MOD-p HOMOTOPY DECOMPOSITIONS OF LOOPED STIEFEL MANIFOLDS

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Abstract. Let $W_{n,k}$ be the Stiefel manifold $U(n)/U(n-k)$. For odd primes p and for $k \leq (p-1)(p-2)$, we give a homotopy decomposition of the based loop space $\Omega W_{n,k}$ as a product of $p-1$ factors, each of which is the based loops on a finite H-space. Similar decompositions are obtained for $Sp(n)/Sp(n-k)$ and $O(n)/O(n-k)$ and upper bounds on the homotopy exponents are obtained.

1. Introduction

Let $V_{n,k} = O(n)/O(n-k)$, $W_{n,k} = U(n)/U(n-k)$ and $X_{n,k} = Sp(n)/Sp(n-k)$ be the real, complex and quaternionic Stiefel manifolds respectively. The topology of Stiefel manifolds is of long-standing interest, and many of their properties have been determined. James' book [J2] on the subject is an excellent exposition of what was done up to the late 1970's. In terms of homotopy decompositions, Miller [M] gave stable decompositions of $W_{n,k}$ and $X_{n,k}$, which were later refined in different ways by Crabb [C] and Yang [Yan]. Unstably, a product decomposition of $W_{n,k}$ or $X_{n,k}$ is unlikely since, in general, Stiefel manifolds are not H-spaces, even when localized at an odd prime. Nevertheless, Hemmi [He] and Yamaguchi [Yam] have determined many cases when $W_{n,k}$ and $X_{n,k}$ are homotopy equivalent to a product of odd dimensional spheres when localized at an odd prime.

It is more reasonable to ask for a product decomposition of the loop spaces $\Omega W_{n,k}$ and $\Omega X_{n,k}$. Mimura, Nishida and Toda's [MNT2] work on mod-p homotopy decompositions of simple, compact Lie groups may lead to mod-p decompositions of $\Omega W_{n,k}$ and $\Omega X_{n,k}$. However, the factors would only be opaquely identified as the homotopy fibres of maps between various factors of the Lie groups. Recently, using a different approach, Beben [B] and Grbić and Zhao [GZ] gave p-local loop space decompositions of $\Omega W_{n,k}$ for $n \leq (p-1)(p-2)$ and $\Omega X_{n,k}$ for $n \leq (p-1)(p-2)/2$, where the factors are better identified as the loops on finite H-spaces.

In this paper we greatly improve on Beben's and Grbić and Zhao's results. We show that if $k \leq (p-1)(p-2)$ then for *any* n there is p-local loop space decomposition of $\Omega W_{n,k}$ as a product of loop spaces on finite H-spaces, and if $k \leq (p-1)(p-2)/2$ then for *any* n there is a p-local loop space decomposition of $\Omega X_{n,k}$ as a product of loop spaces on finite H-spaces.

To state our results explicitly, we introduce some notation. From now on, assume that all spaces and maps have been localized at an odd prime p, and take homology with mod-p coefficients. Recall

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that there is a canonical map $\Sigma\mathbb{C}P^{n-1} \longrightarrow SU(n)$ which induces the inclusion of the generating set in homology. Let $\mathbb{C}P_{n,k} = \mathbb{C}P^{n-1}/\mathbb{C}P^{n-k-1}$ be the stunted projective space. As will be described later, there is an isomorphism $H_*(W_{n,k}) \cong \Lambda(\widetilde{H}_*(\Sigma\mathbb{C}P_{n,k}))$ and a homotopy equivalence $\Sigma\mathbb{C}P_{n,k} \simeq \bigvee_{i=2}^{p} C^i_{n,k}$, where $H_*(C^i_{n,k})$ consists of those elements in $H_*(\Sigma\mathbb{C}P_{n,k})$ in degress of the form $2i-1+2k(p-1)$ for $k \geq 0$.

Theorem 1.1. *Localize at an odd prime p. If $k \leq (p-1)(p-2)$ then there is a homotopy equivalence*

$$\Omega W_{n,k} \simeq \prod_{i=2}^{p} \Omega B^i_{n,k}$$

where $B^i_{n,k}$ is a finite H-space and $H_*(B^i_{n,k}) \cong \Lambda(\widetilde{H}_*(C^i_{n,k}))$.

Similarly, James constructed a quasi-projective space Q^n and a map $Q^n \longrightarrow Sp(n)$ which induces the inclusion of the generating set in homology. Let $Q_{n,k} = Q^n/Q^{n-k}$ be the stunted quasi-projective space. Then there is an isomorphism $H_*(X_{n,k}) \cong \Lambda(\widetilde{H}_*(Q_{n,k}))$ and a homotopy equivalence $Q_{n,k} \simeq \bigvee_{i=2}^{p} R^i_{n,k}$, where $H_*(R^i_{n,k})$ consists of those elements in $H_*(Q_{n,k})$ in degress of the form $4i-1+2k(p-1)$ for $k \geq 0$.

