbol{n}$, the scalar-order parameter S and the fluid velocity field $\boldsymbol{v}$ within a nematic liquid crystal. In order to write down equations for these quantities, based on this model, we require knowledge of the *Helmholtz free-energy density* $\phi(\boldsymbol{n}, \nabla\boldsymbol{n}, S, \nabla S, T)$ of the material. This scalar quantity depends on the temperature T of the material and is such that its integral over the domain of the liquid crystal is minimized when the system is in stable equilibrium.

In writing down the equations we shall make some simplifications consistent with the material being sufficiently weakly nematic that, to leading order, the fluid flow is independent of the director field. In other words we look at a liquid crystal for which the equilibrium value of S is small enough that the fluid-flow properties are only weakly anisotropic. In terms of the original paper (**8**) this is equivalent to setting the anisotropic viscosities $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_5 = \alpha_6 = 0$ while retaining the isotropic fluid viscosity $\alpha_4/2$. It follows that $\gamma_1 = \gamma_2 = 0$. We shall also assume that the anisotropic relaxation parameter for the order parameter is negligible and set $\beta_3 = 0$, from which it can be inferred that $\beta_1 = 0$.

Having made these simplifications Ericksen's model reduces to an equation for the momentum of the fluid:

$$\rho \frac{dv_i}{dt} = \sigma_{ij,j}. \tag{1}$$

Here the time derivative is the convective derivative

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\boldsymbol{v} \cdot \nabla),$$

and the stress is defined as follows:

$$\sigma_{ij} = -\hat{p}\delta_{ij} - \frac{\partial \phi}{\partial n_{k,j}} n_{k,i} - S_{,i} \frac{\partial \phi}{\partial S_{,j}} + \frac{\alpha_4}{2} \left(v_{i,j} + v_{j,i} \right). \tag{2}$$

This couples to the incompressibility condition

$$v_{j,j} = 0, \tag{3}$$

an equation describing the evolution of the order parameter

$$\frac{\partial \phi}{\partial S} - \frac{\partial}{\partial x_j} \left(\frac{\partial \phi}{\partial S_{,j}} \right) + \beta_2 \frac{dS}{dt} = 0 \tag{4}$$

(here β_2 is a positive relaxation parameter for S), and to the two equations describing the evolution of the director field

$$|\boldsymbol{n}| = 1, \tag{5}$$

$$\boldsymbol{\pi} \wedge \boldsymbol{n} = 0, \tag{6}$$

where

$$\pi_i = \frac{\partial}{\partial x_j} \left(\frac{\partial \phi}{\partial n_{i,j}} \right) - \frac{\partial \phi}{\partial n_i}. \tag{7}$$

Introducing the modified pressure

$$p = \hat{p} + \phi$$

and using the relation $n_j n_{j,i} = 0$ we rewrite equation (1) as

$$\rho \frac{d\boldsymbol{v}}{dt} = -\nabla p - \beta_2 \nabla S \frac{dS}{dt} + \frac{\alpha_4}{2} \nabla^2 \boldsymbol{v}. \tag{8}$$

It remains to specify the Helmholtz free-energy density and, in order to give the reader some idea why the various terms are included, we split this up into three parts,

$$\phi(\boldsymbol{n}, \nabla \boldsymbol{n}, S, \nabla S, T) = \phi_1(S, T) + \phi_2(S, \boldsymbol{n}, \nabla \boldsymbol{n}, T) + \phi_3(\nabla S, T). \tag{9}$$

The first term ϕ_1 gives the free energy for a bulk transition, that is to say, one for which $S = S(T)$ and $\boldsymbol{n} = \text{const}$. Following de Gennes (**7**) we write this as a quartic in S:

$$\phi_1(S, T) = Q(T) S^2 \left(\frac{S^2}{4} - (S^*(T) + \overline{S}(T)) \frac{S}{3} + \frac{S^*(T)\overline{S}(T)}{2} \right), \qquad S^*(T) > \overline{S}(T), \tag{10}$$

this being the simplest functional form that can be used to model a first-order phase transition. The parameters S^* and $\overline{S}$ are, respectively, the values of S at which the energy ϕ_1 has a local minimum and a local maximum. There is also another local minimum at $S = 0$, corresponding to the isotropic state. The values of ϕ_1 at the two local minima determine whether the isotropic state $S = 0$ or the nematic state $S = S^*$ is preferred (that is, is a global minimizer of the energy).

The second term in the energy ϕ_2 gives the distortional energy. It is required that this decreases to zero at $S = 0$ and that it be consistent with the distortional energy used in the Frank–Oseen theory. We employ the so-called one constant approximation of the Frank–Oseen energy (see deGennes (**7**)) and a natural choice for the distortional energy is thus

$$\phi_2(S, \boldsymbol{n}, \nabla\boldsymbol{n}, T) = \tfrac{1}{2}K(T)S^2 n_{i,j}n_{i,j},$$

where $K(T)$ is a temperature-dependent function of proportionality which gives a measure of the contribution of gradients in the director field to the energy.

The last term in the energy ϕ_3 serves to penalize gradients in the order parameter

$$\phi_3(\nabla S, T) = \tfrac{1}{2}D(T)|\nabla S|^2,$$

the function of proportionality $D(T)$ giving a measure of the contribution of gradients in S to the energy.

We now make the isothermal non-dimensionalization

$$S = S^*S', \quad \boldsymbol{x} = L\boldsymbol{x}', \qquad \boldsymbol{v} = U\boldsymbol{v}',$$
$$\mu = \frac{\alpha_4}{2}, \quad t = \frac{Lt'}{U}, \quad p = \frac{\mu U p'}{L},$$

where L is a typical lengthscale of the problem (usually the size of the apparatus or a typical inter-disclination distance) and U is the (as yet undetermined) typical fluid velocity of the problem.† Then, on substitution of the non-dimensionalized Helmholtz free-energy density (9) into the non-dimensionalized versions of the governing equations (3) to (8), we find the following dimensionless system of equations describing the evolutionary behaviour of a nematic liquid crystal:

$$|\boldsymbol{n}| = 1, \tag{11}$$

$$\boldsymbol{\pi}' = 2S'(\nabla'S'.\nabla')\boldsymbol{n} + S'^2\nabla'^2\boldsymbol{n}, \tag{12}$$

$$\boldsymbol{\pi}' \wedge \boldsymbol{n} = \boldsymbol{0}, \tag{13}$$

$$-\tilde{\varepsilon}\tilde{\beta}_2\frac{dS'}{dt'} + \tilde{D}\nabla'^2S' = \frac{1}{\epsilon^2}S'(S'-1)(S'-b) + S'|\nabla'\boldsymbol{n}|^2, \tag{14}$$

$$\mathrm{Re}\frac{d\boldsymbol{v}'}{dt'} = -\nabla'p' - \tilde{\beta}_2\nabla'S'\frac{dS'}{dt'} + \nabla'^2\boldsymbol{v}', \tag{15}$$

$$\nabla\cdot\boldsymbol{v}' = 0, \tag{16}$$

† We shall show that the fluid velocity scale depends upon the the velocity of the disclination which itself is dependent on local gradients in the director field.

with corresponding dimensionless parameters, which include the Reynolds number Re and the Ericksen number $\tilde{\varepsilon}$, defined as follows:

$$\mathrm{Re} = \frac{\rho L U}{\mu}, \qquad \tilde{\varepsilon} = \frac{\mu L U}{K S^{*2}}, \quad \tilde{D} = \frac{D}{K},$$
$$\epsilon = \frac{1}{L S^*}\left(\frac{K}{Q}\right)^{1/2}, \quad \tilde{\beta_2} = \frac{\beta_2 S^{*2}}{\mu}, \quad b = \frac{\overline{S}}{S^*}.$$

Henceforth the primed superscript, denoting the dimensionless variable, is dropped.

It is of interest to note the similarity between (11) to (16) and the time-dependent Ginzburg–Landau (TDGL) equations of superconductivity (see (**14**, **16**)). This is most marked when the director field is constrained to lie in the (x, y)-plane (say) and so may be written as $\boldsymbol{n} = (\cos\psi, \sin\psi, 0)$. It is not surprising therefore that there should be some similarity between the derivation of the law of motion for a line defect (vortex) in a superconductor (**13**, **14**) and that for a line defect (disclination) in a nematic liquid crystal. However, there is an important difference between these two sets of equations, namely that whereas the time derivatives in the TDGL equations are partial derivatives those in (11) to (16) are convective derivatives. Thus while there is a special frame of reference for the TDGL equations (being the frame of the superconducting medium), equations (11) to (16) are invariant under rigid-body translation. This causes some notable differences between the calculation of superconducting vortex and nematic disclination velocities and makes the latter calculation, perhaps, rather more subtle.

2.1 *Estimating the size of the dimensionless parameters in the model*

The dimensional parameters D, Q and β_2 have not, to our knowledge, been measured for any nematic liquid crystal. Despite this we can make sensible guesses as to the relative sizes of some of the more important dimensionless parameters. It is known for instance that the core of a disclination (that is the region in which the order parameter deviates significantly from S^*) is extremely small. In his book Chandresekhar (**17**) quotes a figure of 10^{-6} cm. As we shall show in section 2.2, ϵ gives a measure of the radius of the disclination core to the typical lengthscale L of the problem. Taking an experimental lengthscale in the range $L = 10^{-4}$ cm to 1 cm gives $\epsilon = 10^{-6}$to 10^{-2}.

