

# Reduced Order Unsteady Aerodynamic Model of a Rigid Aerofoil in Gust Encounters

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## Abstract

Predicting gust loads using computational fluid dynamics is prohibitively expensive and unrealistic for parametric searches. This work presents the development, implementation, and demonstration of a reduced order model which balances accuracy and speed. The model builds on a proper orthogonal decomposition representation of the linearised time-domain equations and achieves a further reduction in size through a balanced truncation. The novelty of the work lies in the mechanism to introduce any arbitrary gust shape within the reduced order model framework. The methodology combines an analytical formulation, loosely based on the Küssner function, and a numerical approach to identify, or optimise, the unknown parameters of the analytical ansatz. A model problem is investigated for various gust shapes for incompressible and transonic flows. It is found that: (i) the generation of the reduced order model is equivalent to about two steady-state analyses;

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(ii) the predictions of the reduced order model are in good to excellent agreement with the reference solution; and (iii) the reduced order model achieves a consistent speed-up of about 300 times compared to time integrating the original equations. The reduced order model is parametric with respect to the gust disturbance, and may be employed for the worst-case-gust search without extra costs.

*Keywords:* gust loads, computational fluid dynamics, reduced order model, unsteady aerodynamics

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## 1 Nomenclature

$a$	speed of sound
$\mathbf{A}(\alpha)$	volume of fluid cell
$\mathbf{A}, \mathbf{B}, \mathbf{C}$	matrices of the linearised time domain equations
$b$	reference semi-chord
$c$	parameter to be identified in the reduced model
$\mathbf{F}$	inviscid flux vector
$\mathbf{H}$	Jacobian matrix of $\mathbf{F}$ with respect to $\mathbf{w}$
$H_g$	non-dimensional gust wavelength normalised by the reference semi-chord
$m$	dimension of the full order model
$M$	freestream Mach number
$n$	dimension of the reduced order model
$r$	dimension of the reduced order model after balanced truncation
$s$	non-dimensional time based on reference semi-chord
$t^*$	non-dimensional time based on reference chord
$t_f^*$	non-dimensional final time based on reference chord

$U$	freestream speed
$u_g, w_g$	horizontal and vertical components of the gust velocity
$w_{g0}$	maximum intensity of vertical component of the gust velocity
$\boldsymbol{w}$	vector of conservative flow variables
$\boldsymbol{x}$	state vector of the linearised time domain equations

## 2 *Greek*

$\alpha$	angle of attack
$\beta$	parameter to be identified in the reduced model
$\Delta$	perturbation variable
$\varepsilon_1, \varepsilon_2, \Phi_1, \Phi_2$	coefficients in the Wagner approximation
$\varepsilon_3, \varepsilon_4, \Psi_1, \Psi_2$	coefficients in the Küssner approximation
$\lambda$	Lagrange multiplier, and advance ratio
$\Phi, \Psi$	subspaces in the proper orthogonal decomposition

## 3 *Abbreviations*

BT	balance truncation
CFD	computational fluid dynamics
DoF(s)	degree(s) of freedom
POD	proper orthogonal decomposition
ROM	reduced order model

## 4 1. Introduction

5 Specific aircraft performance parameters are monitored and stored by  
6 the digital flight data recorder. These data include, among more than 900  
7 parameters, the aircraft body rates and attitude, speed, altitude, and load  
8 factors, which can be used to reconstruct the properties of any turbulence  
9 encountered by in-flight transport aircraft (1). The reasons this dataset is  
10 valuable to aircraft manufacturers and airlines are two-fold. First, with an  
11 adequate dynamic model of the aircraft, loads on any point of the airframe  
12 can be reconstructed, providing indications on whether design target loads  
13 were reached during operation. The second reason is that knowledge of the  
14 actual loads experienced by the aircraft during flight allows optimising the  
15 time interval of inspections and grounding, and targeting specific structural  
16 components of the airframe.

17 The ability to perform the above mentioned tasks relies on the availabil-  
18 ity of a model of the aircraft dynamics, which includes the effects of atmo-  
19 spheric turbulence and gusts on the rigid and flexible body dynamics. To  
20 make progress in this direction, the work carried out in this paper presents  
21 the implementation of a rapid gust loads analysis capability based on compu-  
22 tational fluid dynamics (CFD) and its application to arbitrary-shaped gust  
23 encounters.

24 For an incompressible, irrotational and two-dimensional flow around a  
25 flat plate, Theodorsen (2) provides an analytical formulation of the unsteady  
26 aerodynamic loads. The Wagner function describes the indicial built-up of  
27 the circulatory part of the lift, including the influence of the shed wake.  
28 Although known exactly in terms of Bessel functions, the Wagner function

29 is practically expressed as an exponential approximation

$$\phi(s) = 1 - \Phi_1 e^{-\varepsilon_1 s} - \Phi_2 e^{-\varepsilon_2 s} \quad (1)$$

30 with the coefficients  $\Phi_1 = 0.165$ ,  $\Phi_2 = 0.335$ ,  $\varepsilon_1 = 0.0455$ , and  $\varepsilon_2 = 0.3$   
31 from R. T. Jones (3). It is interesting to observe that at the initial time  
32  $\phi(0) = 0$ . The Küssner function, on the other hand, gives the lift built-up  
33 for the penetration into a sharp-edged gust. Like the Wagner function, it is  
34 known exactly in terms of Bessel functions, but an exponential approximation  
35 is generally employed for calculations

$$\psi(s) = 1 - \Psi_1 e^{-\varepsilon_3 s} - \Psi_2 e^{-\varepsilon_4 s} \quad (2)$$

36 where the coefficients  $\Psi_1 = 0.5792$ ,  $\Psi_2 = 0.4208$ ,  $\varepsilon_3 = 0.1393$ , and  
37  $\varepsilon_4 = 1.802$  are taken from Ref. (4). Note that  $\psi(0) = 0$ , and that the  
38 non-dimensional time used in the analytical formulation,  $s = tU_\infty/b$ , is  
39 normalised by the reference semi-chord,  $b$ . This is in contrast to the conven-  
40 tion generally adopted in CFD, where the non-dimensional time is defined  
41 as  $t^* = tU_\infty/2b$ . For a moving sharp-edged gust with variable propagation  
42 speed, in which the gust front moves towards or away from the aerofoil, an  
43 analytical formulation is given in Ref. (5). Exact analytical expressions of  
44 the indicial response to a step change in angle of attack, a step change in  
45 pitch rate, and for the penetration into a sharp-edged gust in subsonic com-  
46 pressible flow were obtained by Lomax (6). For a compressible flow, there  
47 are no exact closed-form analytical solutions for all times. By use of linear  
48 piston theory (7), the initial values of the indicial response can be obtained,  
49 and the final values are given by a steady-state method.

## 50 **2. Gust Analysis using Computational Fluid Dynamics**

51 Indicial aerodynamics is a powerful mathematical tool based upon the  
52 assumption of linearity. Once the indicial response is known, the system  
53 response to any arbitrary input signal can be calculated by the convolution  
54 or Duhamel's integral. The predictive capabilities of indicial aerodynamics  
55 are, however, restricted to conditions where the principle of superposition  
56 is valid. To overcome this issue, researchers investigated a number of alter-  
57 natives. The first attempts to directly determine the indicial response by  
58 CFD date back to 1990s (8). This approach has received widespread use  
59 but presents a number of challenges (9; 10). Considering that an unsteady  
60 time-domain CFD analysis is needed to compute the indicial response, large  
61 computational resources are expected making any iterative process impracti-  
62 cal. A second major difficulty is that an unsteady CFD analysis is sensitive to  
63 the spatial and temporal resolutions, grid quality, and the dissipation of the  
64 numerical scheme. More recently, a combined analytical-numerical approach  
65 was proposed in Ref. (11), whereby the predictive capabilities of CFD are  
66 employed to revise and extend the (classic) analytical formulations. In that  
67 work, aerodynamic indicial functions for a compressible subsonic flow were  
68 obtained by modifying those for an incompressible flow using the Prandtl-  
69 Glauert transformation for the circulatory part of the lift, and using the  
70 piston theory for the non-circulatory part. The coefficients of the revised  
71 formulation were then fine-tuned with the CFD response for aerofoils and  
72 elliptic wings. Reference (12) extended that approach to applications on  
73 trapezoidal wings for different aspect ratios and wing sweep angles, for Mach  
74 numbers between 0.3 and 0.7. A number of assessments were carried out,

75 revealing the computational efficiency and general predictive capability of  
76 the new analytical formulation.

