2017 IEEE 56th Annual Conference on Decision and Control (CDC)
December 12-15, 2017, Melbourne, Australia

Multiple Model Switched Repetitive Control

Chris Freeman

***Abstract*— A multiple model switched repetitive control (RC) framework is developed for a general class of system and widely used RC update structure. This guarantees stability and robust performance under the assumption that the true plant model belongs to a plant uncertainty set specified by the designer. A comprehensive design procedure for the candidate model set and RC update is presented based on novel application of gap metric analysis to RC, and switching of the corresponding RC schemes is achieved efficiently using a bank of Kalman filters.**

I. INTRODUCTION

Repetitive Control (RC) is a design technique for high performance trajectory tracking or set-point regulation in the case where the reference trajectory and/or system disturbances are periodic, see, for example, [1]. There is no resetting action between periods, and the aim is to use error data from previous periods to modify the control input such that the tracking error is reduced. Expanding RC robustness to uncertain plant dynamics or disturbance periods has been studied for many years. For example, frequency based methods to expand robustness for non-harmonic frequencies are considered in [2], and reducing sensitivity to period variations is addressed in [3]. A range of adaptive RC schemes have also focused on unknown or variable periods, see e.g. [4], [5] and references therein.

In [6] a switching mechanism is formulated for a class of SISO first order nonlinear systems, which may include an integrator chain, with a disturbance of unknown period. This approach is extended in [7] by employing a switched adaptive RC design to provide stabilization when both the parametrized nonlinearity and period are unknown. However current adaptive RC schemes inevitably restrict the underlying plant structure and the uncertainty. Performance and ease of application and are also limited by permissible RC forms.

This paper extends the gap metric analysis of [8] to RC, in order to develop a rigorous adaptive framework that provides comprehensive robustness guarantees for a general class of RC update. The class of estimation based multiple model RC (EMMRC) is introduced, and comprises a bank of plant models, covering a region of plant-space in which the 'true' plant lies. An estimation process based on past observations ranks the models' ability to explain the observations, and the best fitting model is then used to determine the RC update for each trial. EMMRC is an extension of the framework of [9], [10], an axiomatic approach concerned with feedback control stabilization. To apply this approach to RC we first embed the 'lifted' RC control action within its structure, and also incorporate tracking by considering stabilization around a shifted equilibrium corresponding to the noise free response to the tracking input. Finally we derive robust stability results to provide a comprehensive EMMRC framework for design.

C. Freeman is with Electronics and Computer Science, University of Southampton, UK, cf@ecs.soton.ac.uk

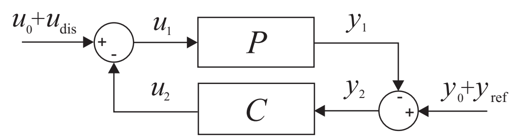

Fig. 1. Closed loop $[P, C]$ with reference $y_{\rm ref}$ and disturbance $u_{\rm dis}$.

II. PRELIMINARIES

Let $\mathcal{S} := map(\mathbb{Z}, \mathbb{R}^h)$ denote the collection of all maps, where $h \in \mathbb{N}$. Define the restriction of a signal $v \in \mathcal{S}$ over the interval $I = [a, b]$ by $v|_I := (v(a), \cdots, v(b))$ where $a \leq b,\ a, b \in \mathbb{Z}$, and similarly for $I = [a, b)$. Let $\mathscr{T}_t : \mathcal{S} \cup_{b\in\mathbb{Z}} \mathcal{S}|_{[0,b]} \to \mathcal{S},\ t \in \mathbb{Z}$ denote the truncation operator:

$$(\mathscr{T}_t v)(\tau) = \{v(\tau) \ \text{ if } \ \tau \in \text{dom}(v),\ \tau \leq t, \ \ 0 \ \text{ otherwise}\ .$$

Define signal space $\mathcal{V} \subset \mathcal{S}$ and extended signal space $\mathcal{V}_e \subset \mathcal{S}$:

$$\mathcal{V} := \{v \in \mathcal{S} \ | \ v(-t) = 0,\ \forall t \in \mathbb{N};\ \|v\| < \infty\} \quad (1)$$
$$\mathcal{V}_{[a,b]} := \left\{v \in \mathcal{S}|_{[a,b]} \,\middle|\, \exists x \in \mathcal{V}\ s.t.\ v = x|_{[a,b]}\right\}$$
$$\mathcal{V}_e := \{v \in \mathcal{S} \ | \ \forall t \in \mathbb{Z} : \mathscr{T}_t v \in \mathcal{V}\} \quad (2)$$

where $\|\cdot\| = \|\cdot\|_2$. The input and output signal spaces are defined as: $\mathcal{U} := \mathcal{V} \times \cdots \times \mathcal{V} = \mathcal{V}^m, \mathcal{Y} := \mathcal{V} \times \cdots \times \mathcal{V} = \mathcal{V}^o$, and let $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$. Given a plant $P : \mathcal{U}_e \to \mathcal{Y}_e$ satisfying $P(0) = 0$ and a controller $C : \mathcal{Y}_e \to \mathcal{U}_e$ satisfying $C(0) = 0$, the closed-loop system $[P, C]$ in Fig. 1 is defined by

$$y_1 = Pu_1, \qquad u_0 = u_1 + u_2 - u_{\rm dis} \quad (3)$$
$$y_0 = y_1 + y_2 - y_{\rm ref}, \quad u_2 = Cy_2. \quad (4)$$

Here $y_{\rm ref} \in \mathcal{Y}_e$ and $u_{\rm dis} \in \mathcal{U}_e$ are externally applied biases, and $w_i = (u_i, y_i)^\top \in \mathcal{W}_e$ represents the plant input and output ($i = 1$), disturbances ($i = 0$) and observations ($i = 2$). Given bias $\bar{w} = (\bar{u}, \bar{y})^\top \in \mathcal{W}_e$, define biased signal spaces

$$\mathcal{U}_{\bar{u}} = \{u \in \mathcal{U}_e \ | \ u - \bar{u} \in \mathcal{U}\}, \mathcal{Y}_{\bar{y}} = \{y \in \mathcal{Y}_e \ | \ y - \bar{y} \in \mathcal{Y}\},$$

and $\mathcal{W}_{\bar{w}} = \mathcal{U}_{\bar{u}} \times \mathcal{Y}_{\bar{y}}$. We associate $\mathcal{U}_{\bar{u}}, \mathcal{Y}_{\bar{y}}, \mathcal{W}_{\bar{w}}$ with norms

$$\|u\|_{\bar{u}} = \|u - \bar{u}\|, \quad \|y\|_{\bar{y}} = \|y - \bar{y}\|, \quad \|w\|_{\bar{w}} = \|w - \bar{w}\|$$

and will use notation $\|\mathscr{T}_t w\|_{\bar{w}} = \|\mathscr{T}_t(w - \bar{w})\|$, $\|w|_I\|_{\bar{w}} = \|(w - \bar{w})|_I\|$. Denoting $\bar{w}_0 = (u_{\rm dis}, y_{\rm ref})^\top$ as the external bias, we can define the maps from external to internal signals

$$\Pi_{P//C} : \mathcal{W}_{\bar{w}_0} \to \mathcal{W}_e : (w_0 + \bar{w}_0) \mapsto w_1 \quad (5)$$

and $\Pi_{C//P} = I - \Pi_{P//C}$. $[P, C]$ is well-posed if for all $w_0, \bar{w}_0 \in \mathcal{W}_e$ there exists a unique $(w_1, w_2) \in \mathcal{W}_e \times \mathcal{W}_e$ such that (3)-(4) hold. We use the following concept of stability:

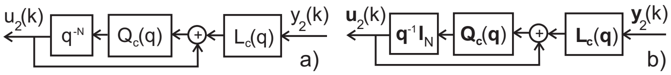

Fig. 2. RC controller: a) non-lifted form C_c and b) lifted equivalent $\boldsymbol{C_c}$.

Definition 1: Let closed loop system $[P, C]$ be well-posed. $[P, C]$ is said to be gain stable with respect to the external bias $\bar{w}_0 \in \mathcal{W}_e$ if there exists a scalar $0 < M < \infty$ such that:

$$\|\Pi_{P//C}\|_{\bar{w}_0} = \sup_{\substack{w \in \mathcal{W}_{\bar{w}_0}, \tau > 0 \\ \|\mathscr{T}_\tau(w - \bar{w}_0)\| \neq 0}} \frac{\|\mathscr{T}_\tau(\Pi_{P//C}w - \Pi_{P//C}\bar{w}_0)\|}{\|\mathscr{T}_\tau(w - \bar{w}_0)\|} < M \quad (6)$$

and similarly for $\|\Pi_{C//P}\|_{\bar{w}_0}$. Let $\mathcal{P}$ parametrize a collection of plant operators, P_p. For example, $\mathcal{P}_{\rm LTI}(m, o) \in \mathcal{P}$ is the set of all $(A, B, C, D) \in \cup_{n \geq 1} \mathbb{R}^{n_p \times n_p} \times \mathbb{R}^{n_p \times m} \times \mathbb{R}^{o \times n_p} \times \mathbb{R}^{o \times m}$ such that (A, B, C, D) is minimal and

$$\begin{aligned} &P_p : \mathcal{U}_e \to \mathcal{Y}_e,\ u_1 \mapsto y_1,\ p = (A_p, B_p, C_p, D_p) \\ &\quad x_p(k+1) = A_p x_p(k) + B_p u_1(k) \\ &\quad y_1(k) = C_p x_p(k) + D_p u_1(k),\ x_p(-k) = 0,\ k \in \mathbb{N} \end{aligned} \quad (7)$$

Hence $y_1(-k) = (P_p u_1)(-k) = 0$ for all $k \in \mathbb{N}$. Also define $\bar{\mathcal{P}}_{LTI}(m, o) := \{(A, B, C, D) \in \mathcal{P} \mid D = 0\}$. Analogously, define $\mathcal{C}_{\rm LTI}(m, o)$ to be the set of all control operators, C_c, parametrized by $(A, B, C, D) \in \cup_{n \geq 1} \mathbb{R}^{n_c \times n_c} \times \mathbb{R}^{n_c \times o} \times \mathbb{R}^{m \times n_c} \times \mathbb{R}^{m \times o}$ such that (A, B, C, D) is minimal.

