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G. Richardson and J. Rubinstein

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A one-dimensional model for superconductivity in a thin wire of slowly varying cross-section

By G. Richardson and J. Rubinstein

Faculty of Mathematics, Technion, Haifa 32000, Israel

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Using formal asymptotics, a one-dimensional Ginzburg–Landau model describing superconductivity in a thin wire of arbitrary shape and slowly varying cross-section is derived. The model is valid for all magnetic fields and for temperatures T, such that the thickness of the wire is much less than the coherence length $\xi(T)$. The model is used to calculate the normal–superconducting transition curves for closed wire loops of different cross-sections, as functions of temperature and the magnetic flux cutting the loop. This shows a periodic dependence on flux, superimposed on a parabolic background.

Keywords: superconductivity; Ginzburg–Landau model; thin wire

1. Introduction

When cooled below a certain critical temperature, so-called superconducting materials undergo a phase transition that takes them from a normal state, in which they behave like conventional metals, to a superconducting state, in which they can support electric currents without resistance. In this state, they also exhibit macroscopic behaviour typically associated with quantum mechanical systems. In this context, we note (i) the *Josephson effect*, whereby two pieces of superconducting material separated by a non-superconductor interact via quantum mechanical tunnelling; and (ii) the quantization of magnetic flux, exemplified by a circulating current structure, termed a *vortex*, which is associated with one quantum of magnetic flux. The important role that the magnetic flux quantum plays in superconductivity was first demonstrated experimentally by Little & Parks (1962). They found that the behaviour of a thin cylindrical superconducting ring in an axial applied magnetic field depends crucially on whether it encloses an integer or non-integer number of magnetic flux quanta. In the latter case, the superconducting state is penalized (particularly when the ring encloses an integer-plus-half number of flux quanta) and the critical temperature for the onset of superconductivity is lowered.

Recently there has been considerable interest in a closely related set-up consisting of a thin loop of wire set in an applied magnetic field (see, for example, Fomin *et al*. 1997; Moshchalkov *et al*. 1995). More exotic experiments along similar lines have been conducted on networks of connected wire loops (see Bruyndoncx *et al*. 1996) with the goal of eventually using such networks as electronic devices.

In this work, we set out to derive a one-dimensional model, based on the Ginzburg–Landau equations (see Ginzburg & Landau 1950), which is capable of describing the behaviour of a thin-wire loop (or network of loops) of arbitrary smooth shape and with slowly varying cross-section in magnetic fields ranging in magnitude from zero to sizes great enough to destroy superconductivity in the wire. One-dimensional models of this kind (in the limit as the cross-section tends to zero) have been derived for a variety of problems in classical mechanics (see, for example, Ciarlet 1990), and even in quantum mechanics and superconductivity (see Pannetier 1991; Chapman *et al*. 1996; Rubinstein & Schatzman 1997). The one-dimensional models for a thin superconducting film and wire derived, respectively, in Rubinstein & Schatzman (1997) and Pannetier (1991), are based on the assumption that the applied magnetic field is small, in the sense that the flux threading through the surface bounded by the wire/film is an $O(1)$ multiple of the fundamental flux quantum. In this case, it is found that the supercurrents induced by the magnetic field do not appreciably depress the level of superconductivity (as measured by the modulus of the Ginzburg–Landau order parameter). Using such models, it has recently been shown by Berger & Rubinstein (1995, 1998) and Richardson (1998) that the order parameter exhibits unusual behaviour whenever the cross-section of the wire is not exactly uniform. For example, when the cross-section of the loop *is* exactly uniform, the only solutions have constant modulus of the order parameter. For non-uniform cross-sections, it has been shown that there is some range of temperatures where the order parameter has a non-trivial point zero when the flux (measured in units of the fundamental flux quantum) is exactly an integer-plus-half. Horane *et al*. (1996) have proposed an ad hoc one-dimensional model that includes a term that takes account of the effect of large magnetic fields on the order parameter. They use their model to argue that the non-trivial zero set of the order parameter occurs in a region of the magnetic flux–temperature phase space; this is in contrast to the prediction of the high-field model derived in this paper, which is that the zero set occurs along a line segment in the phase space.

In the next section, we introduce the Ginzburg–Landau equations and non-dimensionalize them appropriately. In § 3, the steady-state model is derived. This is the main result of the paper and is given in equations (3.31) and (3.32). It may be applied to any network of thin wires (of whatever topology). However, the difficulty of applying this model directly to non-simply connected wire geometries motivates us to consider the reformulation of the model in terms of a complex order parameter. In § 4, we illustrate this reformulation by considering a closed wire loop (topologically equivalent to a torus); the resulting model is given in equations (4.5)–(4.7). In § 5, we show how the model may be generalized to include time dependence. The model is then used, in § 6, to calculate the position of the normal–superconducting transition, as a function of temperature and magnetic field, for uniform loops of different cross-sections. This curve displays a periodic dependence on the magnetic flux through the loop superimposed on a parabolic background, which is precisely the sort of behaviour observed in experiment. The two specific examples considered in detail illustrate the dependence of the transition curve on the shape of the wire; roughly speaking, the greater the area of wire presented perpendicular to the magnetic field, the lower the critical field required to destroy superconductivity in the wire. Finally, in § 7, we present our conclusions.

2. The Ginzburg–Landau model

We consider a loop of thin superconducting wire, whose centreline has length $2\pi l$, held at temperature T. Our starting point is the Ginzburg–Landau equations (see

Ginzburg & Landau 1950), which model the behaviour inside the superconductor (in V)

$$(1/m_s)(\hbar\nabla - \mathrm{i}e_s\boldsymbol{A})^2\psi - a(T)\psi - b(T)|\psi|^2\psi = 0, \tag{2.1}$$

$$\nabla \wedge \left(\frac{\boldsymbol{B}}{\mu}\right) = -\frac{\mathrm{i}e_s\hbar}{m_s}(\psi^*\nabla\psi - \psi\nabla\psi^*) - \frac{2e_s^2}{m_s}|\psi|^2\boldsymbol{A}, \tag{2.2}$$

$$\boldsymbol{B} = \nabla \wedge \boldsymbol{A}. \tag{2.3}$$

Here, ψ is a complex order parameter defined such that the number density of superconducting charge carriers is proportional to $|\psi|^2$; $\boldsymbol{A}$ is the magnetic vector potential; $\boldsymbol{B}$ is the magnetic field; and e_s and m_s are, respectively, the charge and the effective mass of the superconducting charge carriers. The ratio of the function $a(T)$ to the positive function $b(T)$ determines the magnitude of the order parameter as a function of temperature. In particular, superconductivity is only possible if $a(T) < 0$. The Ginzburg–Landau equations couple to Maxwell's equations

$$\nabla \wedge \boldsymbol{B} = \boldsymbol{j}_{\mathrm{driv}}, \qquad \nabla \cdot \boldsymbol{B} = 0,$$
$$\boldsymbol{B} \to \boldsymbol{0}, \quad \text{as } |\boldsymbol{x}| \to \infty,$$

in the exterior domain V^{c} via the following jump and boundary conditions on ∂V:

$$\boldsymbol{N} \cdot (\hbar\nabla - \mathrm{i}e_s\boldsymbol{A})\psi = 0, \qquad [\boldsymbol{B} \cdot \boldsymbol{N}] = 0, \qquad [(1/\mu)\boldsymbol{B} \wedge \boldsymbol{N}] = 0. \tag{2.4}$$

Here, $\boldsymbol{j}_{\mathrm{driv}}$ is a current density used to model the device producing the magnetic field (for example a solenoid), $\boldsymbol{N}$ is the outward normal to V, and μ is the magnetic permeability. In de Gennes (1966), it is proposed that where the superconductor lies adjacent to a non-insulator, the natural boundary condition on ψ $(2.4)_1$ should be replaced by

$$\boldsymbol{N} \cdot (\hbar\nabla - \mathrm{i}e_s\boldsymbol{A})\psi = -\hbar\psi/d, \tag{2.5}$$

where d ranges from 0 to ∞ depending on the properties of the adjacent material.

