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AN INQUIRY INTO THE NATURE OF GRAVITATIONAL SINGULARITIES

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ABSTRACT

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We provide different mathematical frameworks to describe singularities in General Relativity. The main idea is to regard singularities as obstructions to the dynamics of different matter models.

The first part of the thesis initiates our examination of spacetime by probing spacetime with matter that can be modelled as a point-particle. In particular, we discuss the case of two-dimensional Lorentzian metrics. We give concrete applications of the framework in the case of the Minkowski spacetime (which is regular) and the Friedmann-Robertson-Walker spacetime (which is geodesically incomplete).

In the second part of the thesis, we probe the geometry of spacetime with classical scalar fields. The general motivation is to redefine a singularity in spacetime not as an obstruction to geodesics or curves but as an obstruction to the dynamics of test fields. We discuss curve-integrable spacetimes, spacetimes with surface layers, impulsive gravitational waves and brane-world scenarios, and spacetimes that contain string-like singularities.

In the third part of the thesis, we present the outline of a framework to analyse the geometry of the spacetime by probing it with quantum scalar fields. The main focus of this chapter is to describe what is meant by a quantisation in a spacetime with finite differentiability.

The fourth part of the thesis presents future outlooks and some open problems.

Contents

Declaration of Authorship

I, Yafet Erasmo Sanchez Sanchez, declare that the thesis entitled *An Inquiry into the Nature of Gravitational Singularities"* and the work presented in the thesis are both my own, and have been generated by me as the result of my own original research. I confirm that:

- this work was done wholly or mainly while in candidature for a research degree at this University;
- where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
- where I have consulted the published work of others, this is always clearly attributed;
- where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
- parts of this work have been published as:
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Chapter 1

Introduction

The nature of space and time has fascinated the human mind for centuries. Works in literature, philosophy and science have approached the relationship between space, time and our empirical experience each in their own way. In the physical sciences, the best description of space and time is provided by the theory of General Relativity (GR) [1].

GR models space and time as a single geometrical entity called spacetime. To be precise, spacetime is a 4-dimensional manifold, $\mathcal{M}$, with a Lorentzian metric, g_{ab}, which is related to the energy-momentum content of matter, T_{ab}, by Einstein's field equations,

$$R_{ab} - \tfrac{1}{2}R\, g_{ab} + \Lambda g_{ab} = \frac{8\pi G}{c^4} T_{ab}.$$

Moreover, the theory has been successful in predicting the perihelion precession of Mercury and the bending of light near a massive object. More recently, the direct measurement of gravitational waves has been confirmed which was one of the most striking predictions of GR [2]. Despite these aspects, we know that the quest for a full physical understanding of space and time is not over yet.

One of the biggest surprises that GR has given us is that under certain circumstances the theory predicts its own limitations. There are two physical situations where we expect the theory to break down. The first is the gravitational collapse of certain massive stars when their nuclear fuel is spent. The second one is the far past of the universe when the density and temperature were extreme. In both cases, we expect that the geometry of spacetime will show some pathological behaviour or singularities.

The definition of a gravitational singularity is a delicate issue. It might be tempting to define a gravitational singularity following other physical theories (like electromagnetism) as the locations where the physical relevant quantities are undefined. However, in the gravitational case, this prescription cannot work. This is due to the identification of the spacetime background with the gravitational field. Hence, only where the gravitational field is defined it is meaningful to talk about locations.

If one thinks of a singularity in classical Newtonian gravity, the statement that the gravitational field is singular at a certain location is unambiguous. As an example, take the gravitational potential of a spherical mass

$$V(t, x, y, z) = \frac{GM}{\sqrt{x^2 + y^2 + z^2}}$$

with a singularity at the point $x = y = z = 0$ for any time t in $\mathbb{R}$. The location of the

singularity is well defined because the coordinates have an intrinsic character which is independent of V.
However, this prescription does not work in GR. Consider the spacetime with the line element

$$ds^2 = -\frac{1}{t^2}dt^2 + dx^2 + dy^2 + dz^2.$$

defined on the manifold $\{(t, x, y, z) \in \mathbb{R}\backslash\{0\} \times \mathbb{R}^3\}$. If we say that there is a singularity at the point $t = 0$, we might be speaking too soon for two reasons. The first is that $t = 0$ is not part of the manifold. It makes no sense to talk about $t = 0$ as a location where the field diverges. The second thing is that the lack of an intrinsic meaning of the coordinates in GR must be taken seriously. By making the coordinate transformation $\tau = \log(t)$, we obtain the line element

$$ds^2 = -d\tau^2 + dx^2 + dy^2 + dz^2,$$

on $\mathbb{R}^4$ which is an isometric extension of the previously defined spacetime. This spacetime is, of course, Minkowski spacetime which is non-singular [3].
Another idea might be trying to define a singularity in terms of invariant quantities such as scalar invariants. These scalars are usually constructed from contractions of the Riemann tensor and its derivatives. The reason is that if these quantities diverge it matches our physical idea that objects must suffer stronger and stronger deformations. Unfortunately, the scalars are not well-suited to define the complete geometry. Consider the metric

$$ds^2 = dudv + H_{ij}(u)x^i x^j du^2 - dx^i dx^i$$

given by coordinates (u, v, x^1, x^2) and where $H(u)$ is C^1. This spacetime is known as a *pp*-wave spacetime and it can be shown that every curvature scalar polynomial vanishes,despite the fact that the spacetime is not flat in general [4].
A more troublesome feature of using scalars to define singularities is that they are too local in the sense that they are evaluated at points. Therefore, if the point is removed, the scalar cannot be computed directly and we need an approximation procedure. In this case, the statement we must formalise is the following: "The scalar invariant diverges as we approach a point that has been cut out of the manifold." A mathematical precise way to add the "missing points" is to complete the manifold using convergent sequences of points in the manifold. Then the formal statement is: The sequence $\{R(x_n)\}$ diverges as the sequence $\{x_n\}$ converges to y, where $R(x_n)$ is some curvature scalar invariant evaluated at x_n in $\mathcal{M}$ and y is some point not necessarily in $\mathcal{M}$.
In Riemannian geometry, the notion of a distance allows us to define Cauchy sequences $\{x_n\}$. Moreover, if every Cauchy sequence converges in $\mathcal{M}$ then every geodesic can be extended indefinitely. That means we can take the domain of every geodesic to be $\mathbb{R}$. In this case, we say that $\mathcal{M}$ is *geodesically complete.* In fact, also the converse is true, that is if $\mathcal{M}$ is geodesically complete then $\mathcal{M}$ is metrically complete, i.e. every Cauchy sequence converges to a point in $\mathcal{M}$ [5]. So we can use as our sequences of points Cauchy sequences or sequences of points along geodesics.
The Riemannian case is a useful example, but as soon as we move to Lorentzian geometry, which is the correct geometric setting for GR, the previous discussion cannot be used as stated. The reason is that Lorentzian metrics do not define a distance function and therefore Cauchy sequences cannot be defined. Thus, one is restricted to the notion of geodesically complete manifolds in the Lorentzian setting.

Moreover, the existence of three kinds of vectors available in any Lorentzian metric define three nonequivalent notions of geodesic completeness depending on the character of the tangent vector of the curve: spacelike completeness, null completeness and timelike completeness. Unfortunately, they are not equivalent. It is possible to construct spacetimes with the following characteristics [6, 7, 8]:

- timelike complete, spacelike and null incomplete
- spacelike complete, timelike and null incomplete
- null complete, timelike and spacelike incomplete
- timelike and null complete, spacelike incomplete
- spacelike and null complete, timelike incomplete
- timelike and spacelike complete, null incomplete

Furthermore, there are examples of a geodesically null, timelike and spacelike complete spacetimes with an inextendible timelike curve of finite length [7, 8]. A particle following this trajectory will experience bounded acceleration and in a finite amount of proper time its spacetime location would stop being represented as a point in the manifold. This example shows that using points on geodesics to test the geometry cannot be enough to characterise missing points in a manifold. Much less, to probe if there are divergent scalar invariants as one approaches these missing points.
This leaves the question of how we can characterise a singularity in GR? The purpose of this thesis is an inquiry into the nature of gravitational singularities. Our inquiry consists of a systematic description of spacetime which is examined with three different probes: point-particles, classical fields and quantum fields. The general idea is that a spacetime has a singularity if there are obstructions to the dynamics of the probe.
We have divided the thesis into three main parts. In each part, we give a physical motivation of the choice of the probe, a geometric description of the spacetimes we are considering and applications. Our endeavour is descriptive rather than explanatory. We will not explain the mechanism that triggers the formation of the singularity, instead, we will show different mathematical frameworks that allow us to talk about gravitational singularities in different scenarios.
The first part of the thesis initiates our examination of spacetime by probing spacetime with matter that can be modelled as a point-particle. This requires us to introduce the b-boundary construction. This mathematical technique allows us first to detect if a spacetime is incomplete (for all kinds of curves, not only geodesics) and in that case it provides a boundary, $\partial\mathcal{M}$, to the incomplete spacetime, $\mathcal{M}$. Moreover, the construction allows us characterise some information about the nature of the singularities. We give a classification of the possible behaviour of the singularities that might be obtained by studying the b-boundary. In particular, we discuss the case of two-dimensional Lorentzian metrics. We give concrete applications of the framework in the case of the Minkowski spacetime (which is regular) and the Friedmann-Robertson-Walker spacetime (which is geodesically incomplete).
In the second part of the thesis, we probe the geometry of spacetime with classical scalar fields. The general motivation is to redefine a singularity in spacetime not as an obstruction to geodesics or curves but as an obstruction to the dynamics of test fields.

This means that a singularity is present in a region of a spacetime if the initial value problem is not well-posed. A well-posed problem has three ingredients:

- There exists a solution.
- The solution is unique.
- The solution depends continuously on the initial data.

A spacetime $\mathcal{M}$ that has well-posed field equations can be considered as a spacetime where the geometry is behaved well enough so that the fields remain deterministic. Moreover, we require that the solution is regular enough so that the energy-momentum tensor of the solution is finite as an integrable function.

The second part of the thesis is divided into three chapters in which we focus on different spacetimes. In the first chapter, we present a general argument by Clarke [9] which allows one to define the dynamics of the scalar field in spacetimes with square integrable Christoffel symbols and curve integrable curvature. In the second chapter, we prove well-posedness in spacetimes with low regularity that include spacetimes with surface layers, impulsive gravitational waves and brane-world scenarios. In the third chapter, we consider spacetimes that contain string-like singularities, modelling cosmic strings.

Although, we mention in the introduction of the first part how certain regularity of the metric is necessary for the singularity theorems, we want to point out that the regularity of the metric takes a central role in the second part of the thesis. The concept of regularity can be intuitively understood as the number of smooth or non-singular derivatives. We call spacetimes with finite regularity rough spacetimes. An example of how this concept is key follows from the following example. Let us consider a spacetime where the metric is C^∞ except at a point where the regularity is only C^0. Then, the geodesic equation is not well posed there and also the curvature (which depends on the second derivative of the metric) may blow-up. Therefore, a description of a singularity based on curvature blow up or the notion of geodesics would require us to remove this point of spacetime. Nevertheless, one can still have well-posedness of test field in the spacetime which includes the point with low regularity. We show this sort of situations is the case in different scenarios.

The third part of the thesis presents the outline of a framework to analyse the geometry of the spacetime by probing it with quantum scalar fields. The main focus of this chapter is to describe what is meant by a quantisation in a rough curved spacetime. We describe the algebraic quantisation procedure and also the more common Hilbert space representation. It is remarkable, that even in the smooth case, a full geometric understanding of the physical Hilbert space representation on a generally globally hyperbolic space is still missing. Therefore, we also describe the different ways classical dynamics can be implemented in the quantum case. At the end, we explicitly quantise a scalar field in a rough background and implement the dynamics as unitary operators.

The fourth part of the thesis present future outlooks and some open problems.

Part I

The point-particle probe

Chapter 2

Introduction

In this first part of the thesis, we develop a framework that will probe spacetime with point-particle objects. This approach characterises gravitational singularities as world lines which are incomplete. Moreover, this is the characterisation used in the classical singularity theorems. Different descriptions of gravitational singularities will be given in the second and third part of the thesis where we probe spacetime with classical and quantum test-fields.

2.1 Singularity Theorems

The first modern definition of a gravitational singularity comes from Penrose and Hawking in their seminal theorems. They show that General Relativity breaks down under certain conditions (see [3], Chapter 8). The general structure of the theorems establishes that if a spacetime $(\mathcal{M}, g_{ab})$ with enough differentiability satisfies:

- an energy condition,
- a global causal condition,
- and an appropriate initial or boundary condition

then $(\mathcal{M}, g_{ab})$ must be geodesically incomplete [10].
The differentiability of the spacetime is a subtle point. Originally, a C^2 condition on the metric condition was needed, so classical theorems from calculus of variations hold [3]. However, recent developments in differential geometry in low regularity and in analysis have managed to prove the singularity theorems with just a $C^{1,1}$ [1]. condition on the metric [11, 12]. This seems to be the natural regularity of the theorems as this is the threshold for existence and uniqueness of geodesics. However, from the point of view of spacetimes modelling collapsing stars scenarios [15], spacetimes with cosmic strings [16], brane world cosmologies [17] (see examples in later chapters) and considering Einstein's equations as an initial value system [18], one would like to work with metrics with less regularity. These regularity conditions will be developed and analysed in more detail in part two of this thesis.
We describe now each of these conditions. The energy conditions are general inequalities that relate the energy momentum tensor of matter, T_{ab}, or equivalently (via the

[1]A function f on an open set $\mathcal{U}$ of $\mathbb{R}^n$ is said to be Lipschitz or $C^{0,1}$ if there is some constant K such that for each pair of points $p, q \in \mathcal{U}$, $|f(p) - f(q)| \leqslant K|p - q|$, where $|p|$ denotes the usual Euclidean distance. We denote by $C^{k,1}$ those functions where the kth derivative is a Lipschitz function.

Einstein's equations) the Ricci tensor, R_{ab}, with a certain class of vector fields. For example, the weak energy condition requires that $T_{ab}u^a u^b \geqslant 0$ for any timelike vector u^a (by continuity this will then also be true for any null vector v^a), this condition ensures that all observers see nonnegative energy density. The dominant energy condition requires that in any orthonormal basis the energy density dominates all the other terms, $T^{00} \geqslant |T^{\alpha\beta}|$, which ensures that all observers see a causal flux of energy-momentum. The strong energy condition states that $T_{ab}u^a u^b \geqslant T^c_c$ for any unit timelike vector u^a. All the energy conditions guarantee a focusing effect, which encodes the attractive nature of gravity and is important for the formation of conjugate points.

The global causal conditions come in various different forms. The idea of a chronological spacetime is that there are no closed timelike curves. A strongly causal spacetime satisfies the condition that for every point $p \in \mathcal{M}$ there is a neighbourhood $\mathcal{V}$ of p which no non-spacelike curve intersects more than once. Finally, one can require that there is an spacelike surface S which is intersected by every causal curve, exactly once. This surface then is called a Cauchy Surface (see Appendix for details). The importance of this condition has two aspects: first, to prevent the possibility of time travel to the past and second to ensure the existence of geodesics of maximal proper length between events. In particular, such geodesics will not contain conjugate points.

A boundary or initial condition is the missing link to relate the focusing effect of the energy conditions and the existence of maximal geodesics to geodesic incompleteness which is the conclusion of the theorems. One example of such a condition is the existence of a closed trapped surface, $\mathcal{T}$. By this is meant a C^2 closed spacelike 2-surface such that the two families of null geodesics orthogonal to $\mathcal{T}$ are converging. This is the formal description of the intuitive idea that the gravitational field is becoming so strong in some region that light rays (and so all the other forms of matter as well) are trapped inside a succession of 2-surfaces of smaller and smaller area. Another example of a initial condition is the requirement that there is a spacelike surfaces S where all the unit normals are everywhere diverging either to the past or the future.

The notion of geodesic incompleteness was discussed briefly in the introduction and can be better understood by defining what we mean by geodesic completeness. A geodesically complete spacetime is one where any geodesic admits an extension to arbitrarily large parameter values. Then, a spacetime that is not geodesically complete, must be geodesically incomplete. Geodesic incompleteness describes intuitively that there is an obstruction to free falling observers to continue travelling through spacetime. In some sense, they have reached the edge of spacetime in a finite amount of time; they have encountered a singularity.

The conclusion of the singularity theorems, despite the generic nature of the hypothesis, is not as precise as one would wish. That is because the size, place and shape of the singularities can not be straightforwardly characterised by any physical measurement. Sometimes, one cannot even know whether it is in the future or past. All one knows is that there exists at least one incomplete causal geodesic.

The structure of the remainder of the thesis is as follows. Below in Section 2.2 we give an outline of the proof of a particular version of the singularity theorems. In Section 3.1 we give some general descriptions about boundary techniques in GR. In Section 3.2 we describe the b-boundary. Then in Section 3.3 we give the classification of singularities using the b-boundary. In Section 3.4, we discuss the b-boundary construction for $1+1$ spacetimes and provide two examples. The first one shows in detail the geometry of the Schmidt metric when one is dealing with a general $1+1$ metric written in

conformally flat form. We then calculate the geometry of the frame bundle in the case of $1+1$ dimensional Minkowski spacetime. To our knowledge, such a calculation has not been done before and might provide a better geometric understanding of the b-boundary construction. The second example is the b-boundary completion of a Friedmann-Robertson-Walker spacetime (FRW) where it is shown that the initial singularity collapses to a point. In Section 3.5 we discuss the topology of the spacetime when one includes the b-boundary.
We attach an appendix at the end of the thesis containing the general background in Differential Geometry and Causality theory required in this part.

2.2 Proof of the Singularity theorems.

Below we give a precise mathematical formulation of one of the singularity theorems and give the outline of the proof. This will explicitly show how all the hypotheses of the theorem are used and how the geodesic incompleteness conclusion is obtained. Moreover, it is useful to see how the regularity of the metric is needed. There are other more complex versions of the theorems for which formulations and proofs can be found in [3, 13] and low regularity versions can be found in [11, 12]. The formal statements of the results in calculus of variations and causality theory used for this proof are given in the Appendix.

Theorem 1 *[13] A spacetime (M, g_{ab}) that satisfies the following conditions:*

- $R_{ab}K^aK^b \geqslant 0$ *for all null vectors* K^a*,*
- *there is a compact Cauchy surface S in $\mathcal{M}$,*
- *all the unit normals to S are diverging, i.e, the congruence of geodesics meeting S orthogonally have $\theta > 0$ at every point of S.*

cannot be geodesically complete.

2.2.1 Outline of the proof.

The proof consists of the following steps:

1. We start assuming that the spacetime (M, g_{ab}) is geodesically complete.
2. The assumption of geodesic completeness together with the energy condition $R_{ab}K^aK^b \geqslant 0$ guarantees that every geodesic γ is focusing.
3. The initial condition, that the congruence of geodesics meeting S orthogonally have $\theta > 0$ at every point of S, guarantees that the focusing effect will produce conjugate points. Therefore every past geodesic γ, that extends arbitrarily into the past, contains a futuremost point q conjugate to S. This is Theorem 15 in the Appendix.
4. Since S is compact and such conjugate points move continuously provided the metric is at least C^2 there is an upper bound B to the distance, along geodesics from q to S. This is a step in the proof where the regularity of the metric is critical. The condition that the conjugate points move in a C^0 form depends on the coefficients of the Raychaudhuri equation, which includes the curvature tensor and therefore it must be C^0 [14].

5. However, if γ extends arbitrarily into the past, there is a point w on γ with distance to S bigger than B. Thus, q lies between w and S.

6. Using standard results from the calculus of variations the geodesics from w to S can not be maximal since they have conjugate points. This is the content of Theorem 16 in the Appendix and the main reason why we required the energy condition and the boundary/initial condition as hypotheses.

7. However since S is a Cauchy surface the point w belongs to the past domain of dependence, $D^{-}(S)$, which is globally hyperbolic.

8. However, the existence of the Cauchy surface, S, which is equivalent to global hyperbolicity, guarantees the existence of a maximal geodesic from w to S. This is Theorem 18 in the Appendix.

9. However, such geodesic must pass through a conjugate point and therefore cannot be not maximal which leads to a contradiction with the initial assumption. ⊡

Chapter 3

The b-boundary

3.1 The boundary of spacetime

The procedure to attach a boundary to a Lorentzian manifold can be done in several nonequivalent ways. In this chapter we will focus on the b-boundary. The reason for this is that it allows a classification of singularities in terms of parallel propagated frames, distinguishes between points at infinity and points at a finite length, and generalises the idea of affine length to all curves, being geodesic or not. Other common techniques to attach boundaries to Lorentzian manifolds are conformal boundaries [3, 19], causal boundaries [3] and approaches that mix several techniques like the a-boundary [20].
We give now below a description of such techniques.
The conformal boundary allows us to study the structure at "infinity" of the metric. The idea of conformal compactification is to bring points at "infinity" on a non-compact pseudo-Riemannian manifold $(\mathcal{M}, g_{ab})$ to a finite distance (in a new metric) by a conformal rescaling of the metric $\tilde{g}_{ab} = \Omega^2 g_{ab}$. The precise definition only applies to an asymptotically simple spacetime. $\mathcal{M}$ is asymptotically simple, if there is another smooth Lorentz manifold $(\tilde{\mathcal{M}}, \tilde{g}_{ab})$ such that:

- $\mathcal{M}$ is an open sub-manifold of $\tilde{\mathcal{M}}$ with smooth boundary $\partial\mathcal{M}$ called the conformal boundary;

- there exists a smooth scalar field Ω on $\tilde{\mathcal{M}}$ such that $\tilde{g}_{ab} = \Omega^2 g_{ab}$ on $\mathcal{M}$, and so $\Omega = 0, d\Omega \neq 0$ on $\partial\mathcal{M}$;

- every null geodesic in $\mathcal{M}$ acquires a future and a past endpoint on $\partial\mathcal{M}$.

This technique has the evident drawback that it can only be applied for this kind of spacetimes [19]. Moreover, notice that for example in the Minkowski spacetime, $\partial M = \mathscr{I}^- \cup \mathscr{I}^+$ (which corresponds to null past and future infinity) while i^o, i^+, i^- which correspond to spacelike infinity and future and past timelike infinity do not belong to the conformal boundary (see [3] for a precise definition of $\mathscr{I}^-, \mathscr{I}^+, i^o, i^+, i^-$). The reason for this is because ∂M is not a smooth manifold at these points. Despite this, the conformal boundary has been successfully applied to study isolated systems in General Relativity [19] and more recently to the AdS-CFT correspondence [21].
Giving a causal boundary to a Lorentzian manifold is a technique to attach a boundary to spacetime that depends only on the causal structure. However, this implies the

construction is not able to distinguish between boundary points and points at infinity. Moreover one has to assume that (M, g_{ab}) is strongly causal.
The construction relies on indecomposable past sets (IP) and indecomposable future sets (IF) which we now define. A set U that is open, satisfies $I^-(U) \subset U$ and cannot be expressed as the union of two proper subsets V, W which are open and satisfy $I^-(V) \subset V$, $I^-(W) \subset W$ is an IP. One can define IF, similarly using I^+. The class of IP's can be divided into two classes: proper IP's (PIPs) which are of the form $I^-(p)$ for $p \in M$, and terminal IP's (TIPs) which are not formed by the past of any point in $\mathcal{M}$. We shall denote by $\overbrace{M}$ the set of all IPs of the space $(\mathcal{M}, g_{ab})$ and $\underbrace{M}$ the set of all IFs of the space (M, g_{ab}). Then defining a suitable topology on $\overbrace{M}, \underbrace{M}$ to identify IFs and IPs (see [3]) one can form a space $M^* = M \bigcup \Delta$ where Δ is called the causal boundary.
The b-boundary is a method developed by Schmidt that allows one to attach a boundary $\partial\mathcal{M}$ called the *b-boundary* to any incomplete spacetime $\mathcal{M}$ (or indeed any manifold with a connection). The procedure consists in constructing a Riemannian metric on the frame bundle $L(\mathcal{M})$ or the orthonormal bundle $O(\mathcal{M})$ called the Schmidt metric. This metric is then used to generalise the idea of affine length to all curves. This generalisation is important because it helps to unify some elements of Riemannian geometry with Lorentzian geometry. For example, while geodesic completeness implies that every curve is complete only in the Riemannian case; the notion of b-completeness implies completeness of every curve in both signatures. The definition of a curve we are using here is a piecewise-C^1 curve (see the Appendix for a formal definition). Also, the b-boundary has given some results that link the geometry of principal bundles with that of the base manifold [22].
The relationship between the conformal boundary, the causal boundary and the b-boundary is not yet understood.

