Consensus for linear agents with unknown parameters

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Abstract: We solve consensus problems for networks of identical linear agents whose dynamics is unknown except for the order. We estimate the unknown dynamics via an "extended state-observer", and cancel it with a suitably designed control. The agent dynamics is thus approximated by that of an n-integrator, and standard protocols for networks of such type can then be used to achieve consensus for the network of original unknown agents.

Keywords: Multi-agent systems, extended state-observers, consensus

1. INTRODUCTION

We consider N identical SISO agents with *unknown* but *fixed* dynamics described by the equations

$$\frac{d}{dt}x_k = \underbrace{\begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \dots & \ddots & \vdots \\ 0 & \dots & \dots & 1 \\ -\overline{\alpha}_0 & \dots & \dots & -\overline{\alpha}_{n-1} \end{bmatrix}}_{=:A} x_k + \underbrace{\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{=:B} u_k$$

$$y_k = \underbrace{[1\ 0 \dots 0]}_{=:C} x_k \ . \qquad (1)$$

Such agents are interconnected through a *known*, *connected*, *undirected graph* $\mathcal{G}$ with Laplacian $\mathcal{L}$. In the following, we use the notation $\text{col}(\overline{y}_i)_{i=0,\dots,n-1} := [\overline{y}_0 \ \cdots \ \overline{y}_{n-1}]^\top$.

Definition 1. The network achieves *consensus* if

$$\lim_{t\to\infty} |y_k(t) - y_j(t)| = 0 \ , k, j = 1, \dots, N \ .$$

If $\exists\, \overline{y} := \text{col}(\overline{y}_i)_{i=0,\dots,n-1} \in \mathbb{R}^n$ such that $\lim_{t\to\infty} |y_k^{(i)}(t) - \overline{y}_i| = 0$, $k = 1, \dots, N$, $i = 0, \dots, n-1$, then $\overline{y}$ is called a *consensus value*. Let $y^* : \mathbb{R} \to \mathbb{R}$ be an n-times differentiable function; the network achieves *group-reference velocity consensus* if $\lim_{t\to\infty} |y_k^{(j)}(t) - y_i^{(j)}(t)| = 0$, $k, i = 1, \dots, N$, $j = 0, \dots, n-1$, and $\lim_{t\to\infty} |y_k^{(n)}(t) - y^{*(n)}(t)| = 0$, $k = 1, \dots, N$.

Assume that only the order n of (1) is known. To solve our problem one can use system identification to model the agent dynamics, and then apply standard protocols. We develop instead a *model-free* approach [2] . We first design an *extended state observer* (*ESO*s) for the unknown dynamics (1), i.e. a linear $(n+1)$-th order observer whose state variable $\widehat{x}_{n+1}^k$ is an estimate of the unknown dynamics

$$y_k^{(n)} - u_k = \underbrace{-\overline{\alpha}_0 y_k - \dots - \overline{\alpha}_{n-1} y_k^{(n-1)}}_{=:\nu_k} \ . \qquad (2)$$

We then *approximately* cancel the unknown dynamics by feeding back $u_k = -\widehat{x}_{n+1}^k + v_k$ to the k-th agent, where v_k is an external input:

$$y_k^{(n)} = \underbrace{-\overline{\alpha}_0 y_k - \dots - \overline{\alpha}_{n-1} y_k^{(n-1)} - \widehat{x}_{n+1}^k}_{=:\zeta_k} + v_k \ . \qquad (3)$$

If the estimate $\widehat{x}_{n+1}^k$ is accurate, then the *combined k-th agent*, i.e. the interconnection of the actual dynamics and the ESO under quasi-cancellation feedback, approximates an *n-integrator*; several protocols are known for networks of such agents (see e.g. Jiang et al. (2009); Ren (2007); Ren and Atkins (2007); Ren (2008); Xie and Wang (2006)).

Recently ESOs have been used in consensus problems when dealing with uncertain nonlinearities (see Qin et al. (2014); Yang et al. (2015)) and disturbances (see Cao et al. (2015); Yang et al. (2014)). By concentrating on a simpler class (linear dynamics, no disturbances) we show that consensus is achieved *exactly* and *asymptotically*, rather than only approximately (i.e. the agents' outputs remaining in a bounded neighbourhood of the reference) as in the more general cases considered in the literature.

