Reduction of discrete linear repetitive processes to nonsingular Roesser models via elementary operations ⋆

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Abstract: A method based on the elementary operations algorithm (EOA) is developed that reduces a system matrix describing a discrete linear repetitive process to a 2-D nonsingular Roesser form such that all the input-output properties, including the transfer-function matrix, are preserved. Some areas for possible future use/application of the developed results will also be briefly discussed.

Keywords: Linear repetitive processes, 2-D discrete systems, System matrix, 2-D non-singular Roesser form, Input-output equivalence.

1. INTRODUCTION

Various state-space models, e.g., (Roesser, 1975) and (Fornasini and Marchesini, 1976), have been proposed for 2D discrete linear systems and used to develop systems theoretic properties, such as controllability and observability. Other work has extended these models to 2D discrete singular linear systems, e.g., (Zak, 1984) based on the Roesser model and (Kaczorek, 1988) based on the Fornasini-Marchesini model. Furthermore, Galkowski (Galkowski, 2001) developed an algorithm for reducing 2D, with an extension to nD ($n \geq 2$), rational polynomial matrices to those that arise for singular or nonsingular Roesser forms. More recent research reported on this approach includes (Xu et al., 2012) and the cited references.

Repetitive processes make a series of sweeps through dynamics defined over a finite duration. The terms pass and pass length are used in the literature to denote in the sweep number and its associated duration respectively. On each pass an output, termed the pass profile, is produced and the 2D systems structure of these processes arises because the pass profile on any pass acts as a forcing function on the dynamics produced on the next pass, with the result that oscillations can appear that increase in amplitude from pass-to-pass. These oscillations cannot be removed by standard, or 1D, control action.

Background on repetitive processes, including references to the original work on the modeling of physical processes, can be found in (Rogers et al., 2007). A stability theory for these processes has also been developed and this has been extended to control law design, including experimental verification when applied to the design of iterative learning control laws, see. e.g., (Rogers et al., 2007; Paszke et al., 2016). This application to iterative learning control is an example where systems theory for repetitive processes can be employed to solve problems in other areas.

The most commonly studied 2D linear systems have dynamics that are defined over the complete upper-right quadrant of the 2D plane whereas repetitive process dynamics evolve over a subset of this quadrant. Hence the possibility that systems theory developed for repetitive processes can be employed in the analysis of 2D systems and vice versa. This general approach has resulted in the development of conditions under which discrete linear repetitive processes have the local reachability/obdervability property, which both have well defined physical physical meaning, by transforming the repetitive process dynamics to an equivalent Roesser or Fornasini Marchesini model description (Galkowski et al., 1998). Related results can be found in (Galkowski et al., 1999; Rogers et al., 2007).

Of particular interest in this paper is the systems matrix for these two classes of linear systems. Previous work (Boudellioua et al., 2017) developed a direct method for the reduction of the polynomial matrix description for linear repetitive processes (Rogers et al., 2007) to an equivalent form that is the system matrix for a 2D singular Roesser state-space model. The results in this paper extend the analysis to obtain a system matrix corresponding to a nonsingular Roesser model, including

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the structure of the transformation in the form of input-output equivalence. Input-output equivalence has been extensively studied in the 2D/nD linear systems literature for nonsingular models, e.g., (Levy, 1981), (Johnson, 1993) and (Pugh et al., 1996, 1998) and hence the possibly of solving open questions for repetitive processes by exploiting the equivalence established in this paper.

2. DISCRETE LINEAR REPETITIVE PROCESSES

The state-space model of a discrete linear repetitive process as given by (Rogers et al., 2007), has the following form over the two indeterminates p and k where $0 \leq p \leq \alpha - 1,\ k \geq 0$

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + B_0 y_k(p) + Bu_{k+1}(p), \\ y_{k+1}(p) &= Cx_{k+1}(p) + D_0 y_k(p) + Du_{k+1}(p), \end{aligned} \tag{1}$$

where $\alpha < +\infty$ denotes the finite pass length and on pass $k \geq 0$, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k(p) \in \mathbb{R}^m$ is the pass profile vector, which also serves as a system output, and $u_k(p) \in \mathbb{R}^l$ is the input vector. The system description is completed by specifying the boundary conditions, i.e., the state initial vector on each pass and the initial pass profile (i.e. on the 0th pass).