77 High-fidelity models based on CFD are needed for transonic as well as  
78 massively separated flows. The cost of time-domain calculations makes the  
79 routine use of CFD in gust response analysis unrealistic, and reduced or-  
80 der models (ROMs) have been used as a cheaper alternative. For example,  
81 Ref. (13) investigated the identification of four ROMs for the response of a  
82 rigid three-dimensional wing to a travelling gust and its coupling with the  
83 aeroelastic equations of motion. The ROMs included a convolution model  
84 identified from a CFD-based sharp-edged gust response, a parametric au-  
85 toregressive moving average and state space model, and a frequency-response  
86 model. The training data for the latter three ROMs were for a random gust  
87 excitation with Gaussian distribution. The models were found computation-  
88 ally efficient and reproduced the CFD response accurately in the subsonic  
89 flow regime. Then, Ref. (14) generated a ROM of the unsteady aerody-  
90 namics based on the autoregressive with exogenous input model. The test  
91 case was for the BAH wing (15) using one rigid body mode and five flexible  
92 modes. After the ROM was generated, which is the most expensive step,  
93 good predictions for a single flight point were reported. Reference (16) pre-  
94 sented the development of a CFD-based gust model for the Boeing truss  
95 braced wing aircraft. The ROM combines the convolution of a sharp-edged  
96 gust with a model constructed from the proper orthogonal decomposition  
97 (POD) of the covariance matrix of the sharp-edged gust unsteady pressure  
98 coefficients. The ROM was found in good agreement with the CFD solution  
99 for a "one-minus-cosine" gust, and was then employed to compute at no

100 extra costs a family of "one-minus-cosine" gust responses.

101 Further applications of the POD method include the generation of CFD-  
102 based ROMs for control design synthesis. Reference (17) performed active  
103 flutter suppression and control design for gust loads alleviation using a POD  
104 ROM, whereas a further reduction in the ROM size using a balanced trun-  
105 cation (BT) method was demonstrated in Ref. (18).

106 System identification models, like those reviewed above and the references  
107 therein, suffer from the usual limitations related to the necessity for a set of  
108 training data closely related to the final application cases, and the difficulty of  
109 accounting for the non-linearity in the reduced model. A systematic and cost  
110 effective approach to developing ROMs capable of describing both linear and  
111 nonlinear effects for a range of cases based on limited development cost has, to  
112 date, proved elusive. A novel approach to the reduction of nonlinear models  
113 for gust loads prediction was first introduced in Ref. (19). The method uses  
114 information on the eigenspectrum of the coupled system Jacobian matrix and  
115 projects the full order model through a series expansion onto a small basis of  
116 eigenvectors which is capable of representing the full order model dynamics.  
117 Linear and nonlinear ROMs derived from linear unsteady aerodynamics/CFD  
118 and linear/nonlinear structural models were generated. The application to  
119 the Goland wing was documented in Ref. (20), and that to a complete aircraft  
120 configuration in Ref. (21). The method has several strengths: (i) it exploits  
121 information from the stability (flutter) calculation for the development of a  
122 ROM for dynamic response analyses; (ii) linear or nonlinear reduced models  
123 can be developed within the same framework; and (iii) the reduced model  
124 can be parametrised to avoid ROM regeneration.



125 The main contribution of this paper is the introduction of gust terms  
126 in a CFD-based ROM which allows arbitrary-shaped gust encounters to be  
127 analysed. While limited to a thorough investigation of the gust loads around  
128 a rigid aerofoil, the methods were demonstrated adequate for predictions  
129 around a three-dimensional flexible configuration (22). This extension will  
130 be reported in a future manuscript.

131 The paper continues in Section 3 with a description of the CFD solver  
132 and the ROM generation. Section 4 provides a thorough validation of CFD  
133 predictions for gust loads with available data in the literature. Then, Sec-  
134 tion 5 compares the gust loads computed using the ROM with those using the  
135 CFD solver, evaluating the accuracy and the computational costs. Finally,  
136 conclusions are given in Section 6.

### 137 **3. Formulation**

#### 138 *3.1. Computational Fluid Dynamics Solver*

139 The research CFD solver used herein is a cell-centred finite volume code  
140 solving the unsteady Reynolds-averaged Navier-Stokes equations (URANS)  
141 on block structured grids. The spatial discretisation is based on the second  
142 order Van Leer scheme (23). For time marching the governing equations,  
143 the Lower Upper Symmetric Gauss-Seidel (LU-SGS) method is used. The  
144 accuracy of the baseline CFD code, both steady and unsteady, has been pre-  
145 viously validated and more information about the numerical details may be  
146 found in Refs. (24; 25). Results in this paper are for the Euler equations. This  
147 is a reasonable assumption for the high Reynolds number of the conditions  
148 considered, where viscous effects can be neglected.

149 The discrete system of non-linear equations in time-domain using an  
150 Arbitrary Lagrangian-Eulerian method is

$$\frac{d(\mathbf{A}(\alpha) \mathbf{w})}{dt^*} + \mathbf{F}(\mathbf{w}, \alpha) = \mathbf{0} \quad (3)$$

151 where symbols are defined in the Nomenclature. The time domain equations  
152 depend generally on the motion of the aircraft. In this work, the parameter  
153  $\alpha(t^*)$  is taken to be the pitch angle of the aerofoil relative to the direction  
154 of the freestream speed, but the concept is easily generalised to arbitrary  
155 motions (22).

### 156 3.1.1. Gust Treatment

157 A methodology for CFD-based gust analysis was introduced in Ref. (8)  
158 and later practised in Ref. (26). The approach, referred to as the field ve-  
159 locity approach, is also exploited in this work. The gust is introduced into  
160 the CFD solver by modification of the velocity of grid points during the un-  
161 steady motion of the aircraft. An advantage of the field velocity approach  
162 is that it overcomes the problems associated with the numerical dissipation  
163 of the disturbance, and no requirements on the spatial discretisation exist.  
164 A disadvantage is that the gust is assumed frozen, and the influence of the  
165 structural response on the gust is neglected. This assumption loses valid-  
166 ity when second order effects, which occur in extreme flight conditions at  
167 the edges of the envelope, become important. The approach, however, has  
168 received widespread use because of the lack of alternative methods.

### 169 3.2. Reduced Order Model

170 The approach here presented to the reduction of a large non-linear CFD  
171 model for gust analysis consists of three steps: (i) linearisation of the gov-

172 ernal equations around a non-linear steady-state flow condition, discussed  
 173 in Section 3.2.1; (ii) generation of a POD basis for the projection of the lin-  
 174 earised flow equations, overviewed in Section 3.2.2; and (iii) identification of  
 175 the optimal coefficients that are used to introduce gust perturbations within  
 176 the ROM framework, discussed in Section 3.2.3.

### 177 3.2.1. Linearised Time Domain Equations

178 The linearised time domain equations are obtained by linearising Eq. (3)  
 179 around a non-linear steady-state solution,  $(\mathbf{w}_0, \alpha_0)$ . Assuming an unsteady  
 180 motion with a small amplitude, the unsteady terms can be expressed as a  
 181 superposition of a steady mean state and a perturbation

$$\begin{aligned}
 \mathbf{w}(t^*) &\approx \mathbf{w}_0 + \Delta\mathbf{w}(t^*), \quad \alpha(t^*) \approx \alpha_0 + \Delta\alpha(t^*) \\
 \|\Delta\alpha\| &\ll \|\alpha_0\| \\
 \dot{\alpha}(t^*) &\approx \Delta\dot{\alpha}(t^*)
 \end{aligned}
 \tag{4}$$

182 When linearizing about the (non-linear) steady mean state, Eq. (3) results  
 183 in the following system of equations

$$\mathbf{A}_0 \Delta\dot{\mathbf{w}} + \mathbf{H} \Delta\mathbf{w} + \mathbf{G} \Delta\alpha + (\mathbf{E} + \mathbf{L}) \Delta\dot{\alpha} = \mathbf{0}
 \tag{5}$$

184 where

$$\begin{aligned}
 \mathbf{A}_0 &= \mathbf{A}(\alpha_0) \\
 \mathbf{H} &= \frac{\partial \mathbf{F}}{\partial \mathbf{w}}(\mathbf{w}_0, \alpha_0) \\
 \mathbf{G} &= \frac{\partial \mathbf{F}}{\partial \alpha}(\mathbf{w}_0, \alpha_0) \\
 \mathbf{E} &= \frac{\partial \mathbf{A}}{\partial \alpha}(\alpha_0) \mathbf{w}_0 \\
 \mathbf{L} &= \frac{\partial \mathbf{F}}{\partial \dot{\alpha}}(\mathbf{w}_0, \alpha_0)
 \end{aligned}
 \tag{6}$$