III. REPETITIVE CONTROL FRAMEWORK

Let $P_p,\ p = (A_p, B_p, C_p, D_p) \in \bar{\mathcal{P}}_{LTI}(m, o)$, be the plant model, and consider the system shown in Fig. 1 and defined by (3)-(4). Suppose $u_{\rm dis}$ and $y_{\rm ref}$ are N-periodic, i.e. $y_{\rm ref}(k) = y_{\rm ref}(k+N)$ and $u_{\rm dis}(k) = u_{\rm dis}(k+N)$ for a given $N > 0$ and all $k \in \mathbb{N}$. The RC tracking problem is to design controller C_c, $c \in \mathcal{C}_{LTI}(m, o)$ such that when $w_0 = 0$ the plant output asymptotically converges to the reference, i.e.

$$\lim_{k \to \infty} y_1(kN + i) = y_{\rm ref}(i), \quad i = 0, \cdots, N-1 \quad (8)$$

RC design is typically conducted using one-sample advance operator q. The most common RC update structure is

$$u_2(k+N) = Q_c(q)(u_2(k) + L_c(q) y_2(k)), \quad \forall\, k \in \mathbb{N} \quad (9)$$

where $P_p(q) = C_p(Iq - A_p)^{-1} B_p$ and the filters have form

$$Q_c(q) = \frac{b^{Q_c}_{-m_{Q_c}} q^{m_{Q_c}} + \cdots + b^{Q_c}_{n_{Q_c}} q^{-n_{Q_c}}}{a^{Q_c}_0 + \cdots + a^{Q_c}_{n_{Q_c}} q^{-n_{Q_c}}}, \quad (10)$$

$$L_c(q) = \frac{b^{L_c}_{-m_{L_c}} q^{m_{L_c}} + \cdots + b^{L_c}_{n_{L_c}} q^{-n_{L_c}}}{a^{L_c}_0 + \cdots + a^{L_c}_{n_{L_c}} q^{-n_{L_c}}}, \quad (11)$$

where $n_{Q_c} + n_{L_c} \leq N$. The nominal control evolution is

$$u_2(k{+}N) = G_{p,c}(q) u_2(k) {+} Q_c(q) L_c(q)(y_{\rm ref}(k) {-} P_p(q) u_{\rm dis}(k))$$

where $G_{p,c}(q) = Q_c(q)(I + L_c(q) P_p(q))$. This yields the well-known sufficient condition for closed-loop stability of

$$\|G_{p,c}(q)\|_\infty < 1. \quad (12)$$

In particular, taking $Q_c = I$ guarantees that (8) holds when $w_0 = 0$. The RC control structure (9) is shown in Fig. 2a).

A. Lifted Setting Representation

The proposed EMMRC framework operates in the so-called 'lifted' domain, in which signals are segmented as

$$\begin{aligned} \boldsymbol{u}_i(k) &= [u_i(kN),\ u_i(kN+1), \cdots u_i((k+1)N-1)]^\top \in \mathbb{R}^{mN}, \\ \boldsymbol{y}_i(k) &= [y_i(kN),\ y_i(kN+1), \cdots y_i((k+1)N-1)]^\top \in \mathbb{R}^{oN}, \end{aligned}$$

with $\boldsymbol{w}_i = (\boldsymbol{u}_i, \boldsymbol{y}_i)^\top \in \boldsymbol{\mathcal{W}}_e = \mathcal{W}_e^N$ for $i \in \{0, 1, 2\}$. Also

$$\begin{aligned} \boldsymbol{u}_{\rm dis} &:= \boldsymbol{u}_{\rm dis}(k) = [u_{\rm dis}(0),\ u_{\rm dis}(1), \cdots u_{\rm dis}(N-1)]^\top \in \mathbb{R}^{mN}, \\ \boldsymbol{y}_{\rm ref} &:= \boldsymbol{y}_{\rm ref}(k) = [y_{\rm ref}(0),\ y_{\rm ref}(1), \cdots y_{\rm ref}(N-1)]^\top \in \mathbb{R}^{oN}. \end{aligned}$$

The plant dynamics are therefore equivalently expressed by

$$\begin{aligned} &\boldsymbol{P_p} : \boldsymbol{\mathcal{U}}_e \to \boldsymbol{\mathcal{Y}}_e : \boldsymbol{u}_1 \to \boldsymbol{y}_1,\ \boldsymbol{p} = (\boldsymbol{A_p}, \boldsymbol{B_p}, \boldsymbol{C_p}, \boldsymbol{D_p}) \\ &\underbrace{x_p((k+1)N)}_{\boldsymbol{x_p}(k+1)} = \underbrace{A_p^N}_{\boldsymbol{A_p}} \underbrace{x_p(kN)}_{\boldsymbol{x_p}(k)} + \underbrace{[A_p^{N-1} B_p, \cdots B_p]}_{\boldsymbol{B_p}} \boldsymbol{u}_1(k) \\ &\boldsymbol{y}_1(k) = \underbrace{\begin{bmatrix} C_p \\ C_p A_p \\ \vdots \\ C_p A_p^{N-1} \end{bmatrix}}_{\boldsymbol{C_p}} \underbrace{x_p(kN)}_{\boldsymbol{x_p}(k)} + \underbrace{\begin{bmatrix} D_p & 0 & \cdots & 0 \\ C_p B_p & D_p & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ C_p A_p^{N-2} B_p & \cdots & C_p B_p & D_p \end{bmatrix}}_{\boldsymbol{D_p}} \boldsymbol{u}_1(k) \end{aligned} \quad (13)$$

where $\boldsymbol{x_p}(k) \in \mathbb{R}^{n_p}$, and $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}_{\rm LTI}(o, m) := \{(A, B, C, D) \in \mathcal{P}_{\rm LTI}(oN, mN)\}$. If causal ($m_{Q_c} = m_{L_c} = 0$) lifted operators L_c and Q_c can similarly be realized as

$$\begin{aligned} &\boldsymbol{Q_c} : \boldsymbol{\mathcal{U}}_e \to \boldsymbol{\mathcal{U}}_e : (\boldsymbol{v} + \boldsymbol{u}_2) \to \boldsymbol{z} : \\ &\quad \boldsymbol{x_{Q_c}}(k+1) = A_{\boldsymbol{Q_c}} \boldsymbol{x_{Q_c}}(k) + B_{\boldsymbol{Q_c}}(\boldsymbol{v} + \boldsymbol{u}_2)(k) \\ &\quad \boldsymbol{z}(k) = C_{\boldsymbol{Q_c}} \boldsymbol{x_{Q_c}}(k) + D_{\boldsymbol{Q_c}}(\boldsymbol{v} + \boldsymbol{u}_2)(k) \end{aligned} \quad (14)$$

where $\boldsymbol{z}(k) = \boldsymbol{u}_2(k+1)$, $\boldsymbol{x_{Q_c}}(k) \in \mathbb{R}^{n_{Q_c}}$, and

$$\begin{aligned} &\boldsymbol{L_c} : \boldsymbol{\mathcal{Y}}_e \to \boldsymbol{\mathcal{U}}_e : \boldsymbol{y}_2 \to \boldsymbol{v} : \\ &\quad \boldsymbol{x_{L_c}}(k+1) = A_{\boldsymbol{L_c}} \boldsymbol{x_{L_c}}(k) + B_{\boldsymbol{L_c}} \boldsymbol{y}_2(k) \\ &\quad \boldsymbol{v}(k) = C_{\boldsymbol{L_c}} \boldsymbol{x_{L_c}}(k) + D_{\boldsymbol{L_c}} \boldsymbol{y}_2(k) \end{aligned} \quad (15)$$

with $\boldsymbol{x_{L_c}}(k) \in \mathbb{R}^{n_{L_c}}$. Hence controller C_c has lifted form

