Observer-based iterative learning control design in the repetitive process setting ⋆

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Abstract: This paper considers the design of an observer-based iterative learning control law for discrete linear systems using repetitive process stability theory. The resulting design produces a stabilizing feedback controller in the time domain and a PD-type of feedforward controller that guarantees monotonic convergence in the trial-to-trial domain. Furthermore, the new design procedure includes limited frequency range specifications, which will be of particular interest in some applications. All design computations required for the new results in this paper can be completed using linear matrix inequalities. A simulation example is given to illustrate the theoretical developments.

Keywords: Iterative learning control, Linear repetitive processes, Linear matrix inequalities.

Iterative learning control (ILC) arose from research for systems that repeat the same finite duration task over and over again. Each repetition is termed a trial or pass in the literature and the trial (or pass) length is the name given to the finite duration of each trial. The notation for variables used in this paper is $q_k(p)$, $0 \leq p \leq \alpha - 1$, where q is the scalar or vector valued variable under consideration, $k \geq 0$, is the trial number and $\alpha < \infty$ is the number of samples along the trial for discrete dynamics (α times the sampling period gives the trial length).

One each trial is complete, all information generated is available and if stored can be used to compute the control input for the next trial. Hence it is possible to implement ILC laws that use information that would be non-causal in the standard sense, provided it has been generated on a previous trial, e.g., at sample instant p information at $p+\lambda$, $\lambda > 0$, can be used. The inclusion of such information is the distinguishing feature of ILC.

The original work on ILC (Arimoto et al., 1984) considered a derivative, or D type, law for an electric motor and since then this design method has remained as a significant area of control systems research with many algorithms experimentally verified in the research laboratory and applied in industrial applications, see, e.g., (Ahn et al., 2007; Bristow et al., 2006) as starting points for the literature and (Wang et al., 2009) is a starting point for the literature on ILC applications in the chemical process industries. Particular application examples include industrial robotics, see, e.g., (Norrlöf, 2002), where the pick and place operation common in many mass manufacturing processes is an immediate fit, and marine systems, see, e.g., (Sornmo et al., 2016). More recently, ILC algorithms first developed in the engineering domain have been used in robotic-assisted upper limb stroke rehabilitation with supporting clinical trials (Freeman et al., 2009, 2015).

Suppose, in the single-input single-output case with an immediate generalization to the multiple-input multiple-output examples, a reference signal $y_d(p)$ is available for an application. Then the error on trial k is $e_k(p) = y_d(p) - y_k(p)$, $0 \leq p \leq \alpha - 1$, where $y_k(p)$ is the output. The ILC design problem is to construct a control input sequence such that the error sequence converges from trial-to-trial. In formal terms, the requirement is to construct a control sequence $\{u_k\}$ such that

$$\lim_{k\to\infty} ||e_k|| = 0, \ \lim_{k\to\infty} ||u_k - u_\infty|| = 0, \tag{1}$$

where $||\cdot||$ is a signal norm in a suitably chosen function space with a norm-based topology and u_∞ is termed the learned control.

The finite trial length means that trial-to-trial (in k) error convergence can be achieved even if the system is unstable and hence the presence of 'growth' terms in the transient response (in p) along the trials. One option in such cases is to first design a stabilizing control law and then apply ILC to the resulting controlled system. A commonly used setting for ILC design for discrete dynamics is the lifting setting, which is based on the use of so-called supervectors. Consider again the single-input single-output case. Then since the trial length is finite the sampled values of, e.g., the output can be assembled into a column vector where the entries correspond to their sample instants along the trial. Applying this to all variables enables the error dynamics to be written as a difference equation in k, to which standard results can be applied to design the ILC law.

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An alternative approach to ILC design is to use the two-dimensional(2D)/repetitive systems setting (Rogers et al., 2007, 2015) i.e., systems that propagate information in two independent directions, where for ILC these directions are from trial-to-trial and along each trial respectively. Repetitive processes evolve over a subset of the upper right quadrant of the 2D plane and make repeated sweeps through dynamics defined over a finite duration. Hence, they are a more natural setting for ILC analysis than other 2D systems models. A particular advantage of repetitive process based ILC design is that trial-to-trial error convergence and the dynamics along the trial can be considered in one setting and also the analysis extends to differential dynamics whereas the lifting approach does not.

A common form of ILC law is the proportional plus derivative, or PD-type, consisting of proportional and derivative gains acting on the tracking error. This form of ILC law is one of the earliest developed and as with the three term control laws for standard systems has seen many implementations, see-(Bristow et al., 2006) as a starting point for further details and literature review.