185 The terms  $\mathbf{G}$ ,  $\mathbf{E}$  and  $\mathbf{L}$  are calculated using a first order finite difference.  
 186 Only the semi-discrete flux Jacobian matrix  $\mathbf{H}$  requires special attention,  
 187 and more details may be found in Ref. (27). Equation (5) is easily recast in  
 188 a classic state space form

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A} \mathbf{x} + \mathbf{B} \mathbf{u} \\ \mathbf{y} = \mathbf{C} \mathbf{x} \end{cases} \quad (7)$$

189 where  $\mathbf{x} = \Delta \mathbf{w}$ ,  $\mathbf{A} = -\mathbf{A}_0^{-1} \mathbf{H}$ ,  $\mathbf{B} = -\mathbf{A}_0^{-1} [\mathbf{G}, \mathbf{E} + \mathbf{L}]$ ,  $\mathbf{u} = [\Delta \alpha, \Delta \dot{\alpha}]^T$ ,  
 190  $\mathbf{y} = \Delta \mathbf{f}^{\text{ext}}$ , and  $\mathbf{C} = \partial \mathbf{f}^{\text{ext}} / \partial \mathbf{w}(\mathbf{w}_0, \alpha_0)$ . The second equation in Eq. (7)  
 191 relates the flow solution to the aerodynamic quantities of interest (pressure  
 192 coefficient, generalised forces and moments, etc.).

193 It is not uncommon for a CFD model to contain several million degrees  
 194 of freedom. Whereas Eq. (7) has a lower degree of complexity than the non-  
 195 linear counterpart in Eq. (3), it retains the same size of the original problem.  
 196 The POD technique, detailed in the next section, is employed to achieve a  
 197 significant reduction on the size of the model by projecting the linearised  
 198 time domain equations onto a small basis of POD modes.

### 199 3.2.2. Proper Orthogonal Decomposition

200 The POD technique is used to extract a small basis of modes that repre-  
 201 sent the dynamics of the original model, Eq. (7). The first step is to create a  
 202 set of snapshots of the full order model,  $\mathbf{X} = [\mathbf{x}^1, \dots, \mathbf{x}^m] \in \mathbb{R}^{n \times m}$  where  
 203  $m$  indicates the number of snapshots. Here,  $\mathbf{x}^k$  represents the  $k$ -th snapshot  
 204 spanning the  $n$ -dimensional space. For an arbitrarily chosen subspace,  $\Phi$ ,  
 205 that satisfies  $\Phi^T \Phi = \mathbf{I}$ , the POD technique searches an  $m$ -dimensional

206 proper orthogonal subspace,  $\Psi \in \mathbb{R}^{n \times m}$ , that minimises the error

$$G = \min_{\Phi} \sum_{k=1}^m \left\| \mathbf{x}^k - \Phi \Phi^T \mathbf{x}^k \right\| = \sum_{k=1}^m \left\| \mathbf{x}^k - \Psi \Psi^T \mathbf{x}^k \right\| \quad (8)$$

207 The above minimisation problem is equivalent to

$$H = \max_{\Phi} \sum_{k=1}^m \frac{\langle (\mathbf{x}^k, \Phi)^2 \rangle}{\|\Phi\|^2} = \sum_{k=1}^m \frac{\langle (\mathbf{x}^k, \Psi)^2 \rangle}{\|\Psi\|^2} \quad (9)$$

208 where the symbol  $\langle \cdot, \cdot \rangle$  indicates an averaging operation, and  $(\cdot, \cdot)$  indicates  
 209 an inner product operation. Since  $\Phi$  has to satisfy  $\Phi^T \Phi = \mathbf{I}$ , the problem  
 210 is reformulated as a constrained optimisation with the Lagrange multiplier,  
 211  $\lambda$ ,

$$J(\Phi) = \sum_{k=1}^m (\mathbf{x}^k, \Phi)^2 - \lambda (\|\Phi\|^2 - 1) \quad (10)$$

212 By taking the partial derivative of  $J(\Phi)$  with respect to the unknown sub-  
 213 space,  $\Phi$ ,

$$\frac{d}{d\Phi} J(\Phi) = 2 \mathbf{X} \mathbf{X}^T \Phi - 2 \lambda \Phi \quad (11)$$

214 the solution that minimises Eq. (8) is also the solution of the following equa-  
 215 tion

$$\left( \mathbf{X} \mathbf{X}^T - \lambda \mathbf{I} \right) \Psi = \mathbf{0} \quad (12)$$

216 Equation (12) is a large dimensional eigenvalue problem for the POD  
 217 kernel,  $\mathbf{K} = \mathbf{X} \mathbf{X}^T$ . To avoid the complexity to numerically solve a large  
 218 eigenvalue problem (28), and by observing that the eigenvalues of  $\mathbf{X} \mathbf{X}^T$   
 219 are the same as those of  $\mathbf{X}^T \mathbf{X}$ , the solution can be found from a smaller  
 220 ( $m$ -dimensional) eigenvalue problem

$$\begin{cases} \mathbf{X}^H \mathbf{X} \mathbf{V} = \mathbf{V} \Lambda \\ \Psi = \mathbf{X} \mathbf{V} \Lambda^{-1/2} \end{cases} \quad (13)$$

221 where  $\Psi = [\Psi_1, \dots, \Psi_m]$  and  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$  is a diagonal matrix  
 222 where the eigenvalues are sorted for decreasing values,  $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m$ .  
 223 As the singular values bring information on the energy contained in the  
 224 system, the number of POD modes is often truncated to match, with the  
 225 least number of modes, a desired energy level of the original model. Define  
 226 the total energy level contained by a POD basis with  $r$  modes as:

$$\eta(r) = \frac{\sum_{i=1}^r \lambda_i}{\sum_{i=1}^m \lambda_i} \quad (14)$$

227 In this work, the number of POD modes retained ensures capturing 99.99%,  
 228 or  $\eta(r) = 0.9999$ , of the energy of the original full order model.

229 Once the POD basis is created by retaining  $r$  modes, a transformation of  
 230 coordinates between the full order model,  $\mathbf{x}$ , and the reduced order model,  
 231  $\mathbf{x}_r \in \mathbb{R}^r$ , is established

$$\mathbf{x} = \Psi_r \mathbf{x}_r \quad (15)$$

232 Finally, the linearised time domain equations, Eq. (7), are projected onto the  
 233 subspace  $\Psi_r = [\Psi_1, \dots, \Psi_r] \in \mathbb{R}^{n \times r}$

$$\begin{cases} \dot{\mathbf{x}}_r = \Psi_r^T \mathbf{A} \Psi_r \mathbf{x}_r + \Psi_r^T \mathbf{B} \mathbf{u} \\ \mathbf{y} = \mathbf{C} \Psi_r \mathbf{x}_r \end{cases} \quad (16)$$

234 Equation (16) represents the dynamic equations of the ROM derived from  
 235 the linearised time domain equations. The advantage of the reduced order  
 236 model is that it is significantly smaller in size than Eq. (7). Disclosing results  
 237 from Section 5, the size of the original model is  $n \approx \mathcal{O}(10^5)$  whereas that  
 238 of the ROM is  $r \approx \mathcal{O}(10^1)$ .

239 *3.2.3. Gust Treatment in the Reduced Order Model*

240 To begin, the gust disturbance is introduced in the ROM equations through  
 241 the term  $\mathbf{u} = [\Delta\alpha, \Delta\dot{\alpha}]^T$ . The methodology herein proposed builds on two  
 242 requirements: speed and simplicity. The first consideration is to avoid de-  
 243 grading the computational performance of the ROM for gust analysis. For  
 244 example, several methods proposed in Ref. (19) require manipulating large  
 245 matrices, as large as the degrees of freedom of the CFD model, considerably  
 246 slowing down the time integration of the ROM equations. The second con-  
 247 sideration of simplicity is inspired by the Küssner function that gives the lift  
 248 built-up for a sharp-edged gust (recall Eq. (2)).