It is natural to choose a scaling for the magnetic field that corresponds to the ring enclosing of the order of one quantum of magnetic flux; this motivates the following isothermal non-dimensionalization:

$$\boldsymbol{B} = \frac{\hbar}{e_s l^2}\boldsymbol{B}', \qquad \boldsymbol{x} = l\boldsymbol{x}', \qquad \boldsymbol{A} = \frac{\hbar}{e_s l}\boldsymbol{A}', \qquad \psi = \psi' \qquad \boldsymbol{j}_{\mathrm{driv}} = \frac{\hbar}{e_s l^3}\boldsymbol{j}'_{\mathrm{driv}}.$$

On dropping the primes, equations (2.1)–(2.3) and jump and boundary conditions $(2.4)_2$, $(2.4)_3$ and (2.5) become

$$(\nabla - \mathrm{i}\boldsymbol{A})^2\psi = \Gamma(\mathcal{D}|\psi|^2 - 1)\psi, \tag{2.6}$$

$$\nabla \wedge (\nabla \wedge \boldsymbol{A}) = -(\mathcal{D}\Gamma/\kappa^2)(|\psi|^2\boldsymbol{A} + \tfrac{1}{2}\mathrm{i}(\psi^*\nabla\psi - \psi\nabla\psi^*)), \tag{2.7}$$

$$\boldsymbol{B} = \nabla \wedge \boldsymbol{A}, \tag{2.8}$$

$$[\boldsymbol{B} \cdot \boldsymbol{N}]_{\partial V} = 0, \qquad [(1/\mu)\boldsymbol{B} \wedge \boldsymbol{N}]_{\partial V} = 0, \tag{2.9}$$

$$\boldsymbol{N} \cdot (\nabla - \mathrm{i}\boldsymbol{A})\psi\,|_{\partial V} = -\beta\psi\,|_{\partial V}, \tag{2.10}$$

where

$$\Gamma = -\frac{l^2 m_s a(T)}{\hbar^2}, \qquad \kappa = \frac{m_s}{e_s\hbar}\sqrt{\frac{b(T)}{2\mu}}, \qquad \mathcal{D} = -\frac{b(T)}{a(T)}, \qquad \beta = \frac{l}{d}.$$

Here, the dimensionless parameter κ is the Ginzburg–Landau parameter, $1/\mathcal{D}$ is the square of the equilibrium density of superconducting electrons, β gives the ratio of the chosen length-scale to the de Gennes distance, and Γ is related to the coherence length $\xi(T)$ by the following:

$$\Gamma = l^2/\xi^2.$$

Close to T_{c}, the critical temperature, below which the superconducting state is energetically favourable in the absence of magnetic field, Γ can be approximated by $\Gamma = k(T_{\mathrm{c}} - T)$, where k is a positive constant.

It is possible to reduce the number of dependent variables in (2.6)–(2.8) by the introduction of the gauge invariant variables

$$\psi = f\mathrm{e}^{\mathrm{i}\chi}, \qquad \boldsymbol{Q} = \boldsymbol{A} - \nabla\chi. \tag{2.11}$$

This leads to the following non-dimensional system of equations in $\boldsymbol{V}$:

$$\nabla^2 f = \Gamma(\mathcal{D}f^3 - f) + f|\boldsymbol{Q}|^2, \tag{2.12}$$

$$\nabla \wedge \boldsymbol{B} = -(\mathcal{D}\Gamma/\kappa^2)(f^2\boldsymbol{Q}), \tag{2.13}$$

$$\boldsymbol{B} = \nabla \wedge \boldsymbol{Q}, \tag{2.14}$$

which couple to Maxwell's equations

$$\left.\begin{aligned} \nabla \wedge \boldsymbol{B} = \boldsymbol{j}_{\mathrm{driv}}, \qquad \nabla \cdot \boldsymbol{B} = 0,\\ \boldsymbol{B} \to \boldsymbol{0}, \quad \text{as } |\boldsymbol{x}| \to \infty, \end{aligned}\right\} \tag{2.15}$$

in the exterior domain V^{c} via the jump conditions (2.9) and the boundary conditions

$$\boldsymbol{Q} \cdot \boldsymbol{N}\,|_{\partial\boldsymbol{V}} = 0, \qquad \left.\frac{\partial f}{\partial N}\right|_{\partial\boldsymbol{V}} = -\beta f. \tag{2.16}$$

3. Derivation of the model

We consider a wire of small aspect ratio, such that the typical thickness of the wire, in non-dimesional terms, is ϵ, where $\epsilon \ll 1$. Then, in order to apply equations (2.12)–(2.15) together with the jump and boundary conditions (2.9) and (2.16) to the problem, we introduce a local coordinate system about the wire.

Consider the centreline of the wire $\boldsymbol{x} = \boldsymbol{r}(s)$ (i.e. the centre of mass of a cross-section of the wire). Any point $\boldsymbol{x}$ inside (or sufficiently close to) the wire can be represented in terms of the coordinates (X, Y, s) by the following relation:

$$\boldsymbol{x} = \boldsymbol{r}(s) + \epsilon X\boldsymbol{n}(s) + \epsilon Y\boldsymbol{b}(s), \tag{3.1}$$

where $\boldsymbol{n}$ and $\boldsymbol{b}$ are, respectively, the principal normal and the binormal to the curve $\boldsymbol{x} = \boldsymbol{r}(s)$, and s is the arclength along the centreline of the wire. The coordinate system so defined is not orthogonal, in the sense that the unit vectors

$$\boldsymbol{e}_X = \frac{\partial\boldsymbol{x}/\partial X}{|\partial\boldsymbol{x}/\partial X|}, \qquad \boldsymbol{e}_Y = \frac{\partial\boldsymbol{x}/\partial Y}{|\partial\boldsymbol{x}/\partial Y|}, \qquad \boldsymbol{e}_s = \frac{\partial\boldsymbol{x}/\partial s}{|\partial\boldsymbol{x}/\partial s|},$$

are not all mutually orthogonal themselves. However, since these unit vectors are close to being orthogonal, that is to say

$$\boldsymbol{e}_X \cdot \boldsymbol{e}_Y = 0, \qquad \boldsymbol{e}_X \cdot \boldsymbol{e}_s = -\epsilon\tau Y + O(\epsilon^2), \qquad \boldsymbol{e}_Y \cdot \boldsymbol{e}_s = \epsilon\tau X + O(\epsilon^2),$$

where τ is the torsion of the curve $\boldsymbol{x} = \boldsymbol{r}(s)$, the coordinate system itself is termed nearly orthogonal. As a consequence, we can find an asymptotic expression for the gradient of a function F in terms of these coordinates

$$\nabla F = \frac{1}{\epsilon}\left(\boldsymbol{e}_X \frac{\partial F}{\partial X} + \boldsymbol{e}_Y \frac{\partial F}{\partial Y}\right) + \boldsymbol{e}_s \frac{\partial F}{\partial s}(1 + \epsilon C X + O(\epsilon^2)). \tag{3.2}$$