This paper continues the development of the repetitive process setting for ILC design, starting with a new result on the design of PD-type ILC laws where the state feedback control is used. Implementation of such control laws, however, requires the availability for measurement of all state variables. Since this will often not be the case an logical step is to consider the use of a state observer to estimate the current trial state vector entries. The analysis shows that the design problem can be completed by formulation as a stability problem for discrete linear repetitive processes, leading to design based on Linear Matrix Inequality (LMI) computations. An extension to obtain LMI based conditions for ILC design over finite frequency regions, allowing the use of different performance specifications over particular finite frequency ranges, see also Paszke et al. (2016), is developed. This analysis is based on the generalized Kalman-Yakubovich-Popov (KYP) lemma.

Throughout this paper, the null and identity matrices with compatible dimensions are denoted by 0 and I respectively. Moreover, $\mathrm{sym}(X)$ is used to denote $X + X^T$ and $X^{\perp}$ denotes the orthogonal complement. The notation $X \succ Y$ (respectively $X \prec Y$) means that the symmetric matrix $X - Y$ is positive definite (respectively negative definite).The symbol $(\star)$ denotes block entries in symmetric matrices and $\rho(\cdot)$ and $\overline{\sigma}(\cdot)$ denote the spectral radius and maximum singular value, respectively, of their matrix arguments.

Use will also be made of the following results, where the first is the generalized KYP lemma and the second the Elimination (or Projection) Lemma.

Lemma 1. Iwasaki and Hara (2005) Consider matrices $\mathbb{A}$, $\mathbb{B}_0$, Θ and

$$\Phi = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \Psi = \begin{bmatrix} 0 & \mathrm{e}^{\mathrm{j}\omega_c} \\ \mathrm{e}^{-\mathrm{j}\omega_c} & -2\cos(\omega_d) \end{bmatrix}, \tag{2}$$

with $\omega_c = (\omega_l + \omega_u)/2$, $\omega_d = (\omega_u - \omega_l)/2$ and ω_l, ω_u satisfying $-\pi \leq \omega_l \leq \omega_u \leq \pi$. Suppose also that $\det(\mathrm{e}^{\mathrm{j}\omega} I - \mathbb{A}) \neq 0$ for all $\omega \in [\omega_l, \omega_u]$. Then the following statements are equivalent.

i) $\forall \omega \in [\omega_l, \omega_u]$

$$\begin{bmatrix} (\mathrm{e}^{\mathrm{j}\omega} I - \mathbb{A})^{-1}\mathbb{B}_0 \\ I \end{bmatrix}^* \Theta \begin{bmatrix} (\mathrm{e}^{\mathrm{j}\omega} I - \mathbb{A})^{-1}\mathbb{B}_0 \\ I \end{bmatrix} \prec 0. \tag{3}$$

ii) There exist $\mathcal{Q} \succ 0$ and a symmetric $\mathcal{P}$ such that

$$\begin{bmatrix} \mathbb{A} & \mathbb{B}_0 \\ I & 0 \end{bmatrix}^\top (\Phi \otimes \mathcal{P} + \Psi \otimes \mathcal{Q}) \begin{bmatrix} \mathbb{A} & \mathbb{B}_0 \\ I & 0 \end{bmatrix} + \Theta \prec 0. \tag{4}$$

Lemma 2. Gahinet and Apkarian (1994) Given a symmetric matrix $\Gamma \in \mathbb{R}^{p \times p}$ and two matrices Λ, Σ of column dimension p, there exists a matrix $\mathcal{W}$ such that the following inequality holds

$$\Gamma + \mathrm{sym}\{\Lambda^\top \mathcal{W} \Sigma\} \prec 0, \tag{5}$$

if, and only if the following two projection inequalities with respect to $\mathcal{W}$ are satisfied

$$\Lambda^{\perp\top} \Gamma \Lambda^{\perp} \prec 0, \; \Sigma^{\perp\top} \Gamma \Sigma^{\perp} \prec 0, \tag{6}$$

where $\Lambda^{\perp}$ and $\Sigma^{\perp}$ are arbitrary matrices whose columns form a basis of the null spaces of Λ and Σ respectively.

Next the required background on discrete linear repetitive processes is given.

1. DISCRETE LINEAR REPETITIVE PROCESSES

The state-space model of a discrete linear repetitive process has the following form (Rogers et al., 2007) over $0 \leq p \leq \alpha - 1$, $k \geq 0$

$$\begin{aligned} x_{k+1}(p+1) &= \mathcal{A} x_{k+1}(p) + \mathcal{B} u_{k+1}(p) + \mathcal{B}_0 y_k(p), \\ y_{k+1}(p) &= \mathcal{C} x_{k+1}(p) + \mathcal{D} u_{k+1}(p) + \mathcal{D}_0 y_k(p), \end{aligned} \tag{7}$$

where $\alpha < +\infty$ denotes the pass length and on pass k, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the pass profile (output) vector and $u_k(p) \in \mathbb{R}^l$ is the control input vector. The terms $\mathcal{B}_0 y_k(p)$ and $\mathcal{D}_0 y_k(p)$ represent the contribution of the previous pass profile to the current pass state and pass profile vectors respectively.