In this paper, it is assumed that the state initial vector at the start of each new pass is of the form $x_{k+1}(0) = d_{k+1},\ k \geq 0$ and $y_0(p) = f(p),\ 0 \leq p \leq \alpha - 1$ where the $n \times 1$ vector d_{k+1} has known constant entries and those in the $m \times 1$ vector $f(p)$ are known functions of p.

In the Roesser 2D state-space model (Roesser, 1975), a state vector is defined for axes, commonly termed vertical and horizontal respectively. Denoting these vectors by $x^h(i,j) \in \mathbb{R}^n$ and $x^v(i,j) \in \mathbb{R}^m$, and introducing the output vector $y(i,j) \in \mathbb{R}^q$ and the input vector $u(i,j) \in \mathbb{R}^l$, the Roesser state-space model has the structure:

$$\begin{aligned} \begin{bmatrix} x^h(i+1,j) \\ x^v(i,j+1) \end{bmatrix} &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(i,j), \\ y(i,j) &= [\, C_1 \ C_2 \,] \begin{bmatrix} x^h(i,j) \\ x^v(i,j) \end{bmatrix} + \mathbb{D}u(i,j). \end{aligned} \tag{2}$$

Boundary conditions are defined as $x^h(0,j) = f(j),\ j \geq 0$ and $x^v(i,0) = d(i),\ i \geq 0$, where the $n \times 1$ vector $f(j)$ and the $m \times 1$ vector $d(i)$ have known constant entries.

In a repetitive process the 2D systems structure arises from the influence of the previous pass profile on the current pass state and pass profile vectors, i.e., from the terms $B_0 y_k(p)$ and $D_0 y_k(p)$ in (1) respectively. The updating structures of the Roesser model and its counterpart for a discrete linear repetitive process is shown in Fig. 1. These are different and therefore it is to be expected that systems theory for one of them will always apply to the other but where it can be established that this is possible then benefit may result.

3. SYSTEM EQUIVALENCE

The concept of a polynomial system matrix is given in, e.g., (Rosenbrock, 1970) for standard, also termed 1D in some of the nD systems literature, linear systems. The natural generalization to 2D linear systems is by a polynomial system description of the dynamics in the form, under zero boundary conditions,

$$\begin{aligned} T(z_1,z_2)x &= U(z_1,z_2)u, \\ y &= V(z_1,z_2)x + W(z_1,z_2)u, \end{aligned} \tag{3}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^p$ is the input vector and $y \in \mathbb{R}^m$ is the output vector, T, U, V and W are polynomial matrices with elements in $\mathbb{R}[z_1, z_2]$ of dimensions $n \times n, n \times p, m \times n$ and $m \times p$ respectively. The operators z_1 and z_2 have various meanings depending on the type of system considered, e.g., in delay-differential systems z_1 represents a differential operator and z_2 a delay-operator. For 2D discrete systems, z_1 and z_2 represent horizontal and vertical shift operators, respectively, which is the only case considered in this paper.

The system (3) with zero boundary conditions gives rise to the system matrix:

$$P(z_1,z_2) = \begin{bmatrix} T(z_1,z_2) & U(z_1,z_2) \\ -V(z_1,z_2) & W(z_1,z_2) \end{bmatrix}, \tag{4}$$

where

$$P(z_1,z_2) \begin{bmatrix} x \\ -u \end{bmatrix} = \begin{bmatrix} 0 \\ -y \end{bmatrix}. \tag{5}$$

If $T(z_1,z_2)$ is invertible, the system matrix in (4) is said to be regular. The transfer-function matrix corresponding to the system matrix in (4) is given by:

$$G(z_1,z_2) = V(z_1,z_2)T^{-1}(z_1,z_2)U(z_1,z_2) + W(z_1,z_2). \tag{6}$$

One of the most fundamental requirements for the equivalence of two system matrices is the preservation of the input-output (I/O) behavior. Invariant I/O behavior is ensured by the equality of the transfer function matrices.

