A Generalised Nonlinear Isolator-Elastic Beam Interaction Analysis for Extremely Low or High Supporting Frequency

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Summary. This paper presents an integrated analysis of a nonlinear isolator-elastic beam interaction system to obtain extremely low or high supporting frequency. The nonlinear suspension unit consists of two vertical inclined springs and two horizontal springs, of which the vertical ones are to generate nonlinear effect while the horizontal one to provide a physical mean for realising required horizontal forces in reported nonlinear isolation systems. The dynamic equations of the system are derived, based on which three reduced models are obtained by introducing the related conditions. The nonlinear dynamic behavour on equilibria and stablities of the system are investigated. The dynamic interaction mechanism between the nonlinear suspension system and the eleastic structures are revealed. It is investigated the two application cases: one for aircraft ground vibration tests requiring an extreme low suporting frequency and another involving structure dynamic tests in laboratory where a rigid supporting foundation is expected.

High performance vibration isolators with very low or very high stiffness are widely required in engineering. For ground vibration tests (GVT) of aircrafts, the supporting frequency have to be lower than one third of its first elastic natural frequency for flutter analysis. The weight of a large aircraft is huge but its first elastic natural frequency is quite low so that the stiffness of suportor must have a big static stiffness to support the large weight and also a very low dynamic stiffness for a very low supporting frequency [1]. In laboratories, dynamic tests of structures are often expected to be fixed on a rigid foundation, for which the stiffenss of supportor must be extremelly high. Experiments show a quite “rigid” foundation for static tests could be very soft for high frequency dynamic tests. To design this type of supportors, one approach is using active feedback controls in passive systems to modify its dynamic stiffness [2-3], which requires energy supply, so that it is difficult if the required energy is huge. Another approach is useing nonlinear spring [1]. The investigations on nonlinear isolatorss were reported [4-6] and their behaviour on stabilities, birfurcations and chaos are also given [7-8]. Available publications seem not tackling nonlinear isolator-structure interactions. As for structure-control interactions [9], the dynamic characteristics of both structures and control system are affected each other. To assess the efficiency of a nonlinear isolator, interactions analysis is necessary. This paper intends to discuss this problem.

Mathematical model of an integrated nonlinear isolator-beam interaction system

Fig. 1 shows an interaction system in which an elastic beam, of length $2S$, mass density ρ and bending stiffness $\Psi = EI$ is supported by a nonlinear isolator, with a mass $2M$ (supporting frame weight) at its middle point. The beam deflection is $Y(\xi,t)$ and $Y(0,t) = y(t)$ is the displacement of mass. The mass $2M$, subjected to a harmonic force $2F_0 \cos\Omega_0 t$ and connected to two linear inclined springs of stiffness k and unstreched length l, moves in y direction. The other ends of two inclined springs are connected to two carts A and B of mass m, positioning at x, allowing horizontal motions. Two horizontal springs K_1 of length L_1 and two dampers C_1 connected to carts A and B. The system is symmetrical to axis $O-y$ along which a spring-damper set of spring stiffness $2K$, length L and damper $2C$ supports mass $2M$. This system redeuces to the SD [7] when no beam and A, B fixed or irrational one [8] when very rigid spring k and constant pressure gas spring K_1 adopted.

Beam equation. It is linear elastic and its motion is represented by a mode summation

$$\begin{gathered}\mathbf{m}\ddot{\mathbf{\Phi}} + \mathbf{k}\mathbf{\Phi} = \mathbf{Y}^T(0) f_{bs} - \mathbf{G}, \mathbf{m} = \mathrm{diag}(M_{nn}), \mathbf{k} = \mathrm{diag}(K_{nn}), \mathbf{G} = \rho g \int_0^S \mathbf{Y}^T d\xi, \mathbf{\Lambda}^2 = \mathrm{diag}(\hat{\Omega}_n^2), Y(\xi,t) = \mathbf{Y}(\xi)\mathbf{\Phi}(t), \\ \mathbf{Y} = \begin{bmatrix} Y_1 & \cdots & Y_N \end{bmatrix}, \mathbf{\Phi} = \begin{bmatrix} \phi_1 & \cdots & \phi_N \end{bmatrix}^T, \int_0^S Y_n'' EI Y_m'' d\xi = \begin{cases} 0, n \neq m \\ K_{nn}, n = m \end{cases}, \int_0^S Y_n \rho Y_m d\xi = \begin{cases} 0, n \neq m \\ M_{nn}, n = m \end{cases}, \hat{\Omega}_n = \sqrt{\frac{K_{nn}}{K_{nn}}}.\end{gathered} \tag{1}$$