249 The main contribution of this paper is the calculation of the gust terms  
 250 in the ROM. We propose to model the angle of attack due to a moving  
 251 sharp-edge gust of intensity  $\hat{w}_{g0} = w_{g0}/U$  as:

$$\alpha_{\text{SE}}(t^*) = \hat{w}_{g0} c \left(1 - e^{-\beta t^*}\right) \quad \text{for } 0 < t^* < \frac{H_g}{U_\infty \lambda^{-1}} \quad (17)$$

252 where  $c$  and  $\beta$  are parameters to be identified. The similarity with the  
 253 Küssner function is evident. One exponential is retained as it provides good  
 254 approximations. More exponentials can be added to refine the approximation  
 255 if needed, without affecting the convergence of the series. The advance ratio<sup>1</sup>,  
 256  $\lambda$ , is defined in the usual way as:

$$\lambda = \frac{U}{U + u_g} = \frac{M}{M + u_g/a} \quad (18)$$

257 The angle of attack for any arbitrary gust input is obtained using the con-

---

<sup>1</sup>Note that the Lagrange multiplier is commonly denoted in the literature by  $\lambda$ , as in Eq. (10). For the remainder of the work,  $\lambda$  is used to indicate the advance ratio.

258 evolution integral:

$$\alpha(t^*) = \int_0^{t^*} \frac{d\hat{w}_g(\tau)}{d\tau} \alpha_{\text{SE}}(t^* - \tau) d\tau \quad (19)$$

259 which is used to introduce the gust terms in the ROM equations (recall  
260 Eq. (16)).

261 Equation (17) contains two parameters,  $c$  and  $\beta$ , that need to be identified  
262 once during the ROM generation. The parameter  $c$  is identified from the  
263 relation

$$c = \frac{C_L(t_f^*)}{C_L^{\text{ROM}}(t_f^*, \beta = 0, c = 1)} \quad (20)$$

264 where the steady-state response (at the final time  $t_f^* = N \Delta t$ ) to a sharp-  
265 edged gust computed using the original CFD equations,  $C_L$ , is compared with  
266 that computed using the ROM,  $C_L^{\text{ROM}}$ . The parameter  $\beta$  rules the transient  
267 response and is calculated from a minimisation problem carried out in time:

$$f = \min_{\beta} \sum_{i=0}^N \left( C_{L_i} - C_{L_i}^{\text{ROM}}(\beta) \right)^2 \quad (21)$$

## 269 4. Validation Problems

270 Numerical results are compared to exact closed-form expressions obtained  
271 for a flat plate (6). To reduce the effects of thickness, a NACA 0012 aerofoil  
272 was used. A preliminary study was made to test that solutions presented are  
273 independent of the grid used, see Fig. 1. Three sets of grids were generated.  
274 The two-dimensional domain extends fifty chords from the solid wall to the  
275 farfield. The coarsest grid had a total number of 7,220 cells with 130 nodes  
276 around the aerofoil. A medium grid consisted of 11,020 cells where 230 nodes



277 were distributed on the aerofoil. The finest grid was obtained with a total  
 278 of 14,820 cells with 330 nodes around the aerofoil. Boundary conditions  
 279 on the aerofoil surface are set to inviscid wall, and those at the farfield set  
 280 to Riemann farfield boundary conditions. The computational mesh for the  
 281 solution of the Euler equations on the medium grid was found adequate, and  
 282 results presented hereafter are for the medium grid shown in Fig. 1(b).

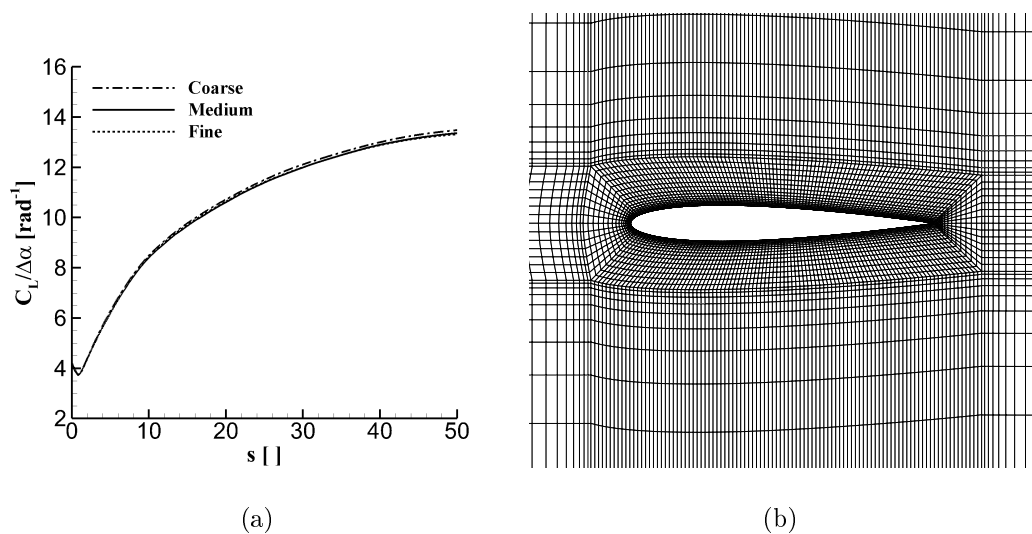


Figure 1: Spatial convergence study: (a) indicial response of lift coefficient for a step change in angle of attack ( $\Delta\alpha = 2.0$  deg,  $M = 0.8$ ); (b) medium grid used in this study

283 For validation of the CFD gust analysis, three test cases are analysed: the  
 284 sharp-edged gust response in Section 4.1, the step change in angle of attack  
 285 response in Section 4.2, and the response to discrete gusts in Section 4.3.

#### 286 4.1. Sharp-edged Gust Response

287 Firstly, a time step convergence study was performed. Three values of  
 288 the time step ( $\Delta s = 0.01, 0.02$ , and  $0.04$ ) were used. Figure 2(a) shows

289 the lift response. The calculations for the smaller time steps are identical,  
 290 and hereafter the unsteady simulations used a non-dimensional time step of  
 291  $\Delta s = 0.02$ . A consideration is for the high frequency oscillations visible for  
 292 the smallest time step. Whereas this is related to the numerical process, as  
 293 commented in the following paragraphs, it is worth noting that the time step  
 294 size acts as a filter dampening out the oscillations for increasing step size.

295 Then, the spatially and temporally converged CFD-based response was  
 296 compared to the analytical solution of Küssner function. Figure 2(b) shows  
 297 the comparison. Overall, a good agreement is observed between the analytical  
 298 and numerical solutions.

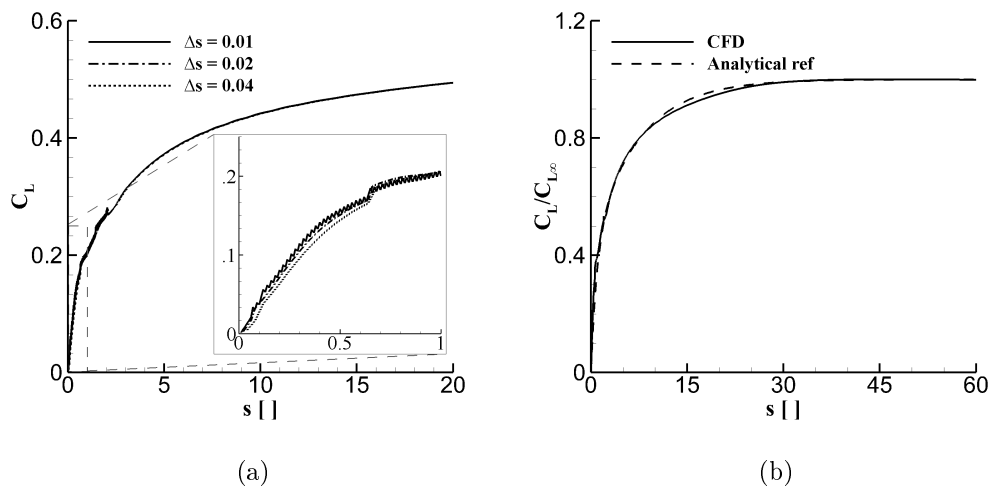


Figure 2: Indicial response of lift coefficient for a sharp-edged gust normalised by the asymptotic value ( $\hat{w}_{g0} = 0.08$ ,  $M = 0.2$ ): (a) temporal convergence study; (b) analytical reference from Ref. (4)

299 For small times, a closed-form expression was given by Lomax (6) for the

300 penetration into a sharp-edged gust as

$$\frac{C_L}{\hat{w}_{g0}} = \frac{2s}{\sqrt{M}} \quad \text{for } 0 \leq s \leq \frac{2M}{1+M} \quad (22)$$

301 Figure 3 shows the computed and analytical results for small times. At lower  
 302 Mach numbers, results are virtually identical. At the largest Mach number,  
 303 differences are attributable to compressibility effects not accounted for in the  
 304 analytical formulation, and the effects of finite thickness of the NACA0012  
 305 aerofoil compared to a flat plate. Decreasing the Mach number results in  
 306 the appearance of oscillations of small entity in the numerical solution as the  
 307 gust travels over the aerofoil surface. This phenomenon has been reported in  
 308 a previous study (10; 13), and attributed to the convergence process as the  
 309 gust moves between two adjacent grid points.