We can estimate sizes of the Ericksen and Reynolds number, up to the undetermined velocity scaling U, using typical values of ρ, μ and $K S^{*2}$ (the elastic constant) taken from (**7**)

$$\rho = 1{\cdot}0\, g\,\mathrm{cm}^{-3}, \qquad \mu = 0{\cdot}25\, g\,\mathrm{cm}^{-1}\,\mathrm{sec}^{-1}, \qquad K S^{*2} = 10^{-6}\,\mathrm{dyn}. \tag{17}$$

It is harder to say much about the parameters $\tilde{D}$ and $\tilde{\beta_2}$. All we can say for certain is that if $\tilde{D}$ exceeds a certain $O(1)$ size it leads to an unphysical infinite energy density (though not infinite energy) along the disclination line (see section 2.2). Fortunately though, the method used to calculate the disclination velocity is fairly robust with respect to the size of these parameters; the only necessary requirement is that $0 < \tilde{\beta_2} \ll 1/\epsilon$.

2.2 *An isolated disclination*

Disclinations are characterized by a singularity in the director field along a line and a zero of the order parameter S along the same line. The singularity is such that as a circuit is made about the disclination line the director $\boldsymbol{n}$ rotates through an integer multiple of π (say N). Although this does not ensure the single-valuedness of $\boldsymbol{n}$ away from the disclination line it is consistent with $\boldsymbol{n}$ being indistinguishable from $-\boldsymbol{n}$. We now find a disclination solution for equations (11) to (15) making the solution ansatz

$$
\begin{aligned}
S &= S(r),\\
\boldsymbol{n} &= \cos\left(c + \tfrac{1}{2}N\theta\right)\boldsymbol{e_x} + \sin\left(c + \tfrac{1}{2}N\theta\right)\boldsymbol{e_y}\\
&= \cos\left(\left(\tfrac{1}{2}N - 1\right)\theta + c\right)\boldsymbol{e_r} + \sin\left(\left(\tfrac{1}{2}N - 1\right)\theta + c\right)\boldsymbol{e_\theta},\\
\boldsymbol{v} &= \boldsymbol{0},
\end{aligned}
$$

where N is an integer, r and θ are polar coordinates. We find that S satisfies

$$\tilde{D}\frac{1}{r}\frac{d}{dr}\left(r\frac{dS}{dr}\right) = \frac{1}{\epsilon^2}S(S-1)(S-b) + \frac{N^2 S}{4r^2} \tag{18}$$

and, in order that the Helmholtz free energy be finite, the solution must satisfy the boundary conditions

$$S(0) = 0, \tag{19}$$

$$dS/dr \to 0 \qquad \text{as } r \to \infty. \tag{20}$$

Far from the origin S asymptotes to a constant with one of three possible values 0, b and 1. In order to ascertain which of these possible values gives rise to a physically pertinent solution we must briefly consider their stability. Any solution for which $S \to b$ as $r \to \infty$ must be unstable since $S = b$ gives a maximum of the bulk energy ϕ_1 (see equation (10)). The asymptotic behaviour $S \to 0$ as $r \to \infty$ will be stable if ϕ_1 has a global minimum for $S = 0$ but gives rise to a rather dull solution $S \equiv 0$ corresponding to a uniform isotropic state. However, if $S = 1$ is the global minimizer of ϕ_1 one can expect to find stable disclination solutions with asymptotic behaviour

$$S \to 1 \qquad \text{as } r \to \infty. \tag{21}$$

We rescale distance with $\epsilon|N|$, the typical width of the disclination, such that $r = \epsilon|N|R$. Under such a rescaling the system (18), (19) and (21) becomes

$$\Lambda^2\left(\frac{d^2S}{dR^2} + \frac{1}{R}\frac{dS}{dR}\right) = S(S-1)(S-b) + \frac{S}{4R^2}, \tag{22}$$

where $\Lambda = \tilde{D}^{1/2}/|N|$. In order that this match to the outer solution and satisfy (19) and (21) we require the boundary conditions

$$S(0) = t0, \tag{23}$$

$$S \to 1 \quad \text{as} \quad R \to \infty. \tag{24}$$

We have performed several numerical solutions of this problem, using a NAG routine, for different values of Λ and b (see Fig. 1). A proof of the existence of a solution to a very similar problem, describing the core structure of a superconducting vortex, has been given by Berger and Chen (**18**).

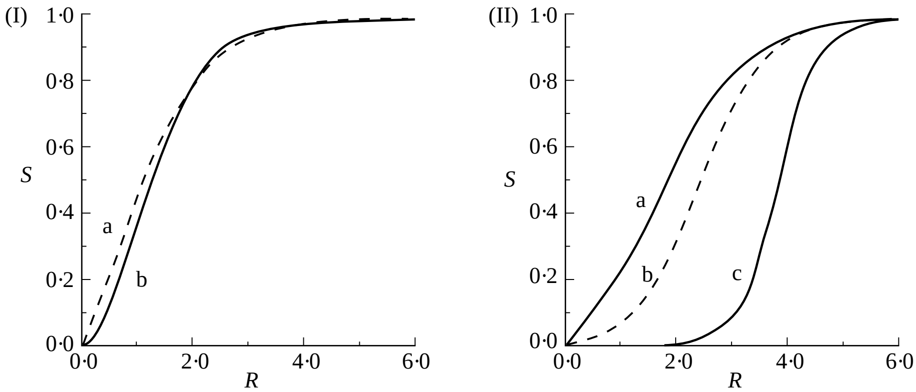

Fig. 1 Solutions for the order parameter close to an isolated disclination. In (I)a $(b, \Lambda) = (0{\cdot}3, 0{\cdot}5)$, (I)b $(b, \Lambda) = (0{\cdot}3, 0{\cdot}35)$, (II)a $(b, \Lambda) = (0{\cdot}5, 0{\cdot}5)$, (II)b $(b, \Lambda) = (0{\cdot}5, 0{\cdot}375)$, (II)c $(b, \Lambda) = (0{\cdot}5, 0{\cdot}25)$

REMARK. As $R \to 0$ the solution of (18) has the asymptotic behaviour $S \sim kR^{1/(2\Lambda)}$ (where k is some constant). This leads to a singularity in the energy density at the origin if $\Lambda > \frac{1}{2}$. The total energy of the disclination, though, remains bounded provided Λ is positive. There is something inherently unsatisfactory in the infinite-energy-density scenario which leads us to suggest that physically relevant values of Λ lie in the range $\Lambda \leqslant \frac{1}{2}$.

3. Motion of a rectilinear disclination

We shall consider a two-dimensional setup in which disclinations and boundaries lie parallel to the z-axis and look for solutions to (11) to (16) which are independent of z and which have director fields of the form

$$\boldsymbol{n} = (\cos\psi(x, y, t), \sin\psi(x, y, t), 0).$$

We choose as our outer lengthscale L the lengthscale for typical separations between disclinations and boundaries. This also is the lengthscale for typical variations in the director field.

Inner region. We define a local coordinate system (ς, θ) about one of the disclination cores

$$\boldsymbol{x} - \boldsymbol{q}(t) = \varsigma\cos\theta\boldsymbol{e_1} + \varsigma\sin\theta\boldsymbol{e_2}, \tag{25}$$

where $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$ are conveniently chosen unit base vectors in the (x, y)-plane and $\boldsymbol{q}(t)$ is the position of the disclination at time t. Inner coordinates are obtained by introducing the stretched variable R, such that $\varsigma = \epsilon|N|R$. Provided that the velocity of the fluid relative to the disclination is not too large (that is, $\tilde{\varepsilon}\tilde{\beta}_2 \ll 1/\epsilon$) the solution to (11) to (16), at leading order in the inner regions, is the same as that for the isolated disclination since the convective time derivatives do not enter at this order; thus we find the leading-order inner solution has the following form:

$$\begin{aligned}\psi_i &= c + \tfrac{1}{2}N\theta + o(1),\\ S_i &= S_i^{(0)}(R) + o(1),\\ \boldsymbol{v}_i &= o(1/\epsilon),\end{aligned}$$

where $S_i^{(0)}(R)$ satisfies (22) to (24) and c is a constant determined by matching to the outer solution.

Outer region. In the outer region, away from the disclination core, we cannot expect the leading-order solution to retain cylindrical symmetry yet we require it to match to the inner region. We thus expand in the outer region as follows:

$$\psi_o = \psi_o^{(0)} + O(\epsilon^2),$$
$$S_o = 1 + O(\epsilon^2),$$
$$\boldsymbol{v}_o = o(1/(\epsilon^2\tilde{\varepsilon}\tilde{\beta}_2)).$$

Substituting this expansion into the governing equations (11) to (15) yields the leading-order outer equation

$$\nabla^2\psi_o^{(0)} = 0. \tag{26}$$

Matching to the inner solution using Van Dyke's matching principle (**19**) gives a boundary condition on $\psi_o^{(0)}$

$$\psi_o^{(0)} \sim c + \tfrac{1}{2}N\theta \qquad \text{as } \varsigma \to 0. \tag{27}$$

Given boundary conditions on the edge of the liquid-crystal domain and the positions and strength of any other disclination we can solve exactly for $\psi_o^{(0)}$. This will enable us to match back to the inner solution at next order and hence determine the disclination velocity. However, instead of solving for $\psi_o^{(0)}$ in some particular domain we seek to relate the next few terms in its expansion for small ς,

$$\psi_o^{(0)} = c + \tfrac{1}{2}N\theta + d_1\varsigma\cos\theta + d_2\varsigma\sin\theta + \cdots, \tag{28}$$

to the disclination velocity. We will thus arrive at a law of motion, relating the expansion of $\psi_o^{(0)}$ about the disclination to its velocity, which can be used to investigate the motion of disclinations in arbitary two-dimensional geometries. In section 4 we illustrate its use by considering three examples: (I) the interaction of two disclinations in an infinite medium; (II) the interaction of a disclination with a parallel planar wall on which the natural boundary condition $\partial\psi_o^{(0)}/\partial n = 0$ is imposed and (III) the interaction of a disclination with a parallel planar wall on which the 'strong anchoring' boundary condition $\psi_o^{(0)} = \text{const.}$ is imposed.