Here, $Y_n(\xi)$, $(n = 1,2,..., N)$ are normalized mode functions, ϕ_n generalised coordinates; $\hat{\Omega}_n$, K_n and M_n are the *n-th* natural frequency, generalised stiffness and mass; f_{bs} represents a shearing force acting on beam section $\xi = 0$ by mass M. For free-free beams, its first rigid mode $\hat{\Omega}_1 = 0$ and $Y_1 = 1$.

Dynamic equations of nonlinear isolator

$$\begin{gathered}\mathbf{M}\ddot{\mathbf{x}} + (\mathbf{C} + \mathbf{C}_c)\dot{\mathbf{x}} + (\mathbf{K} + \mathbf{K}_k)\mathbf{x} = \begin{bmatrix} K_1\Delta_1 & f_{sb} + F_0 \cos\Omega t - Mg - K\Delta \end{bmatrix}^T, \quad \mathbf{M} = \mathrm{diag}(m, M), \quad \mathbf{C} = \mathrm{diag}(C_1, C), \\ \mathbf{C}_c = c\mathbf{x}\mathbf{x}^T / \mu_x^2, \quad \mathbf{K} = \mathrm{diag}(K_1 + k, K + k), \quad \mathbf{K}_k = -kl\mathbf{I} / \mu_x, \mathbf{x} = \begin{bmatrix} x & y \end{bmatrix}^T, \Delta_1 = X_0 - L_1, \mu_x = \sqrt{x^2 + y^2}, \Delta = Y_0 - L.\end{gathered} \tag{2}$$

Here, Δ_1 and Δ respectively are the static extensions of K_1 and $2K$, when their moving ends are at origin O.

Interaction conditions. On section $\xi = 0$, an equilibrium condition and geomentrical constraint condition are required,

$$\text{Equilib.: } f_{bs} + f_{sb} = 0, \; -f_{bs} = f_{sb} = f; \quad \text{Const.: } Y(0,t) = y(t); \quad \text{Matrix form: } \mathbf{Y}(0)\mathbf{\Phi} = \mathbf{Y}_0 y, \quad \mathbf{Y}_0 = \mathbf{Y}(0). \tag{3}$$

Introduce the following non-dimensional parameters,

$$\begin{gathered}\bar{x} = x/l, \; \bar{y} = y/l, \; \bar{\Delta}_1 = \Delta_1/l, \; \bar{\Delta} = \Delta/l, \; \bar{t} = \Omega_0 t, \; \bar{Y} = Y/l, \; \bar{\xi} = \xi/l, \; \bar{S} = S/l, \; \bar{\omega} = \omega/\Omega_0, \gamma_k = k/(k+K_1), \gamma_K = K_1/(k+K_1), \\ \gamma = k/K_1, \chi_c = c/C_1, \bar{K}_1 = K_1/(m\Omega_0^2), \gamma_k + \gamma_K = 1, \omega = \sqrt{(k+K_1)/m}, \varepsilon = C_1/(2m\omega), \Gamma_k = k/(k+K), \Gamma_K = K/(k+K), \Gamma = K/K_1, \\ \mathrm{X}_c = c/C, \bar{K} = K/(M\Omega_0^2), \bar{\Omega} = \Omega/\Omega_0, \Omega = \sqrt{(k+K)/M}, \mathrm{E} = C/(2M\Omega), \Gamma_k + \Gamma_K = 1, \bar{F}_0 = F_0/(M\Omega_0^2 l), \bar{g} = g/(\Omega_0^2 l), \bar{f}_1 = \bar{K}_1\bar{\Delta}_1, \\ \bar{F} = \bar{K}\bar{\Delta}, \bar{f} = f_{sb}/(M\Omega_0^2 l), \; \bar{\rho} = \rho l/M, \; \bar{\mathbf{m}} = \mathbf{m}/M, \; \bar{\mathbf{\Phi}} = \mathbf{\Phi}/l, \; \bar{\mathbf{\Lambda}}^2 = \mathbf{\Lambda}^2/\Omega_0^2, \; \bar{\mathbf{G}} = \mathbf{G}/(M\Omega_0^2 l) = \bar{\rho}\bar{g}\int_0^{\bar{S}} \mathbf{Y}^T d\bar{\xi},\end{gathered} \tag{4}$$