Here, C is the curvature of the centreline. Similarly, we can find asymptotic expressions for the divergence and curl of the vector $\boldsymbol{P} = P_1 \boldsymbol{e}_X + P_2 \boldsymbol{e}_Y + P_3 \boldsymbol{e}_s$,

$$\nabla \cdot \boldsymbol{P} = \frac{1}{\epsilon}\left(\frac{\partial P_1}{\partial X} + \frac{\partial P_2}{\partial Y}\right) + \frac{\partial P_3}{\partial s} - C P_1 + O(\epsilon |\boldsymbol{P}|), \tag{3.3}$$

$$\begin{aligned}\nabla \wedge \boldsymbol{P} = {} & \frac{1}{\epsilon}\left(\left(\frac{\partial P_2}{\partial X} - \frac{\partial P_1}{\partial Y}\right)\boldsymbol{e}_s + \frac{\partial P_3}{\partial Y}\boldsymbol{e}_X - \frac{\partial P_3}{\partial X}\boldsymbol{e}_Y\right) \\ & + \left(\tau Y\left(\frac{\partial P_2}{\partial X} - \frac{\partial P_1}{\partial Y}\right) - \frac{\partial P_2}{\partial s}\right)\boldsymbol{e}_X + \left(\frac{\partial P_1}{\partial s} + C P_3 + \tau X\left(\frac{\partial P_1}{\partial Y} - \frac{\partial P_2}{\partial X}\right)\right)\boldsymbol{e}_Y \\ & + \left(X \frac{\partial P_3}{\partial X} + Y \frac{\partial P_3}{\partial Y}\right)\boldsymbol{e}_s + O(\epsilon |\boldsymbol{P}|).\end{aligned} \tag{3.4}$$

In the following analysis, it will prove helpful to recall that the unit vectors $\boldsymbol{e}_X$ and $\boldsymbol{e}_Y$ are directed, respectively, along the normal and the binormal to the curve $\boldsymbol{x} = \boldsymbol{r}(s)$. Finally, we write an expression for the Laplacian of F by combining (3.2) and (3.3):

$$\nabla^2 F = \frac{1}{\epsilon^2}\left(\frac{\partial^2 F}{\partial X^2} + \frac{\partial^2 F}{\partial Y^2}\right) - \frac{C}{\epsilon}\frac{\partial F}{\partial X} + \frac{\partial^2 F}{\partial s^2} + O\left(\frac{\partial F}{\partial X}, \frac{\partial F}{\partial Y}, \epsilon \frac{\partial F}{\partial s}\right). \tag{3.5}$$

Let the surface of the wire be given by $G(X, Y, s) = 0$, with the function G taking negative values in the interior V, and positive values in the exterior V^{c}. The cross-section of the wire $\Omega(s_0)$ about the point $\boldsymbol{x} = \boldsymbol{r}(s_0)$ is given by

$$\Omega(s_0) = \{(X, Y) : G(X, Y, s_0) < 0\}.$$

It follows, from this definition and the requirement that $\boldsymbol{x} = \boldsymbol{r}(s)$ is the centre of mass of the cross-section, that

$$\int_{\Omega(s)} X \,\mathrm{d}X \mathrm{d}Y = \int_{\Omega(s)} Y \,\mathrm{d}X \mathrm{d}Y = 0, \quad \forall s. \tag{3.6}$$

We are interested in the response of the wire to magnetic fields large enough to appreciably delay the onset of superconductivity as the temperature is lowered (and Γ increases). On examination of equation (2.12) and boundary condition $(2.16)_2$, one can see that $\boldsymbol{Q}$ must be $O(1)$ to significantly lower the critical temperature at which the wire becomes superconducting. We then note that $\boldsymbol{Q}$ changes in response to a magnetic field according to equation (2.14). This leads us to conclude that the regime of interest is $|\boldsymbol{B}| = O(1/\epsilon)$, since this produces an $O(1)$ change of $\boldsymbol{Q}$ across the wire. We therefore investigate the behaviour of the system in response to a large current density $\boldsymbol{j}_{\mathrm{driv}} = \tilde{\boldsymbol{j}}_{\mathrm{driv}}/\epsilon$ in the device producing the applied magnetic field, and seek an asymptotic solution to (2.12)–(2.13) of the form

$$\left.\begin{aligned}\boldsymbol{B} = (\boldsymbol{B}^{(0)}/\epsilon) + \cdots, \quad \boldsymbol{Q} = \boldsymbol{Q}^{(0)} + \cdots, \quad f = f^{(0)} + \cdots, \quad \text{in } V, \\ \boldsymbol{B} = (\boldsymbol{B}^{(0)}_{\mathrm{ext}}/\epsilon) + \cdots, \quad \text{in } V^{\mathrm{c}}.\end{aligned}\right\} \tag{3.7}$$

Throughout the calculation we will use the local coordinates defined in (3.1) together with the expansions of the Laplacian and the vector operators' gradient, divergence and curl found in (3.2)–(3.5). Substituting the solution ansatz made in (3.7) into (2.13), and noting that the magnetic field is divergence free, leads to governing equations for the leading-order magnetic field in V:

$$\frac{\partial B_2^{(0)}}{\partial X}-\frac{\partial B_1^{(0)}}{\partial Y}=0,\qquad \frac{\partial B_3^{(0)}}{\partial Y}=0,$$
$$\frac{\partial B_1^{(0)}}{\partial X}+\frac{\partial B_2^{(0)}}{\partial Y}=0,\qquad \frac{\partial B_3^{(0)}}{\partial X}=0.$$

The current carried by the wire is of $O(\epsilon^2)$ and, therefore, does not influence the leading-order magnetic field exterior to the wire in V^{c}. We thus solve for $\boldsymbol{B}_{\mathrm{ext}}^{(0)}$ in the whole of $\mathbb{R}^3$ using

$$\left.\begin{array}{c}\nabla\wedge\boldsymbol{B}_{\mathrm{ext}}^{(0)}=\tilde{\boldsymbol{j}}_{\mathrm{driv}},\qquad \nabla\cdot\boldsymbol{B}_{\mathrm{ext}}^{(0)}=0,\\ \boldsymbol{B}_{\mathrm{ext}}^{(0)}\to\boldsymbol{0},\quad \text{as } |\boldsymbol{x}|\to\infty.\end{array}\right\}\tag{3.8}$$

Applying the jump conditions $(2.9)_{1,2}$ and solving for $\boldsymbol{B}^{(0)}$, we see that the leading-order magnetic field in $\boldsymbol{V}$ is determined by the field in $\boldsymbol{V}^{\mathrm{c}}$ as follows:

$$B_1^{(0)}(s)=\boldsymbol{B}_{\mathrm{ext}}^{(0)}(\boldsymbol{r}(s))\cdot\boldsymbol{n}(s),\qquad B_2^{(0)}(s)=\boldsymbol{B}_{\mathrm{ext}}^{(0)}(\boldsymbol{r}(s))\cdot\boldsymbol{b}(s),\tag{3.9}$$

$$B_3^{(0)}(s)=(1/\mu_{\mathrm{ext}})\boldsymbol{B}_{\mathrm{ext}}^{(0)}(\boldsymbol{r}(s))\cdot\boldsymbol{t}(s),\tag{3.10}$$

where $\boldsymbol{t}(s)$ is the tangent to the curve $\boldsymbol{x}=\boldsymbol{r}(s)$.