3.2 The Schmidt metric

The procedure for constructing the Schmidt metric consists of building a Riemannian metric on the frame bundle $L(\mathcal{M})$. We will use the soldering form θ on $L(\mathcal{M})$ and the connection form ϖ on $L(\mathcal{M})$ associated to the Levi-Civita connection ∇ on $\mathcal{M}$ to do this. Explicitly, the Schmidt metric is given by

$$\overline{g}(X, Y) = \theta(X) \cdot \theta(Y) + \varpi(X) \bullet \varpi(Y) \tag{3.1}$$

where $X, Y \in T_{\overline{p}}P$ and $\cdot, \bullet$ are the inner products in $\mathbb{R}^n$ and $\mathfrak{g} \cong \mathbb{R}^{n^2}$. It can be shown that despite the freedom in the choice of the inner product, all the metrics constructed in this way are equivalent metrics [26].
Let $\gamma : [a, b] \to \mathcal{M}$ be a piecewise-C^1 curve through p in $\mathcal{M}$. A curve $\overline{\gamma} : [a, b] \to L(\mathcal{M})$ in $L(\mathcal{M})$ is called a *lift of the curve* γ if it satisfies $\pi(\overline{\gamma}) = \gamma$ and $D\pi(\dot{\overline{\gamma}}) = \dot{\gamma}$. The length of $\overline{\gamma}$ with respect to the Schmidt metric given by

$$L_{\overline{\gamma}}(b) = \int_a^b \|\dot{\overline{\gamma}}\|_{\overline{g}} d\tau$$

is called the *generalised affine length* of γ. We can then use this length to reparametrize γ. When γ is a geodesic, then if it is parametrized by $L(t)$, it is parametrized with respect to an affine parameter (see Appendix for a definition of affine parameter in the case of a geodesic curve). If every curve in a spacetime $\mathcal{M}$ that has finite affine generalised length has endpoints, we call the spacetime *b-complete*. If it is not b-complete we call the spacetime *b-incomplete*.

Notice that if there is a curve γ in $\mathcal{M}$ that has finite affine length and no endpoint then the lift curve $\overline{\gamma}$ can not have an endpoint. Otherwise, if $\overline{p}$ is the endpoint of $\overline{\gamma}$, $\pi(\overline{p}) = p$ would be an endpoint of γ contradicting the incompleteness of γ. This observation shows that geodesic incompleteness implies b-incompleteness. The converse is not true as Geroch's example [7] shows a b-incomplete spacetime that is geodesically complete. Therefore b-incompleteness is a generalisation of geodesic incompleteness.

Now given an incomplete spacetime $\mathcal{M}$, using the Riemannian metric $\overline{g}$ on $L(\mathcal{M})$, we can Cauchy complete $L(\mathcal{M})$. Let us denote by $\overline{L(\mathcal{M})}$ the Cauchy completion of $L(\mathcal{M})$. Then, we define the quotient space $\overline{\mathcal{M}} = \overline{L(\mathcal{M})}/G^+$, where G^+ is the connected component of the identity of $GL(n;\mathbb{R})$ under the equivalence of orbits, i.e., $(\overline{p}, g) \in \overline{L(\mathcal{M})} \sim (\overline{q}, g') \in \overline{L(\mathcal{M})}$ if $\overline{p} = \overline{q}$ and there is $h \in GL(n;\mathbb{R})$ such that $g = hg'$. This quotient induces a topology in $\overline{\mathcal{M}}$ by requiring that the map $\pi : \overline{L(\mathcal{M})} \to \overline{\mathcal{M}}$ is continuous and therefore $\overline{\mathcal{M}}$ is a topological space. However, it does not imply that $\overline{\mathcal{M}}$ is a manifold. Finally we can characterise the b-boundary as the set $\partial\mathcal{M} = \overline{\mathcal{M}} \backslash \mathcal{M}$.

We can repeat the same construction for subgroups of $GL(n;\mathbb{R})$. In particular, a common choice in the Lorentzian case is the the subgroup of all Lorentz transformations preserving both orientation and direction of time, called the proper orthochronous Lorentz group, and denoted by $SO^+(1,n)$. In a completely analogous way we can form the quotient $\overline{\mathcal{M}} = \overline{L(\mathcal{M})}/SO^+(1,n;\mathbb{R})$ and define the b-boundary as the set $\partial\mathcal{M} = \overline{\mathcal{M}} \backslash \mathcal{M}$. The advantage of this construction is that $SO^+(1,n;\mathbb{R})$ is a manifold of dimension $n + \frac{n(n-1)}{2}$ instead of the $n + n^2$ dimensions of $L(\mathcal{M})$. Also, the construction can be carried on a manifold with a Riemannian metric, in which case $\overline{\mathcal{M}}$ is homeomorphic to the Cauchy completion of $\mathcal{M}$ and a natural choice for the subgroup is the special orthogonal group $SO(n;\mathbb{R})$. This reinforces the conviction that the b-boundary is a natural way to attach boundaries to manifolds with connections.

3.3 The classification of singularities

The notion of b-incomplete spaces allows us to describe incomplete curves in manifolds with connections. Our initial motivation to study this, was to develop the language to describe pathologies in the geometry as we approach points that in some sense are "boundary points" of the manifold. In this section we describe how the main manifestation of gravity in General Relativity, the curvature of the manifold, can behave along b-incomplete curves. This is the scheme proposed by Ellis and Schmidt to classify singularities [23, 24].

Suppose that p is a point of the b-boundary $\partial\mathcal{M}$ of $(\mathcal{M}, g_{ab})$. Then:

1. p is a $C^r(r \geqslant 0)$ *regular boundary point* if there is a spacetime $(\mathcal{M}', g'_{ab})$ which contains $(\mathcal{M}, g_{ab})$ as a sub-manifold and such that the Riemann tensor of $(\mathcal{M}', g_{ab})$ exists and is C^r at p. We call $(\mathcal{M}', g'_{ab})$ an *extension* of $(\mathcal{M}, g_{ab})$.

2. p is a *singular boundary point* otherwise.

If p is a singular point then the following scenarios can occur:

- p is a *parallelly propagated (p.p.) curvature singularity* if, for some curve γ with endpoint p, at least one component of the Riemann tensor, with respect to a parallel propagated orthonormal basis along γ, is not continuous. If the r-covariant derivative of some component of the Riemann tensor is not continuous, we call the singularity a C^r *p.p.* curvature singularity.

- If p is a singular boundary point and is not a C^r *p.p.* curvature singularity, then it is a C^r *quasi-regular singularity.* A useful example to visualise such singularities is the singularity one encounters in the apex of a cone or spacetimes containing cosmic strings. In this cases the obstruction to extending the spacetime is a topological one.

- If p is a singular boundary point at the end of the curve γ in which some polynomial constructed from the tensors g_{ab}, R^a_{bcd} and r-covariant derivatives of R^a_{bcd} does not behave in a C^0 way, then p is a C^r *scalar singularity.* Those singularities are necessarily C^r curvature singularities.

- If p is a C^r curvature singularity, but not a scalar singularity, then it is a C^r *non scalar singularity.*

Furthermore, we can classify the curvature singularities in the following subclasses:

- p is a matter singularity if the Ricci tensor diverges at p.

- p is a conformal singularity if the Weyl tensor diverges at p.

- p is a divergent singularity if the curvature components are unbounded at p.

- p is a oscillatory singularity if the curvature components are bounded at p.

The above list gives a description of gravitational singularities in which the physically interesting cases can be described by the behaviour of the curvature along a curve ending at a point p in the b-boundary $\partial\mathcal{M}$.

3.4 The b-boundary for 1+1 spacetimes

The b-boundary is a mathematically elegant construction to attach a boundary to an incomplete spacetime. However, working directly with it is cumbersome because of the high dimensionality of the bundles involved. In addition, the link between the topology and geometry of the bundle is not easily related to that of the base manifold. In this section, we construct the Schmidt metric for general $1+1$ spacetimes locally and then give two explicit examples. The first example is an explicit calculation of the geometric properties of the Schmidt metric $(O(\mathcal{M}), \overline{g})$ in the case of 2-dimensional Minkowski spacetime. The second example follows closely the calculations of [25] and [26] to show that the initial singularity in the FRW case collapses to a point.

3.4.1 The Schmidt metric for 1+1 conformal spacetimes

Let $\mathcal{M}$ be a manifold of dimension two with a Lorentzian metric g_{ab} with orthonormal bundle $O(\mathcal{M})$. Then, we can find coordinates (v, w) which transform locally the line element of the metric g_{ab} to the following form [27]:

$$ds^2 = \Omega^2(v,w)(-dv^2 + dw^2). \tag{3.2}$$

An orthonormal basis is given by the vector fields

$$E_1 = \frac{1}{\Omega}\frac{\partial}{\partial v} \tag{3.3}$$
$$E_2 = \frac{1}{\Omega}\frac{\partial}{\partial w} \tag{3.4}$$

Any other orthonormal basis is of the form

$$\tilde{E}_1 = \cosh\chi\frac{1}{\Omega}\frac{\partial}{\partial v} + \sinh\chi\frac{1}{\Omega}\frac{\partial}{\partial w} \tag{3.5}$$
$$\tilde{E}_2 = \cosh\chi\frac{1}{\Omega}\frac{\partial}{\partial w} + \sinh\chi\frac{1}{\Omega}\frac{\partial}{\partial v} \tag{3.6}$$

for some $\chi(v,w) \in \mathbb{R}$.
We notice that the coefficients of such a basis with respect to $\frac{\partial}{\partial v}, \frac{\partial}{\partial w}$ define a unique matrix β and its inverse β^{-1}.

$$\beta = \frac{1}{\Omega}\begin{pmatrix} \cosh\chi & \sinh\chi \\ \sinh\chi & \cosh\chi \end{pmatrix}$$

$$\beta^{-1} = \Omega\begin{pmatrix} \cosh\chi & -\sinh\chi \\ -\sinh\chi & \cosh\chi \end{pmatrix}$$

Then we can use these matrices to define local coordinates on $O(\mathcal{M})$:

$$\{v, w, \frac{1}{\Omega}\left(\cosh\chi\frac{\partial}{\partial v} + \sinh\chi\frac{\partial}{\partial w}\right), \frac{1}{\Omega}\left(\cosh\chi\frac{\partial}{\partial w} + \sinh\chi\frac{\partial}{\partial v}\right) |(v,w) \in \mathcal{M}, \chi \in \mathbb{R}\} \tag{3.7}$$

The Schmidt metric $\overline{g}$ on $O(\mathcal{M})$ is given by:

$$\overline{g}(X,Y) : \varpi(X)\cdot\varpi(Y) + \theta(X)\cdot\theta(Y) \tag{3.8}$$

for $X, Y \in TO(\mathcal{M})$ where ϖ is the connection form on $O(\mathcal{M})$ and θ the soldering form. In coordinates the connection form ϖ is written as:

$$\varpi^a_b = (\beta^{-1})^a_c d\beta^c_b + (\beta^{-1})^a_c \Gamma^c_{ml}\beta^l_b dx^m \tag{3.9}$$

where $v = x^0, w = x^1$, β^a_c are the components of the matrix β and Γ^a_{bc} are the Christoffel symbols of the metric g_{ab}.
The soldering form θ is given by:

$$\theta^a = (\beta^a_c)^{-1} dx^c \tag{3.10}$$

Next, consider a curve $\gamma(s)$ in $O(\mathcal{M})$ given by $\gamma : s \in [a,b] \mapsto (v(s), w(s), \beta^a_c(s))$ and evaluate $\theta(\dot{\gamma})$ and $\tilde{\Omega}(\dot{\gamma})$. Explicitly we have:

$$\theta(\dot{\gamma}) = \Omega\begin{pmatrix} \dot{v}\cosh\chi - \dot{w}\sinh\chi \\ -\dot{v}\sinh\chi + \dot{w}\cosh\chi \end{pmatrix}$$

and

$$\varpi(\dot\gamma) = \begin{pmatrix} 0 & \dot\chi + \frac{1}{\Omega}((\partial_v\Omega)\dot w + (\partial_w\Omega)\dot v) \\ \dot\chi + \frac{1}{\Omega}((\partial_v\Omega)\dot w + (\partial_w\Omega)\dot v) & 0 \end{pmatrix}$$

giving the line element for the Schmidt metric:

$$\begin{aligned} ds^2 = \Omega^2(v,w)(\cosh(2\chi)(dv^2+dw^2) - \sinh(2\chi)dvdw) \\ + \left(d\chi + \frac{1}{\Omega}\left(\frac{\partial\Omega}{\partial v}dw + \frac{\partial\Omega}{\partial w}dv\right)\right)^2 \end{aligned} \tag{3.11}$$

Therefore, in principle one can use this line element and *X-tensor* or similar algebraic computing packages to give an expression for the curvature tensor $\overline{R}^a_{bcd}$ in $O(\mathcal{M})$ for a general $1+1$ Lorentzian manifolds $\mathcal{M}$ in terms of Ω. However, this is not very illuminating so instead we look at two simple examples below.

3.4.2 The Schmidt metric of Minkowski spacetime

We now develop here in detail the geometric properties of the Schmidt metric for the case of Minkowski spacetime which is the simplest model of a Lorentzian manifold. The spacetime can be characterised by the condition $R^a_{bcd} = 0$, i.e. Minkowski spacetime is the only flat Lorentzian manifold. This spacetime is also the spacetime of Special Relativity, which although relativistic does not include the curvature effects produced by matter or gravity. In the 1+1 case, the line element takes the form $ds^2 = -dt^2 + dx^2$ defined in $\mathbb{R}^2$. Using (3.11) we have that the Schmidt metric takes the form:

$$ds^2 = (dt^2+dx^2)\cosh(2\chi) - 2dtdx\sinh(2\chi) + d\chi^2 \tag{3.12}$$

Now if we consider the change of coordinates: $t = u + \tilde v, x = u - \tilde v$ we get:

$$ds^2 = 2(\cosh 2\chi + \sinh 2\chi)du^2 + 2(\cosh 2\chi - \sinh 2\chi)d\tilde v^2 + d\chi^2 \tag{3.13}$$

so can rewrite the line element as:

$$ds^2 = 2e^{2\chi}du^2 + 2e^{-2\chi}d\tilde v^2 + d\chi^2 \tag{3.14}$$

We are going to use now Cartan's method [28] to calculate the curvature.
So choosing as an orthonormal coframe:

$$\begin{aligned} l^0 &= \sqrt{2}e^{\chi}du \\ l^1 &= \sqrt{2}e^{-\chi}d\tilde v \\ l^2 &= d\chi \end{aligned}$$

then

$$\begin{aligned} dl^0 &= \sqrt{2}e^{\chi}d\chi\wedge du = -l^0\wedge l^2 \\ dl^1 &= -\sqrt{2}e^{-\chi}d\chi\wedge d\tilde v = l^1\wedge l^2 \\ dl^2 &= 0 \end{aligned} \tag{3.15}$$

Now we find the connection one form using equation (3) in the Appendix gives

$$
\begin{aligned}
dl^0 &= -l^0 \wedge l^2 \\
&= -\varpi^0{}_0 \wedge l^0 - \varpi^0{}_1 \wedge l^1 - \varpi^0{}_2 \wedge l^2 \\
dl^1 &= -l^1 \wedge l^2 \\
&= -\varpi^1{}_0 \wedge l^0 - \varpi^1{}_1 \wedge l^1 - \varpi^1{}_2 \wedge l^2 \\
dl^2 &= 0 \\
&= -\varpi^2{}_0 \wedge l^0 - \varpi^2{}_1 \wedge l^1 - \varpi^2{}_2 \wedge l^2
\end{aligned}
$$

Then we have:

$$
\begin{aligned}
\varpi^0{}_2 &= l^0 = -\varpi^2{}_0 \\
\varpi^1{}_2 &= -l^1 = -\varpi^2{}_1
\end{aligned}
\tag{3.16}
$$

where all the other components are zero.
Using Equation (3.16) and Equation (10.4) in the Appendix we obtain:

$$
\begin{aligned}
\Omega^0{}_2 &= -l^0 \wedge l^2 \\
\Omega^0{}_1 &= l^0 \wedge l^1 \\
\Omega^1{}_2 &= -l^1 \wedge l^2
\end{aligned}
$$

Because the geometry is three-dimensional the Weyl tensor vanishes [3] and therefore all the information of the curvature is in the Ricci tensor which is given by $\overline{R}_{ab} = \overline{R}^c_{acb}$. So we have using (10.5)

$$\overline{R}_{22} = -2 \tag{3.17}$$

where all the other components are zero.
and the Ricci scalar $\overline{R}$ given by $\overline{R}^a_a$ is

$$\overline{R} = -2. \tag{3.18}$$

Hence, the geometry in the bundle is not flat even if Minkowski spacetime is flat. This result illustrates that the geometry in $O(\mathcal{M})$ depends in a subtle way on the geometry of the fibre when the metric is Lorentzian. Notice that in the Riemannian case for the flat metric we have

$$\theta(\dot{\gamma}) = \Omega \begin{pmatrix} \dot{v}\cos\chi - \dot{w}\sin\chi \\ \dot{v}\sin\chi + \dot{w}\cos\chi \end{pmatrix}$$

and then the line element takes the form

$$ds^2 = dv^2 + dw^2 + d\chi^2 \tag{3.19}$$

which is just the flat metric in $O(\mathcal{M})$.

3.4.3 The Schmidt metric of a FRW spacetime.

The FRW spacetime corresponds to the model of a non stationary, spatially homogeneous and isotropic universe. The observation of the cosmic microwave background [29] and the red shift of type Ia supernovas [30] correspond to the strongest evidence to sustain a model of a non-stationary, homogeneous and isotropic universe.
For a universe with topology $\mathbb{R} \times S^3$ the FRW metric is given by

$$ds^2 = -dt^2 + a^2(t)\left(d\sigma^2 + \sin^2\sigma\left(d\vartheta^2 + \sin^2\vartheta d\varrho^2\right)\right) \tag{3.20}$$

where $\sigma, \vartheta, \varrho$ are polar coordinates, $t \in \mathbb{R}^+$ is the temporal coordinate and $a(t)$ is called the scale factor.
The physical requirements of positive matter density, nonnegative pressure and the observed recession of galaxies leads to the behaviour $a(t) \to 0$ as $t \to 0$. For a radiation-dominated universe the evolution of the scale factor is $a(t) \propto t^{1/2}$. For a matter dominated universe the evolution of the scale factor is $a(t) \propto t^{2/3}$. For a dark-energy-dominated universe, the evolution of the scale factor is $a(t) \propto \exp(Ht)$. Here, the coefficient H in the exponential, the Hubble constant, is $H = \sqrt{\Lambda/3}$ where Λ is the cosmological constant.
In the early universe, after inflation the universe was radiation dominated. So let us consider that the scale factor takes the form $a(t) = t^\rho(\rho > 0)$ as $t \to 0$. By rescaling the time coordinate we can put the metric in the form

$$ds^2 = -a^2(\tau)d\tau^2 + a^2(\tau)\left(d\sigma^2 + \sin^2\sigma\left(d\vartheta^2 + \sin^2\vartheta d\varrho^2\right)\right) \tag{3.21}$$

with $a^2(\tau) = \tau^q(q > 0)$.
For simplicity, we are going to consider the case of the $1+1$ FRW cosmological model which we can obtain from the 4-dimensional one by suppressing two polar coordinates. This can be seen by considering the injection map into

$$h : (\tau, \sigma) \to (\tau, \sigma, \vartheta_0, \varrho_0) : \mathcal{M} \to \mathcal{N} = \mathcal{M} \times \Sigma \tag{3.22}$$

where Σ is a suitable two dimensional manifold. Then, from (3.21) we obtain that the $1+1$ FRW spacetime takes the form

$$ds^2 = \tau^q(-d\tau^2 + d\sigma^2) \tag{3.23}$$

for $q > 0$. Moreover, from (3.11) we obtain that the Schmidt metric in $O(\mathcal{M})$ takes the form

$$ds^2 = \tau^q\left(\cosh(2\chi)(d\tau^2 + d\sigma^2) - \sinh(2\chi)d\tau d\sigma\right) + \left(d\chi + \frac{q}{\tau}d\sigma\right)^2 \tag{3.24}$$

This injection gives a natural smooth injection $h : O(\mathcal{M}) \to O(\mathcal{N})$. Moreover, we have the following Lemma [26]:

Lemma 1 *For all $Y \in TO(\mathcal{M})$. $||Y|| = ||Dh(Y)||$*
where the norms are induced by the Schmidt metric for $O(\mathcal{M})$ and $O(\mathcal{N})$.

This result implies that Cauchy sequences in $O(\mathcal{M})$ correspond to Cauchy sequences in $O(\mathcal{N})$ under the injection map, which then can be used to show that degeneracy of fibres in the $1+1$ case implies degeneracy in the 4 dimensional one.
Now we show below the procedure to treat the cosmological singularity at $\tau = 0$.
First we give a non convergent Cauchy sequence $\{s_n\}_{n=1}^{\infty}$ in $O(\mathcal{M})$ to determine a point in the completion of $O(\mathcal{M})$. As a consequence of the previous lemma, $\{h(s_n)\}_{n=0}^{\infty}$ will be a non convergent Cauchy sequence and therefore will also determine a point in the completion of $O\mathcal{N}$.
Fix σ_o and consider the sequence

$$s : \mathbb{N} \to O\mathcal{M} : n \to (\tau_n, \sigma_o, 0) \tag{3.25}$$

for some $\tau_n \in (0, T)$ with $\tau_n \to 0$ as $n \to \infty$. Then for all n greater that some $N_0 \in \mathbb{N}$ we can find $\{\tilde{t}_n\}_{n=1}^{\infty}$ such that $\tilde{t}_n = 1 - \tau_n$ and $\tilde{t}_n \to 1$. This sequence eventually lies on the curve

$$c : [0, 1) \to O(\mathcal{M}) : \tilde{t} \to (1 - \tilde{t}, \sigma_o, 0) \tag{3.26}$$

Then from (3.24) we obtain

$$||\dot{c}||_{\overline{g}} = (1 - \tilde{t})^{\frac{q}{2}} \tag{3.27}$$

Therefore, for large enough k, n

$$||s(n) - s(k)||_{\overline{g}} \leqslant \left| \int_{\tilde{t}_k}^{\tilde{t}_n} ||\dot{c}||_{\overline{g}} d\tilde{t} \right| \sim \left| \int_{\tau_k}^{\tau_n} \tau^{\frac{q}{2}} d\tau \right| \tag{3.28}$$

Since $\tau^{\frac{q}{2}} \to 0$ as $n \to \infty$ for all $\epsilon > 0$ we can find $N_\epsilon \in \mathbb{N}$ such that for $k, n \geqslant N_\epsilon$

$$||s(k) - s(n)||_{\overline{g}} \leqslant \epsilon \tag{3.29}$$

Thus $\{s\}_{n=1}^{\infty}$ is a Cauchy sequence and the equivalence class determines a point $\overline{p}$ in $\overline{O(\mathcal{M})}$. This of course projects to a point $p \in \partial\mathcal{M}$ which we can identify as the singularity.
Notice similarly that the sequence:

$$s^1 : \mathbb{N} \to O(\mathcal{M}) : n \to (\tau_n, w_o + \delta_n, \chi_1) \tag{3.30}$$

with $\tau_n \to 0, \delta_n \to 0$ as $n \to \infty$ is also a Cauchy sequence that determines a point $\overline{q} \in O(\mathcal{M})$ and satisfies $\pi(\overline{q}) = \pi(\overline{p}) = p$.
We now show that the fibre in $\overline{O(\mathcal{M})}$ over $\tau = 0$ degenerates to a point and therefore $\overline{O(\mathcal{M})}$ is not a fibre bundle.
The procedure to do this is to show that the distance between $\overline{p}$ and $\overline{q}$ can be made arbitrarily small. Let $a_n = a(\tau_n)$, $\dot{a}_n = \frac{d}{d\tau}a(\tau)|_{\tau_n}$ and $\delta_n = -\chi_1 \frac{a_n}{\dot{a}_n}$.
Then the curve

$$\gamma_n : [0, \delta_n] \to \overline{O(\mathcal{M})} : l \to (\tau_n, \sigma_0 + l, -\frac{\dot{a}_n}{a_n} l)$$

satisfies $\gamma_n(0) = \overline{p}$ and $\gamma_n(\delta_n) = \overline{q}$.
Now notice that the length of the curve is given by

$$L_{\gamma_n} = \left| a_n \int_0^{\delta_n} \cosh\left(-2\frac{\dot{a}_n}{a_n} l\right)^{\frac{1}{2}} dl \right| = \left| \frac{a_n^2}{\dot{a}_n} \int_0^{\chi_1} \cosh(2\chi)^{\frac{1}{2}} dl \right| \tag{3.31}$$

Now we have the following classical inequalities:

$$0 < \cosh\chi < (\cosh\chi^2 + \sinh\chi^2)^{\frac{1}{2}} = (\cosh 2\chi)^{\frac{1}{2}} \leqslant 2^{\frac{1}{2}}\cosh\chi \tag{3.32}$$

Since $\tau_n \to 0$ as $n \to \infty$ we obtain:

$$L_{\gamma_n} \leqslant \left|\frac{a_n^2}{\dot{a_n}} 2^{\frac{1}{2}}\sinh\chi\right| = \left|2^{\frac{1}{2}}\sinh\chi\tau_n^{\frac{q}{2}+1}\right| \to 0 \tag{3.33}$$

Therefore L_{γ_n} goes to zero as $n \to \infty$. Hence, $\overline{p} = \overline{q}$.
Since σ_0, χ_1 were arbitrary, we can conclude that any fibre over the singularity at $\tau = 0$ is degenerate and collapses to a point.

