Second-order perturbation theory: the problem of infinite mode coupling

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Second-order self-force computations, which will be essential in modeling extreme-mass-ratio inspirals, involve two major new difficulties that were not present at first order. One is the problem of large scales, discussed in [Phys. Rev. D 92, 104047 (2015)]. Here we discuss the second difficulty, which occurs instead on small scales: if we expand the field equations in spherical harmonics, then because the first-order field contains a singularity, we require an arbitrarily large number of first-order modes to accurately compute even a single second-order mode. This is a generic feature of nonlinear field equations containing singularities, allowing us to study it in the simple context of a scalar toy model in flat space. Using that model, we illustrate the problem and demonstrate a robust strategy for overcoming it.

I. INTRODUCTION AND SUMMARY

Gravitational self-force theory [? ? ?] has proven to be an important tool in efforts to model compact binary inspirals. It is currently the only viable method of accurately modeling extreme-mass-ratio inspirals (EMRIs) [? ?], it is a potentially powerful means of modeling intermediate-mass-ratio inspirals, and by interfacing with other methods, it can even be used to validate and improve models of comparable-mass binaries [? ? ? ? ? ?]. However, the self-force model is based on an asymptotic expansion in the limit $m/M \to 0$, where m and M are the two masses in the system. The model's accuracy is hence limited by the perturbative order at which it is truncated. Unfortunately, although numerous concrete self-force computations of binary dynamics have been performed (see the reviews [? ? ?] and Refs. [? ? ? ? ? ? ?] for some more recent examples), until now they have been restricted to first perturbative order, limiting their capacity to assist other models and rendering them insufficiently accurate to model EMRIs [?].

In recent years, substantial effort has gone into overcoming this limitation [? ? ? ? ? ? ? ? ? ? ? ?]. The foundations of second-order self-force theory are now established [? ? ? ?], the key analytical ingredients are in place [?], and at least in some scenarios, practical formulations of the second-order field equations have been developed [? ? ?]. However, concrete solutions to the field equations have remained elusive.

There have been two major obstacles to finding these solutions. The first is the problem of large scales, described in Ref. [?], which manifests in spurious unbounded growth and ill-defined retarded integrals. As demonstrated in a simple toy model in Ref. [?], this obstacle can be overcome by utilizing multiscale and matched-expansion techniques; full descriptions of these techniques in the gravitational problem will be given in future papers. The second major obstacle arises in the opposite extreme: rather than a problem on large scales, it is a problem on small ones.

To introduce the problem, we refer to the Einstein equations through second order, which we can write as

$$\delta G_{\mu\nu}[h^1] = 8\pi T_{\mu\nu}, \tag{1}$$

$$\delta G_{\mu\nu}[h^2] = -\delta^2 G_{\mu\nu}[h^1, h^1]. \tag{2}$$

Here the metric has been expanded as $g_{\mu\nu} + (m/M)h^1_{\mu\nu} + (m/M)^2 h^2_{\mu\nu} + \mathcal{O}(m^3)$; $T_{\mu\nu}$ is the stress-energy of a point particle, representing the leading approximation to the smaller object m on the background $g_{\mu\nu}$; $\delta G_{\mu\nu}$ is the linearized Einstein tensor (in some appropriate gauge [?]); and $\delta^2 G_{\mu\nu}[h^1, h^1]$ is the second-order Einstein tensor, which has the schematic form $h^1\partial^2 h^1 + \partial h^1 \partial h^1$. Because $h^1_{\mu\nu}$ is singular at the particle, Eq. (??) is only valid at points away from the particle's worline [?], but that suffices for our purposes here.

Equations (??)–(??) can in principle be solved in four dimensions (4D). However, in practice it is desirable to reduce their dimension by decomposing them into a basis of harmonics. For illustration let us use some basis of tensor harmonics $Y^{ilm}_{\mu\nu}$; here we use the notation of Barack-Lousto-Sago [? ?], with $i = 1, \ldots, 10$, but the particular choice of basis, whether spherical or spheroidal (for example), is immaterial. We have

$$h^n_{\mu\nu} = \sum_{ilm} h^n_{ilm} Y^{ilm}_{\mu\nu} \tag{3}$$

and

$$\delta G_{ilm}[h^1] = 8\pi T_{ilm}, \tag{4}$$

$$\delta G_{ilm}[h^2] = -\delta^2 G_{ilm}[h^1, h^1]. \tag{5}$$

Now consider the source term $\delta^2 G_{ilm}$. Substituting the expansion (??) into $\delta^2 G_{\mu\nu}$ leads to a mode-coupling formula with the schematic form

$$\delta^2 G_{ilm} = \sum_{\substack{i_1 l_1 m_1 \\ i_2 l_2 m_2}} \mathscr{D}^{i_1 l_1 m_1 i_2 l_2 m_2}_{ilm} [h^1_{i_1 l_1 m_1}, h^1_{i_2 l_2 m_2}], \tag{6}$$