Definition 1. (Johnson et al. (1996)). Let $\mathbb{T}(m,n)$ denote the class of $(r+m) \times (r+n)$ rational matrices where $r > \min(m,n)$. The subset $\mathbb{P}(m,n)$ of $\mathbb{T}(m,n)$ obtained by requiring $r > 0$ represents the 2D rational system matrices. Two system matrices $P_1(z_1,z_2)$ and $P_2(z_1,z_2) \in \mathbb{P}(m,n)$, are said to be I/O equivalent if they have the same transfer function matrix, i.e.,

$$G_1(z_1,z_2) = G_2(z_1,z_2). \tag{7}$$

The following is a characterization of I/O equivalence for the case considered.

Lemma 1. (Johnson et al. (1996)). Two system matrices $P_1(z_1,z_2)$ and $P_2(z_1,z_2) \in \mathbb{P}(m,n)$, are I/O equivalent if and only if there exist rational matrices $M(z_1,z_2)$, $N(z_1,z_2)$, $X(z_1,z_2)$, and $Y(z_1,z_2)$ such that

$$\begin{aligned} &\begin{bmatrix} M(z_1,z_2) & 0 \\ X(z_1,z_2) & I_m \end{bmatrix} \begin{bmatrix} T_2(z_1,z_2) & U_2(z_1,z_2) \\ -V_2(z_1,z_2) & W_2(z_1,z_2) \end{bmatrix} \\ &= \begin{bmatrix} T_1(z_1,z_2) & U_1(z_1,z_2) \\ -V_1(z_1,z_2) & W_1(z_1,z_2) \end{bmatrix} \begin{bmatrix} N(z_1,z_2) & Y(z_1,z_2) \\ 0 & I_p \end{bmatrix}. \end{aligned} \tag{8}$$

4. SYSTEM MATRICES OF LINEAR REPETITIVE PROCESSES

In this section linear repetitive processes described by (1) are considered under the assumption of zero boundary conditions. Introduce the forward shift operators z_1 in the pass-to-pass direction and z_2 in the along the pass direction, i.e.,

$$z_1 s_k(p) = s_{k+1}(p), \quad z_2 s_k(p) = s_k(p+1) \tag{9}$$

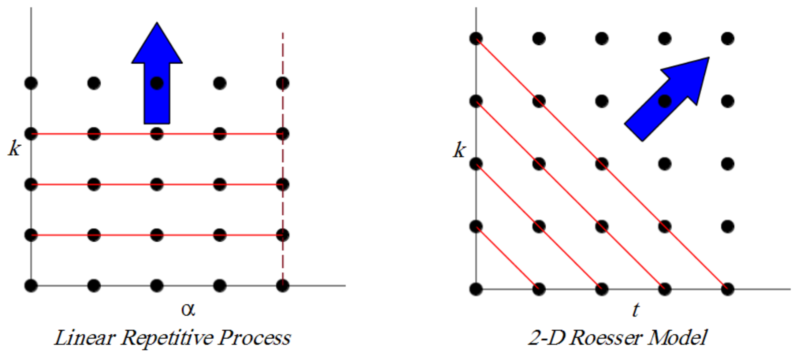

Fig. 1. Evolution of the dynamics of a discrete linear repetitive process and the Roesser model

respectively, where the signal $s_k(p)$ represents $x_k(p)$ or $y_k(p)$ as appropriate. The process dynamics can now be written in polynomial matrix form as:

$$P_{RP}(z_1,z_2)\left[\begin{array}{c} x_k(p) \\ y_k(p) \\ \hline -u_k(p) \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \\ \hline -y_k(p) \end{array}\right], \tag{10}$$