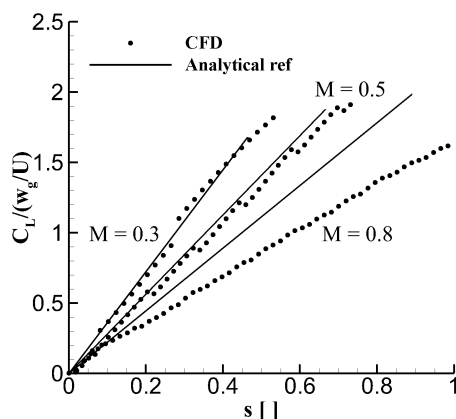


Figure 3: Indicial response of lift coefficient for a sharp-edged gust at small times ( $\hat{w}_{g0} = 0.08$ ); analytical reference from Ref. (6)

310 Finally, results for a moving sharp-edged gust were also computed. The  
 311 lift coefficient responses for several values of the advance ratio are shown in

312 Figure 4. The gust front moves downstream at a speed  $U_\infty / \lambda$  relative to the  
 313 aerofoil: (a) for  $\lambda < 1$ , the gust moves downstream faster than the freestream  
 314 speed; (b) for  $\lambda = 1$ , the gust has the same speed of the surrounding flow;  
 315 and (c) for  $\lambda > 1$ , the gust moves downstream slower than the freestream  
 316 speed. Results are in excellent agreement with those presented in Ref. (10).

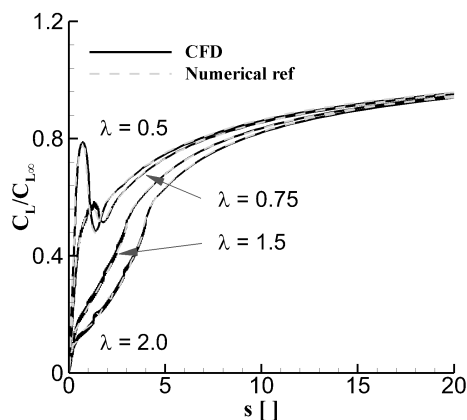


Figure 4: Indicial response of lift coefficient for a moving sharp-edged gust normalised by the asymptotic value ( $\hat{w}_{g0} = 0.08$ ,  $M = 0.2$ ); numerical reference from Ref. (10)

#### 317 4.2. Step Change in Angle of Attack Response

318 The indicial response of the lift coefficient for a step change in angle of  
 319 attack,  $\Delta\alpha = 2.0$  deg, is shown in Figure 5. The indicial response consists  
 320 of two distinct regions, separated by an intermediate overlapping area. The  
 321 initial part of the response is representative of the impulsive motion of the  
 322 body, and of the resulting pressure difference between the upper and lower  
 323 surfaces where the formation of a compression and expansion wave is ob-  
 324 served. The effects of the initial non-circulatory loading are confined within  
 325 a few chord lengths of the distance travelled.

326 After the decay of initial transients, the response converges asymptotically  
 327 to the steady-state value corresponding to the effective angle of attack caused  
 328 by the step change. The asymptotic value of the circulatory loading can be  
 329 obtained by linearized quasi-steady theory

$$\lim_{s \rightarrow \infty} \frac{C_L(s)}{\Delta\alpha} = \frac{2\pi}{\sqrt{1-M^2}} \quad (23)$$

330 Table 1 summarises analytical and numerical results of the asymptotic values  
 331 of the indicial response. Increased differences for the highest Mach number  
 332 are likely because of non-linear compressibility effects captured in the CFD  
 calculations.

Table 1: Asymptotic values of the indicial response of lift coefficient for a step change in angle of attack

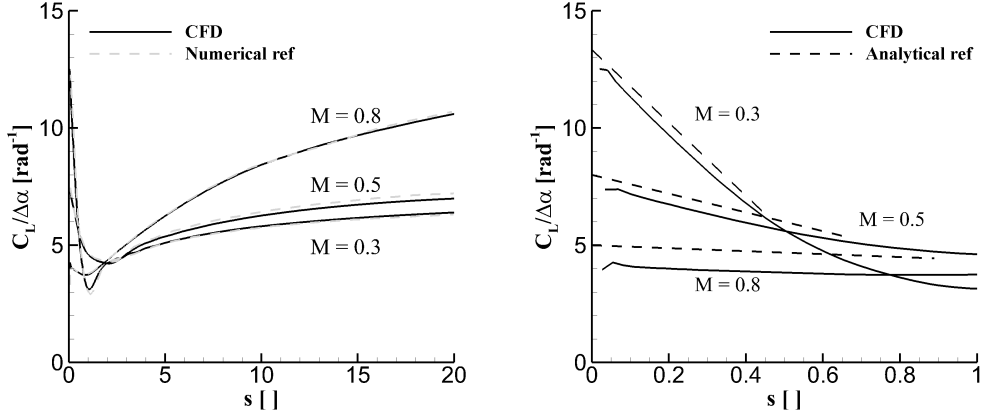
Mach	Quasi-steady Theory	CFD
0.3	6.58	6.79
0.5	7.25	7.49
0.8	10.47	12.34

333

334 An exact analytical expression was obtained by Lomax (6) for a flat plate  
 335 in a linearised compressible flow. The expression is valid for small times, less  
 336 than a chord of the distance travelled, and gives

$$\frac{C_L(s)}{\Delta\alpha} = \frac{4}{M} \left( 1 - \frac{1-M}{2M} s \right) \quad \text{for } 0 \leq s \leq \frac{2M}{1+M} \quad (24)$$

337 Figure 5(b) shows the comparison between the analytical expression and  
 338 the numerical results. The qualitative trends are well captured, with some  
 339 differences becoming more apparent for increasing Mach number.



(a) All times; reference from Ref. (26)

(b) Small times; reference from Ref. (6)

Figure 5: Indicial response of lift coefficient for a step change in angle of attack ( $\Delta\alpha = 2.0$  deg)

#### 340 4.3. Discrete Gust Response

341 The last validation test case is for a family of discrete gusts. The one-  
 342 minus-cosine gust has the form

$$\hat{w}_g(t, x_g) = \frac{\hat{w}_{g0}}{2} \left( 1 - \cos \left( \frac{2\pi x_g}{H_g} \right) \right), \quad \text{for } 0 \leq x_g \leq H_g \quad (25)$$

343 where  $x_g$  is the position of the aircraft in the spatial description of the gust  
 344 relative to a convenient fixed frame of reference. Similarly, the sinusoidal  
 345 gust is given by

$$\hat{w}_g(t, x_g) = \hat{w}_{g0} \sin \left( \frac{2\pi x_g}{H_g} \right), \quad \text{for } 0 \leq x_g \leq H_g \quad (26)$$

346 Figure 6 illustrates the lift coefficient response to both discrete gust  
 347 shapes. The source of comparison is Ref. (29), and the Mach number and

348 the gust intensity are accordingly set to  $M = 0.2$  and  $\hat{w}_{g0} = 0.0349$ , re-  
 349 spectively. The response to a one-minus-cosine gust was computed for two  
 350 gust wavelengths,  $H_g = 10$  and 50. There is a good agreement for the lift  
 351 build-up, but small differences were found when the lift coefficient achieves  
 352 the maximum value. These differences are likely attributed to the different  
 353 grid used in Ref. (29) from that employed in this work. Qualitatively, similar  
 354 considerations hold valid for the response to a sinusoidal gust.