from which it follows

$$\ddot{\bar{\mathbf{\Phi}}}+\bar{\mathbf{\Lambda}}^2\bar{\mathbf{\Phi}}=\mathbf{R}\bar{\mathbf{f}}-\bar{\mathbf{m}}^{-1}\bar{\mathbf{G}},\qquad \mathbf{R}=\begin{bmatrix}\mathbf{0} & -\bar{\mathbf{m}}^{-1}\mathbf{Y}_0^T\end{bmatrix},\qquad \mathbf{Y}_0\bar{\mathbf{\Phi}}=\bar{y},$$
$$\ddot{\mathbf{q}}+2\bar{\boldsymbol{\omega}}\boldsymbol{\varepsilon}[\mathbf{I}+\tilde{\mathbf{C}}(\mathbf{q})]\dot{\mathbf{q}}+\bar{\boldsymbol{\omega}}^2[\mathbf{I}+\tilde{\mathbf{K}}(\mathbf{q})]\mathbf{q}=\bar{\mathbf{F}}_0+\bar{\mathbf{f}}-\mathbf{F}_g+\bar{\mathbf{F}}_1,\quad \tilde{\mathbf{C}}(\mathbf{q})=\boldsymbol{\chi}_c\mathbf{q}\mathbf{q}^T/\mu^2,\quad \tilde{\mathbf{K}}(\mathbf{q})=-\boldsymbol{\gamma}_k/\mu,$$
$$\boldsymbol{\gamma}_k=\mathrm{diag}(\gamma_k,\Gamma_k),\quad \boldsymbol{\gamma}_K=\mathrm{diag}(\gamma_K,\Gamma_K),\quad \bar{\boldsymbol{\omega}}=\mathrm{diag}(\bar{\omega},\bar{\Omega}),\quad \mathbf{I}=\mathrm{diag}(1,1),\quad \boldsymbol{\chi}_c=\mathrm{diag}(\chi_c,\mathrm{X}_c),\quad \boldsymbol{\varepsilon}=\mathrm{diag}(\varepsilon,\mathrm{E}),$$
$$\bar{\mathbf{f}}=\begin{bmatrix}0 & \bar{f}\end{bmatrix}^T,\quad \bar{\mathbf{F}}_0=\begin{bmatrix}0 & \bar{F}_0\cos\bar{t}\end{bmatrix}^T,\quad \mathbf{F}_g=\begin{bmatrix}0 & \bar{g}\end{bmatrix}^T,\quad \mathbf{F}_1=\begin{bmatrix}\bar{f}_1 & -\bar{F}\end{bmatrix}^T,\quad \mathbf{q}=\begin{bmatrix}q_1 & q_2\end{bmatrix}^T=\begin{bmatrix}\bar{x} & \bar{y}\end{bmatrix}^T,\quad \mu=\sqrt{\mathbf{q}^T\mathbf{q}}. \tag{5}$$

$\tilde{\mathbf{C}}(\mathbf{q})$ and $\tilde{\mathbf{K}}(\mathbf{q})$ represent nonlinear damping and stiffness matrices. Eq. 6 is re-written in the integrated coupling form

