KEK-TH-1780

Quark mixing from $\Delta(6N^2)$ family symmetry

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Abstract

We consider a direct approach to quark mixing based on the discrete family symmetry $\Delta(6N^2)$ in which the Cabibbo angle is determined by a residual $Z_2 \times Z_2$ subgroup to be $|V_{us}| = 0.222521$, for N being a multiple of 7. We propose a particular model in which unequal smaller quark mixing angles and CP phases may occur without breaking the residual $Z_2 \times Z_2$ symmetry. We perform a numerical analysis of the model for $N = 14$, where small $Z_2 \times Z_2$ breaking effects of order 3% are allowed by model, allowing perfect agreement within the uncertainties of the experimentally determined best fit quark mixing values.

1 Introduction

Non-Abelian discrete groups have been extensively used as family symmetries in the lepton sector, in order to account for the large leptonic mixing angles [1] (for reviews see e.g. [2, 3, 4, 5].) In the direct approach, a non-Abelian family symmetry in the lepton sector is assumed. Following the determination of a Cabibbo-sized reactor angle, the only viable class appears to be $\Delta(6N^2)$ for large N values [6, 7, 8, 9]. Then, such a symmetry is broken to $Z_2 \times Z_2$ in the neutrino sector (the so called Klein symmetry) and Z_3 in the charged lepton sector, with the mixing angles determined from symmetry.

An analogous approach based on $\Delta(6N^2)$ has also been considered in the quark sector [10, 11]. In the quark sector one may envisage a residual $Z_n \times Z_m$ symmetry of the quark mass matrices, where this is a subgroup of the $\Delta(6N^2)$ family symmetry. However, in the quark sector, this approach is more challenging due to the small mixing angles. Nevertheless, earlier work showed that the Cabibbo angle could emerge from a residual $Z_2 \times Z_2$ symmetry, arising as a subgroup of the dihedral family symmetry D_7 [12, 13], D_{12} [14], or D_{14} [15, 16, 17]. Then, more general analyses based on larger discrete family symmetry groups was considered [18, 10]. Some authors have speculated that both the lepton mixing angles and the Cabibbo angle may arise from some common discrete family symmetry group [17, 18]. Note that only the Cabibbo angle is determined, since the residual $Z_2 \times Z_2$ symmetry only fixes the upper 2×2 block of the mixing matrix. The Cabibbo angle is predicted by $\theta_C = \pi n/N$ where n and N are integers relating to the family symmetry. A complementary approach to deriving the Cabibbo angle of $\theta_C \approx 1/4$ at leading order was recently considered in an indirect model based on a vacuum alignment $(1, 4, 2)$ without any residual symmetry [19].

It is clear that the residual $Z_2 \times Z_2$ symmetry is insufficient by itself to determine all the small quark mixing angles. Moreover, it is not even sufficient to fully determine the structure of the CKM matrix, since the eigenvalues of Z_2 are ± 1, hence at least two eigenvalues of the 3×3 generators should be the same. In order to break the degeneracy, it is necessary to consider concrete models. In a recent paper [11], a realistic model of quarks was proposed based on the discrete family symmetry $\Delta(6N^2)$, where the residual symmetry for the quark sector was assumed to be $Z_2 \times Z_2$ symmetry, corresponding to a Z_2 symmetry in each of the up and down sectors. However, a drawback of that model was that, the resulting structure of the CKM matrix required $\theta_{23} = \theta_{13}$, in the $Z_2 \times Z_2$ symmetry limit. The purpose of the present paper is to consider an alternative direct model of quarks based on $\Delta(6N^2)$ in which an alternative $Z_2 \times Z_2$ subgroup is preserved which allows $\theta_{23} \neq \theta_{13}$. As in the previous model, the present model will provide a qualitative explanation for the smaller mixing angles, although their quantitative values must be fitted to experimental values, rather than being predicted.

This paper is organized as follows. In section 2, we discuss the $Z_n \times Z_m$ symmetry of the quark mass matrices and the relation with the CKM matrix. In section 3, we present a brief review of the group theory of the $\Delta(6N^2)$ series and identify suitable $Z_2 \times Z_2$ subgroups which may be preserved in the quark sector, leading to a successful determination of the Cabibbo angle. In section 4, we present a model of quarks based on $\Delta(6N^2)$. We construct the quark mass matrices and resulting CKM mixing and derive the vacuum alignments that are required. In section 5, we perform a full numerical analysis of the model for $N = 14$ and show that all the quark masses, CKM mixing angles and the unitarity triangle are accommodated. Section 6 is devoted to the summary.

2 CKM matrix and $Z_n \times Z_m$ symmetry of quark mass matrices

The quark mass matrices, M_u and M_d, are defined in a general RL basis by

$$-\mathcal{L} = \begin{pmatrix}\overline{u} & \overline{c} & \overline{t}\end{pmatrix}_R M_u \begin{pmatrix} u \\ c \\ t \end{pmatrix}_L + \begin{pmatrix}\overline{d} & \overline{s} & \overline{b}\end{pmatrix}_R M_d \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L + H.c.. \tag{2.1}$$

We write the mass matrices in the diagonal basis with hats, where,

$$M_u = V_u' \hat{M}_u V_u^\dagger \quad \text{and} \quad M_d = V_d' \hat{M}_d V_d^\dagger. \tag{2.2}$$

Hence,

$$M_u^\dagger M_u = V_u \hat{M}_u^\dagger \hat{M}_u V_u^\dagger \quad \text{and} \quad M_d^\dagger M_d = V_d \hat{M}_d^\dagger \hat{M}_d V_d^\dagger. \tag{2.3}$$

Thanks to $Z_n \times Z_m$ symmetry, the quark mass matrices in the diagonal basis are invariant under $\hat{Q}$ and $\hat{A}$ transformations,

$$\hat{Q}^\dagger \left(\hat{M}_u^\dagger \hat{M}_u\right) \hat{Q} = \hat{M}_u^\dagger \hat{M}_u \quad \text{and} \quad \hat{A}^\dagger \left(\hat{M}_d^\dagger \hat{M}_d\right) \hat{A} = \hat{M}_d^\dagger \hat{M}_d, \tag{2.4}$$

where $\hat{Q}$ and $\hat{A}$ are elements of Z_n and Z_m, respectively, given by

$$\hat{Q} = \begin{pmatrix} e^{2\pi i n_u/n} & 0 & 0 \\ 0 & e^{2\pi i n_c/n} & 0 \\ 0 & 0 & e^{2\pi i n_t/n} \end{pmatrix}, \quad \hat{A} = \begin{pmatrix} e^{2\pi i m_d/m} & 0 & 0 \\ 0 & e^{2\pi i m_s/m} & 0 \\ 0 & 0 & e^{2\pi i m_b/m} \end{pmatrix}, \tag{2.5}$$

where $n_{u,c,t}$ and $m_{d,s,b}$ are integers. It then follows that in the original (non-diagonal) basis that the mass matrices are invariant under Q and A transformations,

$$Q^\dagger \left(M_u^\dagger M_u\right) Q = M_u^\dagger M_u \quad \text{and} \quad A^\dagger \left(M_d^\dagger M_d\right) A = M_d^\dagger M_d, \tag{2.6}$$

where

$$Q = V_u \hat{Q} V_u^\dagger, \quad A = V_d \hat{A} V_d^\dagger. \tag{2.7}$$

In the non-diagonal basis they also satisfy $Q^n = A^m = e$. Since the CKM matrix is given by $V_u^\dagger V_d$, up to phase transformations, it can be determined from the matrices which diagonalise Q and A,

$$Q = V_Q \hat{Q} V_Q^\dagger, \quad A = V_A \hat{A} V_A^\dagger, \tag{2.8}$$

where we identify $V_u = V_Q$ and $V_d = V_A$.