$$\begin{aligned} &\boldsymbol{C_c} : \boldsymbol{\mathcal{Y}}_e \to \boldsymbol{\mathcal{U}}_e : \boldsymbol{y_2} \to \boldsymbol{u}_2,\ \boldsymbol{c} = (\boldsymbol{A_c}, \boldsymbol{B_c}, \boldsymbol{C_c}, \boldsymbol{D_c}) \\ &\underbrace{\begin{bmatrix} \boldsymbol{x_{L_c}} \\ \boldsymbol{x_{Q_c}} \\ \boldsymbol{u}_2 \end{bmatrix}}_{\boldsymbol{x_c}}(k+1) = \underbrace{\begin{bmatrix} A_{\boldsymbol{L_c}} & \boldsymbol{0} & \boldsymbol{0} \\ B_{\boldsymbol{Q_c}} C_{\boldsymbol{L_c}} & A_{\boldsymbol{Q_c}} & B_{\boldsymbol{Q_c}} \\ D_{\boldsymbol{Q_c}} C_{\boldsymbol{L_c}} & C_{\boldsymbol{Q_c}} & D_{\boldsymbol{Q_c}} \end{bmatrix}}_{\boldsymbol{A_c}} \begin{bmatrix} \boldsymbol{x_{L_c}} \\ \boldsymbol{x_{Q_c}} \\ \boldsymbol{u}_2 \end{bmatrix}(k) + \underbrace{\begin{bmatrix} B_{\boldsymbol{L_c}} \\ B_{\boldsymbol{Q_c}} D_{\boldsymbol{L_c}} \\ D_{\boldsymbol{Q_c}} D_{\boldsymbol{L_c}} \end{bmatrix}}_{\boldsymbol{B_c}} \boldsymbol{y}_2(k) \\ &\boldsymbol{u}_2(k) = \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}}_{\boldsymbol{C_c}} \begin{bmatrix} \boldsymbol{x_{L_c}} \\ \boldsymbol{x_{Q_c}} \\ \boldsymbol{u}_2 \end{bmatrix}(k) + \underbrace{\boldsymbol{0}}_{\boldsymbol{D_c}} \boldsymbol{y}_2(k) \end{aligned} \quad (16)$$

where $\boldsymbol{x_c}(k) \in \mathbb{R}^{n_c}$, $n_c = n_{L_c} + n_{Q_c} + mN$, and $\boldsymbol{c} \in \boldsymbol{\mathcal{C}}_{\rm LTI}(o, m) := \{(A, B, C, D) \in \mathcal{C}_{\rm LTI}(oN, mN)\}$. This is shown Fig. 2b) where $\boldsymbol{q}$ is the trial domain one-sample advance. Likewise, the disturbance-free RC update relationship (9) has the lifted representation $\boldsymbol{G_{p,c}} : \boldsymbol{\mathcal{U}}_e \to \boldsymbol{\mathcal{U}}_e : \boldsymbol{u}_2 \to \boldsymbol{z}$:

$$\begin{aligned} &\underbrace{\begin{bmatrix} \boldsymbol{x_p} \\ \boldsymbol{x_{L_c}} \\ \boldsymbol{x_{Q_c}} \end{bmatrix}}_{\boldsymbol{x_{G_{p,c}}}}(k+1) = \underbrace{\begin{bmatrix} A_{\boldsymbol{p}} & \boldsymbol{0} & \boldsymbol{0} \\ -B_{\boldsymbol{L_c}} C_{\boldsymbol{p}} & A_{\boldsymbol{L_c}} & \boldsymbol{0} \\ -B_{\boldsymbol{Q_c}} D_{\boldsymbol{L_c}} C_{\boldsymbol{p}} & B_{\boldsymbol{Q_c}} C_{\boldsymbol{L_c}} & A_{\boldsymbol{Q_c}} \end{bmatrix}}_{A_{\boldsymbol{G_{p,c}}}} \begin{bmatrix} \boldsymbol{x_p} \\ \boldsymbol{x_{L_c}} \\ \boldsymbol{x_{Q_c}} \end{bmatrix}(k) + \\ &\Bigg[\underbrace{\begin{bmatrix} -B_{\boldsymbol{p}} \\ B_{\boldsymbol{L_c}} D_{\boldsymbol{p}} \\ B_{\boldsymbol{Q_c}}(\boldsymbol{I} + D_{\boldsymbol{L_c}} D_{\boldsymbol{p}}) \end{bmatrix}}_{B_{\boldsymbol{G_{p,c}}}}, \underbrace{\begin{bmatrix} B_{\boldsymbol{p}} & \boldsymbol{0} \\ -B_{\boldsymbol{L_c}} D_{\boldsymbol{p}} & B_{\boldsymbol{L_c}} \\ -B_{\boldsymbol{Q_c}} D_{\boldsymbol{L_c}} D_{\boldsymbol{p}} & B_{\boldsymbol{Q_c}} D_{\boldsymbol{L_c}} \end{bmatrix}}_{B_{0,\boldsymbol{G_{p,c}}}}\Bigg] \begin{bmatrix} \boldsymbol{u}_2 \\ \bar{\boldsymbol{w}}_0 \end{bmatrix}(k) \end{aligned}$$

Stabilizing controller design procedure

$$K : \boldsymbol{\mathcal{P}} \to \boldsymbol{\mathcal{C}} \tag{18}$$

Estimator

$$X : \boldsymbol{\mathcal{W}}_e \to map(\mathbb{N}, map(\boldsymbol{\mathcal{P}}, \mathbb{R}^+)) \\ : \boldsymbol{w}_2 \mapsto [k \to (\boldsymbol{p} \mapsto r_{\boldsymbol{p}}[k])] \tag{19}$$

Minimising operator

$$M : (map(\mathbb{N}, map(\boldsymbol{\mathcal{P}}, \mathbb{R}^+))) \to map(\mathbb{N}, \boldsymbol{\mathcal{P}}^*) \\ : [k \mapsto (\boldsymbol{p} \mapsto r_{\boldsymbol{p}}[k])] \mapsto [k \mapsto s_f(k)] \tag{20}$$

$$s_f(k) := \operatorname*{argmin}_{\boldsymbol{p} \in G} r_{\boldsymbol{p}}[k], \ \forall k \in \mathbb{N} \tag{21}$$

Delay transition operator

$$D : map(\mathbb{N}, \boldsymbol{\mathcal{P}}^*) \to map(\mathbb{N}, \boldsymbol{\mathcal{P}}^*) : [k \mapsto s_f(k)] \mapsto [k \mapsto s(k)] \tag{22}$$

$$s(k) := \begin{cases} s_f(k) & \text{if} \quad k - k_s(k) \geq \Delta(s(k_s(k))) \\ s(k_s(k)) & \text{else} \end{cases} \tag{23}$$

$$\Delta : \boldsymbol{\mathcal{P}} \to \mathbb{N} \tag{24}$$

$$k_s(k) := \max\{i \in \mathbb{N} \mid 0 \leq i \leq k, \ s(i) \neq s(i-1)\} \tag{25}$$

Overall switching operator

$$S : \boldsymbol{\mathcal{W}}_e \to map(\mathbb{N}, \boldsymbol{\mathcal{P}}^*) : \boldsymbol{w}_2 \mapsto s \tag{26}$$

$$S = DM(X, G) \tag{27}$$

Controller

$$\boldsymbol{C} : \boldsymbol{\mathcal{Y}}_e \to \boldsymbol{\mathcal{U}}_e : \boldsymbol{y}_2 \mapsto \boldsymbol{u}_2 \tag{28}$$

$$\boldsymbol{u}_2(k) = \boldsymbol{C}_{K(s(k))}(\boldsymbol{y}_2 - \mathscr{T}_{k_s(k)-1}\boldsymbol{y}_2)(k) \tag{29}$$

TABLE I

EQUATIONS SPECIFYING THE EMMRC ALGORITHM.

$$\boldsymbol{z}(k) = \overbrace{\begin{bmatrix} -D_{\boldsymbol{Q_c}} D_{\boldsymbol{L_c}} C_{\boldsymbol{p}} & D_{\boldsymbol{Q_c}} C_{\boldsymbol{L_c}} & C_{\boldsymbol{Q_c}} \end{bmatrix}}^{C_{\boldsymbol{G_{p,c}}}} \begin{bmatrix} \boldsymbol{x_p} \\ \boldsymbol{x_{L_c}} \\ \boldsymbol{x_{Q_c}} \end{bmatrix}(k) \\ + \Big[\underbrace{D_{\boldsymbol{Q_c}} D_{\boldsymbol{L_c}} D_{\boldsymbol{p}}}_{D_{\boldsymbol{G_{p,c}}}}, \underbrace{\begin{bmatrix} -D_{\boldsymbol{Q_c}} D_{\boldsymbol{L_c}} D_{\boldsymbol{p}} & D_{\boldsymbol{Q_c}} D_{\boldsymbol{L_c}} \end{bmatrix}}_{D_{0,\boldsymbol{G_{p,c}}}}\Big] \begin{bmatrix} \boldsymbol{u}_2 \\ \bar{\boldsymbol{w}}_0 \end{bmatrix}(k) \tag{17}$$

where $\boldsymbol{x_{G_{p,c}}}(k) \in \mathbb{R}^{n_{G_{p,c}}}$, $n_{G_{p,c}} = n_p + n_{L_c} + n_{Q_c}$. If L_c and Q_c are non-causal, parts of the one period delay contained in (9) must be split between these operators, as shown in [11]. These calculations are straight-forward and yield the same forms of $\boldsymbol{C_c}$, $\boldsymbol{G_{p,c}}$ on which subsequent analysis is based.