Theorem 1.2. *Localize at an odd prime p. If $k \leq (p-1)(p-2)/2$ then there is a homotopy equivalence*

$$\Omega X_{n,k} \simeq \prod_{i=2}^{(p-1)/2} \Omega D^i_{n,k}$$

where $D^i_{n,k}$ is a finite H-space and $H_*(D^i_{n,k}) \cong \Lambda(\widetilde{H}_*(R^i_{n,k}))$.

In the next to last section of the paper we combine Theorem 1.2 with Harris' odd primary decompositions $SO(2n+1) \simeq Sp(n)$ and $SO(2n) \simeq S^{2n-1} \times Sp(n-1)$ in order to give analogous odd primary homotopy decompositions for $\Omega V_{n,k}$. Finally, in the last section of the paper we use the decompositions in Theorems 1.1 and 1.2 in order to deduce upper bounds on the homotopy exponents of $V_{n,k}$, $W_{n,k}$ and $X_{n,k}$ for appropriate values of k.

2. Preliminary information on Stiefel manifolds

The material in this section will be valid for either $W_{n,k}$ or $X_{n,k}$, so we will denote them commonly by $Y_{n,k}$. There is a canonical fibration

$$Y_{n-1,k-1} \xrightarrow{\epsilon} Y_{n,k} \xrightarrow{\pi} S^{2dn-1}$$

where $d = 1$ in the complex case and $d = 2$ in the quaternionic case. Fixing k, James [J1] showed that there is an integer $m > k$ with the property that the map $Y_{m,k} \xrightarrow{\pi} S^{2dm-1}$ has a cross-section $\theta\colon S^{2dm-1} \longrightarrow Y_{m,k}$. Using this, Hemmi [He] defined the map

$$J_k\colon Y_{n,k} \longrightarrow \Omega^{2dm} Y_{m+n,k}$$

as the adjoint of the composite

$$S^{2dm} \wedge Y_{n,k} \simeq S^{2dm-1} * Y_{n,k} \xrightarrow{\theta * 1} Y_{m,k} * Y_{n,k} \xrightarrow{h} Y_{m+n,k},$$

where h is James' intrinsic join. Then in the proof of [He, Lemma 3.7 (ii)], he showed the following.

Lemma 2.1. *Let $Y_{n,k}$ be either $W_{n,k}$ or $X_{n,k}$. There is a homotopy fibration diagram*

$$\begin{array}{ccccc}
Y_{n-1,k-1} & \xrightarrow{\epsilon} & Y_{n,k} & \xrightarrow{\pi} & S^{2dn-1} \\
\big\downarrow{\scriptstyle J_{k-1}} & & \big\downarrow{\scriptstyle J_k} & & \big\downarrow{\scriptstyle J_1} \\
\Omega^{2dm} Y_{m+n-1,k-1} & \xrightarrow{\Omega^{2dm}\epsilon} & \Omega^{2dm} Y_{m+n,k} & \xrightarrow{\Omega^{2dm}\pi} & \Omega^{2dm} S^{2d(m+n)-1}
\end{array}$$

where J_1 is the adjoint of the identity map on $S^{2d(m+n)-1}$. □

We wish to determine the connectivity of the homotopy fibre of the map $Y_{n,k} \xrightarrow{J_k} \Omega^{2dm} Y_{m+n,k}$. Let $E^2 \colon S^{2n-1} \longrightarrow \Omega^2 S^{2n+1}$ be the double suspension, which is adjoint to the identity map on S^{2n+1}. It is well known that the homotopy fibre of E^2 is $(2np-4)$-connected.

Lemma 2.2. *The homotopy fibre of the map $Y_{n,k} \xrightarrow{J_k} \Omega^{2dm} Y_{m+n,k}$ is $(2(d(n-k)+1)p-4)$-connected.*

Proof. Consider the homotopy fibration diagram in Lemma 2.1. Observe that J_1 is the composite $\Omega^{2dm-2}E^2 \circ \cdots \circ E^2$. Therefore, the connectivity of the homotopy fibre of E^2 implies that the homotopy fibre of J_1 is $(2(d(m+n)-1)p-4)$-connected. This implies that the homotopy fibres of J_{k-1} and J_k are homotopy equivalent in dimensions strictly less than $2(d(m+n)-1)p-4$. Downward induction on k then implies that the homotopy fibre of J_k is homotopy equivalent to that of $S^{2d(n-k)+1} \xrightarrow{J_1} \Omega^{2dm} S^{2d(m+n-k)+1}$ in dimensions strictly less than $2(d(n-k+1)+1)p-4$. Note this is map J_1 is different in dimension from the previous one, but as before, J_1 is an iterated suspension, so its homotopy fibre is $(2(d(n-k)+1)p-4)$-connected. Therefore, the homotopy fibre of J_k has the same connectivity. □