3 The group $\Delta(6N^2)$ and Z_2 symmetry

Let us briefly review the discrete group $\Delta(6N^2)$ [3], which is isomorphic to $(Z_N^c \times Z_N^d) \rtimes S_3$. The group S_3 is isomorphic to $Z_3^a \rtimes Z_2^b$, where we denote the generators of Z_3^a and Z_2^b as a and b and we write the generators of Z_N^c and Z_N^d as c and d. These generators satisfy

$$\begin{aligned} &a^3 = b^2 = (ab)^2 = c^N = d^N = e, \quad cd = dc, \\ &aca^{-1} = c^{-1}d^{-1}, \quad ada^{-1} = c, \\ &bcb^{-1} = d^{-1}, \quad bdb^{-1} = c^{-1}. \end{aligned} \tag{3.1}$$

Using them, all of $\Delta(6N^2)$ elements are written as

$$g = a^k b^\ell c^m d^n, \tag{3.2}$$

for $k = 0, 1, 2$, $\ell = 0, 1$ and $m, n = 0, 1, 2, \cdots, N-1$.

For $N/3 \neq$integer, irreducible representations are $\mathbf{1}_{0,1}$, $\mathbf{2}$, $\mathbf{3}_{1k}$, $\mathbf{3}_{2k}$, and $\mathbf{6}_{[[k],[\ell]]}$. Tensor products relating to doublet and triplets are

$$\begin{aligned} \mathbf{3}_{1k} \times \mathbf{3}_{1k'} = \mathbf{3}_{1(k+k')} + \mathbf{6}_{[[k],[-k']]}, \quad \mathbf{3}_{1k} \times \mathbf{3}_{2k'} = \mathbf{3}_{2(k+k')} + \mathbf{6}_{[[k],[-k']]}, \\ \mathbf{3}_{2k} \times \mathbf{3}_{2k'} = \mathbf{3}_{1(k+k')} + \mathbf{6}_{[[k],[-k']]}, \quad \mathbf{3}_{1k} \times \mathbf{2} = \mathbf{3}_{1k} + \mathbf{3}_{2k}, \\ \mathbf{3}_{2k} \times \mathbf{2} = \mathbf{3}_{1k} + \mathbf{3}_{2k}, \quad \mathbf{2} \times \mathbf{2} = \mathbf{1}_0 + \mathbf{1}_1 + \mathbf{2}. \end{aligned} \tag{3.3}$$

Some triplets and sextet are reducible, precisely $\mathbf{3}_{10} = \mathbf{1}_0 + \mathbf{2}$, $\mathbf{3}_{20} = \mathbf{1}_1 + \mathbf{2}$, and $\mathbf{6}_{[[-k],[k]]} = \mathbf{3}_{1k} + \mathbf{3}_{2k}$. If their representations are explicitly given, they are $(x_1, x_2, x_3)_{\mathbf{3}_{10}} = (x_1 + x_2 + x_3)_{\mathbf{1}_0} + (\omega x_1 + x_2 + \omega^2 x_3, \omega^2 x_1 + x_2 + \omega x_3)_{\mathbf{2}}$, $(x_1, x_2, x_3)_{\mathbf{3}_{20}} = (x_1 + x_2 + x_3)_{\mathbf{1}_1} + (\omega x_1 + x_2 + \omega^2 x_3, \omega^2 x_1 + x_2 + \omega x_3)_{\mathbf{2}}$, and $(x_1, x_2, x_3, x_4, x_5, x_6)_{\mathbf{6}_{[[-k],[k]]}} = (x_1 + x_6, x_2 + x_5, x_3 + x_4)_{\mathbf{3}_{1k}} + (-x_1 + x_6, -x_2 + x_5, -x_3 + x_4)_{\mathbf{3}_{2k}}$.

In a particular matrix representation, the irreducible triplet generators are,

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad b = \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad c = \begin{pmatrix} \eta^k & 0 & 0 \\ 0 & \eta^{-k} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^k & 0 \\ 0 & 0 & \eta^{-k} \end{pmatrix}, \tag{3.4}$$

for the triplet $\mathbf{3}_{1k}$ with plus sign and for $\mathbf{3}_{2k}$ with minus sign where $\eta = e^{2\pi i/N}$.

Let us consider $Q = abc^x$ and $A = abc^y$, i.e.

$$Q = \begin{pmatrix} 0 & \eta^{-kx} & 0 \\ \eta^{kx} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & \eta^{-ly} & 0 \\ \eta^{ly} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.5}$$

for $\mathbf{3}_{1k}$ to Q and $\mathbf{3}_{1l}$ to A. Because of the degeneracy of the two eigenvalues $+1$ for the above matrices,we generally have

$$Q = V_Q \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \mp 1 & 0 \\ 0 & 0 & +1 \end{pmatrix} V_Q^\dagger, \quad A = V_A \begin{pmatrix} \pm 1 & 0 & 0 \\ 0 & \mp 1 & 0 \\ 0 & 0 & +1 \end{pmatrix} V_A^\dagger, \tag{3.6}$$

which corresponds to having a $+1$ eigenvalue in the (3,3) position and the other two eigenvalues ± 1 being in all possible places, with the trace equal to $+1$. The position of these eigenvalues is not fixed by symmetry arguments alone since they may be interchanged by further (1,2) unitary rotations, with each choice being consistent with Q, A in Eq. (3.5). A particular model will resolve the degeneracy. For example in the model in [11], the ordering chosen was,

$$Q = V_Q \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_Q^\dagger, \quad A = V_A \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_A^\dagger, \tag{3.7}$$

This ordering was responsible for the unwanted prediction $\theta_{23} = \theta_{13}$, as discussed in [11].