IV. THE EMMRC STRUCTURE

The EMMRC structure is shown in Fig. 3, with operators summarized in Table I. Here K is a control design procedure that assigns a stabilising controller $\boldsymbol{C_c}, \boldsymbol{c} \in \boldsymbol{\mathcal{C}}$ to each plant $\boldsymbol{P_p}, \boldsymbol{p} \in \boldsymbol{\mathcal{P}}$, such that $[\boldsymbol{P_p}, \boldsymbol{C_c}]$ is gain stable. The powerset of $\boldsymbol{\mathcal{P}}$ is denoted $\boldsymbol{\mathcal{P}}^*$ and $G \subset \boldsymbol{\mathcal{P}}$ is a constant set of candidate plant models thought to represent the true plant. For each $\boldsymbol{p} \in G$ we implement an estimator X which uses observations $\boldsymbol{w}_2$ to generate a residual $r_{\boldsymbol{p}}[k]$ at sample k. These are fed to minimization operator M, which returns the index, s_f, of the plant with minimal residual. Operator D delays the free switching signal s_f to prevent instability due to rapid switching, by associating a minimum delay $\Delta(\boldsymbol{p})$ with every plant which must elapse before another is permitted. The signal s then determines the atomic controller choice $\boldsymbol{C}_{K(s(k))}$ corresponding to the selected plant. These components comprise switching operator S shown in Fig. 3.

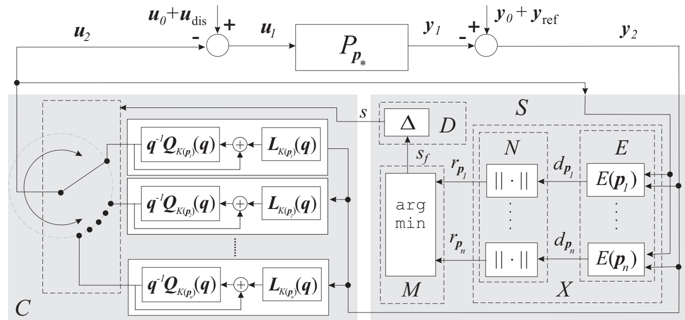

Fig. 3. EMMRC structure: switch S outputs signal s which determines atomic controller choice $\boldsymbol{C}_{K(s(k))}$. Delay operator D generates s from free switching signal s_f, which in turn is generated from minimization operator M acting on residuals $r_{\boldsymbol{p}}[k]$ from estimator X. $\boldsymbol{P}_{\boldsymbol{p}_*}$ denotes the true plant.

V. ATOMIC RC PROPERTIES

Proposition 2: If update (9) satisfies $n_{L_c} + n_{Q_c} < mN$, then the minimum interval length, $\sigma(\boldsymbol{c})$, that $(\boldsymbol{u}_2^{\boldsymbol{c}}, \boldsymbol{y}_2^{\boldsymbol{c}})^\top$ must be observed to uniquely determine the initial conditions of $\boldsymbol{C_c}$ is $\sigma(\boldsymbol{c}) = 2$. Here $\boldsymbol{u}_2^{\boldsymbol{c}} = \boldsymbol{C_c}\boldsymbol{y}_2^{\boldsymbol{c}}$, $\boldsymbol{y}_1^{\boldsymbol{p}} = \boldsymbol{P_p}\boldsymbol{u}_1^{\boldsymbol{p}}$, and

$$\boldsymbol{u}_0^{\boldsymbol{p}} = \boldsymbol{u}_1^{\boldsymbol{p}} + \boldsymbol{u}_2 - \boldsymbol{u}_{\text{dis}}, \ \boldsymbol{y}_0^{\boldsymbol{p}} = \boldsymbol{y}_1^{\boldsymbol{p}} + \boldsymbol{y}_2 - \boldsymbol{y}_{\text{ref}}, \tag{30}$$

$$\sigma(\boldsymbol{c}) = \min \left\{ k \geq 0 : \begin{array}{c} \forall l \geq 0, \\ \boldsymbol{u}_2^{\boldsymbol{c}} = \boldsymbol{C_c}\boldsymbol{y}_2^{\boldsymbol{c}}, \ \hat{\boldsymbol{u}}_2^{\boldsymbol{c}} = \boldsymbol{C_c}\hat{\boldsymbol{y}}_2^{\boldsymbol{c}}, \\ (\boldsymbol{u}_2^{\boldsymbol{c}}, \boldsymbol{y}_2^{\boldsymbol{c}})^\top|_{[l,l+k]} = (\hat{\boldsymbol{u}}_2^{\boldsymbol{c}}, \hat{\boldsymbol{y}}_2^{\boldsymbol{c}})^\top|_{[l,l+k]}, \\ \boldsymbol{y}_2^{\boldsymbol{c}} = \hat{\boldsymbol{y}}_2^{\boldsymbol{c}} \Rightarrow \boldsymbol{u}_2^{\boldsymbol{c}} = \hat{\boldsymbol{u}}_2^{\boldsymbol{c}} \end{array} \right\}$$

Similarly, if $\boldsymbol{p}$ satisfies $n_p \leq oN$, then $\sigma(\boldsymbol{p}) = 1$.

Using this information, we can then define the properties that will be required of the set of atomic closed loop RC systems:

Assumption 3: There exist functions $\alpha, \beta : \boldsymbol{\mathcal{P}} \times \boldsymbol{\mathcal{C}} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ such that the following holds:

1) (Linear growth of $[\boldsymbol{P_p}, \boldsymbol{C_c}]$): Let $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}$, $\boldsymbol{c} \in \boldsymbol{\mathcal{C}}$ and the closed-loop system $[\boldsymbol{P_p}, \boldsymbol{C_c}]$ be well-posed. Let $l_1, l_1, l_2, l_3, l_4 \in \mathbb{N}$, $l_1 < l_2 \leq l_3 < l_4$ and $I_1 = [l_1, l_2), I_2 = [l_2, l_3), I_3 = [l_3, l_4)$. Suppose $\boldsymbol{w}_2, \boldsymbol{w}_2^{\boldsymbol{c}}, \boldsymbol{w}_1^{\boldsymbol{p}} \in \boldsymbol{\mathcal{W}}_e, \ \boldsymbol{w}_0^{\boldsymbol{p}} \in \boldsymbol{\mathcal{W}}$ satisfy equations (30) on $I_1 \cup I_2 \cup I_3$. Suppose either

$$\begin{aligned} \boldsymbol{w}_2^{\boldsymbol{c}}|_{I_1} = 0, \ \boldsymbol{w}_2^{\boldsymbol{c}}|_{I_2 \cup I_3} &= \boldsymbol{w}_2|_{I_2 \cup I_3} \quad \text{or} \\ \boldsymbol{w}_2^{\boldsymbol{c}}|_{I_1 \cup I_2 \cup I_3} &= \boldsymbol{w}_2|_{I_1 \cup I_2 \cup I_3} \end{aligned}$$

where $|I_1| = l_2 - l_1 \geq \max\{\sigma(\boldsymbol{p}), \sigma(\boldsymbol{c})\}$. Then, in both cases:

$$\begin{aligned} \|\boldsymbol{w}_2|_{I_3}\|_{\bar{\boldsymbol{w}}_2} \leq\ & \alpha(\boldsymbol{p}, \boldsymbol{c}, |I_2|, |I_3|)\|\boldsymbol{w}_2|_{I_1}\|_{\bar{\boldsymbol{w}}_2} \\ & + \beta(\boldsymbol{p}, \boldsymbol{c}, |I_2|, |I_3|)\|\boldsymbol{w}_0^p|_{I_1 \cup I_2 \cup I_3}\|. \end{aligned} \tag{31}$$

2) (Stability of $[\boldsymbol{P_p}, \boldsymbol{C}_{K(\boldsymbol{p})}]$): Let $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}$ and $x \in \mathbb{N}$. Then

$$\alpha(\boldsymbol{p}, K(\boldsymbol{p}), a, x) \to 0 \text{ as } a \to \infty \tag{32}$$

and α is monotonic in a.

These properties are underpinned by projection map $\Pi_{C//P}$:

$$\underbrace{\begin{bmatrix} \boldsymbol{x_p} \\ \boldsymbol{x_c} \end{bmatrix}}_{\boldsymbol{x_{p,c}}}(k+1) = \underbrace{\begin{bmatrix} A_{\boldsymbol{G_{p,c}}} & B_{\boldsymbol{G_{p,c}}} \\ C_{\boldsymbol{G_{p,c}}} & D_{\boldsymbol{G_{p,c}}} \end{bmatrix}}_{A_{\boldsymbol{p,c}}} \begin{bmatrix} \boldsymbol{x_p} \\ \boldsymbol{x_c} \end{bmatrix}(k) + \underbrace{\begin{bmatrix} B_{0,\boldsymbol{G_{p,c}}} \\ D_{0,\boldsymbol{G_{p,c}}} \end{bmatrix}}_{B_{\boldsymbol{p,c}}}(\boldsymbol{w}_0 + \bar{\boldsymbol{w}}_0)(k)$$

$$\boldsymbol{w}_2(k) = \underbrace{\begin{bmatrix} \boldsymbol{0} & C_{\boldsymbol{c}} \\ -C_{\boldsymbol{p}} & D_{\boldsymbol{p}}C_{\boldsymbol{c}} \end{bmatrix}}_{C_{\boldsymbol{p,c}}} \begin{bmatrix} \boldsymbol{x_p} \\ \boldsymbol{x_c} \end{bmatrix}(k) + \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ -D_{\boldsymbol{p}} & \boldsymbol{I} \end{bmatrix}}_{D_{\boldsymbol{p,c}}}(\boldsymbol{w}_0 + \bar{\boldsymbol{w}}_0)(k) \tag{33}$$

