SEMI-ANALYTICAL PROPAGATION WITH DRAG COMPUTATION AND FLOW EXPANSION USING DIFFERENTIAL ALGEBRA

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Efficient long-term propagation of orbits is needed for e.g. the design of disposal orbits and analysis of their stability. Semi-analytical methods are suited for this as they combine accuracy and efficiency. However, the semi-analytical modelling of non-conservative forces is challenging and in general numerical quadrature is required to accurately average their effects, which reduces the efficiency of semianalytical propagation. In this work we apply Differential Algebra (DA) for efficient evaluation of the mean element rates due to drag. The effect of drag is computed numerically in the DA arithmetic such that in subsequent integration steps the drag can be calculated by only evaluating a DA expansion. The method is tested for decaying low Earth and geostationary transfer orbits and it is shown that the method can provide accurate propagation with reduced computation time with respect to nominal semi-analytical and numerical propagation. Furthermore, the semi-analytical propagator is entirely implemented in DA to enable higherorder expansion of the flow that can be used for efficient propagation of initial conditions. The approach is applied to expand the evolution of a Galileo disposal orbit. The results show a large validity domain of the expansion which represents a promising result for the application of the method for e.g. stability analysis.

INTRODUCTION

The present international concern in space situational awareness (SSA) has produced a renewed interest in efficient methods for the propagation of catalogues of orbital objects. In particular, for studying the long-term evolution of orbits, e.g. for de-orbiting or graveyard orbit analysis, efficient propagation techniques are needed. Moreover, the discovery of instability of the orbits of Global Navigation Satellite Systems and their graveyard orbits,^{1,2,3} has increased the need for stability analyses. These kind of studies require propagation techniques that are not only efficient, but also sufficiently accurate to capture all important features of the orbital motion. The techniques developed thus far for propagating perturbed orbits fall into three broad categories: analytical, numerical, and semi-analytical.

Analytical closed form solutions to the equations of motion can be obtained using perturbation theory based on series expansion.^{4,5,6,7,8,9,10} These solutions are explicit functions of time, initial conditions, and problem parameters. The state of an object at any epoch can therefore be computed by a single evaluation of the explicit functions. In addition, the analytical solutions are valid for large

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ranges or all initial conditions. However, due to simplifications required to obtain the analytical solution, such as e.g. simplified perturbation models or low-order approximations, the accuracy of the solution is limited.

In the numerical approach the equations of motion are numerically integrated. No approximations are required in the equations of motion and consequently very high accuracies can be achieved. In addition, perturbations can generally be included in the equations of motion easily, which makes numerical propagation a straightforward technique. However, the solutions depend explicitly on the initial conditions and therefore the propagation has to be performed again if the initial conditions change. Furthermore, if the perturbations cause short time-scale changes in the orbit, the step size of the integration needs to be small to ensure the propagation error remains sufficiently limited.¹¹

The term semi-analytical is used for propagation methods that combine analytical and numerical integration techniques. To this aim, short-periodic motion is filtered out and the remaining secular and long-period dynamics are integrated numerically with long time steps. The filtering of the highest frequencies of motion is traditionally done via averaging procedures, either analytical or numerical. Analytic averaging can be performed directly over the variation of parameters equations of motion using the generalised method of averaging,^{12, 13, 14} or in the Hamiltonian formulation of the dynamics using canonical perturbation theory.^{15, 16, 17, 18} In contrast, averaging can be carried out numerically to avoid the need to obtain analytical expressions first.^{19, 20, 21, 22} In all three approaches the equations of motion are expanded with respect to a small parameter and higher-order terms are neglected to simplify the equations and the averaging process. The remaining terms are subsequently averaged to obtain the equations of motion in mean elements. In addition, approximate equations can be found for the short-periodic motion which enables the osculating elements to be reconstructed from the mean elements.^{23,24,25}

The advantage of semi-analytical propagation over numerical propagation is that large time steps can be used for the integration. This however comes at the cost of not knowing the osculating states directly and possibly missing the effect of coupling between short-periodic perturbations. The need for approximations is however much less compared to analytical techniques²⁶ and secondand higher-order dynamics can therefore be included in semi-analytical methods more easily.²³ As a result, for many applications the attainable accuracy of semi-analytical methods is nearly as good as fully numerical propagation.²⁶ Because of their combination of accuracy and efficiency semi-analytical methods are the most promising methods for use in the field of SSA. However, the modelling of non-conservative forces is challenging, because in general their effect can not be averaged analytically or only in an inaccurate way, e.g. using simplified atmospheric models.²³ This means that the semi-analytical modelling of non-conservative forces is either not highly accurate or not very efficient which reduces the applicability of semi-analytical propagators.