In order to simplify the ensuing calculation we now set

$$d_1 = 0, \tag{29}$$

by making a judicious choice of the unit base vectors $\boldsymbol{e_1}$ and $\boldsymbol{e_2}$.

Inner expansion. Matching the asymptotic behaviour of $\psi_o^{(0)}$ for small ς to ψ_i it becomes clear how to proceed with the inner expansion:

$$S_i = S_i^{(0)}(R) + \epsilon S_i^{(1)} + \cdots,$$
$$\psi_i = c + \tfrac{1}{2}N\theta + \epsilon N\psi_i^{(1)} + \cdots,$$
$$\boldsymbol{v}_i = |N|\boldsymbol{v}_i^{(0)} + \cdots,$$
$$p_i = \frac{1}{\epsilon}p_i^{(0)} + \cdots.$$

Substituting this expansion into the governing equations (11) to (15) and refering to the definition of the inner coordinate system (25) we find the following relations for $S_i^{(1)}$ and $\psi_i^{(1)}$:

$$\Lambda^2\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial S_i^{(1)}}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2 S_i^{(1)}}{\partial\theta^2}\right)-\tilde{\varepsilon}\tilde{\beta}_2\left(\boldsymbol{v}_i^{(0)}\cdot\boldsymbol{e_R}\right)S_i^{(0)\prime}=$$
$$\left(3S_i^{(0)2}-2(1+b)S_i^{(0)}+b+\frac{1}{4R^2}\right)S_i^{(1)}+\frac{S_i^{(0)}}{R^2}\frac{\partial\psi_i^{(1)}}{\partial\theta}, \tag{30}$$

$$S_i^{(0)2}\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi_i^{(1)}}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2\psi_i^{(1)}}{\partial\theta^2}\right)+\frac{S_i^{(0)}}{R^2}\frac{\partial S_i^{(1)}}{\partial\theta}+2S_i^{(0)}S_i^{(0)\prime}\frac{\partial\psi_i^{(1)}}{\partial R}=0, \tag{31}$$

together with a relation for the leading-order velocity $\boldsymbol{v}_i^{(0)}$

$$\nabla_i^4\mu_i^{(0)}=\tilde{\beta}_2\frac{S_i^{(0)\prime 2}}{R^2}\frac{\partial^2\mu_i^{(0)}}{\partial\theta^2} \tag{32}$$

in terms of the stream function $\mu_i^{(0)}$ defined by

$$\boldsymbol{v}_i^{(0)}=\boldsymbol{e_R}\frac{1}{R}\frac{\partial\mu_i^{(0)}}{\partial\theta}-\boldsymbol{e_\theta}\frac{\partial\mu_i^{(0)}}{\partial R}. \tag{33}$$

Here we have made the reasonable assumption that $\mathrm{Re}=o(1/\epsilon)$ which allows us to neglect the inertial terms in the fluid-momentum equation at leading order. In order to match ψ_i to the asymptotic behaviour of the outer solution (28), (29) we require $\psi_i^{(1)}\sim d_2R\sin\theta$ as $R\to\infty$. With this motivation we look for a solution to (30) to (33) of the form

$$\psi_i^{(1)}=m(R)\sin\theta,$$
$$S_i^{(1)}=a(R)\cos\theta,$$
$$\mu_i^{(0)}=g(R)\sin\theta.$$

The system of equations (30) to (33) then reduces to

$$\Lambda^2\left(a''+\frac{a'}{R}-\frac{a}{R^2}\right)-\tilde{\varepsilon}\tilde{\beta}_2\frac{S_i^{(0)\prime}g}{R}=$$
$$a\left(3S_i^{(0)2}-2(1+b)S_i^{(0)}+b+\frac{1}{4R^2}\right)+\frac{S_i^{(0)}m}{R^2}, \tag{34}$$

$$S_i^{(0)2}\left(m''+\frac{m'}{R}-\frac{m}{R^2}\right)-\frac{S_i^{(0)}a}{R^2}+2S_i^{(0)}S_i^{(0)\prime}m'=0, \tag{35}$$

$$\frac{1}{R^2}\frac{d}{dR}\left(R^2g'''-3g'+\frac{3g}{R}\right)=-\tilde{\beta}_2\frac{(S_i^{(0)\prime})^2g}{R^2}. \tag{36}$$

We now proceed to look for a solution to (36). In the limit $R\to 0$, $S_i^{(0)}$ and g have the following asymptotic behaviour:

$$\left.\begin{array}{c} S_i^{(0)} \sim k_s R^{1/(2\Lambda)} \\ g \sim G_1(R^{-1} + O(R^{-1+1/\Lambda})) + G_2(R\log R + O(R^{1+1/\Lambda}\log R)) \\ +G_3(R + O(R^{1+1/\Lambda})) + G_4(R^3 + O(R^{3+1/\Lambda})) \end{array}\right\} \text{as } R \to 0, \tag{37}$$

and, if the fluid velocity $\boldsymbol{v}_i$ is to remain finite at the origin, we require $G_1 = G_2 = 0$. For large R the asymptotic behaviour is given by

$$\left.\begin{array}{c} S_i^{(0)} \sim 1 - \dfrac{1}{4(1-b)R^2} + \cdots \\ g \sim \Gamma_1(R^{-1} + O(R^{-5})) + \Gamma_2(R\log R + O(R^{-3}\log R)) \\ +\Gamma_3(R + O(R^{-3})) + \Gamma_4(R^3 + O(R^{-1})) \end{array}\right\} \text{as } R \to \infty. \tag{38}$$

In order to match to the outer expansion we must set $\Gamma_4 = 0$. It now remains to determine one more of the the unspecified constants G_3, G_4, Γ_1, Γ_2 and Γ_3 to complete the boundary-value problem for g. We do this by obtaining a solvability condition on the system (34) to (36). Taking the derivative with respect to R of (22), the equation for $S_i^{(0)}$ gives

$$\Lambda^2\left(S_i^{(0)'''} + \frac{S_i^{(0)''}}{R} - \frac{S_i^{(0)'}}{R^2}\right) = S_i^{(0)'}\left(3S_i^{(0)2} - 2(1+b)S_i^{(0)} + b + \frac{1}{4R^2}\right) - \frac{S_i^{(0)}}{2R^3}. \tag{39}$$

Multiplying this by aR and subtracting the result from (34) multiplied by $RS_i^{(0)'}$ results in a relation which may, after some manipulations which make use of (35) and (36), be written as an exact differential. Integrating this between $R = \delta$ and $R = \lambda$ we find

$$\Lambda^2\left[R\left(a'S_i^{(0)'} - aS_i^{(0)''}\right)\right]_\delta^\lambda + \tilde{\varepsilon}\left[R^2 g''' - 3g' + \frac{3g}{R}\right]_\delta^\lambda = \frac{1}{2}\left[\frac{mS_i^{(0)2}}{R} + S_i^{(0)2}m'\right]_\delta^\lambda. \tag{40}$$

The idea now is to match $\psi_i^{(1)}$ and $S_i^{(1)}$ to the outer solution, thereby obtaining the asymptotic behaviour of a and m as $R \to \infty$, and then find the solvability condition on (34) to (36) by taking the limits $\lambda \to \infty$ and $\delta \to 0$ in (40). In other words we use the Fredholm alternative to show that a and m have solutions satisfying appropriate boundary data if and only if g has a certain asymptotic behaviour at infinity. In order to do this we must first establish the possible asymptotic behaviours of (35), (36) as $R \to 0$ and $R \to \infty$. In the former limit we find

$$a \sim \frac{k_s}{\Lambda}\left[c_0\left(R^{1+1/(2\Lambda)} + \cdots\right) + c_1\left(R^{-1-1/(2\Lambda)} + \cdots\right) + c_2\left(R^{1-1/(2\Lambda)} + \cdots\right)\right.$$
$$\left. + c_3\left(R^{1/(2\Lambda)-1} + \cdots\right) + \frac{\tilde{\varepsilon}\tilde{\beta}_2 G_3}{4\Lambda}\left(\frac{\log(R)\, R^{1+1/(2\Lambda)}}{1+2\Lambda} + \cdots\right)\right] \quad \text{as } R \to 0,$$
$$m \sim \left[c_0\,(R + \cdots) + c_1\left(R^{-1-1/\Lambda} + \cdots\right) - c_2\left(R^{1-1/\Lambda} + \cdots\right)\right.$$
$$\left. - c_3\left(R^{-1} + \cdots\right) + \frac{\tilde{\varepsilon}\tilde{\beta}_2 G_3}{4\Lambda}\left(\frac{R\,(\log R - 1 - 2\Lambda)}{1+2\Lambda}\right)\right] \quad \text{as } R \to 0,$$

whilst in the latter

$$a \sim \left[e_1 \left(R^{-1/2} \exp\left(\frac{(1-b)^{1/2}}{\Lambda} R \right) + \cdots \right) - e_2 \left(\frac{1}{(1-b)R} + \cdots \right) - e_3 \left(\frac{1}{R^3(1-b)} + \cdots \right) \right.$$

$$\left. + e_4 \left(R^{-1/2} \exp\left(-\frac{(1-b)^{1/2}}{\Lambda} R \right) + \cdots \right) \right] - \left(\frac{\tilde{\varepsilon}\tilde{\beta}_2 \Gamma_1}{2(1-b)^2} \frac{\log R}{R^3} + \cdots \right) \quad \text{as } R \to \infty,$$

$$m \sim \left[e_1 \left(\frac{\Lambda^2 R^{-5/2}}{(1-b)} \exp\left(\frac{(1-b)^{1/2}}{\Lambda} R \right) + \cdots \right) + e_2 (R + \cdots) + e_3 \left(\frac{1}{R} + \cdots \right) \right.$$

$$\left. + e_4 \left(\frac{\Lambda^2 R^{-5/2}}{(1-b)} \exp\left(-\frac{(1-b)^{1/2}}{\Lambda} R \right) + \cdots \right) \right] - \left(\frac{\tilde{\varepsilon}\tilde{\beta}_2 \Gamma_1}{18(1-b)^2} \frac{\log R}{R^3} + \cdots \right) \quad \text{as } R \to \infty.$$