where

$$P_{RP}(z_1,z_2) = \left[\begin{array}{cc|c} z_1z_2I_n - z_1A & -B_0 & z_1B \\ -z_1C & z_1I_m - D_0 & z_1D \\ \hline 0_{m,n} & -I_m & 0_{m,p} \end{array}\right] \tag{11}$$

is the system matrix associated with (1). Alternatively the dynamics of (1) can be represented in transfer-function matrix form as:

$$Y(z_1,z_2) = G_{RP}(z_1,z_2)U(z_1,z_2), \tag{12}$$

where

$$\begin{aligned} &G_{RP}(z_1,z_2) \\ &= [\,0\ \ I_m\,]\begin{bmatrix} z_1z_2I - z_1A & -B_0 \\ -z_1C & z_1I - D_0 \end{bmatrix}^{-1}\begin{bmatrix} z_1B \\ z_1D \end{bmatrix}. \end{aligned} \tag{13}$$

The following example illustrates the computation of the system matrix and 2D transfer-function matrix.

Example 1. Consider the discrete repetitive process with state-space model matrices:

$$\begin{aligned} A &= \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \\ C &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad D_0 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}. \end{aligned} \tag{14}$$

The polynomial system matrix corresponding to (14) is:

$$P_{RP}(z_1,z_2) \tag{15}$$

$$= \left[\begin{array}{cccc|cc} z_1z_2 - z_1 & z_1 & 0 & -1 & z_1 & 0 \\ -2\,z_1 & z_1z_2 & -2 & -1 & z_1 & z_1 \\ 0 & 0 & z_1+1 & 0 & z_1 & 0 \\ 0 & -z_1 & 0 & z_1 & 0 & -z_1 \\ \hline 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \end{array}\right] \tag{16}$$

and the 2-D transfer-function matrix is:

$$G_{RP}(z_1,z_2) = \begin{bmatrix} \dfrac{z_1}{z_1+1} & 0 \\ \dfrac{z_1\,(z_1z_2 + z_1 + 3\,z_2 - 1)}{(z_1+1)\,(z_1{z_2}^2 - z_1z_2 + 2\,z_1 - z_2 - 1)} & -\dfrac{z_1\,({z_2}^2 - 2\,z_2 + 3)}{z_1{z_2}^2 - z_1z_2 + 2\,z_1 - z_2 - 1} \end{bmatrix}. \tag{17}$$

5. REDUCTION OF A DISCRETE LINEAR REPETITIVE PROCESS TO NONSINGULAR ROESSER FORM

In (Boudellioua et al., 2017) an equivalence between discrete linear repetitive processes and a singular 2D Roesser model was established, which is central to the analysis in the remainder of this paper. Consider the polynomial system matrix $P_{RP}(z_1,z_2)$ of (11) which describes the dynamics of the discrete linear repetitive process (1). Introducing the new state vector

$$\nu(k,p) = [x_k^T(p)\ \ y_k^T(p)]^T \tag{18}$$

$P_{RP}(z_1,z_2)$ can be written in the modified 2D singular form

$$\begin{aligned} &\tilde{P}_{RP}(z_1,z_2) \\ &= \left[\begin{array}{c|c} z_1z_2\tilde{E} - z_1\tilde{A}_1 - z_2\tilde{A}_2 - \tilde{A}_0 & z_1\tilde{B}_1 + z_2\tilde{B}_2 + \tilde{B}_0 \\ \hline -\tilde{C} & \tilde{D} \end{array}\right], \end{aligned} \tag{19}$$

where

$$\begin{aligned} &\tilde{E} = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \tilde{A}_1 = \begin{bmatrix} A & 0 \\ C & -I_m \end{bmatrix}, \tilde{A}_2 = 0, \tilde{A}_0 = \begin{bmatrix} 0 & B_0 \\ 0 & -D_0 \end{bmatrix}, \\ &\tilde{B}_1 = \begin{bmatrix} B \\ D \end{bmatrix}, \ \tilde{B}_2 = \tilde{B}_0 = 0, \ \tilde{C} = [\,0\ \ I_m\,], \ \tilde{D} = 0. \end{aligned} \tag{20}$$