In the present paper we propose a model which selects the following ordering,

$$Q = V_Q \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_Q^\dagger, \quad A = V_A \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} V_A^\dagger, \tag{3.8}$$

where

$$\begin{aligned} V_Q &= \frac{1}{\sqrt{2}} \begin{pmatrix} \eta^{-kx} & -\eta^{-kx} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta e^{i\alpha} \\ 0 & 1 & 0 \\ -\sin\theta e^{-i\alpha} & 0 & \cos\theta \end{pmatrix}, \\ V_A &= \frac{1}{\sqrt{2}} \begin{pmatrix} -\eta^{-ly} & \eta^{-ly} & 0 \\ 1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta' & \sin\theta' e^{i\beta} \\ 0 & -\sin\theta' e^{-i\beta} & \cos\theta' \end{pmatrix}. \end{aligned} \tag{3.9}$$

For simplicity, we consider $\alpha = \beta = 0$ in this section. As noted above, the CKM matrix is given by $V_{\rm CKM} = V_Q^\dagger V_A$ up to phase transformations so that

$$V_{\rm CKM} = \frac{1}{2} \begin{pmatrix} (1-\eta^{kx-ly})c & (1+\eta^{kx-ly})cc' + 2ss' & -2sc' + (1+\eta^{kx-ly})cs' \\ 1+\eta^{kx-ly} & (1-\eta^{kx-ly})c' & (1-\eta^{kx-ly})s' \\ (1-\eta^{kx-ly})s & (1+\eta^{kx-ly})sc' - 2cs' & 2cc' + (1+\eta^{kx-ly})ss' \end{pmatrix}, \tag{3.10}$$

where $c = \cos\theta$, $s = \sin\theta$, $c' = \cos\theta'$, and $s' = \sin\theta'$. If we take $N = 7$, $kx - ly = 5$ with $s = -0.0021$ and $s' = 0.042$, we obtain $|V_{us}| = 0.222$, $|V_{cb}| = 0.0409$, $|V_{ub}| = 0.00911$ and $J = {\rm Im}(V_{us}V_{cb}V_{ub}^* V_{cs}^*) = 1.81 \times 10^{-5}$. Detail numerical discussions will be presented based on our model in section 5.

4 Model building

4.1 Particle contents and charge assignment

Let us present the model, which realizes the quark mass matrices with the symmetric property in the section 3. As seen Table 1, we suppose the charge assignment of the

	(q_1,q_2,q_3)	(u^c,c^c)	t^c	(d^c,s^c)	b^c	h_u,h_d	χ_u	χ'_u	χ''_u	χ_d	χ'_d	χ''_d
$\Delta(6N^2)$	$\mathbf{3}_{1k}$	$\mathbf{2}$	$\mathbf{1}_0$	$\mathbf{2}$	$\mathbf{1}_0$	$\mathbf{1}_0$	$\mathbf{3}_{1(-k)}$	$\mathbf{3}_{1(-k)}$	$\mathbf{1}_0$	$\mathbf{3}_{1(-k)}$	$\mathbf{3}_{1(-k)}$	$\mathbf{1}_0$
Z_{N+1}	0	1	1	0	0	0	-1	-1	0	1	0	-1
Z'_{N+1}	0	0	0	1	1	0	1	0	-1	-1	-1	0
$U(1)_R$	1	1	1	1	1	0	0	0	0	0	0	0

Table 1: Particle contents and charge assignment of the flavor symmetry for fermions and scalar fields χ's.

quarks and scalar fields χs in the flavor symmetry $\Delta(6N^2)$ and Z_{N+1} where $N/3$ is not integer.

The superpotential for the quark sector is

$$\begin{aligned}
w_q =& y_{u1}((u^c+\omega^2c^c)q_1\chi_{u1}+(u^c+c^c)\omega q_2\chi_{u2}+(\omega^2u^c+c^c)q_3\chi_{u3})h_u\chi''_u/\Lambda^2 \\
&+y_{u2}((u^c+\omega^2c^c)q_1\chi'_{u1}+(u^c+c^c)\omega q_2\chi'_{u2}+(\omega^2u^c+c^c)q_3\chi'_{u3})h_u/\Lambda \\
&+y_{u3}t^c(q_1\chi_{u1}+q_2\chi_{u2}+q_3\chi_{u3})h_u\chi''_u/\Lambda^2+y_{u4}t^c(q_1\chi'_{u1}+q_2\chi'_{u2}+q_3\chi'_{u3})h_u/\Lambda \\
&+y_{d1}((d^c+\omega^2s^c)q_1\chi_{d1}+(d^c+s^c)\omega q_2\chi_{d2}+(\omega^2d^c+s^c)q_3\chi_{d3})h_d\chi''_d/\Lambda^2 \\
&+y_{d2}((d^c+\omega^2s^c)q_1\chi'_{d1}+(d^c+s^c)\omega q_2\chi'_{d2}+(\omega^2d^c+s^c)q_3\chi'_{d3})h_d/\Lambda \\
&+y_{d3}b^c(q_1\chi_{d1}+q_2\chi_{d2}+q_3\chi_{d3})h_d\chi''_d/\Lambda^2+y_{d4}b^c(q_1\chi'_{d1}+q_2\chi'_{d2}+q_3\chi'_{d3})h_d/\Lambda.
\end{aligned} \tag{4.1}$$

Multiplication rule of the group $\Delta(6N^2)$ is based on the review [3]. For instance, the term of y_{u1} is given by using $(x_1,x_2,x_3)_{3_{1k}}\times(y_1,y_2,y_3)_{3_{1k}}=(x_1y_1+x_2y_2+x_3y_3)_{1_0}+(\omega x_1y_1+x_2y_2+\omega^2x_3y_3,\omega^2x_1y_1+x_2y_2+\omega x_3y_3)_2+(x_3y_2,x_1y_3,x_2y_1,x_1y_2,x_3y_1,x_2y_3)_{6_{[k,k]}}$ and $(x_1,x_2)_2\times(y_1,y_2)_2=(x_1y_2+x_2y_1)_{1_0}+(x_1y_2-x_2y_1)_{1_1}+(x_2y_2,x_1y_1)_2$, where ω is the cubic root of one. The vacuum alignment is taken as

$$\begin{aligned}
\langle\chi_u\rangle &= \begin{pmatrix} u_u \\ -u_u\eta^x \\ 0\end{pmatrix}, \quad \langle\chi'_u\rangle = \begin{pmatrix} 0 \\ 0 \\ u'_u\end{pmatrix}, \quad \langle\chi''_u\rangle = u''_u, \\
\langle\chi_d\rangle &= \begin{pmatrix} u_d \\ -u_d\eta^y \\ 0\end{pmatrix}, \quad \langle\chi'_d\rangle = \begin{pmatrix} 0 \\ 0 \\ u'_d\end{pmatrix}, \quad \langle\chi''_d\rangle = u''_d.
\end{aligned} \tag{4.2}$$

We will discuss how to get this vacuum alignment in subsection 4.2. The minus signs of the vacuum expectation values (VEV's) $\langle\chi_u\rangle$ and $\langle\chi_d\rangle$ are important to get the stable vacuum in the potential analysis, and those can be given only when N is even. Although, $N=7$ is the minimum number to get the Cabibbo angle $\theta_{12}\approx 0.22$, we have to take $N=14$ as the minimum to realize the stable vacuum in our model. In this paper, we assume VEV's are real. By choosing proper Q and A, we can obtain $Q\langle\chi_u\rangle=\langle\chi_u\rangle$, $Q\langle\chi'_u\rangle=\langle\chi'_u\rangle$, $A\langle\chi_d\rangle=\langle\chi_d\rangle$, and $A\langle\chi'_d\rangle=\langle\chi'_d\rangle$ from the Eqs. (3.5) when $kx=x+N/2$ and $ly=y+N/2$. Then we have residual symmetry $Z_2\times Z_2$ for mass matrices of quarks.