3. Preliminary information on stunted projective spaces

There is a homeomorphism $U(n)/U(n-k) \cong SU(n)/SU(n-k)$, so we may regard $W_{n,k}$ as $SU(n)/SU(n-k)$. Recall that there is a Hopf algebra isomorphism $H_*(SU(n)) \cong \Lambda(x_3, \ldots, x_{2n-1})$ where $|x_{2i-1}| = 2i-1$, and the quotient map $SU(n) \longrightarrow W_{n,k}$ induces a projection of coalgebras onto $H_*(W_{n,k}) \cong \Lambda(x_{2(n-k)+1}, \ldots, x_{2n-1})$. It is well known that there is a map

$$\gamma_n \colon \Sigma\mathbb{C}P^{n-1} \longrightarrow SU(n)$$

which induces the inclusion of the primitive generating set in homology, and this map is natural in the sense that there is a commutative diagram

$$\begin{array}{ccc} \Sigma\mathbb{C}P^{n-k} & \longrightarrow & \Sigma\mathbb{C}P^{n-1} \\ \downarrow{\scriptstyle \gamma_{n-k-1}} & & \downarrow{\scriptstyle \gamma_n} \\ SU(n-k) & \longrightarrow & SU(n). \end{array}$$

Notice that the homology property of γ_n implies that there is a Hopf algebra isomorphism $H_*(SU(n)) \cong \Lambda(\widetilde{H}_*(\Sigma\mathbb{C}P^{n-1}))$.

Let $\mathbb{C}P_{n,k} = \mathbb{C}P^{n-1}/\mathbb{C}P^{n-k-1}$ be the stunted projective space. The previous diagram implies that there is an induced map

$$\gamma_{n,k} \colon \Sigma\mathbb{C}P_{n,k} \longrightarrow SU(n)/SU(n-k) = W_{n,k}.$$

Since γ_n induces an inclusion onto the generating set in homology and the quotient map $SU(n) \longrightarrow W_{n,k}$ is a coalgebra projection in homology, we immediately have the following.

Lemma 3.1. *There is a coalgebra isomorphism $H_*(W_{n,k}) \cong \Lambda(\widetilde{H}_*(\Sigma\mathbb{C}P_{n,k}))$ and $\gamma_{n,k}$ induces the inclusion of the primitives.* □

Next, by [MNT1, Corollary 9.5], there is a homotopy equivalence

$$\Sigma\mathbb{C}P^{n-1} \simeq \bigvee_{i=2}^{p} A_i$$

where $H_*(A_i)$ consists of all those elements in $H_*(\Sigma\mathbb{C}P^{n-1})$ in degrees of the form $2i-1+2k(p-1)$ for $k \geq 0$. Since the map $\mathbb{C}P^{n-k-1} \longrightarrow \mathbb{C}P^{n-1}$ is homotopy equivalent to the inclusion of the $(2(n-k))$-skeleton, the homotopy decompositions for $\Sigma\mathbb{C}P^{n-k-1}$ and $\Sigma\mathbb{C}P^{n-1}$ can be made compatible as follows. For $2 \leq i \leq p$, let A_i' be the $(2(n-k)+1)$-skeleton of A_i and let $j_i \colon A_i' \longrightarrow A_i$ be the skeletal inclusion. Then by connectivity the composite $\bigvee_{i=2}^{p} A_i' \xrightarrow{\bigvee_{i=2}^{p} j_i} \bigvee_{i=2}^{p} A_i \xrightarrow{\simeq} \Sigma\mathbb{C}P^{n-1}$ lifts through the skeletal inclusion $\Sigma\mathbb{C}P^{n-k-1} \longrightarrow \Sigma\mathbb{C}P^{n-1}$ and this lift induces an isomorphism in homology and so is a homotopy equivalence. That is, there is a homotopy commutative diagram

$$\begin{array}{ccc} \bigvee_{i=2}^{p} A_i' & \xrightarrow{\bigvee_{i=2}^{p} j_i} & \bigvee_{i=2}^{p} A_i \\ \downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle \simeq} \\ \Sigma\mathbb{C}P^{n-k-1} & \longrightarrow & \Sigma\mathbb{C}P^{n-1}. \end{array}$$

For $2 \leq i \leq p$, let $C_{n,k}^i = A_i/A_i'$. Then by taking homotopy cofibres in the previous diagram we obtain the following.

Lemma 3.2. *There is a homotopy equivalence $\Sigma\mathbb{C}P_{n,k} \simeq \bigvee_{i=2}^{p} C_{n,k}^i$ where $H_*(C_{n,k}^i)$ consists of all those elements in $H_*(\Sigma\mathbb{C}P_{n,k})$ in degrees of the form $2i-1+2k(p-1)$ for $k \geq 0$.* □

Combining Lemmas 3.1 and 3.2 we obtain the following.