This polynomial system matrix is zero coprime equivalent to a Roesser singular type system matrix

$$\begin{aligned} &\tilde{P}_{SR}(z_1,z_2) \\ &= \left[\begin{array}{cccc|c} I_{n+m} & -z_1I_{n+m} & 0_{n+m,p} & 0_{n+m,m} & 0_{n+m,p} \\ z_2\tilde{E} - \tilde{A}_1 & -\tilde{A}_0 & z_1\tilde{B}_1 & 0_{n+m,m} & 0_{n+m,p} \\ 0_{m,n+m} & -\tilde{C} & 0_{m,p} & I_m & 0_{m,p} \\ 0_{p,n+m} & 0_{p,n+m} & -I_p & 0_{p,m} & I_p \\ \hline 0_{m,n+m} & 0_{m,n+m} & 0_{m,p} & -I_m & 0_{m,p} \end{array}\right], \end{aligned} \tag{21}$$

by applying appropriate block elementary operations (Boudellioua et al., 2017). Also the extended polynomial matrix $\begin{bmatrix} I_{n+m} & 0 \\ 0 & \tilde{P}_{SR}(z_1,z_2) \end{bmatrix}$ can be transformed by use of appropriate block elementary operations to the singular Roesser form

$$\begin{aligned}&\widehat{P}_{SR}(z_1,z_2)\\&=E\left[\begin{array}{cc|c} z_2I_M-H_{11} & -H_{12} & -H_{13}\\ -H_{21} & z_1I_N-H_{22} & -H_{23}\\ \hline -H_{31} & -H_{32} & -H_{33}\end{array}\right]\end{aligned} \tag{22}$$

and hence is also zero coprime equivalent to the repetitive process polynomial description $P_{RP}(z_1,z_2)$ of (11). The indeterminate k (p) in the repetitive process description refers to the indeterminate j (i) in the Roesser model and hence due to (9) z_1 and z_2 are exchanged. An alternative way of deriving this result is to use a non–block procedure, i.e., the EOA algorithm (Galkowski, 2001).

It was shown in (Galkowski, 2001) that, in some cases, the singular 2D Roesser model can be reduced to one that is nonsingular by using the inverses of the variables z_1 and z_2, i.e., the substitutions $z_1=\widehat{z_1}^{-1}$ and $z_2=\widehat{z_2}^{-1}$. However, the inverted variables $\widehat{z}_i,\ \ i=1,2$ correspond to a reversed time shift and the resulting system would be non-causal and hence not physically realizable

To avoid this last situation from arising, the route in this paper is to first invert variables in the considered transfer-function matrix, i.e., make the substitutions $z_1=\widehat{z_1}^{-1}$ and $z_2=\widehat{z_2}^{-1}$, then obtain the auxiliary singular Roesser model represented by a polynomial system matrix of the form $\widehat{P}_{SR}(z_1,z_2)$ of (22) (with variables replaced by their inverses), using the results in (Boudellioua et al., 2017) and (Galkowski, 2001). Next, form the following block matrices

$$\widehat{H}_{11}=\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix},\ \widehat{H}_{12}=\begin{bmatrix}H_{13}\\ H_{23}\end{bmatrix},$$
$$\widehat{H}_{21}=\left[H_{31}\ \ H_{32}\right],\ \widehat{H}_{22}=H_{33}.$$

and apply the substitutions $z_1=\widehat{z_1}^{-1}$ and $z_2=\widehat{z_2}^{-1}$, i.e. return to the original variables $z_1,\ z_2$. Then the nonsingular Roesser state-space model matrices can be obtained from this starting point as follows, see (Galkowski, 2001)