where $\boldsymbol{x}_{\boldsymbol{p},\boldsymbol{c}}(k) \in \mathbb{R}^{n_p+n_c}$. In addition define for $a \geq 0$:

$$K^a_{\boldsymbol{p},\boldsymbol{c}} = \left[A^{a-1}_{\boldsymbol{p},\boldsymbol{c}}, \cdots\ A_{\boldsymbol{p},\boldsymbol{c}}B_{\boldsymbol{p},\boldsymbol{c}},\ B_{\boldsymbol{p},\boldsymbol{c}}\right],\ O^a_{\boldsymbol{p},\boldsymbol{c}} = \begin{bmatrix} C_{\boldsymbol{p},\boldsymbol{c}} \\ C_{\boldsymbol{p},\boldsymbol{c}}A_{\boldsymbol{p},\boldsymbol{c}} \\ \vdots \\ C_{\boldsymbol{p},\boldsymbol{c}}A^{a-1}_{\boldsymbol{p},\boldsymbol{c}} \end{bmatrix}$$

$$T^a_{\boldsymbol{p},\boldsymbol{c}} = \begin{bmatrix} D_{\boldsymbol{p},\boldsymbol{c}} & 0 & \cdots & 0 \\ C_{\boldsymbol{p},\boldsymbol{c}}B_{\boldsymbol{p},\boldsymbol{c}} & D_{\boldsymbol{p},\boldsymbol{c}} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ C_{\boldsymbol{p},\boldsymbol{c}}A^{a-2}_{\boldsymbol{p},\boldsymbol{c}}B_{\boldsymbol{p},\boldsymbol{c}} & C_{\boldsymbol{p},\boldsymbol{c}}A^{a-3}_{\boldsymbol{p},\boldsymbol{c}}B_{\boldsymbol{p},\boldsymbol{c}} & \cdots & D_{\boldsymbol{p},\boldsymbol{c}} \end{bmatrix} \quad (34)$$

We can now state the EMMRC control design procedure:

Proposition 4: Let control design $K: \boldsymbol{\mathcal{P}} \to \boldsymbol{\mathcal{C}}$ be such that given $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}$, $\boldsymbol{c} = K(\boldsymbol{p}) \in \boldsymbol{\mathcal{C}}$ is the lifted realization of (9) with the filter $Q_c(q)$ and $L_c(q)$ forms (10), (11) chosen to satisfy (12). Then Assumptions 3(1) and 3(2) hold, with

$$\begin{aligned}\alpha(\boldsymbol{p},\boldsymbol{c},|I_2|,|I_3|) &= \rho(A_{\boldsymbol{p},\boldsymbol{c}})^{|I_2|}\|O^{|I_3|}_{\boldsymbol{p},\boldsymbol{c}}\|\|V_{\boldsymbol{p},\boldsymbol{c}}\|\|V^{-1}_{\boldsymbol{p},\boldsymbol{c}}\|(Y_{\boldsymbol{p}}+Y_{\boldsymbol{c}}) \\ \beta(\boldsymbol{p},\boldsymbol{c},|I_2|,|I_3|) &= \rho(A_{\boldsymbol{p},\boldsymbol{c}})^{|I_2|}\|O^{|I_3|}_{\boldsymbol{p},\boldsymbol{c}}\|\|V_{\boldsymbol{p},\boldsymbol{c}}\|\|V^{-1}_{\boldsymbol{p},\boldsymbol{c}}\|Y_{\boldsymbol{p}} \\ &\quad + \left\|\left[O^{|I_3|}_{\boldsymbol{p},\boldsymbol{c}}K^{|I_2|}_{\boldsymbol{p},\boldsymbol{c}},\ T^{|I_3|}_{\boldsymbol{p},\boldsymbol{c}}\right]\right\| \end{aligned} \quad (35)$$

where $\rho(\cdot)$ is the spectral radius, $V_{\boldsymbol{p},\boldsymbol{c}}$ contains the eigenvectors of $A_{\boldsymbol{p},\boldsymbol{c}}$, and $Y_{\boldsymbol{p}} = \left\|\left[B_{\boldsymbol{p}} - A_{\boldsymbol{p}}C^{\dagger}_{\boldsymbol{p}}D_{\boldsymbol{p}}, A_{\boldsymbol{p}}C^{\dagger}_{\boldsymbol{p}}\right]\right\|$, $Y_{\boldsymbol{c}} = \left\|\left[\left[A_{\boldsymbol{c}}B_{\boldsymbol{c}}, B_{\boldsymbol{c}}\right] - A^2_{\boldsymbol{c}}\begin{bmatrix} C_{\boldsymbol{c}} \\ C_{\boldsymbol{c}}A_{\boldsymbol{c}}\end{bmatrix}^{\dagger}\begin{bmatrix} D_{\boldsymbol{c}} & 0 \\ C_{\boldsymbol{c}}B_{\boldsymbol{c}} & D_{\boldsymbol{c}}\end{bmatrix}, A^2_{\boldsymbol{c}}\begin{bmatrix} C_{\boldsymbol{c}} \\ C_{\boldsymbol{c}}A_{\boldsymbol{c}}\end{bmatrix}^{\dagger}\right]\right\|$

Moreover, if $\boldsymbol{w}_0 = \boldsymbol{0}$, convergence follows

$$\lim_{k\to\infty}\left(\Pi_{\boldsymbol{C_c}//\boldsymbol{P_p}}\bar{\boldsymbol{w}}_0\right)(k) = C_{\boldsymbol{p},\boldsymbol{c}}\left(I - A_{\boldsymbol{p},\boldsymbol{c}}\right)^{-1}B_{\boldsymbol{p},\boldsymbol{c}} \quad (36)$$

and, if $Q_c(q) = I$, satisfies convergence requirement (8).

Proof: The proof is omitted for brevity. ■

Assumptions 3(1) and 3(2) are the biased, lifted extensions to the framework developed in [9]. The next result shows how to design RC controllers to minimize $\alpha(\cdot)$, which will later be seen to be critical to maximize robust performance.

Theorem 5: Let the RC update (9) be designed such that $\|G_{p,c}(q)\|_\infty < \gamma \leq 1$ with the poles of $G_{p,c}(q)$ having magnitude less than $\gamma^{\frac{1}{N}}$. Then

$$\rho\left(A_{\boldsymbol{p},\boldsymbol{c}}\right) < \gamma \quad (37)$$

Proof: The proof is based on application of structured singular value analysis to (33), but is omitted for brevity. ■

Note: (i) setting $\gamma = 1$ in Theorem 5 retrieves stability condition (12), and (ii) in the case of causal $Q_c(q)$, $L_c(q)$, the poles of $G_{p,c}(q)$ equal those of $P_p(q)$, $Q_c(q)$ and $L_c(q)$.

VI. Robust Performance of RC

We next establish a region of robust stability around each atomic RC controller, measured using the gap metric.

Theorem 6: Let $\boldsymbol{p},\boldsymbol{p}_1 \in \boldsymbol{\mathcal{P}}_{\mathrm{LTI}}$, and suppose the control design $K: \boldsymbol{\mathcal{P}}_{\mathrm{LTI}} \to \boldsymbol{\mathcal{C}}_{\mathrm{LTI}}$ is such that $\boldsymbol{C_c}$, $\boldsymbol{c} = K(\boldsymbol{p})$, is the lifted realization of (9) in which $Q_{\hat{c}}, L_{\hat{c}}$ are chosen to satisfy the conditions of Theorem 5. Then

$$\|\Pi_{\boldsymbol{P_p}//\boldsymbol{C_c}}\|_{\bar{\boldsymbol{w}}_0} \leq b^{-1}_{\boldsymbol{P_p},\boldsymbol{C_c}} \quad (38)$$

where

$$b_{\boldsymbol{P_p},\boldsymbol{C_c}} = \left(\left\|I - D_{\boldsymbol{p},\boldsymbol{c}}\right\| + \frac{\left\|C_{\boldsymbol{p},\boldsymbol{c}}V_{\boldsymbol{p},\boldsymbol{c}}\right\|\left\|V^{-1}_{\boldsymbol{p},\boldsymbol{c}}B_{\boldsymbol{p},\boldsymbol{c}}\right\|}{1-\gamma}\right)^{-1}. \quad (39)$$

If the gap metric, $\vec{\delta}(\boldsymbol{p},\boldsymbol{p}_1)$, between lifted plants satisfies

$$\vec{\delta}(\boldsymbol{p},\boldsymbol{p}_1) < b_{\boldsymbol{P_p},\boldsymbol{C_c}} \quad (40)$$

then the closed loop system $[\boldsymbol{P}_{\boldsymbol{p}_1}, \boldsymbol{C_c}]$ is gain stable.