Matching to the leading-order outer solution for $\psi_o^{(0)}$ (whose expansion close to the inner region is given in (28), (29)) we find that

$$e_1 = 0 \qquad \text{and} \qquad e_2 = d_2 \operatorname{sgn}(N),$$

where d_2 is an order-one constant determined by solving for $\psi_o^{(0)}$ in the outer region. Consider now the expansions for a and m as $R \to 0$. It is clear that unless $c_1 = c_3 = 0$ the expansions of S_i and ψ_i will break down close to the origin. For similar reasons, where $\Lambda < 1$ we also require

$$c_2 = 0. \tag{41}$$

As remarked upon in section 2.2, unless $\Lambda < \frac{1}{2}$ the energy density develops a singularity at the origin. Ruling out this singular behaviour thus enforces the boundary condition (41). Even where one does not rule out this possibility the only sensible choice for the remaining boundary condition on (34), (35) is that the solution pair (a, m) has the lowest singularity possible at the origin, which is equivalent to requiring (41) be satisfied. Finally, by taking the limits $\delta \to 0$, $\lambda \to \infty$ in (40), we can show that solutions of (34), (35) for a and m exist if and only if the remaining boundary condition on (36) is

$$\Gamma_2 = -\frac{d_2 \operatorname{sgn}(N)}{4\tilde{\varepsilon}}. \tag{42}$$

We now have sufficient boundary data to find a unique solution for g where $\tilde{\beta}_2 < 0$ (see Appendix B for a proof of the existence of such a solution). Examples of such solutions are given in Figs 2 and 3 (the boundary condition $\Gamma_2 = -1$ is used and the constant $\tilde{\Gamma}$ which is evaulated from these calculations is the corresponding value of Γ_3).

The result given in equation (42) also enables us to determine the velocity scaling U; our assumption that $\mathbf{v}_i^{(0)}$ is order one necessitates that $\tilde{\varepsilon}$ be also order one. A suitable choice for U is therefore

$$U = \frac{KS^{*2}}{4\mu L} \Rightarrow \tilde{\varepsilon} = \frac{1}{4}.$$

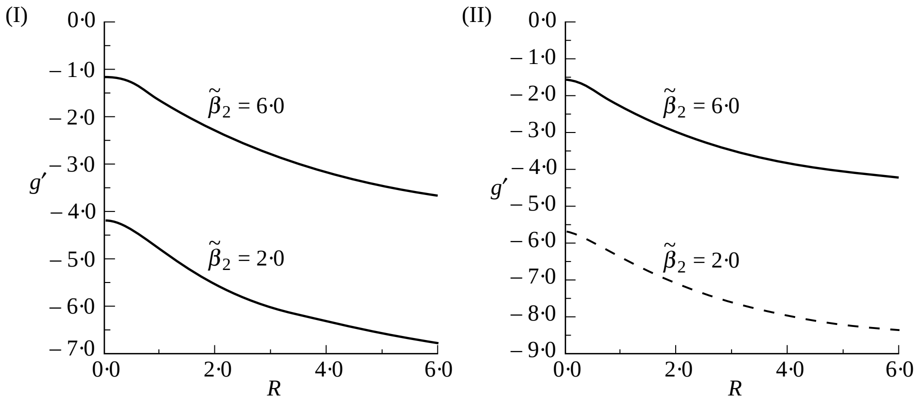

Fig. 2 This plot shows the results of a numerical calculation for g' where we impose the boundary condition $g \sim \text{const.}R$ as $R \to 0$ and $g \sim -R \log R$ as $R \to \infty$. After solving we take a further term in the large R expansion $g \sim -R \log R + \tilde{\Gamma} R$ and evaluate $\tilde{\Gamma}$. In plot (I) $b = 0{\cdot}3$, $\Lambda = 0{\cdot}35$; with $\tilde{\beta_2} = 6{\cdot}0$, $\tilde{\Gamma} = -0{\cdot}90$; with $\tilde{\beta_2} = 2{\cdot}0$, $\tilde{\Gamma} = -4{\cdot}23$. In plot (II) $b = 0{\cdot}3$, $\Lambda = 0{\cdot}5$; with $\tilde{\beta_2} = 6{\cdot}0$, $\tilde{\Gamma} = -1{\cdot}66$; with $\tilde{\beta_2} = 2{\cdot}0$, $\tilde{\Gamma} = -5{\cdot}55$

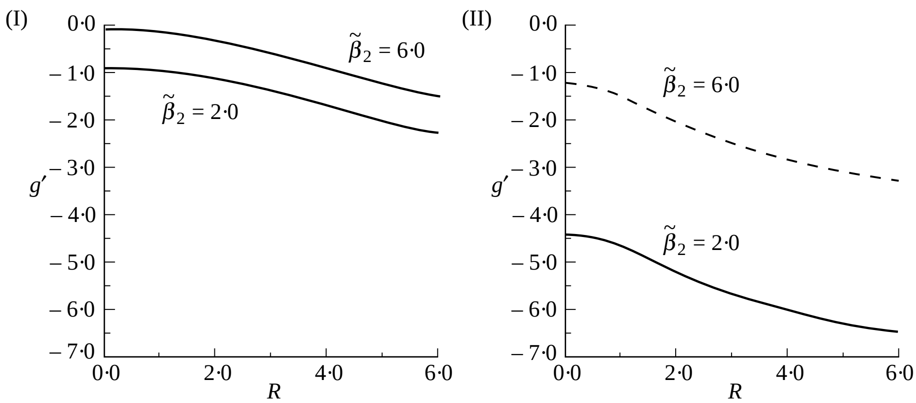

Fig. 3 This plot shows the results of a numerical calculation for g' with the same boundary conditions as above. In plot (I) $b = 0{\cdot}5$, $\Lambda = 0{\cdot}25$; with $\tilde{\beta_2} = 6{\cdot}0$, $\tilde{\Gamma} = 1{\cdot}22$; with $\tilde{\beta_2} = 2{\cdot}0$, $\tilde{\Gamma} = 0{\cdot}43$. In plot (II) $b = 0{\cdot}5$, $\Lambda = 0{\cdot}5$; with $\tilde{\beta_2} = 6{\cdot}0$, $\tilde{\Gamma} = -0{\cdot}52$; with $\tilde{\beta_2} = 2{\cdot}0$, $\tilde{\Gamma} = -3{\cdot}70$

3.1 *Matching the inner velocity to the far field*

It remains to match the velocity field in the inner region to the outer and hence to determine the velocity of the disclination $\dot{\boldsymbol{q}}$. This is complicated by the fact that the fluid velocity does not decay over the inner region but has instead a logarithmic singularity as $R \to \infty$,

$$\boldsymbol{v}_i^{(0)} \sim -N d_2 \left((\log R - \tilde{\Gamma}) \boldsymbol{e_1} - \sin\theta \boldsymbol{e_\theta} \right) \qquad \text{as } R \to \infty. \tag{43}$$

Here d_2 is the magnitude of the gradient of the regular part of the phase of the director field at the disclination, $\mathbf{e_1}$ is the vector orthogonal to this (see (28) and (29)), and $\tilde{\Gamma} = \Gamma_3/(\text{sgn}(N) d_2)$.

The parameter $\tilde{\Gamma}$ is thus independent of N and d_2 and may be obtained directly by solving (36) with boundary conditions $g \sim \text{const.}R$ as $R \to 0$ and $g \sim -R \log R$ as $R \to \infty$; $\tilde{\Gamma}$ is then obtained by taking a further term in the large R expansion of g

$$g \sim -R \log R + \tilde{\Gamma} R + \cdots \qquad \text{as } R \to 0. \tag{44}$$

We expect this parameter to be $O(1)$ (see Figs 2 and 3) except where $\tilde{\beta_2} \ll 1$ and therefore lie close to the eigenvalue $\tilde{\beta_2} = 0$. In such cases we can use an asymptotic method to calculate an approximate value for $\tilde{\Gamma}$ (see Appendix C):

$$\tilde{\Gamma} = -\frac{4}{\tilde{\beta_2} \int_0^\infty R S_i^{(0)'^2} dR} + O(1), \qquad \tilde{\beta_2} \ll 1.$$

If one now tries to match the leading-order inner-velocity field (43) to an intermediate region in which inertial terms in the fluid-velocity equation are negligible one finds that the velocity necessarily grows unboundedly at infinity; this is the Stokes paradox. In the case of a rigid body moving in an infinite viscous medium, the Stokes paradox is resolved by taking the scale of the far-field region (in our case the intermediate region) large enough so that inertial terms appear in the far-field equations (see Oseen (**20**) and Proudman and Pearson (**21**)). As far as this work is concerned, however, an intermediate region is necessary (case I) only if inertial terms become comparable with viscous terms on a lengthscale much smaller than the outer (the latter is on the lengthscale of inter-disclination and disclination–wall separations). There are two other possible scenarios.

Case II: this occurs when inertial terms are insignificant on the outer lengthscale, so that we must match the singularity in the inner solution onto the Stokes equations, which we then solve in the bounded outer region.