3.5 The topology of $\overline{\mathcal{M}}$

We have shown in Section 3.4 that if one calculates the Schmidt metric of the two-dimensional flat Riemannian space one obtains that the Schmidt metric is simply the flat metric in the orthonormal bundle. This example manifestly shows that the geometric relationship between the base manifold and the orthonormal bundle is more subtle when the metric is Lorentzian. Moreover, in the Riemannian case, $\overline{\mathcal{M}}$ is always equal to the Cauchy completion of $\mathcal{M}$ [26]. This shows important differences between the signatures.
In our exposition about the b-boundary, we obtained equation (3.11) which is the line element of the Schmidt metric for all $1+1$ Lorentzian manifolds $\mathcal{M}$ which determines, via the curvature, all the local isometric invariants. If $\partial M = \emptyset$, then the 3-manifold corresponds to the orthonormal bundle where the fibres of the bundle are $SO^+(1,1) \cong \mathbb{R}$. Therefore, $SO^+(M)$ is not compact. If $\partial M \neq \emptyset$, then the 3-manifold is not necessarily a G-bundle (the group may not act freely or transitively). This is, for example, the case when $\mathcal{M}$ is the Friedmann-Robertson-Walker spacetime. This affects the topology of $\overline{\mathcal{M}}$, which in the Friedmann-Robertson-Walker case is no longer Hausdorff [26]. Then $\overline{O(\mathcal{M})}$ is not a G-bundle as the fibre over the singularity is a point instead of a copy of $SO(1,1)^+(M)$. Nevertheless, $\overline{O(\mathcal{M})}$ is still a 3-dimensional Riemannian non-compact manifold. In general, the topology of $\overline{\mathcal{M}}$ is poorly understood. It is known that in four dimensions the Schwarzschild and the Friedmann-Robertson-Walker b-completions result in non-Hausdorff spaces. But it is still a conjecture that in the first case ∂M is a line while in the second ∂M is a point [26, 25]. Therefore, in order to understand the topology of $\overline{\mathcal{M}}$, one can study first the geometry of the Riemannian manifold $\overline{O(\mathcal{M})}$. In the particular case of the 3-manifold $\overline{O(\mathcal{M})}$ in the Riemannian case, one may use the Geometrization conjecture which establishes that all 3-dimensional closed manifolds are the connected sum of prime 3-manifolds. And each of these prime 3-manifolds can be cut along tori, so that they correspond to one of the eight Thurston geometric structures [31]. This, of course, is not suitable in the Lorentzian case because as mentioned above $\overline{O(\mathcal{M})}$ is not compact. Nevertheless, whenever the Schmidt metric has finite volume the geometric structure can be almost obtained through the fundamental group $\pi_1(\overline{O(\mathcal{M})})$ (see [32] for a precise definition of π_1 and [33] for the relationship between π_1 and the finite volume condition). For higher dimensions, the dimension of $\overline{O(\mathcal{M})}$ is now $n + \frac{n(n-1)}{2} > 5$ and no classification is possible [34].

Part II

The classical test-field probe

Chapter 4

Introduction

In the first part of this thesis we have given a framework to probe spacetime with point-particle objects, and in the third part part we will develop tools to probe spacetime with quantum test-fields. In this part, we enquire how gravitational singularities can be characterised, as the obstruction to well-posedness, when spacetime is probed using a classical test field.

4.1 Test-field Singularities

An important requirement of any classical physical theory is that given suitable initial data one can determine the evolution of the system. Within the theory of general relativity the condition of global hyperbolicity [35] therefore plays a key role, because this condition is a sufficient, but not necessary, condition for the well-posedness of hyperbolic equations for several physically important fields several tensor fields (see e.g.[3] for details). Mathematically a spacetime region $\mathcal{N}$ is said to be globally hyperbolic if the causality condition is satisfied, and for any two points $p, q \in \mathcal{N}$ the causal diamond $J^+(p) \cap J^-(q)$ is compact and contained in $\mathcal{N}$ [36].
One way of interpreting the compactness of $J^+(p) \cap J^-(q)$ is that this set "does not contain any points on the edge of spacetime, i.e. at infinity or at a singularity" [3, §6.6]. In this context one has to choose a definition of singularity and therefore choose the regularity for the metric in the region $\mathcal{N}$. If one considers the existence and uniqueness of geodesics, then a sufficient condition to ensure this is that the metric is, $C^{1,1}$. Moreover, this choice of regularity is the threshold in which one can characterise a gravitational singularity in terms of geodesic incompleteness. This seems to be the natural regularity for the singularity theorems [12].
However, there are other criteria for regularity which may seem equally reasonable on physical grounds. For example, the $C^{0,1}$ regularity can be seen as a threshold for some basic aspects of causal structure. Metrics with lower regularity, $C^{0,\alpha} \quad \alpha \in (0,1)$, exist in which lightcones are no longer manifolds of codimension one. In addition, all the results of causal theory for smooth metrics that do not require the use of normal neighbourhoods can be transferred to the $C^{0,1}$ case such as the existence of domains of dependence that admit Cauchy time functions [37].
Another important criterion of regularity can be formed on the basis of considering Einstein's field equations as a hyperbolic evolution system. It has been shown that local well-posedness follows from having enough control over the L^2 norm of the curvature on the spatial foliation and the radius of injectivity [38]. From this point of view, the

relevant condition to characterise a singularity is a L^2 type condition on the metric and its derivatives.

Closely related, one can choose a criterion of regularity using a precise formalisation of the "Strong Cosmic Censorship Conjecture" (SCCC). This conjecture can be loosely stated as follows: given generic initial data the maximal global Cauchy development is regular. The precise mathematical meaning of "generic" and "regular" is part of the content of the Strong Cosmic Censorship Conjecture [39]. One example is the characteristic initial value problem for the spherically symmetric Einstein-Maxwell-scalar field equations: If one requires that a regular extension must have C^0 curvature then the SCCC holds, while a criterion for regularity that only requires a C^0 extension of the metric does not support the SCCC [40]. A common choice of the required regularity of the spacetime is that the solution should be a weak solution to Einstein's equations, e.g. the metric is C^0 with Christoffel symbols in L^2 [39]. In particular, Luk [41] has developed a series of spacetimes with null singularities where the spacetime cannot be extended as a weak solution. He conjectured that this kind of weak null singularity will generically develop in gravitational collapse and form in the interior of black holes.

In this thesis, we would like to explore a further option. We will consider a singularity as an obstruction to the evolution of a scalar test-field. This point of view was called generalised hyperbolicity by Clarke [9]. It involves regarding certain traditional singularities as interior points in a spacetime with low regularity and then proving local and global well-posedness of the wave equation in the rough extension.

Clarke's definition of $\Box_g$*-globally hyperbolicity* was motivated by two things. Firstly, there are a number of spacetimes in which points are removed due to the presence of weak singularities and that are therefore not globally hyperbolic but still of physical interest. These include spacetimes with thin shells of matter [42], impulsive gravitational waves [44] and shell-crossing singularities [45]. The second motivating factor was that of using test fields (given by solutions of the wave equation) rather than test particles (given by solutions of the geodesic equation) to probe the structure of singularities. Some work to examine the physical effect of gravitational singularities, has been done by using a 3-parameter family of test particles (see for example [46]). However, an advantage of using test fields is that the behaviour of the naturally defined energy-momentum tensor of the field gives a direct measurement of the physical effect of the singularity which is not easily obtained when considering families of test particles.

There have been previous approaches studying the nature of the singularities using test fields, which involved considering self-adjoint extensions of the (spatial) Laplace-Beltrami operator and applying boundary conditions (see e.g. [47, 48, 49, 50]). All this work has focused on the case of static spacetimes and additionally most of these self-adjoint extensions are in L^2 (except [50] who consider finite energy solutions). However, if one wishes to consider the energy-momentum tensor of the test field then one is required to consider solutions in other function spaces. A natural condition in this context is to require that the solutions lie in the Sobolev space H^1_{loc}, which ensures that the energy-momentum tensor is well-defined as a distribution.

Earlier work by Wald [47] gave a prescription to define dynamics in static, non-globally hyperbolic spacetimes. Based on similar techniques, Kay and Studer [49] determined the boundary conditions for quantum scalar fields on singular spacetimes with conical singularities representing cosmic strings. Subsequently, Horowitz and Marolf [48] used

Wald's approach to study the theory of quantum free particles in static spacetimes with timelike singularities. They used the term *quantum regular* if the evolution of any state was uniquely defined for all times. The main technique employed was to notice that if the spatial Laplace-Beltrami operator is essentially self-adjoint in $L^2(\Sigma', \nu_h)$ (where Σ' is a three dimensional geodesically incomplete manifold and ν_h the volume form of the induced metric on Σ'), then using standard properties of self-adjoint operators, a unique evolution of the wave function is obtained. Moreover, the classical singularities disappear in the sense that there is no freedom in the boundary conditions to define the state evolution. Later work by Ishibashi and Hasoya [50] used similar techniques to investigate the evolution of the wave equation $\Box_g \phi = 0$ in static singular spacetimes by focusing on changing the function space from $L^2(\Sigma', \nu_h)$ to $H^1(\Sigma', \nu_h)$. The main reason for this is that finiteness of the energy states implies the finiteness of the H^1 norm.

Finally, Ishibashi and Hasoya used the term *wave regular* if the initial value of the wave equation has unique solutions in the whole spacetime with no arbitrariness in the choice of boundary conditions. Vickers and Wilson [58] also studied the problem of conical singularities from Clarke's perspective and Wilson [56] showed that one could obtain dynamic evolution subject to constraints on the initial data and a flux condition on the singularity.

The link between the concepts of $\Box$-globally hyperbolicity, quantum regularity and wave regularity is to redefine a singularity in spacetime not as an obstruction to geodesics or curves but as an obstruction to the dynamics of test fields. Nevertheless, each concept has it own characteristics. While quantum regularity probes spacetime with a quantum free particle, the notion of $\Box$-globally hyperbolic and wave regular uses the classical wave equation. Furthermore, quantum regularity and wave regularity look at singularities in terms of boundary conditions, whereas a $\Box$-globally hyperbolic approach identifies the singularity as an interior point in a spacetime with low differentiability. In an heuristic manner, one can refer to *test-field singularities* as the approach to identify and characterise gravitational singularities as an obstruction to the evolution of test fields. This is in contrast to the standard approach where one uses geodesic incompleteness (which describes an obstruction to the evolution of a test particle) to identify singularities.

One would also like to know the extent to which the concept of field regularity can be applied to the interior of black holes. The behaviour of test fields at the interior of black holes has been studied in the Kerr and Reissner-Nordström spacetimes [51, 52] for the case of a scalar field. There, it has been shown that the assumptions on the decay rate of the field at the event horizon are crucial to the existence of an H^1_{loc} solution at the interior up to the Cauchy Horizon. Thus, to understand the generic case one would need to analyse the well-posedness of the test fields in spacetimes with low regularity. Furthermore, several astrophysical scenarios like the Oppenheimer-Snyder model of a collapsing star have jumps in the matter variables and are therefore at best $C^{1,1}$, while shells of matter have even lower regularity.

4.2 Well-posedness and energy estimates

In this section, we establish the meaning of well-posedness, weak solutions and energy estimates which will be key concepts used in this part of the thesis.

A problem in the theory of partial differential equations (PDE) defined in a manifold

M is said to be *locally well posed* in the sense of Hadamard if each point p contains a neighbourhood $\mathcal{U}$ such that it satisfies the following criteria [53]:

- There exist a solution in $\mathcal{U}$.
- The solution is unique in $\mathcal{U}$.
- The solution depends continuously on the initial data.

The existence of the solution requires us to choose a space of functions.
We start by considering the case of a $C^\infty(M)$ metric in a globally hyperbolic spacetime $M = \mathbb{R} \times \Sigma$. In this situation, there is a $C^\infty(M)$ solution to the problem:

$$\Box_g \phi = f \tag{4.1}$$

in M with $f \in C_0^\infty(M)$ and initial data $(\varphi, \pi) \in C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$.

$$\phi|_{\Sigma_0} = \varphi \tag{4.2}$$

$$\frac{\partial \phi}{\partial t}|_{\Sigma_0} = \pi \tag{4.3}$$

Then, multiplying (4.1) by a test field $\rho \in C_0^\infty(M)$ vanishing on $\partial M \backslash \Sigma_0$ and integrating by parts gives:

$$\int_M \frac{\partial \phi}{\partial x^i} \left(\frac{\partial \rho}{\partial x^j} g^{ij} \right) \nu_g = -\int_M \rho f \nu_g + \int_{\Sigma_0} \rho \pi \nu_h \tag{4.4}$$

$$\phi|_{\Sigma_0} = \varphi \tag{4.5}$$

Clarke defines a weak solution to (4.1), (4.2) and (4.3) in M for a low differentiable metric to be a function ϕ that satisfies condition (4.4) $\forall \rho \in C_0^\infty(M)$ with initial value (4.5). Notice that the expression is well defined for $g^{ij} \in C^0(M)$, $\phi \in H^1_{loc}(M, \nu_g)$, $\varphi \in L^2_{loc}(\Sigma_0, \nu_h)$, $\pi \in L^2_{loc}(\Sigma_0, \nu_h)$ and $f \in L^2_{loc}(M, \nu_g)$. Therefore, using Clarke's definition, the well-posedness of the problem should be shown in the function space $H^1_{loc}(M, \nu_g)$ with initial data in $L^2_{loc}(\Sigma_0, \nu_h) \times L^2_{loc}(\Sigma_0, \nu_h)$ and source function in $L^2_{loc}(M, \nu_h)$.
As we can see the concept of a weak solution allows us to define a solution when the differentiability of the solution or the metric does not have all the regularity needed to interpret the differential equation $\Box_g \phi = f$ point-wise. Now, our solution is not defined at points, but in the domain of integration. There exist however several different definitions of a weak solution which will be studied in detail in the following chapters. In each chapter, we will employ a specific definition and discuss the corresponding methods to prove the existence of the weak solution. Nonetheless, all our definitions of a weak solution satisfy the heuristic conditions a good weak solution must satisfy. Firstly, if there are solutions that satisfy the differential equation point-wise then those solutions are also weak solutions. Secondly, under certain conditions on the regularity of the initial data, the source function and the metric, there exist bootstrapping arguments that improve the regularity of the weak solutions. This means that our weak solutions satisfy certain so called "energy estimates". These estimates are fundamental to show uniqueness and stability and allow us to control the H^1 norm of the solutions in terms of the initial data and the source function f.

In order to motivate the name of "energy estimates", the problem of a vibrating string in the $1+1$ case is illustrative. If we consider the vibration of a string with constant linear density, ρ, and tension magnitude, T, the equations describing the motion of the string with fixed endings are given by

$$\rho\phi_{tt} = T\phi_{xx} \text{ in } [0,T]\times[0,l] \tag{4.6}$$
$$\phi(t,0) = \phi(t,l) = 0 \tag{4.7}$$
$$\phi(0,x) = g(x) \tag{4.8}$$
$$\phi_t(0,x) = h(x) \tag{4.9}$$

and the kinetic energy and potential energy of the string is given by:

$$E_K(t) = \frac{1}{2}\rho\int_0^l \phi_t(t,x)^2 dx \tag{4.10}$$
$$E_P(t) = \frac{1}{2}T\int_0^l \phi_x(t,x)^2 dx \tag{4.11}$$

Notice that conservation of energy states that $E(t) = E_K(t) + E_P(t)$ is a constant. This can be obtained by considering $\frac{dE}{dt}$ and using the fact that u is a solution. We use now the $L^2 \oplus L^2$ inner product to write the conservation of energy as:

$$2E(t) = T||\phi_x(t)||^2_{L^2([0,l])} + \rho||\phi_t(t)||^2_{L^2([0,l])} \tag{4.12}$$
$$= T\int_0^l \phi_x(t,x)^2 dx + \rho\int_0^l \phi_t(t,x)^2 dx \tag{4.13}$$
$$= T\int_0^l g_x^2 dx + \rho\int_0^l h^2 dx \tag{4.14}$$
$$= 2E(0) \tag{4.15}$$

This implies

$$||\phi_x(t)||^2_{L^2([0,l])} + ||\phi_t(t)||^2_{L^2([0,l])} = ||\phi_x(0)||^2_{L^2([0,l])} + ||\phi_t(0)||^2_{L^2([0,l])} \tag{4.16}$$

Therefore, we have a L^2 control of the derivatives of the solution at future times $t > 0$ in terms of the derivatives at the initial time. A remarkable characteristic of inhomogeneous wave equations is that similar estimates hold in higher dimensional curved spacetimes, i.e. the problem (4.1),(4.2) and (4.3) admits an a priori estimate

$$(\tilde{||}\phi||^1_{\Sigma_t})^2 \leqslant C\left((\tilde{||}\phi||^1_{\Sigma_0})^2 + (||f||_{L^2(M,\nu_g)})^2\right) \tag{4.17}$$

for some constant C and where we have introduced the norm

$$\tilde{||}\phi||^1_{\Sigma_t} = \left[\int_{\Sigma_t}\left(\frac{\partial\phi}{\partial t}\right)^2 + \sum_{i=1}^{3}\left(\frac{\partial\phi}{\partial x^i}\right)^2 + \phi^2\nu_h\right]^{\frac{1}{2}} \tag{4.18}$$

As we can see the structure of the energy estimates works naturally where the function and the derivatives satisfy L^2 conditions. In fact, the reason for using Hilbert spaces, such as H^k spaces or $L^2((0,T),H^k)$ rather than C^k spaces in the analysis of first order and second order hyperbolic PDEs is because of the existence of this kind of estimate. We also notice that although there is a natural norm in the C^k spaces given by

$$||\phi||_{C^k(\mathcal{U})} = \sum_{|\alpha|\leqslant k} \sup_{x\in\mathcal{U}} \left|\frac{\partial^\alpha f}{\partial x^\alpha}(x)\right|,$$

where we have used multi-index notation, the use of this norm misses the point as there is no simple relation between the C^k norm of the function at the initial time and the C^k norm at future time steps. Although, for elliptic problems similar estimates hold for Holder's spaces [54].
Moreover, subject only to the regularity of the initial data, the source function and the coefficients of $\Box_g$, one can obtain "higher energy estimates" of the form

$$(\tilde{\|}\phi\|^k_{\Sigma_t})^2 \leqslant C((\tilde{\|}\phi\|^k_{\Sigma_0})^2 + (\|f\|^{k-1}_{\Sigma_{[0,T]}})^2) \tag{4.19}$$

which correspond to norms that include derivatives of k-th order. For big enough k, one can guarantee the continuity of the function in terms of the continuity of the initial data and the continuity of f using Sobolev embedding theorems.
The existence of solutions satisfying this type of estimate implies uniqueness and stability. This can be seen by taking the difference of two weak solutions, $\phi = \phi_1 - \phi_2$, and noticing that it satisfies the homogeneous problem with vanishing initial data. The stability of the solution can be shown by using the fact that the norm of the difference between two solutions is bounded from above by the difference of the norm of the initial data and the norm of the source function. Therefore, the proof of uniqueness and stability will be similar in all the different settings.
The general structure of Part II of this thesis is as follows. In the next three chapters we define regions of spacetime $\Sigma_{[0,T]} = [0,T]\times\Sigma$ with different geometric conditions on the regularity of the metric and we find a triple $(\mathcal{P}, \mathcal{Q}, \mathcal{R}\times\mathcal{S})$ such that the initial value problem for the wave equation $\Box_g\phi = f$ on $\Sigma_{[0,T]}$ is well posed in the following sense:

- There exists a weak solution in the function space $\mathcal{P}$.
- The solution is unique in the function space $\mathcal{Q}$.
- The solutions in the space $\mathcal{Q}$ depend continuously with respect to initial data in function space $\mathcal{R}\times\mathcal{S}$.

In all the cases, the function space $\mathcal{P}$ allows us to define the energy-momentum tensor of the weak solution as a distribution. We will also require that $\mathcal{P} = \mathcal{Q}$, which means that there is no need for extra constraints to choose a unique solution and in fact we will prove the slightly stronger condition of Lipschitz dependence of the solution with respect to initial data.
We include in the appendix of this part of the thesis the relevant definitions and theorems on functional analysis, measure theory and regularity theory.
Notation. We denote the derivative of a function u with respect all coordinates by using roman letters a, b, c. If only the spatial part is consider we use middle roman letter i, j, k and use u_i or $\partial_i u$. When we differentiate with respect time we use $\partial_t u$ or when a function $d(t)$ depends only on time or is a Banach space valued function we denote the derivative by $\dot{d}(t)$. Additionally to avoid cumbersome notation we will not always explicitly use $\sum$ to denote a sum and we use instead Einstein's summation

notation, with roman letters $a, b \ldots$ etc used for summations over $0 \cdots n$ and $i, j \ldots$ etc used for summations over $1 \cdots n$. The signature of the metric will be chosen as $(+, -, -, ..., -)$.

Chapter 5

Curve Integrable Spacetimes

5.1 The main results

In this chapter, we provide full details of Clarke's work presented in [9] on, what he calls, curve integrable spacetimes defined below and a refinement of his arguments. Specifically, this will involve a detailed description of the techniques needed to show that solutions to the wave equation exist and are unique. Roughly speaking, a curve integrable spacetime is one in which the integrals of both the connection and curvature along a curve are bounded (see condition (4) of the main theorem for a precise description). From a physical point of view a spacetime is a curve integrable spacetime if there is a set C that defines a range of timelike directions which are transverse to any shock or caustic that may be present. The theorem proved by Clarke [9] requires both the quadratic and linear part of the Riemann tensor (in terms of the Christoffel symbols) to be separately integrable along the timelike directions. In addition, we also show continuity with respect to the source function.

5.1.1 The general setting

Following Clarke [9], we introduce an enlarged notion of a solution for curve integrable spacetimes. The general geometric background we use to define a generalised solution is the existence of a *lens-shaped* domain $\Sigma_{[T',T]}$ properly contained in an open bounded subset of $\mathbb{R}^4$.
This means that there is a smooth map $\Theta : \Sigma \times (-a, a) \to M$ where Σ is a compact, C^1 co-dimension one sub-manifold with boundary that has the following properties:

- $\Theta(\cdot\,, 0) : \Sigma \to \Sigma_0$ is the identity map.
- $\Theta(x, T) = \Theta(x, s)$ for any $x \in \partial\Sigma$, $T, s \in (-a, a)$.
- For any fixed $s \in (-a, a)$, $\Theta(\Sigma, s)$ is an 3-dimensional spacelike hypersurface.
- Away from $\partial\Sigma \times (-a, a)$, Θ is a diffeomorphism.

We denote the region from 0 to $T \geqslant 0$ by $\Sigma_{[0,T]}$ and from $T' \leqslant 0$ to 0 by $\Sigma_{[T',0]}$. Notice that given coordinates x^i on Σ, Θ provides coordinates (t, x^i) for the region $\Sigma_{[0,T]}$ away from the image of $\partial\Sigma$. We will therefore always choose charts such that the time coordinate coincides with the time coordinate given by Θ when working in coordinates.