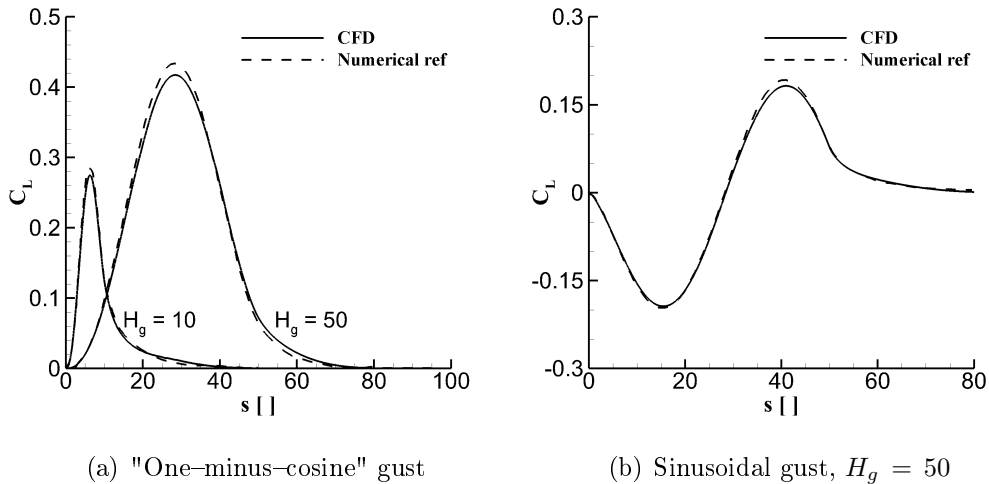


Figure 6: Lift coefficient response to discrete gusts ( $\hat{w}_{g0} = 0.0349$ ,  $M = 0.2$ ); numerical reference from Ref. (29)

## 355 5. Results

### 356 5.1. Generation of the Reduced Order Model

357 The generation of the ROM consists of the following steps.

- 358 1. The linearised CFD equations are evaluated for a steady-state flow
- 359 condition. In this work, the linearisation is carried out for all cases at

360 a freestream angle of attack  $\alpha_\infty = 0$  deg. The validity of the ROM  
361 predictions around different mean angles of attack holds valid until  
362 second order effects are negligible.

363 2. A number of POD modes is extracted from an unsteady time–domain  
364 analysis. The training data were obtained imposing an impulse function  
365 for the angle of attack,  $\alpha$ , and its time derivative,  $\dot{\alpha}$ . The impulse func-  
366 tion was chosen because it allows exciting the relevant unsteady aero-  
367 dynamic characteristics. After a sensitivity analysis, 160 POD modes  
368 were used given the broad frequency range of the response.

369 3. A smaller basis of POD modes is formed, extracting a small subset  
370 of modes from the original basis and ensuring that  $\eta(r) = 0.9999$   
371 is achieved. For the aerofoil test case, this was met by selecting 50 POD  
372 modes ( $r = 50$ ). Finally, the LTD model is projected onto the basis of  
373 50 POD modes.

374 Table 2 summarises the computational cost incurred in the generation  
375 of the ROM. To provide general insights on the process, CPU times are  
376 normalised by the CPU time needed for a steady–state solution to converge  
377 9 orders of magnitude. The breakdown conveys that: (i) the linearisation  
378 of the Euler equations is an inexpensive step; (ii) the calculation of the  
379  $m = 160$  POD snapshots is nearly as costly as a steady–state solution; as  
380 already observed in Ref. (30), the reason for this is the slow convergence rate  
381 of the GMRES linear solver in the LTD equations; and (iii) the identification  
382 of the POD basis consisting of 50 modes requires calculating the singular  
383 value decomposition of a large–order snapshot matrix.

384 The identification of the optimal parameters,  $c$  and  $\beta$ , appearing in the



Table 2: Computational cost of the ROM generation ( $m = 160$  and  $r = 50$ ); CPU times are normalised by the CPU time needed for a steady-state analysis

ROM Generation Step	CPU Time
Steady-state analysis	1.00
Linearisation of the time domain equations	0.11
Generation of $m$ POD modes	0.89
Extraction of $r$ POD modes and ROM projection	0.58

385 ROM gust formulation, Eq. (17), is straightforward. Table (3) reports the  
386 values obtained for a sharp-edged gust of intensity  $\hat{w}_{g0} = 0.0349$  for several  
387 Mach numbers. The values of the parameter  $c$  are around unity, and a low  
388 dependency on the Mach number was found. This is not unexpected as the  
389 parameter relates the intensity of the gust input,  $\hat{w}_{g0}$ , to the output,  $\alpha_{SE}$ ,  
390 and is related to steady-state conditions. The transients are ruled by the  
391 parameter  $\beta$ . The increasing trend of the parameter for increasing Mach  
392 number reflects the faster dynamics occurring at the higher Mach numbers.

Table 3: Parameters of the gust formulation in the ROM, see Eq. (17)

Mach Number	$c$	$\beta$
0.3	0.9574	0.44
0.5	0.9613	0.75
0.8	1.0335	0.92

393 Considering that it is good in practise to initialise an unsteady time-  
394 domain analysis from a well-converged steady-state solution, the overall cost  
395 of the ROM generation equals 1.72 times the CPU cost of a steady-state run,

396 Table 2. Furthermore, the ROM provides the following advantages: (i) once  
397 it is generated, the ROM may be employed to parametrically investigate the  
398 effects of arbitrary gust profiles on the dynamic response; and (ii) the ROM is  
399 of small size, therefore predictions are performed at virtually no extra costs.

#### 400 *5.2. Step Change in Angle of Attack Response*

401 Figure 7 shows the indicial response of the lift coefficient for a step change  
402 in angle of attack,  $\Delta\alpha = 2.0$  deg. Three Mach numbers representative  
403 of incompressible, subsonic, and transonic flows are evaluated. The first  
404 comparison in Figure 7(a) indicates a good agreement between the nonlinear  
405 CFD equations and the LTD counterpart, consisting both of 55,100 degrees  
406 of freedom (DoFs). In Figure 7(b), the ROM predictions are superimposed on  
407 the LTD solution, revealing an excellent agreement. Here, the ROM consists  
408 of 50 DoFs. The comparison provides a first indication on the accuracy of  
409 the proposed ROM.

410 The minimum number of POD modes for accurate predictions is not  
411 known a priori, but an estimate of the error can be computed a posteriori.  
412 The sensitivity of the ROM results on the number of POD modes was in-  
413 vestigated for the  $M = 0.8$  case of Fig. 7. A number of snapshots (80,  
414 120 and 160) were computed using the LTD solver and then used to form  
415 the basis vectors for the POD projection (31). The ROM solutions were  
416 computed using the three sets of basis vectors, and compared to the LTD  
417 response. Overall, the present technique is in good agreement with the LTD  
418 model. For 80 and 120 POD modes, the behaviour in the region connecting  
419 the initial part of the impulsive motion with the circulatory part is not well  
420 predicted, as shown in Fig. 8. For 160 POD modes, the predicted unsteady

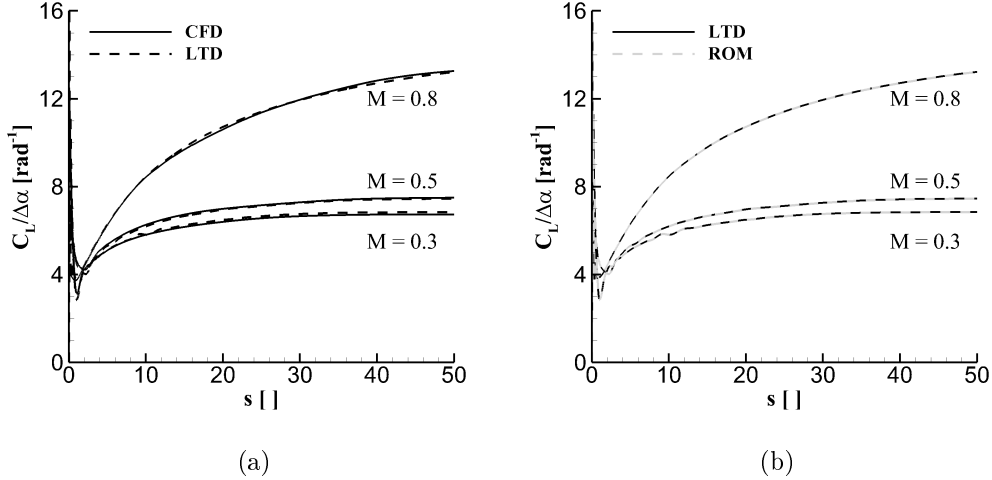


Figure 7: Indicial response of lift coefficient for a step change in angle of attack ( $\Delta\alpha = 2.0$  deg); the CFD and LTD models have 55,100 DoFs, the ROM has 50 DoFs

421 solution is in excellent agreement with the LTD response. The BT technique  
 422 was then used to reduce the 160 POD modes into a 50 DoFs ROM that is  
 423 used for the remainder of this work.