Proof: The proof is omitted for brevity. ■

Theorem 6 establishes a robust stability bound for any lifted plant $\boldsymbol{p}_1$ within a ball of radius $b_{\boldsymbol{P_p},\boldsymbol{C_c}}$ around the nominal lifted plant, $\boldsymbol{p}$, as measured using the gap metric. This results also holds for the original unlifted plant descriptions $p, p_1 \in \mathcal{P}_{\mathrm{LTI}}$, since it can be shown that $\vec{\delta}(\boldsymbol{p},\boldsymbol{p}_1) \leq \vec{\delta}(p,p_1)$, and hence condition (40) can be replaced by $\vec{\delta}(p,p_1) < b_{\boldsymbol{P_p},\boldsymbol{C_c}}$.

VII. Estimators

The required EMMRC estimator properties are biased, lifted extensions of those in [9], [10], and can be stated as:

Assumption 7: Let $\lambda \in \mathbb{R}$ be given. The residual operator X factorises $X = NE$ with the norm operator N and the estimation operator E given in Table II, and:

1) (Causality): E is causal.

2) (Weak consistency): For all $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}_{LTI}$ there exists a map $\Phi_\lambda : map(\mathbb{N},\mathbb{R}^h) \to \mathbb{R}^{(m+o)N(\lambda+1)}$, such that for all $(\boldsymbol{w}_2, \bar{\boldsymbol{w}}_2, \bar{\boldsymbol{w}}_0) \in \boldsymbol{\mathcal{W}}_e \times \boldsymbol{\mathcal{W}}_e \times \boldsymbol{\mathcal{W}}_e$ and for all $k \in \mathbb{N}$,

$$\begin{aligned}\|\Phi_\lambda E(\boldsymbol{w}_2 - \bar{\boldsymbol{w}}_2 + \bar{\boldsymbol{w}}_0)(k)(\boldsymbol{p})\| &\leq \|\mathscr{R}_{\lambda,k}E(\boldsymbol{w}_2 - \bar{\boldsymbol{w}}_2 + \bar{\boldsymbol{w}}_0)(k)(\boldsymbol{p})\|, \\ \Phi_\lambda E(\boldsymbol{w}_2 - \bar{\boldsymbol{w}}_2 + \bar{\boldsymbol{w}}_0)(k)(\boldsymbol{p}) &\in \mathcal{N}^{[k-\lambda,k]}_{\boldsymbol{p},\bar{\boldsymbol{w}}_2}(w_2). \end{aligned} \quad (41)$$

where the set of weakly consistent disturbances for plant $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}_{\mathrm{LTI}}$ and observation $\boldsymbol{w}_2 = (\boldsymbol{u}_2, \boldsymbol{y}_2)^\top$ is $\mathcal{N}^{[a,b]}_{\boldsymbol{p},\bar{\boldsymbol{w}}_0}(\boldsymbol{w}_2) :=$

$$\begin{aligned}\Big\{\boldsymbol{v} \in \boldsymbol{\mathcal{W}}|_{[a,b]} \,\Big|\, \exists (\boldsymbol{u}^{\boldsymbol{p}}_0, \boldsymbol{y}^{\boldsymbol{p}}_0)^\top \in \boldsymbol{\mathcal{W}}_e : \boldsymbol{v} = \left(\mathscr{R}_{b-a,b}\boldsymbol{u}^{\boldsymbol{p}}_0, \mathscr{R}_{b-a,b}\boldsymbol{y}^{\boldsymbol{p}}_0\right) \\ \mathscr{R}_{b-a,b}\boldsymbol{P_p}\left(\boldsymbol{u}^{\boldsymbol{p}}_0 - \boldsymbol{u}_2 + \boldsymbol{u}_{\mathrm{dis}}\right) = \mathscr{R}_{b-a,b}\left(\boldsymbol{y}^{\boldsymbol{p}}_0 - \boldsymbol{y}_2 + \boldsymbol{y}_{\mathrm{ref}}\right)\Big\} \end{aligned} \quad (42)$$

where $a \leq b$, $a,b \in \mathbb{Z}$, and restriction operator $\mathscr{R}_{\sigma,t} : \mathcal{S} \to \mathbb{R}^{h(\sigma+1)}$ is given by $\mathscr{R}_{\sigma,t}\boldsymbol{v} := (\boldsymbol{v}(t-\sigma), \cdots, \boldsymbol{v}(t))$.

3) (Monotonicity): For all $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}_{\mathrm{LTI}}$, for all $k, l \in \mathbb{N}$ with $0 \leq k \leq l$ and for all $(\boldsymbol{w}_2, \bar{\boldsymbol{w}}_2, \bar{\boldsymbol{w}}_0) \in \boldsymbol{\mathcal{W}}_e \times \boldsymbol{\mathcal{W}}_e \times \boldsymbol{\mathcal{W}}_e$,

$$\|E(\boldsymbol{w}_2 - \bar{\boldsymbol{w}}_2 + \bar{\boldsymbol{w}}_0)(k)(\boldsymbol{p})\| \leq \|\mathscr{T}_k E(\boldsymbol{w}_2 - \bar{\boldsymbol{w}}_2 + \bar{\boldsymbol{w}}_0)(l)(\boldsymbol{p})\|.$$

4) (Minimality): There exists $\mu > 0$ such that for all $k \geq 0$, for $\boldsymbol{p} \in \boldsymbol{\mathcal{P}}_{\mathrm{LTI}}$ and for all $(\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2, \bar{\boldsymbol{w}}_2, \bar{\boldsymbol{w}}_0) \in \boldsymbol{\mathcal{W}} \times \boldsymbol{\mathcal{W}}_e \times \boldsymbol{\mathcal{W}}_e \times \boldsymbol{\mathcal{W}}_e \times \boldsymbol{\mathcal{W}}_e$ satisfying equations (3) and (4),

$$\|E(\boldsymbol{w}_2 - \bar{\boldsymbol{w}}_2 + \bar{\boldsymbol{w}}_0)(k)(\boldsymbol{p})\| \leq \mu\|\mathscr{T}_k \boldsymbol{w}_0\|. \quad (43)$$

5) (Continuity): For all $k \in \mathbb{N}$, $\boldsymbol{p},\boldsymbol{p}_1 \in \boldsymbol{\mathcal{P}}_{\mathrm{LTI}}$, $(\boldsymbol{w}_2, \bar{\boldsymbol{w}}_2, \bar{\boldsymbol{w}}_0) \in \boldsymbol{\mathcal{W}}_e \times \boldsymbol{\mathcal{W}}_e \times \boldsymbol{\mathcal{W}}_e$

$$\begin{aligned}&\|E(\boldsymbol{w}_2 - \bar{\boldsymbol{w}}_2 + \bar{\boldsymbol{w}}_0)(k)(\boldsymbol{p}) - E(\boldsymbol{w}_2 - \bar{\boldsymbol{w}}_2 + \bar{\boldsymbol{w}}_0)(k)(\boldsymbol{p}_1)\| \\ &\qquad \leq \vec{\delta}(\boldsymbol{p},\boldsymbol{p}_1)\|\mathscr{T}_k \boldsymbol{w}_2\|_{\bar{\boldsymbol{w}}_2} \end{aligned} \quad (44)$$

Proposition 8: The biased infinite horizon operator:

$$\begin{aligned} X^A_{\bar{\boldsymbol{w}}_0}(\boldsymbol{w}_2)(k)(\boldsymbol{p}) &= r^A_{\boldsymbol{p},\bar{\boldsymbol{w}}_0}[k] \\ &= \inf\{r \geq 0 \mid r = \|\boldsymbol{v}_0\|,\ \boldsymbol{v}_0 \in \mathcal{N}^{[0,k]}_{\boldsymbol{p},\bar{\boldsymbol{w}}_0}(\boldsymbol{w}_2)\}. \end{aligned} \quad (50)$$

where $k \in \mathbb{N}$ and $\boldsymbol{w}_2 \in \boldsymbol{\mathcal{W}}_e$, satisfies Assumption 7.

Proof: The proof is omitted for brevity. ■

Operator (50) can be efficiently implemented as follows:

Estimator

$$X : \mathcal{W}_e \to map(\mathbb{N}, map(\mathcal{P}_{\rm LTI}, \mathbb{R}^+)) \\ : \boldsymbol{w}_2 \mapsto [k \to (\boldsymbol{p} \mapsto r_{\boldsymbol{p}}[k])] \quad (45) \\ X = NE$$

Estimation Operator

$$E : \mathcal{W}_e \to map(\mathbb{N}, map(\mathcal{P}_{\rm LTI}, map(\mathbb{N}, \mathbb{R}^h))) \quad (46)$$
$$: \boldsymbol{w}_2 \mapsto \big[k \mapsto (\boldsymbol{p} \mapsto d_{\boldsymbol{p}}[k])\big] \quad (47)$$

$$d_{\boldsymbol{p}}[k] : \mathbb{N} \to map(\mathbb{N}, \mathbb{R}^h)$$
$$d_{\boldsymbol{p}}[k] = (d_{\boldsymbol{p}}[k](0), d_{\boldsymbol{p}}[k](1), \dots, d_{\boldsymbol{p}}[k](k), 0, \cdots)$$

Norm Operator

$$N : map(\mathbb{N}, map(\mathcal{P}_{\rm LTI}, map(\mathbb{N}, \mathbb{R}^h))) \\ \to map(\mathbb{N}, map(\mathcal{P}_{\rm LTI}, \mathbb{R}^+)) \quad (48)$$
$$: \big[k \mapsto (\boldsymbol{p} \mapsto d_{\boldsymbol{p}}[k])\big] \mapsto \big[k \mapsto (\boldsymbol{p} \mapsto \|d_{\boldsymbol{p}}[k]\| = r_{\boldsymbol{p}}[k])\big] \quad (49)$$