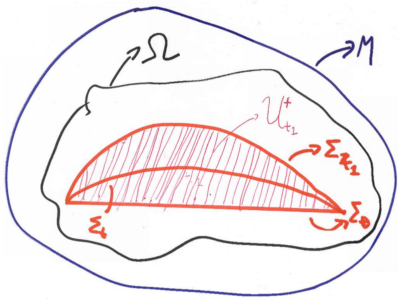


Figure 5.1: The general geometric setting.

We now state the main conditions that a curve integrable spacetime has to satisfy and therefore the precise regularity of the spacetime we are considering.

Geometric Conditions 1 (Curve integrable spacetimes) *Let* $(\Sigma_{[0,T]}, g)$ *be a lens-shaped domain and* p *a point in the domain satisfying:*

1. *The components* g_{ab} *and* g^{ab} *are* C^0 *;*
2. $g_{ab,c}$ *exist and are* $L^p_{loc}(\Sigma_{(0,T)})$ *with* $p \geqslant 3$*;*
3. *The components* g_{ab} *are* C^2 *in* $\Sigma_{(0,T)} \backslash J^+(p)$*;*
4. *There is a non-empty set,* $C \subset \mathbb{R}^4$*, and positive functions* $M, N : \mathbb{R}^+ \to \mathbb{R}^+$ *such that, if* γ *is a curve with* $d\gamma/ds \in C$ *for all* s *then*
   - γ *is future timelike;*
   - *the integrals*
   $$I_\gamma(a) := \int_0^a |\Gamma^a_{bc}(\gamma(s))|^2 ds \tag{5.1}$$
   and
   $$J_\gamma(a) := \int_0^a |R^a_{bcf}(\gamma(s))| ds \tag{5.2}$$
   (where Γ *is defined using the weak derivatives of* g *and* R *is understood in the sense of distributions) are finite for all* $a < T$*, with*
   $$I_\gamma(a) < M(a), J_\gamma(a) < N(a) \tag{5.3}$$
   and $M(a), N(a) \to 0$*, as* $a \to 0$.

Then $\Sigma_{[0,T]}$ *is a curve integrable spacetime.*

The first and second condition are the minimal requirements needed to recover the curvature tensor as a distribution defined in terms of g^{ab} and $g_{ab,c}$ [60]. The third and fourth conditions formalise our statement that a curve integrable spacetime admits a set C that defines a range of timelike directions which are transverse to any shock or caustic that may be present. Moreover the integrals of both the connection and curvature along the integral curves of this timelike directions are bounded.

5.1.2 Weak solutions and the main theorem

In this subsection we define an appropriate definition of a weak solution suitable for the low regularity geometric setting.
Our main motivation will be the fact that $\Box_g$ is formally self-adjoint in $\Sigma_{(0,T)}$. That is for $v, w \in C_0^\infty(\Sigma_{(0,T)})$, we have

$$(v, \Box_g w)_{L^2(\Sigma_{(0,T)},\nu_g)} = (\Box_g v, w)_{L^2(\Sigma_{(0,T)},\nu_g)} \tag{5.4}$$

This leads us to define a weak solution of

$$\Box_g u = f \tag{5.5}$$

in the region $\Sigma_{(0,T)}$ where g_{ab} is a curve integrable spacetime, $f \in L^2(\Sigma_{(0,T)})$ and $f|_{\Sigma_{t<0}} = 0$ satisfying

$$u|_{\Sigma_{t<0}} = 0 \tag{5.6}$$

$$\frac{\partial u}{\partial t}|_{\Sigma_{t<0}} = 0 \tag{5.7}$$

as follows

Definition 1 (Weak solution) *We say a function:*

$$u \in H^1(\Sigma_{(0,T)}, \nu_g)$$

is a local weak solution of the initial value problem (5.5), (5.6) and (5.7) provided that: For all $v \in C^\infty(\Sigma_{(0,T)})$ with supp$(v) \subset (0,T) \times \{\Sigma \setminus \partial\Sigma\}$

$$\int_{\Sigma_{(0,T)}} u \Box_g v \nu_g = \int_{\Sigma_{(0,T)}} f v \nu_g \tag{5.8}$$

where $f \in L^2(\Sigma_{(0,T)})$ and $f|_{\Sigma_{t<0}} = 0$ and u satisfies

$$u|_{\Sigma_{t<0}} = 0 \tag{5.9}$$

$$\frac{\partial u}{\partial t}|_{\Sigma_{t<0}} = 0 \tag{5.10}$$

As mentioned, in the introduction, we will require an energy estimate to show uniqueness and stability. This naturally leads to the following definition.

Definition 2 (Regular wave solution) *We say a weak solution $u \in H^1(\Sigma_{(0,T)}, \nu_g)$ is* **regular** *if u satisfies the energy estimate*

$$(\tilde{\|}u\|^1_{\Sigma_t})^2 \leqslant C\|f\|^2_{L^2(\Sigma_{(0,T)},\nu_g)} \tag{5.11}$$

where

$$\tilde{\|}u\|^1_{\Sigma_t} = \left[\int_{\Sigma_t} \left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^{3} \left(\frac{\partial u}{\partial x^i}\right)^2 \nu_h\right]^{\frac{1}{2}} \tag{5.12}$$

for all $0 \leqslant t \leqslant T$ and ν_h is the volume form associated to the induced metric on Σ_t

We now state our main result concerning the solutions of wave equations in curve integrable spacetimes.

Theorem 2 *Given the wave equation*

$$\Box_g u = f \tag{5.13}$$

in the region $\Sigma_{(0,T)}$ where g_{ab} is a curve integrable spacetime, $f \in L^2(\Sigma_{(0,T)})$ and $f|_{\Sigma_{t<0}} = 0$ satisfying

$$u|_{\Sigma_{t<0}} = 0 \tag{5.14}$$

$$\frac{\partial u}{\partial t}|_{\Sigma_{t<0}} = 0 \tag{5.15}$$

then there exists a unique regular weak solution $u \in H^1(\Sigma_{(0,T)}, \nu_g)$. Furthermore this solution is continuous with respect the source function.

5.2 Proof of the main theorem

5.2.1 Outline of the proof

The proof of Theorem 2 follows the classical method of proving well posedness of the wave equation by using an energy inequality as shown for example in Hawking and Ellis [3], Clarke [9] and Wilson [56]. The main steps of the proof are as follows:

1. First, we show that the regularity of the spacetime is enough to define a timelike vector field with bounded covariant derivative.
2. Second, using this timelike vector field we obtain an energy inequality of the form
$$(\tilde{\|}v\|^1_{\Sigma_T})^2 \leqslant C\|\Box_g v\|^2_{L^2(\Sigma_{(0,T)},\nu_g)} \tag{5.16}$$
for all $v \in C^\infty_0(\Sigma_{(0,T)})$.
3. Third, we obtain that the wave operator $\Box_g$ is densely defined and symmetric with respect to two different spaces of smooth functions denoted by V_{Σ_0} and V_{Σ_t}.
4. Fourth, applying the Hahn-Banach theorem and the Riesz representation theorem we show the existence of an element u in $L^2(\Sigma_{(0,T)}, \nu_g)$ that satisfies condition (5.8), (5.9) and (5.10).
5. Fifth, we show uniqueness and stability using the energy inequality (5.11).
6. Sixth, using commutator estimates we obtain that in fact u is a weak solution and therefore is in $H^1(\Sigma_{(0,T)}, \nu_g)$.

5.2.2 Existence of a timelike vector field with bounded covariant derivative

In the next proposition we show that there is a congruence of timelike geodesics whose tangent vector has an essentially bounded weak derivative. This is a key requirement to make sense of the energy inequalities because in a general low differentiable spacetime the covariant derivative may be unbounded and the argument breaks down.

Proposition 1 *Let $(\mathcal{M}, g)$ be a Lorentzian manifold and p a point in an open subset $\Omega \subset M$ with compact closure such that there is a lens-shaped domain $\Sigma_{[-t,t]}$ satisfying the Geometric Condition 1.*
Then there exists a congruence of timelike geodesics whose tangent vector, Υ^a, has an essentially bounded weak covariant derivative.

Summary of the proof Proposition 1. The proof will consist of eight steps.

1. The first step defines the geometric setting and defines the mollification used through the proof.
2. The second step focuses on establishing a majorizing ordinary differential equation that helps to uniformly bound the norm of the tangent vectors of the mollified geodesics.
3. The third step establishes a uniform time for existence of the mollified geodesics such that the tangent vectors of the mollified geodesics, γ_n, are contained inside the set C (see hypothesis 4 of the Geometric Conditions 2).
4. The fourth step focuses on establishing a majorizing ordinary differential equation that helps to essentially bound the geodesic connecting vector Y.
5. The fifth step establishes that Y is essentially bounded.
6. The sixth step uses the Arzela-Ascoli theorem and the bounds previously obtained to show that the limit $\lim_{n\to\infty} \gamma_n$ is well-defined and gives meaning to the notion of a geodesic and its tangent vector.
7. The seventh step establishes the distributional nature of the weak covariant derivative of the tangent vector.
8. The eighth step shows the essential boundedness of the weak covariant derivative.

First Step. Let $\Sigma_{[-t,t]}$ be a lens-shaped domain contained in a compact set Ω that contains p. Also assume there is an appropriate choice of coordinates $x^i = \{t, x^\alpha\}$ on $\Sigma_{(0,T)}$ such that $0 \leqslant t \leqslant T$ (where T is defined in the second step) and the constant values of t correspond to the spacelike surfaces Σ_t, $\Sigma = \Sigma_0$. Now we choose a vector $\vec{V} \in C \subset \mathbb{R}^4$. Then define the vector $\vec{V_q}$ as the vector at $q \in \Sigma$ whose components in the vector basis of the coordinates $\{t, x^\alpha\}$ are the same as the components of $\vec{V}$ in the canonical basis in $\mathbb{R}^4$.
The regularity of curve integrable spacetimes only allows us to define the connection as a distribution. In order to use the differential equations in the classical sense, we will use convolutions and then take limits.

For convenience we state the definition of a convolution here. However, for details see Appendix. For any $f \in L^1_{loc}$ we call the *convolution* of f with a function ρ the function given by

$$\rho * f(x) = \int_{B(x,d\rho)} f(x-y)\rho(y)dy \tag{5.17}$$

where $B(x,r)$ denotes the open ball of radius r around x and we denote by $d\rho$ the *diameter of the support* of ρ , i.e. $d\rho := sup\{|x| : x \in supp(\rho)\}$.
An important property of convolution is that convolution commutes with derivatives, i.e. $\partial_x(\rho * f) = \rho * \partial_x f$ and that the convolution is smooth.
Now, we define $\Gamma^{a(n)}_{bc} := \rho_n * \Gamma^a_{bc}$ where $\{\rho_n\}$ is a strict delta net (see Appendix) and for each point q in Σ we consider a family of geodesics $\{\gamma^q_n(s)\}$ that satisfies:

$$\frac{d^2{\gamma^q_n}^a}{ds^2} = -\Gamma^{a(n)}_{bc}\frac{d{\gamma^q_n}^b}{ds}\frac{d{\gamma^q_n}^c}{ds} \tag{5.18}$$

with the initial conditions

$$\gamma^q_n(0) = q \tag{5.19}$$

$$\frac{d\gamma^q_n}{ds}(0) = \vec{V}_q \tag{5.20}$$

Second Step.
We now show that there is some time s_1 (uniform in n) such that for $|s| < s_1$ we have $\frac{d\gamma_n}{ds}(s) \in C$. We will drop the q in the notation until it is needed.
First we define:

$$l = \sup\left\{\|\vec{V}\|_{L^2(\Sigma_{(0,T)})} : \vec{V} \in C\right\} \tag{5.21}$$

$$r = \inf\left\{dist(\frac{d\gamma_n}{ds}(0), C^c) : q \in \Sigma\right\} \tag{5.22}$$

where *dist* means the euclidean distance and where C^c is the complement of C. We first uniformly bound $\left\|\frac{d\gamma_n}{ds}\right\|$ in terms of s and the initial value where $\|V\|$ is the euclidean norm of V in $\mathbb{R}^4$.
Now notice that:

$$\left(\left\|\frac{d\gamma_n}{ds}\right\|\right)\frac{d}{ds}\left\|\frac{d\gamma_n}{ds}\right\| = \frac{1}{2}\frac{d}{ds}\left\|\frac{d\gamma_n}{ds}\right\|^2 \tag{5.23}$$

$$= \frac{d}{ds}\left(\frac{d\gamma_n}{ds}\right)\cdot\frac{d\gamma_n}{ds} \tag{5.24}$$

$$\leqslant \left\|\frac{d}{ds}\left(\frac{d\gamma_n}{ds}\right)\right\|\left\|\frac{d\gamma_n}{ds}\right\| \tag{5.25}$$

where $u\cdot v$ is the dot product in $\mathbb{R}^4$.
Hence

$$\frac{d}{ds}\left\|\frac{d\gamma_n}{ds}\right\| \leqslant \left\|\frac{d}{ds}\left(\frac{d\gamma_n}{ds}\right)\right\| \tag{5.26}$$

Consider now the following inequalities:

$$\left\| \frac{d}{ds}\left(\frac{d\gamma_n}{ds}\right) \right\| \leqslant 2 \sup_{a=0,..,3} | \left| \frac{d}{ds}\left(\frac{d\gamma_n^a}{ds}\right) \right| \quad (5.27)$$

$$= \sup_{a=0,..,3} \left| -\Gamma^{a(n)}_{bc} \frac{d\gamma^{q\ b}_n}{ds} \frac{d\gamma^{q\ c}_n}{ds} \right| \quad (5.28)$$

$$\leqslant 32 \sup_{i,j,k} \left| -\Gamma^{a(n)}_{bc} \right| \sup_{j,k=0,..,3} \left| \frac{d\gamma^{q\ b}_n}{ds} \frac{d\gamma^{q\ c}_n}{ds} \right| \quad (5.29)$$

$$\leqslant 32 \sup_{i,j,k} \left| -\Gamma^{a(n)}_{bc} \right| \left\| \frac{d\gamma_n}{ds} \right\|^2 \quad (5.30)$$

which using (5.26) gives:

$$\frac{d}{ds} \left\| \frac{d\gamma_n}{ds} \right\| \leqslant 32 \sup_{a,b,c} \left| -\Gamma^{a(n)}_{bc} \right| \left\| \frac{d\gamma_n}{ds} \right\|^2 \quad (5.31)$$

So $\left\| \frac{d\gamma_n}{ds} \right\|$ is bounded by the majorizing equation

$$\frac{dx}{ds} = \lambda^n x^2 \quad (5.32)$$

subject to the initial condition $x(0) < l$ and where $\lambda^n(s) := 32 \sup_{i,j,k} \left| -\Gamma^{a(n)}_{bc}(\gamma_n(s)) \right|$

Then we have:

$$\frac{dx}{ds} \leqslant \lambda^n x^2 \quad (5.33)$$

$$\Rightarrow \quad -\frac{d}{ds}\left(\frac{1}{x}\right) \leqslant \lambda^n(s) \quad (5.34)$$

$$\Rightarrow \quad \frac{1}{x(s)} - \frac{1}{x(0)} \leqslant -\int_0^s \lambda^n(s')ds' \quad (5.35)$$

$$\Rightarrow \quad x(s) < \frac{x(0)}{1 - x(0)\int_0^s \lambda^n(s')ds'} < kx(0) \quad (5.36)$$

for $k > \frac{1}{1-x(0)\int_0^s \lambda^n(s')ds'} > 1$

Third Step. We now use the result above to show the existence of a time interval for which $\dot{\gamma}$ remains in C.
Notice that

$$\int_0^s |\Gamma^{a(n)}_{bc}|ds = \int_0^s |\rho_n * \Gamma^a_{bc}|ds' \quad (5.37)$$

$$= \int_0^s | \int_{\mathbb{R}^4} \rho_n(z)\Gamma^a_{bc}(\gamma(s') + z)dz|ds' \quad (5.38)$$

$$\leqslant \int_0^s \int_{\mathbb{R}^4} |\rho_n(z)||\Gamma^a_{bc}(\gamma(s') + z)|dzds' \quad (5.39)$$

$$\leqslant \int_{\mathbb{R}^4} |\rho_n(z)| \int_0^s |\Gamma^a_{bc}(\gamma(s') + z)|ds'dz \quad (5.40)$$

$$\leqslant M(s)^{\frac{1}{2}} s^{\frac{1}{2}} \int_{\mathbb{R}^4} |\rho_n(z)|dz \quad (5.41)$$

$$\leqslant M(s)^{\frac{1}{2}} s^{\frac{1}{2}} (1 + \eta) \quad (5.42)$$

for $\eta > 0$ when $n \geqslant N_0$ where we have use (5.36) with $k = 1 + \frac{r}{x(0)}$ so that $\dot{\gamma} \in C$. Then $g^n(s) = \int_0^s \lambda^n(s')ds'$ is a non-decreasing function that satisfies $\lim_{s^+ \to 0} g^n(s) = 0$ for all n as a consequence of property 4 of the Geometric Conditions 1 and (5.42). Moreover, there is a smallest time $s_0 \neq 0$ such that $k = 1 + \frac{r}{x(0)} = \frac{1}{1 - x(0) g^{n_0}(s_0)}$ for some n_0. Then choosing $s < s_0$ guarantees that $1 + \frac{r}{x(0)} > \frac{1}{1 - l \int_0^s \lambda^n(s')ds'}$ for all n. Notice that s_0 can not be 0 because that would imply $k = 1$ contradicting $k > 1$. Then for $s < s_0$, integrating and taking absolute values on the geodesic equation (5.18) we have that

$$\left| \frac{d\gamma_n}{ds}^a(s) - \frac{d\gamma_n}{ds}^a(0) \right| \leqslant \int_0^s |\Gamma^{a(n)}_{bc} \frac{d\gamma_n}{ds}^j \frac{d\gamma_n}{ds}^k |ds' \tag{5.43}$$

$$\leqslant 16(r+l)^2 \int_0^s |\Gamma^{a(n)}_{bc}|ds' \tag{5.44}$$

$$\leqslant 16(r+l)^2 M(s)^{\frac{1}{2}} s^{\frac{1}{2}} (1+\eta) \tag{5.45}$$

Now because $M(s) \to 0$ as $s \to 0$ we can make the difference as small as we want. So taking explicitly $s_1 \leqslant s_0$ such that

$$M(s_1)^{\frac{1}{2}} s_1^{\frac{1}{2}} < \frac{r}{32(r+l)^2(1+\eta)} \tag{5.46}$$

ensures that:

$$\left\| \frac{d\gamma_n}{ds}(s) - \frac{d\gamma_n}{ds}(0) \right\| \leqslant 2 \sup_{a=0,..,3} \left| \frac{d\gamma_n}{ds}^a(s) - \frac{d\gamma_n}{ds}^a(0) \right| \tag{5.47}$$

$$\leqslant 32(r+l)^2 M(s)^{\frac{1}{2}} s^{\frac{1}{2}} (1+\eta) \tag{5.48}$$

$$\leqslant r \tag{5.49}$$

for all $s \leqslant s_1$ (where s_1 is independent of n). This implies that $\frac{d\gamma_n}{ds}(s) \in C$. Then we can choose T sufficiently close to 0 to ensure that p is covered by the curves up to s_1.

Fourth Step. We now consider a 1-parameter family of initial conditions

$$\gamma^q_n(0,u) = q(u) \tag{5.50}$$

$$\frac{d\gamma^q_n}{ds}(0,u) = \vec{V}_{q(u)} \tag{5.51}$$

and the corresponding family of geodesics, $\{\gamma_n^{q(u)}(s)\}_u$. Now let Y be the connecting vector of this family. For simplicity later on, we are going to use a parallel propagated co-frame on γ^q_n, $\{e^{d(n)} = e^{d(n)}_a dx^a\}_{d=0,1,2,3}$ coinciding with the coordinate basis at $s = 0$. So we define:

$$Y^d_u := \frac{\partial \gamma^{q\ a}_n}{\partial u} e^{d(n)}_a := Y^a_u e^{d(n)}_a \tag{5.52}$$

Moreover if $q(u)$ is a coordinate function of Σ, i.e. $q(u) = x^\alpha$, we will denote the connecting vector as Y_α with frame components Y^d_α. Now we have

$$\frac{D^2 Y^d_\alpha}{Ds^2} = \frac{D}{Ds}\left(\frac{dY^d_\alpha}{ds}\frac{\partial}{\partial s}\right) \tag{5.53}$$
$$= \frac{d^2 Y^d_\alpha}{ds^2}\frac{\partial}{\partial s} \tag{5.54}$$

where we have used the fact that Y^d_α is a scalar and $\frac{D}{Ds}\left(\frac{\partial}{\partial s}\right)$ is the geodesic equation where we have defined $\frac{D}{Ds} := \nabla^{(n)}_{\frac{\partial}{\partial s}}$.
Now notice that in frame components we have

$$\frac{D^2 Y^d_\alpha}{Ds^2} = \frac{D}{Ds}\left(\frac{DY^a_\alpha}{Ds}e^{d(n)}_a + Y^a_\alpha \frac{De^{d(n)}_a}{Ds}\right) \tag{5.55}$$
$$= \frac{D}{Ds}\left(\frac{dY^a_\alpha}{ds}e^{d(n)}_a \frac{\partial}{\partial s} + Y^a_\alpha \frac{De^{d(n)}_a}{Ds}\right) \tag{5.56}$$
$$= \frac{D^2 Y^a_\alpha}{Ds^2}e^{d(n)}_a \frac{\partial}{\partial s} \tag{5.57}$$

where we have used the fact that Y^d_α is a scalar and $e^{d(n)}_a$ are parallel propagated coefficients.
Now using the geodesic deviation equation, (5.57) and (5.54) we have:

$$\frac{d^2 Y^d_\alpha}{ds^2} = \frac{D^2 Y^d_\alpha}{Ds^2} \tag{5.58}$$
$$= \frac{D^2 Y^a_\alpha}{Ds^2}e^{d(n)}_a \tag{5.59}$$
$$= e^{d(n)}_a R^{a(n)}_{bcf} \frac{d\gamma_n}{ds}^b \frac{d\gamma_n}{ds}^c Y^e_\alpha e^{f(n)}_e \tag{5.60}$$

with the initial conditions:

$$Y^d_\alpha(0) = Y^a_\alpha(0)e^{d(n)}_a(0) \tag{5.61}$$
$$= \delta^a_\alpha \delta^d_a \tag{5.62}$$
$$= \delta^d_\alpha \tag{5.63}$$

where the first δ follows by noting that at $s = 0$ we have $\frac{\partial \gamma^{qi}_n}{\partial x^\alpha} = \frac{\partial (x^i)^q_n}{\partial x^\alpha}$ and the second one as a consequence of the initial alignment with the coordinates.
Also notice that

$$\frac{dY^d_\alpha}{ds}e^{(n)}_d = \left(\nabla_{\frac{\partial}{\partial s}} Y^d_\alpha\right) e^{(n)}_d \tag{5.64}$$
$$= \left(\nabla_{\frac{\partial}{\partial s}} Y^d_\alpha\right) e^{(n)}_d + Y^d_\alpha \nabla_{\frac{\partial}{\partial s}} e^{(n)}_d \tag{5.65}$$
$$= \nabla_{\frac{\partial}{\partial s}} Y^d_\alpha e^{(n)}_d \tag{5.66}$$
$$= \nabla_{Y^d_\alpha e^{(n)}_d} \frac{\partial}{\partial s} \tag{5.67}$$
$$= Y^d_\alpha \nabla_{e^{(n)}_d} \frac{\partial}{\partial s} \tag{5.68}$$

where the torsion free condition has been used in rewriting (5.67). Now if we evaluate at $s = 0$, we obtain the second initial condition through the following calculation

$$
\begin{aligned}
\frac{dY^d_\alpha}{ds}(0)e^{(n)}_d &= Y^d_\alpha(0)\nabla_{e^{(n)}_d}\frac{\partial}{\partial s}|_0 && (5.69)\\
&= \delta^a_\alpha\nabla_{e^{(n)}_d(0)}\frac{\partial}{\partial s}|_0 && (5.70)\\
&= \nabla_{\frac{\partial}{\partial x^\alpha}}\frac{\partial}{\partial s}|_0 && (5.71)\\
&= \nabla_{\frac{\partial}{\partial x^\alpha}}\vec{V}_q && (5.72)\\
&:= \vartheta && (5.73)
\end{aligned}
$$

where (5.73) is a definition.
Now notice that:

$$\nabla_{\frac{\partial}{\partial s}} e^{d(n)} = 0 \tag{5.74}$$

as a consequence of being parallel propagated.
Then in coordinates we have:

$$\frac{de^{d(n)}_a}{ds} = \Gamma^{f(n)}_{ab} e^{d(n)}_f \frac{d\gamma_n}{ds}^b \tag{5.75}$$

So using again a majorizing equation

$$\frac{dy}{ds} = c\lambda^n y$$

with initial condition $z(0) = 1$, c a constant and applying similar arguments as the ones used in Step 2 and 3, we can conclude that $e^{d(n)}_a$ is uniformly bounded in terms of $M(s_1)$.
It follows then by using (5.60) and the uniform bounds on $e^{d(n)}_a$ and $\frac{d\gamma_n}{ds}$ that:

$$\left|\frac{d^2Y^d_\alpha}{ds^2}\right| \leqslant C\sigma\|Y^d_\alpha\|_{L^\infty} \tag{5.76}$$

where C is a suitable constant and

$$\sigma := \sup_{a,b,c,f} |R^{(n)a}_{bcf}(\gamma_n(s))| \tag{5.77}$$

Next we consider the majorizing equation

$$\frac{d^2z}{ds^2} = C\sigma z \tag{5.78}$$

with initial conditions

$$z(0) = 1, \quad \frac{dz(0)}{ds} = \sup|\vartheta| \tag{5.79}$$

where the supremum is taken with respect all indices that appear in ϑ. Notice that bounding z, implies a bound of $\|Y^d_\alpha\|_{L^\infty}$.