Actually, the mass matrices are expressed by

$$\begin{aligned}(M_u)_{RL} &= \frac{v_u}{\Lambda^2}\begin{pmatrix} y_{u1}u_u u''_u & -\omega y_{u1}u_u u''_u\eta^x & \omega^2 y_{u2}u'_u\Lambda \\ \omega^2 y_{u1}u_u u''_u & -\omega y_{u1}u_u u''_u\eta^x & y_{u2}u'_u\Lambda \\ y_{u3}u_u u''_u & -y_{u3}u_u u''_u\eta^x & y_{u4}u'_u\Lambda \end{pmatrix}, \\ (M_d)_{RL} &= \frac{v_d}{\Lambda^2}\begin{pmatrix} y_{d1}u_d u''_d & -\omega y_{d1}u_d u''_d\eta^y & \omega^2 y_{d2}u'_d\Lambda \\ \omega^2 y_{d1}u_d u''_d & -\omega y_{d1}u_d u''_d\eta^y & y_{d2}u'_d\Lambda \\ y_{d3}u_d u''_d & -y_{d3}u_d u''_d\eta^y & y_{d4}u'_d\Lambda \end{pmatrix}.\end{aligned} \tag{4.3}$$

They satisfy $Q^\dagger M_u^\dagger M_u Q = M_u^\dagger M_u$ and $A^\dagger M_d^\dagger M_d A = M_d^\dagger M_d$.

Mass matrices in LL basis become

$$\begin{aligned}&V_{12}^{u\dagger}M_u^\dagger M_u V_{12}^u \\ &= \frac{v_u^2}{\Lambda^4}\begin{pmatrix} (|y_{u1}|^2+2|y_{u3}|^2)u_u^2u_u''^2 & 0 & -\sqrt{2}(y_{u1}^*y_{u2}-y_{u3}^*y_{u4})u_u u'_u u''_u\Lambda \\ 0 & 3|y_{u1}|^2u_u^2u_u''^2 & 0 \\ -\sqrt{2}(y_{u1}y_{u2}^*-y_{u3}y_{u4}^*)u_u u'_u u''_u\Lambda & 0 & (2|y_{u2}|^2+|y_{u4}|^2)u_u'^2\Lambda^2 \end{pmatrix}, \\ &V_{12}^{d\dagger}M_d^\dagger M_d V_{12}^d \\ &= \frac{v_d^2}{\Lambda^4}\begin{pmatrix} 3|y_{d1}|^2u_d^2u_d''^2 & 0 & 0 \\ 0 & (|y_{d1}|^2+2|y_{d3}|^2)u_d^2u_d''^2 & \sqrt{2}(y_{d1}^*y_{d2}-y_{d3}^*y_{d4})u_d u'_d u''_d\eta^{-y} \\ 0 & \sqrt{2}(y_{d1}y_{d2}^*-y_{d3}y_{d4}^*)u_d u'_d u''_d\Lambda\eta^y & (2|y_{d2}|^2+|y_{d4}|^2)u_d'^2\Lambda^2 \end{pmatrix},\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}V_{12}^u &= \begin{pmatrix} 1 & \eta^x & 0 \\ -\eta^{-x} & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\eta^{-kx-x} & 0 \\ 0 & 0 & 1 \end{pmatrix} V_Q \begin{pmatrix} \eta^{kx} & 0 & 0 \\ 0 & -\eta^{kx+x} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \\ V_{12}^d &= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -\eta^y & 0 \\ \eta^{-y} & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \eta^{-ly-y} & 0 \\ 0 & 0 & 1 \end{pmatrix} V_A \begin{pmatrix} \eta^{ly} & 0 & 0 \\ 0 & \eta^{ly+y} & 0 \\ 0 & 0 & 1 \end{pmatrix},\end{aligned} \tag{4.5}$$

where V_Q and V_A are the ones of Eq. (3.9) (where the CKM matrix is only specified by the symmetry up to phase transformations). Each mass matrix contains four parameters, we can obtain three masses and additional mixing angle in general.

4.1.1 Masses and mixing

Masses and mixing angles can be obtained by diagonalizing the mass matrices. Masses are expressed by

$$\begin{aligned} m_u^2 &= \frac{v_u^2}{2\Lambda^4}(m_{u22}^4+m_{u33}^4-\sqrt{(m_{u22}^4-m_{u33}^4)^2+4m_{u23}^8}\ ), \quad m_c^2 = \frac{3|y_{u1}|^2v_u^2u_u^2u_u''^2}{\Lambda^4}, \\ m_t^2 &= \frac{v_u^2}{2\Lambda^4}(m_{u22}^4+m_{u33}^4+\sqrt{(m_{u22}^4-m_{u33}^4)^2+4m_{u23}^8}\ ), \\ m_d^2 &= \frac{3|y_{d1}|^2v_d^2u_d^2u_d''^2}{\Lambda^4}, \quad m_s^2 = \frac{v_d^2}{2\Lambda^4}(m_{d22}^4+m_{d33}^4-\sqrt{(m_{d22}^4-m_{d33}^4)^2+4m_{d23}^8}\ ), \\ m_b^2 &= \frac{v_d^2}{2\Lambda^4}(m_{d22}^4+m_{d33}^4+\sqrt{(m_{d22}^4-m_{d33}^4)^2+4m_{d23}^8}\ ),\end{aligned} \tag{4.6}$$

where $m^4_{\alpha 22} = (|y_{\alpha 1}|^2 + 2|y_{\alpha 3}|^2) u^2_\alpha u''^2_\alpha$, $m^4_{\alpha 23} = \sqrt{2}|(y^*_{\alpha 1} y_{\alpha 2} - y^*_{\alpha 3} y_{\alpha 4})| u_\alpha u'_\alpha u''_\alpha \Lambda$, and $m^4_{\alpha 33} = (2|y_{\alpha 2}|^2 + |y_{\alpha 4}|^2) u'^2_\alpha \Lambda^2$ with $\alpha = u, d$. Similarly, mixing matrices are

$$V^u = V^u_{12} \begin{pmatrix} \cos\theta_u & 0 & -e^{i\phi_u}\sin\theta_u \\ 0 & 1 & 0 \\ e^{-i\phi_u}\sin\theta_u & 0 & \cos\theta_u \end{pmatrix}, \quad V^d = V^d_{12} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_d & -e^{i\phi_d}\sin\theta_d \\ 0 & e^{-i\phi_d}\sin\theta_d & \cos\theta_d \end{pmatrix} \tag{4.7}$$