Case III: this ocurs when inertial terms are comparable with viscous terms on the outer lengthscale and the outer velocity problem is fully nonlinear.

In order to determine which of these possibilities occurs we must find the lengthscale $\bar{L}$ at which the corresponding Reynolds number $\bar{\mathrm{Re}}$ becomes comparable with one. In dimensional terms the fluid velocity $\boldsymbol{v}$ generated by the disclination at a distance $\bar{L}$ from its core is of order

$$|\boldsymbol{v}| = O\left(\frac{KS^{*2}}{4\mu L} \log\left(\frac{1}{\epsilon}\right)\right) \qquad \text{for } \epsilon \ll \frac{\bar{L}}{L} \ll \frac{1}{\epsilon}.$$

The Reynolds number is thus of order

$$\bar{\mathrm{Re}} = O\left(\rho \frac{KS^{*2}}{4\mu^2} \left(\frac{\bar{L}}{L}\right) \log\left(\frac{1}{\epsilon}\right)\right),$$

from which we find that inertial terms become significant on a lengthscale much less than the outer (and we must match to an intermediate Oseen region) if

$$\rho \frac{KS^{*2}}{4\mu^2} \log\left(\frac{1}{\epsilon}\right) \gg 1, \qquad \text{case I,} \tag{45}$$

or are insignificant in the outer region if

$$\rho \frac{KS^{*2}}{4\mu^2} \log\left(\frac{1}{\epsilon}\right) \ll 1, \qquad \text{case II.} \tag{46}$$

In practice we only expect to observe case II; using the typical figures given in (17) and taking the lengthscale $L = 10^{-4}$ cm to 1 cm gives $\mathrm{Re} = 4{\cdot}6 \times 10^{-6}$ to $5{\cdot}5 \times 10^{-5}$. We shall, therefore, only treat case II here, relegating discussion of case I to Appendix A.

3.2 *Disclinations with interacting velocity fields* (*case* II)

We now treat the scenario in which inertial terms are insignificant on the outer lengthscale as determined by typical disclination–disclination and disclination–wall separations. In such a regime it is natural to match the inner velocity straight on to an equation for the outer velocity. For convenience, however, we choose to write down the outer coordinates moving with the disclination, distinguishing the outer variables written in the moving frame with a hat:

$$\begin{aligned} \mathrm{Re}(\hat{\boldsymbol{v}}_o \cdot \nabla)\hat{\boldsymbol{v}}_o &= -\nabla \hat{p}_o + \nabla^2 \hat{\boldsymbol{v}}_o - \tilde{\beta}_2 \nabla \hat{S}_o \frac{d\hat{S}_o}{dt}, \\ \nabla \cdot \hat{\boldsymbol{v}}_o &= 0, \\ \hat{\boldsymbol{v}}_o &= -\dot{\boldsymbol{q}} \qquad \text{on } \partial\Omega(t), \end{aligned}$$

where

$$\mathrm{Re} = \frac{\rho K S^{*2}}{4\mu^2} \ll 1$$

and $\partial\Omega(t)$ is the boundary of the domain in the moving coordinates. Matching to the inner region we see that expansions for $\hat{\boldsymbol{v}}_o$, $\hat{S}_o$, $\hat{p}_o$ and $\dot{\boldsymbol{q}}$ proceed as follows:

$$\begin{aligned} \hat{\boldsymbol{v}}_o &= \log(1/|\epsilon N|)\hat{\boldsymbol{v}}_o^{(0)} + \hat{\boldsymbol{v}}_o^{(1)} + \cdots, \\ \hat{S}_o &= 1 + O(\epsilon^2), \\ \hat{p}_o &= \log(1/|\epsilon N|)\hat{p}_o^{(0)} + \hat{p}_o^{(1)} + \cdots, \\ \dot{\boldsymbol{q}} &= \log(1/|\epsilon N|)\dot{\boldsymbol{q}}^{(\mathbf{0})} + \dot{\boldsymbol{q}}^{(\mathbf{1})} + \cdots, \end{aligned}$$

such that the leading-order fluid velocity satisfies the Stokes equations for viscous flow

$$\begin{aligned} -\nabla \hat{p}_o^{(0)} + \nabla^2 \hat{\boldsymbol{v}}_o^{(0)} &= 0, \\ \nabla \cdot \hat{\boldsymbol{v}}_o^{(0)} &= 0, \end{aligned}$$

together with the boundary conditions

$$\begin{aligned} \hat{\boldsymbol{v}}_o^{(0)} &\sim -N d_2 \boldsymbol{e}_1, \qquad \varsigma \to 0, \\ \hat{\boldsymbol{v}}_o^{(0)} &= -\dot{\boldsymbol{q}}^{(\mathbf{0})} \qquad \text{on } \partial\Omega(t). \end{aligned}$$

This has the unique solution

$$\hat{\boldsymbol{v}}_o^{(0)} = -N d_2 \boldsymbol{e}_1 \quad \Longrightarrow \quad \dot{\boldsymbol{q}}^{(\mathbf{0})} = N d_2 \boldsymbol{e}_1.$$

At next order we again find Stokes equations, which we write in terms of a stream function $\hat{\mu}_o^{(1)}$,

$$\nabla^4 \hat{\mu}_o^{(1)} = 0,$$
$$\hat{\boldsymbol{v}}_o^{(1)} = \frac{1}{\varsigma}\frac{\partial \hat{\mu}_o^{(1)}}{\partial \theta}\boldsymbol{e}_\varsigma - \frac{\partial \hat{\mu}_o^{(1)}}{\partial \varsigma}\boldsymbol{e}_\theta,$$

but here the boundary conditions are non-trivial:

$$\hat{\mu}_o^{(1)} \sim C - Nd_2\varsigma \log\varsigma \sin\theta + Nd_2\tilde{\Gamma}\varsigma \sin\theta \qquad \text{as } \varsigma \to 0,$$
$$\frac{1}{\varsigma}\frac{\partial \hat{\mu}_o^{(1)}}{\partial \theta}\boldsymbol{e}_\varsigma - \frac{\partial \hat{\mu}_o^{(1)}}{\partial \varsigma}\boldsymbol{e}_\theta = -\dot{\boldsymbol{q}}^{(1)} \qquad \text{on } \partial\Omega(t),$$

where C is an unknown constant. We now rewrite the above in terms of coordinates that are stationary with respect to the boundary $\partial\Omega$:

$$\nabla^4 \mu_o^{(1)} = 0, \tag{47}$$
$$\mu_o^{(1)} \sim C - Nd_2\varsigma \log\varsigma \sin\theta + (Nd_2\tilde{\Gamma} + (\dot{\boldsymbol{q}}^{(1)} \cdot \boldsymbol{e}_1))\varsigma \sin\theta - (\dot{\boldsymbol{q}}^{(1)} \cdot \boldsymbol{e}_2)\varsigma \cos\theta \qquad \text{as } \varsigma \to 0, \tag{48}$$
$$\left.\begin{array}{r} \mu_o^{(1)} = 0 \\ \partial\mu_o^{(1)}/\partial n = 0 \end{array}\right\} \text{ on } \partial\Omega, \tag{49}$$

where $\mu_o^{(1)}$ is the stream function for the first-order velocity written in the stationary frame, namely $\hat{\boldsymbol{v}}_o^{(1)} + \dot{\boldsymbol{q}}^{(1)}$. We may rewrite (47) together with the singular part of (48) as

$$\nabla^4 \mu_o^{(1)} = -4\pi N\, d_2(\boldsymbol{e_2} \cdot \nabla)(\delta(x - q_1)\delta(y - q_2)), \tag{50}$$

which can then be solved with boundary condition (49) to obtain the unknown terms in (48), namely C, $(Nd_2\tilde{\Gamma} + (\dot{\boldsymbol{q}}^{(1)} \cdot \boldsymbol{e}_1))$ and $(\dot{\boldsymbol{q}}^{(1)} \cdot \boldsymbol{e}_2)$.

REMARK. Even where $\tilde{\beta}_2 \ll 1$ and $\tilde{\Gamma}$ is thus $O(1/\tilde{\beta}_2)$ the matching procedure described above gives the correct disclination velocity provided we evaluate it up to and including the '$O(1)$ terms' (which now contain a term of $O(1/\tilde{\beta}_2)$ arising from $\tilde{\Gamma}$).

3.2.1 *Summary.* Consider a situation where there are a number of disclinations inside the liquid-crystal domain with strengths N_j and at positions $\boldsymbol{q_j}$. In order to find the velocities of the disclinations we must first solve

$$\nabla^2 \psi_o^{(0)} = 0, \tag{51}$$
$$\nabla\psi_o^{(0)} \sim \frac{N_j}{2}\left(\frac{(x - q_{1,j})\boldsymbol{e_y} - (y - q_{2,j})\boldsymbol{e_x}}{(x - q_{1,j})^2 + (y - q_{2,j})^2}\right) \qquad \text{as } |\boldsymbol{x} - \boldsymbol{q_j}| \to 0, \tag{52}$$

with appropriate boundary conditions on $\partial\Omega$, to find the phase of the director. We then determine the regular part of the gradient of the phase at the jth disclination, $d_{2,j}\boldsymbol{e_{2,j}}$, by expanding the solution for ${\psi_o}^{(0)}$ about the disclination

$$\nabla\psi_o^{(0)} \sim \frac{N_j}{2}\left(\frac{(x - q_{1,j})\boldsymbol{e_y} - (y - q_{2,j})\boldsymbol{e_x}}{(x - q_{1,j})^2 + (y - q_{2,j})^2}\right) + d_{2,j}\boldsymbol{e_{2,j}} + \cdots \qquad \text{as } |\boldsymbol{x} - \boldsymbol{q_j}| \to 0. \tag{53}$$

We must now distinguish between case I, in which the criterion (45) is satisfied, and case II, in which the criterion (46) is satisfied.