Recently, Wittig et al.^{27,28} developed a new technique called the high-order transfer map (HOTM) method. This technique is based on automatically expanding the solution of the equations of motion up to high order through differential algebra (DA). The equations of motion are numerically propagated over a single orbital revolution in DA arithmetic to obtain a HOTM, i.e. a high-order analytical approximation of the true transfer map. This map is then used to compute the state after two orbital periods by evaluating it at the state after one revolution. This analytical evaluation of the HOTM can be repeated for several orbital periods to efficiently obtain the future state. This method shares the mathematical simplicity of numerical techniques as the HOTM is obtained by the DA-based numerical integration of the equations of motion for a single revolution. On the other hand, the major part of the orbital propagation is achieved by multiple evaluations of the HOTM

that is carried out efficiently.

In this work we present the development of a new approach that combines semi-analytical techniques with DA for efficient orbital propagation. As discussed, non-conservative forces represent a main limitation for semi-analytical techniques because the averaging of their effects requires either to perform numerical quadrature, which limits the computational efficiency, or to adopt simplified models, which reduces the accuracy. For accurate propagation, numerical quadrature is required to accurately compute the mean element rates due to non-conservative forces. This reduces the advantage of semi-analytical over numerical propagation techniques in terms of efficiency and therefore limits the applicability of semi-analytical techniques. In this work we use semi-analytical techniques to propagate trajectories and apply DA for efficient repetitive evaluation of the mean element rates due to drag. Here, the conservative force effects are taken into account using a semi-analytical propagator. The effect of drag, on the other hand, is computed numerically in the DA arithmetic such that in subsequent integration steps the drag can be calculated by only evaluating a DA expansion.

In addition, to enable efficient propagation of initial conditions, the semi-analytical propagator is entirely implemented in DA arithmetic such that the flow can be expanded to higher-order with respect to initial conditions and parameters. The computation of the higher-order flow expansion allows for efficient stability analysis that generally requires the propagation of many different initial conditions.

The remainder of the paper is organized as follows. First a brief introduction of the DA techniques is given and their use for high-order expansion of the solution of ordinary differential equations (ODEs) with respect to initial conditions and parameters is explained. This is followed by a description of the semi-analytical propagator used in this work and of the numerical propagator used for verification. After that the method for computing mean element rates due to drag using DA is explained and the higher-order flow expansion for semi-analytical propagation is introduced. Finally, techniques are tested and their performance and results are discussed.

INTRODUCTION TO DIFFERENTIAL ALGEBRA

DA techniques are based on the observation that it is possible to extract more information from a function rather than its mere values. The basic idea is to treat functions and the operations on them in a computer environment in a similar way as real numbers are treated there. Real numbers are transformed to floating point representations in order to operate on them in a computer environment. Any operation on real numbers is defined as an adjoint operation on floating point (FP) numbers such that the results are their equivalents. In other words, transforming the real numbers into their FP representation and operating on them in the set of FP numbers returns the same result as carrying out the operation in the set of real numbers and then transforming the achieved result in its FP representation, see Figure 1.

In a similar way, suppose we have real functions in n variables that are k times differentiable. These functions can be represented in a computer environment by their k-th order Taylor expansions (with respect to a reference point). Similar to the transformation of real numbers in their FP representation, functions are converted by extracting their k-th order Taylor expansions. For each operation in the space of k times differentiable functions, an adjoint operation in the space of Taylor polynomials is defined so that the resulting functions are their equivalents. In other words, extracting the Taylor expansions of real functions and operating on them in the space of Taylor polynomials (labelled as kD_n) returns the same result as operating on the original functions in the original space



Figure 1. Analogy between the floating point representation $(\overline{a}, \overline{b})$ of real numbers (a, b) in a computer environment (left) and the introduction of the algebra of Taylor polynomials (F, G) for functions (f, g) in the differential algebraic framework (right).

and then extracting the Taylor expansion of the resulting function, see Figure 1.