where

$$\tan 2\theta_u = \frac{2m^4_{u23}}{m^4_{u33} - m^4_{u22}}, \quad \tan 2\theta_d = \frac{2m^4_{d23}}{m^4_{d33} - m^4_{d22}}, \tag{4.8}$$

and $\phi_{u,d}$ are given by phases of Yukawa coupling and η^y. The CKM matrix is given by $V_{\rm CKM} = V^\dagger_u V_d$ so that

$$\begin{aligned} &V_{\rm CKM} \\ &= \frac{1}{2} \begin{pmatrix} (1-\eta^{x-y})c_u & -(\eta^x+\eta^y)c_u c_d + 2e^{i(\phi_u-\phi_d)} s_u s_d & 2e^{i\phi_u} s_u c_d + (\eta^x+\eta^y)e^{i\phi_d} c_u s_d \\ \eta^{-x}+\eta^{-y} & (1-\eta^{-x+y})c_d & -(1-\eta^{-x+y})e^{i\phi_d} s_d \\ -(1-\eta^{x-y})e^{-i\phi_u} s_u & (\eta^x+\eta^y)e^{-i\phi_u} s_u c_d + 2e^{-i\phi_d} c_u s_d & 2c_u c_d - (\eta^x+\eta^y)e^{-i(\phi_u-\phi_d)} s_u s_d \end{pmatrix}, \end{aligned} \tag{4.9}$$

where $s_u = \sin\theta_u$, $c_u = \cos\theta_u$, $s_d = \sin\theta_d$, and $c_d = \cos\theta_d$. For example, if we take $N = 7$, $x - y = 4$, $s_u = -0.0021$ and $s_d = 0.042$ with real Yukawa couplings, we obtain the desired values $|V_{us}| = 0.22$ and $|V_{cb}| = 0.041$, but undesired one $|V_{ub}| = 0.0091$, which is the predicted lower bound of $|V_{ub}|$.

In addition, they obtain the unitarity triangle with three angles $\alpha = 90°$, $\beta = 77°$, and $\gamma = 13°$, which is an unfavored triangle. Therefore, we need to take complex Yukawa couplings in order to get the proper $|V_{ub}|$ and CP phase.

4.2 Potential analysis

	χ_u	χ'_u	χ''_u	χ_d	χ'_d	χ''_d	Φ_1	Φ_2	Φ_3	Φ_4	Φ_5	Φ_6	Φ_7	Φ_8
$\Delta(6N^2)$	$\mathbf{3}_{1(-k)}$	$\mathbf{3}_{1(-k)}$	$\mathbf{1}_0$	$\mathbf{3}_{1(-k)}$	$\mathbf{3}_{1(-k)}$	$\mathbf{1}_0$	$\mathbf{1}_0$	$\mathbf{1}_0$	$\mathbf{1}_0$	$\mathbf{1}_0$	$\mathbf{1}_0$	$\mathbf{1}_0$	$\mathbf{1}_0$	$\mathbf{1}_0$
Z_{N+1}	-1	-1	0	1	0	-1	3	3	3	-3	-1	0	-1	1
Z'_{N+1}	1	0	-1	-1	-1	0	-3	-1	0	3	3	3	1	-1
$U(1)_R$	0	0	0	0	0	0	2	2	2	2	2	2	2	2

Table 2: Particle contents and charge assignment of the flavor symmetry and $U(1)_R$ for the flavon fields χ's and driving fields Φ_i.

In order to get desired vacuum expectation values of χ's, we introduce the driving fields Φ_i with the $U(1)_R$ symmetry in the framework of the supersymmetry. The charge assignment for the scalar fields χ's and driving fields Φ_i is given in Table 2. Then, the leading order of the superpotential is given by

$$\begin{aligned} w =& \frac{\lambda_1}{\Lambda}\chi^3_u \Phi_1 + \frac{\lambda_2}{\Lambda}\chi_u \chi'^2_u \Phi_2 + \frac{\lambda_3}{\Lambda}\chi'^3_u \Phi_3 + \frac{\lambda_4}{\Lambda}\chi^3_d \Phi_4 + \frac{\lambda_5}{\Lambda}\chi_d \chi'^2_d \Phi_5 + \frac{\lambda_6}{\Lambda}\chi'^3_d \Phi_6 \\ &+ \sum_n \left(\frac{\lambda_{7n}}{\Lambda^{2n-1}}\chi^n_u \chi^{n+1}_d\right)\Phi_7 + \frac{\lambda'_7}{\Lambda^{N-2}}\chi^N_u \Phi_7 + \sum_n \left(\frac{\lambda_{8n}}{\Lambda^{2n-1}}\chi^{n+1}_u \chi^n_d\right)\Phi_8 + \frac{\lambda'_8}{\Lambda^{N-2}}\chi^N_d \Phi_8. \end{aligned} \tag{4.10}$$

They can be explicitly written as

$$
\begin{aligned}
w =&\frac{\lambda_1}{\Lambda}\chi_{u1}\chi_{u2}\chi_{u3}\Phi_1 + \frac{\lambda_2}{\Lambda}(\chi_{u1}\chi'_{u2}\chi'_{u3} + \chi_{u2}\chi'_{u1}\chi'_{u3} + \chi_{u3}\chi'_{u1}\chi'_{u2})\Phi_2 + \frac{\lambda_3}{\Lambda}\chi'_{u1}\chi'_{u2}\chi'_{u3}\Phi_3 \\
&+ \frac{\lambda_4}{\Lambda}\chi_{d1}\chi_{d2}\chi_{d3}\Phi_4 + \frac{\lambda_5}{\Lambda}(\chi_{d1}\chi'_{d2}\chi'_{d3} + \chi_{d2}\chi'_{d1}\chi'_{d3} + \chi_{d3}\chi'_{d1}\chi'_{d2})\Phi_5 + \frac{\lambda_6}{\Lambda}\chi'_{d1}\chi'_{d2}\chi'_{d3}\Phi_6 \\
&+ \sum_{n_1,n_2,n_1\geq n_2} \frac{\lambda_{7n_1,n_2}}{\Lambda^{12n_1-6n_2+1}}(\chi_{u1}\chi_{u2}\chi_{u3})^{n_1}(\chi_{d1}\chi_{d2}\chi_{d3})^{n_2} \\
&\quad\times(\chi_{u1}\chi_{d2}\chi_{d3} + \chi_{u2}\chi_{d1}\chi_{d3} + \chi_{u3}\chi_{d1}\chi_{d2})^{3n_1-3n_2+1}\Phi_7 \\
&+ \sum_{n_1,n_2,n_1\geq n_2} \frac{\lambda_{8n_1,n_2}}{\Lambda^{12n_1-6n_2+1}}(\chi_{u1}\chi_{u2}\chi_{u3})^{n_1}(\chi_{d1}\chi_{d2}\chi_{d3})^{n_2} \\
&\quad\times(\chi_{u1}\chi_{u2}\chi_{d3} + \chi_{u2}\chi_{u3}\chi_{d1} + \chi_{u3}\chi_{u1}\chi_{d2})^{3n_1-3n_2+1}\Phi_8 \\
&+ \frac{\lambda'_7}{\Lambda^{N-2}}(\chi_{u1}^N + \chi_{u2}^N + \chi_{u3}^N)\Phi_7 + \frac{\lambda'_8}{\Lambda^{N-2}}(\chi_{d1}^N + \chi_{d2}^N + \chi_{d3}^N)\Phi_8.
\end{aligned}
\tag{4.11}
$$