TABLE II

FACTORISATION OF THE RESIDUAL OPERATOR

Theorem 9: Suppose a Kalman filter is applied to unlifted plant $p = (A_p, B_p, C_p, 0) \in \mathcal{P}_{\rm LTI}$ with interconnections (3),(4), using equations

$$\begin{aligned} \hat{x}(t+1/2) &= \hat{x}(t) - \Sigma(t) C_p^\top [C_p \Sigma(t) C_p^\top + I]^{-1} \\ &\qquad \cdot [y_2(t) - y_{\rm ref}(t) + C_p \hat{x}(t)] \\ \Sigma(t+1/2) &= \Sigma(t) - \Sigma(t) C_p^\top [C_p \Sigma(t) C_p^\top + I]^{-1} C_p \Sigma(t) \\ \hat{x}(t+1) &= A_p \hat{x}(t+1/2) + B_p(u_2(t) - u_{\rm dis}(t)) \\ \Sigma(t+1) &= A_p \Sigma(t+1/2) A_p^\top + B_p B_p^\top, \end{aligned} \quad (51)$$

where $\hat{x} : [0, \tau] \mapsto \mathbb{R}^{n_p}$, $\tau \in \mathbb{N}$, $\Sigma : \mathbb{N} \mapsto \mathbb{R}^{n_p \times n_p}$. Then

$$r^A_{\boldsymbol{p}, \bar{\boldsymbol{w}}_0}[k] = r_p^{\rm KF}((k+1)N - 1) \quad (52)$$

where the residual $r : \mathbb{N} \to \mathbb{R}^+$ for $0 \le \tau$ is

$$r_p^{\rm KF}(\tau) = \Big[\textstyle\sum_{t=0}^{\tau} \|y_2(t) - y_{\rm ref}(t) + C_p \hat{x}(t)\|^2_{[C_p \Sigma(t) C_p^\top + I]^{-1}}\Big]^{\frac{1}{2}}.$$

Proof: This is an extension to analysis in [9], [10]. ■

Hence, from (50) and (52), estimator X can be efficiently implemented for each candidate plant $\boldsymbol{p}$ using a standard Kalman filter applied to unlifted plant p.

VIII. NOMINAL STABILITY & GAIN BOUNDS

Let $U \subset \mathcal{P}_{LTI}$ be the uncertainty set we seek to control, which contains the true plant. Let $G \subset \mathcal{P}_{LTI}$ be a suitable sampling of U specifying the available candidate plant set.

Definition 10: $C(U, K, \Delta, G, X)$ is a standard EMMRC controller if the following conditions are satisfied:

- RC design $K : \mathcal{P}_{\rm LTI} \to \mathcal{C}_{\rm LTI}$ satisfies Proposition 4.
- Δ is a delay satisfying $\forall \boldsymbol{p} \in U$ that $\Delta(\boldsymbol{p}) > 2 + \log_\gamma\Big(2\|O^2_{\boldsymbol{p},K(\boldsymbol{c})}\|\|V_{\boldsymbol{p},K(\boldsymbol{c})}\|\|V^{-1}_{\boldsymbol{p},K(\boldsymbol{c})}\|\big(Y_{\boldsymbol{p}} + Y_{K(\boldsymbol{c})}\big)\Big)$.
- Estimator $X = NE$ is defined by (45)-(52).
- Switching operator $S(X, G)$ is given by (19)-(27).
- Switching controller C is defined by (28)-(29).

The gain bounds that follow depend on the size and geometry of a 'cover' of the plant uncertainty set U, rather than on the plant uncertainty set itself. The notion of the cover is as follows. Let $H \subset \mathcal{P}_{LTI}$ be a plant set and let $\nu := map(\mathcal{P}_{\rm LTI}, \mathbb{R}^+)$ be given. Now define for $\boldsymbol{p} \in \mathcal{P}_{\rm LTI}$

$$B_{\vec{\delta}}(\boldsymbol{p}, \nu(\boldsymbol{p})) := \{\boldsymbol{p}\} \cup \big\{\boldsymbol{p}_1 \in \mathcal{P}_{\rm LTI} \mid \vec{\delta}(\boldsymbol{p}, \boldsymbol{p}_1) < \nu(\boldsymbol{p})\big\} \cap U,$$

to be the set of plants that reside within a neighbourhood of radius $\nu(\boldsymbol{p})$, as measured by $\vec{\delta}$, around $\boldsymbol{p}$. For an appropriate choice of (H, ν), the union of the corresponding neighbourhoods in U then leads to a cover for U, so that

Definition 11: (H, ν) is a cover for uncertainty set U if $U \subset R$ where $R := \cup_{\boldsymbol{p} \in H} B_{\vec{\delta}}(\boldsymbol{p}, \nu(\boldsymbol{p}))$.

The main result that follows gives gain bounds for the interconnection of 'true' plant $\boldsymbol{p}_*$ with the EMMRC structure.

Theorem 12: Let $\boldsymbol{p}_* \in U \subset \mathcal{P}_{\rm LTI}$ where U is an uncertainty set with finite cover (H, ν). Suppose $C(U, K, \Delta, G, X)$ is a standard EMMRC scheme with $G \subset U$. Suppose $(\boldsymbol{w}_0, \boldsymbol{w}_1, \boldsymbol{w}_2, \bar{\boldsymbol{w}}_0)$ satisfy (3),(4), and

$$\exists\, \boldsymbol{p} \in G, \quad \vec{\delta}(\boldsymbol{p}, \boldsymbol{p}_*) < \varepsilon\ \chi_\nu(H, \nu) \quad (53)$$

for $\varepsilon > 0$. If

$$\pi(U, H, \nu, \varepsilon, \boldsymbol{p}_*) > 0, \quad (54)$$

then:

$$\|\mathscr{T}_k \boldsymbol{w}_2\|_{\bar{\boldsymbol{w}}_2^{\boldsymbol{p}_*}} \le \hat{\gamma}(U, H, \nu, \varepsilon, \boldsymbol{p}_*) \|\boldsymbol{w}_0\| \quad (55)$$

where the ideal control input for the true plant is

$$\bar{\boldsymbol{w}}_2^{\boldsymbol{p}_*} = \lim_{k \to \infty} \big[\Pi_{\boldsymbol{C}_{K(\boldsymbol{p}_*)} // \boldsymbol{P}_{\boldsymbol{p}_*}} \bar{\boldsymbol{w}}_0\big](k),$$

and χ_ν, π, $\hat{\gamma}$ are defined in Table III.

Proof: This follows the structure of the unbiased case ([10], Theorem 7) but embeds a range of biases on the various signals, estimators, metrics and operators. ■

A. The EMMRC Design Procedure

Theorem 12 enables transparent selection of model set G: (54) specifies a maximum value of $\rho = \varepsilon \chi_\nu(H, \nu)$, which is used in (53) to specify a maximum distance between elements in G. Therefore G can be designed by constructing a covering of U by gap balls of radius ρ with centre $\boldsymbol{p}_i \in G$. Since $\vec{\delta}(\boldsymbol{p}, \boldsymbol{p}_1) \le \vec{\delta}(p, p_1)$, this can be designed transparently in the unlifted plant space.

Both (54) and gain bound (55) are invariant to the number of elements in either U or G, but instead scale (exponentially) with the number of elements in H. The use of (H, ν) thereby embeds substantial improvement on previous bounds within the field of multiple model adaptive control, which scaled exponentially with the number of elements in the candidate plant set (i.e. with G).

Control design K must associate each $\boldsymbol{p} \in G$ with a stable controller. To choose candidate model set G either:

1. Include as many plant models as hardware permits, to ensure (54) is satisfied (this does not degrade performance bounds since they depend only on the cover), or
2. Define (H, ν) as a minimal controller set that stablises U using the robust stability margin of Theorem 6. Then compute a more conservative set G using equations (53) and (54).