Remark 3.1. We could equally have chosen the driving current to be $O(1)$, such that $\boldsymbol{j}=\tilde{\boldsymbol{j}}_{\mathrm{driv}}$ and $\boldsymbol{B}=\boldsymbol{B}_{\mathrm{ext}}^{(0)}+\cdots$ in V^{c}, and $\boldsymbol{B}=\boldsymbol{B}^{(0)}+\cdots$ in V, without altering the conclusion that the leading-order magnetic fields, $\boldsymbol{B}^{(0)}$ and $\boldsymbol{B}_{\mathrm{ext}}$, are given by equations (3.8)–(3.10).

Next we write the boundary conditions in (2.16) in terms of the function G by noting that the normal to ∂V is given by $\nabla G/|\nabla G|$:

$$G_Xf_X+G_Yf_Y+\epsilon^2G_sf_s+\epsilon^2\beta_0f(G_X^2+G_Y^2)^{1/2}\,|_{\partial V}=O(\epsilon^2f_X,\epsilon^2f_Y,\epsilon^3f_s),\tag{3.11}$$

$$G_XQ_1+G_YQ_2+\epsilon G_sQ_3\,|_{\partial V}=O(\epsilon^2|\boldsymbol{Q}|).\tag{3.12}$$

Here, we investigate the case $\beta=O(\epsilon)$ by writing $\beta=\epsilon\beta_0$, where β_0 is an $O(1)$ function of position along the surface of the wire. It will be shown *a posteriori* that this is the canonical scaling for β.

Leading order. We now seek to determine the leading-order behaviour of f and $\boldsymbol{Q}$. By referring to equations (2.12) and (3.11), one can see that f satisfies the following problem at leading order:

$$f_{XX}^{(0)}+f_{YY}^{(0)}=0,\quad (X,Y)\in\Omega(s),\tag{3.13}$$

$$G_Xf_X^{(0)}+G_Yf_Y^{(0)}\,|_{\partial\Omega(s)}=0.\tag{3.14}$$

This has the solution

$$f^{(0)}=f^{(0)}(s).\tag{3.15}$$

The governing equations for $\boldsymbol{Q}^{(0)}$ are obtained from equation (2.14) and the divergence of (2.13); they are

$$\left.\begin{aligned} Q^{(0)}_{1X} + Q^{(0)}_{2Y} &= 0, & Q^{(0)}_{2X} - Q^{(0)}_{1Y} &= B^{(0)}_3, \\ Q^{(0)}_{3Y} &= B^{(0)}_1, & Q^{(0)}_{3X} &= -B^{(0)}_2, \end{aligned}\right\} \tag{3.16}$$

from which it immediately follows that $Q^{(0)}_3$ is of the form

$$Q^{(0)}_3 = B^{(0)}_1(s)Y - B^{(0)}_2(s)X + q(s). \tag{3.17}$$

Boundary conditions on the other components of $\boldsymbol{Q}^{(0)}$ are determined from (3.12); they are

$$G_X Q^{(0)}_1 + G_Y Q^{(0)}_2 \,|_{\partial\Omega(s)} = 0. \tag{3.18}$$

We solve for these components by introducing the stream function $\Upsilon(X, Y, s)$, such that

$$Q^{(0)}_1 = B^{(0)}_3(s)\Upsilon_Y, \qquad Q^{(0)}_2 = -B^{(0)}_3(s)\Upsilon_X, \tag{3.19}$$

which, on substitution into $(3.16)_2$ and (3.18), gives rise to the following problem for Υ:

$$\Upsilon_{XX} + \Upsilon_{YY} = -1, \quad (X, Y) \in \Omega(s), \tag{3.20}$$

$$\Upsilon \,|_{\partial\Omega(s)} = 0. \tag{3.21}$$

In order to find a relationship between the unknown quantities $f^{(0)}(s)$ and $q(s)$, we must proceed to higher orders in the expansions of f and $\boldsymbol{Q}$:

$$\left.\begin{aligned} f &= f^{(0)}(s) + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \cdots, \\ \boldsymbol{Q} &= \boldsymbol{Q}^{(0)} + \epsilon \boldsymbol{Q}^{(1)} + \cdots. \end{aligned}\right\} \tag{3.22}$$

First order. At first order, equations (2.12) and (3.11) lead to a problem for $f^{(1)}$ identical to that satisfied by $f^{(0)}$, namely (3.13)–(3.14), from which it follows that

$$f^{(1)} = f^{(1)}(s).$$

The following equation for the first and second components of $\boldsymbol{Q}$ is obtained by taking the divergence of (2.13) and expanding in powers of ϵ:

$$\frac{\partial}{\partial s}(f^{(0)2} Q^{(0)}_3) - C f^{(0)2} Q^{(0)}_1 + f^{(0)2}\left(\frac{\partial Q^{(1)}_1}{\partial X} + \frac{\partial Q^{(1)}_2}{\partial Y}\right) = 0. \tag{3.23}$$

The corresponding boundary condition results from an expansion in ϵ of (3.12):

$$Q^{(1)}_1 G_X + Q^{(1)}_2 G_Y + Q^{(0)}_3 G_s \,|_{\partial\Omega(s)} = 0. \tag{3.24}$$

Substituting for $\boldsymbol{Q}^{(0)}$ in (3.23), using (3.17) and (3.19), then integrating over $\Omega(s)$, and applying the boundary condition (3.24), we find

$$D(s)\frac{\partial}{\partial s}(f^{(0)2} q) - C f^{(0)2} B^{(0)}_3 \int_{\partial\Omega(s)} \frac{\Upsilon G_Y}{\sqrt{G_X^2 + G_Y^2}}\,\mathrm{d}l - f^{(0)2} \int_{\partial\Omega(s)} \frac{G_s(B^{(0)}_1(s)Y - B^{(0)}_2(s)X + q(s))}{\sqrt{G_X^2 + G_Y^2}}\,\mathrm{d}l = 0, \tag{3.25}$$

where we define $D(s)$, a measure of the thickness of the wire, to be

$$D(s) = \iint_{\Omega(s)} \mathrm{d}X\mathrm{d}Y. \tag{3.26}$$

The first integral in equation (3.25) vanishes since $\Upsilon\,|_{\partial\Omega} = 0$. The remaining integral may be evaluated with the aid of the following identity:

$$\frac{\mathrm{d}}{\mathrm{d}s}\iint_{\Omega(s)} \theta(X,Y,s)\,\mathrm{d}X\mathrm{d}Y = \iint_{\Omega(s)} \frac{\partial\theta}{\partial s}\,\mathrm{d}X\mathrm{d}Y - \int_{\partial\Omega(s)} \frac{G_s\theta}{\sqrt{G_X^2+G_Y^2}}\,\mathrm{d}l, \tag{3.27}$$

which holds for any smooth function $\theta(X,Y,s)$. When we apply this result, together with (3.6), to equation (3.25), we obtain the following relation between $f^{(0)}$, and q:

$$\frac{1}{D(s)}\frac{\partial}{\partial s}(D(s)f^{(0)2}q) = 0. \tag{3.28}$$

Second order. In order to derive a further relation between these quantities, we must proceed to second order in f. At this order, (2.12) and (3.11) give the following equation for $f^{(2)}$:

$$\frac{\partial^2 f^{(2)}}{\partial X^2} + \frac{\partial^2 f^{(2)}}{\partial Y^2} + \frac{\partial^2 f^{(0)}}{\partial s^2} = \Gamma(\mathcal{D}f^{(0)^3} - f^{(0)}) + f^{(0)}|\boldsymbol{Q}^{(0)}|^2, \tag{3.29}$$

together with the boundary condition

$$G_X f_X^{(2)} + G_Y f_Y^{(2)} + G_s f_s^{(0)}\,|_{\partial\Omega(s)} = -\beta_0 f^{(0)}(G_X^2+G_Y^2)^{1/2}\,|_{\partial\Omega(s)}\,. \tag{3.30}$$