The straightforward implementation of differential algebra in a computer enables the computation of the Taylor coefficients of a function up to a specified order k, along with the function evaluation, with a fixed amount of effort. The Taylor coefficients of order n for sums and products of functions, as well as scalar products with reals, can be computed from those of summands and factors; therefore, the set of equivalence classes of functions can be endowed with well-defined operations, leading to the so-called truncated power series algebra.^{29,30} Similarly to the algorithms for floating point arithmetic, algorithms for functions followed. These included methods to perform composition of functions and to treat common elementary operations, but also to invert functions and to solve nonlinear systems explicitly.^{31,32} Finally, in addition to these algebraic operations, the DA framework is endowed with differentiation and integration operators.

High-order Expansion of the Solution of an ODE

Taylor differential algebra allows one to compute the derivatives of any function f of n variables up to arbitrary order k by representing the function by its k-th order Taylor expansions (with respect to a reference point). This technique can be applied when numerically integrating an ODE using an arbitrary integration scheme. As any integration scheme is based on algebraic operations, which involve the evaluation of the right hand side of the ODE at several integration points, all evaluations can be carried out in the DA framework. This allows one to compute the arbitrary order expansion of the flow of an ODE with respect to the initial condition automatically using DA.

Now consider the scalar initial value problem (IVP):

$$\begin{cases} \dot{x} = f(x,t) \\ x(t_0) = x_0 \end{cases}$$
(1)

We want to show that, starting from the DA representation of the initial condition x_0 , differential algebra allows us to compute the Taylor expansion of the IVP with respect to the initial condition at the final time t_f .

First, the point initial condition x_0 is replaced by the DA representation of its identity function up to order k, which is the collection of (k + 1) Taylor coefficients. The first Taylor coefficient, the constant part, is equal to x_0 . The second Taylor coefficient corresponds to the first derivative and all other coefficients are zero. This DA variable, $[x_0]$, can be written as $x_0 + \delta x_0$, in which x_0 is the reference point for the expansion. Now, if all the operations of the numerical integration scheme are carried out in the framework of differential algebra, the solution x_i at each fixed time step t_i is not a scalar but a Taylor expansion in x_0 .

For the sake of clarity, consider the forward Euler's scheme

$$x_{i} = x_{i-1} + f(x_{i-1})\Delta t$$
(2)

and substitute the initial value with the DA identity $[x_0] = x_0 + \delta x_0$. At the first time step we have

$$[x_1] = [x_0] + f([x_0]) \cdot \Delta t.$$
(3)

If the function f is evaluated in the DA framework, the output of the first step, $[x_1]$, is the k-th order Taylor expansion of the solution of the IVP in x_0 at $t = t_1$. Note that, as the DA evaluation of $f([x_0])$ may involve non-linear operations, the coefficients corresponding to high-order terms in δx_0 may become non zero. This procedure can be repeated for subsequent time steps, such that we finally obtain the k-th order Taylor expansion of the solution in x_0 at the final time t_f . The solution of the IVP can thus be obtained, at each time step t_i , as a k-th order Taylor expansion in x_0 . This Taylor map or polynomial approximates the solution of the IVP for initial conditions close to x_0 . In the remainder of the paper this result is expressed as $[x_i] = \mathcal{M}_{x_0}(\delta x_0)$, in which the square brackets indicate that the output is a DA variable, \mathcal{M} indicates the Taylor map or polynomial, the subscript refers to the variables of the Taylor expansion, and the δ indicates that the Taylor expansion is function of the variation with respect to the reference values.

It should be noted that the expansion of the solution of the IVP can be easily obtained also with respect to any parameter q that appears in the dynamics model. In this case the parameter q is also initialized as a DA variable, i.e. $[q] = q + \delta q$, and the solution at time t_i is $[x_i] = \mathcal{M}_{x_0,q}(\delta x_0, \delta q)$.

The main advantage of the DA-based approach is that there is no need to write and integrate variational equations in order to obtain high order expansions of the flow. This result is basically obtained by the substitution of operations between real numbers with those on DA variables. The DA technique can therefore be applied to many different problems independent of the ODE.

The DA software used in the work is the DA Computational Engine (DACE) developed by DI-NAMICA.³³ This engine includes all core DA functionality and a C++ interface.