To complete the process description, it is necessary to specify the boundary conditions, i.e., the state initial vector on each pass and the initial pass profile (i.e., on pass 0). For the purposes of this paper, it is assumed that the state initial vector at the start of each new pass is of the form $x_{k+1}(0) = d_{k+1}$, $k \geq 0$, where the $n \times 1$ vector d_{k+1} has known constant entries. Also it is assumed that the entries in initial pass profile vector $y_0(p)$ are known functions of p over the pass length.

Let $\{y_k\}$ denote the pass profile sequence generated by a repetitive process. Then the unique control problem is that this pass profile sequence can contain oscillations that increase in amplitude in the pass-to-pass direction (k). Hence the stability theory for linear repetitive processes Rogers et al. (2007) requires that a bounded initial pass profile produces a bounded sequence of pass profiles $\{y_k\}$, where the bounded is defined in terms of the norm on the underlying function space.

This stability property can be enforced over the finite pass length of an example or uniformly, i.e., for all possible values of the pass length. The former property is termed asymptotic stability and the latter stability along the pass. The finite pass length means that an example can be asymptotically stable but produce dynamics with unacceptable dynamics along the pass and hence it is stability along the pass that is used in this paper, for which the following result is a starting point.

Lemma 3. Rogers et al. (2007) Suppose that the pair $\{\mathcal{A}, \mathcal{B}_0\}$ is controllable and the pair $\{\mathcal{C}, \mathcal{A}\}$ observable. Then a discrete linear repetitive process described by (7) is stable along the pass if and only if

i) $\rho(\mathcal{D}_0) < 1$,
ii) $\rho(\mathcal{A}) < 1$,
iii) all eigenvalues of $\mathcal{G}(z) = \mathcal{C}(zI - \mathcal{A})^{-1}\mathcal{B}_0 + \mathcal{D}_0$, $\forall |z| = 1$ have modulus strictly less than unity.

In terms of checking the conditions of the above result, the first two are just the standard linear systems stability condition. The third requires computations for all points on the unit circle and hence computational problems could arise. Also this condition requires frequency attenuation of the previous pass profile over the complete frequency spectrum which, by analogy with the standard linear systems case, could be very stringent.

In terms of design for performance, a physically motivated approach is to specify a reference vector and then attempt control law design to force the sequence of pass profiles to track this vector to within a specified tolerance after a number of passes have elapsed with acceptable transient dynamics along the passes. In some cases it will only be possible, or required, to ensure stability and performance over finite frequency ranges of particular interest. The choice of such ranges and the performance specifications in addition to stability can only be decided based on knowledge of the particular application under consideration.

Recently, it has been shown, by making extensive use of generalized KYP lemma, that the third condition in Lemma 3 can be replaced by LMIs. To proceed, divide this frequency range into H intervals (not necessarily containing the same number of frequencies) such that

$$[0, \pi] = \bigcup_{h=1}^{H} [\omega_{h-1}, \omega_h], \tag{8}$$

where $\omega_0 = 0$ and $\omega_H = \pi$. Then Lemma 1 can be applied at any frequency in any of these intervals and gives the following result.

Lemma 4. Paszke et al. (2016) Suppose that the entire frequency range is arbitrarily divided into H possible different frequency intervals as in (8). Then a discrete linear repetitive process described by (7) is stable along the pass if there exist matrices $S \succ 0$, W, $Q_h \succ 0$ and symmetric P_h such that the following matrix inequalities are feasible

$$\begin{bmatrix} S - W - W^T & W^T\mathcal{A} \\ \mathcal{A}^T W & -S \end{bmatrix} \prec 0, \tag{9}$$

$$\begin{bmatrix} -P_h & \mathrm{e}^{\mathrm{j}\omega_{ch}}Q_h - W & 0 & 0 \\ (\star) & \Upsilon_1 & W^T\mathcal{B}_0 & \mathcal{C}^T \\ (\star) & (\star) & -I & \mathcal{D}_0^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix} \prec 0, \tag{10}$$

for all $h = 1, \ldots, H$, where

$$\begin{aligned} \omega_{ch} &= \frac{\omega_{h-1} + \omega_h}{2},\ \omega_{dh} = \frac{\omega_h - \omega_{h-1}}{2}, \\ \Upsilon_1 &= P_h - 2\cos(\omega_{dh})Q_h + \mathcal{A}^T W + W^T\mathcal{A}. \end{aligned} \tag{11}$$