We then substitute for $\boldsymbol{Q}^{(0)}$ in (3.29) using (3.17) and (3.19); integrate over $\Omega(s)$; apply the boundary condition (3.30); and make use of the result (3.27) to obtain the following relation between $f^{(0)}$ and q:

$$\begin{aligned}\frac{1}{D(s)}\frac{\partial}{\partial s}\left(D(s)\frac{\partial f^{(0)}}{\partial s}\right) - \Gamma(\mathcal{D}f^{(0)^3} - f^{(0)})& \\ = f^{(0)}q^2 + f^{(0)}(B_1^{(0)2}M_{22}(s) &+ B_2^{(0)2}M_{11}(s) \\ &- 2B_1^{(0)}B_2^{(0)}M_{12}(s) + B_3^{(0)2}L(s) + P(s)).\end{aligned} \tag{3.31}$$

Here, the functions $M_{11}(s)$, $M_{22}(s)$, $M_{12}(s)$, $L(s)$ and $P(s)$ depend on the shape of the wire through the definitions

$$\left.\begin{aligned} M_{11}(s) &= \frac{1}{D(s)}\iint_{\Omega(s)} X^2\,\mathrm{d}X\mathrm{d}Y, \\ M_{12}(s) &= \frac{1}{D(s)}\iint_{\Omega(s)} XY\,\mathrm{d}X\mathrm{d}Y, \\ M_{22}(s) &= \frac{1}{D(s)}\iint_{\Omega(s)} Y^2\,\mathrm{d}X\mathrm{d}Y, \\ P(s) &= \frac{1}{D(s)}\int_{\partial\Omega(s)} \beta_0\,\mathrm{d}S, \\ L(s) &= \frac{1}{D(s)}\iint_{\Omega(s)} \left(\frac{\partial\Upsilon}{\partial X}\right)^2 + \left(\frac{\partial\Upsilon}{\partial Y}\right)^2 \mathrm{d}X\mathrm{d}Y = \frac{1}{D(s)}\iint_{\Omega(s)} \Upsilon\,\mathrm{d}X\mathrm{d}Y, \end{aligned}\right\} \tag{3.32}$$

where $\Upsilon(X,Y,s)$ is a solution to the problem given in (3.20)–(3.21) and $D(s)$, the thickness of the wire, is given by (3.26).

4. Complex formulation of the model for a closed wire loop

When considering networks of wires it is more convenient to work with a complex formulation of the model, and, in order to illustrate this formulation, we shall consider a closed wire loop. However, it is not difficult to extend this method to more complicated topologies.

We need first to relate $q(s)$ to the vector potential $\boldsymbol{A}$. Across the surface of the wire we impose the jump condition

$$[\boldsymbol{n} \wedge \boldsymbol{A}]_V = \boldsymbol{0},$$

which has the consequence of ensuring that $(2.9)_1$ is satisfied. Then, making the assumption that the minimal surface spanned by the wire loop has area of order 1 (recall that distances have been non-dimensionalized with the length of the wire), we see that the magnitude of A_3 is $O(1/\epsilon)$ in V. Next we expand $\boldsymbol{A}$ about the centreline of the wire $\boldsymbol{x} = \boldsymbol{r}(s)$

$$\boldsymbol{A} = (\boldsymbol{a}(s)/\epsilon) + X\boldsymbol{a}_{\boldsymbol{X}}(s) + Y\boldsymbol{a}_{\boldsymbol{Y}}(s) + O(\epsilon).$$

We make use of the definition of $\boldsymbol{A}$ made in (2.8) and the expansion of the curl operator found in (3.4) to show that

$$A_3 = (a_3(s)/\epsilon) + (a_1'(s) + Ca_3(s) - B_2^{(0)}(s))X + (B_1^{(0)}(s) + a_2'(s))Y + O(\epsilon).$$

We then expand $\boldsymbol{Q}$, as defined in (2.11), about the centreline of the wire

$$\begin{aligned}
Q_1 &= \frac{1}{\epsilon}\left(a_1(s) - \frac{\partial \chi}{\partial X}\right) + O(1),\\
Q_2 &= \frac{1}{\epsilon}\left(a_2(s) - \frac{\partial \chi}{\partial Y}\right) + O(1),\\
Q_3 &= \left(\frac{a_3(s)}{\epsilon} - \frac{\partial \chi}{\partial s}\right) + X\left(a_1'(s) - B_2^{(0)}(s) + C\left(a_3(s) - \epsilon\frac{\partial \chi}{\partial s}\right)\right)\\
&\qquad + Y(B_1^{(0)}(s) + a_2'(s)) + O\left(\epsilon^2\frac{\partial \chi}{\partial s}, \epsilon\right).
\end{aligned}$$

Comparing this expansion with the leading-order behaviour of $\boldsymbol{Q}$, derived in equations (3.17) and (3.19), we calculate the following expansion for $\partial\chi/\partial s$:

$$\frac{\partial \chi}{\partial s} = \frac{a_3(s)}{\epsilon} - q(s) + a_2'(s)Y + a_1'(s)X + O(\epsilon).$$

It is apparent that $q(s)$ can only be of $O(1)$ if $a_3(s)/\epsilon$ and $\partial\chi/\partial s$ differ by an $O(1)$ quantity. The regime of interest is, therefore,

$$q(s) = \frac{a_3(s)}{\epsilon} - \left.\frac{\partial \chi}{\partial s}\right|_{X=Y=0} = O(1).$$

In order that ψ remains single valued, χ must satisfy the constraint

$$\oint_{\tilde{\gamma}} \frac{\partial \chi}{\partial s}\,\mathrm{d}s = 2n\pi, \tag{4.1}$$

where n is an integer, $\tilde{\gamma}$ is the closed loop formed by the centreline of the wire. It follows that, in general, $q(s)$ cannot adopt the most energetically favoured state $q \equiv 0$, even when the system has attained equilibrium.