In sect. 2 we introduce the ESO and describe the combined dynamics of the actual agent and the ESO. In sect. 3 we prove some structural properties of the network of combined dynamics under standard consensus protocols. We exploit such properties in sect. 4 to obtain our main results. In sect. 5 we give an example of our approach to solve average consensus problems using a heuristic design procedure aided by a graphic user interface specially designed for the purpose. Sect. 6 contains the conclusions and a discussion of current research.

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[2] We use the term "model-free" less restrictively than in Fliess and Join (2013), where not even the order of the dynamics is known.

2. THE COMBINED K-TH AGENT DYNAMICS

2.1 The extended state observer (ESO)

ESOs were introduced in Han (2009, 1999) in more general terms; we discuss them only for the problem at hand.

Define the observer of McMillan degree $n+1$:

$$\frac{d}{dt}\begin{bmatrix}\widehat{x}_1^k\\ \vdots\\ \widehat{x}_n^k\\ \widehat{x}_{n+1}^k\end{bmatrix} = \underbrace{\begin{bmatrix}-\widehat{\alpha}_n & 1 & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ -\widehat{\alpha}_1 & \dots & \dots & 1\\ -\widehat{\alpha}_0 & 0 & \dots & 0\end{bmatrix}}_{=:\widehat{A}}\underbrace{\begin{bmatrix}\widehat{x}_1^k\\ \vdots\\ \widehat{x}_n^k\\ \widehat{x}_{n+1}^k\end{bmatrix}}_{=:\widehat{x}_k} + \underbrace{\begin{bmatrix}0\\ \vdots\\ 1\\ 0\end{bmatrix}}_{=:\widehat{B}}u_k + \underbrace{\begin{bmatrix}\widehat{\alpha}_n\\ \vdots\\ \widehat{\alpha}_1\\ \widehat{\alpha}_0\end{bmatrix}}_{=:\widehat{F}}y_k$$

$$\widehat{y}_k = \underbrace{\begin{bmatrix}1 & 0 & \dots & 0\end{bmatrix}}_{=:\widehat{C}}\begin{bmatrix}\widehat{x}_1^k\\ \vdots\\ \widehat{x}_{n+1}^k\end{bmatrix}, \tag{4}$$

where $\widehat{\alpha}_j \in \mathbb{R}$, $j = 0, \dots, n$.

Theorem 1. Let $(u_k(\cdot), y_k(\cdot))$ be an i/o trajectory of (1). Assume that $\frac{d}{dt}\left(\sum_{j=0}^{n-1}\overline{\alpha}_j y_k^{(j)}(\cdot)\right)$ is bounded and that $\widehat{\alpha}(s) := s^{n+1} + \ldots + \widehat{\alpha}_0$ of (4) is Hurwitz. $\exists\ T_k, T_k' \in \mathbb{R}$ and $M_k, M_k' \in \mathbb{R}$ depending on $u_k(\cdot)$, $y_k(\cdot)$, $x_k(0)$ and $\widehat{x}_k(0)$, such that $|\widehat{y}_k(t) - y_k(t)| < M_k$ for $t > T_k$ and $|\zeta_k(t)| = |-\widehat{x}_{n+1}^k(t) - \sum_{j=0}^{n-1}\overline{\alpha}_j y_k^{(j)}(t)| < M_k'$ for $t > T_k'$.

Proof. A standard argument, see e.g. Guo and Zhao (2011), which we summarise to make the paper self-contained. Define $\epsilon_k := y_k - \widehat{x}_1^k$ and ν_k as in (2); we show

$$\epsilon_k^{(n+1)} + \widehat{\alpha}_n\epsilon_k^{(n)} + \ldots + \widehat{\alpha}_1\epsilon_k^{(1)} + \widehat{\alpha}_0\epsilon_k = \nu_k^{(1)}\ . \tag{5}$$

It follows from (4) that $\widehat{x}_1^{k(j)} = \widehat{\alpha}_n\epsilon_k^{(j-1)} + \ldots + \widehat{\alpha}_{n-j+1}\epsilon_k + \widehat{x}_{j+1}^k$, $j = 1, \dots, n-1$. Differentiate the last of such equations and substitute $\frac{d}{dt}\widehat{x}_k$ from (4):

$$\begin{aligned}\widehat{x}_1^{k(n)} &= \widehat{\alpha}_n\epsilon_k^{(n-1)} + \ldots + \widehat{\alpha}_1\epsilon_k + \frac{d}{dt}\widehat{x}_n^k\\ &= \widehat{\alpha}_n\epsilon_k^{(n-1)} + \ldots + \widehat{\alpha}_1\epsilon_k + \widehat{x}_{n+1}^k + u_k\ .\end{aligned}$$