Case I: in this scenario the result obtained in (53) is sufficient to obtain both the leading-order and the first-order terms in the velocity; we find

$$\dot{\boldsymbol{q}}_{\boldsymbol{j}} = N_j\, d_{2,j} \left(\lambda(\epsilon, N_j) + \log \left| \frac{4}{N_j\, d_{2,j}} \right| + 1 - \gamma - \tilde{\Gamma} \right) (\boldsymbol{e}_{\boldsymbol{2,j}} \wedge \boldsymbol{e}_{\boldsymbol{z}}) + o\,(1)\,. \tag{54}$$

Here $\lambda(\epsilon, N_j)$ is defined by the transcendental equation

$$\lambda = \log \left(\frac{1}{\epsilon \mathrm{Re} |N_j|} \right) - \log \lambda.$$

Case II: here we can use the result obtained in (53) to find the leading-order velocity:

$$\dot{\boldsymbol{q}}_{\boldsymbol{j}} = N_j\, d_{2,j} \log(1/|\epsilon N_j|) \boldsymbol{e}_{\boldsymbol{2,j}} \wedge \boldsymbol{e}_{\boldsymbol{z}} + O(1). \tag{55}$$

The next term is found by solving an auxiliary problem for the stream function $\mu_o^{(1)}$ of the fluid velocity

$$\nabla^4 \mu_o^{(1)} = -4\pi \sum_j N_j\, d_{2,j} (\boldsymbol{e}_{\boldsymbol{2,j}} \cdot \nabla)(\delta(x - q_{1,j})\delta(y - q_{2,j})), \tag{56}$$

$$\left. \begin{array}{r} \mu_o^{(1)} = 0 \\ \partial \mu_o^{(1)} / \partial n = 0 \end{array} \right\} \text{ on } \partial\Omega, \tag{57}$$

and expanding the solution about each disclination core

$$\begin{aligned} \mu_o^{(0)} \sim{} & -N_j\, d_{2,j} \log \left(\left((x - q_{1,j})^2 + (y - q_{2,j})^2 \right)^{1/2} \right) (\boldsymbol{e}_{\boldsymbol{2,j}} \cdot \nabla) \left(\frac{(x - q_{1,j})^2 + (y - q_{2,j})^2}{2} \right) \\ & + C + (\boldsymbol{P} \cdot \nabla) \left(\frac{(x - q_{1,j})^2 + (y - q_{2,j})^2}{2} \right) + O((x - q_{1,j})^2 + (y - q_{2,j})^2). \end{aligned} \tag{58}$$

The two-term dimensionless velocity law for the jth disclination follows from this expansion and that made in (53); it is

$$\begin{aligned} \dot{\boldsymbol{q}}_{\boldsymbol{j}} ={} & N_j\, d_{2,j} \log(1/|\epsilon N_j|) \boldsymbol{e}_{\boldsymbol{2,j}} \wedge \boldsymbol{e}_{\boldsymbol{z}} \\ & - \left(\boldsymbol{P} \cdot (\boldsymbol{e}_{\boldsymbol{2,j}} \wedge \boldsymbol{e}_{\boldsymbol{z}}) \right) \boldsymbol{e}_{\boldsymbol{2,j}} + \left(\boldsymbol{P} \cdot \boldsymbol{e}_{\boldsymbol{2,j}} - N_j d_{2,j} \tilde{\Gamma} \right) \boldsymbol{e}_{\boldsymbol{2}} \wedge \boldsymbol{e}_{\boldsymbol{z}}) + O(\epsilon \log \epsilon). \end{aligned} \tag{59}$$

4. Examples of disclination motion

In this section we apply the law of motion obtained in the previous section to some particular examples. For brevity we restrict our discussion to the physically relevant scenario in which the Reynolds number is small (that is, case II).

4.1 *Disclination–disclination interaction*

Consider two disclinations of strengths N_1 and N_2 lying in an infinite medium at positions $\boldsymbol{x} = (Q_1, 0)$ and $\boldsymbol{x} = (Q_2, 0)$ respectively. Solving for $\psi_o^{(0)}$ from (51) and (52) we find

$$\nabla\psi_o^{(0)} = \frac{N_1}{2}\frac{(x - Q_1)\boldsymbol{e_y} - y\boldsymbol{e_x}}{(x - Q_1)^2 + y^2} + \frac{N_2}{2}\frac{(x - Q_2)\boldsymbol{e_y} - y\boldsymbol{e_x}}{(x - Q_2)^2 + y^2}. \tag{60}$$

By comparing our solution with equation (53) we calculate the regular parts of the gradient of the phase, $d_{2,1}\mathbf{e}_{2,1}$ and $d_{2,2}\mathbf{e}_{2,2}$, on the disclinations at $\boldsymbol{x} = (Q_1, 0)$ and $\boldsymbol{x} = (Q_2, 0)$ respectively. Substituting these results into the law of motion (55) we obtain the velocity of the disclination at leading order

$$\dot{Q}_1 = \frac{N_1 N_2}{2(Q_1 - Q_2)}\log\left|\frac{1}{\epsilon N_1}\right| + O(1), \qquad \dot{Q}_2 = -\frac{N_1 N_2}{2(Q_1 - Q_2)}\log\left|\frac{1}{\epsilon N_2}\right| + O(1).$$

If we require the next term in the velocity we must solve the auxiliary problem for the fluid velocity (57)

$$\nabla^4\mu_o^{(1)} = \frac{N_1 N_2}{2(Q_1 - Q_2)}\left(\delta(x - Q_2)\delta'(y) - \delta(x - Q_1)\delta'(y)\right).$$

Requiring that the velocity $\nabla \wedge (\mu_o^{(1)}\boldsymbol{e_z})$ be bounded at infinity allows us to determine a unique solution for $\mu_o^{(1)}$, namely

$$\mu_o^{(1)} = \frac{N_1 N_2}{2(Q_1 - Q_2)}\left(y\log\left(\left(y^2 + (x - Q_2)^2\right)^{1/2}\right) - y\log\left(\left(y^2 + (x - Q_1)^2\right)^{1/2}\right)\right);$$

from this and (60) we obtain the disclination velocities

$$\begin{aligned}
\dot{Q}_1 &= \frac{N_1 N_2}{2(Q_1 - Q_2)}\left(\log\left|\frac{Q_1 - Q_2}{\epsilon N_1}\right| - \tilde{\Gamma}\right) + O(\epsilon\log\epsilon),\\
\dot{Q}_2 &= -\frac{N_1 N_2}{2(Q_1 - Q_2)}\left(\log\left|\frac{Q_1 - Q_2}{\epsilon N_2}\right| - \tilde{\Gamma}\right) + O(\epsilon\log\epsilon).
\end{aligned}$$

REMARK. Thus same-sign disclinations repel each other, moving along their line of centres, whilst opposite-sign disclinations are attracted to each other along their line of centres.

4.2 *Disclination–wall interaction*

Consider a situation in which a disclination at $\boldsymbol{x} = (q_1, 0, 0)$ lies parallel to an infinite planar wall stretching along the (y, z)-plane.

Natural boundary conditions. Firstly we shall investigate the motion of the disclination where the natural boundary condition $\partial\psi_o^{(0)}/\partial x = 0$ on $x = 0$ is imposed on the wall. Solving (51) and (52) for $\psi_o^{(0)}$ we find

$$\nabla\psi_o^{(0)} = \frac{N}{2}\left(\frac{(x - q_1)\boldsymbol{e_y} - y\boldsymbol{e_x}}{(x - q_1)^2 + y^2} - \frac{(x + q_1)\boldsymbol{e_y} - y\boldsymbol{e_x}}{(x + q_1)^2 + y^2}\right).$$

We evaluate the regular part of the gradient of the phase on the disclination, which we find to be $d_2\boldsymbol{e}_{\boldsymbol{2},\boldsymbol{j}} = -N\boldsymbol{e}_{\boldsymbol{y}}/(4q_1)$, and hence deduce the velocity at leading order:

$$\dot{q}_1 = -\frac{N^2}{4q_1}\log\left(\frac{1}{\epsilon|N|}\right) + \cdots.$$

To obtain the first-order correction to the velocity we must first solve for the stream function of the fluid flow which satisfies

$$\nabla^4\mu_o^{(1)} = \frac{\pi N^2}{q_1}\delta(x-q_1)\delta'(y),$$
$$\left.\begin{array}{r}\mu_o^{(1)} = 0\\ \partial\mu_o^{(1)}/\partial x = 0\end{array}\right\} \text{ on } x = 0.$$

The solution to this problem is

$$\mu_o^{(1)} = \frac{N^2}{4q_1}\left(y\log\left(\left((x-q_1)^2+y^2\right)^{1/2}\right) - y\log\left(\left((x+q_1)^2+y^2\right)^{1/2}\right)\right)$$
$$+\frac{N^2}{2\pi}\int_{-\infty}^{\infty}\frac{s}{q_1^2+s^2}\left(\frac{x^2}{(x^2+(y-s)^2)}\right)ds.$$

Expanding the stream function about the disclination, comparing the result with (58) and then referring to (59) we find the disclination velocity, correct to first order,

$$\dot{q}_1 = -\frac{N^2}{4q_1}\left(\log\left(\frac{2q_1}{\epsilon|N|}\right) - \tilde{\Gamma}\right) + \frac{N^2}{\pi}\int_{-\infty}^{\infty}\frac{s^2q_1^2}{(q_1^2+s^2)^3}ds + O(\epsilon\log\epsilon).$$