We consider a loop of dimensionless length 2π, and define a new $O(1)$ function $\tilde{\alpha}(s)$ by the following relation:

$$\tilde{\alpha}(s) = \frac{a_3(s)}{\epsilon} - \frac{\omega(s)}{\epsilon} = O(1).$$

Here, $\omega(s)$ is chosen such that $\tilde{\alpha}$ is of $O(1)$ and such that

$$\frac{1}{\epsilon}\int_0^{2\pi} \omega(s)\,\mathrm{d}s = 2\pi j,$$

where j is some $(O(1/\epsilon))$ integer. We then define new variables

$$\tilde{\chi}(s) = \chi\,|_{X=Y=0} - \frac{1}{\epsilon}\int_0^s \omega(s)\,\mathrm{d}s,$$
$$\tilde{\psi}(s) = f^{(0)}\exp(\mathrm{i}\tilde{\chi}(s)),$$

and note that this definition implies that $\tilde{\psi}$ is periodic on $(0, 2\pi)$, since $\partial\chi/\partial s|_{X=Y=0}$ satisfies the relation (4.1). In terms of these new variables, the model, given by equations (3.28) and (3.31), is

$$\left.\begin{aligned}\left(\frac{\partial}{\partial s} - \mathrm{i}\tilde{\alpha}(s)\right)^2\tilde{\psi} + \frac{1}{D}\frac{\partial D}{\partial s}\left(\frac{\partial\tilde{\psi}}{\partial s} - \mathrm{i}\tilde{\alpha}(s)\tilde{\psi}\right) = \mathit{\Gamma}\tilde{\psi}(\mathcal{D}|\tilde{\psi}|^2 - 1) + \tilde{\psi}\mathcal{G}(s),\\ \tilde{\psi}\ \text{periodic on}\ (0, 2\pi).\end{aligned}\right\} \tag{4.2}$$

In order to simplify this model further, we make the following gauge transformation:

$$\tilde{\psi}(s) = \hat{\psi}(s)\exp\left(\mathrm{i}\left[\int_0^s \tilde{\alpha}(s) - \hat{\alpha}\,\mathrm{d}s\right]\right), \tag{4.3}$$

where

$$\hat{\alpha} = \frac{1}{2\pi}\int_0^{2\pi}\tilde{\alpha}(s)\,\mathrm{d}s = \frac{1}{2\pi}\int_0^{2\pi}\frac{a_3(s)}{\epsilon}\,\mathrm{d}s - j, \tag{4.4}$$

and j is some integer chosen such that $\tilde{\alpha}$ and $\hat{\alpha}$ are $O(1)$. The integral of $a_3(s)$ around the ring is related to the magnetic flux cutting the ring by the expression

$$\mathcal{F} = \frac{\mathcal{F}^{(0)}}{\epsilon} = \int_0^{2\pi}\frac{a_3}{\epsilon}\,\mathrm{d}s.$$

It follows that $\hat{\alpha}$ may be expressed in terms of the flux by the relation

$$\hat{\alpha} = (\mathcal{F}^{(0)}/2\pi\epsilon) - j. \tag{4.5}$$

Substitution of (4.3) and (4.4) into (4.2) leads to the following, more tractable, model, in which the function $\tilde{\alpha}(s)$ is replaced by the constant $\hat{\alpha}$:

$$\left.\begin{aligned}\left(\frac{\partial}{\partial s} - \mathrm{i}\hat{\alpha}\right)^2\hat{\psi} + \frac{1}{D}\frac{\partial D}{\partial s}\left(\frac{\partial\hat{\psi}}{\partial s} - \mathrm{i}\hat{\alpha}\hat{\psi}\right) = \mathit{\Gamma}\hat{\psi}(\mathcal{D}|\hat{\psi}|^2 - 1) + \hat{\psi}\mathcal{G}(s),\\ \hat{\psi}\ \text{periodic on}\ (0, 2\pi).\end{aligned}\right\} \tag{4.6}$$

Here,

$$\mathcal{G}(s) = (B_1^{(0)2}M_{22}(s) + B_2^{(0)2}M_{11}(s) - 2B_1^{(0)}B_2^{(0)}M_{12}(s) + B_3^{(0)2}L(s) + P(s)), \tag{4.7}$$

and the functions M_{11}, M_{12}, M_{22}, L and P are defined in (3.32).

5. Inclusion of time dependence

It is worth generalizing the steady-state thin-wire model to include time dependence as this provides an easy way of investigating the stability of solutions to the equilibrium problem. Our starting point is the time-dependent generalization of the Ginzburg–Landau equations considered by Schmid (1966) and Gor'kov & Eliashburg (1968). These hold in V and, in non-dimensional form, are as follows:

$$\left(\frac{\partial\psi}{\partial t} + \mathrm{i}\Phi\psi\right) = (\nabla - \mathrm{i}\boldsymbol{A})^2\psi - \Gamma(\mathcal{D}|\psi|^2 - 1)\psi, \tag{5.1}$$

$$\boldsymbol{j} = \nabla\wedge(\nabla\wedge\boldsymbol{A}) = -\frac{\mathcal{D}\Gamma}{\kappa^2}(|\psi|^2\boldsymbol{A} + \tfrac{1}{2}\mathrm{i}(\psi^*\nabla\psi - \psi\nabla\psi^*)) - \sigma\left(\frac{\partial\boldsymbol{A}}{\partial t} + \nabla\Phi\right), \tag{5.2}$$

$$\boldsymbol{E} = -\left(\nabla\Phi + \frac{\partial\boldsymbol{A}}{\partial t}\right). \tag{5.3}$$

Here, $\boldsymbol{E}$ is the electric field, Φ the scalar potential, and σ the scaled normal conductivity of the material. Where the material immediately adjacent to the wire (in V^{c}) is insulating, these couple to Maxwell's equations

$$\begin{gathered}\nabla\wedge\boldsymbol{B} = 0, \qquad \nabla\cdot\boldsymbol{B} = \boldsymbol{j}_{\mathrm{driv}},\\ \boldsymbol{B}_t + \nabla\wedge\boldsymbol{E} = 0, \qquad \nabla\cdot\boldsymbol{E} = 0,\\ \boldsymbol{B}\to\boldsymbol{0}, \quad \boldsymbol{E}\to\boldsymbol{0}, \quad \text{as } |\boldsymbol{x}|\to\infty,\end{gathered}$$

via the jump and boundary conditions

$$\left.\begin{gathered}\boldsymbol{N}\cdot(\nabla - \mathrm{i}\boldsymbol{A})\psi + \beta\psi\,|_{\partial V} = 0, \qquad \boldsymbol{j}\cdot\boldsymbol{N}\,|_{\partial V} = 0,\\ [\boldsymbol{B}\cdot\boldsymbol{N}]_{\partial V} = 0, \qquad [(1/\mu)\boldsymbol{B}\wedge\boldsymbol{N}]_{\partial V} = 0, \qquad [\boldsymbol{E}\wedge\boldsymbol{N}]_{\partial V} = 0.\end{gathered}\right\} \tag{5.4}$$

Since we are primarily interested in the stability of the solutions to the steady-state model, we make the assumption that the current density in the device producing the applied magnetic field is time independent, such that

$$\boldsymbol{j}_{\mathrm{driv}} = (\tilde{\boldsymbol{\jmath}}_{\mathrm{driv}}(\boldsymbol{x})/\epsilon) + \cdots, \quad \text{in } V^{\mathrm{c}}.$$

It follows that, to leading order, the magnetic fields in V^{c}, $\boldsymbol{B}_{\mathrm{ext}} = \boldsymbol{B}^{(0)}_{\mathrm{ext}}/\epsilon + \cdots$, and in V, $\boldsymbol{B} = \boldsymbol{B}^{(0)}/\epsilon + \cdots$, are both time independent and satisfy (3.8) and (3.9)–(3.10), respectively.