Lemma 3.3. *The composite* $\bigvee_{i=2}^{p} C^i_{n,k} \xrightarrow{\simeq} \Sigma\mathbb{C}P_{n,k} \xrightarrow{\gamma_{n,k}} W_{n,k}$ *induces the inclusion of the primitives in homology. In particular, there is a coalgebra isomorphism* $H_*(W_{n,k}) \cong \otimes_{i=2}^{p}\Lambda(\widetilde{H}_*(C^i_{n,k}))$. □

Similarly, there is a Hopf algebra isomorphism $H_*(Sp(n)) \cong \Lambda(x_3,\ldots,x_{4n-1})$, and James [J2] showed that there is a quasi-projective space Q_n and a map $\delta_n\colon Q_n \longrightarrow Sp(n)$ which induces the inclusion of the primitive generating set in homology. This inclusion is compatible with the inclusion of $Sp(n-k)$ into $Sp(n)$ so we obtain a stunted quasi-projective space $Q_{n,k} = Q_n/Q_k$ and a map $\delta_{n,k}\colon Q_{n,k} \longrightarrow X_{n,k}$ which induces the inclusion of the primitives in homology. By [MNT1], there is a homotopy equivalence $Q_n \simeq \bigvee_{i=2}^{(p-1)/2} P_i$ where $H_*(P_i)$ consists of those elements in $H_*(Q_n)$ in degrees of the form $4i-1+2k(p-1)$ for $k \geq 0$, and for skeletal reasons this homotopy equivalence is compatible with the inclusion of Q_{n-k} into Q_n. So if P'_i is the $(4(n-k)+1)$-skeleton of P_i and $R^i_{n,k} = P_i/P'_i$ we obtain the analogue of Lemma 3.3.

Lemma 3.4. *The composite* $\bigvee_{i=2}^{(p-1)/2} R^i_{n,k} \xrightarrow{\simeq} Q_{n,k} \xrightarrow{\delta_{n,k}} X_{n,k}$ *induces the inclusion of the primitives in homology. In particular, there is a coalgebra isomorphism* $H_*(X_{n,k}) \cong \otimes_{i=2}^{(p-1)/2}\Lambda(\widetilde{H}_*(R^i_{n,k}))$. □

4. Loop space decompositions of Stiefel manifolds

In what follows, we will describe a homotopy decomposition for $\Omega W_{n,k}$ and then indicate alterations for $\Omega X_{n,k}$. For $2 \leq i \leq p$, let s_i be the composite

$$s_i\colon C^i_{n,k} \hookrightarrow \bigvee_{i=2}^{p} C^i_{n,k} \xrightarrow{\simeq} \Sigma\mathbb{C}P_{n,k} \xrightarrow{\gamma_{n,k}} W_{n,k}.$$

Ideally, one would like to: (i) construct a space $B^i_{n,k}$ with the property that $H_*(B^i_{n,k}) \cong \Lambda(\widetilde{H}_*(C^i_{n,k}))$; (ii) extend s_i to a map $t_i\colon B^i_{n,k} \longrightarrow W_{n,k}$; and (iii) multiply the maps t_i together to obtain a map $\prod_{i=2}^{p} B^i_{n,k} \longrightarrow W_{n,k}$ which, by Lemma 3.3, induces an isomorphism in homology and so is a homotopy equivalence. However, $W_{n,k}$ is not an H-space in general so step (iii) is likely out of the question, and steps (i) and (ii) may also be out of reach in general. Instead, we will carry out steps (i) and (ii) for a range of values of k, and then loop in order to multiply.

To begin, we construct the space $B^i_{n,k}$ for a range of values of k. Cohen and Neisendorfer [CN] gave a construction of finite p-local H-spaces satisfying many useful properties. The ones we need are listed below.

Theorem 4.1. *Fix a prime* p. *Let* A *be a CW-complex consisting of* ℓ *odd dimensional cells, where* $\ell < p-1$. *Then there is a finite* H*-space* $M(A)$ *with the following properties:*

- (a) *there is an isomorphism of Hopf algebras* $H_*(M(A)) \cong \Lambda(\widetilde{H}_*(A))$;
- (b) *there is a map* $j\colon A \longrightarrow M(A)$ *which induces the inclusion of the generating set in homology.* □

Suppose that $k \leq (p-1)(p-2)$. The number k corresponds to the number of generators in $H_*(W_{n,k})$. By Lemma 3.1, this corresponds to the number of cells in $\Sigma\mathbb{C}P_{n,k}$. So by Lemma 3.2, each $C^i_{n,k}$ has at most $p-2$ cells. Theorem 4.1 therefore implies that there is an H-space $B^i_{n,k} = M(C^i_{n,k})$ with the property that $H_*(B^i_{n,k}) \cong \Lambda(\widetilde{H}_*(C^i_{n,k}))$.

Next, we wish to extend the map $C^i_{n,k} \xrightarrow{s_i} W_{n,k}$ to a map $B^i_{n,k} \xrightarrow{t_i} W_{n,k}$. As an intermediate step, we use a universal property of the Cohen-Neisendorfer H-spaces $M(A)$ proved in [T1].