SEMI-ANALYTICAL PROPAGATOR

The semi-analytical (SA) propagator used in this work applies basically the same perturbation model as implemented in HEOSAT,³⁴ which was developed to study the long-term evolution of satellites in Highly Elliptical Orbits (HEO). The model takes into account the gravitational effects due to zonal terms and lunisolar perturbations and solar radiation pressure (SRP) and atmospheric drag. The gravitational terms are expressed in Hamiltonian form to obtain the mean elements' equations of motion using Deprit's perturbation algorithm¹⁶ based on Lie transformations. The equations of motion due to SRP and drag perturbations are averaged over the mean anomaly via Gauss equations. The averaging techniques applied for developing HEOSAT are described by Lara et al.³⁴ The main characteristics of the perturbation model and averaging procedures are as follows:

• The zonal-term Hamiltonians are simplified by removing parallactic terms (via Elimination of the parallax^{35,36,37}) and short-periodic terms are eliminated by Delaunay normalization. This is carried out up to second order of the second zonal harmonic, J_2 , and to first order for $J_3 - J_{10}$.

- The disturbing potentials of the Sun and Moon (point-mass approximation) are expanded using Legendre series to obtain the Hamiltonians for averaging out the short-periodic terms.³⁸ Second- and sixth-order Legendre polynomials are taken for the Sun and Moon potentials, respectively.
- Averaged equations of motion due to SRP are obtained by assuming a spherical satellite and constant solar flux along the orbit (i.e. no shadow). Then, Kozai's analytical expressions for the perturbations due to SRP³⁹ are used to average Gauss equations over the mean anomaly analytically.
- Mean element rates due to atmospheric drag are computed by *numerically* averaging Gauss equations over the mean anomaly assuming a spherical satellite and a rotating atmosphere. The atmospheric density is taken from the Harris-Priester atmospheric density model,⁴⁰ which is implemented with modifications to account for the diurnal bulge.⁴¹

The propagation is carried out in the True of Date reference system and NASA's SPICE toolbox* is used for both Moon and Sun ephemerides (DE405 kernels) and reference frame transformations (True of Date, Mean of Date and J2000 reference frame kernels). For future work tesseral resonance effects will be added to the perturbation model and the Harris-Priester density model will be replaced by the NRLMSISE-00 model.

Averaging atmospheric drag effects

The mean element rates due to drag are computed by averaging the drag effect over one orbital period by numerical integration. The numerical quadrature is carried out by keeping the orbital elements constant and integrating in mean anomaly from 0 to 2π . At each integration point the perturbing acceleration due to drag is computed as follows:

$$\mathbf{f}_{drag} = -\frac{1}{2}C_d \frac{A}{m}\rho V \mathbf{V} \tag{4}$$

where C_d is the drag coefficient, A/m the area-to-mass ratio, ρ the atmospheric density and V the velocity vector with respect to the atmosphere and V its magnitude. The element rates due to this drag acceleration are then evaluated using Gauss equations of motion:

$$\frac{da}{dt} = 2\frac{a^2}{\sqrt{\mu p}} \left[f_r e \sin \theta + f_\theta \frac{p}{r} \right]$$

$$\frac{de}{dt} = \frac{1}{\sqrt{\mu p}} \left[f_r p \sin \theta + f_\theta \left((p+r) \cos \theta + re \right) \right]$$

$$\frac{di}{dt} = f_h \frac{r}{\sqrt{\mu p}} \cos \left(\omega + \theta \right)$$

$$\frac{d\Omega}{dt} = f_h \frac{r}{\sqrt{\mu p} \sin i} \sin \left(\omega + \theta \right)$$

$$\frac{d\omega}{dt} = -\sqrt{\frac{p}{\mu}} \left[f_h \frac{r}{p} \cot i \sin \left(\omega + \theta \right) + \frac{1}{e} \left\{ f_r \cos \theta - f_\theta \left(1 + \frac{r}{p} \right) \sin \theta \right\} \right]$$

$$\frac{dM}{dt} = n - f_r \left[\frac{2r}{\sqrt{\mu a}} - \frac{1 - e^2}{e} \sqrt{\frac{a}{\mu}} \cos \theta \right] - f_\theta \frac{1 - e^2}{e} \sqrt{\frac{a}{\mu}} \left(1 + \frac{r}{p} \right) \sin \theta$$
(5)

*https://naif.jpl.nasa.gov/naif/index.html

where $(a, e, i, \Omega, \omega, M)$ are the Keplerian orbital elements, p is the semi-latus rectum, r the radial distance, μ the gravitational parameter of the Earth, θ the true anomaly, and f_r , f_{θ} and f_h are the components of the perturbing acceleration in radial, normal and out-of-plane directions, respectively.