By solving the potential minimum conditions, we obtain the vacuum expectation values as follows:

$$
\langle\chi_u\rangle = \begin{pmatrix} u_u \\ -u_u\eta^x \\ 0 \end{pmatrix}, \quad \langle\chi'_u\rangle = \begin{pmatrix} 0 \\ 0 \\ u'_u \end{pmatrix}, \quad \langle\chi_d\rangle = \begin{pmatrix} u_d \\ -u_d\eta^y \\ 0 \end{pmatrix}, \quad \langle\chi'_d\rangle = \begin{pmatrix} 0 \\ 0 \\ u'_d \end{pmatrix}, \tag{4.12}
$$

where N is taken to be even otherwise the minus sign does not appear. These VEV's present desirable vacuum alignments.

4.3 Z_2 breaking terms

Z_2 breaking terms for the Yukawa couplings are highly suppressed. The leading order for the breaking is

$$
\begin{aligned}
\Delta w_q =&y_{b1}((u^c + \omega^2 c^c)q_1\chi_{d1} + (u^c + c^c)\omega q_2\chi_{d2} + (\omega^2 u^c + c^c)q_3\chi_{d3})h_u\chi_u''^{N-1}\chi_d''^2/\Lambda^{N+2} \\
&+ y_{b2}t^c(q_1\chi_{d1} + q_2\chi_{d2} + q_3\chi_{d3})h_u\chi_u''^{N-1}\chi_d''^2/\Lambda^{N+2} \\
&+ y_{b3}((d^c + \omega^2 s^c)q_1\chi_{u1} + (d^c + s^c)\omega q_2\chi_{u2} + (\omega^2 d^c + s^c)q_3\chi_{u3})h_d\chi_u''^2\chi_d''^{N-1}/\Lambda^{N+2} \\
&+ y_{b4}b^c(q_1\chi_{u1} + q_2\chi_{u2} + q_3\chi_{u3})h_d\chi_u''^2\chi_d''^{N-1}/\Lambda^{N+2}.
\end{aligned}
\tag{4.13}
$$

For the superpotential of scalar fields, the leading order of Z_2 breaking terms appears as

$$
\Delta w =\frac{\lambda_{b1}}{\Lambda^{N-1}}\chi_u'^N\chi_u''\Phi_7 + \frac{\lambda_{b2}}{\Lambda^{N-1}}\chi_d'^N\chi_d''\Phi_8. \tag{4.14}
$$

The VEV's of χ_u and χ_d are deviated by these terms. Then, the vacuum alignment is deviated by

$$
\langle\chi_u\rangle = \begin{pmatrix} u_u + \mathcal{O}(u_u'^N u_u''/\Lambda^N) \\ -u_u\eta^x + \mathcal{O}(u_u'^N u_u''/\Lambda^N) \\ 0 \end{pmatrix}, \quad \langle\chi_d\rangle = \begin{pmatrix} u_d + \mathcal{O}(u_d'^N u_d''/\Lambda^N) \\ -u_d\eta^y + \mathcal{O}(u_d'^N u_d''/\Lambda^N) \\ 0 \end{pmatrix}, \tag{4.15}
$$

and alignment of other fields are highly suppressed. With this deviation, the mass matrix is modified as

$$
\begin{aligned}
(M_u)_{RL} &= \frac{v_u}{\Lambda^2}\begin{pmatrix} y_{u1}u_u u''_u & -\omega y_{u1}u_u u''_u \eta^x & \omega^2 y_{u2}u'_u\Lambda \\ \omega^2 y_{u1}u_u u''_u & -\omega y_{u1}u_u u''_u\eta^x & y_{u2}u'_u\Lambda \\ y_{u3}u_u u''_u & -y_{u3}u_u u''_u\eta^x & y_{u4}u'_u\Lambda \end{pmatrix} + \frac{v_u}{\Lambda^{N+2}}\begin{pmatrix} \mathcal{O}(u'^N_u u''^2_u) & \mathcal{O}(u'^N_u u''^2_u) & 0 \\ \mathcal{O}(u'^N_u u''^2_u) & \mathcal{O}(u'^N_u u''^2_u) & 0 \\ \mathcal{O}(u'^N_u u''^2_u) & \mathcal{O}(u'^N_u u''^2_u) & 0 \end{pmatrix}, \\
(M_d)_{RL} &= \frac{v_d}{\Lambda^2}\begin{pmatrix} y_{d1}u_d u''_d & -\omega y_{d1}u_d u''_d \eta^y & \omega^2 y_{d2}u'_d\Lambda \\ \omega^2 y_{d1}u_d u''_d & -\omega y_{d1}u_d u''_d\eta^y & y_{d2}u'_d\Lambda \\ y_{d3}u_d u''_d & -y_{d3}u_d u''_d\eta^y & y_{d4}u'_d\Lambda \end{pmatrix} + \frac{v_d}{\Lambda^{N+2}}\begin{pmatrix} \mathcal{O}(u'^N_d u''^2_d) & \mathcal{O}(u'^N_d u''^2_d) & 0 \\ \mathcal{O}(u'^N_d u''^2_d) & \mathcal{O}(u'^N_d u''^2_d) & 0 \\ \mathcal{O}(u'^N_d u''^2_d) & \mathcal{O}(u'^N_d u''^2_d) & 0 \end{pmatrix}.
\end{aligned}
\tag{4.16}
$$

Thus, the magnitude of Z_2 breaking terms for the mass matrix is of order $\mathcal{O}(u'^N_u u''_u/u_u\Lambda^N)$ for up-type quarks and $\mathcal{O}(u'^N_d u''_d/u_d\Lambda^N)$ for down-quarks, respectively.

5 Numerical analysis

When the subgroup $Z_2 \times Z_2$ is preserved and the phase of VEV's is fixed, the number of parameters is four in each mass matrix. Then we can obtain three masses and one mixing angle as free parameters. For the symmetry and phases, we choose $N = 14$ and $x - y = 6$ then we predict $\sin\theta_{12} = 0.222521$ at the leading order. [1] This is to be compared to the experimental value at the weak scale of $|V_{us}| = 0.225 \pm 0.001$.