We write the equations (5.1)–(5.3) and boundary conditions (5.4) in terms of gauge invariant variables f, $\boldsymbol{Q}$ and Θ, where the latter is defined by

$$\Theta = \Phi + \frac{\partial\chi}{\partial t}.$$

Then, by making the assumption that the system lies close enough to equilibrium so that the time derivatives do not appear at leading order, we can attempt a derivation of a one-dimensional model similar to that carried out in the steady-state case. Under such an assumption, the expansions for the variables f, $\boldsymbol{Q}$ and Θ are, at leading order, as follows:

$$\begin{gathered} f = f^{(0)}(s,t) + \cdots, \qquad \Theta = \Theta^{(0)}(s,t) + \cdots,\\ \boldsymbol{Q} = (B^{(0)}_1(s)Y - B^{(0)}_2(s)X + q(s,t))\boldsymbol{e}_s + B^{(0)}_3(s)\Upsilon_Y\boldsymbol{e}_X - B^{(0)}_3(s)\Upsilon_X\boldsymbol{e}_Y + \cdots.\end{gathered}$$

Proceeding to higher orders leads to the following closed system of equations for $f^{(0)}$, q and $\Theta^{(0)}$:

$$-\frac{\partial f^{(0)}}{\partial t}+\frac{1}{D(s)}\frac{\partial}{\partial s}\left(D(s)\frac{\partial f^{(0)}}{\partial s}\right)-\Gamma(\mathcal{D}{f^{(0)}}^3-f^{(0)})=f^{(0)}\mathcal{G}(s)+f^{(0)}q^2, \tag{5.5}$$

$$f^{(0)2}\Theta^{(0)}+\frac{1}{D(s)}\frac{\partial}{\partial s}(D(s)f^{(0)2}q)=0, \tag{5.6}$$

$$\frac{1}{D(s)}\frac{\partial}{\partial s}\left(D(s)\left(\frac{\partial\Theta^{(0)}}{\partial s}+\frac{\partial q}{\partial t}\right)\right)=\frac{\mathcal{D}\Gamma}{\sigma\kappa^2}f^{(0)2}\Theta^{(0)}. \tag{5.7}$$

For a single closed loop of wire, we may rewrite these equations in terms of the complex variable $\hat{\psi}$, defined in equation (4.3), the constant $\hat{\alpha}$, defined in (4.4), and the scalar $\hat{\Phi}$, defined by

$$\hat{\Phi}=\Theta^{(0)}-\left.\frac{\partial\hat{\chi}}{\partial t}\right|_{X=Y=0}.$$

In terms of these new variables, the model is as follows:

$$-\left(\frac{\partial\hat{\psi}}{\partial t}+\mathrm{i}\hat{\Phi}\hat{\psi}\right)+\left(\frac{\partial}{\partial s}-\mathrm{i}\hat{\alpha}\right)^2\hat{\psi}+\frac{1}{D}\frac{\partial D}{\partial s}\left(\frac{\partial\hat{\psi}}{\partial s}-\mathrm{i}\hat{\alpha}\hat{\psi}\right)=\Gamma\hat{\psi}(\mathcal{D}|\hat{\psi}|^2-1)+\hat{\psi}\mathcal{G}(s), \tag{5.8}$$

$$\frac{\partial}{\partial s}\left(D(|\hat{\psi}|^2\hat{\alpha}+\tfrac{1}{2}\mathrm{i}\left(\hat{\psi}^*\frac{\partial\hat{\psi}}{\partial s}-\hat{\psi}\frac{\partial\hat{\psi}^*}{\partial s}\right)\right)+\frac{\sigma\kappa^2}{\mathcal{D}\Gamma}\frac{\partial}{\partial s}\left(D\left(\frac{\partial\hat{\Phi}}{\partial s}+\frac{\partial\hat{\alpha}}{\partial t}\right)\right)=0, \tag{5.9}$$

where $\mathcal{G}(s)$ is as given in equation (4.7) and $\hat{\psi}$ is periodic on $(0,2\pi)$ in order to satisfy the constraint (4.1).

6. An example: the transition curve for a uniform loop in a uniform magnetic field

We now proceed to calculate the normal–superconducting transition line for a loop of uniform cross-section, whose centreline is planar, which is subjected to a uniform magnetic field $\boldsymbol{B}=B_2^{(0)}\boldsymbol{e_Y}/\epsilon$ perpendicular to the plane of the loop (see figure 1).

Since the loop has uniform thickness, the functions $D(s)$ and $M_{11}(s)$ are constant, with values that are determined by the shape of the wire through (3.26) and (3.32). The function $P(s)$ depends both on the composition of the adjacent materials and on their surface area in contact with the wire. We consider the case where neither of these properties varies along the length of the wire so that $P(s)$ is also constant (see equation (3.32)). The function $\mathcal{G}(s)=B_2^{(0)2}M_{11}+P$, which appears in (4.6), is, thus, also constant along the loop. Under these conditions, the time-independent model is

$$\left.\begin{aligned}\hat{\psi}''-2\mathrm{i}\hat{\alpha}\hat{\psi}'-\hat{\alpha}^2\hat{\psi}+\Gamma(1-\mathcal{D}|\hat{\psi}|^2)\hat{\psi}-({B_2^{(0)}}^2M_{11}+P)\hat{\psi}=0,\\ \hat{\psi}(s)\text{ periodic on }(0,2\pi),\end{aligned}\right\} \tag{6.1}$$

where primes denote differentiation with respect to s.

Equation (6.1) has solution $\hat{\psi}=0$, corresponding to the normal state. We search for a bifurcation from this solution to a superconducting solution by linearizing about $\hat{\psi}=0$:

$$\hat{\psi}=\delta\hat{\psi}_1+\cdots,\quad \delta\ll 1,$$

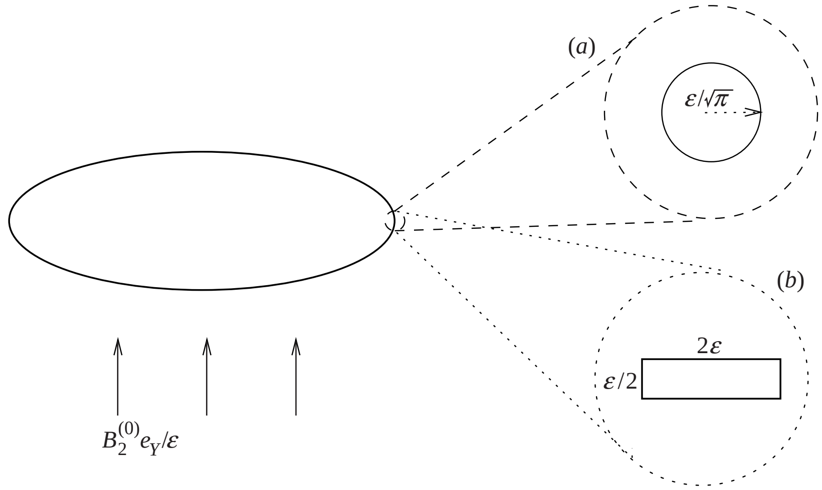

Figure 1. The two different set-ups for which we calculate the normal–superconducting transition curve. These are plotted in figure 2.