The phase of the director constant on the boundary. We now investigate the motion of the disclination where the boundary condition

$$\psi_o^{(0)} = \text{const.} \qquad \text{on } x = 0,$$

is imposed on the wall. With this boundary condition we find that the gradient of $\psi_o^{(0)}$ satisfies

$$\nabla\psi_o^{(0)} = \frac{N}{2}\left(\frac{(x-q_1)\boldsymbol{e}_{\boldsymbol{y}} - y\boldsymbol{e}_{\boldsymbol{x}}}{(x-q_1)^2+y^2} + \frac{(x+q_1)\boldsymbol{e}_{\boldsymbol{y}} - y\boldsymbol{e}_{\boldsymbol{x}}}{(x+q_1)^2+y^2}\right).$$

Evaluating the regular part of the gradient of the phase on the disclination we find $d_2\boldsymbol{e}_2 = N\boldsymbol{e}_{\boldsymbol{y}}/(4q_1)$. It follows that the leading-order velocity is equal and opposite to that in the previous example (natural boundary conditions imposed on the wall). It is easy to show that the stream function for the fluid flow is also the same, except for a sign change, as for the previous example. The disclination velocity, correct to first order, is thus

$$\dot{q}_1 = \frac{N^2}{4q_1}\left(\log\left(\frac{2q_1}{\epsilon|N|}\right) - \tilde{\Gamma}\right) - \frac{N^2}{\pi}\int_{-\infty}^{\infty}\frac{s^2q_1}{(q_1^2+s^2)^3}ds + O(\epsilon\log\epsilon).$$

Summary. We find that a disclination is attracted, along the shortest path, towards a planar boundary on which the natural boundary condition $\partial\psi_o^{(0)}/\partial n = 0$ holds, whereas, if the angle of the director $\psi_o^{(0)}$ is specified equal to some constant along the boundary, the disclination is repelled from the boundary along a path perpendicular to the boundary.

5. Conclusion

We have derived a law of motion for a line disclination in a nematic liquid crystal from an asymptotic analysis of the Ericksen model. We found that the disclination velocity $\dot{\boldsymbol{q}}$ is determined by a local expansion of the phase of the director field about the disclination. Where $\psi_o^{(0)}$ has the expansion (with θ defined in the standard way about the point $(x, y) = (q_1, q_2)$)

$$\psi_o^{(0)} \sim \frac{N\theta}{2} + c + d_2\,((y - q_2)\cos\nu - (x - q_1)\sin\nu) + \cdots \qquad \text{as } (x, y) \to (q_1, q_2),$$

such that the local behaviour of the director field as $(x, y) \to (q_1, q_2)$ is

$$\begin{aligned}\boldsymbol{n} \sim &\left(\cos\left(\tfrac{1}{2}N\theta + c\right)\boldsymbol{e_x} + \sin\left(\tfrac{1}{2}N\theta + c\right)\boldsymbol{e_y}\right)\\ &+ d_2\,(y\cos\nu - x\sin\nu)\left(\cos\left(\tfrac{1}{2}N\theta + c\right)\boldsymbol{e_y} - \sin\left(\tfrac{1}{2}N\theta + c\right)\boldsymbol{e_x}\right) + \cdots,\end{aligned}$$

the leading-order dimensional disclination velocity $\dot{\boldsymbol{q}}$ is

$$\dot{\boldsymbol{q}} = d_2\frac{NKS^{*2}}{4\mu L}\log(1/\epsilon)\left(\cos\nu\boldsymbol{e_x} + \sin\nu\boldsymbol{e_y}\right) + O\left(\frac{NKS^{*2}d_2}{4\mu L}\right). \tag{61}$$

Remarkably the leading-order disclination velocity depends only on the structure of the core through the dimensionless parameter ϵ which gives a measure of its radius in comparison to a typical lengthscale of the problem. Moreover the other parameters which appear in the velocity law, KS^{*2} (the elastic constant) and μ (the viscosity), are parameters occurring in the standard Leslie–Ericksen model in which it is assumed that $S = \text{const.} = S^*$.

In cases where the Reynolds number of the fluid flow generated by the moving disclination is either very large or very small (experimental data suggests that it is almost always very small) we were able to calculate the fluid flow field and, from this, the next term in the expansion of the disclination velocity. We note that it is possible to generalize our analysis to situations in which there is a fluid flow in the liquid crystal, resulting from sources other than the motion of the disclination itself, and that in such circumstances the disclination is transported with a velocity equal to the sum of the velocities arising from the regular part of the gradient of the director field and of the flow field on the disclination line.

Finally we draw the reader's attention to some areas of future work arising from the work in this paper. The present work uses almost the simplest continuum model with which it is possible to describe the motion of a disclination. It is therefore of interest to examine the effects of the terms (describing the action of the director field on the fluid flow and vice versa) which have been neglected in our treatment of Ericksen's model. These terms lead to anisotropic flow properties and, in order to examine their effects on the disclination velocity, it is necessary to solve a linear partial differential equation for the leading-order fluid velocity field in the inner region. Again the velocity field will grow like $\log R$ for large values of R (here R is the distance from the centre of the disclination on the inner lengthscale). The direction and magnitude of the logarithmic term can be obtained by looking for a solvability condition on the inner equations at first order. In essence, therefore, it is necessary only to find two 'basis' solutions for the leading-order inner velocity field (using some numerical method) and then derive a solvability condition to find the appropriate coefficients of the basis solutions. We also think that it is possible to extend our analysis to look at the effects that biaxiality may have on the motion.

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APPENDIX A

Matching the inner solution to an intermediate solution and determination of the disclination velocity in case I

Consider now the scenario where the Oseen lengthscale is much smaller than the outer lengthscale, determined by typical disclination–disclination and disclination–boundary separations (the regime given by (45)). In line with Proudman and Pearson (**21**), who were the first to satisfactorily resolve the Stokes paradox, we introduce an intermediate Oseen region that moves with the disclination and has polar length coordinate $\hat{r}$; this is defined by the scaling

$$\varsigma = \frac{\bar{L}}{L}\hat{r} = \epsilon|N|R, \quad \text{where} \quad \frac{\bar{L}}{L} = \frac{4\mu^2}{\rho K S^{*2}\lambda} = \frac{1}{\mathrm{Re}\lambda(\epsilon, N)}, \tag{A1}$$

and $\lambda(\epsilon, N)$ is an, as yet undetermined, large parameter. In this region (15), (16) become

$$\frac{1}{\lambda(\epsilon, N)}\frac{d\boldsymbol{v}_{\mathrm{int}}}{dt} = -\left(\frac{\bar{L}}{L}\right)\nabla p_{\mathrm{int}} + \nabla^2\boldsymbol{v}_{\mathrm{int}} - \tilde{\beta}_2\frac{dS_{\mathrm{int}}}{dt}\nabla S_{\mathrm{int}}, \tag{A2}$$

$$\nabla\cdot\boldsymbol{v}_{\mathrm{int}} = 0, \tag{A3}$$

$$\boldsymbol{v}_{\mathrm{int}} \sim -\dot{\boldsymbol{q}} \qquad \text{as } \hat{r}\to\infty, \tag{A4}$$

where the subscript int denotes an intermediate variable. Matching to the inner region at leading order we see that

$$\boldsymbol{v}_{\mathrm{int}} \sim -d_2 N\log\left(\frac{\bar{L}}{L\epsilon|N|}\right)\boldsymbol{e_1} - d_2 N\left(\left(\log\hat{r} - \tilde{\Gamma}\right)\boldsymbol{e_1} - \sin\theta\boldsymbol{e_\theta}\right) \qquad \text{as } \hat{r}\to 0.$$

The expansions for $\boldsymbol{v}_{\mathrm{int}}$, S_{int}, p_{int} and $\dot{\boldsymbol{q}}$ thus proceed as follows:

$$\boldsymbol{v}_{\mathrm{int}} = \lambda(\epsilon, N)\boldsymbol{v}^{(0)}_{\mathrm{int}} + \boldsymbol{v}^{(1)}_{\mathrm{int}} + \cdots,$$

$$S_{\mathrm{int}} = 1 + O\left(\left(\frac{L\epsilon}{\bar{L}}\right)^2\right),$$

$$p_{\mathrm{int}} = \frac{L}{\bar{L}}\left(\lambda(\epsilon, N)p^{(0)}_{\mathrm{int}} + p^{(1)}_{\mathrm{int}} + \cdots\right),$$

$$\dot{\boldsymbol{q}} = \lambda(\epsilon, N)\dot{\boldsymbol{q}}^{(\mathbf{0})} + \dot{\boldsymbol{q}}^{(\mathbf{1})} + \cdots,$$

where the unknown parameter λ and the unknown lengthscale $\bar{L}$ are determined by matching the velocity to the inner solution and by the rescaling carried out in (A1). This leads to a transcendental equation for λ:

$$\lambda = \log\left(\frac{1}{\epsilon\mathrm{Re}|N|}\right) - \log\lambda.$$

Substituting the expansion of the intermediate variables into the governing equations (A2) to (A4) and matching to the inner solution gives the following problem for the leading-order velocity:

$$\left(\boldsymbol{v}^{(0)}_{\mathrm{int}}\cdot\nabla\right)\boldsymbol{v}^{(0)}_{\mathrm{int}} = -\nabla p^{(0)}_{\mathrm{int}} + \nabla^2\boldsymbol{v}^{(0)}_{\mathrm{int}},$$

$$\nabla\cdot\boldsymbol{v}^{(0)}_{\mathrm{int}} = 0,$$

$$\boldsymbol{v}^{(0)}_{\mathrm{int}} \sim -N\,d_2\boldsymbol{e_1} \qquad \text{as } \hat{r}\to 0,$$

$$\boldsymbol{v}^{(0)}_{\mathrm{int}} \sim -\dot{\boldsymbol{q}}^{(\mathbf{0})} \qquad \text{as } \hat{r}\to\infty.$$