Subtracting such expression from (2) yields

$$\begin{aligned}\epsilon_k^{(n)} &= -\widehat{\alpha}_n\epsilon_k^{(n-1)} - \widehat{\alpha}_{n-1}\epsilon_k^{(n-2)} - \ldots - \widehat{\alpha}_1\epsilon_k\\ &\quad -\widehat{x}_{n+1}^k - \overline{\alpha}_0 y_k - \ldots - \overline{\alpha}_{n-1}y_k^{(n-1)}\ .\end{aligned} \tag{6}$$

Differentiating (6) and using the last of the state equations in (4) yields (5). Since $\widehat{\alpha}(s)$ is Hurwitz, the system (6) is BIBO-stable. The first part of the claim follows from (5) and the assumption on the boundedness of $\frac{d}{dt}\nu_k$. To prove the second part, rewrite (6) as $\epsilon_k^{(n)} + \widehat{\alpha}_n\epsilon_k^{(n-1)} + \widehat{\alpha}_{n-1}\epsilon_k^{(n-2)} + \ldots + \widehat{\alpha}_1\epsilon_k = \nu_k - \widehat{x}_{n+1}^k$; since $\epsilon_k(\cdot)$ is bounded and $\widehat{\alpha}(s)$ is Hurwitz, so is $\nu_k(\cdot) - \widehat{x}_{n+1}^k(\cdot) = \zeta_k(\cdot)$. □

The design of $\widehat{\alpha}_i$, typically involving high-gain, is discussed in Guo and Zhao (2011); Zheng et al. (2007). In the latter source an ESO design sufficient for our situation is shown, where only the "observer bandwidth" needs to be tuned. In such case, a bound for the convergence rate of the estimation (depending on the unknown linear dynamics (1)) is given in the proof of Th. 1 p. 3502.

2.2 Interconnection of ESO and agent

The result of Th. 1 implies that defining

$$u_k(\cdot) := -\widehat{x}_{k,n+1}(\cdot) + v_k(\cdot)\ , \tag{7}$$

where $v_k : \mathbb{R} \to \mathbb{R}$ is an additional control input, the k-th agent dynamics (1) is described by the *endogenously perturbed* higher-order integrator dynamics

$$y_k^{(n)} = v_k + \zeta_k\ , \tag{8}$$

with the "disturbance" ζ_k defined by (3). Denote the i-th element of the canonical basis of $\mathbb{R}^n$ by e_i, and that of $\mathbb{R}^{n+1}$ by e_i'; the *combined dynamics* of the actual (unknown) k-th agent with the corresponding ESO is described by

$$\frac{d}{dt}\begin{bmatrix}x_k\\ \widehat{x}_k\end{bmatrix} = \underbrace{\begin{bmatrix}A & -Be_{n+1}'\\ \widehat{F}e_1 & \widehat{A} - \widehat{B}e_{n+1}'\end{bmatrix}}_{=:A^e}\begin{bmatrix}x_k\\ \widehat{x}_k\end{bmatrix} + \underbrace{\begin{bmatrix}B\\ \widehat{B}\end{bmatrix}}_{=:B^e}v_k$$

$$y_k = \underbrace{\begin{bmatrix}C & 0_{1\times(n+1)}\end{bmatrix}}_{=:C^e}\begin{bmatrix}x_k\\ \widehat{x}_k\end{bmatrix}\ . \tag{9}$$

We write the combined dynamics order as $n_e := n + (n+1)$.

Since $\widehat{x}_{n+1}^k$ converges asymptotically to ν_k, the feedback $u_k = -\widehat{x}_{n+1}^k + v_k$ cancels the unknown dynamics, and the combined system (9) asymptotically behaves as an n-th order integrator. In section 4 we show that standard consensus protocols for networks of higher-order integrators can be used also for networks of agents (9). Before that, in the next section we establish some spectral- and eigenvector properties of the network consisting of the combined dynamics (9) under standard consensus protocols for *standard* and *group reference-input consensus*.

