A Generalised Nonlinear Isolator-Elastic Beam Interaction Analysis for Extremely Low or High Supporting Frequency

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Summary. This paper presents an integrated analysis of a nonlinear isolator-elastic beam interaction system to obtain extremely low or high supporting frequency. The nonlinear suspension unit consists of two vertical inclined springs and two horizontal springs, of which the vertical ones are to generate nonlinear effect while the horizontal one to provide a physical mean for realising required horizontal forces in reported nonlinear isolation systems. The dynamic equations of the system are derived, based on which three reduced models are obtained by introducing the related conditions. The nonlinear dynamic behaviour on equilibria and stabilities of the system are investigated. The dynamic interaction mechanism between the nonlinear suspension system and the elastic structures are revealed. It is investigated the two application cases: one for aircraft ground vibration tests requiring an extremely low supporting frequency and another involving structure dynamic tests in laboratory where a rigid supporting foundation is expected.

High performance vibration isolators with very low or very high stiffness are widely required in engineering. For ground vibration tests (GVT) of aircrafts, the supporting frequency have to be lower than one third of its first elastic natural frequency for flutter analysis. The weight of a large aircraft is huge but its first elastic natural frequency is quite low so that the stiffness of supporer must have a big static stiffness to support the large weight and also a very low dynamic stiffness for a very low supporting frequency [1]. In laboratories, dynamic tests of structures are often expected to be fixed on a rigid foundation, for which the stiffness of supporer must be extremely high. Experiments show a quite “rigid” foundation for static tests could be very soft for high frequency dynamic tests. To design this type of supports, one approach is using active feedback controls in passive systems to modify its dynamic stiffness [2-3], which requires energy supply, so that it is difficult if the required energy is huge. Another approach is using nonlinear spring [1]. The investigations on nonlinear isolator were reported [4-6] and their behaviour on stabilities, bifurcations and chaos are also given [7-8]. Available publications seem not tackling nonlinear isolator-structure interactions. As for structure-control interactions [9], the dynamic characteristics of both structures and control system are affected each other. To assess the efficiency of a nonlinear isolator, interactions analysis is necessary. This paper intends to discuss this problem.

Mathematical model of an integrated nonlinear isolator-beam interaction system

Fig. 1 shows an interaction system in which an elastic beam, of length 2L, mass density ρ and bending stiffness Ψ = EI is supported by a nonlinear isolator, with a mass 2M (supporting frame weight) at its middle point. The beam deflection is Y(ξ, t) and Y(0, t) = y(t) is the displacement of mass. The mass 2M, subjected to a harmonic force 2F0 cos Ωt and connected to two linear inclined springs of stiffness k and unstretched length l, moves in y direction. The other ends of two inclined springs are connected to two bars and B of mass m; positioning at x3, allowing horizontal motions. Two horizontal springs Kc of length L and two dampers C connected to bars A and B. The system is symmetrical to axis O - y along which a spring-damper set of spring stiffness 2K, length L and damper 2C supports mass 2M. This system reduces to the SD [7] when no beam and A, B fixed or irrational one [8] when very rigid spring k and constant pressure gas spring Ks adopted.

Beam equation. It is linear elastic and its motion is represented by a mode summation

\[ m\Phi + k\Phi = Y(0) \text{ (0)} - G \text{. m} = \text{diag}(M_{nn}), k = \text{diag}(K_{nn}), G = \rho \gamma l^* Y_{n} d\xi, \Lambda^2 = \text{diag} (\Omega_n^2), Y(\xi, t) = Y(\xi) \Phi(t), \]

\[ Y = \begin{bmatrix} Y_1 & \cdots & Y_N \end{bmatrix}, \Phi = \begin{bmatrix} \phi_1 & \cdots & \phi_N \end{bmatrix}, \frac{1}{M_{nn}} Y_{n} \Phi_{n} d\xi = \begin{bmatrix} 0, n \times m \end{bmatrix} M_{nn} \frac{1}{m} Y_{n} \Phi_{n} d\xi = \frac{1}{K_{nn}} \Omega_n^2 \]

Here, Y(n), (n = 1, 2, ..., N) are normalized mode functions; \( \phi_n \) generalised coordinates; \( \Omega_n, k, \) and \( M_n \) are the n-th natural frequency, generalised stiffness and mass; \( f_{bs} \) represents a shearing force acting on beam section \( \xi = 0 \) by mass \( M \). For free-free beams, its first rigid mode \( \Omega_1 = \Omega_0 = 1 \).