Fifth Step. We now obtain a bound on z in terms of the curvature tensor. We notice the following

$$R^{a(n)}_{bcf} = \partial_c \Gamma^{a(n)}_{fb} - \partial_f \Gamma^{a(n)}_{cb} + \Gamma^{a(n)}_{c\lambda}\Gamma^{\lambda(n)}_{fb} - \Gamma^{a(n)}_{f\lambda}\Gamma^{\lambda(n)}_{cb} \tag{5.80}$$

and that

$$\rho_n * R^a_{bcf} = \partial_c \Gamma^{a(n)}_{fb} - \partial_f \Gamma^{a(n)}_{cb} + \rho_n * \left(\Gamma^a_{c\lambda}\Gamma^\lambda_{fb}\right) - \rho_n * \left(\Gamma^a_{f\lambda}\Gamma^\lambda_{cb}\right) \tag{5.81}$$

We will bound the integral of $R^{a(n)}_{bcf}$ along γ_n. Notice that we have that

$$\int_0^s |\partial_c \Gamma^{a(n)}_{fb} - \partial_f \Gamma^{a(n)}_{cb}| ds' = \int_0^s |\rho_n * R^a_{bcf} + \rho_n * \left(\Gamma^a_{f\lambda}\Gamma^\lambda_{cb} - \Gamma^a_{c\lambda}\Gamma^\lambda_{fb}\right)| ds' \tag{5.82}$$

Now we have

$$\begin{aligned}
\int_0^s |\rho_n * R^a_{bcf}| ds' &= \int_0^s |\int \rho_n(z) R^a_{bcf}(\gamma(s') + z) dz| ds' \\
&\leqslant \int_{\mathbb{R}^4} |\rho_n(z)| \int_0^s |R^a_{bcf}(\gamma(s') + z)| ds' dz \\
&\leqslant N(s) \int_{\mathbb{R}^4} |\rho_n(z)| dz \\
&\leqslant N(s)
\end{aligned}$$

and

$$\begin{aligned}
\int_0^s |\rho_n * \left(\Gamma^a_{c\lambda}\Gamma^\lambda_{fb}\right)| ds' &= \int_0^s |\int \rho_n(z) \Gamma^a_{c\lambda}(\gamma(s') + z) \Gamma^\lambda_{fb}(\gamma(s') + z) dz| ds' \\
&\leqslant \int_{\mathbb{R}^4} |\rho_n(z)| \int_0^s |\Gamma^a_{c\lambda}(\gamma(s') + z) \Gamma^\lambda_{fb}(\gamma(s') + z)| ds' dz \\
&\leqslant K_1 M(s) \int_{\mathbb{R}^4} |\rho_n(z)| dz \\
&\leqslant K_1 M(s)
\end{aligned}$$

where K_1 is a suitable constant and therefore (5.82) which corresponds to the linear part of the curvature tensor satisfies the inequality

$$\int_0^s |\partial_c \Gamma^{a(n)}_{fb} - \partial_f \Gamma^{a(n)}_{cb}| ds' \leqslant K_2(M(s) + N(s)) \tag{5.83}$$

where K_2 is a suitable constant.
Now we bound the non linear part as follows

$$\begin{aligned}
\int_0^s |\left(\Gamma^{a(n)}_{c\lambda}\Gamma^{\lambda(n)}_{fb}\right)| ds' &= \int_0^s |\int \rho_n(z) \int \rho_n(z') \Gamma^a_{c\lambda}(\gamma(s') + z) \Gamma^\lambda_{fb}(\gamma(s') + z') dz dz'| ds' \\
&\leqslant \int_{\mathbb{R}^4} |\rho_n(z) \int_{\mathbb{R}^4} |\rho_n(z')| \int_0^s |\Gamma^a_{c\lambda}(\gamma(s') + z) \Gamma^\lambda_{fb}(\gamma(s') + z')| ds' dz dz' \\
&\leqslant K_3 M(s) \int_{\mathbb{R}^4} |\rho_n(z)| dz \int_{\mathbb{R}^4} |\rho_n(z')| dz' \\
&\leqslant K_3 M(s)
\end{aligned}$$

where K_2 is a suitable constant.

and therefore we have an estimate of the form

$$\int_0^s |\left(\Gamma^{a(n)}_{c\lambda}\Gamma^{\lambda(n)}_{fb}\right)|ds' \leqslant K_3 M(s) \tag{5.84}$$

where K_3 is a suitable constant.
Then using (5.83) and (5.84) we conclude that

$$\int_0^s |R^{(n)a}_{bcf}|ds \quad\leqslant\quad \int_0^s \sigma ds' \leqslant K_4(L(s)) \tag{5.85}$$

where K_4 is a suitable constant and $L(s)$ is a linear combination of $M(s)$ and $N(s)$.

Note: Provided that the metric is C^1 (see the discusion of H^1 solutions below) we can bound $\int_0^s |R^{(n)a}_{bcf}|ds'$ in an alternative way using the following mild version of Friedrich Lemma:

Theorem 3 Friedrich's Lemma *[12]*
Let $a \in C^0, b \in L^\infty_{loc}$. Then $\rho_n * (ab) - (\rho_n * a)(\rho_n * b) \to 0$ locally uniformly.

Using Friedrich's Lemma we can guarantee that given $\epsilon > 0$ we have that $|R^{(n)a}_{bcf}| \leqslant |\rho_n * R^a_{bcf}| + \epsilon$ for all n close enough to zero provided that the metric is at least C^1. If this is the case then we will have the following bound

$$\int_0^s |R^{(n)a}_{bcf}|ds \leqslant \int_0^s |\rho * R^a_{bcf}| + \epsilon ds' \leqslant N(s) + \epsilon s \tag{5.86}$$

for all n close enough to zero. However, with this regularity, the existence of a timelike vector field with bounded covariant derivative follows easily anyway.
To estimate a solution to (9.88) we notice that if s_2 is the first value of Σ at which $|z| = 2$ (possibly $s_2 = \infty$) then before s_2, one has

$$\frac{d^2 z}{ds^2} \quad = \quad C\sigma z \tag{5.87}$$

$$\leqslant \quad C\sigma 2 \tag{5.88}$$

which is a consequence of the initial conditions and the continuity of z.
Now by integrating twice both sides, considering (5.85) and the initial conditions on z, we get

$$z \leqslant 1 + \sup|\vartheta|s + 2C\int_0^s L(s')ds' \tag{5.89}$$

for $0 \leqslant s \leqslant s_2 \leqslant s_1$
Now the right side is an increasing function that starts at zero. So there is a s_3 such that

$$|\vartheta|s_3 + 2C\int_0^{s_3} L(s')ds' \leqslant 1 \tag{5.90}$$

then we will have $z \leqslant 2$ up to s_3, and hence

$$\|Y^d_\alpha\|_{L^\infty} \leqslant 2 \tag{5.91}$$

in this interval.
Sixth Step. The Arzela-Ascoli theorem guarantees that given a sequence of equicontinuous and uniformly bounded functions there is a sub-sequence that converges uniformly. Now the functions

$$\{\gamma^n : (q,s) \to \gamma_n^q(s)\}$$

are equicontinuous and uniformly bounded. This can be seen by noting that the functions are defined in a bounded domain and the fact that $\frac{\partial \gamma_n^q}{\partial x^j}$ and $\frac{\partial \gamma_n^q}{ds}$ are uniformly bounded.
So by Arzela-Ascoli theorem there is a sub-sequence of $\{\gamma^n\}$ that gives meaning to the idea of a geodesic γ with tangent vector Υ^a.

Seventh Step.
We now define the b-th component of $\left(\nabla_a \frac{d\gamma_n}{ds}\right)$ as:

$$\left(\left(\nabla_a \frac{d\gamma_n}{ds}\right)^b, \phi\right) = \int_{\Sigma_{(0,T)}} \left(\nabla_a \frac{d\gamma_n}{ds}\right)^b \phi \nu_g \tag{5.92}$$

$$= \int_{\Sigma_{(0,T)}} \left(\frac{\partial}{\partial x^a}\left(\frac{d\gamma_n}{ds}^b\right) + \Gamma_{ca}^{(n)b}\frac{d\gamma_n}{ds}^c\right)\phi\nu_g \tag{5.93}$$

$$= -\int_{\Sigma_{(0,T)}} \frac{d\gamma_n}{ds}^c \left(\frac{\partial \phi}{\partial x^a}\delta_c^b - \Gamma_{ca}^{(n)b}\phi\right)\nu_g \tag{5.94}$$

where (5.94) is obtained by integration by parts.
Notice that the right hand side converges in $\mathbb{R}$ as n tends to infinity for every $\phi \in \mathcal{D}(\Sigma_{(0,T)})$. Hence, the expression converges in the sense of distributions to the distributional covariant derivative of Υ^a (see [53], p. 134).

Eighth Step. We now establish the essential boundedness of the weak covariant derivative of the tangent vector. Consider a basis for $T_p\Sigma_{(0,T)}$ is $\{\frac{d\gamma_n}{ds}, Y_\alpha^a \frac{\partial}{\partial x^a}\}_{\alpha=1,2,3}$. So any vector X can be written as a linear combination of those.
If the a-th component of X is $X^a = X^\alpha Y_\alpha^a + X^0 \frac{d\gamma_n}{ds}^a$ then

$$\nabla_X \frac{d\gamma_n}{ds} = \nabla_{X^a \frac{\partial}{\partial x^a}} \frac{d\gamma_n}{ds} \tag{5.95}$$

$$= \nabla_{X^\alpha Y_\alpha^a \frac{\partial}{\partial x^a}} \frac{d\gamma_n}{ds} + \nabla_{X^0 \frac{d\gamma_n}{ds}^a \frac{\partial}{\partial x^a}} \frac{d\gamma_n}{ds} \tag{5.96}$$

$$= X^\alpha \nabla_{Y_\alpha} \frac{d\gamma_n}{ds} + X^0 \nabla_{\frac{d\gamma_n}{ds}} \frac{d\gamma_n}{ds} \tag{5.97}$$

$$= X^\alpha \nabla_{\frac{d\gamma_n}{ds}} Y_\alpha \tag{5.98}$$

$$= X^\alpha \nabla_{\frac{d\gamma_n}{ds}} Y_\alpha^d e_d^{(n)} \tag{5.99}$$

$$= X^\alpha \frac{dY_\alpha^d}{ds} e_d^{(n)} \tag{5.100}$$

where we have used the linearity of the covariant derivative, the torsion free condition and the vanishing Lie bracket between Y_α, and $\frac{d\gamma_n}{ds}$ along with the geodesic equation. Now integrating (5.60) once we have:

$$\frac{dY_\alpha^d}{ds} = \frac{d\gamma_n^c}{ds}(0)\Gamma_{\alpha c}^{d(n)}(\gamma_n(0)) + \int_0^s e_a^{d(n)} R_{bcf}^{a(n)} \frac{d\gamma_n}{ds}^b \frac{d\gamma_n}{ds}^c Y_\alpha^g e_g^{f(n)} ds' \tag{5.101}$$

Now this implies that $\left(\nabla_a \frac{d\gamma_n}{ds}\right)^b$ is bounded and the bound is independent of n. This can be seen by using (5.47), (5.75),(5.91) to bound $\frac{dY^d_\alpha}{ds}$ and the fact that X^α are continuous functions in a compact set. So a bound exist.
Now using the boundedness of $\left(\nabla_a \frac{d\gamma_n}{ds}\right)^b$ and (5.92) we have an estimate of the form:

$$\left(\left(\nabla_a \frac{d\gamma_n}{ds}\right)^b, \phi\right) \leqslant B\|\phi\|_{L^1(\Sigma_{(0,T)})} \tag{5.102}$$

where B is a constant.
Moreover, (5.102) allows us to define $\left(\nabla_a \frac{d\gamma_n}{ds}\right)^b$ as a functional over the space of integrable functions, $L^1(\Sigma_{(0,T)}, \nu_g)$. Taking the limit as $n \to \infty$ we obtain

$$\lim_{n\to\infty} \left(\left(\nabla_a \frac{d\gamma_n}{ds}\right)^b, \phi\right) \leqslant B\|\phi\|_{L^1(\Sigma_{(0,T)})} \tag{5.103}$$

because all the bounds hold in the limit.
So $\lim_{n\to\infty} \left(\nabla_a \frac{d\gamma_n}{ds}\right)^b$ converges in the dual space (the space of linear functionals) of integrable functions. This space is isomorphic to $L^\infty(\Sigma_{(0,T)})$ because $\Sigma_{(0,T)}$ is compact. Then $\left(\nabla_a \frac{d\gamma_n}{ds}\right)^b \in L^\infty(\Sigma_{(0,T)})$ and it is a uniformly bounded function. Finally we have that $\lim_{n\to\infty} \nabla_a \dfrac{d\gamma_n}{ds}$ is essentially bounded because each component is an essentially bounded function.

5.2.3 Energy inequality

The energy inequality gives an integral of the function and its derivatives at a future time bounded above by an integral of the function and its derivatives at the initial time and an integral of the source function over the region between the initial time and the future time. We will first assume that g_{ab} is C^2. Then at the end we will give the extra requirements that the metric must satisfy in order for the energy inequality to be still valid when the differentiability is below this regularity. Consider u satisfying $\Box_g u = f$ and C^∞, with energy-momentum tensor T^{ab} given by:

$$T^{ab}[u] = \left(g^{ac}g^{bd} - \frac{1}{2}g^{ab}g^{cd}\right) \frac{\partial u}{\partial x^c}\frac{\partial u}{\partial x^d} \tag{5.104}$$

Now choosing a smooth timelike vector field, Υ^a, as in Proposition 1 we define the energy integral:

$$E(t) = \int_{\Sigma_t} T^{ab}\Upsilon_a n_b \nu_h \tag{5.105}$$

where n^a is a future pointing vector normal to Σ_t.
Then we use the divergence theorem on the domain $\overline{\Sigma_{(0,T)}}$:

$$\int_{\Sigma_{(0,T)}} \text{div}\left(T^{ab}\Upsilon_a\right)\nu_g = \int_{\partial\Sigma_{(0,T)}} T^{ab}\Upsilon_a n_b \nu_h \tag{5.106}$$

The left hand side takes the explicit form:

$$\int_{\Sigma_{(0,T)}} \left(g^{ab}\frac{\partial u}{\partial x^b}\Upsilon_a\right)[f - u] + T^{ab}\nabla_b\Upsilon_a N\sqrt{h}d^4x \tag{5.107}$$

where $f = \Box_g u$, N is the lapse function which satisfies $Ndt = n_a$ and $\sqrt{h}$ is the scalar density associated to the volume form given by the induced metric h_{ij} on the hypersurface Σ.
The right hand side then takes the form:

$$\left(\int_{\Sigma_t} - \int_{\Sigma_0}\right) T^{ab}\Upsilon_a n_b \nu_h. \tag{5.108}$$

Now we introduce the following norm:

$$\tilde{|}\!|u\|^1_{\Sigma_t} = \left[\int_{\Sigma_t}\left(\left(\frac{\partial u}{\partial t}\right)^2 + \sum_{i=1}^{3}\left(\frac{\partial u}{\partial x^i}\right)^2\right)\nu_h\right]^{\frac{1}{2}} \tag{5.109}$$

The main reason for introducing this norm, which is naturally related to Sobolev type norms is the following result which can be found in [56]

$$C_1 E(t) \leqslant (\tilde{|}\!|u\|^1_{\Sigma_t})^2 \leqslant C_2 E(t) \tag{5.110}$$

for constants $C_1, C_2 \geqslant 0$.
Also notice that:

$$\|\phi\|_{H^1(\Sigma_{(0,T)})} \leqslant C\left(\int_0^t \left(\tilde{|}\!|\phi\|^1_{\Sigma_{t'}}\right)^2 dt'\right)^{\frac{1}{2}} \tag{5.111}$$

where C is a constant that depends on g_{ab} and the interval $[0, T]$.
Now we can obtain the following bounds for all the terms in (5.107)

$$\begin{aligned}&\int_0^t \left(\int_{\Sigma_{t'}}\left(g^{ab}\frac{\partial u}{\partial x^a}\Upsilon_b\right) fN\nu_h\right) dt' \\ &\leqslant K_1((\|u\|_{H^1(\Sigma_{(0,T)},\nu_g)})^2 + (\|f\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2)\end{aligned} \tag{5.112}$$

$$\int_0^t \left(\int_{\Sigma_{t'}}\left(g^{ab}\frac{\partial u}{\partial x^b}\Upsilon_a\right) uN\nu_h\right) dt' \leqslant K_2(\|u\|_{H^1(\Sigma_{(0,T)},\nu_g)})^2 \tag{5.113}$$

where K_1, K_2 are constants that depend on g_{ab} and Υ^a.
The last term in (5.107) is bounded by:

$$\int_{\Sigma_{(0,T)}} T^{ab}\nabla_a\Upsilon_b \leqslant K_3(\|u\|_{H^1(\Sigma_{(0,T)},\nu_g)})^2 \tag{5.114}$$

where K_3 is a constant that depends on g_{ab} and $\nabla_b\Upsilon_a$.
Estimating all the terms in (5.106) by the bounds available (5.112), (5) and (5.114) gives the inequality:

$$E(t) \leqslant E(0) + k_0(\|f\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2 + k_1(\|u\|_{H^1(\Sigma_{(0,T)},\nu_g)})^2 \tag{5.115}$$

where k_0, k_1 are positive constants that depend on the metric g_{ab}, the vector field Υ_a and the covariant derivative $\nabla_b\Upsilon_a$.
Now rewriting (5.115) using (5.110) and (5.111) we find

$$E(t) \leqslant E(0) + k_0(\|f\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2 + k_2 \int_0^t E(t')dt' \tag{5.116}$$

Using Gronwall's inequality the desired energy inequality is obtained:

$$E(t) \leqslant K_4(E(0) + (\|f\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2 \text{ for all } t \leqslant T \tag{5.117}$$

where K_4 is a positive constant that depends on the chosen finite time T, the metric g_{ab}, the vector field Υ_a and the covariant derivative $\nabla_b \Upsilon_a$.
In term of the Sobolev norms we obtain the expression:

$$(\tilde{\|}u\|^1_{\Sigma_t})^2 \leqslant K((\tilde{\|}u\|^1_{\Sigma_0})^2 + (\|f\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2 \tag{5.118}$$

for some constant K.
We now look for the conditions required to obtain again the energy inequalities in the low differentiable setting. The basic requirement we need is that we can apply Stokes' Theorem. To our knowledge the optimum results are given by the following Theorem 14 in the Appendix.
Now we check the analytical conditions needed for the theorem to apply. First, we need the metric and its metric volume element to be continuous and the metric to satisfy $g_{ab} \in W^{1,3}_{loc}(\Omega)$. This is enough to satisfy the hypothesis of Theorem 14 and allows us to apply Stokes' theorem. Of course we would like to have the same expression as in (5.107). This requires the existence of a metric connection, i.e. $\nabla g_{ab} = 0$. In [57], it is stated that sufficient conditions for the existence of a Levi-Civita connection are the existence of a connection $\nabla \in L^2_{loc}(\Omega, \nu_g)$ and that $g^{ab} \in L^\infty_{loc}(\Omega)$. We say $\nabla \in L^2_{loc}(\Omega, \nu_g)$ if $\nabla_{X_a} Y^a \in L^2_{loc}(\Omega, \nu_g)$ for any pair X_a, Y_a of C^∞ vector fields. If the Christoffel symbols satisfy $\Gamma^a_{bc} \in L^2_{loc}(\Omega, \nu_g)$ then they define a Levi-Civita connection. For example, the Geroch-Traschen class of metrics satisfies the above conditions. It can be seen by direct inspection that the other inequalities require only that $g^{ab} \in L^\infty_{loc}(\Omega)$ and $g_{ab} \in L^\infty_{loc}(\Omega)$ which is also enough to maintain the results (5.110) and (5.111). All these conditions are satisfied by the Geometric Conditions 1.
For clarity, we state the result as a Lemma:

Lemma 2 *Let $\Omega \subset \mathbb{R}^4$ be an open set with compact closure such that there is a lens-shaped domain, $\Sigma_{(0,T)}$, in Ω satisfying the Geometric Conditions 1. Then for all $u \in C^\infty(\Omega)$ such that $\Box_g u = f$ with $f \in L^2(\Sigma_{(0,T)}, \nu_g)$ we have that*

$$(\tilde{\|}u\|^1_{\Sigma_t})^2 \leqslant C\left((\tilde{\|}u\|^1_{\Sigma_0})^2 + (\|f\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2\right) \tag{5.119}$$

for all $0 \leqslant t \leqslant T$

It is also natural to make $t = T$ as this is the time range in which we have shown the existence of a timelike vector field with bounded covariant derivatives.

5.2.4 Self-adjointness and existence

The next step is to establish that the formal adjoint of the operator $\Box_g$ in $L^2(\Sigma_{(0,T)}, \nu_g)$ is equal to $\Box_g$. Then we show the existence of a solution as in equation 5.8 in Definition 1.
Consider the $L^2(\Sigma_{(0,T)}, \nu_g)$ norm.

$$(\psi, \omega)_{L^2(\Sigma_{(0,T)},\nu_g)} = \int_{\Sigma_{(0,T)}} \psi\omega\nu_g \tag{5.120}$$

We shall single out two important subspaces of $H^1(\Sigma_{(0,T)}, \nu_g)$:

$$\begin{aligned} V_{\Sigma_0} &= \{\psi \in C^\infty(\Omega) \text{ s. t. } \psi|_{\Sigma_0} = n^a\psi_{,a}|_{\Sigma_0} = 0 \text{ and } \Box_g\psi \in L^2(\Sigma_{(0,T)}, \nu_g)\} \\ V_{\Sigma_T} &= \{\omega \in C^\infty(\Omega) \text{ s. t. } \omega|_{\Sigma_T} = n^a\omega_{,a}|_{\Sigma_T} = 0 \text{ and } \Box_g\omega \in L^2(\Sigma_{(0,T)}, \nu_g)\} \end{aligned}$$

Then the condition that g_{ab} is C^0 and that $Z_a = \omega\frac{\partial\psi}{\partial x^a} \in C^\infty(\Omega)$ is enough to apply Theorem 14 and obtain:

$$\int_{\Sigma_{(0,T)}} g^{ab}\psi_b\omega_a\nu_g + \int_{\Sigma_{(0,T)}} \omega\Box_g\psi\nu_g = 0 \tag{5.121}$$

$$\int_{\Sigma_{(0,T)}} g^{ab}\psi_b\omega_a\nu_g + \int_{\Sigma_{(0,T)}} \psi\Box_g\omega\nu_g = 0 \tag{5.122}$$

So combining the above equations give

$$\int_{\Sigma_{(0,T)}} \Box_g\psi\omega\nu_g = \int_{\Sigma_{(0,T)}} \psi\Box_g\omega\nu_g \tag{5.123}$$

and we obtain

$$(\Box_g\psi, \omega)_{L^2(\Sigma_{(0,T)},\nu_g)} = (\psi, \Box_g\omega)_{L^2(\Sigma_{(0,T)},\nu_g)} \tag{5.124}$$

which proves the symmetry of $\Box_g$ for $\psi \in V_{\Sigma_0}, \omega \in V_{\Sigma_T}$.