3. THE NETWORK OF COMBINED AGENTS

A typical protocol for networks of n-integrators is

$$v_k = -cx_k - \sum_{j\in\mathcal{N}_k} a_{kj}\gamma_k\ell(x_k - x_j)\ , \tag{10}$$

where $c = [0\ c_1\ \dots\ c_{n-1}] \in \mathbb{R}^{1\times n}$ is the *absolute information gain*, $\ell \in \mathbb{R}^{1\times n}$ and γ_k are the *relative information gains*, $A = [a_{kj}]$ is the adjacency matrix, and $\mathcal{N}_k$ is the set of neighbouring nodes, see e.g. Jiang et al. (2009); Xie and Wang (2006). In group-reference consensus the protocol is

$$\begin{aligned}v_k &= y^{*(n)} - c\left(x_k - \mathrm{col}\left(y^{*(k)}\right)_{k=0,\dots,n-1}\right)\\ &\quad - \sum_{j\in\mathcal{N}_k} a_{kj}\gamma_k\ell(x_k - x_j)\ ,\end{aligned} \tag{11}$$

where y^* is a given reference signal in $\mathfrak{C}^n(\mathbb{R},\mathbb{R})$, see e.g. sect. V of Ren (2008). The difference with (10) is the injection of a linear combination of y^* and its derivatives. The analysis of such case is analogous to that of (10), and in the following we concentrate mainly on the latter.

We now describe the dynamics of the network consisting of the combined dynamics (9) under the protocol (10). Define

$$A_{cl}^e(c) := \begin{bmatrix}A - Bc & -Be_{n+1}'\\ \widehat{F}e_1 - \widehat{B}c & \widehat{A} - \widehat{B}e_{n+1}'\end{bmatrix} \in \mathbb{R}^{n_e\times n_e}$$

$$K(\ell) := \begin{bmatrix}B\\ \widehat{B}\end{bmatrix}\begin{bmatrix}\ell & 0_{1\times(n+1)}\end{bmatrix}\ ,\ \Gamma := \mathrm{diag}(\gamma_k)_{k=1,\dots,N} \tag{12}$$

Proposition 1. $A^e_{cl}(c)$ defined in (12) is singular. Let $\beta, \delta \in \mathbb{R}$, and define

$$v^*(\beta,\delta) := \beta \begin{bmatrix} 1 & 0_{1\times(n-1)} & 1 & 0_{1\times(n-1)} & -\delta \end{bmatrix}^\top . \quad (13)$$

Then for every $\beta \in \mathbb{R}$, $v^*(\beta, \overline{\alpha}_0) \in \ker\ A^e_{cl}(c)$.

Proof. A straightforward check from (12), (1), and (4).

Define the *network extended-state vector* by $x^{e\top} := \text{col}\left(\begin{bmatrix} x_k^\top & \widehat{x}_k^\top \end{bmatrix}^\top\right)_{k=1,\ldots,N}$; the dynamics of the network of combined systems is described by

$$\frac{d}{dt}x^e = \underbrace{[\text{blockdiag}(A^e_{cl}(c))_{k=1,\ldots,N} + \Gamma\mathcal{L} \otimes K(\ell)]}_{=:\Omega(c,\Gamma,\ell)} x^e . \quad (14)$$

Proposition 2. Let $\beta, \delta \in \mathbb{R}$, and define

$$v^*_e(\beta,\delta) := \begin{bmatrix} v^*(\beta,\delta)^\top & \ldots & v^*(\beta,\delta)^\top \end{bmatrix}^\top \in \mathbb{R}^{Nn_e} \quad (15)$$

where $v^*(\beta,\delta)$ is defined by (13). Then $v^*_e(\beta,\overline{\alpha}_0) \in \ker\ \Omega(c,\Gamma,\ell)$.

Proof. It follows from Prop. 1 that for all $\beta \in \mathbb{R}$, $v^*_e(\beta,\overline{\alpha}_0) \in \ker \text{blockdiag}(A^e_{cl}(c))_{k=1,\ldots,N}$. Now observe that $K(\ell)v^*(\beta,\overline{\alpha}_0) = \beta\ell_1 \,\text{col}(e_n, e'_n) \in \mathbb{R}^{n_e}$. Now recall that $\mathcal{L}\mathbf{1}_N = 0$, where all entries of $\mathbf{1}_N \in \mathbb{R}^N$ equal 1. □

4. MAIN RESULTS

We now prove that consensus protocols exist for the *combined* agents (9), achieving the *same* consensus as that of a network of *n-integrators*. The first result is a consequence of Prop. 2.

Theorem 2. The network of combined dynamics (14) reaches consensus under (10) iff $\exists\ c \in \mathbb{R}^{1\times n}$, $\ell \in \mathbb{R}^{1\times n}$, and $\gamma_k \in \mathbb{R}$, $k = 1,\ldots,N$, such that $\Omega(c,\Gamma,\ell)$ has all its eigenvalues in $\mathbb{C}_-$, except the fixed one at zero.