Finally, the mean element rates are computed by evaluating the following integral using the trapezoidal rule:

$$\frac{\overline{d\alpha}}{dt} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\alpha}{dt} dM$$
(6)

The drag model was improved for this work by adding a J_2 short-periodic correction to the radial distance r to account for the J_2 -drag coupling. This so-called J_2 height correction adjusts the altitude that is used to determine the atmospheric density and is given as follows:⁴²

$$\Delta r = \frac{J_2 R_E^2}{4(1-e^2)a} \left[\sin^2 i \cos 2(\theta+\omega) + (3\sin^2 i - 2) \left(1 + \frac{e\cos\theta}{1+\sqrt{1-e^2}} + \frac{2\sqrt{1-e^2}}{1+e\cos\theta} \right) \right]$$
(7)

Validation

To test the validity of the SA propagator, a Molniya orbit has been propagated for 35 years and compared with two-line element (TLE) data. Figure 2 shows the orbital elements of the Molniya 1-32 satellite according to TLE data and SA propagation and their differences over 35 years (the initial conditions are shown in Table 1). The root mean square (RMS) differences between the orbital elements according to TLE and SA propagation are: $\Delta a_{rms} = 4.46$ km, $\Delta e_{rms} = 1.03e^{-3}$, $\Delta i_{rms} = 0.034$ deg, $\Delta \Omega_{rms} = 3.14$ deg, $\Delta \omega_{rms} = 0.17$ deg. These differences are very small considering the propagation time and the change in orbital elements. In comparison to the TLE data the largest drift is in the right ascension of the ascending node Ω . The source of this drift is under investigation. Finally, the time needed for propagating the orbit for 35 years was 130 s. In case the drag perturbation is not taken into account, the propagation time is only 17 s. The drag calculations thus significantly reduce the computational speed, especially considering the fact that only 29% of the time the perigee of the orbit was inside the atmosphere.

NUMERICAL PROPAGATOR

To verify the output of the semi-analytical propagator in this study a numerical propagator was used. For this the Accurate Integrator for Debris Analysis (AIDA) was used; a high-precision numerical propagator using up-to-date perturbation models. AIDA includes the following force models:⁴³ geopotential acceleration computed using the EGM2008 model (10x10), atmospheric drag modelled using the NRLMSISE-00 air density model, solar radiation pressure with dual-cone shadow model and third body perturbations from Sun and Moon. NASA's SPICE toolbox is used both for Moon and Sun ephemerides (DE405 kernels) and for reference frame and time transformations (ITRF93 and J2000 reference frames and leap-seconds kernel). Space weather data are obtained from CelesTrak^{*}.

As the semi-analytical propagator uses the Harris-Priester density model, the NRLMSISE-00 density model was replaced by the Harris-Priester model for the purpose of verification in this work^{\dagger}.

^{*}http://www.celestrak.com/SpaceData/sw19571001.txt

[†]The output of AIDA shown in this work is therefore less accurate (considering drag effects) than attainable by the unmodified version of AIDA.



Figure 2. Comparison of SA propagation with TLE data for Molniya 1-32 satellite (NORAD ID 8601); mean orbital elements according to TLE data and SA propagation (left) and their difference (right).



Figure 3. Semi-major axis according to SA propagation with and without J_2 height correction for drag computation and according to AIDA (osculating and numerical average) (JD₀=2451545.5, a_0 =6878 km, e_0 =0.01, i_0 =30 deg, $\Omega_0 = \omega_0 = M_0$ =0 deg).

AIDA has for example been used to check the effect of the J_2 height correction in the semianalytical drag model. In Figure 3 an example of the effect of the J_2 height correction on the semi-major axis for a LEO orbit at 500 km is shown in comparison to numerical propagation with AIDA. For the comparison, the average of the osculating orbital elements from AIDA have been computed per orbital revolution by quadrature, similar to Eq. (6). Figure 3 shows that the J_2 -drag correction improves the accuracy of the SA propagation and is therefore used throughout this paper.

DRAG MODELLING USING DIFFERENTIAL ALGEBRA

During SA propagation the quadrature in Eq. (6) must be computed numerically at each integration point. This process is time consuming and therefore reduces the efficiency of the semianalytical propagation. In this section we develop a new technique to efficiently compute the mean element rates due to drag. This method is based on the philosophy of the HOTM to expand the flow and then use this expansion to efficiently propagate the orbit. However, instead of building a transfer map that relates an initial state to a final state, we construct a Taylor expansion of the mean element rates due to drag with respect to the initial mean elements. As long as the mean elements change little the expansion can be used to accurately compute mean element rates due to drag without evaluating numerical quadratures.