Obviously, the above result guarantees

$$\overline{\sigma}(\mathcal{G}(\mathrm{e}^{\mathrm{j}\omega})) < 1,\ \forall \omega \in [\omega_{h-1}, \omega_h], h = 1, \ldots, H,$$

where

$$\mathcal{G}(\mathrm{e}^{\mathrm{j}\omega}) = \mathcal{C}(\mathrm{e}^{\mathrm{j}\omega}I - \mathcal{A})^{-1}\mathcal{B}_0 + \mathcal{D}_0.$$

Since $\overline{\sigma}(\mathcal{G}(\mathrm{e}^{\mathrm{j}\omega})) < 1$ holds in each frequency interval then it follows from (8) that $\overline{\sigma}(\mathcal{G}(\mathrm{e}^{\mathrm{j}\omega})) < 1$ holds over entire frequency range. Moreover, the convergence speed of the sequence of pass profiles generated by a discrete linear repetitive process is also governed by $\overline{\sigma}(\mathcal{G}(\mathrm{e}^{j\theta})) < 1$ and can be increased by minimizing parameter $0 < \gamma < 1$ subject to $\overline{\sigma}(\mathcal{G}(\mathrm{e}^{j\theta})) < \gamma$ over all frequency ranges. This fact is used again later in this paper.

2. EMBEDDING ILC IN A REPETITIVE PROCESS SETTING

This paper considers the case when the process dynamics can be modeled by the following discrete linear time-invariant state-space model written in the ILC setting as

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p), \\ y_{k+1}(p) &= Cx_{k+1}(p), \end{aligned} \tag{12}$$

where on trial k, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the output vector and $u_k(p) \in \mathbb{R}^l$ is the control input vector. Also the standard forward shift operator z along discrete-time axis (p - axis) is defined as

$$zx_{k+1}(p) = x_{k+1}(p+1),$$

see Bristow et al. (2006) for the details of how the z-transform can be applied over the finite trial length without errors arising from the basic definition of this transform over an infinite interval. Hence (12) can, under the assumption that the pair $\{A, B\}$ is controllable and the pair $\{C, A\}$ observable, be equivalently represented by the transfer-function matrix

$$G(z) = C(zI - A)^{-1}B$$

and hence

$$e_{k+1}(z) = G(z)e_k(z),\ \forall k \geq 0.$$

From this point onwards, pass is used instead to emphasize the repetitive process setting for design.

A commonly used ILC strategy is to construct the current pass input as that used on the previous pass plus a correction term, i.e., a control law of the form

$$u_{k+1}(p) = u_k(p) + \Delta u_{k+1}(p),$$

where $\Delta u_{k+1}(p)$ denotes the correction term to be designed. In this paper the correction term is selected as

$$\begin{aligned} \Delta u_{k+1}(p) = &Kx_{k+1}(p) + K_1 e_k(p) \\ &+ K_2(e_k(p+1) - e_k(p)), \end{aligned} \tag{13}$$

where K, K_1 and K_2 are matrices to be designed. The control law correction term is the sum of state feedback control on the current pass plus a learning term based on the previous pass error (e_k). However, in practical applications, all of the entries in the current pass state vector may not be available for measurement and one option is to use a suitably designed state observer and hence the aim is to implement the following control law

$$\begin{aligned}\Delta u_{k+1}(p) &= K\tilde{x}_{k+1}(p)+K_1e_k(p)\\ &\quad + K_2(e_k(p+1)-e_k(p)),\\ \tilde{x}_{k+1}(p+1) &= A\tilde{x}_{k+1}(p)+Bu_{k+1}(p)\\ &\quad + L(y_{k+1}(p)-\tilde{y}_{k+1}(p)),\\ \tilde{y}_{k+1}(p) &= C\tilde{x}_{k+1}(p),\end{aligned} \tag{14}$$

where $\tilde{x}$ is the estimated state vector and $\tilde{y}$ the resulting pass profile vector.

Application of (14) to (12) results in controlled process dynamics described by

$$\begin{aligned}\hat{x}_{k+1}(p+1) &= \widehat{A}+\widehat{B}\widehat{K}\widehat{C}\hat{x}_{k+1}(p)+\widehat{B}_1Me_k(p)\\ &\quad +\widehat{B}_1K_2e_k(p+1)+\widehat{B}_1u_k(p),\\ y_{k+1}(p) &= \widehat{C}_1\hat{x}_{k+1}(p),\end{aligned} \tag{15}$$

where $M=K_1-K_2$ and

$$\hat{x}_{k+1}(p)=\begin{bmatrix} x_{k+1}(p)\\ x_{k+1}(p)-\tilde{x}_{k+1}(p)\end{bmatrix}, \widehat{B}_1=\begin{bmatrix} B\\ 0\end{bmatrix}, \widehat{C}=\begin{bmatrix} I & -I\\ 0 & C\end{bmatrix},$$

$$\widehat{K}=\begin{bmatrix} K & 0\\ 0 & L\end{bmatrix}, \widehat{A}=\begin{bmatrix} A & 0\\ 0 & A\end{bmatrix}, \widehat{B}=\begin{bmatrix} B & 0\\ 0 & -I\end{bmatrix}, \widehat{C}_1=[C,\ 0].$$