If a consensus state is achieved, it lies on the line in $\mathbb{R}^{Nn_e}$ spanned by $v^*_e(\beta,\overline{\alpha}_0)$ defined by (15).

Proof. It follows from Prop. 2 that independently of the absolute and the relative gains, the combined network dynamics (14) admits at least one eigenvector in the direction of $v^*(\beta,\overline{\alpha}_0)$ defined in (15), associated with the zero eigenvalue. Consequently, consensus is achieved if and only if c, Γ and ℓ can be chosen so that all other eigenvalues of $\Omega(c,\Gamma,\ell)$ are in the open left half-plane. □

We remark on some consequences of Th. 2.

Remark 1. Observe that if a protocol exists achieving consensus for the network of combined agents, then the the error between the agents' and the consensus state is *asymptotically zero*; in the case of nonlinear dynamics (see Qin et al. (2014); Yang et al. (2015); Cao et al. (2015); Yang et al. (2014)), only uniform ultimate boundedness of the error is achieved.

Remark 2. If consensus is achieved for the combined network, then the projection of the consensus state on the variables x_k, $k = 1,\ldots,N$ lies in the same direction as the consensus state for the "nominal" n-integrator agents, see e.g. Lemma 2 of Jiang et al. (2009) and Lemma 2 of Xie and Wang (2006). The combined dynamics (9) only *approximates* the integrator dynamics (see (8)), but the *network* of combined agents achieves consensus in the *same* direction as the network of "nominal" ones.

Remark 3. Note that the asymptotic consensus value depends on the left eigenvectors of the network dynamics matrix (14), and the initial conditions of the agents *and* of those of the associated ESOs. Consequently the consensus state is not related *only* to the initial conditions of the agents, as in the case of n-th order integrators (see Th. 1 of Jiang et al. (2009) or the proof of Th. 1 of Xie and Wang (2006)).

The following is a refinement of Th. 2.

Theorem 3. Denote the eigenvalues of $\mathcal{L}$ by λ_i, $i = 1,\ldots,N$; since the graph is connected, $\lambda_1 = 0$ and $\lambda_i > 0$, $i = 2,\ldots,N$. Assume that $\gamma_i = \gamma$, $i = 1,\ldots,N$. The dynamics (14) reaches consensus if and only if there exist c and ℓ such that for $i = 2,\ldots,N$ $A^e_{cl}(c) + \gamma\lambda_i K(\ell)$ is Hurwitz, and $A^e_{cl}(c)$ has all its eigenvalues in the open left half-plane except the fixed one at zero.

Proof. The statement on the eigenvalues of $\mathcal{L}$ follows from standard graph theory. To prove the second statement, consider a factorisation $\mathcal{L} = V\Lambda V^\top$ with $V^\top \in \mathbb{R}^{N\times N}$ the orthogonal matrix of right eigenvectors, and $\Lambda := \text{diag}(\lambda_i) \in \mathbb{R}^{N\times N}$ the matrix of eigenvalues of $\mathcal{L}$. Using standard identities about Kronecker products (see Loan (2000)), conclude that $(V^\top \otimes I_{n_e}) = (V \otimes I_{n_e})^{-1}$: thus pre- and post-multiplying $\Omega(c,\Gamma,\ell)$ respectively by $(V^\top \otimes I_{n_e})$ and $(V \otimes I_{n_e})$ corresponds to a similarity transformation.

Once again using a standard Kronecker product identity it follows that $(V^\top \otimes I_{n_e})(\Gamma\mathcal{L} \otimes K(\ell))(V \otimes I_{n_e})$ equals $(V^\top\Gamma\mathcal{L} \otimes K(\ell))(V \otimes I_{n_e})$, equivalently

$$(V^\top\Gamma\mathcal{L}V \otimes K(\ell)) = \gamma(V^\top\mathcal{L}V \otimes K(\ell)) = \gamma(\Lambda \otimes K(\ell)) .$$

Since $\text{blockdiag}(A^e_{cl}(c)) = (I_N \otimes A^e_{cl}(c))$ we conclude that

$$(V^\top \otimes I_{n_e})\,\text{blockdiag}(A^e_{i,cl}(c)(V \otimes I_{n_e}) = (I_N \otimes A^e_{i,cl}(c)) .$$

Consequently, the similarity transformation brings $\Omega(c,\Gamma,\ell)$ to $(I_N \otimes A^e_{i,cl}(c)) + \gamma(\Lambda \otimes K(\ell))$, which equals

$$\text{blockdiag}\left(A^e_{i,cl}(c) + \gamma\lambda_i K(\ell)\right)_{i=1,\ldots,N} ,$$

from which the claim follows using Prop. 2. □

We now prove the existence of protocols achieving consensus for the network (14) consisting of combined agents.