Suppose that the flavor symmetry exists at the scale of the grand unified theory (GUT). Then, we should fit the quark masses and mixing angles at the GUT scale with the supersymmetry. Inputting experimental data at the low energy scale, the renormalization group runnings give us following values [20]:

$$
\begin{aligned}
\theta_{12} \approx 0.2276, \quad 2.9\times 10^{-3} \le \theta_{13} \le 3.4\times 10^{-3}, \quad 3.3\times 10^{-2} \le \theta_{23} \le 3.9\times 10^{-2}, \\
4.8\times 10^{-6} \le \frac{m_u}{m_t} \le 5.4\times 10^{-6}, \quad 2.3\times 10^{-3} \le \frac{m_c}{m_t} \le 2.6\times 10^{-3}, \\
6.3\times 10^{-4} \le \frac{m_d}{m_b} \le 8.9\times 10^{-4}, \quad 1.8\times 10^{-2} \le \frac{m_s}{m_b} \le 1.2\times 10^{-2}.
\end{aligned}
\tag{5.1}
$$

We reproduce these mass and mixing angles by scattering our model parameters while $N = 14$ and $x - y = 6$ are fixed.

In the Figures 1 and 2, we show the scattering plots to see the consistency with experiments. Giving random values for all the Yukawa couplings with phases and VEV's of flavons, we get quark masses and mixing angles by diagonalising mass matrices of up- and down-type quarks, which are constrained by the observed values in Eq. (5.1). The physical values are actually three up-quark masses, three-down quark masses, three mixing angles, and CP phase. Since the third generation masses can be determined independently, we fit the mass ratios.

For the case of the $Z_2 \times Z_2$ invariant quark mass matrices, we plot the CKM matrix elements, the CP angles (α, β, γ) and the mass ratios in Figure 1, where red and blue

[1] This is identical to the example in the Introduction for $N = 7$ as well as the prediction of the previous model for $N = 28$ [11].

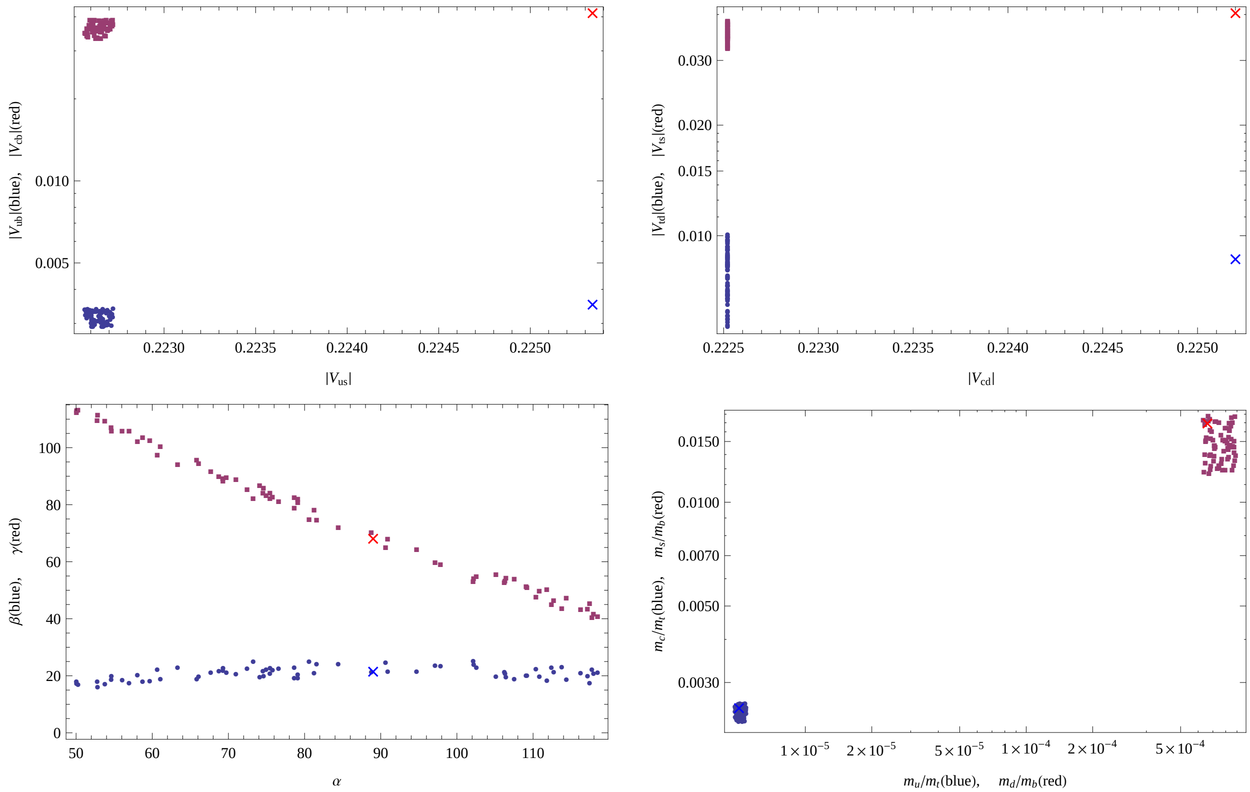

Figure 1: Scattering plots among the CKM matrix elements, the angles of the unitarity triangle and the mass ratios in the case of the $Z_2 \times Z_2$ invariant mass matrices. Cross marks denote the experimental central values.

cross marks denote the experimental central values at the weak scale [1] since the running effect is small.

As discussed above, $|V_{us}|$ ($|V_{cd}|$) is predicted to be in the very narrow range even if the next leading terms are added to the leading term $|\eta^x + \eta^y|/2$. The CKM elements $|V_{cb}|$ and $|V_{ts}|$ are reproduced due to the parameter θ_d. The $|V_{ub}|$ and $|V_{td}|$ depend on both θ_d and θ_u. Due to the phases of Yukawa couplings, these elements are fitted well. The three angles of the unitarity triangle and the quark mass ratios are also reproduced.

In order to fit the mixing angles perfectly, especially $|V_{us}|$, Z_2 breaking terms are required. As seen in Eq. (4.16), the mass matrices are modified due to the deviation of VEV's. Comparing to the leading terms that preserve Z_2, the magnitude of breaking terms is of order $u_u'^N u_u''/u_u\Lambda^N$. We show the scattering plot of the CKM matrix elements including the Z_2 breaking effect at 3% level in Figure 2, where red and blue cross marks also denote the experimental central values at the weak scale [1]. As seen in this figure, we can reproduce the experimental values of the mixing angles perfectly if Z_2 is broken of order 3%.