Also, the transfer-function matrix description describing the coupling e_k to y_{k+1} in this last state-space model is

$$\begin{aligned}H(z) &= \widehat{C}_1(zI-(\widehat{A}+\widehat{B}\widehat{K}\widehat{C}))^{-1}\widehat{B}_1(M+zK_2)\\ &= \widehat{C}_1(zI-(\widehat{A}+\widehat{B}\widehat{K}\widehat{C}))^{-1}\widehat{B}_1M\\ &\quad +\widehat{C}_1z(zI-(\widehat{A}+\widehat{B}\widehat{K}\widehat{C}))^{-1}\widehat{B}_1K_2.\end{aligned} \tag{16}$$

Assuming $(zI-(\widehat{A}+\widehat{B}\widehat{K}\widehat{C}))^{-1}$ exists

$$(zI-(\widehat{A}+\widehat{B}\widehat{K}\widehat{C}))^{-1}(zI-(\widehat{A}+\widehat{B}\widehat{K}\widehat{C}))=I,$$

and hence and hence

$$z(zI-(\widehat{A}+\widehat{B}\widehat{K}\widehat{C}))^{-1}=I+(zI-(\widehat{A}+\widehat{B}\widehat{K}\widehat{C}))^{-1}(\widehat{A}+\widehat{B}\widehat{K}\widehat{C}).$$

Therefore

$$\begin{aligned}H(z) &= \widehat{C}_1(zI-(\widehat{A}+\widehat{B}\widehat{K}\widehat{C}))^{-1}\widehat{B}_1M\\ &\quad +\widehat{C}_1(I+(zI-(\widehat{A}+\widehat{B}\widehat{K}\widehat{C}))^{-1}(\widehat{A}+\widehat{B}\widehat{K}\widehat{C}))\widehat{B}_1K_2\\ &= \widehat{C}_1(zI-(\widehat{A}+\widehat{B}\widehat{K}\widehat{C}))^{-1}(\widehat{B}_1M+(\widehat{A}+\widehat{B}\widehat{K}\widehat{C})\widehat{B}_1K_2)\\ &\quad +\widehat{C}_1\widehat{B}_1K_2.\end{aligned}$$

Also the tracking error on pass k is given by

$$E_k(z)=Y_d(z)-Y_k(z)=Y_d(z)-G(z)U_k(z),$$

where $Y_d(z)$ is the z-transform of the reference trajectory. Hence $E_{k+1}(z)$ can be written as

$$E_{k+1}(z)-E_k(z)=-G(z)\left(U_{k+1}(z)-U_k(z)\right). \tag{17}$$

To focus on pass-to-pass error convergence introduce $M(z)$ as

$$M(z)=-H(z)+I,$$

where $H(z)$ is defined in (16) and then the propagation of the error from pass-to-pass is given by

$$E_{k+1}(z)=M(z)E_k(z).$$

Hence the tracking error converges as $k\to\infty$, if and only if all eigenvalues of $M(z)$ are less than one in magnitude, i.e.

$$\rho\left(M(e^{j\omega})\right)<1,\ \forall\omega\in[-\pi,\pi]. \tag{18}$$

This last condition is not a convenient basis for control law design but can be reformulated in an $\mathcal{H}_\infty$ setting, starting from the sufficient condition for monotonic pass-to-pass error convergence

$$\overline{\sigma}(M(\mathrm{e}^{j\omega}))<1,\ \forall\omega\in[-\pi,\pi]. \tag{19}$$

Also

$$\|M(z)\|_\infty\triangleq\max_{\omega\in[-\pi,\pi]}\overline{\sigma}(M(\mathrm{e}^{j\omega})),$$

and hence the pass-to-pass error convergence problem for ILC schemes can be reformulated as an $\mathcal{H}_\infty$ control problem. Moreover, since

$$\|e_{k+1}(p)-e_\infty(p)\|_2=\|E_{k+1}-E_\infty\|_2\le\|M\|_\infty\|E_k-E_\infty\|_2$$

with $\|M\|_\infty<1$, $\|e_{k+1}(p)-e_\infty(p)\|_2$ converges monotonically to zero as $k\to 0$.