### 424 5.3. Moving Sharp-edged Gust Response

425 The case of a moving sharp-edged gust is analysed in Figure 9 for three  
 426 Mach numbers and three advance ratios. It was found that the ROM re-  
 427 produces the physics responsible for the decreasing over-shoot of the lift  
 428 coefficient for increasing Mach number. For the higher  $\lambda$ , the delay in the  
 429 lift-up is predicted in the ROM with some discrepancies at the intermediate  
 430 times compared to the CFD solutions. It was found that the addition of a  
 431 second exponential in Eq. (17) improves the ROM predictions at small to in-  
 432 termediate times. However, it is worth observing that the present differences  
 433 between the CFD solver and the ROM are similar, if not smaller, to those

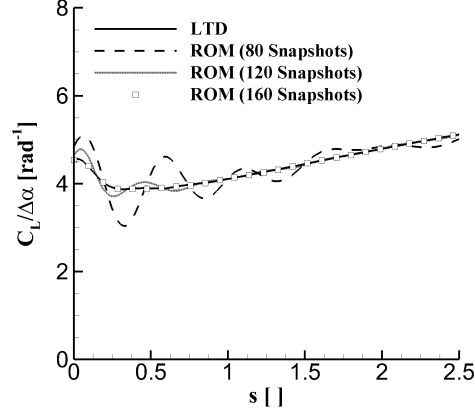


Figure 8: Sensitivity of the ROM on the number of POD modes for the indicial response of lift coefficient for a step change in angle of attack ( $\Delta\alpha = 2.0$  deg,  $M = 0.8$ )

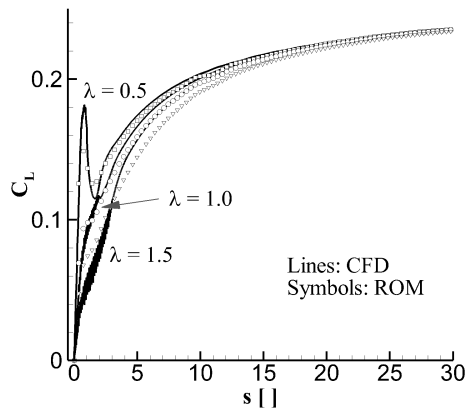
434 observed when using two different CFD codes, as discussed in Ref. (12).

#### 435 5.4. Moving Discrete Gust Response

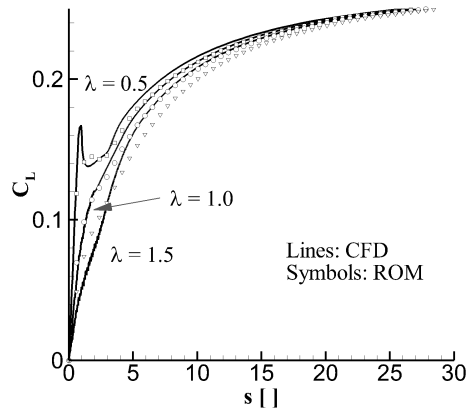
436 The final test case is for a family of moving "one-minus-cosine" gusts.  
 437 The ROM predictions were obtained using the convolution integral proposed  
 438 in Eq. (19) as an input to the model.

439 First, Figure 10 compares the response of the lift coefficient at  $M = 0.8$   
 440 and  $\hat{w}_{g0} = 0.0349$ . Three gust wavelengths ( $H_g = 10, 20,$  and  $50$ ) are  
 441 considered. The ROM parameters were identified from the response to a  
 442 sharp-edged gust with the same intensity as above,  $\hat{w}_{g0} = 0.0349$ . The  
 443 ROM predictions follow closely the FOM response for all gust wavelengths  
 444 tested. This confirms the adequacy of the ROM gust treatment and the  
 445 ability to predict the loads response to arbitrary gust shapes.

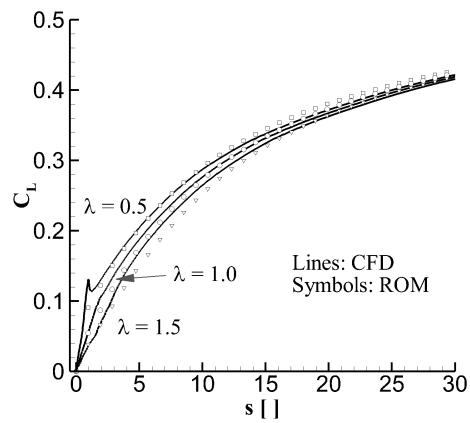
446 Then, the ability of the ROM to predict the response to gust intensities  
 447 ( $\hat{w}_{g0} = 0.0698$  and  $0.1047$ ) other than that used for its generation ( $\hat{w}_{g0} =$



(a)  $M = 0.3$



(b)  $M = 0.5$



(c)  $M = 0.8$

Figure 9: Response of lift coefficient for a moving sharp-edged gust ( $\hat{w}_{g0} = 0.0349$ )

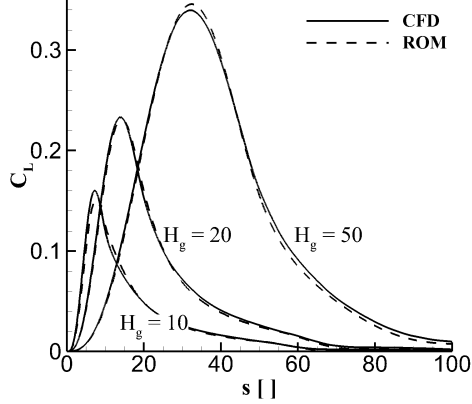


Figure 10: Response of lift coefficient for a "one-minus-cosine" gust ( $\hat{w}_{g0} = 0.0349$ ,  $M = 0.8$ )

448 0.0349) was assessed. In Figure 11, the comparison is carried out at  $M = 0.8$   
 449 for a gust wavelength  $H_g = 20$ . To quantify the model accuracy, a fit  
 450 parameter is introduced

$$\eta = 1 - \sqrt{\frac{\sum_{i=1}^N (C_{L_i} - C_{L_i}^{\text{ROM}})^2}{\sum_{i=1}^N (C_{L_i})^2}} \quad (27)$$

451 A fit parameter equal unity indicates a perfect match of the ROM results with  
 452 the CFD results. The fit coefficient is reported for the three gust intensities  
 453 in Table 4. It is not unexpected that the error increases for increasing gust  
 454 intensity because the flow field characteristics depart more markedly from  
 455 the reference steady-state solution. For example, Figure 12 compares the  
 456 pressure coefficient distribution at  $M = 0.8$  for two flow conditions. The  
 457 first is for the steady solution at  $\alpha = 0$  deg, which was used to generate the  
 458 ROM, featuring a symmetric shock wave. The second condition depicts the  
 459 instantaneous flow solution when the lift coefficient reaches the highest peak

460 for  $\hat{w}_{g0} = 0.1047$  in Figure 11. The shock disappears from the lower surface  
 461 and becomes stronger on the upper surface.

462 Similarly, a study was conducted to assess the validity of the ROM around  
 463 mean angles of attack different from the one used for its generation,  $\alpha = 0.0$   
 464 deg. For the transonic case,  $M = 0.8$ , with a "one-minus-cosine" gust of  
 465 intensity  $\hat{w}_{g0} = 0.0349$  in Figure 11, second order effects start to become  
 466 significant above 2 deg mean angle of attack.