$$\begin{aligned}&\begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}=\widehat{H}_{11}^{-1}E',\ \begin{bmatrix}B_1\\ B_2\end{bmatrix}=-\widehat{H}_{11}^{-1}\widehat{H}_{12},\\ &[C_1\ \ C_2]=\widehat{H}_{21}\widehat{H}_{11}^{-1}E',\ \ \mathbb{D}=\widehat{H}_{22}-\widehat{H}_{21}\widehat{H}_{11}^{-1}E'\widehat{H}_{12},\end{aligned} \tag{23}$$

provided that $\widehat{H}_{11}$ is nonsingular. Also the system matrix for the nonsingular Roesser form associated with the repetitive process (11) is given by:

$$P_{RO}=\left[\begin{array}{cc|c} z_2I_M-A_{11} & -A_{12} & B_1\\ -A_{21} & z_1I_N-A_{22} & B_2\\ \hline -C_1 & -C_2 & \mathbb{D}\end{array}\right], \tag{24}$$

where $N\geq n,\ M\geq m$.

Theorem 1. Let $P_{RP}(z_1,z_2)$ be the $[(n+m)+m]\times[(n+m)+p]$ polynomial system matrix given by (11). Then $P_{RP}(z_1,z_2)$ is I/O equivalent to a Roesser nonsingular system matrix of the form (24), provided that this representation exists, i.e.,

$$S_1(z_1,z_2)P_{RP}(z_1,z_2)=P_{RO}(z_1,z_2)S_2(z_1,z_2), \tag{25}$$

where

$$\begin{aligned}S_1(z_1,z_2)&=\begin{bmatrix}M(z_1,z_2) & 0\\ X(z_1,z_2) & I_m\end{bmatrix},\\ S_2(z_1,z_2)&=\begin{bmatrix}M(z_1,z_2) & Y(z_1,z_2)\\ 0 & I_p\end{bmatrix}\end{aligned} \tag{26}$$

are matrices with elements in $\mathbb{R}(z_1,z_2)$.

Proof 1. As shown in (Boudellioua et al., 2017), a polynomial system matrix associated with a discrete repetitive process is zero coprime system equivalent and hence I/O equivalent to a polynomial system matrix associated with a Roesser model described by (22). Conversely, the double inversion of variables preserves a transfer-function matrix. Finally, (25) and (26) follow from Lemma 1.

The example given below uses the EOA (Galkowski, 2001), where the relevant steps of this algorithm are now given, starting from a square 2D transfer function matrix

$$F(z_1,z_2)=\begin{bmatrix}f_{11}(z_1,z_2) & \cdots & f_{1m}(z_1,z_2)\\ \vdots & \ddots & \vdots\\ f_{m1}(z_1,z_2) & \cdots & f_{mm}(z_1,z_2)\end{bmatrix}, \tag{27}$$

with the elements that are transfer-functions

$$f_{ij}(z_1,z_2)=\frac{b_{ij}(z_1,z_2)}{a_{ij}(z_1,z_2)}. \tag{28}$$

Next, the transfer function matrix (27) is written in terms the least common multiple of the denominators in each column (or equivalently in rows), i.e., in the form

$$f_{ij}(z_1,z_2)=\frac{B_{ij}(z_1,z_2)}{A_j(z_1,z_2)}. \tag{29}$$

Finally, an additional indeterminate, say z is added to construct an auxiliary polynomial matrix of the form:

$$\begin{aligned}&A_F(z_1,z_2,z)\\&=\begin{bmatrix}zA_1(z_1,z_2)-B_{11}(z_1,z_2) & \cdots & -B_{1m}(z_1,z_2)\\ \vdots & \ddots & \vdots\\ -B_{m1}(z_1,z_2) & \cdots & zA_m(z_1,z_2)-B_{mm}(z_1,z_2)\end{bmatrix}\end{aligned} \tag{30}$$

and a series of matrix augmentations of the form

$$Augment(X)=\begin{bmatrix}1 & 0\\ 0 & X\end{bmatrix} \tag{31}$$

and then the series of elementary row and column operations are applied in order to separate the polynomial variables and transform to the form with only one variable first degree monomials.