$$\widehat{\mathbf{M}}\ddot{\mathbf{Q}}+2\hat{\bar{\mathbf{m}}}\mathbf{T}^T\bar{\boldsymbol{\omega}}\boldsymbol{\varepsilon}(\mathbf{I}+\mathbf{q}\mathbf{q}^T/\mu^2)\mathbf{T}\dot{\mathbf{Q}}+\hat{\bar{\mathbf{m}}}\{\mathrm{diag}(0,\bar{\mathbf{m}}\bar{\mathbf{\Lambda}}^2)+\mathbf{T}^T\bar{\boldsymbol{\omega}}^2(\mathbf{I}-\boldsymbol{\gamma}_k/\mu)\mathbf{T}\}\mathbf{Q}=\widehat{\mathbf{F}}_0-\widehat{\mathbf{F}}_G-\widehat{\mathbf{F}}_g+\widehat{\mathbf{F}}_1,$$
$$\mathbf{Q}=\begin{bmatrix}q_1\\ \bar{\mathbf{\Phi}}\end{bmatrix},\widehat{\mathbf{M}}=\begin{bmatrix}1 & \mathbf{0}^T\\ \mathbf{0} & \mathbf{I}+\mathbf{m}_N\end{bmatrix},\hat{\bar{\mathbf{m}}}=\begin{bmatrix}1 & 0\\ \mathbf{0} & \bar{\mathbf{m}}^{-1}\end{bmatrix},\mathbf{T}=\begin{bmatrix}1 & 0\\ \mathbf{0} & \mathbf{Y}_0^T\end{bmatrix}^T,\widehat{\mathbf{F}}_G=\begin{bmatrix}0\\ \bar{\mathbf{m}}^{-1}\bar{\mathbf{G}}\end{bmatrix},\widehat{\mathbf{F}}_0=\hat{\bar{\mathbf{m}}}\mathbf{T}^T\bar{\mathbf{F}}_0,\widehat{\mathbf{F}}_g=\hat{\bar{\mathbf{m}}}\mathbf{T}^T\mathbf{F}_g,\widehat{\mathbf{F}}_1=\hat{\bar{\mathbf{m}}}\mathbf{T}^T\mathbf{F}_1. \tag{6}$$

$$\begin{cases}\dot{\mathbf{Q}}=\mathbf{P}\\ \dot{\mathbf{P}}=\widehat{\mathbf{M}}^{-1}\{\widehat{\mathbf{F}}-(\widehat{\mathbf{C}}^L+\widehat{\mathbf{C}}^N)\mathbf{P}-(\widehat{\mathbf{K}}^L-\widehat{\mathbf{K}}^N)\mathbf{Q}\}\end{cases},\widehat{\mathbf{C}}^N=2\hat{\bar{\mathbf{m}}}\mathbf{T}^T\bar{\boldsymbol{\omega}}\boldsymbol{\varepsilon}(\mathbf{q}\mathbf{q}^T/\mu^2)\mathbf{T},\widehat{\mathbf{C}}^L=2\hat{\bar{\mathbf{m}}}\mathbf{T}^T\bar{\boldsymbol{\omega}}\boldsymbol{\varepsilon}\mathbf{T},$$
$$\widehat{\mathbf{K}}^L=\hat{\bar{\mathbf{m}}}\{\mathrm{diag}(0,\bar{\mathbf{m}}\bar{\mathbf{\Lambda}}^2)+\mathbf{T}^T\bar{\boldsymbol{\omega}}^2\mathbf{T}\},\quad \widehat{\mathbf{K}}^N=\hat{\bar{\mathbf{m}}}\mathbf{T}^T\bar{\boldsymbol{\omega}}^2(\boldsymbol{\gamma}_k/\mu)\mathbf{T},\quad \widehat{\mathbf{F}}=\widehat{\mathbf{F}}_0-\widehat{\mathbf{F}}_G-\widehat{\mathbf{F}}_g+\widehat{\mathbf{F}}_1. \tag{7}$$

Indices "*L*" and "*N*" identify linear and nonlinear parts of matrices. Based on Eq. 7, equilibrium points, stabilities, nonlinear behavior and energy flow characteristics [10] can be investigated.