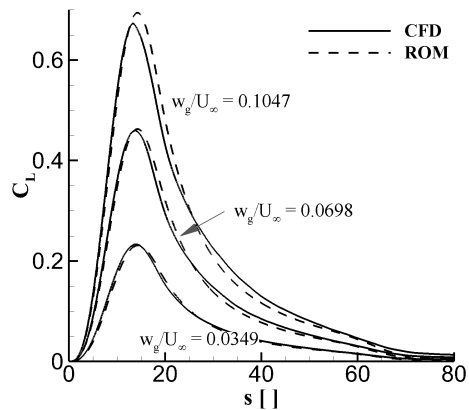
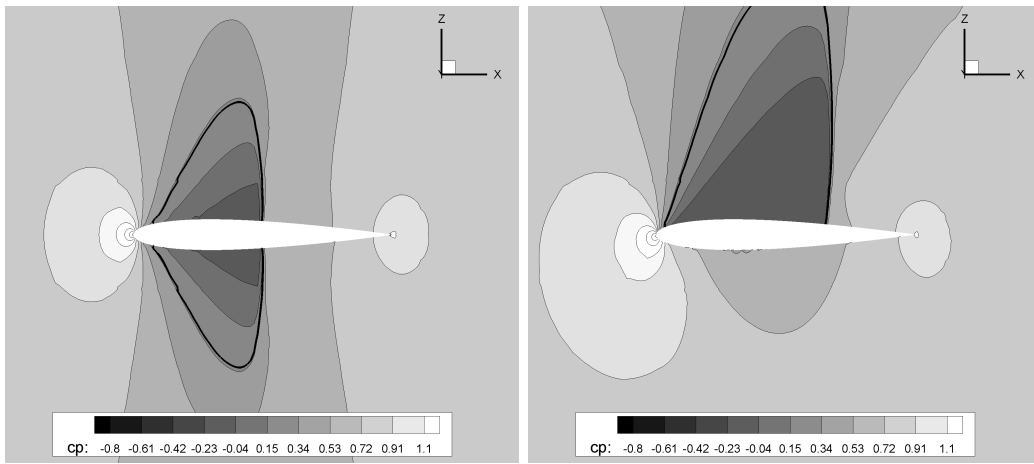


Figure 11: Response of lift coefficient for a "one-minus-cosine" gust ( $H_g = 20$ ,  $M = 0.8$ )

Table 4: Fit parameter of the lift coefficient for a "one-minus-cosine" gust ( $H_g = 20$ ,  $M = 0.8$ )

$\hat{w}_g$	$\eta$
0.0349	0.979
0.0698	0.961
0.1047	0.940



(a) Steady,  $\alpha = 0$  deg

(b) Instantaneous,  $\alpha \approx 6$  deg

Figure 12: Pressure coefficient distribution during the response to a "one-minus-cosine" gust ( $\hat{w}_{g0} = 0.1047$ ,  $H_g = 20$ ,  $M = 0.8$ ): (a) steady solution at  $\alpha = 0$  deg, (b) snapshot corresponding to maximum  $C_L$  for  $\hat{w}_{g0} = 0.1047$  in Figure 11; the solid thick line indicates  $M = 1.0$



467 Finally, the case of a moving "one-minus-cosine" gust is analysed in  
468 Figure 13 at  $M = 0.8$  and with gust parameters  $\hat{w}_{g0} = 0.0349$  and  $H_g = 20$ .  
469 An excellent agreement is observed in all cases. For increasing advance ratio,  
470 the horizontal speed of the gust in the downstream direction decreases. As  
471 the total time of the gust-aerofoil interaction increases, so does the effect of  
472 the gust on the aerodynamic loads.

473 Results in Figure 13 illustrate the impact of the advance ratio on the aero-  
474 dynamic loads around a rigid configuration. For a gust of given intensity and  
475 wavelength, as in the test case herein considered, the action of the advance  
476 ratio translates into an apparent change of both intensity and wavelength.  
477 This aspect of gusts, which has a low to null coverage in the open literature,  
478 is more critical for low speed flight, as typical of high altitude long endurance  
479 aircraft. To date, however, certification requirements for structural sizing do  
480 not include any relevant information in this regard. For a flexible structure,  
481 the horizontal velocity component of the gust is likely to affect the aeroelas-  
482 tic response to a higher degree compared to a rigid configuration. An initial  
483 work in this direction is given in Ref. (22).

#### 484 5.5. Computational Efficiency

485 The computational cost of the ROM generation was addressed in Sec-  
486 tion 5.1, while this Section is concerned with the computational efficiency  
487 of time integrating the ROM equations. The settings of the unsteady time-  
488 domain analysis (time step size, final simulation time) are identical between  
489 the CFD and the ROM. Two cases are evaluated: (i) the response for the  
490 moving sharp-edge gust shown in Figure 9; and (ii) the response for the  
491 "one-minus-cosine" gust of Figure 10. Table 5 summarises the computa-

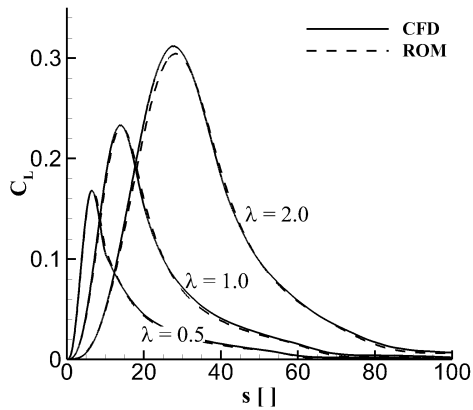


Figure 13: Response of lift coefficient for a moving "one-minus-cosine" gust ( $\hat{w}_{g0} = 0.0349$ ,  $H_g = 20$ ,  $M = 0.8$ )

492 tional efficiency of the ROM compared to the CFD solver. Values are nor-  
 493 malised by the CPU time of the ROM to complete the unsteady analysis. It  
 494 was found that the computational efficiency of the ROM maintains nearly  
 495 constant for both cases, with a speed-up of about 300 times. For reference,  
 496 the ROM calculations were performed in about one second on a standard  
 497 computer.

Table 5: Computational cost comparison between CFD and ROM for various gust shapes

	Sharp-edged gust (see Figure 9)	"One-minus-cosine" gust (see Figure 10)
CFD time integration, Eq. (3)	310.0	296.8
ROM time integration, Eq. (16)	1.0	1.0

## 498 6. Conclusions

499 The novelty of the proposed work is on the gust treatment within a re-  
500 duced order model, which is based on a proper orthogonal decomposition  
501 representation of the linearised time-domain equations. The approach builds  
502 on two requirements, speed and simplicity, and combines within the same  
503 framework analytical and numerical evaluations. The analytical component  
504 is inspired by the Küssner function which models the lift built-up for a  
505 sharp-edged gust through exponential functions. Unknown parameters of  
506 the analytical component are then identified from an appropriate training  
507 response. To demonstrate the application of the new methodology, a model  
508 problem for the inviscid flow around a standard aerofoil section is thoroughly  
509 evaluated.

510 First, an extensive validation of the computational fluid dynamics solver  
511 for gust loads is carried out using available data in the open literature. Then,  
512 the reduced order model is demonstrated for a variety of test cases, including  
513 the step change in angle of attack, moving sharp-edged gust, and moving dis-  
514 crete gusts both in the incompressible and compressible flow regimes. Three  
515 assessments were performed to draw conclusions on the reduced order model:  
516 (i) cost of the reduced order model generation; (ii) accuracy of the reduced  
517 order model predictions; and (iii) computational efficiency of the reduced  
518 order model. The cost of the reduced order model generation is critical, but  
519 shall be balanced against the accuracy of predictions. It was found that  
520 the model generation was performed at the equivalent cost of less than two  
521 steady-state analyses, and that the most expensive step was identified to be  
522 the generation of the proper orthogonal modes. The predictions from the

523 reduced order model were found in good to excellent agreement with the  
524 computational fluid dynamics simulations for all gust shapes and flow condi-  
525 tions. Finally, the computational efficiency of time integrating the reduced  
526 order model compared to the computational fluid dynamics equations was  
527 assessed. Employing the same numerical settings, the reduced order model  
528 achieved a speed-up of about 300 times consistently for two test cases.

529 In conclusion, the proposed methodology to introduce gust effects in the  
530 reduced order model is deemed accurate and relatively inexpensive, open-  
531 ing up a potential application in the parametric search of the worst-case  
532 loads. Whereas this work has focused on the methodology demonstrated on  
533 a model problem, the application to three-dimensional aeroelastic problems  
534 has already been performed successfully and will be reported separately.

535 Lastly, a consideration on the impact of the advance ratio on the aero-  
536 dynamic loads is worth mentioning. For a gust of given intensity and wave-  
537 length, the action of the advance ratio translates into an apparent change  
538 of both intensity and wavelength. This aspect of gust loads analysis seems  
539 absent in the literature and, most importantly, is not included in current  
540 certification requirements for structural sizing. Certainly, this becomes more  
541 critical for low speed flight, as typical of high altitude long endurance aircraft  
542 and it is expected to have a large impact on flexible structures.

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