The controlled dynamics can be written as

$$\begin{aligned}x_{k+1}(p+1) &= \mathcal{A}x_{k+1}(p)+\mathcal{B}_0e_k(p),\\ e_{k+1}(p) &= \mathcal{C}x_{k+1}(p)+\mathcal{D}_0e_k(p),\end{aligned} \tag{20}$$

where

$$\begin{aligned}&\mathcal{A}=\widehat{A}+\widehat{B}\widehat{K}\widehat{C},\ \mathcal{B}_0=(\widehat{B}_1M+(\widehat{A}+\widehat{B}\widehat{K}\widehat{C})\widehat{B}_1K_2),\\ &\mathcal{C}=-\widehat{C}_1, \mathcal{D}_0=I-\widehat{C}_1\widehat{B}_1K_2.\end{aligned} \tag{21}$$

The state-space model (20) is a discrete linear repetitive process of the form (7) where on pass $k+1$, $x_{k+1}(p)$ is the state vector $e_k(p)$ is the previous pass term and there is no input term. Stability along the pass for this repetitive process model requires that $\{e_k\}$ converges to zero as $k\to\infty$ independent of the pass length or in ILC terms the monotonic pass-to-pass error convergence to zero occurs independent of the pass length.

3. LMI BASED DESIGN OVER FINITE FREQUENCY DOMAINS

Given the condition for monotonic convergence, the main result of the paper is developed and guarantees monotonic pass-to-pass error convergence under finite frequency design specifications. The possibility of specifying different performance specifications has considerable practical significance since common performance issues occur over different frequency ranges. For example, the pass-to-pass error convergence rate is in the 'low' frequency range whereas low sensitivity to disturbances and sensor noise are in the 'high' frequency range.

Direct application of Lemma 4 is not possible in this case since the resulting inequalities are not LMIs as they are bilinear in W and the matrices defining the controller $\widehat{K}$. To overcome problems with converting (9) and (10) into LMIs one approach is to assume that $W=\mathrm{diag}\{W_1,W_1\}$, i.e. W has a block diagonal structure that may introduce some level of conservativeness. Moreover it is required to find a matrix X such that

$$CW_1=XC. \tag{22}$$

It is known that the above equality is equivalent to imposing a block diagonal structural constraint on the matrix variable W. To proceed, assume that C is full rank. Then the singular value decomposition (SVD) of C can be written as $C=U[R\ 0]V^T$ where U and V are unitary matrices and R is a diagonal matrix with positive diagonal elements in decreasing order. Next, take

$$W_1=V\begin{bmatrix} W_{11} & 0\\ W_{21} & W_{22}\end{bmatrix}V^T, \tag{23}$$

and suppose X is chosen as $X = URW_{11}R^{-1}U^T$. Then

$$XC = URW_{11}[I\ 0]V^T = UR[I\ 0]V^TV\begin{bmatrix} W_{11} & 0 \\ W_{21} & W_{22}\end{bmatrix}V^T = CW_1.$$

Also the matrix structure as in (23) enables the result of Lemma 4 to reformulated as an LMI-based characterization for the existence of an observer L and controllers K, K_1, K_2 of (14).

Theorem 1. Consider an ILC scheme described as a discrete linear repetitive process of the form (20) and (21). Furthermore, suppose that the entire frequency range is arbitrarily divided into H possible different frequency intervals as in (8). Then a discrete linear repetitive process described by (7) is stable along the pass and hence monotonic pass-to-pass error convergence occurs if there exist matrices $\mathcal{W}_1$ of the form (23), $\mathcal{S} \succ 0$, $\mathcal{Q}_h \succ 0$, N_1, N_2, M, K_2, and symmetric $\mathcal{P}_h$ such that the following LMIs are feasible

$$\begin{bmatrix} \mathcal{S}-\mathcal{W}-\mathcal{W}^T & \widehat{A}\mathcal{W}+\widehat{B}N\widehat{C} \\ (\widehat{A}\mathcal{W}+\widehat{B}N\widehat{C})^T & -\mathcal{S}\end{bmatrix} \prec 0, \tag{24}$$

$$\begin{bmatrix} -\mathcal{P}_h & e^{j\omega_{ch}}\mathcal{Q}_h-\mathcal{W} & 0 & 0 & 0 \\ (\star) & \Upsilon_2 & \widehat{B}_1M & \mathcal{W}\widehat{C}^T & \widehat{A}\mathcal{W}+\widehat{B}N\widehat{C} \\ (\star) & (\star) & -I & (I+\widehat{C}\widehat{B}_1K_2)^T & K_2^T\widehat{B}_1^T \\ (\star) & (\star) & (\star) & -I & 0 \\ (\star) & (\star) & (\star) & 0 & -\mathcal{W}-\mathcal{W}^T \end{bmatrix} \prec 0, \tag{25}$$

for all $h = 1, \ldots, H$, where ω_{ch}, ω_{dh} are defined in (11) and

$$\Upsilon_2 = \mathcal{P}_h - 2\cos(\omega_{dh})\mathcal{Q}_h + \mathrm{sym}(\widehat{A}\mathcal{W}+\widehat{B}N\widehat{C}),$$
$$N = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix},\ \mathcal{W} = \begin{bmatrix} \mathcal{W}_1 & 0 \\ 0 & \mathcal{W}_1 \end{bmatrix}.$$

If the above set of LMIs is feasible, the required control law matrices K, L, K_1 and K_2 in (14) can be computed directly from the solution of this set as

$$K = N\mathcal{W}^{-1},\ L = N_2X^{-1},\ K_1 = K_2 + M,$$

where $X = UR\mathcal{W}_{11}R^{-1}U^T$ and U, R are computed by SVD of C given above (23).