Theorem 4. Consider the dynamics (9). Assume that $\widehat{\alpha}(s) := \widehat{\alpha}_0 + \ldots + s^{n+1}$ of the ESO is Hurwitz. There exists $\overline{k} \in \mathbb{R}^{1\times n}$ such that $A^e - B^e \begin{bmatrix} \overline{k} & 0_{1\times(n+1)} \end{bmatrix}$ is Hurwitz.

Proof. Denote the i-th component of $\overline{k}$ by $\overline{k}_{i-1}$, $i = 1,\ldots,n$. Write $A^e - B^e \begin{bmatrix} \overline{k} & 0_{1\times(n+1)} \end{bmatrix}$:

$$\begin{bmatrix}
0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
0 & \ldots & 0 & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \vdots & \ldots & \vdots & \vdots \\
0 & \ldots & 1 & 0 & \ldots & 0 & 0 \\
-\overline{\alpha}_0 - \overline{k}_0 & \ldots & -\overline{\alpha}_{n-1} - \overline{k}_{n-1} & 0 & \ldots & 0 & -1 \\
\widehat{\alpha}_n & \ldots & 0 & -\widehat{\alpha}_n & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ldots & \ddots & \vdots \\
\widehat{\alpha}_1 - \overline{k}_0 & \ldots & -\overline{k}_{n-1} & -\widehat{\alpha}_1 & 0 & \ldots & 0 \\
\widehat{\alpha}_0 & \ldots & 0 & -\widehat{\alpha}_0 & 0 & \ldots & 0
\end{bmatrix} . \quad (16)$$

Define $\widehat{A}$ as in (4), and

$$T := \begin{bmatrix} I_n & 0_{n\times n} & 0_{1\times n} \\ I_n & -I_n & 0_{n\times 1} \\ 0_{1\times n} & 0_{1\times n} & -1 \end{bmatrix} \ , \ \overline{\alpha} := [-\overline{\alpha}_0 \ \dots \ -\overline{\alpha}_{n-1}] \ .$$

A straightforward check shows that $T^{-1} = T$ and that

$$T(A^e - B^e \left[\overline{k} \ 0_{1\times(n+1)}\right])T^{-1} = \begin{bmatrix} A - B\overline{k} & Be'_{n+1} \\ -\widehat{B}\overline{\alpha} & \widehat{A} \end{bmatrix} \ . \quad (17)$$

This is the state matrix of the interconnection of

$$\frac{d}{dt}x = (A - B\overline{k})x + Bu_1 \quad (18)$$

and

$$\frac{d}{dt}\widehat{x} = \widehat{A}\widehat{x} + \widehat{B}u_2 \quad (19)$$

under the constraints $u_1 = e_{n+1}^\top \widehat{x} + v_1$, $u_2 = -\overline{\alpha}x + v_2$, with v_j additional control inputs, $j = 1, 2$. It is a matter of straightforward verification to check that such interconnection with inputs u_1 and u_2 and output x_1 is observable and controllable. Since the ESO characteristic polynomial $\widehat{\alpha}(s)$ and the actual agent characteristic polynomial $\overline{\alpha}(s)$ are fixed (and finite), the system (19) has a fixed finite ∞-gain. Consequently, the ∞-gain of (18) can be made arbitrarily small by suitably choosing the gain $\overline{k}$. A small-gain argument yields asymptotic stability of the interconnection. □

Define $\overline{k}_i := c - \gamma\lambda_i\ell$, $i = 2, \dots, N$; from the dynamics (12), the condition of Th. 3 and the result of Th. 4 it follows that a protocol (10) exists for the network of combined agents. Based as it is on a small-gain argument that only assumes that the unknown dynamics are fixed and consequently bounded, such result is of an existential nature and the issue arises how to design a protocol. We have implemented a graphic tool to assist in devising such control law, which we briefly describe in sect. 5. Such an approach however falls short of providing a workable answer for real-life applications, due to the inherent complexity of the problem.

Remark 4. We illustrate some of the difficulties inherent in the design of protocols for the case of polytopic uncertainties in the agent dynamics, i.e. when the state matrix in (1) is of the form $A(\delta) := \sum_{i=1}^{q} \delta_i A_i$, where A_i are known companion matrices, $i = 1, \dots, q$, and δ is in the unit simplex $\Delta_q := \{\text{col}(\delta_i)_{i=1,\dots,q} \mid \delta_i \geq 0 \quad \text{and} \ \sum_{i=1}^{q} \delta_i = 1\}$.