B. Example EMMRC Application

Consider electrical stimulation of muscle for stroke rehabilitation, considered in, for example [12]. The angle of a joint in response to electrical stimulation can be modeled by

$$P_{p_*}(q) = \frac{(q - e^{T_s/a})}{q^2 - 1.845q + 0.86} \cdot \frac{0.015}{1 - e^{T_s/a}} \tag{56}$$

where a is uncertain, but known to be within $[-0.01, 0.01]$. Selecting sampling time $T_s = 0.01$, and setting $N = 125$, defines uncertainty set $U \subset \boldsymbol{\mathcal{P}}_{\mathrm{LTI}}$. The update employed by each atomic EMMRC controller is chosen to be gradient-based RC [13]. Control design procedure $\boldsymbol{c} = K(\boldsymbol{p})$ is then defined by Proposition 4 with $L_c(q) = bP_p(q)^*$ and $Q_c(q)$ a zero-phase filter with cut-off frequency 3 Hz. Choosing $b = 0.5$ then satisfies Theorem 5 with $\gamma = 0.76$. The robust stability margin of Theorem 6 (equation (40)) shows that stabilisation of U can be achieved using a minimum of two controllers. In particular, a suitable set $\{P_{p_1}, P_{p_2}\}$ is

$$\frac{(q - e^{-2})\frac{0.015}{1-e^{-2}}}{q^2 - 1.845q + 0.86} \cdot \frac{0.015}{1 - e^{-2}}, \quad \frac{(q - e^{2})\frac{0.015}{1-e^{2}}}{q^2 - 1.845q + 0.86} \cdot \frac{0.015}{1 - e^{2}}.$$

We hence define cover set $H = \{\boldsymbol{P}_{\boldsymbol{p}_1}, \boldsymbol{P}_{\boldsymbol{p}_2}\}$. Choosing $\Delta = 10$, equations (53), (54) yield the radius for G

$$\rho \le (2^{1/2}(1 + \bar{\gamma}_1^2(H, U))\gamma_4(U))^{-1} = 4.376 \times 10^{-4}.$$

The radius of the uncertainty set U is 0.1435, hence 327 estimators are sufficient. Plants with parameter $a \in \{-9.94 \times 10^{-3}, -9.88 \times 10^{-3}, \cdots, 0\}$ are associated with controller $K(\boldsymbol{p}_1)$ and plants with parameter $a \in \{0.06 \times 10^{-3}, \cdots, 9.88 \times 10^{-3}, 9.94 \times 10^{-3}\}$ are associated with controller $K(\boldsymbol{p}_2)$. Equation (55) guarantees the gain bound

$$\|\mathscr{T}_k \boldsymbol{w}_2\|_{\bar{\boldsymbol{w}}_2^{\boldsymbol{p}_*}} \le (8.45 \times 10^4)\|\boldsymbol{w}_0\|.$$

is satisfied when EMMRC is applied to the true plant.

Controller bounds with nominal plant

$$\alpha_{OP}(U) = 4 \sup_{\boldsymbol{p}_1 \in U} \alpha^2(\boldsymbol{p}_1, K(\boldsymbol{p}_1), \Delta(\boldsymbol{p}_1) - \sigma, \sigma)$$

$$\beta_{OP}(U) = 4 \sup_{\substack{\Delta(\boldsymbol{p}_1) \le x \le 2\Delta(\boldsymbol{p}_1) \\ \boldsymbol{p}_1 \in U}} \beta^2(\boldsymbol{p}_1, K(\boldsymbol{p}_1), x - \sigma, \sigma)$$

$$\alpha_{OS}(U) = 4 \sup_{\substack{\Delta(\boldsymbol{p}_1) \le x \le 2\Delta(\boldsymbol{p}_1) \\ \boldsymbol{p}_1 \in U}} \alpha^2(\boldsymbol{p}_1, K(\boldsymbol{p}_1), 0, x - \sigma)$$

$$\beta_{OS}(U) = 4 \sup_{\substack{\Delta(\boldsymbol{p}_1) \le x \le 2\Delta(\boldsymbol{p}_1) \\ \boldsymbol{p}_1 \in U}} \beta^2(\boldsymbol{p}_1, K(\boldsymbol{p}_1), 0, x - \sigma)$$

$$\gamma_3(U) = (1 + \alpha_{OS}^{\frac{1}{2}}(U)) \left(\frac{\alpha_{OP}(U)}{1 - \alpha_{OP}(U)} \right)^{\frac{1}{2}} + \alpha_{OS}^{\frac{1}{2}}(U) \tag{57}$$

$$\gamma_4(U) = (1 + \alpha_{OS}^{\frac{1}{2}}(U)) \left(\frac{\beta_{OP}(U)}{1 - \alpha_{OP}(U)} \right)^{\frac{1}{2}} + \beta_{OS}^{\frac{1}{2}}(U) \tag{58}$$

Controller bounds with real plant

$$\begin{aligned} \gamma_1(\boldsymbol{p}, \boldsymbol{p}_*) &= 1 + \sup_{\Delta(\boldsymbol{p}) \le x \le 2\Delta(\boldsymbol{p})} \alpha(\boldsymbol{p}_*, K(\boldsymbol{p}), 0, x) \\ \gamma_2(\boldsymbol{p}, \boldsymbol{p}_*) &= \sup_{\Delta(\boldsymbol{p}) \le x \le 2\Delta(\boldsymbol{p})} \beta(\boldsymbol{p}_*, K(\boldsymbol{p}), 0, x), \\ \bar{\gamma}_i(Q_1, Q_2) &= \sup_{\boldsymbol{p}_1 \in Q_1} \sup_{\boldsymbol{p}_2 \in Q_2} \gamma_i(\boldsymbol{p}_1, \boldsymbol{p}_2), \ i = 1, 2 \end{aligned}$$

Gain bound

$$\hat{\gamma}(U, H, \nu, \varepsilon, \boldsymbol{p}_*) = \left(\sum_{\boldsymbol{p} \in H} \gamma_2(\boldsymbol{p}, \boldsymbol{p}_*) + \frac{\gamma_4(U)\eta(H, \nu, \varepsilon, \boldsymbol{p}_*)}{\pi(U, H, \nu, \varepsilon, \boldsymbol{p}_*)} \right) \times \phi(U, H, \nu, \varepsilon, \boldsymbol{p}_*) \tag{59}$$

where

$$\pi(U, H, \nu, \varepsilon, \boldsymbol{p}_*) = 1 - 2^{\frac{1}{2}} \varepsilon \chi_\nu(H, v)(1 + \bar{\gamma}_1^2(H, U))\gamma_4(U)$$

$$\eta(H, \nu, \varepsilon, \boldsymbol{p}_*) = 2^{\frac{1}{2}} (\mu + \varepsilon \chi_\nu(H, \nu) \bar{\gamma}_2(H, \boldsymbol{p}_*)(1 + \bar{\gamma}_1(H, \boldsymbol{p}_*)))$$

$$\phi(U, H, \nu, \varepsilon, \boldsymbol{p}_*) = \left(\frac{1 + \gamma_3(U)}{\pi(U, H, \nu, \varepsilon, \boldsymbol{p}_*)} \right)^{|H|} \prod_{\boldsymbol{p} \in H} \gamma_1(\boldsymbol{p}, \boldsymbol{p}_*)$$

$$\chi_\nu(H, \nu) = 2 \sup_{\boldsymbol{p} \in H} \nu(\boldsymbol{p}) \text{ where } \nu : \boldsymbol{\mathcal{P}}_{\mathrm{LTI}} \to \mathbb{R}^+ \tag{60}$$

TABLE III

FUNCTIONS SPECIFYING THE EMMRC GAIN BOUND.

IX. CONCLUSIONS

A multiple model switched adaptive approach to RC has been developed, providing guaranteed robustness and performance bounds for an unknown true plant. By applying gap metric based analysis to RC, the EMMRC framework addresses the problems of performance degradation and instability that have been key issues in RC research.

REFERENCES

[1] S. Hara, Y. Yamamoto, T. Omata, and M. Nakano, "Repetitive control system: a new type servo system for periodic exogenous signals," *IEEE Transactions on Automatic Control*, vol. AC-33, pp. 402–411, 1988.

[2] G. Pipeleers, B. Demeulenaere, and S. Sewers, "Robust high order repetitive control: Optimal performance trade offs," *Automatica*, vol. 44, pp. 2628–2634, 2008.

[3] M. Steinbuch, S. Weiland, and T. Singh, "Design of noise and period-time robust high order repetitive control, with application to optical storage," *Automatica*, vol. 43, pp. 2086–2095, 2007.

[4] G. Fuentes, J. Miras, and R. Castello, "A survey of repetitive control in varying frequency conditions," *Ingenieria e Investigacion*, vol. 31, pp. 29–37, 08 2011.

[5] W.-S. Yao, "Adaptive repetitive control with two nonsynchronized sampling," *Journal of Dynamic Systems, Measurement, and Control*, vol. 137, pp. 1–8, 2015.

[6] Y. Xudong, "Nonlinear adaptive learning control with disturbance of unknown periods," *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1269–1273, 2012.

[7] M. Yu and D. Huang, "Switching adaptive learning control for non-linearly parameterized systems with disturbance of unknown periods," *International journal of robust and nonlinear control*, vol. DOI: 10.1002/rnc.3144, 2014.

[8] T. T. Georgiou and M. C. Smith, "Robustness analysis of nonlinear feedback systems: An input-output approach," *IEEE Transactions on Automatic Control*, vol. 42, no. 9, pp. 1200–1221, 1997.

[9] D. Buchstaller and M. French, "Robust stability and performance for multiple model adaptive control: Part i - the framework," *IEEE Transactions on Automatic Control*, vol. 61, no. 3, pp. 677–692, 2016.

[10] ——, "Robust stability and performance for multiple model adaptive control: Part ii - gain bounds," *IEEE Transactions on Automatic Control*, vol. 61, no. 3, pp. 693–708, 2016.

[11] G. Pipeleers, B. Demeulenaere, and J. Swevers, *Optimal Linear Controller Design for Periodic Inputs*. Spring-Verlag, London, U. K., 2009.

[12] O. W. Brend, C. T. Freeman, and M. French, "Multiple model adaptive control of functional electrical stimulation," *IEEE Transactions on Control System Technology*, vol. 23, no. 5, pp. 1901–1913, 2015.

[13] J. J. Hätönen, C. T. Freeman, D. H. Owens, P. L. Lewin, and E. Rogers, "A gradient-based repetitive control algorithm combining ILC and pole placement," *European Journal of Control*, vol. 12, no. 3, pp. 278–292, 2006.