Proof. Suppose that the LMIs (24) and (25) are feasible. Then $\mathcal{W}+\mathcal{W}^T \succ \mathcal{S} \succ 0$, which implies that $\mathcal{W}$ is invertible. Next post- and pre-multiply (24) by diag$\{\mathcal{W}^{-1}, \mathcal{W}^{-1}\}$ to obtain (9) on also setting $W = \mathcal{W}^{-1}$ and $S = \mathcal{W}^{-T}\mathcal{S}\mathcal{W}^{-1}$. Next, introduce the notation

$$\Omega_h = \begin{bmatrix} -\mathcal{P}_h & e^{j\omega_{ch}}\mathcal{Q}_h-\mathcal{W} & 0 & 0 \\ (\star) & \Upsilon_2 & \widehat{B}_1M & \mathcal{W}\widehat{C}^T \\ (\star) & (\star) & -I & (I+\widehat{C}\widehat{B}_1K_2)^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix}, N = \widehat{K}\mathcal{W},$$

$$\mathcal{U}^T = \begin{bmatrix} 0 & (\widehat{A}+\widehat{B}\widehat{K}\widehat{C})^T & 0 & 0 \end{bmatrix}, \mathcal{J} = \begin{bmatrix} 0 & 0 & K_2^T\widehat{B}_1^T & 0 \end{bmatrix},$$

and rewrite (25) as

$$\begin{bmatrix} \Omega_h & \mathcal{J} \\ \mathcal{J}^T & 0 \end{bmatrix} + \mathrm{sym}\left\{\begin{bmatrix} 0 \\ I \end{bmatrix}\mathcal{W}^T\begin{bmatrix}\mathcal{U}^T & -I\end{bmatrix}\right\} \prec 0.$$

Hence, by the result of Lemma 2, (25) holds if and only if

$$\Omega_h + \mathcal{U}\mathcal{J}^T + \mathcal{J}\mathcal{U}^T \prec 0,$$

and this last inequality can be rewritten as

$$\begin{bmatrix} -\mathcal{P}_h & e^{j\omega_{ch}}\mathcal{Q}_h-\mathcal{W} & 0 & 0 \\ (\star) & \Upsilon_2 & \mathcal{B}_0 & \mathcal{W}\widehat{C}^T \\ (\star) & (\star) & -I & \mathcal{D}_0^T \\ (\star) & (\star) & (\star) & -I \end{bmatrix} \prec 0. \tag{26}$$

Next, post- and pre-multiply the above inequality by diag$\{\mathcal{W}^{-1}, \mathcal{W}^{-1}, I, I\}$ and its transpose respectively. Finally, set $P_h = \mathcal{W}^{-T}\mathcal{P}_h\mathcal{W}^{-1}$ and $Q_h = \mathcal{W}^{-T}\mathcal{Q}_h\mathcal{W}^{-1}$ to establish that (26) is equivalent to (10) and the proof is complete.

In physical applications, the effects of high frequency noise and non-repeating disturbances degrade the performance of an ILC design. When the tracking error is attenuated, any high frequency noise present may be amplified and one solution to this problem is Q-filtering to limit the frequency range of the learning for stability and noise attenuation. In such a situation, the result of Theorem 1 can bee used to design both the feedback and learning controllers for error convergence and performance in prescribed frequency ranges. These frequency ranges can be chosen by inspection of frequency spectrum of the signal to be tracked. Then Q-filter has to be chosen as a low-pass filter with (ideally) unity magnitude for low for the frequency range where reference tracking is required and zero at all other frequencies.

4. APPLICATION CASE STUDY

In this section a numerical example is considered that illustrate the effectiveness and application of the new result in this paper. The example uses a model of laboratory servomechanism system that consists of a DC motor and an inertial mass that are connected through the rigid shaft. Rotational motion of the mass is exited by the DC motor, where the rotational speed of the mass is considered as the output and the armature voltage as the input. Hence the transfer-function model

$$G(s) = \frac{\dot{\Theta}(s)}{V(s)} = \frac{K}{(Js+b)(Ls+R)+K^2}, \tag{27}$$

where K represents both the motor torque constant (K_t) and the back emf constant (K_e), J is the total moment of inertia of the rotor and the mass, b is the motor viscous friction constant, L is the electrical inductance and R denotes the electrical resistance. Choosing $J = 0.001118$, $b = 3.5077 \cdot 10^{-6}$, $K = 0.056$, $R = 2$ and $L = 0.001$ gives

$$G(s) = \frac{0.056}{1.118\cdot 10^{-6}s^2 + 0.002236s + 0.003143}. \tag{28}$$

This transfer-function has been discretized with a sampling time of $T_s = 0.001$ seconds to give a discrete linear state-space model of the form (12) with

$$A = \begin{bmatrix} 1.1341 & -0.2707 \\ 0.5000 & 0 \end{bmatrix},\ B = \begin{bmatrix} 0.1250 \\ 0 \end{bmatrix}, C = [0.1137\ 0.1190].$$