Proposition 4.2. *Let A, $M(A)$ and j be as in Theorem 4.1. Let Z be a homotopy associative, homotopy commutative H-space. Then any map $f\colon A \longrightarrow Z$ can be extended to an H-map $\overline{f}\colon M(A) \longrightarrow Z$ and this is the unique H-map such that $\overline{f} \circ j \simeq f$.* □

Since $\Omega^{2m} W_{m+n,k}$ is homotopy associative and homotopy commutative, Proposition 4.2 implies that there is a homotopy commutative diagram

$$\begin{array}{ccc} C^i_{n,k} & \xrightarrow{j} & B^i_{n,k} \\ \downarrow{\scriptstyle s_i} & & \downarrow{\scriptstyle \overline{s}_i} \\ W_{n,k} & \xrightarrow{J_k} & \Omega^{2m} W_{m+n,k} \end{array} \qquad (1)$$

where $\overline{s}_i$ is the unique H-map such that $j \circ \overline{s}_i \simeq J_k \circ s_i$. In Proposition 4.3 we give conditions for when $\overline{s}_i$ lifts through J_k.

Notice that k records the number of generators in $H_*(W_{n,k})$. As $H_*(W_{n,k}) \cong \Lambda(\widetilde{H}_*(\Sigma\mathbb{C}P^n_{n-k}))$, this is equivalent to recording the number of cells in $\Sigma\mathbb{C}P^n_{n-k}$. The decomposition $\Sigma\mathbb{C}P^n_{n-k} \simeq \bigvee_{i=2}^p C^i_{n,k}$ partitions the cells of $\Sigma\mathbb{C}P^n_{n-k}$ cyclically to the $C^i_{n,k}$'s as dimension increases. So if $k \leq (p-1)(p-2)$ then each $C^i_{n,k}$ has at most $p-2$ cells. Therefore, provided $k \leq (p-1)(p-2)$, we can apply Theorem 4.1 to each $C^i_{n,k}$.

Suppose that $C^i_{n,k}$ has r cells, for some $r < p-1$. Then $H_*(B^i_{n,k}) \cong \Lambda(\widetilde{H}_*(C^i_{n,k}))$, so $B^i_{n,k}$ has rank r (where *rank* refers to the number of spheres appearing in a rational decomposition of $B^i_{n,k}$).

Proposition 4.3. *Let $k \leq (p-1)(p-2)$. Suppose that $B^i_{n,k}$ has rank r, for some $r < p-1$. If $n \geq (p-1)r$ then the map $B^i_{n,k} \xrightarrow{\overline{s}_i} \Omega^{2m} W_{m+n,k}$ lifts through J_k to $W_{n,k}$.*

Proof. Step 1: The dimension of $B^i_{n,k}$. If the highest dimensional cell of $C^i_{n,k}$ is in dimension $2t-1$, then the other cells are in dimensions $2t-1-2(p-1), \ldots, 2t-1-2(r-1)(p-1)$. Since $H_*(B^i_{n,k}) \cong \Lambda(\widetilde{H}_*(C^i_{n,k}))$, we therefore have the dimension of $B^i_{n,k}$ being

$$\begin{aligned} \dim B^i_{n,k} &= (2t-1) + (2t-1-2(p-1)) + \cdots + (2t-1-2(r-1)(p-1)) \\ &= (2t-1)r - 2(p-1)(1+\cdots+r-1) \\ &= (2t-1)r - 2(p-1)(r-1)r/2 \\ &= (2t-1)r - (p-1)(r-1)r. \end{aligned}$$

Since the dimension of $C^i_{n,k}$ is at most $2n-1$, for any $B^i_{n,k}$ of rank r we obtain

$$\dim B^i_{n,k} \leq (2n-1)r - (p-1)(r-1)r.$$

Step 2: The connectivity of J_k. Lemma 2.2 shows that the connectivity of $W_{n,k} \xrightarrow{J_k} \Omega^{2m}W_{m+n,k}$ is the same as the connectivity of the iterated suspension $S^{2(n-k)+1} \longrightarrow \Omega^{2m}S^{2m+2(n-k)+1}$, where $S^{2(n-k)+1}$ is the bottom cell of $W_{n,k}$. So to determine the connectivity of J_k in the rank r case, we need to identify the least dimension d of the bottom cell in any $B^i_{n,k}$ of rank r. This is the same as the least dimension of the bottom cell in any $C^i_{n,k}$ with r cells.