Influence of the nonlinear isolator on the beam

Eliminating the interaction force vector $\bar{\mathbf{f}}$ in Eq. 5, we obtain

$$(\mathbf{I}+\mathbf{m}_N)\ddot{\bar{\mathbf{\Phi}}}+\mathbf{c}_N\dot{\bar{\mathbf{\Phi}}}+[\bar{\mathbf{\Lambda}}^2+\mathbf{k}_N]\bar{\mathbf{\Phi}}=-2\bar{\Omega}\mathrm{E}\tilde{C}_{21}\bar{\mathbf{m}}^{-1}\mathbf{Y}_0^T\dot{q}_1+\bar{\mathbf{m}}^{-1}\mathbf{Y}_0^T(\bar{F}_0\cos\bar{t}-\bar{g}-\bar{F})-\bar{\mathbf{m}}^{-1}\bar{\mathbf{G}},\quad \tilde{K}_{22}=\Gamma_k/\mu,$$
$$\mathbf{m}_N=\bar{\mathbf{m}}^{-1}\bar{\mathbf{Y}},\quad \mathbf{k}_N=\bar{\Omega}^2(1-\tilde{K}_{22})\mathbf{m}_N,\quad \mathbf{c}_N=2\bar{\Omega}\mathrm{E}(1+\tilde{C}_{22})\mathbf{m}_N,\quad \tilde{C}_{22}=\mathrm{X}_c q_2^2/\mu^2,\quad \tilde{C}_{21}=\mathrm{X}_c q_2 q_1/\mu^2, \tag{8}$$

describing the beam dynamics influenced by the nonlinear isolator. $\mathbf{m}_N$, $\mathbf{k}_N$ and $\mathbf{c}_N$ denote an additional mass, stiffness and damping matrices added to the beam by the nonlinear isolator. Fig.2 shows the suspension frequency affected by the supporting stiffness.

Extremely low supporting frequency. A very low supporting frequency required by GVT is designed

$$\Omega_{NSA}=\hat{\bar{\Omega}}_2^{(1)}=\sqrt{(1-\Gamma_k/\mu_0^{(1)})/(1+\tilde{\rho}\tilde{S})}\bar{\Omega},\quad \gamma_k+\bar{f}_1/\bar{\omega}^2=\mu_0^{(1)},$$
$$1-\Gamma_k/\mu_0^{(1)}=1-\tilde{k}(1+\tilde{k}_1)/[(1+\tilde{k})(\tilde{k}_1+\bar{\Delta}_1)],\quad \tilde{k}=k/K,\quad \tilde{k}_1=k/K_1,\quad \bar{\Delta}_1=(X_0-L_1)/l. \tag{9}$$

by choosing nonlinear term as a negative stiffness satisfying

$$\Gamma_k/\mu_0^{(1)}>0,\qquad \tilde{k}_1+\bar{\Delta}_1=k/K_1+(X_0-L_1)/l>0,\qquad 1>\tilde{k}(1+\tilde{k}_1)/[(1+\tilde{k})(\tilde{k}_1+\bar{\Delta}_1)]>0, \tag{10}$$

A rigid supporting platform. This design is to obtain an extreme high frequency, it is necessary

$$\Gamma_k/\mu_0^{(1)}=\Gamma_k/(\gamma_k+\bar{f}_1/\bar{\omega}^2)=\tilde{k}(1+\tilde{k}_1)/[(1+\tilde{k})(\tilde{k}_1+\bar{\Delta}_1)]<0,\quad \tilde{k}_1+\bar{\Delta}_1=k/K_1+(X_0-L_1)/l<0, \tag{11}$$

where the nonlinear term plays a positive stiffness.

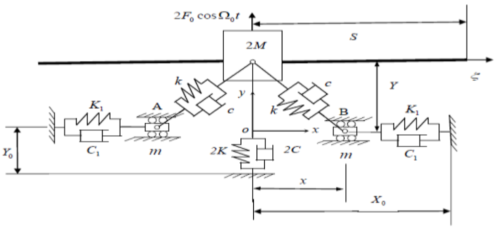

Fig.1 Integrated interaction system.

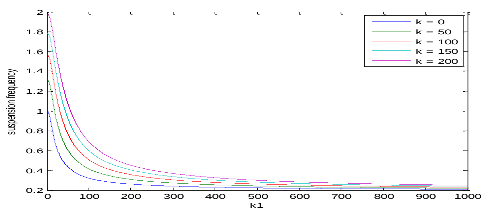

Fig.2 Suspension frequency affected by supporting stiffness

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