This system has uniform solution

$$\boldsymbol{v}^{(0)}_{\mathrm{int}} = -N\,d_2\boldsymbol{e_1}, \quad p^{(0)}_{\mathrm{int}} = 0 \quad\Longrightarrow\quad \dot{\boldsymbol{q}}^{(\mathbf{0})} = N\,d_2\boldsymbol{e_1}.$$

At first order we obtain a non-trivial problem for the velocity, which we formulate in terms of $\mu^{(1)}_{\mathrm{int}}$, the stream function for $\boldsymbol{v}^{(1)}_{\mathrm{int}}$, as follows:

$$\left(\nabla^2 + Nd_2(\boldsymbol{e_1}\cdot\nabla)\right)\nabla^2\mu^{(1)}_{\mathrm{int}} = 0, \tag{A5}$$

$$\mu^{(1)}_{\mathrm{int}} \sim -N\,d_2(\hat{r}\log\hat{r} - \tilde{\Gamma})\sin\theta \qquad \text{as } \hat{r}\to 0, \tag{A6}$$

$$\frac{1}{\hat{r}}\frac{\partial\mu^{(1)}_{\mathrm{int}}}{\partial\theta}\boldsymbol{e_{\hat{r}}} - \frac{\partial\mu^{(1)}_{\mathrm{int}}}{\partial\hat{r}}\boldsymbol{e_\theta} \sim -\dot{\boldsymbol{q}}^{(\mathbf{1})} \qquad \text{as } \hat{r}\to\infty. \tag{A7}$$

By imposing the boundedness of $\partial\mu^{(1)}_{\text{int}}/\partial\hat{r}$ at infinity we can find a unique solution of (A5), (A6). Referring to (**21**, **22**) we see that this solution is

$$\mu^{(1)}_{\text{int}} = -Nd_2 \sum_{n=1}^{\infty} \phi_n \left(\frac{|Nd_2|\hat{r}}{2}\right) \frac{\hat{r}\sin(n(\theta+\pi))}{n} + m\hat{r}\sin\theta \qquad \text{for } Nd_2 > 0, \tag{A8}$$

$$\mu^{(1)}_{\text{int}} = Nd_2 \sum_{n=1}^{\infty} \phi_n \left(\frac{|Nd_2|\hat{r}}{2}\right) \frac{\hat{r}\sin(n\theta)}{n} + m\hat{r}\sin\theta \qquad \text{for } Nd_2 < 0, \tag{A9}$$

where

$$\phi_n(\cdot) = 2K_1(\cdot)I_n(\cdot) + K_0(\cdot)(I_{n+1}(\cdot) + I_{n-1}(\cdot)),$$

and I_n and K_n are modified Bessel functions of the first and second orders respectively. In order that the matching to (A6) be complete we require $m = d_2N(\gamma - 1 - \log(4/(|Nd_2|)) + \tilde{\Gamma})$, where γ is Euler's constant. By considering the asymptotic behaviour of (A8) in the limit $\hat{r} \to \infty$ and the boundary condition (A7) we may obtain the contribution to the disclination velocity at this order:

$$\dot{\boldsymbol{q}}^{(\mathbf{1})} = N\,d_2\left(1 + \log\left(\frac{4}{|Nd_2|}\right) - \gamma - \tilde{\Gamma}\right)\boldsymbol{e}_{\mathbf{1}}.$$

APPENDIX B

The existence of a solution to the inner-fluid-velocity problem

We write equation (36) in the following way:

$$\frac{d}{dR}\left(\frac{1}{R}\frac{d}{dR}(R\Omega(R))\right) = -\tilde{\beta_2}\eta(R)g(R), \tag{B1}$$

$$\frac{d}{dR}\left(\frac{1}{R}\frac{d}{dR}(Rg(R))\right) = \Omega(R), \tag{B2}$$

where $\eta(R)$ is the positive function $\eta = (S_i^{(0)'})^2/R^2$. Multiplying (B1) by Rg and integrating by parts twice we find the following relation:

$$\left[g\frac{d}{dR}(R\Omega) - \Omega\frac{d}{dR}(Rg)\right]_\delta^\lambda + \int_\delta^\lambda R\Omega^2 + \tilde{\beta_2}\eta Rg^2 dR = 0.$$

Taking the limits $\delta \to 0$ and $\lambda \to \infty$ and referring to the asymptotic behaviour of g given in (37) and (38) we see that, where $\tilde{\beta_2} > 0$, no non-trivial solution exists to the homogeneous problem given by equation (B1) and the boundary conditions $G_1 = G_2 = \Gamma_2 = \Gamma_4 = 0$.

The rest of the proof is standard. We write g in terms of basis functions

$$g(R) = Ag_1(r) + Bg_2(R) + Cg_3(R) + Dg_4(R),$$

where the g_i are solutions to (B1) with the asymptotic behaviours

$$g_1 \sim R^{-1}, \quad g_2 \sim R\log R, \quad g_3 \sim R, \quad g_4 \sim R^3,$$

as $R \to 0$. Thus where g satisfies the boundary conditions $G_1 = G_2 = 0$ we find that $A = B = 0$. As $R \to \infty$ we make the following series expansions of g_3 and g_4:

$$g_3 = \alpha_1(R^{-1} + O(R^{-5})) + \alpha_2(R\log R + O(R^{-3}\log R)) + \alpha_3(R + O(R^{-3})) + \alpha_4(R^3 + O(R^{-1})),$$
$$g_4 = \beta_1(R^{-1} + O(R^{-5})) + \beta_2(R\log R + O(R^{-3}\log R)) + \beta_3(R + O(R^{-3})) + \beta_4(R^3 + O(R^{-1})).$$

If no non-trivial solution exists to (B1) with the homogeneous boundary conditions $G_1 = G_2 = \Gamma_2 = \Gamma_4 = 0$ it follows that $\alpha_4\beta_2 - \alpha_2\beta_4 \neq 0$. Hence a unique solution exists to (B1) with the non-homogeneous boundary conditions $G_1 = G_2 = \Gamma_4 = 0$, $\Gamma_2 \neq 0$; it is $g = \Gamma_2(g_3\beta_4 - g_4\alpha_4)/(\alpha_2\beta_4 - \alpha_4\beta_2)$.

APPENDIX C

Calculation of $\tilde{\Gamma}$ *for* $\tilde{\beta_2}$ *small*

In order to obtain $\tilde{\Gamma}$ we must solve equation (36) together with the boundary conditions

$$g \sim \text{const.}R \quad \text{as } R \to 0, \qquad g \sim -R\log R \quad \text{as } R \to \infty. \tag{C1}$$

Then $\tilde{\Gamma}$ is obtained by taking a further term in the large R expansion of g (see equation (44)). However, no solution to this problem can be found where $\tilde{\beta_2} = 0$ since a solution exists to the homogeneous problem given by equation (36) and the boundary conditions $g \sim \text{const.}R$ as $R \to 0$ and $g \sim \text{const.}R$ as $R \to \infty$; namely $g = \text{const.}R$. Where $\tilde{\beta_2} \ll 1$ we can calculate an approximate solution to the problem by writing (36) in the form (B1) and (B2) and expanding g and Ω in powers of $\tilde{\beta_2}$:

$$g = \frac{g_0}{\tilde{\beta_2}} + g_1 + \cdots, \qquad \Omega = \frac{\Omega_0}{\tilde{\beta_2}} + \Omega_1 + \cdots.$$

Substituting this expansion into (B1), (B2) and (C1) and taking the leading term we find

$$\frac{d}{dR}\left(\frac{1}{R}\frac{d}{dR}(R\Omega_0)\right) = 0, \qquad \frac{d}{dR}\left(\frac{1}{R}\frac{d}{dR}(Rg_0)\right) = \Omega_0, \tag{C2}$$

with boundary condition $g_0 \sim \text{const.}R$ as $R \to 0$ and $g_0 \sim \text{const.}R$ as $R \to \infty$. This has solution

$$g_0 = AR, \qquad \Omega_0 = 0. \tag{C3}$$

In order to determine the coefficient A we proceed to next order in the expansion where we find

$$\frac{d}{dR}\left(\frac{1}{R}\frac{d}{dR}(R\Omega_1)\right) = -\frac{(S_i^{(0)'})^2}{R^2}g_0, \qquad \frac{d}{dR}\left(\frac{1}{R}\frac{d}{dR}(Rg_1)\right) = \Omega_1, \tag{C4}$$

with boundary conditions

$$g_1 \sim \text{const.}R \quad \text{as } R \to 0, \qquad g_1 \sim R\log R \quad \text{as } R \to \infty. \tag{C5}$$

Multiplying $(\text{C4})_1$ by Rg_0 and subtracting $(\text{C2})_1$ multiplied by Rg_1 and then integrating by parts twice results in the following relation:

$$\left[g_0\frac{d}{dR}(R\Omega_1) - g_1\frac{d}{dR}(R\Omega_0) - \Omega_1\frac{d}{dR}(Rg_0) + \Omega_0\frac{d}{dR}(Rg_1)\right]_\delta^\lambda = -\int_\delta^\lambda \frac{(S_i^{(0)'})^2}{R}g_0^2\,dR.$$

We then substitute the result obtained for g_0, take the limits $\delta \to 0$ and $\lambda \to \infty$, and apply the boundary condition (C5) to find a condition on A; this is

$$A = -\frac{4}{\int_0^\infty R(S_i^{(0)'})^2 dR}.$$

It follows that

$$\tilde{\Gamma} = -\frac{4}{\tilde{\beta_2}\int_0^\infty R(S_i^{(0)'})^2 dR} + O(1) \qquad \text{for } \tilde{\beta_2} \ll 1.$$