Assume that the dynamics of the ESO have been fixed; we use the matrix in eq. (17) in the proof of Th. 4, to which the combined agent dynamics under the protocol are reduced by nonsingular transformation. The Hurwitzianity of such parameter-dependent matrix can be assessed by determining if a parameter-dependent $P(\delta) = P(\delta)^\top > 0$ exists such that for all $\delta \in \Delta_q$ the LMI

$$\begin{bmatrix} A(\delta)^\top - \overline{k}^\top B^\top & -\overline{\alpha}^\top \widehat{B}^\top \\ e'^\top_{n+1} B^\top & \widehat{A}^\top \end{bmatrix} P(\delta) \quad (20)$$
$$+ P(\delta) \begin{bmatrix} A(\delta) - B\overline{k} & Be'_{n+1} \\ -\widehat{B}\overline{\alpha} & \widehat{A} \end{bmatrix} < 0 \ ,$$

holds. For a given $\overline{k}$, algorithms to solve such parameter-dependent matrix inequalities are well-known (see e.g. Oliveira and Peres (2007)); however, our design problem is made more complex by the fact that $\overline{k}$ is itself a to-be-designed parameter.

Non-linear matrix inequalities as (20) appear also in some sufficient conditions for the existence of ESO-based protocols for the nonlinear case, see e.g. Th. 1 of Yang et al. (2014). □

We remarked in sect. 3 that the only difference between the standard and the group-reference consensus problem is the injection to each agent of a suitable linear combination of the derivatives of a given reference signal. Using arguments similar to those of Th.s 3 and 4, and standard linear system theory, it is straightforward to show that consensus protocols (11) exist that solve the group-reference problem. Due to space limitations we do not enter in the details.

5. EXAMPLE

In section IV-B of Jiang et al. (2009) a protocol design procedure for the case of n-th order integrators was proposed, based on the following formula (see (5) *ibid.*) for the characteristic polynomial of the closed-loop network:

$$\Pi_{i=1}^{N} \left(s^m + c_{m-1}s^{m-1} + \dots + c_1 s + \gamma_i\lambda_i \right) \ , \quad (21)$$

where λ_i is the i-th eigenvalue of the Laplacian, $\lambda_1 = 0$. We have developed a graphics user interface to aid in the design of a protocol for systems of order $n = 2$ to $n = 4$, using the following procedure. Using computations based on the Routh-Hurwitz stability test, we choose values for the consensus parameters c_j, $j = 1, \dots, n-1$, and γ_i, $i = 1, \dots, N$ that guarantee asymptotic consensus for the ideal case of n-integrator dynamics. After having fixed the value of c_j, $j = 1, \dots, n-1$, in this way, the existence and robustness of the consensus protocol as a function of the parameters $k_i = \gamma_i\lambda_i$, $i = 1, \dots, N$ can be checked using a root locus plot approach. By ordering the eigenvalues of the Laplacian in non-decreasing order the test can be reduced to a single root locus analysis for the case $i = N$, i.e. $k = \gamma\lambda_N$.

By analysing the root locus plot we can choose the parameters γ_i more conservatively, to accommodate for the discrepancy between the actual dynamics of the combined agent (i.e. unknown agent dynamics and extended state observer), and the ideal one of the n-th order integrator for which the original design was aimed. Such graphical choice is automatically translated into values for γ_i, $i = 1, \dots, N$. Finally, the graphical user interface offers the possibility of simulating the protocol designed in this way on the unknown agents initialised with random initial conditions, to verify its performance.

As an example to illustrate our design procedure and the basics of the graphical user interface, we consider the case of 4 unknown heterogeneous agents of order $n = 3$ whose dynamics are generated in a random way using `Matlab`. The agents are connected in the graph shown on the left-hand side of the screen shot shown in Fig. 1. Such graph has been chosen as completely connected in this case, but can be chosen randomly if required.