In the decomposition $\Sigma\mathbb{C}P^n_{n-k} \simeq \bigvee_{i=2}^p C^i_{n,k}$ the homology classes of $\Sigma\mathbb{C}P^n_{n-k}$ are distributed to the $C^i_{n,k}$'s cyclically as dimension increases. So the wedge summand with the least dimensional bottom cell is $C^2_{n,k}$. Further, the dimension of $C^2_{n,k}$ is at least $2n-1-2(p-2)$. So the smallest value for the dimension d of the bottom cell in $C^2_{n,k}$ is

$$\begin{aligned} d &= (2n-1-2(p-2)) - 2(r-1)(p-1) \\ &= 2n-1+2-2r(p-1) \\ &= 2(n-r(p-1)+1)-1. \end{aligned}$$

Therefore, by Lemma 2.2, the connectivity of J_k satisfies

$$\operatorname{conn} J_k \geq 2(n-r(p-1)+1)p-4.$$

Step 3: comparing $\dim B^i_{n,k}$ and $\operatorname{conn} J_k$. Let $c = 2(n-r(p-1)+1)p-4$. Since the map $W_{n,k} \xrightarrow{J_k} \Omega^{2m}W_{m+n,k}$ is c-connected, if X is any CW-complex of dimension $\leq c+1$ then there is an epimorphism $[X, W_{n,k}] \xrightarrow{(J_k)_*} [X, \Omega^{2m}W_{m+n,k}]$. Consequently, if the dimension of $B^i_{n,k}$ is $\leq c+1$ then the map $B^i_{n,k} \xrightarrow{\overline{s}_i} \Omega^{2m}W_{m+n,k}$ will lift through J_k.

By Step 1, $\dim B^i_{n,k} \leq (2n-1)r-(p-1)(r-1)r$, so aim for conditions on n and r so that

$$(2n-1)r-(p-1)(r-1)r \leq c+1 = 2(n-(r(p-1)+1)p-3. \tag{2}$$

Reorganizing, this inequality is the same as

$$2(p-1)(p-r)r - r - p + 3 \leq 2n(p-r). \tag{3}$$

Since we are assuming that all spaces and maps are localized at $p \geq 3$, we have $3-p-r<0$. So inequality (3) will hold provided that

$$2(p-1)(p-r)r \leq 2n(p-r), \tag{4}$$

that is, inequality (3) holds provided that

$$(p-1)r \leq n. \tag{5}$$

Therefore, if $n \geq (p-1)r$ then (2) holds, implying that $\dim B^i_{n,k} \leq \operatorname{conn} J_k + 1$, and so $\overline{s}_i$ lifts through J_k. □

By Proposition 4.3, the map $B^i_{n,k} \xrightarrow{\overline{s}_i} \Omega^{2m} W_{n,k}$ lifts through J_k to a map

$$t_i \colon B^i_{n,k} \longrightarrow W_{n,k}.$$

Lemma 4.4. *In homology, the map $B^i_{n,k} \xrightarrow{t_i} W_{n,k}$ induces the coalgebra inclusion of $\Lambda(\widetilde{H}_*(C^i_{n,k}))$ into $H_*(W_{n,k}) \cong \otimes_{i=2}^p \Lambda(\widetilde{H}_*(C^i_{n,k}))$.*

Proof. Consider the diagram

$$\begin{array}{ccc} C^i_{n,k} & \xrightarrow{\ j\ } & B^i_{n,k} \\ \Big\downarrow{\scriptstyle s_i} & \overset{t_i}{\swarrow} & \Big\downarrow{\scriptstyle \overline{s}_i} \\ W_{n,k} & \xrightarrow{\ J_k\ } & \Omega^{2m} W_{m+n,k}. \end{array}$$

The outer square homotopy commutes by (1) and the lower right triangle homotopy commutes since t_i is a lift of $\overline{s}_i$ through J_k. For the upper left triangle, observe that $C^i_{n,k}$ has dimension at most $2n-1$, which is less than the connectivity of J_k by Lemma 2.2, so J_k is a homotopy equivalence in dimensions $\leq 2n-1$. Therefore, the upper right triangle homotopy commutes because its composition with J_k homotopy commutes. Thus the entire diagram homotopy commutes. This implies that t_i is an extension of s_i.

In homology, $(s_i)_*$ is the inclusion of $\widetilde{H}_*(C^i_{n,k})$ into $H_*(W_{n,k})$. Therefore the same is true of the restriction of $(t_i)_*$ to the generating set of $H_*(B^i_{n,k}) \cong \Lambda(\widetilde{H}_*(C^i_{n,k}))$. A standard argument using the reduced diagonal and inducting on monomial length therefore implies that $(t_i)_*$ is a coalgebra isomorphism from $H_*(B^i_{n,k})$ onto the sub-coalgebra $\Lambda(\widetilde{H}_*(C^i_{n,k}))$ of $H_*(W_{n,k})$. □

Now we turn to homotopy decompsitions and the proof of Theorem 1.1.