The expansion is computed by initializing the mean orbital elements, α_0 , as DA variables: $[\alpha_0] = \alpha_0 + \delta \alpha_0$ (except for M, which is the independent variable in the quadrature). Then the mean element rates due to drag are computed by numerical integration in the DA framework. The result is a Taylor expansion of the mean element rates, $\left\lceil \frac{d\alpha}{dt} \right\rceil$, with respect to the initial mean elements:

$$\left[\frac{\overline{d\boldsymbol{\alpha}}}{\overline{dt}}\right] = \mathcal{M}_{a_0,e_0,i_0,\Omega_0,\omega_0}(\delta a_0,\delta e_0,\delta i_0,\delta\Omega_0,\delta\omega_0) \tag{8}$$

where \mathcal{M} is the DA expansion and $(a_0, e_0, i_0, \Omega_0, \omega_0)$ are the initial mean orbital elements.

Once the expansion is computed, it can be used in subsequent integration steps to compute the mean element rates. First the differences of the current mean elements with respect to the reference

values, δa_0 , δe_0 , δi_0 , $\delta \Omega_0$, $\delta \omega_0$, are computed and these are then used to evaluate the expansion, \mathcal{M} , to obtain the mean element rates efficiently.

As the mean orbital elements change over time, their differences with respect to the reference values grow and the expansion becomes less accurate. At some point the expansion needs to be recomputed to ensure sufficiently accurate results. The mean element rates due to drag depend on the drag model and the atmospheric model, which means that mean element rates change quickly with changing perigee radius and to a lesser extent with changing orbit shape. In addition, because of the Earth's oblateness the mean element rates depend on the orbit's orientation and since the implemented Harris-Priester density model accounts for the diurnal bulge the mean element rates also depend on time. The expansion of mean element rates due to drag is therefore expected to depend on all parameters and time, but most on the semi-major axis and eccentricity.

Indeed, it was found that the DA expansion of the mean element rates due to drag quickly becomes inaccurate with changing perigee altitude. On the other hand, for orbits with nearly constant perigee height, the expansion degrades with changing apogee altitude. Besides, when the orbit changes little, the accuracy of the expansion decreases with time as the diurnal bulge rotates. Therefore, the expansion was recomputed whenever the mean perigee altitude changed by more than 1 km, or when time or the mean apogee altitude changed by more than 15 days or 100 km, respectively. This means that for orbits with a quickly changing perigee altitude the expansion needs to be recomputed frequently which reduces the efficiency of the method. It should be noted that the thresholds for recomputing the expansion are based on experience and that they may be too strict considering efficiency or too loose regarding accuracy for different cases. Therefore in the future these thresholds will be replaced by indicators that use direct measures of the accuracy of the expansion, such as the radius of convergence of the expansion.⁴⁴

Finally, the expansion diverges quickly with changing perigee radius r_p , which depends on the semi-major axis and eccentricity as $r_p = a(1-e)$. Therefore the accuracy of the expansion strongly depends on the change in semi-major axis and eccentricity. On the other hand, the apogee radius, $r_a = a(1 + e)$, is also a function of a and e, but the expansion does not diverge quickly with changing apogee. It was found that expanding the mean element rates due to drag with respect to r_p and r_a instead of a and e results in a DA expansion that diverges less quickly with changing apogee. Basically, the dependency of the expansion on r_p that was distributed over a and e is now concentrated in the r_p only. Therefore, the expansion used in this work depends on the initial r_p and r_a instead of a and e:

$$\left[\frac{d\boldsymbol{\alpha}}{dt}\right] = \mathcal{M}_{r_{p_0}, r_{a_0}, i_0, \Omega_0, \omega_0}(\delta r_{p_0}, \delta r_{a_0}, \delta i_0, \delta \Omega_0, \delta \omega_0) \tag{9}$$

where r_{p_0} is the initial perigee radius and r_{a_0} the initial apogee radius.

SEMI-ANALYTICAL PROPAGATION IN DA

The speed-up of drag computations is important for propagating orbits with low perigees, e.g. for de-orbiting strategies. However, orbits that requires stability analysis are often not subject to atmospheric drag, e.g. navigation satellite orbits. Traditionally, stability analysis involves the propagation of many different initial conditions to find the conditions that are stable or unstable.^{3,45} To carry out the propagation of different initial conditions efficiently, the semi-analytical propagator has been implemented in the DA framework to obtain high-order expansions of the flow. In the literature various DA implementations of numerical propagators can be found, but this is not the

case for semi-analytical propagators. The possible advantage of using semi-analytical instead of numerical propagation for expanding the flow is that the motion is smoother due to the absence of short-periodic behaviour. As a result, the semi-analytical DA map may have a larger radius of convergence, i.e. the map is accurate for a larger range of initial conditions.