To find values of the protocol parameters that guarantee consensus we apply the Routh-Hurwitz stability criterion on each of the polynomial factors of (21). Such analysis results in the following condition on c_j, $j = 1, 2$ and γ_i, $i = 1, \dots, 4$: $c_1 c_2 > \gamma_i\lambda_4$, for the case of third order integrators. Since the graph is connected, $\lambda_4 = N = 4$

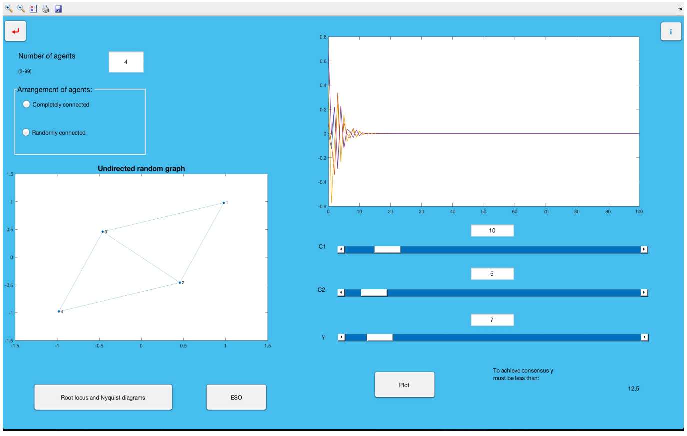

Fig. 1. The graph and the consensus protocol parameters for third-order integrators

in this case. We choose the values $c_1 = 10$, $c_2 = 5$, $\gamma_i = 7$, $i = 1, \dots, 4$. From the result of Lemma 1 of would be sufficient for achieving consensus for the case of third-order integrator dynamics.

The roots locus plot for the given c_i values and the largest eigenvalue $\lambda_4 = 4$ is shown on the left-hand side of Fig. 2. Based on such plot, we choose the value of k corresponding to the point in Fig. 2, i.e. $k = 10.7 = \gamma\lambda_4$, corresponding to the value $\gamma = 2.675 = \gamma_i$, $i = 1, \dots, 4$.

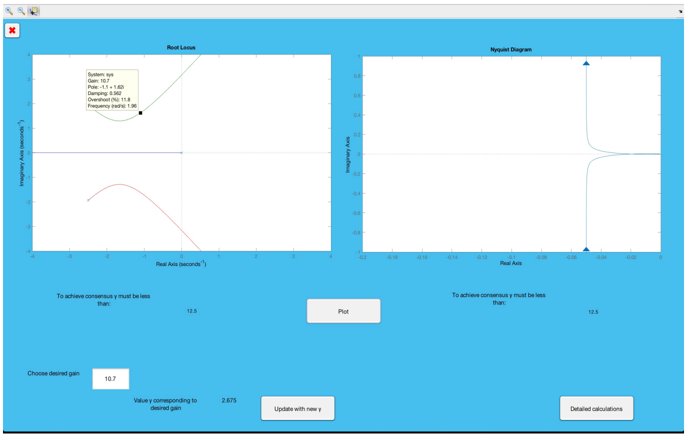

Fig. 2. The root locus for a robust choice of γ.

A simulation of the effects of such protocol for random initial conditions of the agents' dynamics results in the plot shown in Fig. 3.

The program also provides plots for the higher-order derivatives of the agents' outputs, and accommodates the design of nonlinear heterogeneous agent dynamics obtained as random perturbations of a specific nominal one.

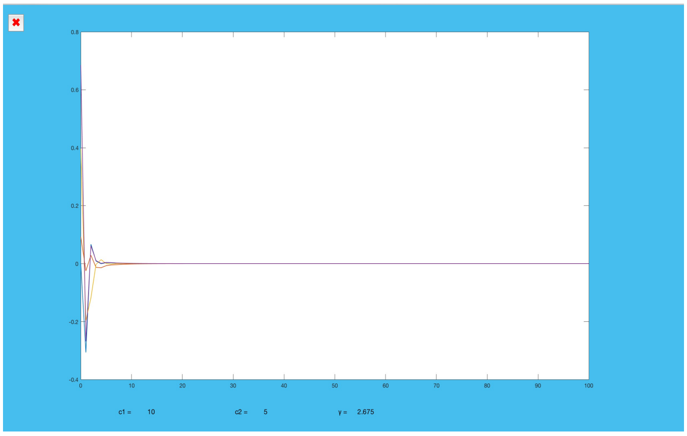

Fig. 3. Outputs of the combined agents reaching asymptotic consensus.

6. CONCLUSIONS

We proposed a (relatively) model-free consensus approach for networks of identical linear systems, based on the estimation and cancellation of the partially known dynamics via an ESO, and the consequent approximation of the unknown agents with higher-order integrators. We proved the existence of protocols for which the network of combined agents achieves consensus in the same direction as that of a network of integrators. We showed an example of heuristic consensus design based on standard protocols for higher-order integrators, using a graphical user interface developed to this purpose.

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