The proof of existence uses the Hahn-Banach Theorem and the Riesz Representation Theorem (see Appendix Theorem 20 and Theorem 21).
We define the functional

$$k_f(\Box_g\omega) = (f, \omega)_{L^2(\Sigma_{(0,T)},\nu_g)} \tag{5.125}$$

$$k_f : \Box V_{\Sigma_T} \to \mathbb{R} \tag{5.126}$$

We show, that the functional is bounded in order to apply Riesz's Theorem. In that way k_f defines an element $u \in L^2(\Sigma_{(0,T)}, \nu_g)$ such that $k_f(\Box_g\omega) = (u, \Box_g\omega)_{L^2(\Sigma_{(0,T)},\nu_g)}$. Now using the energy inequality given by Lemma 2 and Cauchy-Schwartz we obtain:

$$k(\Box_g\omega) = (f, \omega)_{L^2(\Sigma_{(0,T)},\nu_g)} \tag{5.127}$$

$$\leqslant (f, f)_{L^2(\Sigma_{(0,T)},\nu_g)}(\omega, \omega)_{L^2(\Sigma_{(0,T)},\nu_g)} \tag{5.128}$$

$$\leqslant c(\|f\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2(\|\Box_g\omega\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2 \tag{5.129}$$

The Hahn-Banach theorem allows us to extend the functional to the whole $L^2(\Sigma_{(0,T)}, \nu_g)$ without increasing the norm. Moreover, the estimate implies that k_f is bounded in $L^2(\Sigma_{(0,T)}, \nu_g)$.

Then using the Riesz's Theorem there is a $u \in L^2(\Sigma_{(0,T)}, \nu_g)$ such that

$$\int_{\Sigma_{(0,T)}} u \Box_g \omega \nu_g = \int_{\Sigma_{(0,T)}} f \omega \nu_g$$

for all $\omega \in V_{\Sigma_T} \supset C_0^\infty(\Sigma_{(0,T)})$.
Notice that we can extend u to an L^2 function on $(-\infty, T) \times \Sigma$ by setting $u|_{t<0} = 0$ and we can extend f in the same manner. This shows that u satisfies the requirements to be a weak solution.

5.2.5 Uniqueness and continuity with respect to the source function.

The proof of uniqueness follows directly from (5.11). We *assume* that there are two regular solutions u_1 and u_2 as in Definition 7. Therefore, the function $\tilde{u} = u_1 - u_2$ satisfies (5.11) with $f = 0$. This implies:

$$(\|\tilde{u}\|^1_{\Sigma_{(0,T)}})^2 \leqslant 0 \tag{5.130}$$

which implies $(\|\tilde{u}\|^1_{\Sigma_{(0,T)}})^2 = 0$ and hence $\tilde{u} = 0$.
Therefore we can conclude that

$$u_1 = u_2.$$

In a similar way, we prove the continuity of the solution with respect to the source function. We make the concept precise as follows. We say the solution is continuously stable in $H^1(\Sigma_{(0,T)}, \nu_g)$ with respect to the source functions in $L^2(\Sigma_{(0,T)}, \nu_g)$ if for every $\epsilon' > 0$ there is a δ depending on f such that if:

$$(\|f - \tilde{f}\|_{L^2(\Sigma_{(0,T)}, \nu_g)})^2 \leqslant \delta \tag{5.131}$$

for$f \in L_2(\Sigma_{(0,T)}, \nu_g)$ implies that

$$(\|u - \tilde{u}\|^1_{\Sigma_{(0,T)}})^2 \leqslant \epsilon' \tag{5.132}$$

where $\tilde{u}$ is a weak solution of $\Box_g \tilde{u} = \tilde{f}$.
Choosing

$$\delta = \frac{\epsilon}{K'T}$$

and then applying again the energy inequality (5.11) we have:

$$\begin{aligned} \left(\|u - \tilde{u}\|_{H^1(\Sigma_{(0,T)}, \nu_g)}\right)^2 &\leqslant K \int_0^T \left(\|u - \tilde{u}\|^1_{\Sigma_t}\right)^2 dt && (5.133) \\ &\leqslant K' \int_0^T (\|f - \tilde{f}\|_{L^2(\Sigma_{(0,T)}, \nu_g)})^2 dt && (5.134) \\ &\leqslant \epsilon && (5.135) \end{aligned}$$

5.2.6 H^1 Regularity

In the previous section we assume the solution we construct satisfied the energy inequality (5.11) and the regularity condition that $u \in H^1(\Sigma_{(0,T)}, \nu_g)$. In this section we prove both assumptions, however we will need an extra condition on the regularity of the metric which we state below

Geometric Conditions 2 *Let $(\Sigma_{(0,T)}, g)$ be a lens-shaped domain satisfying:*

1. *The components g_{ab} and g^{ab} are $C^{1,1}$.*
2. *The inverse of the metric tensor satisfies*

$$g^{ab}(\mathbf{x}) - g^{ab}(\mathbf{x}') \leqslant C(d_E(\mathbf{x}, \mathbf{x}'))^2$$

where C is a constant and $(d_E(\mathbf{x}, \mathbf{x}'))^2 = \left((t-t')^2 + \sum_{i=1}^{3} |x^i - x^{i'}|^2\right)$
with $\mathbf{x} = (t, x^1, x^2, x^3)$ and $\mathbf{x}' = (t', x^{1'}, x^{2'}, x^{3'})$

This regularity condition is strong enough to guarantee the existence of a timelike vector field with covariant derivative without any problem. Nevertheless, we did not assume this before because to show the existence of L^2 solution it is not needed. This is precisely the content of Clarke's paper [9]. The requirement of H^1-regularity of the solution is not contained in the paper and therefore we will make this extra assumption in order to show the solution has this regularity.
The main idea will be to regularise the solution u using a strict delta net $\{\rho_n\}$ and then show that there are good estimates for the commutator $[\rho_n, \Box_g]u$.
The strict delta net we will use is such that the mollification $\rho_n * u = u_n$ is a smooth function with compact support vanishing at $t \leqslant 0$ and therefore Lemma 2 applies which gives

$$(\tilde{\|}u_n\|^1_{\Sigma_t})^2 \leqslant C(\|f_n\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2 \tag{5.136}$$

for all $0 \leqslant t \leqslant T$.
Now notice that

$$f_n = \Box_g u_n = \rho_n * f + [\Box_g, \rho_n *]u \tag{5.137}$$

by the properties of the strict delta net we have that $(\|\rho_n * f\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2$ converge to $(\|f\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2$. Therefore, the key point is to show that $[\Box_g, \rho_n *]u$ is bounded in $L^2(\Sigma_{(0,T)}, \nu_g)$. We do that in the following lemma.

Lemma 3 *If $u \in L^2(\Sigma_{(0,T)}, \nu_g)$, then the commutator $[\Box_g, \rho_n *]u$ is bounded in $L^2(\Sigma_{(0,T)}, \nu_g)$.*

Proof.
We start by doing the following direct calculation in which for the moment we use $\mathbf{x} := (t, x^i)$ and $\mathbf{y} := (t, y^i)$ for convenience. For the moment, we assume $u \in C_0^\infty(\Sigma_{(0,T)})$ then

$$\begin{aligned}
[\Box_g, \rho_n *]u &= (g^{ab}\rho_n * u_{ab} - \Gamma^{ab}_{\ a}\rho_n * u_b) - \rho_n * (g^{ab}u_{ab} - \Gamma^{ab}_{\ a}u_b) \\
&= \Big(g^{ab}\rho_n * u_{ab} - \rho_n * g^{ab}u_{ab}\Big) - \Big(\Gamma^{ab}_{\ a}\rho_n * u_b - \rho_n * \Gamma^{ab}_{\ a}u_b\Big)
\end{aligned}$$

Now we will analyse both terms.

$$\begin{aligned}
&\Big(g^{ab}\rho_n * u_{ab} - \rho_n * g^{ab}u_{ab}\Big) \\
&\qquad = \int_{\mathbb{R}^4} \rho_n(\mathbf{x}-\mathbf{y}) \Big[g^{ab}(\mathbf{x}) - g^{ab}(\mathbf{y})\Big] \partial_a\partial_b u(\mathbf{x}) dx^4 \\
&\qquad = \int_{\mathbb{R}^4} \partial_a\partial_b\rho_n(\mathbf{x}-\mathbf{y}) \Big[g^{ab}(\mathbf{x}) - g^{ab}(\mathbf{y})\Big] u(\mathbf{x}) dx^4 \\
&\qquad + 2\int_{\mathbb{R}^4} \partial_a\rho_n(\mathbf{x}-\mathbf{y})\partial_b \Big[g^{ab}(\mathbf{x}) - g^{ab}(\mathbf{y})\Big] u(\mathbf{x}) dx^4 \\
&\qquad + \int_{\mathbb{R}^4} \rho_n(\mathbf{x}-\mathbf{y})\partial_a\partial_b \Big[g^{ab}(\mathbf{x}) - g^{ab}(\mathbf{y})\Big] u(\mathbf{x}) dx^4
\end{aligned}$$

Now the first term gives

$$\int_{\mathbb{R}^4} \partial_a\partial_b\rho_n(\mathbf{x}-\mathbf{y}) \Big[g^{ab}(\mathbf{x}) - g^{ab}(\mathbf{y})\Big] u(\mathbf{x}) dx^4 \tag{5.138}$$

$$\leqslant K\int_{\mathbb{R}^4} \partial_a\partial_b\rho_n(\mathbf{x}-\mathbf{y}) \left|\mathbf{x}-\mathbf{y}\right|^2 u(\mathbf{x}) dx^4 \tag{5.139}$$

$$= K' \left[(|\mathbf{x}|^2\partial_a\partial_b\rho_n(\mathbf{x})) * u\right] \tag{5.140}$$

which implies that

$$\left\| \int_{\mathbb{R}^4} \partial_a\partial_b\rho_n(\mathbf{x}-\mathbf{y}) \Big[g^{ab}(\mathbf{x}) - g^{ab}(\mathbf{y})\Big] u(\mathbf{x}) dx^4 \right\|_{L^2(\Sigma_{(0,T)})} \tag{5.141}$$

$$\leqslant K \left\| \Big[|\mathbf{x}|^2\partial_a\partial_b\rho_n(\mathbf{x}) * u\Big] \right\|_{L^2(\Sigma_{(0,T)})} \tag{5.142}$$

$$\leqslant K' \left\| |\mathbf{x}|^2\partial_a\partial_b\rho_n(\mathbf{x}) \right\|_{L^1(\Sigma_{(0,T)})} \|u\|_{L^2(\Sigma_{(0,T)})} \tag{5.143}$$

where we have used Young's inequality and the first condition given by the Geometric Conditions 2.
Notice that

$$\begin{aligned}
\left\| |\mathbf{x}|^2\partial_a\partial_b\rho_n(\mathbf{x}) \right\|_{L^1(\Sigma_{(0,T)})} &= \frac{1}{\epsilon^6}\int_{\Sigma_{(0,T)}} |\mathbf{x}|^2\partial_a\partial_b\rho(\frac{\mathbf{x}}{\epsilon}) dx^4 \\
&= \int_{\Sigma_{(0,T)}} |\mathbf{y}|^2\partial_a\partial_b\rho(\mathbf{y}) dx^4
\end{aligned}$$

which shows that $\left\| |\mathbf{x}|^2\partial_a\partial_b\rho_n(\mathbf{x}) \right\|_{L^1(\Sigma_{(0,T)})}$ is independent of ϵ.
so we can conclude

$$\left\| \int_{\mathbb{R}^4} \partial_a\partial_b\rho_n(\mathbf{x}-\mathbf{y}) \Big[g^{ab}(\mathbf{x}) - g^{ab}(\mathbf{y})\Big] u(\mathbf{x}) dx^4 \right\|_{L^2(\Sigma_{(0,T)})} \tag{5.144}$$

$$\leqslant K \|u\|_{L^2(\Sigma_{(0,T)})} \tag{5.145}$$

We also have that

$$\int_{\mathbb{R}^4} \partial_a \rho_n(\mathbf{x}-\mathbf{y}) \partial_b \left[g^{ab}(\mathbf{x}) - g^{ab}(\mathbf{y}) \right] u(\mathbf{x}) dx^4 \tag{5.146}$$

$$\leqslant K \int_{\mathbb{R}^4} \partial_a \rho_n(\mathbf{x}-\mathbf{y}) \left|\mathbf{x}-\mathbf{y}\right| u(\mathbf{x}) dx^4 \tag{5.147}$$

$$= K' \left[|\mathbf{x}| \partial_a \rho_n(\mathbf{x}) * u \right] \tag{5.148}$$

which implies that

$$\left\| \int_{\mathbb{R}^4} \partial_a \rho_n(\mathbf{x}-\mathbf{y}) \partial_b \left[g^{ab}(\mathbf{x}) - g^{ab}(\mathbf{y}) \right] u(\mathbf{x}) dx^4 \right\|_{L^2(\Sigma_{(0,T)})} \tag{5.149}$$

$$\leqslant K \left\| \left[|\mathbf{x}| \partial_a \partial_b \rho_n(\mathbf{x}) * u \right] \right\|_{L^2(\Sigma_{(0,T)})} \tag{5.150}$$

$$\leqslant K' \left\| |\mathbf{x}| \partial_a \partial_b \rho_n(\mathbf{x}) \right\|_{L^1(\Sigma_{(0,T)})} \left\| u \right\|_{L^2(\Sigma_{(0,T)})} \tag{5.151}$$

where we have used Young's inequality and the Lipschitz condition given in the Geometric Conditions 2.
Notice that

$$\begin{aligned} \left\| |\mathbf{x}| \partial_a \partial_b \rho_n(\mathbf{x}) \right\|_{L^1(\Sigma_{(0,T)})} &= \frac{1}{\epsilon^5} \int_{\Sigma_{(0,T)}} |\mathbf{x}| \partial_a \partial_b \rho(\frac{\mathbf{x}}{\epsilon}) dx^4 \\ &= \int_{\Sigma_{(0,T)}} |\mathbf{y}| \partial_a \partial_b \rho(\mathbf{y}) dx^4 \end{aligned}$$

which shows that $\left\| |\mathbf{x}| \partial_a \partial_b \rho_n(\mathbf{x}) \right\|_{L^1(\Sigma_{(0,T)})}$ is independent of ϵ.
so we can conclude

$$\left\| \int_{\mathbb{R}^4} \partial_a \rho_n(\mathbf{x}-\mathbf{y}) \partial_b \left[g^{ab}(\mathbf{x}) - g^{ab}(\mathbf{y}) \right] u(\mathbf{x}) dx^4 \right\|_{L^2(\Sigma_{(0,T)})} \tag{5.152}$$

$$\leqslant K'' \left\| u \right\|_{L^2(\Sigma_{(0,T)})} \tag{5.153}$$

The last term gives

$$\int_{\mathbb{R}^4} \rho_n(\mathbf{x}-\mathbf{y}) \partial_a \partial_b \left[g^{ab}(\mathbf{x}) - g^{ab}(\mathbf{y}) \right] u(\mathbf{x}) dx^4 \tag{5.154}$$

$$\leqslant K \int_{\mathbb{R}^4} \rho_n(\mathbf{x}-\mathbf{y}) u(\mathbf{x}) dx^4 \tag{5.155}$$

$$\leqslant K''' \left\| u \right\|_{L^2(\Sigma_{(0,T)})} \tag{5.156}$$

where we have used Young's inequality and the Lipschitz condition given in the Geometric Conditions 2.
Finally, using (5.144), (5.152) and (5.154) we have that

$$\left\| \left(g^{ab} \rho_n * u_{ab} - \rho_n * g^{ab} u_{ab} \right) \right\|_{L^2(\Sigma_{(0,T)})} \leqslant K \left\| u \right\|_{L^2(\Sigma_{(0,T)})} \tag{5.157}$$

Now we analyse the second term

$$\begin{aligned}
&\left(\Gamma_a^{ab}\rho_n * u_b - \rho_n * \Gamma_a^{ab}u_b\right) \\
&\qquad = \int_{\mathbb{R}^4} \rho_n(\mathbf{x}-\mathbf{y}) \left[\Gamma_a^{ab}(\mathbf{y}) - \Gamma_a^{ab}(\mathbf{x})\right] \partial_b u(\mathbf{x}) dx^4 \\
&\qquad = \int_{\mathbb{R}^4} \partial_b \rho_n(\mathbf{x}-\mathbf{y}) \left[\Gamma_a^{ab}(\mathbf{y}) - \Gamma_a^{ab}(\mathbf{x})\right] u(\mathbf{x}) dx^4 \\
&\qquad + \int_{\mathbb{R}^4} \rho_n(\mathbf{x}-\mathbf{y}) \partial_b \left[\Gamma_a^{ab}(\mathbf{y}) - \Gamma_a^{ab}(\mathbf{x})\right] u(\mathbf{x}) dx^4
\end{aligned}$$

The first term can be bounded in an analogous manner as in (5.152) while the second term can be bounded as in (5.154). Putting together this results give

$$\left\| \left(\Gamma_a^{ab}\rho_n * u_b - \rho_n * \Gamma_a^{ab}u_b\right) \right\|_{L^2(\Sigma_{(0,T)})} \leqslant K' \|u\|_{L^2(\Sigma_{(0,T)})} \tag{5.158}$$

We can conclude then that

$$\|[\Box_g, \rho_n]u\|_{L^2(\Sigma_{(0,T)})} \leqslant C \|u\|_{L^2(\Sigma_{(0,T)})} \tag{5.159}$$

for all $u \in C_0^\infty(\Sigma_{(0,T)})$.
Therefore the linear functional $F(u) = [\Box_g, \rho_n]u$ is bounded and densely defined in $L^2(\Sigma_{(0,T)})$ with a unique extension to $L^2(\Sigma_{(0,T)})$.
This concludes the proof. ⊡
Therefore using Lemma 3 we can prove now the regularity of the solution as in definition 7.
We begin by noticing that

$$\begin{aligned}
(\|u_n\|_{H^1(\Sigma_{(0,T)},\nu_g)})^2 &\leqslant (\tilde{\|}u_n\|^1_{\Sigma_t})^2 \\
&\leqslant (\|f_n\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2 \\
&= (\|\rho_n * f + [\Box_g, \rho_n]u\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2 \\
&\leqslant \tilde{C}\left(\left(\|f\|_{L^2(\Sigma_{(0,T)},\nu_g)}\right)^2 + \left(\|u\|_{L^2(\Sigma_{(0,T)},\nu_g)}\right)^2\right)
\end{aligned}$$

for all $0 \leqslant t \leqslant T$
Now using Banach-Alaoglu (Theorem 22 in the Appendix), we know there is a subsequence that converge weakly to an element $\tilde{u} \in H^1(\Sigma_{(0,T)})$ and therefore also converge weakly in $L^2(\Sigma_{(0,T)})$. However we know u_n converges strongly to $u \in L^2(\Sigma_{(0,T)})$ and therefore weakly. By uniqueness of limits we have that $\tilde{u} = u$. This proves $u \in H^1(\Sigma_{(0,T)})$
Moreover using that the norm is weakly semi continuous we have that

$$\begin{aligned}
(\|u\|_{H^1(\Sigma_{(0,T)},\nu_g)})^2 &\leqslant \lim\inf_{n\to\infty} (\|u\|_{H^1(\Sigma_{(0,T)},\nu_g)})^2 \\
&\leqslant (\tilde{\|}u_n\|^1_{\Sigma_t})^2 \\
&\leqslant (\|f_n\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2 \\
&= (\|\rho_n * f + [\Box_g, \rho_n]u\|_{L^2(\Sigma_{(0,T)},\nu_g)})^2 \\
&\leqslant \tilde{C}\left(\left(\|f\|_{L^2(\Sigma_{(0,T)},\nu_g)}\right)^2 + \left(\|u\|_{L^2(\Sigma_{(0,T)},\nu_g)}\right)^2\right)
\end{aligned}$$

Using that $u, f \in L^2(\Sigma_{(0,T)}, \nu_g)$ we can conclude that

$$\left(\|u\|_{H^1(\Sigma_{(0,T)},\nu_g)}\right)^2 \quad < \quad \infty$$

5.3 Applications

The techniques used to prove the well posedness and the regularity of the solution requires a level of regularity given by the Geometric Conditions 1 and 2. These conditions are very restrictive and would be relaxed significantly in the following chapters. However, for completeness we mention that $C^{1,1}$ metrics of finite regularity can be constructed to satisfy the Geometric Conditions (2). For example in a manifold M we construct the metric g^{ab}:

$$g^{ab} = \eta^{ab} + \xi_1 h^{ab} \tag{5.160}$$

where η^{ab} is the Minkowski metric, h^{ab} is a $C^{1,1}$ perturbation metric where $h^{ab} \sim C\left((t-t')^{\alpha} + |x-x'|^{\alpha}\right)$ with $\alpha > 2$ and ξ_1 is a positive smooth function equal to 1 in the domain $A = \{(t,x)|0 < t < \frac{1}{2}, 0 < |x'-x| < \frac{1}{2}\}$; 0 in the domain $B = \{(t,x)|t > 1, |x'-x| > 1\}$ and smoothly varying from 0 to 1 in $M\backslash(A \cup B)$.

Chapter 6

$C^{0,1}$ Spacetimes

6.1 The main results

The interest in co-dimension one singular submanifolds covers a variety of different interesting physical phenomena such as surface layers [42], impulsive gravitational waves [44], and shell-crossing singularities [59] all of which fall outside the class of smooth globally hyperbolic spacetimes. Moreover, the mathematical analysis by Geroch and Traschen [55] of what is now called the class of Geroch-Traschen metrics and the subsequent analysis by Steinbauer and Vickers [60] using generalised functions gives co-dimension one singular submanifolds a robust mathematical background. In addition, recent proposals in semi-classical gravity and quantum gravity [61] suggest that the metric near the event horizon must present some loss of regularity. In this section, we present techniques to prove local well-posedness of the wave equation in spacetimes with co-dimension one singularities subject to certain conditions on the metric.
The plan of this chapter is as follows. In the first part, we introduce the general setting for the problem and state the main theorems. An important point is that we write the wave equation as a first order system (see e.g. [62]). This enables us to work with the L^2 energy of the first order system which corresponds to an H^1 energy of the second order system. In this setting we establish local well-posedness for general first order linear symmetric hyperbolic systems with coefficients with low regularity. We show that unique stable solutions exist in $L^2(0,T;L^2(\Sigma,\mathbb{R}^N))$. In the second order formalism this solution corresponds to a finite energy solution in H^1 of the wave equation. Moreover, the main advantage in writing the problem as a first order system is that the existence of a covariantly constant timelike vector field and the condition on the curvature are not needed, which have been key conditions in previous works [56]. Therefore, the new results obtained extend previous results of Vickers and Wilson [58] and Ishibashi and Hosoya [50] by allowing a larger class of non-vacuum time dependent spacetimes. We also establish not only the existence and uniqueness of solutions but also their stability and local well-posedness. The second part of this chapter contains the proofs of the theorems and several applications such as a discontinuity across a hypersurface, impulsive gravitational waves and brane-world cosmologies.

6.1.1 The general setting

The geometric setting considered is a region $\Sigma_{[0,T]} = [0,T] \times \Sigma$, where Σ is either a compact closed n-dimensional manifold or an open, bounded set of $\mathbb{R}^n$ with smooth boundary $\partial\Sigma$. In what follows, for simplicity, we will only consider the former case. The proof in the latter case follows by replacing $H^1(\Sigma)$ by $H^1_0(\Sigma)$ and using the volume form given by dx^n.