Dynamic equations of nonlinear isolator

\[ M\ddot{x} + (C + C)\dot{x} + (K + K_k)x = \left[ K_{\Delta}, f_{bs} + F_0 \cos \Omega t - K_{\Delta} \right], M = \text{diag} (m, M), C = \text{diag} (C, C), C_k = \text{cx} \mu, K = \text{diag} (k, k + k + k + k), K_k = -\mu \Lambda / \mu_x, x = \begin{bmatrix} x & y \end{bmatrix}^T, \Delta_n = X_n - L_n, \mu_x = \sqrt{x^2 + y^2}, \Delta = Y_0 - L \]

Here, \( \Delta_u \) and \( \Delta_r \) are the static and dynamic components, respectively; \( K_{\Delta} \) is the system stiffness, when its moving ends are at origin \( O \).

Interaction conditions. On section \( \xi = 0 \), an equilibrium condition and geometrical constraint condition are required,

\[ \begin{align*}
\text{Equil.:} \quad & f_{bs} + f_{\mu} = 0, \quad -f_{bs} = f_{\mu}; \quad \text{Const.:} \quad Y(0, t) = y(t); \quad \text{Matrix form:} \quad Y(t) = \Phi Y_0, \quad Y_0 = \Phi Y(t)
\end{align*} \]

Introduce the following non-dimensional parameters,

\[ \begin{align*}
\overline{\xi} &= \xi / L, \quad \overline{y} = y / L, \quad \overline{\Delta} = \Delta / L, \quad \overline{\Lambda} = \Delta / L, \quad \overline{\Omega} = \Omega L, \quad \overline{F} = \left[ F_0 - K_{\mu} \right] / M, \quad \overline{x} = \mu / L, \quad \overline{m} = m / M, \quad \overline{\Phi} = \Phi \left[ M_0 \right], \quad \overline{\Delta}^2 = \Delta^2 / L^2, \quad \overline{\Omega}^2 = \Omega^2 L^2, \quad \overline{G} = \left[ K_{\mu} \right] \left[ M_0 \right], \quad \overline{\rho} = \rho l^* / l^2 \end{align*} \]
from which it follows
\[
\begin{align*}
\mathbf{\ddot{\Phi}} + \mathbf{\Lambda} \mathbf{\ddot{\Phi}} &= \mathbf{R} T - \mathbf{m} \mathbf{\ddot{g}}, \\
\mathbf{R} &= [\mathbf{0} - \mathbf{m} \mathbf{\dddot{Y}}] Y^T \mathbf{\ddot{\Phi}} = \gamma, \\
q &= \text{diag}(1 + C(q) q), \\
\mathbf{\dddot{\Phi}} &= \mathbf{\dddot{q}} + \frac{2\pi q}{\sqrt{1 + C(q) q}}. \\
\end{align*}
\]

\[\gamma = \text{diag}^{\mathcal{I}}(\gamma^1, \gamma^2), \quad \mathbf{\ddot{\Phi}} = \text{diag}(\mathcal{I}, \mathcal{I}), \quad \mathbf{\ddot{g}} = \text{diag}(\vec{\omega}, \vec{\omega}), \quad \mathbf{I} = \text{diag}(1, 1), \quad \mathbf{\ddot{X}} = \text{diag}(\ddot{X}, \ddot{X}), \quad \mathbf{\dddot{e}} = \text{diag}(e, e),
\]

\[\mathbf{f} = [f_1, f_2, \ldots, f_n]^T, \quad \mathbf{F}_x = [F_{x1}, F_{x2}, \ldots, F_{xn}]^T,
\]

\[
\mathbf{\ddot{F}}_x = [\mathbf{I} - F_1, \mathbf{I} - F_2, \ldots, \mathbf{I} - F_n]^T.
\]

\[\mathbf{q} = [q_1, q_2, \ldots, q_n]^T, \quad \mathbf{\dddot{q}} = [\dddot{q}_1, \dddot{q}_2, \ldots, \dddot{q}_n]^T.
\]

\[\mathbf{\ddot{F}}_x = [\mathbf{I} - F_1, \mathbf{I} - F_2, \ldots, \mathbf{I} - F_n]^T.
\]

\[
\mathbf{\ddot{F}}_x = [\mathbf{I} - F_1, \mathbf{I} - F_2, \ldots, \mathbf{I} - F_n]^T.
\]

\[\mathbf{C}(q) \text{ and } \mathbf{K}(q) \text{ represent nonlinear damping and stiffness matrices. Eq. 6 is re-written in the integrated coupling form}
\]