TEST CASES

To analyse the performance of a DA expansion for computing the mean element rates due to drag, two test cases were selected:

- LEO orbit at 400 km altitude and e = 0.01
- Molniya-like HEO orbit with perigee at 277 km

These two cases represent two important orbit classes for which the computation of drag is important. The initial orbital elements are shown in Table 1. For these test cases, recalculation of the DA expansion of the mean element rates due to drag was performed at the previous stated thresholds and a third-order expansion with respect to r_{p_0} , r_{a_0} , i_0 , Ω_0 and ω_0 was used.

Test Case	Molniya 1-32	LEO drag	HEO drag	Galileo disposal
Epoch [JD]	2444312.7484	2451545.5	2444312.7484	2457519.1215
a [km]	26620.0	6778.0	26620.0	35959.020
e [-]	0.6898843	0.01	0.75	0.189463033
i [deg]	63.552	63.552	63.552	63.35692344
Ω [deg]	133.7124	0.0	133.7124	320.4395699
ω [deg]	276.2045	0.0	276.2045	5.411200554
M [deg]	83.8185716	0.0	180.0	-17.05583305
A/m [m ² /kg]	0.01	0.001	0.01	_
C_D [–]	2.2	2.2	2.2	_
C_R [–]	1.3	1.3	1.3	—

Table 1. Test case definitions

For comparison of the SA orbit propagations with numerical results from AIDA, all elements were converted to the J2000 reference frame. In addition, the initial conditions stated in Table 1 are mean orbital elements that were used directly for SA propagation in the True of Date (TOD) reference frame. For AIDA, the elements were first converted to osculating elements via a canonical transformation that was obtained during the averaging process using Lie transformations considering only first-order J_2 effects.⁴ After that the elements were converted from the TOD to the J2000 reference for use in AIDA.

Finally, for testing the expansion of the SA propagation in DA, a possible disposal orbit for a Galileo satellite was selected. The disposal starts in an orbit, close to the original orbit of the satellite in the Galileo constellation, that is subject to lunisolar perturbations. As a result, the eccentricity of the orbit increases until atmospheric re-entry takes place. The orbit is thus characterised by large eccentricity growth caused by lunisolar perturbations and represents a possible case study in stability analysis of disposal orbits. The initial orbital elements are shown in Table 1.



Figure 4. Semi-major axis and eccentricity for LEO orbit according to SA propagation with and without using DA for drag computation and according to AIDA (see Table 1 for initial conditions).

RESULTS

Drag modelling using DA

Figure 4 shows the semi-major axis and eccentricity for the LEO orbit case for 817 days until re-entry. The difference between the standard SA propagation and the SA propagation using DA is on average only 9.8 m in a and 1.5×10^{-6} in e. The epoch of re-entry (that occurs at $r_p \leq 100$ km) differs by only 5 minutes. In addition, the orbital evolution according to SA and numerical propagation shows the same behaviour. The re-entry epoch according to numerical propagation by AIDA differs by only 15 hours from the SA results. This indicates that the simple drag model in the SA propagator is able to accurately compute the drag effect for LEO orbits.

Regarding the computational speed of the methods, it was found that the SA propagation using drag calculation with DA was fastest. SA with DA required 88.1 s, whereas default SA took 515.1 s and AIDA needed 1512.2 s for the propagation. The default SA was much slower than SA with DA, because the variable step-size integrator used a small step size close to re-entry. When considering only the propagation time for the first 500 days, SA with DA required 32.1 s, default SA 52.6 s and AIDA 822.5 s. This shows that SA propagation is significantly faster than numerical propagation, but should not be applied close to re-entry.

The semi-major axis and perigee radius according to SA propagation and AIDA for the HEO test case are shown in Figure 5. The difference between the default SA and SA using DA after 250 days is only 2 km in the semi-major axis. However, compared to numerical propagation, the difference in semi-major axis after 250 days is 1125 km. Figure 6 also shows that the difference in perigee radius between numerical and SA propagation grew quickly. Actually, the initial mean perigee radius is already different from the average perigee radius according to AIDA. This difference is caused by the mean-to-osculating-element transformation of the initial state and shows that the transformation can be improved. If the SA propagation starts at the average perigee radius according to AIDA instead, then the error in semi-major axis reduces, but is still 831 km after 250 days. The fact that the SA results differ significantly from the numerical results indicates that the drag model used in



Figure 5. Semi-major axis and eccentricity for HEO orbit according to SA propagation with and without using DA for drag computation and according to AIDA (see Table 1 for initial conditions).

the SA propagator is not accurate for HEO orbits.