Example 2. Consider again the system of Example 1 and make the substitutions $z_1=\widehat{z_1}^{-1}$ and $z_2=\widehat{z_2}^{-1}$ in the transfer-function matrix G_{RP}, resulting in the transfer-function matrix

$$G_{RS}(\widehat{z}_1,\widehat{z}_2)=\begin{bmatrix}\dfrac{1}{\widehat{z}_1+1} & 0\\ \dfrac{\widehat{z}_2\left(\widehat{z}_1\widehat{z}_2-3\widehat{z}_1-\widehat{z}_2-1\right)}{\left(\widehat{z}_1+1\right)\left(\widehat{z}_1\widehat{z}_2^2+\widehat{z}_1\widehat{z}_2-2\widehat{z}_2^2+\widehat{z}_2-1\right)} & \dfrac{3\widehat{z}_2^2-2\widehat{z}_2+1}{\widehat{z}_1\widehat{z}_2^2+\widehat{z}_1\widehat{z}_2-2\widehat{z}_2^2+\widehat{z}_2-1}\end{bmatrix} \tag{32}$$

and apply the following steps of the EOA, see (Galkowski, 2001).

Create the 3D polynomial matrix of (30)

```
H0:=matrix(2,2,
[z*(s+1)*(s*p^2-2*p^2+s*p+p-1)
-(s*p^2-2*p^2+s*p+p-1),0,
-p*(s*p-3*s-p-1),z*(s*p^2-2*p^2+s*p+p-1)
-(3*p^2-2*p+1)]);
```

where $s = \widehat{z}_1$, $p = \widehat{z}_2$ and z is an auxiliary variable. Next, the following augmentation procedure realizes (31)

```
> Augment := proc(MAT::matrix)
>     local M, w, k;
>     w := linalg[rowdim](MAT);
>     k := linalg[coldim](MAT);
>     M := matrix(w + 1, k + 1, 0);
>     M[1,1] := 1;
>     linalg[copyinto](MAT,M,2,2);
>     RETURN(evalm(M));
> end:
```

Apply the following EOA steps:

```
> H1:=Augment(H0):
> H1:=addrow(H1,1,2,z):
> H1:=addcol(H1,1,2,
  -(s+1)*(s*p^2-2*p^2+s*p+p-1));
> H2:=Augment(H1):
> H2:=addrow(H2,1,4,z):
> H3:=Augment(H2):
> H3:=addcol(H3,1,4,
  (s*p^2-2*p^2+s*p+p-1)):
> H3:=addrow(H3,1,3,s+1):
> H3:=addrow(H3,1,4,1);
> H4:=Augment(H3):
> H4:=addcol(H4,1,5,p);
> H5:=Augment(H4):
> H5:=addcol(H5,1,2,s):
> H5:=addrow(H5,3,7,1):
> H5:=addcol(H5,1,2,-2);
> H6:=Augment(H5):
> H6:=addcol(H6,1,8,p):
> H7:=Augment(H6):
> H7:=addrow(H7,1,6,s-2):
> H7:=addrow(H7,1,9,3);
> H8:=Augment(H7):
> H8:=addcol(H8,1,5,p):
> H9:=Augment(H8):
> H9:=addrow(H9,1,5,s):
> H9:=addcol(H9,1,6,-1):
> H9:=mulrow(H9,3,-1);
```

Finally, apply appropriate row and column permutation and ordering to the matrix $H9$ followed by block splitting to obtain the following singular Roesser state-space model matrices.