\[
\begin{align*}
&\mathbf{M} \ddot{q} + \mathbf{2m} \dddot{\mathbf{v}} + \mathbf{Q} \dddot{\mathbf{q}} = \mathbf{\ddot{F}}_x - \mathbf{F}_x - \mathbf{\ddot{F}}_g + \mathbf{F}_g, \\
&\mathbf{Q} = \mathbf{q} - \mathbf{m} \ddot{\mathbf{v}}, \\
&\mathbf{\ddot{F}}_x = [\mathbf{I} - \mathbf{F}_1, \mathbf{I} - \mathbf{F}_2, \ldots, \mathbf{I} - \mathbf{F}_n]^T, \\
&\mathbf{F}_x = \mathbf{[F_{x1}, F_{x2}, \ldots, F_{xn}]^T}, \\
&\mathbf{\dddot{F}}_x = [\mathbf{I} - \mathbf{F}_1, \mathbf{I} - \mathbf{F}_2, \ldots, \mathbf{I} - \mathbf{F}_n]^T.
\end{align*}
\]

\[\mathbf{C}(q) = \mathbf{M}^{-1} [\mathbf{C}(q) + \mathbf{C}(q)^T] \mathbf{P} - (\mathbf{K}(q) - \mathbf{K}(q)^T) \mathbf{Q}, \quad \mathbf{C}(q) = 2 \mathbf{\dddot{\mathbf{v}}} \mathbf{M}^{-1} \mathbf{C}(q), \quad \mathbf{K}(q) = \mathbf{M}^{-1} \mathbf{C}(q),
\]

\[
\begin{align*}
&\mathbf{C}(q) - \mathbf{K}(q) = \mathbf{Q}, \\
&\mathbf{\dddot{F}}_x = [\mathbf{I} - \mathbf{F}_1, \mathbf{I} - \mathbf{F}_2, \ldots, \mathbf{I} - \mathbf{F}_n]^T, \\
&\mathbf{F}_x = \mathbf{[F_{x1}, F_{x2}, \ldots, F_{xn}]^T}, \\
&\mathbf{\dddot{F}}_x = [\mathbf{I} - \mathbf{F}_1, \mathbf{I} - \mathbf{F}_2, \ldots, \mathbf{I} - \mathbf{F}_n]^T.
\end{align*}
\]

Indices “L” and “N” identify linear and nonlinear parts of matrices. Based on Eq. 7, equilibrium points, stabilities, nonlinear energy flow characteristics [10] can be investigated.

**Influence of the nonlinear isolator on the beam**

Eliminating the interaction force vector \( \mathbf{f} \) in Eq. 5, we obtain
\[
\begin{align*}
&\mathbf{M} \ddot{q} + \mathbf{2m} \dddot{\mathbf{v}} + \mathbf{Q} \dddot{\mathbf{q}} = \mathbf{\ddot{F}}_x - \mathbf{F}_x - \mathbf{\ddot{F}}_g + \mathbf{F}_g, \\
&\mathbf{Q} = \mathbf{q} - \mathbf{m} \ddot{\mathbf{v}}, \\
&\mathbf{\ddot{F}}_x = [\mathbf{I} - \mathbf{F}_1, \mathbf{I} - \mathbf{F}_2, \ldots, \mathbf{I} - \mathbf{F}_n]^T, \\
&\mathbf{F}_x = \mathbf{[F_{x1}, F_{x2}, \ldots, F_{xn}]^T}, \\
&\mathbf{\dddot{F}}_x = [\mathbf{I} - \mathbf{F}_1, \mathbf{I} - \mathbf{F}_2, \ldots, \mathbf{I} - \mathbf{F}_n]^T.
\end{align*}
\]

\[\mathbf{C}(q) = \mathbf{M}^{-1} [\mathbf{C}(q) + \mathbf{C}(q)^T] \mathbf{P} - (\mathbf{K}(q) - \mathbf{K}(q)^T) \mathbf{Q}, \quad \mathbf{C}(q) = 2 \mathbf{\dddot{\mathbf{v}}} \mathbf{M}^{-1} \mathbf{C}(q), \quad \mathbf{K}(q) = \mathbf{M}^{-1} \mathbf{C}(q),
\]

\[
\begin{align*}
&\mathbf{C}(q) - \mathbf{K}(q) = \mathbf{Q}, \\
&\mathbf{\dddot{F}}_x = [\mathbf{I} - \mathbf{F}_1, \mathbf{I} - \mathbf{F}_2, \ldots, \mathbf{I} - \mathbf{F}_n]^T, \\
&\mathbf{F}_x = \mathbf{[F_{x1}, F_{x2}, \ldots, F_{xn}]^T}, \\
&\mathbf{\dddot{F}}_x = [\mathbf{I} - \mathbf{F}_1, \mathbf{I} - \mathbf{F}_2, \ldots, \mathbf{I} - \mathbf{F}_n]^T.
\end{align*}
\]