Theorem 4.5. *Let $k \leq (p-1)(p-2)$ and fix $r < p-1$. Suppose that $B^2_{n,k}, \ldots, B^p_{n,k}$ all have rank at most r. If $n \geq (p-1)r$ then there is a homotopy equivalence*

$$\Omega W_{n,k} \simeq \prod_{i=2}^p \Omega B^i_{n,k}.$$

Proof. By Proposition 4.3, each map $B^i_{n,k} \xrightarrow{\overline{s}_i} \Omega^{2m} W_{m+n,k}$ lifts through J_k to a map $B^i_{n,k} \xrightarrow{t_i} W_{n,k}$, whose image in homology induces the coalgebra inclusion of $\Lambda(\widetilde{H}_*(C^i_{n,k}))$ into $H_*(W_{n,k}) \cong \otimes_{i=2}^p \Lambda(\widetilde{H}_*(C^i_{n,k}))$. In general, if X is a space whose homology is a torsion free exterior algebra on a generating set V then applying the Serre spectral sequence to the path-loop fibration shows that ΩX has homology a torsion free polynomial algebra generated by the desuspension of V. In our case, we obtain $H_*(\Omega W_{n,k}) \cong \otimes_{i=2}^p \mathbb{Z}/p\mathbb{Z}[\Sigma^{-1}\widetilde{H}_*(C^i_{n,k})]$ and the image of $(\Omega t_i)_*$ is the Hopf algebra inclusion of $\mathbb{Z}/p\mathbb{Z}[\Sigma^{-1}\widetilde{H}_*(C^i_{n,k})]$. Thus, if μ is the loop multiplication on $\Omega W_{n,k}$, the composite

$$\prod_{i=2}^p \Omega B^i_{n,k} \xrightarrow{\prod_{i=2}^p \Omega t_i} \prod_{i=2}^p \Omega W_{n,k} \xrightarrow{\ \mu\ } \Omega W_{n,k}$$

induces an isomorphism in homology and so is a homotopy equivalence. □

Proof of Theorem 1.1. Let $k \leq (p-1)(p-2)$. We want to show that for any such k, the asserted homotopy decomposition of $\Omega W_{n,k}$ holds for any $n > k$. If $(r-1)(p-1) < k \leq r(p-1)$ for some $r < p-1$, then every $B^i_{n,k}$ has rank either r or $r-1$, so the asserted decomposition of $\Omega W_{n,k}$ holds by Theorem 4.5, provided that $n \geq (p-1)r$.

As the rank of $B^i_{n,k}$ equals the number of cells in $C^i_{n,k}$, having each $C^i_{n,k}$ with r or $r-1$ cells implies that $\Sigma\mathbb{C}P_{n,k}$ must have at least $(p-1)(r-1)+1$ cells. That is, $n \geq (p-1)(r-1)+2$. But Theorem 4.5 only holds when $n \geq (p-1)r$. So there are $p-2$ cases for rank r in the range $(p-1)(r-1)+2 \leq n < (p-1)r$ that are so far unaccounted for.

By different methods, in [B, GZ] it is shown that if $n \leq (p-1)(p-2)$ then the asserted homotopy equivalence holds for $\Omega W_{n,k}$. This range includes the missing $p-2$ cases above. Thus, in the rank r case, for any $n \geq (p-1)(r-1)+2$ we obtain the asserted homotopy equivalence for $\Omega W_{n,k}$.

As this holds for every $1 \leq r < p-1$, we obtain the asserted homotopy equivalence for $\Omega W_{n,k}$ for every $k \leq (p-1)(p-2)$ and $n > k$. □

Proof of Theorem 1.2. This follows exactly as in the proof of Theorem 1.1. Here, we use the stunted quasi-projective space Q^n_{n-k} with $H_*(Q^n_{n-k}) \cong \mathbb{Z}/p\mathbb{Z}\{x_{4(n-k+1)-1}, \ldots, x_{4n-1}\}$ in place of $\Sigma\mathbb{C}P_{n,k}$, and the corresponding homotopy equivalence $Q^n_{n-k} \simeq \bigvee_{i=2}^{(p-1)/2} R^i_{n,k}$, where $H_*(R^i_{n,k})$ consists of those elements in $H_*(Q^n_{n-k})$ in degrees of the form $4i-1+2k(p-1)$ for some $k \geq 0$. If $D^i_{n,k}$ is the H-space $M(R^i_{n,k})$ obtained by applying Theorem 4.1, then arguing as in Proposition 4.3 we obtain:

(i) $\dim D^i_{n,k} \leq (4n-1)r - 2(p-1)(r-1)r$;

(ii) $\operatorname{conn} J_k \geq 4(n - r(p-1)+1)p - 4$;

(iii) inequality (5) is again $(p-1)r \leq n$.