Rather than considering the particular case of a spacetime with a singular hypersurface, where the regularity of the metric drops below C^2, we will consider solutions to the wave equation on a rough spacetime, where the spacetime metric g_{ab} is only Lipschitz. We will show that in this situation one has well-posedness (in a sense made precise below) of the wave equation with weak solutions of regularity $H^1(\Sigma_{[0,T]})$. In order to do this, we will reformulate the wave equation as a first order symmetric hyperbolic system and look for L^2 solutions of this system.
We therefore start by considering the first order initial value problem

$$L\mathbf{u} = A^0\partial_t\mathbf{u} + A^i\partial_i\mathbf{u} + B\mathbf{u} = \mathbf{F} \tag{6.1}$$
$$\mathbf{u}(0,\cdot) = \mathbf{u}_0(\cdot) \tag{6.2}$$

In the above we have employed the Einstein summation convention where i, j, k, etc. range over $1, 2, \ldots, n$. The unknown $\mathbf{u}$ and the source term $\mathbf{F}$ are both $\mathbb{R}^N$ valued functions on $\Sigma_{[0,T]}$, while A^0, A^i and B are $N \times N$ matrix valued functions on $\Sigma_{[0,T]}$. We will assume that A^0 and A^i are symmetric and that in addition A^0 is positive definite.
In order for such a system to correspond to the wave equation given by a Lipschitz metric, we will require that A^0 and A^i have bounded first derivatives and that B is bounded, so that we require

$$A^0 \in W^{1,\infty}(\Sigma_{[0,T]}, \mathbb{R}^{N^2}), \quad A^i \in W^{1,\infty}(\Sigma_{[0,T]}, \mathbb{R}^{N^2}), \quad B \in L^\infty(\Sigma_{[0,T]}, \mathbb{R}^{N^2}).$$

In the analysis below we will be working in spaces such as $L^2(\Sigma)$. Rather than defining this in terms of a particular coordinate system on Σ, we will introduce a background Riemannian metric h_{ij} on Σ and let ν_h be the corresponding volume form. We then define $L^2(\Sigma)$ to be the space of real valued functions g on Σ such that $\int_\Sigma g^2\nu_h < \infty$ and we denote the associated inner product by $(f,g)_{L^2(\Sigma)} = \int_\Sigma fg\nu_h$. Note that since Σ is compact ν_h is bounded from below and above. Furthermore if Σ is parallelizable (which for simplicity we will assume) there is no loss of generality in taking h_{ij} to be the flat metric. Note that in the three dimensional case, which is most relevant to applications in general relativity, it is enough for Σ to be orientable for it to be parallelizable.
For the case of vector valued functions $\mathbf{v}$ on Σ, we define $L^2(\Sigma, \mathbb{R}^N)$ to be those $\mathbf{v}$ such that $\int_\Sigma \mathbf{v}{\cdot}\mathbf{v}\nu_h < \infty$. The corresponding inner product on $L^2(\Sigma, \mathbb{R}^N)$ is then given by

$$(\mathbf{v}, \mathbf{w})_{L^2(\Sigma,\mathbb{R}^N)} = \int_\Sigma \mathbf{v}{\cdot}\mathbf{w}\nu_h$$

Where there is no risk of confusion, we will write both the inner product in $L^2(\Sigma)$ and in $L^2(\Sigma, \mathbb{R}^N)$ simply as $(\cdot,\cdot)_{L^2}$ The Sobolev spaces $H^1(\Sigma, \mathbb{R}^N)$ etc. are defined in a similar manner (see[63] §5.2, [62]).
We also make use of the function space $L^2(\Sigma_{[0,T]})$ which is defined by requiring that functions are square integrable on $[0,T] \times \Sigma$ with respect to the volume form $dt \wedge \nu_h$. However in the analysis below it is often convenient to think of a function $\mathbf{v}(t,x)$ as a

map from $[0,T]$ to a function $\mathbf{v}(t)(\cdot)$ of $x \in \Sigma$ given by $\mathbf{v}(t)(x) = \mathbf{v}(t,x)$. For example $L^2(0,T;L^2(\Sigma,\mathbb{R}^N))$ is the space of functions

$$\mathbf{v} : [0,T] \rightarrow L^2(\Sigma,\mathbb{R}^N) \tag{6.3}$$

$$t \mapsto \mathbf{v}(t) \tag{6.4}$$

such that $\mathbf{v}(t) \in L^2(\Sigma,\mathbb{R}^N)$ and

$$\int_0^T (\mathbf{v},\mathbf{v})_{L^2(\Sigma,\mathbb{R}^N)} dt < \infty \tag{6.5}$$

When thinking of $\mathbf{v}$ in this way, we will denote the time derivative by $\dot{\mathbf{v}}$.

6.1.2 Weak solutions and the main theorems

We will be looking for weak solutions of the initial value problem (6.1). To motivate the definition we proceed as follows. Given a standard C^1 solution $\mathbf{u}$ of the initial value problem, we may first take the dot product of equation (6.1) with a smooth $\mathbb{R}^N$-valued function $\mathbf{v}$ with support in $[0,T) \times \Sigma$ and then integrate over x and t to obtain

$$\int_0^T (L\mathbf{u},\mathbf{v})_{L^2(\Sigma,\mathbb{R}^N)} dt = \int_0^T (\mathbf{F},\mathbf{v})_{L^2(\Sigma,\mathbb{R}^N)} dt \tag{6.6}$$

Integrating the left hand side by parts with respect to x and t we obtain

$$\int_0^T (\mathbf{u},L^*\mathbf{v})_{L^2(\Sigma,\mathbb{R}^N)} dt - \Big(A^0\mathbf{u}|_{t=0},\mathbf{v}(0)\Big)_{L^2(\Sigma,\mathbb{R}^N)} = \int_0^T (\mathbf{F},\mathbf{v})_{L^2(\Sigma,\mathbb{R}^N)} dt \tag{6.7}$$

where L^* is the formal adjoint of L defined below and the second term on the left hand side (LHS) comes from the $t = 0$ boundary term when we integrate by parts with respect to t. This approach results in the following definition:

Definition 3 **(Weak Solution)** *We say a function:*

$$\mathbf{u} \in L^2(0,T;L^2(\Sigma,\mathbb{R}^N))$$

is a local weak solution of the initial value problem (6.1) provided that: For all $\mathbf{v} \in C^\infty(\Sigma_{[0,T]},\mathbb{R}^N)$*, with supp*$(\mathbf{v}) \subseteq [0,T) \times \Sigma$

$$\int_0^T (\mathbf{u},L^*\mathbf{v})_{L^2(\Sigma,\mathbb{R}^N)} dt = \int_0^T (\mathbf{F},\mathbf{v})_{L^2(\Sigma,\mathbb{R}^N)} dt + \Big(A^0(0)\mathbf{u}_0,\mathbf{v}(0)\Big)_{L^2(\Sigma,\mathbb{R}^N)}. \tag{6.8}$$

This definition of a weak solution is the classical one used by Friedrichs [64] but differs from the one used by Evans [63], who does not integrate with respect to t. Note also that the formal adjoint is defined with respect to ν_h. So in the case where we use a general Riemannian metric, there are additional terms involving the derivatives of ν_h in the expression for L^* compared to the flat case. The explicit expression is:

$$L^*\mathbf{w} =: -\partial_t(A^0\mathbf{w}) - \partial_i(A^i\mathbf{w}) + B^T\mathbf{w} - \tilde{\Gamma}^i_{li}A^l\mathbf{w} \tag{6.9}$$

where $\tilde{\Gamma}^i_{jk}$ are the connection coefficients of the smooth Riemannian metric h_{ij}.
In order to prove uniqueness and well-posedness of the initial value problem, we will need to control the L^2 size of the solution. This motivates the following definition.

Definition 4 *(***Regular Weak Solution***) We say a weak solution* $\mathbf{u} \in L^2(0,T;L^2(\Sigma,\mathbb{R}^N))$ *is* **regular** *if* $\mathbf{u}$ *satisfies the energy estimate*

$$||\mathbf{u}||^2_{L^2(0,T;L^2(\Sigma,\mathbb{R}^N))} \leqslant C\left(||\mathbf{u}_0||_{L^2(\Sigma,\mathbb{R}^N)} + \int_0^T ||\mathbf{F}(t,\cdot)||^2_{L^2(\Sigma,\mathbb{R}^N)}dt\right) \tag{6.10}$$

We may now state our main result concerning solutions of low-regularity symmetric hyperbolic systems.

Theorem 4 *Given the linear symmetric hyperbolic system:*

$$L\mathbf{v} = A^0\partial_t\mathbf{u} + A^i\partial_i\mathbf{u} + B\mathbf{u} = \mathbf{F} \tag{6.11}$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0(\cdot) \tag{6.12}$$

where A^0, A^i, B *and* $\mathbf{F}$ *are as above, and the initial data* $\mathbf{u}_0$ *is in* $L^2(\Sigma,\mathbb{R}^N)$*. Then there exists a unique regular weak solution* $\mathbf{u} \in L^2(0,T;L^2(\Sigma,\mathbb{R}^N))$*. Furthermore this solution is stable in the sense that the solution depends continuously on the norm of the initial data in* $L^2(\Sigma,\mathbb{R}^N)$ *and the norm of the source function in* $L^2(\Sigma_{[0,T]},\mathbb{R}^N)$*.*

We may now use the above result to establish the following theorem for the wave equation.

Theorem 5 *Let* g_{ab}, g^{ab} *be in* $C^{0,1}$*, and* f *in* $L^2(\Sigma_{[0,T]})$*. Given initial data* $(u_0,u_1) \in H^1(\Sigma)\times L^2(\Sigma)$ *then the system*

$$\Box_g u + m^2 u = f \tag{6.13}$$

$$u(0,\cdot) = u_0 \tag{6.14}$$

$$\partial_t u(0,\cdot) = u_1 \tag{6.15}$$

has a unique stable solution $u \in H^1(\Sigma_{[0,T]})$*. Moreover, the corresponding energy-momentum tensor* $T_{ab}[u]$ *is in* $L^1_{loc}(\Sigma_{[0,T]})$*.*

Note that the above result is very similar to the one obtained for the homogeneous wave equation in [65]. We have extended these results to the inhomogeneous case and to include mixed terms of the form $\frac{\partial^2}{\partial t\partial x}$.

6.2 Proof of the main theorem

6.2.1 Outline of the proof

The proof of Theorem 5 uses the *vanishing viscosity method* described in §7.3 of Evans [63]. Note however that Evans assumes that the A^0 and A^i have greater regularity than we do and as a result is able to obtain a solution with greater regularity. This explains why our definition of a weak solution has to differ from his. However the essence of the proof is essentially the same. It consists of the following steps:

1. First, we approximate the problem (6.1) by the system of parabolic initial value-problem on $\Sigma_{[0,T]}$ given by

$$\partial_t\mathbf{u}^\epsilon - \epsilon\Delta_h\mathbf{u}^\epsilon + (A^0)^{-1}A^i\partial_i\mathbf{u}^\epsilon + (A^0)^{-1}B\mathbf{u}^\epsilon = (A^0)^{-1}\mathbf{F} \tag{6.16}$$

$$\mathbf{u}^\epsilon(0,x) = \rho^\epsilon * (\mathbf{u}_0(x))$$

where $\{(\rho^\epsilon)\} \in (0,1]$ is a family of mollifiers. Here Δ_h is the Laplace-Beltrami operator on Σ associated with the smooth background Riemannian metric h_{ij}. By adding in the second order Laplace-Beltrami terms we obtain a system with smooth principal symbol. We may then use classical methods of parabolic regularity theory to obtain a solution with better analytic properties than the original hyperbolic system.

2. Second, we obtain the following uniform energy estimate
$$||\mathbf{u}^\epsilon||^2_{L^2(0,T;L^2(\Sigma,\mathbb{R}^N))} \leqslant C\left(||\mathbf{u}_0||_{L^2(\Sigma,\mathbb{R}^N)} + \int_0^T ||\mathbf{F}(t,\cdot)||_{L^2(\Sigma,\mathbb{R}^N)}dt\right). \tag{6.17}$$
where C is independent of ϵ.

3. Third, we take the limit $\epsilon \to 0$ and show convergence in an appropriate weak sense to a regular weak solution as defined above.

4. Fourth, using the energy inequality (6.17) we show uniqueness and stability. This concludes the proof of Theorem 4.

5. Fifth, we rewrite the wave equation as a symmetric hyperbolic problem and show that for a Lipschitz metric the corresponding L satisfies the conditions of Theorem 4.

6. Sixth, we show that the solution of the wave equation obtained via the symmetric hyperbolic problem is in $H^1(\Sigma_{[0,T]})$. This concludes the proof of Theorem 5.

6.2.2 Approximate solutions and energy estimate

The results we obtain make extensive use of the vanishing viscosity method. As explained above, the first step is to show that there exist suitable solutions to (6.16). This step follows directly from the work of Evans ([63], Th. 1 §7.3).

Proposition 2 (Existence of Approximate solutions) *For each $\epsilon > 0$, there exists a unique solution $\mathbf{u}^\epsilon$ of (6.16) with $\mathbf{u}^\epsilon \in L^2(0,T;H^2(\Sigma,\mathbb{R}^N))$ and $\dot{\mathbf{u}}^\epsilon \in L^2(0,T;L^2(\Sigma,\mathbb{R}^N))$.*

Proof. This is a variant of a standard result for parabolic systems. Following Evans set $X = L^\infty(0,T;H^1(\Sigma,\mathbb{R}^N))$ and then for each $\mathbf{w} \in X$, consider the linear system
$$\partial_t\mathbf{u}^\epsilon - \epsilon\Delta_h\mathbf{u}^\epsilon = -(A^0)^{-1}A^i\partial_i\mathbf{w} - (A^0)^{-1}B\mathbf{w} + (A^0)^{-1}\mathbf{F} \tag{6.18}$$
$$\mathbf{u}^\epsilon(0,x) = \mathbf{u}_0^\epsilon(x) \tag{6.19}$$
where $\mathbf{u}_0^\epsilon(x) = \rho^\epsilon(x) * (\mathbf{u}_0(x))$. Notice that the system is formed of N scalar parabolic equations of the form $\partial_t v - \epsilon\Delta_h v = f$. The coefficients are now all smooth and the only loss of regularity comes from the source term on the RHS of (6.18). However as this is bounded in $L^2(0,T;L^2(\Sigma,\mathbb{R}^N))$, we can apply standard results (see e.g. [63] Th. 5 §7.1) to show that there is a unique solution $\mathbf{u}^\epsilon \in L^2(0,T;H^2(\Sigma,\mathbb{R}^N))$ with $\dot{\mathbf{u}}^\epsilon \in L^2(0,T;L^2(\Sigma,\mathbb{R}^N))$.
In the same manner we can choose $\tilde{\mathbf{w}} \in X$ and find $\tilde{\mathbf{u}}^\epsilon$ that solves
$$\partial_t\tilde{\mathbf{u}}^\epsilon - \epsilon\Delta_h\tilde{\mathbf{u}}^\epsilon = -(A^0)^{-1}A^i\partial_i\tilde{\mathbf{w}} - (A^0)^{-1}B\tilde{\mathbf{w}} + (A^0)^{-1}\mathbf{F} \tag{6.20}$$
$$\tilde{\mathbf{u}}^\epsilon(0,x) = \mathbf{u}_0^\epsilon(x) \tag{6.21}$$

Subtracting $\mathbf{u}^\epsilon - \tilde{\mathbf{u}}^\epsilon$, we find $\bar{\mathbf{u}}^\epsilon = \mathbf{u}^\epsilon - \tilde{\mathbf{u}}^\epsilon$ solves

$$\begin{aligned} \partial_t \bar{\mathbf{u}}^\epsilon - \epsilon\Delta_h \bar{\mathbf{u}}^\epsilon &= -(A^0)^{-1}A^i\partial_i\bar{\mathbf{w}} - (A^0)^{-1}B\bar{\mathbf{w}} && (6.22)\\ \bar{\mathbf{u}}^\epsilon(0,x) &= 0 && (6.23) \end{aligned}$$

where $\bar{\mathbf{w}} = \mathbf{w} - \tilde{\mathbf{w}}$. Using standard energy estimates for solutions of parabolic equations we have that $\bar{\mathbf{u}}$ satisfies:

$$\begin{aligned} &\sup_{0\leqslant t\leqslant T} ||\bar{\mathbf{u}}^\epsilon(t)||^2_{H^1(\Sigma,\mathbb{R}^N)} \\ &\leqslant C(T,\epsilon)\left(||(A^0)^{-1}(A^i\partial_i\bar{\mathbf{w}} + B\bar{\mathbf{w}})||^2_{L^2(0,T;L^2(\Sigma,\mathbb{R}^N))}\right) \\ &\leqslant C(T,\epsilon)\left(\sup_{0\leqslant t\leqslant T}||\bar{\mathbf{w}}(t)||^2_{H^1(\Sigma,\mathbb{R}^N)}\right) \end{aligned} \tag{6.24}$$

Thus

$$||\bar{\mathbf{u}}^\epsilon||_{L^\infty(0,T;\Sigma_{[0,T]}(\Sigma,\mathbb{R}^N))} \leqslant C(T,\epsilon)||\bar{\mathbf{w}}||_{L^\infty(0,T;(\Sigma,\mathbb{R}^N))} \tag{6.25}$$

Therefore, if T is small enough such that $C(T,\epsilon) \leqslant \frac{1}{2}$ we obtain that

$$||\bar{\mathbf{u}}^\epsilon||_{L^\infty(0,T;\Sigma_{[0,T]}(\Sigma,\mathbb{R}^N))} \leqslant \frac{1}{2}||\bar{\mathbf{w}}||_{L^\infty(0,T;(\Sigma,\mathbb{R}^N))} \tag{6.26}$$

so that

$$||\mathbf{u}^\epsilon - \tilde{\mathbf{u}}^\epsilon||_{L^\infty(0,T;\Sigma_{[0,T]}(\Sigma,\mathbb{R}^N))} \leqslant \frac{1}{2}||\mathbf{w} - \tilde{\mathbf{w}}||_{L^\infty(0,T;(\Sigma,\mathbb{R}^N))} \tag{6.27}$$

This implies that we have a contraction mapping and the hypothesis of Banach's fixed point theorem is satisfied for the mapping

$$\begin{aligned} M : L^\infty(0,T;(\Sigma,\mathbb{R}^N)) &\to L^\infty(0,T;(\Sigma,\mathbb{R}^N)) \\ \mathbf{w} &\mapsto \mathbf{u}^\epsilon \end{aligned}$$

which therefore has a unique fixed point which solves (6.16). If $C(T,\epsilon) > \frac{1}{2}$ we can choose T_1 small enough such that $C(T_1,\epsilon) \leqslant \frac{1}{2}$ and then repeat the above argument for intervals $[0,T_1],[T_1,2T_1],...,[nT_1,T]$. In either case we obtain a solution $\mathbf{u}^\epsilon$ which solves (6.16) on the interval $[0,T]$. Standard parabolic regularity theory (see e.g. Th. 5 §7.1 [63]) then gives us $\mathbf{u}^\epsilon \in L^2(0,T;H^2(\Sigma,\mathbb{R}^N))$ and that $\dot{\mathbf{u}}^\epsilon \in L^2(0,T;L^2(\Sigma,\mathbb{R}^N))$, which concludes the proof of Theorem 2. ⊡
Note that Evans goes on to use Th. 6 §7.1 [63] to obtain an improved regularity result showing that $\mathbf{u}^\epsilon \in L^2(0,T;H^3(\Sigma,\mathbb{R}^N))$ and that the time derivative $\dot{\mathbf{u}}^\epsilon \in L^2(0,T;H^1(\Sigma,\mathbb{R}^N))$. However we do not need this result.

6.2.3 Energy estimates

The next step is to obtain the uniform energy estimate in ϵ for the solutions $\mathbf{u}^\epsilon$. This is the content of the following proposition.

Proposition 3 *There exists a constant C depending on T, Σ, h_{ij}, $\partial_l h_{ij}$, $\sup\limits_{x\in\Sigma_{[0,T]}}\left(|(A^0)^{-1}|, |A^i|, |B|, |\partial_t(A^0)^{-1}|, |\partial_t A^i|, |\partial_j(A^0)^{-1}|, |\partial_j A^i|\right)$ such that*

$$||\mathbf{u}^\epsilon||^2_{L^2(0,T;L^2(\Sigma,\mathbb{R}^N))} \leqslant C\left(||\mathbf{u}_0||_{L^2(\Sigma,\mathbb{R}^N)} + \int_0^T ||\mathbf{F}(t,\cdot)||_{L^2(\Sigma,\mathbb{R}^N)}dt\right) \tag{6.28}$$

Therefore the estimate is independent of ϵ.

Proof of Proposition 3. Taking the time derivative of $||\mathbf{u}^\epsilon||^2_{L^2(\Sigma,\mathbb{R}^N)}$ gives:

$$\begin{aligned} &\frac{d}{dt}\left(||\mathbf{u}^\epsilon(t)||^2_{L^2(\Sigma,\mathbb{R}^N)}\right) = 2\left(\mathbf{u}^\epsilon, \dot{\mathbf{u}}^\epsilon\right)_{L^2(\Sigma,\mathbb{R}^N)} \\ &= 2\left(\mathbf{u}^\epsilon, \epsilon\Delta_h\mathbf{u}^\epsilon - (A^0)^{-1}A^i\partial_i\mathbf{u}^\epsilon - (A^0)^{-1}B\mathbf{u}^\epsilon + (A^0)^{-1}\mathbf{F}\right)_{L^2(\Sigma,\mathbb{R}^N)} \end{aligned} \tag{6.29}$$

We have estimates for the following terms in (6.29)

$$|(\mathbf{u}^\epsilon, (A^0)^{-1}\mathbf{F})_{L^2(\Sigma,\mathbb{R}^N)}| \leqslant C_1(||\mathbf{u}^\epsilon(t)||^2_{L^2(\Sigma,\mathbb{R}^N)} + ||\mathbf{F}||^2_{L^2(\Sigma,\mathbb{R}^N)}) \tag{6.30}$$

$$(\mathbf{u}^\epsilon, \epsilon\Delta_h\mathbf{u}^\epsilon)_{L^2(\Sigma,\mathbb{R}^N)} = -\epsilon\sum_{l=1}^N\int_\Sigma h^{ij}\partial_i(u^l)\partial_j(u^l)\nu_h \leqslant 0 \tag{6.31}$$

$$\left|(\mathbf{u}^\epsilon, (A^0)^{-1}B\mathbf{u}^\epsilon)\right| \leqslant C_2||\mathbf{u}^\epsilon||^2_{L^2(\Sigma,\mathbb{R}^N)}. \tag{6.32}$$

However we also require an estimate for $(\mathbf{u}^\epsilon, (A^0)^{-1}A^i\partial_i\mathbf{u}^\epsilon)_{L^2(\Sigma,\mathbb{R}^N)}$, which is the remaining term in (6.29). This term can be estimated by applying a suitable integration by parts.