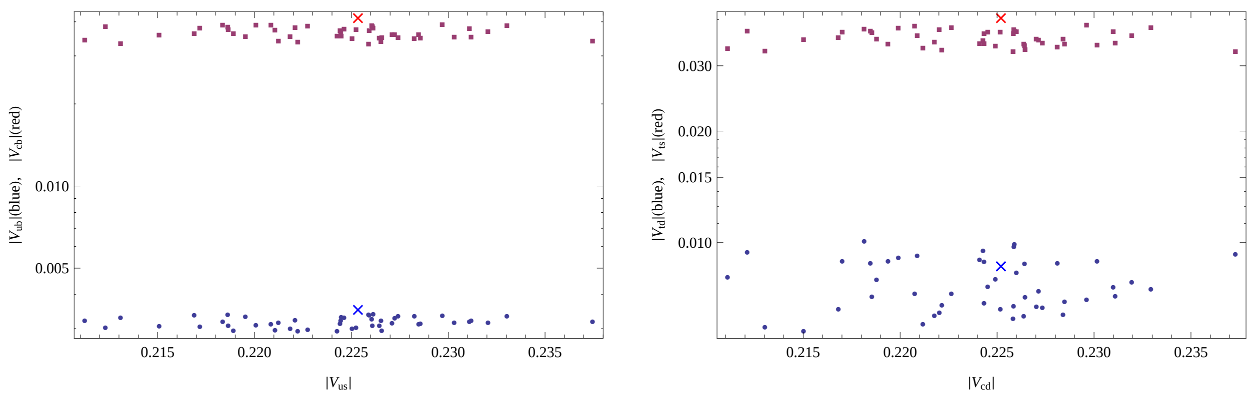

Figure 2: Relations among the CKM matrix elements, where the Z_2 breaking effects of order 3% are introduced. Cross marks denote the experimental central values.

6 Summary

We have considered a direct approach to quark mixing based on the discrete family symmetry $\Delta(6N^2)$ in which the Cabibbo angle is determined by a residual $Z_2 \times Z_2$ subgroup to be $|V_{us}| = 0.222521$, for N being a multiple of 7. This prediction is very close to the experimental value $|V_{us}| = 0.225 \pm 0.001$. We have proposed a particular model in which $|V_{cb}|$, $|V_{ub}|$ and the CP phase may occur without breaking the residual $Z_2 \times Z_2$ symmetry. We performed a numerical analysis of the model for $N = 14$, which realizes the stable vacuum. For the $Z_2 \times Z_2$ invariant quark mass matrices, the CKM matrix elements, the CP angles (α, β, γ) and the mass ratios are accommodated to the experimental data. The small $Z_2 \times Z_2$ breaking effects of order 3% allow perfect agreement within the uncertainties of the experimentally determined best fit quark mixing values.

Finally, it is tempting to speculate that $\Delta(6N^2)$ could be suitable as a candidate family symmetry for a complete model of quark and lepton masses and mixing. In the lepton sector, $\Delta(6N^2)$ has been shown to be the only viable candidate group which can provide a direct symmetry explanation of the lepton mixing, with a preserved Klein symmetry $Z_2 \times Z_2$ in the neutrino sector and a Z_3 in the charged lepton sector, where both symmetries are subgroups of $\Delta(6N^2)$. However no detailed model of leptons has been proposed. Here we have proposed a $\Delta(6N^2)$ model of quarks where a different $Z_2 \times Z_2$ subgroup controls the quark sector, providing an explanation of the Cabibbo angle for N being a multiple of 7, while allowing a good fit to other quark mixing parameters. It might be possible to extend this model to include also leptons, although we leave this idea for future work.

Acknowledgement

This work was supported in part by the Grant-in-Aid for Scientific Research No.24654062 (M.T). SFK acknowledges support from the European Union FP7 ITN-INVISIBLES (Marie Curie Actions, PITN- GA-2011- 289442) and the STFC Consolidated ST/J000396/1 grant.

References

[1] K.A. Olive et al. (Particle Data Group), Chin. Phys. C **38**, 090001 (2014).

[2] G. Altarelli and F. Feruglio, Rev. Mod. Phys. **82**, 2701 (2010) [arXiv:1002.0211 [hep-ph]].

[3] H. Ishimori, T. Kobayashi, H. Ohki, Y. Shimizu, H. Okada and M. Tanimoto, Prog. Theor. Phys. Suppl. **183**, 1 (2010) [arXiv:1003.3552 [hep-th]]; Lect. Notes Phys. **858**, 1 (2012); Fortsch. Phys. **61**, 441 (2013).

[4] S. F. King and C. Luhn, Rep. Prog. Phys. **76**, 056201 (2013) [arXiv:1301.1340 [hep-ph]].

[5] S. F. King, A. Merle, S. Morisi, Y. Shimizu and M. Tanimoto, New J. Phys. **16** (2014) 045018 [arXiv:1402.4271 [hep-ph]].

[6] M. Holthausen, K. S. Lim and M. Lindner, Phys. Lett. B **721** (2013) 61 [arXiv:1212.2411 [hep-ph]].

[7] S. F. King, T. Neder and A. J. Stuart, Phys. Lett. B **726** (2013) 312 [arXiv:1305.3200 [hep-ph]]; S. F. King and T. Neder, Phys. Lett. B **736** (2014) 308 [arXiv:1403.1758 [hep-ph]].

[8] L. Lavoura and P. O. Ludl, Phys. Lett. B **731** (2014) 331 [arXiv:1401.5036 [hep-ph]].

[9] R. M. Fonseca and W. Grimus, JHEP **1409** (2014) 033 [arXiv:1405.3678 [hep-ph]].

[10] T. Araki, H. Ishida, H. Ishimori, T. Kobayashi and A. Ogasahara, Phys. Rev. D **88** (2013) 096002 [arXiv:1309.4217 [hep-ph]].

[11] H. Ishimori and S. F. King, Phys. Lett. B **735** (2014) 33 [arXiv:1403.4395 [hep-ph]].

[12] C. S. Lam, Phys. Lett. B **656** (2007) 193 [arXiv:0708.3665 [hep-ph]].

[13] A. Blum, C. Hagedorn and M. Lindner, Phys. Rev. D **77** (2008) 076004 [arXiv:0709.3450 [hep-ph]].

[14] J. E. Kim and M. -S. Seo, JHEP **1102** (2011) 097 [arXiv:1005.4684 [hep-ph]].

[15] A. Blum and C. Hagedorn, Nucl. Phys. B **821** (2009) 327 [arXiv:0902.4885 [hep-ph]].

[16] A. Blum, C. Hagedorn and A. Hohenegger, JHEP **0803** (2008) 070 [arXiv:0710.5061 [hep-ph]].

[17] C. Hagedorn and D. Meloni, Nucl. Phys. B **862** (2012) 691 [arXiv:1204.0715 [hep-ph]].

[18] M. Holthausen and K. S. Lim, Phys. Rev. D **88** (2013) 033018 [arXiv:1306.4356 [hep-ph]].

[19] S. F. King, JHEP **1401** (2014) 119 [arXiv:1311.3295 [hep-ph]]; S. F. King, JHEP **1408** (2014) 130 [arXiv:1406.7005 [hep-ph]].

[20] S. Antusch and V. Maurer, JHEP **1311** (2013) 115 [arXiv:1306.6879 [hep-ph]].