The subsequent arguments leading to the proof of Theorem 1.1 then carry over *mutatis mutandis.* □

5. Real Stiefel manifolds

The real Stiefel manifold is $V_{n,k} = O(n)/O(n-k) \cong SO(n)/SO(n-k)$. For an odd prime p, Harris [Ha] showed that there are homotopy equivalences:

$$SO(2n+1) \simeq_p Spin(2n+1) \simeq_p Sp(n)$$
$$SO(2n) \simeq_p Spin(2n) \simeq_p Spin(2n-1) \times S^{2n-1}.$$

Therefore, we obtain homotopy equivalences:

$$\Omega(SO(2n+1)/SO(2(n-m)+1)) \simeq_p \Omega(Sp(n)/Sp(n-m))$$
$$\Omega(SO(2n+1)/SO(2(n-m)+2)) \simeq_p S^{2(n-m)+1} \times \Omega(Sp(n)/Sp(n-m))$$
$$\Omega(SO(2n+2)/SO(2(n-m)+1)) \simeq_p \Omega S^{2n+1} \times \Omega(Sp(n)/Sp(n-m))$$
$$\Omega(SO(2n+2)/SO(2(n-m)+2)) \simeq_p S^{2(n-m)+1} \times \Omega S^{2n+1} \times \Omega(Sp(n)/Sp(n-m)).$$

So in each case a homotopy decomposition for $\Omega V_{n,k}$ with $k \leq (p-1)(p-2)$ can be read off from the corresponding homotopy decomposition of $\Omega X_{n,k}$ in Theorem 1.2.

6. Exponents

For a space X, the *homotopy exponent* of X at the prime p is the least power of p that annihilates the p-torsion in $\pi_*(X)$. If this power is a finite number r, write $\exp_p(X) = p^r$. Notice that if X is simply-connected then as looping simply shifts homotopy groups down one dimension, we obtain $\exp_p(\Omega X) = \exp_p(X)$. Also, as $\pi_m(X \times Y) \cong \pi_m(X) \oplus \pi_m(Y)$ for all $m \geq 1$, we have $\exp_p(X \times Y) = \max\{\exp_p(X), \exp_p(Y)\}$. So to find the homotopy exponent of a space X one approach is to establish a homotopy decomposition of ΩX and determine the exponents of the factors. We apply this to find upper bounds for the homotopy exponents of $W_{n,k}$ and $X_{n,k}$.

Consider spaces $A_{m,\ell}$ such that there is a vector space isomorphism (with elements listed by decreasing degree)

$$\widetilde{H}^* A_{m,\ell} \cong \mathbb{Z}/p\mathbb{Z}\{x_{2m-1}, x_{2m-2(p-1)-1}, \ldots, x_{2m-2(\ell-1)(p-1)-1}\}$$

and the action of the Steenrod algebra is given by $\mathcal{P}^j(x_{2r-1}) = \binom{r}{j} x_{2r+jq-1}$. The spaces $C^i_{n,k}$, being retracts of $\Sigma\mathbb{C}P_{n,k}$, satisfy these hypotheses. If $\ell < p-1$, let $B_{m,\ell} = M(A_{m,\ell})$ be the H-space obtained by applying Theorem 4.1 to $A_{m,\ell}$. In [T2, Theorem 5.1] the following was shown.

Proposition 6.1. *If $k < p-2$ then $\exp_p(B_{m,\ell}) \leq p^{(\ell-1)+(m-1)}$.*

It is worth noting that the term $\ell - 1$ in the exponent bound refers to the ℓ cells in $A_{m,\ell}$, while the term $m-1$ refers to the exponent bound for the top dimensional sphere S^{2m-1}. Here, by [CMN], if p is odd then $\exp_p(S^{2n-1}) = p^{n-1}$.

It is also worth noting that the same result should hold true for $k = p-2$, but the methods used in [T2] aimed at a stronger result which used the fact that, if $\ell < p-2$, then $B_{m,\ell}$ is a homotopy associative and homotopy commutative H-space.

For a positive rational number m, let $\lfloor m \rfloor$ be the largest integer smaller than m.

Theorem 6.2. *The following hold:*

(a) *if $k \leq (p-1)(p-3)$ then $\exp_p(W_{n,k}) \leq p^{\lfloor k/(p-1) \rfloor + n - 1}$;*

(b) *if $k \leq (p-1)(p-3)/2$ then $\exp_p(X_{n,k}) \leq p^{\lfloor k/(p-1) \rfloor + 2n - 1}$.*

Proof. Suppose that $k \leq (p-1)(p-3)$. By Theorem 1.1, $\Omega W_{n,k} \simeq \prod_{i=2}^{p} \Omega B^i_{n,k}$ where $B^i_{n,k} = M(C^i_{n,k})$. If $(r-1)(p-1) < k \leq r(p-1)$ then at least one $C^i_{n,k}$ has r cells and the other $C^i_{n,k}$'s have either r or $r-1$ cells. Also, precisely one $C^i_{n,k}$ (possibly a different one) has the top dimensional cell of $\mathbb{C}P_{n,k}$ in dimension $2n-1$. So Proposition 6.1 implies that $\exp_p(B^i_{n,k}) \leq p^{r-1+n-1}$ for $2 \leq i \leq p-1$. Therefore, $\exp_p(W_{n,k}) \leq p^{r+n-2}$. Noting that $r - 1 = \lfloor k/(p-1) \rfloor$, we obtain the asserted exponent bound.

The argument for $X_{n,k}$ is similar. □

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