The propagation time for default SA, SA with DA and AIDA were 36.2 s, 62.5 s and 291.2 s, respectively. As expected, the SA propagations were much faster than numerical propagation. However, the SA propagation using the DA drag model was slower than the default SA, because the DA expansion had to be recomputed many times to ensure accurate results as the perigee altitude changed quickly. Using information about the DA expansion could help to optimize the update frequency of the expansion such that a good accuracy of the propagation is guaranteed while the number of recalculations is as small as possible.

High-order expansion of semi-analytical propagation

The expansion of a semi-analytical propagation using DA has been tested for a possible Galileo disposal orbit. The first 10 years of the orbital evolution of the disposal orbit are shown in Figure 7. In these 10 years the eccentricity grew from 0.189 to 0.319. The semi-major axis is not shown here because it remained constant as only conservative forces acted on the satellite.

The final state of the SA propagation after 10 years was expanded with respect to the initial eccentricity and inclination using DA:

$$[\alpha_f] = \mathcal{M}_{e_0, i_0}(\delta e_0, \delta i_0)$$

where α_f are the final orbital elements and e_0 and i_0 are the initial eccentricity and inclination, respectively, with $e_0 = 0.1895$ and $i_0 = 63.36$ deg.

The final states for different initial eccentricities between 0.0895 and 0.2895 and inclinations between 60.36 and 66.36 deg were computed by SA propagation and by evaluating the DA expansion of the final state. Figure 8(a) shows the final states according to SA propagation and the relative error of the 5th-order DA map for different initial conditions is shown in Figure 8(b).



Figure 6. Difference in semi-major axis and perigee radius between SA propagation and AIDA (numerical average) for HEO orbit (see Table 1 for initial conditions).

The relative error of the DA map is less than 10^{-6} with respect to the final element values for all orbital elements for initial eccentricities between 0.149 and 0.229 and for the entire range of tested inclinations. This means that the DA map has a relative accuracy of 10^{-6} or better over a width of 0.08 in eccentricity and 6 degrees in inclination.

Finally, these results were obtained by propagating 1260 different initial conditions. Performing these propagations pointwise using the SA propagator took 116 min, whereas computing the DA expansion and evaluating it 1260 times took only 6.8 min.

CONCLUSIONS

Semi-analytical techniques have been combined with Taylor Differential Algebra for efficient orbital propagation. First, the computation of the mean element rates due to drag, which requires numerical quadrature, was carried out in DA arithmetic to allow efficient repetitive evaluation of the element rates. It was shown that the approach provided accurate propagation for a decaying LEO orbit with reduced computation time with respect to nominal semi-analytical and numerical propagation. The method can therefore be used to speed-up orbital propagation for e.g. end-of-life de-orbiting analysis of LEO satellites. However, when the drag dynamics changed significantly in the propagation window, e.g. due to variation in perigee altitude, the DA expansion required recalculation to ensure accurate results, which reduced the efficiency of the method.

Secondly, the semi-analytical propagator was entirely implemented in DA arithmetic such that the flow could be expanded to higher-order to enable efficient propagation of initial conditions. The orbital evolution of a Galileo disposal orbit was expanded with respect to the initial eccentricity and inclination, and the relative accuracy of the expansion was 10^{-6} or better over a width of 0.08 in eccentricity and 6 degrees in inclination. The large validity domain of the expansion, due to the use of semi-analytical propagation, is a promising result for the application of the method for SSA. This new approach can for example be used to speed up the stability analysis of graveyard orbits to avoid interference of decommissioned satellites with on-going missions.



Figure 7. Evolution of orbital elements of possible Galileo disposal orbit (see Table 1).

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Figure 8. Final orbital elements and relative error of 5th-order DA map with respect to pointwise SA propagation for different initial eccentricity and inclination of possible Galileo disposal orbit (Ω_0 =320.4 deg and ω_0 =5.4 deg, see Table 1). Relative errors in orders of 10, e.g. yellow colour means relative error between 10⁻⁵ and 10⁻⁴.

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