$$H_{11} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

$$H_{12} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & -1 \\ 0 & -3 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, H_{13} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \end{bmatrix},$$

$$H_{21} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, H_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 & -1 \end{bmatrix},$$

$$H_{31} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & -3 & -1 \end{bmatrix}, H_{32} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 10 & 5 & 1 & 1 & -4 \end{bmatrix}$$

and H_{23} and H_{33} are null matrices of compatible dimensions. Finally, the matrices (23) of the nonsingular Roesser state-space model are:

$$\begin{gathered} A_{11} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \\ B_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ -1 & 0 \end{bmatrix}, \quad A_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \\ A_{22} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -2 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & 0 \end{bmatrix} \\ C_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 3 & -1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & -1 & -1 & 11 \end{bmatrix}, \\ \mathbb{D} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \end{gathered} \tag{33}$$

Also it is easily verified that the transfer-function matrix corresponding to this nonsingular Roesser state-space model is given by:

$$G_{RO}(z_1, z_2) = \begin{bmatrix} \dfrac{z_1}{z_1 + 1} & 0 \\ \dfrac{z_1 \left(z_1 z_2 + z_1 + 3\, z_2 - 1\right)}{\left(z_1 + 1\right)\left(z_1 {z_2}^2 - z_1 z_2 + 2\, z_1 - z_2 - 1\right)} & -\dfrac{z_1 \left({z_2}^2 - 2\, z_2 + 3\right)}{z_1 {z_2}^2 - z_1 z_2 + 2\, z_1 - z_2 - 1} \end{bmatrix} = G_{RP}(z_1, z_2) \tag{34}$$

Remark 1. If the matrix $\widehat{H}_{11}$ is singular the method above is not applicable. In such a case, the bilinear transform is a possible way forward, see (Galkowski, 2001) for the relevant background, which is the subject of ongoing work.

CONCLUSIONS AND OPEN RESEARCH QUESTIONS

In this paper an equivalent representation is obtained in 2D nonsingular Roesser form for a given system matrix arising from a discrete linear repetitive process. The connection between the original system matrix with its corresponding 2D nonsingular Roesser form has been obtained using elementary operations. The I/O behavior of the original system matrix is preserved, making it possible to analyze the polynomial system matrix in terms of its associated 2D nonsingular Roesser form. One motivation for this work is that nonsingular 2D representations are critical to examining certain physically well defined systems theoretic properties and, in particular, local controllability/reachability properties of discrete linear repetitive processes.

One particular feature of repetitive processes is that systems theoretic properties can be directly linked to a physical basis, as is often not the case for other class of 2D systems and many of the controllability/observability properties. Local reachability/controllabiltiy of linear repetitive processes asks the question: does there exist an admissible set of control inputs that will force the process to achieve a certain state or output vector at a specified instant along a specified pass? This property is best characterized in the Roesser model of the process dynamics.

The results in this paper pose a number of questions for further research. Moreover, these questions relate to the basic theory and also applications. In the case of the former, the method developed in this paper is based on explicit use of the inverse of the complex variables z_1 and z_2 and additional constraints occur. If they are not met another approach can be applied, i.e. application of the bilinear transform see (Galkowski, 2001) which is the subject of ongoing work.

The results in this paper show also the possibility of obtaining new equivalent forms of repetitive process description, which may have onward value in terms of the development of a comprehensive systems theory that is also supported by numerically reliable computational algorithms for checking systems theoretic properties, such as controllability, and the design of control laws. Hence, possible future research also include extending the analysis of this paper to more general forms of repetitive process dynamics. For example, in (Rogers et al., 2007) it is established that cases exist where the pass state initial vector sequence is required to contain explicit terms from the previous pass profile, e.g.,

$$x_{k+1}(0) = d_{k+1} + \sum_{j=1}^{M} \sum_{p=0}^{\alpha-1} K_{jp} y_{k+1-j}(p), \quad k \geq 0. \tag{35}$$

If the summation term is removed then the state initial vector sequence assumed in this paper results. However, under-modeling, e.g., assuming that this assumption can always be made is incorrect as the structure of the pass state initial vector sequence alone can destroy stability and also properties such as controllability/reachability. An example demonstrating this fact is given in (Rogers et al., 2007). Another application is for the so-called wave repetitive processes characterized by non-local updating structures, see, e.g. (Cichy et al., 2013).

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