The reference trajectory has been chosen to include intervals when acceleration, constant rotational speed and deceleration are required, see Fig. 1. Examining the reference trajectory in the frequency domain confirms that this signal consists of harmonics between 0 and 2 Hz only. Hence, it is reasonable to choose the cutoff frequency of Q-filter as approximately 2 Hz because frequencies from 0 to 2 Hz have only to be emphasized in the design. Applying Theorem 1 for the frequency range 0 to 2 Hz gives

$$K = [-5.5963\ \ -0.4141], L^T = [9.0648\ 3.4427],$$
$$K_1 = 27.5131, K_2 = 36.9547.$$

The controlled dynamics were simulated over 30 passes and for each one the Root M ean Square (RMS) tracking

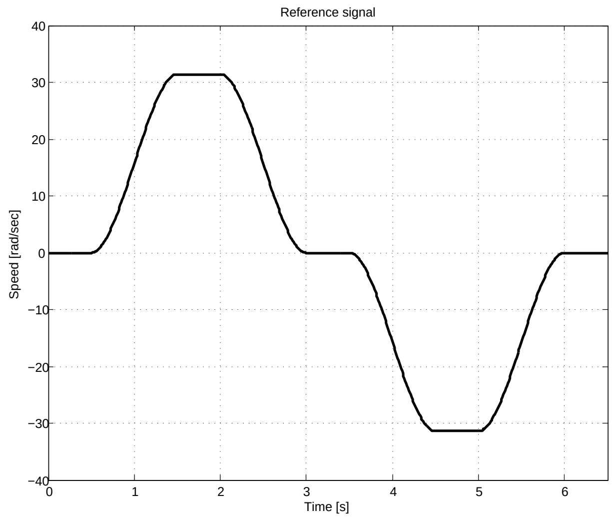

Fig. 1. The reference trajectory.

error was calculated using $\mathrm{RMS}(e) = \sqrt{\frac{1}{\alpha}\sum_{p=1}^{N} e(p)^2}$, where $N = 6500$ for the 1000 Hz samplingfrequency and a pass length of 6.5 seconds. Fig. 2 shows the RMS values of the tracking error as a function of the pass number. Also noise with RMS=10^{-5} is included and the convergence curve stays at the level reached after 10 passes. Simulations,

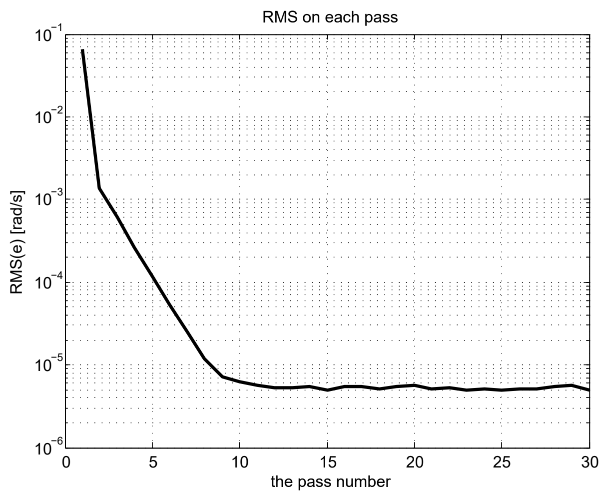

Fig. 2. RMS values of the error over 30 passes.

not shown due to space limitations, of the response of the controlled system confirm that monotonic pass-to-pass error convergence occurs.

5. CONCLUSIONS

This paper has developed a new algorithm for the design of an ILC law for discrete linear systems that combines a PD-type learning function with an observer-based state feedback. The new result also allows design specifications to be imposed over finite frequency domains, where there can vary from one frequency range to another. This is in contrast to other designs where a single specification has to be imposed over the complete frequency range. Furthermore, previous results in the repetitive process setting required storage of current and previous pass state vectors to implement the control law, a requirement that is removed by this new design. The results have been established by using a version of the KYP lemma to transform frequency domain specifications into LMIs and hence are easily computed by numerical software. The new design has been illustrated by a simulation study using a model of laboratory servomechanism system.

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