and substituting into (6.1). At $O(\delta)$, we find that $\hat{\psi}_1$ satisfies the following linear equation:

$$\hat{\psi}_1'' - 2\mathrm{i}\hat{\alpha}\hat{\psi}_1' - \hat{\alpha}^2\hat{\psi}_1 + (\Gamma - ({B_2^{(0)}}^2 M_{11} + P))\hat{\psi}_1 = 0, \tag{6.2}$$

together with periodic boundary conditions on $(0, 2\pi)$. This has the non-trivial eigen-solution

$$\hat{\psi}_1 = \mathrm{e}^{\mathrm{i}ms}, \quad \text{for } \Gamma = (m - \hat{\alpha})^2 + ({B_2^{(0)}}^2 M_{11} + P).$$

Thus, the first eigenvalue to occur as Γ increases is

$$\begin{aligned}\Gamma = \Gamma_{\text{crit}} &= ({B_2^{(0)}}^2 M_{11} + P) + (\text{nint}(\hat{\alpha}) - \hat{\alpha})^2 \\ &= \left(\left(\frac{\mathcal{F}^{(0)}}{\mathcal{S}}\right)^2 M_{11} + P\right) + \left(\text{nint}\left(\frac{\mathcal{F}^{(0)}}{2\pi\epsilon}\right) - \frac{\mathcal{F}^{(0)}}{2\pi\epsilon}\right)^2,\end{aligned}$$

where we define $\text{nint}(\hat{\alpha})$ to be the integer nearest to $\hat{\alpha}$, and $\mathcal{S}$ to be the area of the surface bounded by $\boldsymbol{x} = \boldsymbol{r}(s)$. We can extend the solution to all values of $\Gamma > \Gamma_{\text{crit}}$ by noting that

$$\hat{\psi} = \gamma\mathrm{e}^{\mathrm{i}ms}, \qquad |\gamma|^2 = \frac{1}{\mathcal{D}} - \frac{({B_2^{(0)}}^2 M_{11} + P) + (m - \hat{\alpha})^2}{\mathcal{D}\Gamma},$$

is a solution to the full problem. Using methods similar to those used in Richardson (1998) we can show that the normal solution is stable for $\Gamma < \Gamma_{\text{crit}}$ and unstable for $\Gamma > \Gamma_{\text{crit}}$. We can also show that the superconducting solution which bifurcates at $\Gamma = \Gamma_{\text{crit}}$ is stable in the vicinity of the bifurcation.

In figure 2 we plot the magnetic flux $\mathcal{F} = \mathcal{S}B_2^{(0)}/\epsilon$ versus Γ_{crit} for a uniform circular loop with (a) a circular cross-section, and (b) a rectangular cross-section with width four times its height. In these plots, we take $P = 0$. The figure shows a periodic behaviour superimposed on a parabolic background. Indeed a similar shape of the $\Gamma(\mathcal{F})$ curve was observed in the original experiments of Little & Parks (1962).

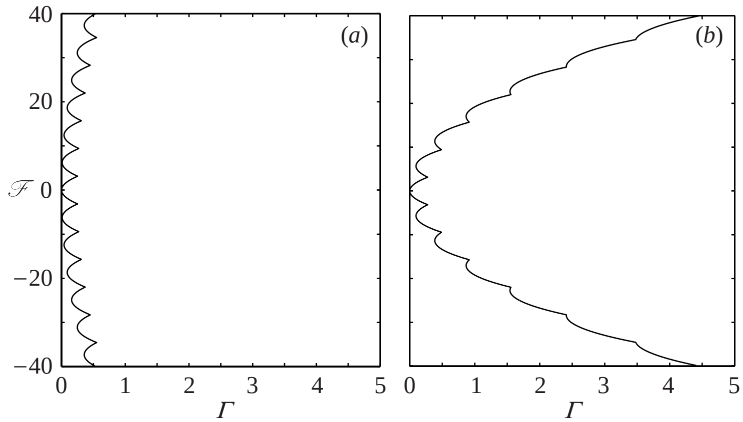

Figure 2. The transition line between the normal and superconducting states for a thin circular loop, of radius 1 and typical width $\epsilon = 0.1$, in a uniform perpendicular field. In (a), the loop has a circular cross-section (see figure 1a) and in (b), it has a rectangular cross-section with width four times its height (see figure 1b).

The periodicity is due to the topological quantization of the phase. The parabolic background is a consequence of the finite thickness of the wire. The importance of the finite thickness of the superconducting domain has been demonstrated by Groff & Parks (1968) in the case of an infinite, thin, circularly symmetric superconducting cylinder. Although we take $P = 0$ in both cases, it is easy to see that the effect of non-zero P is to increase the value of $\Gamma_{\text{crit}}(\mathcal{F})$ by P.

7. Conclusion

In equations (3.31) and (3.32), we have derived a one-dimensional model for a thin wire of slowly varying cross-section that holds for temperatures such that the typical width of the wire $\epsilon l \ll \xi(T)$. The model is similar in form to models previously derived for thin cylinders in axial applied magnetic fields (see Pannetier 1991; Chapman *et al*. 1996; Rubinstein & Schatzman 1997). Where it differs substantively from these it does so because we allow for higher magnetic fields and include the effect of the de Gennes boundary condition. Said differently, the model we derive is the canonical model (although for a wire) and we may retrieve the low-field model simply by setting $\mathcal{G}(s) = 0$ in equation (4.6). The model can be applied to networks of thin wires of any topology. However, the difficulty of applying the model directly to non-simply connected wire geometries leads us to consider the formulation of the model in terms of a complex order parameter ψ. We illustrated the ψ formulation by considering a closed wire loop (it is not difficult to generalize to other topologies) and wrote down the appropriate complex model in (4.5)–(4.7). A noteworthy feature of this formulation is that it is independent of the geometry of the centreline of the wire. In particular, for low fields ($\mathcal{G}(s) = 0$), one can see that the behaviour of a superconducting wire loop is influenced solely by its temperature, the thickness of the wire and the magnetic flux cutting the loop.

In § 6, we used the model to find the normal–superconducting transition for a uniform thin-wire loop as a function of temperature and applied magnetic field; this shows a periodic dependence on the magnetic flux through the loop superimposed on a parabolic background. We mentioned previously (in § 1) that low-field models have been used in Berger & Rubinstein (1995) to predict an isolated zero of the order parameter in a non-uniform ring (in a certain temperature range) whenever an integer-plus-half magnetic flux cuts that ring. We can use our model to show that the same phenomenon occurs for high uniform magnetic fields and again *only* for integer-plus-half fluxes. The effect of the strong uniform magnetic field is just to shift the curve $\Gamma(\mathcal{F})$. The structure of the order parameter, and in particular its zero set, are unchanged by it.

We were also able to conclude that the effect of the de Gennes boundary condition is to depress the temperature at which superconductivity occurs. Indeed the transition temperature is highly sensitive to changes in this boundary condition. In dimensional terms, and for a uniform wire, the important parameter is $\xi^2(T) - \aleph d$, where $\aleph$ is the cross-sectional area of the wire divided by the cross-sectional circumference, d is the de Gennes distance, and ξ the coherence length. When this parameter is positive, the superconducting state is always energetically disadvantageous. This raises the interesting possibility that, for sufficiently thin wires or small de Gennes distance, superconductivity may never be favourable. We hope to treat this subject in more detail in a future work.

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