We first assume that the smooth background Riemannian metric is $h_{ij} = \delta_{ij}$, write $(A^0)^{-1}A^i$ as $\tilde{A}^i$ and estimate $(\mathbf{u}^\epsilon, \tilde{A}^i\partial_i\mathbf{u}^\epsilon)_{L^2(\Sigma,\mathbb{R}^N)}$. Using the fact that the $\tilde{A}^i$ are symmetric, we then have

$$(\mathbf{u}^\epsilon, \tilde{A}^i\partial_i\mathbf{u}^\epsilon)_{L^2(\Sigma,\mathbb{R}^N)} \tag{6.33}$$

$$= \int_\Sigma \mathbf{u}^\epsilon\cdot\left(\tilde{A}^j\partial_j\mathbf{u}^\epsilon\right)d^nx \tag{6.34}$$

$$= \tfrac{1}{2}\int_\Sigma \partial_j\left(\tilde{A}^j\mathbf{u}^\epsilon\right)\cdot\mathbf{u}^\epsilon d^3x - \tfrac{1}{2}\int_\Sigma\left(\partial_j\tilde{A}^j\right)\mathbf{u}^\epsilon\cdot\mathbf{u}^\epsilon d^nx \tag{6.35}$$

$$= -\tfrac{1}{2}\int_\Sigma\left(\partial_j\tilde{A}^j\right)\mathbf{u}^\epsilon\cdot\mathbf{u}^\epsilon d^3x \tag{6.36}$$

So that

$$|(\mathbf{u}, \tilde{A}^i\partial_i\mathbf{u})_{L^2(\Sigma,\mathbb{R}^N)}| \leqslant \frac{1}{2}\left|\int_\Sigma\left(\partial_j\tilde{A}^j\right)\mathbf{u}^\epsilon\cdot\mathbf{u}^\epsilon d^nx\right| \tag{6.37}$$

$$\leqslant C_3||\mathbf{u}^\epsilon||^2_{L^2(\Sigma,\mathbb{R}^N)} \tag{6.38}$$

where the constant C is independent of ϵ and we have used the fact that $\partial_j\tilde{A}^j$ is bounded. In the case of a general background Riemannian metric h_{ij} rather than the ordinary divergence of $\tilde{A}^i$, one obtains the divergence with respect to the background metric h_{ij} and the corresponding result is

$$|(\mathbf{u}, \tilde{A}^i\partial_i\mathbf{u})_{L^2(\Sigma,\mathbb{R}^N)}| \leqslant \frac{1}{2}\left|\int_\Sigma\left((\partial_j + \tilde{\Gamma}^k_{kj})\tilde{A}^j\right)\mathbf{u}^\epsilon\cdot\mathbf{u}^\epsilon\nu_h\right| \tag{6.39}$$

$$\leqslant C_4||\mathbf{u}^\epsilon||^2_{L^2(\Sigma,\mathbb{R}^N)} \tag{6.40}$$

where again C_3 is independent of ϵ. For the case where Σ is an open, bounded set of $\mathbb{R}^n$ with smooth boundary $\partial\Sigma$, one needs a slightly more complicated argument where one approximates $\mathbf{u}$ by smooth functions of compact support (see [63] §7.3 for details). Using all the available estimates (6.30), (6.31), (6.32) and (6.39) in (6.29) we obtain the estimate

$$\frac{d}{dt}\left(||\mathbf{u}^\epsilon(t)||^2_{L^2(\Sigma,\mathbb{R}^N)}\right) \leqslant C_4\left(||\mathbf{u}^\epsilon||^2_{L^2(\Sigma,\mathbb{R}^N)} + ||\mathbf{F}||^2_{L^2(\Sigma,\mathbb{R}^N)}\right) \tag{6.41}$$

Using Gronwall's inequality and the fact that

$$||\mathbf{u}^\epsilon(0,x)||_{L^2(\Sigma,\mathbb{R}^N)} \leqslant ||\mathbf{u}_0(x)||_{L^2(\Sigma,\mathbb{R}^N)} \tag{6.42}$$

we obtain the estimate

$$\sup_{0\leqslant t\leqslant T}||\mathbf{u}^\epsilon(t)||^2_{L^2(\Sigma,\mathbb{R}^N)} \leqslant C_5\left(||\mathbf{u}_0||^2_{L^2(\Sigma,\mathbb{R}^N)} + \int_0^T ||\mathbf{F}||^2_{L^2(\Sigma,\mathbb{R}^N)}dt\right) \tag{6.43}$$

Finally noting that

$$||\mathbf{u}^\epsilon||^2_{L^2(0,T;L^2(\Sigma,\mathbb{R}^N))} \leqslant T\sup_{0\leqslant t\leqslant T}||\mathbf{u}^\epsilon(t)||^2_{L^2(\Sigma,\mathbb{R}^N)} \tag{6.44}$$

and using this in the estimate above we obtain (6.28) which concludes the proof. ⊡

6.2.4 Existence, Uniqueness and Stability

In this section we show the existence and uniqueness of the initial value problem. In Proposition 2 we obtained solutions $\mathbf{u}^\epsilon$ in $L^2(0,T;H^2(\Sigma,\mathbb{R}^N))$ to the parabolic system (6.16). Using the Banach–Alaoglu Theorem (see Theorem 22 in the Appendix) there exists a subsequence $\{\mathbf{u}^{\epsilon_k}\}_{k=1}^\infty$ that converges weakly to a function $\mathbf{u} \in L^2(0,T;L^2(\Sigma,\mathbb{R}^\mathbb{N}))$. We now show that this converges to a weak solution of (6.1).
First we choose a function $\tilde{\mathbf{w}} \in H^2(\Sigma_{[0,T]},\mathbb{R}^N)$ with $\tilde{\mathbf{w}}(T,\cdot) = 0$, take the dot product with equation (6.16) and integrate over t and x. This gives

$$\begin{aligned}&\int_0^T ((A^0)^{-1}L\mathbf{u}^\epsilon,\tilde{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)} - \epsilon(\Delta_h\mathbf{u}^\epsilon,\tilde{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)}dt\\ =&\int_0^T ((A^0)^{-1}\mathbf{F},\tilde{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)}dt\end{aligned} \tag{6.45}$$

Then integrating by parts we have the following results

$$\begin{aligned}\int_0^T (\partial_t\mathbf{u}^\epsilon,\tilde{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)}dt &= -\int_0^T (\mathbf{u}^\epsilon,\partial_t\tilde{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)}dt\\ &+ (\mathbf{u}^\epsilon(0),\tilde{\mathbf{w}}(0))_{L^2(\Sigma,\mathbb{R}^N)}\\ -\epsilon\int_0^T (\Delta_h\mathbf{u}^\epsilon,\tilde{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)}dt &= -\epsilon\int_0^T (\mathbf{u}^\epsilon,\Delta_h\tilde{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)}dt\\ \int_0^T ((A^0)^{-1}B\mathbf{u}^\epsilon,\tilde{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)}dt &= \int_0^T \left(\mathbf{u}^\epsilon,B^T((A^0)^{-1})\tilde{\mathbf{w}}\right)_{L^2(\Sigma,\mathbb{R}^N)}dt\end{aligned} \tag{6.46}$$

$$\int_0^T ((A^0)^{-1}A^i\partial_i \mathbf{u}^\epsilon, \tilde{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)}dt = \; -\int_0^T (\mathbf{u}^\epsilon, \partial_i(A^i(A^0)^{-1}\tilde{\mathbf{w}}))_{L^2(\Sigma,\mathbb{R}^N)}dt + \int_{\Sigma_{[0,T]}} \partial_i \left(\mathbf{u}^\epsilon \cdot A^i(A^0)^{-1}\tilde{\mathbf{w}}\right)\nu_h dt \tag{6.47}$$

Therefore using (6.16) and the fact given by the divergence theorem that

$$\int_\Sigma \partial_i \left(\mathbf{u}^\epsilon \cdot A^i(A^0)^{-1}\tilde{\mathbf{w}}\right)\nu_h = -\int_\Sigma \tilde{\Gamma}^l_{il}\left(\mathbf{u}^\epsilon \cdot A^i(A^0)^{-1}\tilde{\mathbf{w}}\right)\nu_h \tag{6.48}$$

Integrating by parts we obtain

$$\begin{aligned}&\int_0^T (\mathbf{u}^\epsilon, \tilde{L}\tilde{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)} - \epsilon(\mathbf{u}^\epsilon, \Delta_h\tilde{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)}dt \\ &= \int_0^T ((A^0)^{-1}\mathbf{F}, \tilde{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)}dt + (\mathbf{u}^\epsilon(0), \tilde{\mathbf{w}}(0))_{L^2(\Sigma,\mathbb{R}^N)}\end{aligned} \tag{6.49}$$

where the operator $\tilde{L}$ is defined by

$$\tilde{L}\tilde{\mathbf{w}} =: -\partial_t\tilde{\mathbf{w}} - \partial_i(A^i(A^0)^{-1}\tilde{\mathbf{w}}) + B^T((A^0)^{-1})\tilde{\mathbf{w}} - \tilde{\Gamma}^l_{il}A^i(A^0)^{-1}\tilde{\mathbf{w}} \tag{6.50}$$

Then taking the limit $k \to \infty$ and using the weak convergence of $\mathbf{u}^{\epsilon_k} \rightharpoonup \mathbf{u}$ and that $\mathbf{u}^{\epsilon_k}(0) \to (A^0)^{-1}(0)\mathbf{u}_0$ in $L^2(\Sigma,\mathbb{R}^N)$ we obtain

$$\begin{aligned}&\int_0^T (\mathbf{u}, \tilde{L}\tilde{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)}dt \\ &= \int_0^T ((A^0)^{-1}\mathbf{F}, \tilde{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)}dt + (\mathbf{u}_0, \tilde{\mathbf{w}}(0))_{L^2(\Sigma,\mathbb{R}^N)}\end{aligned} \tag{6.51}$$

The above equality was obtained for $\tilde{\mathbf{w}} \in H^2(\Sigma_{[0,T]},\mathbb{R}^N)$, however the equation remains well-defined for $\tilde{\mathbf{w}} \in H^1(\Sigma_{[0,T]},\mathbb{R}^N) \subset C^0([0,T],L^2(\Sigma))$ vanishing at time T. We now show that not only is the equation well-defined, but it remains valid for $\tilde{\mathbf{w}} \in H^1(\Sigma_{[0,T]},\mathbb{R}^N)$. The method involves taking the convolution with a family of mollifiers ρ^δ and then passing to the limit $\delta \to 0$.
Let $\hat{\mathbf{w}} \in H^1(\Sigma_{[0,T]},\mathbb{R}^N)$ such that $\hat{\mathbf{w}} = 0$ for all $t < \tau \leqslant T$ for some $t \in [0,T)$ and define $\hat{\mathbf{w}}^\delta = \rho^\delta * \hat{\mathbf{w}}$ where we have chosen δ close enough to zero such that $\hat{\mathbf{w}}^\delta(T,\cdot) = 0$. Therefore we have

$$\begin{aligned}&\int_0^T (\mathbf{u}, \tilde{L}\hat{\mathbf{w}}^\delta)_{L^2(\Sigma,\mathbb{R}^N)}dt \\ &= \int_0^T ((A^0)^{-1}\mathbf{F}, \hat{\mathbf{w}}^\delta)_{L^2(\Sigma,\mathbb{R}^N)}dt + (\mathbf{u}(0), \hat{\mathbf{w}}^\delta(0))_{L^2(\Sigma,\mathbb{R}^N)}\end{aligned} \tag{6.52}$$

Taking the limit $\delta \to 0$ and using the Schwartz inequality we have the following limits

$$\hat{\mathbf{w}}^\delta_t \;\to\; \hat{\mathbf{w}}_t \text{ in } L^2(0,T;L^2(\Sigma,\mathbb{R}^N)) \tag{6.53}$$

$$-(A^i(A^0)^{-1})\hat{\mathbf{w}}^\delta_i \;\to\; -(A^i(A^0)^{-1})\hat{\mathbf{w}}_i \text{ in } L^2(0,T;L^2(\Sigma,\mathbb{R}^N)) \tag{6.54}$$

$$B^T((A^0)^{-1})\hat{\mathbf{w}}^\delta \;\to\; B^T((A^0)^{-1})\hat{\mathbf{w}} \text{ in } L^2(0,T;L^2(\Sigma,\mathbb{R}^N)) \tag{6.55}$$

$$\tilde{\Gamma}^k_{lk}A^l(A^0)^{-1}\hat{\mathbf{w}}^\delta \;\to\; \tilde{\Gamma}^k_{lk}A^l(A^0)^{-1}\hat{\mathbf{w}} \text{ in } L^2(0,T;L^2(\Sigma,\mathbb{R}^N)) \tag{6.56}$$

$$\hat{\mathbf{w}}^\delta(0) \;\to\; \hat{\mathbf{w}}(0) \text{ in } L^2(\Sigma,\mathbb{R}^N) \tag{6.57}$$

We therefore conclude that

$$\begin{aligned}&\int_0^T (\mathbf{u}, \tilde{L}\hat{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)} dt \\ &= \int_0^T ((A^0)^{-1}\mathbf{F}, \hat{\mathbf{w}})_{L^2(\Sigma,\mathbb{R}^N)} dt + (\mathbf{u}_0, \hat{\mathbf{w}}(0))_{L^2(\Sigma,\mathbb{R}^N)}\end{aligned} \tag{6.58}$$

for $\hat{\mathbf{w}} \in H^1(\Sigma_{[0,T]}, \mathbb{R}^N)$ with $\hat{\mathbf{w}}(T, \cdot) = 0$.

If we now take $\mathbf{w} \in C^\infty(\Sigma_{[0,T]}, \mathbb{R}^N)$, with $supp(\mathbf{w}) \subseteq [0, t < T) \times \Sigma$, and multiply it by A^0 we obtain that $A^0\mathbf{w} \in W^{1,\infty}(\Sigma_{[0,T]}, \mathbb{R}^N) \subset H^1(\Sigma_{[0,T]}, \mathbb{R}^N)$. We may therefore insert $\hat{\mathbf{w}} = A^0\mathbf{w}$ in (6.58) which gives

$$\begin{aligned}&\int_0^T (\mathbf{u}, \tilde{L}A^0\mathbf{w})_{L^2(\Sigma,\mathbb{R}^N)} dt \\ &= \int_0^T ((A^0)^{-1}\mathbf{F}, (A^0)\mathbf{w})_{L^2(\Sigma,\mathbb{R}^N)} dt + \Big(\mathbf{u}_0, (A^0)\mathbf{w}|_{t=0}\Big)_{L^2(\Sigma,\mathbb{R}^N)}\end{aligned} \tag{6.59}$$

which can be rewritten as

$$\int_0^T (\mathbf{u}, L^*\mathbf{w})_{L^2(\Sigma,\mathbb{R}^N)} dt = \int_0^T (\mathbf{F}, \mathbf{w})_{L^2(\Sigma,\mathbb{R}^N)} dt + \Big((A^0(0))\mathbf{u}_0, \mathbf{w}(0)\Big)_{L^2(\Sigma,\mathbb{R}^N)} \tag{6.60}$$

for all $\mathbf{w} \in C^\infty(\Sigma_{[0,T]}, \mathbb{R}^N)$ with $supp(\mathbf{w}) \subseteq [0, t < T) \times \Sigma$, where L^* is the formal adjoint defined by equation (6.9).
We have therefore proved that the $\mathbf{u}$ obtained by taking the limit of the subsequence $\{\mathbf{u}^{\epsilon_k}\}_{k=1}^\infty$ is a weak solution lying in $L^2(0, T; L^2(\Sigma, \mathbb{R}^{\mathbb{N}}))$ with initial data $\mathbf{u}_0$ as in Definition 3.

As the norm function is lower semi continuous, we may take the limit of equation (6.28) to obtain the estimate

$$\begin{aligned}||\mathbf{u}||^2_{L^2(0,T;L^2(\Sigma,\mathbb{R}^N))} &\leqslant \lim_{k\to\infty} ||\mathbf{u}^{\epsilon_k}||^2_{L^2(0,T;L^2(\Sigma,\mathbb{R}^N))} &(6.61)\\ &\leqslant C_6 \left(||\mathbf{u}_0||^2_{L^2(\Sigma,\mathbb{R}^N)} + \int_0^T ||\mathbf{F}||^2_{L^2(\Sigma,\mathbb{R}^N)} dt \right) &(6.62)\end{aligned}$$

So that the solution we have obtained is a regular weak solution.

To show uniqueness we consider two functions $\mathbf{u}_1, \mathbf{u}_2$ which are both regular weak solutions. Then $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ is a regular weak solution with source $\mathbf{F} = 0$ and initial data $\mathbf{u}_0 = 0$. Moreover the solution satisfies the energy estimate (6.10) as shown above. Therefore

$$||\mathbf{u}||_{L^2(0,T;L^2(\Sigma,\mathbb{R}^N))} = 0 \tag{6.63}$$

This implies $\mathbf{u} = 0$ and therefore $\mathbf{u}_1 = \mathbf{u}_2$.

The final step in establishing well-posedness is to prove the stability of the solution with respect to initial data. To make the concept precise we say that the solution $\mathbf{u}$

is continuously stable in $L^2(0,T;L^2(\Sigma,\mathbb{R}^N))$ with initial data $\mathbf{u}_0$ in $L^2(\Sigma,\mathbb{R}^N)$ if given $\epsilon > 0$ there is a δ depending on $\mathbf{u}_0$ such that if $\tilde{\mathbf{u}}_0 \in L^2(\Sigma,\mathbb{R}^N))$ with:

$$\|\tilde{\mathbf{u}}_0 - \mathbf{u}_0\|_{L^2(\Sigma,\mathbb{R}^N)} \leqslant \delta, \tag{6.64}$$

then the corresponding weak solution $\tilde{\mathbf{u}}$ with source function $\mathbf{F}$ satisfies

$$\|\tilde{\mathbf{u}} - \mathbf{u}\|_{L^2(0,T;L^2(\Sigma,\mathbb{R}^N))} \leqslant \epsilon \tag{6.65}$$

The stability results follows from using the energy inequality shown in Theorem 3 for the difference $\tilde{\mathbf{u}} - \tilde{\mathbf{u}}$ which gives

$$\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}\|^2_{L^2(0,T;L^2(\Sigma,\mathbb{R}^N))} \quad \leqslant \quad C||\mathbf{u}_0 - \tilde{\mathbf{u}_0}||^2_{L^2(\Sigma,\mathbb{R}^N)} \tag{6.66}$$

Now choosing $\delta = \frac{\epsilon}{C}$ we obtain the inequality:

$$\|\tilde{\mathbf{u}} - \tilde{\mathbf{u}}\|^2_{L^2(0,T;L^2(\Sigma,\mathbb{R}^N))} \leqslant \epsilon^2 \tag{6.67}$$

which establishes stability with respect to the initial data.
We mention now that previous work by Vickers and Wilson [58] took advantage of the fact that the singularity was concentrated in a hypersurface. This condition was used to obtain the energy estimates via an approximation technique which added a flux term across the singularity in the energy estimates. The flux term vanishes for functions of sufficiently high regularity which allows them to obtain existence of solutions. Now, we have shown how using the first order formalism, and the vanishing viscosity method [63], we are able to establish uniqueness and local well posedness without the requirement for a vanishing flux condition. Therefore, it is not necessary to add any additional boundary condition on the singularity to have local well defined dynamics.
This concludes the proof of Theorem 4.

6.2.5 The wave equation

In order to apply the results of Theorem 1 to applications in general relativity we will show here how the wave equation can be written as a first order linear symmetric problem.
We define

$$\mathbf{v} = (\partial_1 u, ..., \partial_n u, \partial_t u, u)^T = (v^1, ..., v^n, v^{n+1}, v^{n+2})^T \in \mathbb{R}^{n+2} \tag{6.68}$$

and the symmetric $(n+2)\times(n+2)$ matrices A^μ by:

$$A^0 = \begin{pmatrix} g^{11} & g^{12} & \cdots & 0 & 0 \\ g^{21} & g^{22} & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -g^{00} & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

$$A^k = \begin{pmatrix} 0 & 0 & \cdots & g^{1k} & 0 \\ 0 & 0 & \cdots & g^{2k} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ g^{1k} & g^{2k} & \cdots & 2g^{0k} & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We further define the matrix B to be given by

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & 0 \\ g^{ab}\Gamma^1_{ab} & g^{ab}\Gamma^2_{ab} & \cdots & g^{ab}\Gamma^0_{ab} & -m^2 \\ 0 & 0 & \cdots & -1 & 0 \end{pmatrix}.$$

where Γ^c_{ab} are the connection coefficients of the spacetime metric g_{ab}. We also define the $\mathbb{R}^N$-valued vector function $\mathbf{F}$ by $\mathbf{F} = (0, 0, \ldots, -f, 0)^T$.
In this way, we may rewrite the scalar wave equation (6.13) as a first order system which has the form

$$\begin{aligned} A^0\partial_t\mathbf{v} - A^i\partial_i\mathbf{v} + B\mathbf{v} &= \mathbf{F} && (6.69) \\ \mathbf{v}(0,\cdot) &= \mathbf{v}_0(\cdot) && (6.70) \end{aligned}$$

We write below explicitly this assertion for the $1+2$ case where u in (6.68) is C^2.
We have

$$A^0\partial_t\mathbf{v} = \begin{pmatrix} g^{11} & g^{12} & 0 & 0 \\ g^{21} & g^{22} & 0 & 0 \\ 0 & 0 & -g^{00} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \partial_{t1}u \\ \partial_{t2}u \\ \partial_{tt}u \\ \partial_t u \end{pmatrix} = \begin{pmatrix} g^{11}\partial_{t1}u + g^{12}\partial_{t2}u \\ g^{21}\partial_{t1}u + g^{22}\partial_{t2}u \\ -g^{00}\partial_{tt}u \\ \partial_t u \end{pmatrix}$$

$$A^1\partial_1\mathbf{v} = \begin{pmatrix} 0 & 0 & g^{11} & 0 \\ 0 & 0 & g^{21} & 0 \\ g^{11} & g^{21} & 2g^{01} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_{11}u \\ \partial_{12}u \\ \partial_{1t}u \\ \partial_1 u \end{pmatrix} = \begin{pmatrix} g^{11}\partial_{1t}u \\ g^{21}\partial_{1t}u \\ g^{11}\partial_{11}u + g^{21}\partial_{12}u + 2g^{01}\partial_{1t}u \\ 0 \end{pmatrix}$$

$$A^2\partial_2\mathbf{v} = \begin{pmatrix} 0 & 0 & g^{12} & 0 \\ 0 & 0 & g^{22} & 0 \\ g^{12} & g^{22} & 2g^{02} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \partial_{21}u \\ \partial_{22}u \\ \partial_{2t}u \\ \partial_2 u \end{pmatrix} = \begin{pmatrix} g^{12}\partial_{2t}u \\ g^{22}\partial_{2t}u \\ g^{12}\partial_{21}u + g^{22}\partial_{22}u + 2g^{02}\partial_{2t}u \\ 0 \end{pmatrix}$$

$$B\mathbf{v} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ g^{ab}\Gamma^1_{ab} & g^{ab}\Gamma^2_{ab} & g^{ab}\Gamma^0_{ab} & -m^2 \\ 0 & 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \partial_1 u \\ \partial_2 u \\ \partial_t u \\ u \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ 0 \\ g^{ab}\Gamma^1_{ab}\partial_1 u + g^{ab}\Gamma^2_{ab}\partial_2 u + g^{ab}\Gamma^0_{ab}\partial_0 u - m^2 u \\ -\partial_t u \end{pmatrix}$$

which gives using the symmetry of the metric tensor and commuting second derivatives

$$A^0\partial_t\mathbf{v} - A^1\partial_1\mathbf{v} - A^2\partial_2\mathbf{v} + B\mathbf{v} = \begin{pmatrix} 0 \\ 0 \\ -\Box_g u - m^2 u \\ 0 \end{pmatrix}$$

We may then use the Theorem 4 to establish well-posedness of (6.69). To prove existence of solutions to the wave equation, we therefore need to prove that the solution $\mathbf{v}$ of the symmetric hyperbolic system (6.69) has the form $\mathbf{v} = (\partial_1 u, ..., \partial_n u, \partial_t u, u)^T$.
We now mollify our solution $\mathbf{v}$ using a strict delta net (see Appendix for a formal definition) to obtain a sequence of smooth functions $\mathbf{v}^\epsilon = \rho^\epsilon * \mathbf{v} = (v_\epsilon^1, ..., v_\epsilon^{n+2})^T$ that satisfies:

$$\int_0^T (\mathbf{v}^\epsilon, L^*\mathbf{w})_{L^2(\Sigma,\mathbb{R}^N)} dt = \int_0^T (\mathbf{F}^\epsilon, \mathbf{w})_{L^2(\Sigma,\mathbb{R}^N)} dt + (\mathbf{v}^\epsilon(0), \mathbf{w}(0))_{L^2(\Sigma,\mathbb{R}^N)}$$

for a suitable mollified $\mathbf{F}^\epsilon$. Moreover, the regularity of $\mathbf{v}^\epsilon$ allows us to integrate by parts to obtain

$$\int_0^T (\mathbf{F}^\epsilon, w)_{L^2(\Sigma,\mathbb{R}^N)} dt = \int_0^T (L\mathbf{v}^\epsilon, w)_{L^2(\Sigma,\mathbb{R}^N)} dt$$

Then

$$\int_0^T (\mathbf{F}^\epsilon - L\mathbf{v}^\epsilon, w)_{L^2(\Sigma,\mathbb{R}^N)} dt = 0$$

for all $w \in C^\infty(\Sigma_{[0,T]}, \mathbb{R}^N)$ and therefore $L\mathbf{v}^\epsilon = \mathbf{F}^\epsilon$ almost everywhere so we obtain that:

$$\partial_t v_\epsilon^j = \partial_j v_\epsilon^{n+1}$$

and

$$\partial_t v_\epsilon^{n+2} = v_\epsilon^{n+1}$$

almost everywhere.
Also, we choose initial data such that

$$\partial_i v_\epsilon^{n+2}(0,\cdot) = v_\epsilon^i(0,\cdot), \qquad i = 1, 2, \ldots, n \tag{6.71}$$

We now define $u_\epsilon = v_\epsilon^{n+2} = \rho^\epsilon * v^{n+2}$ and obtain

$$\begin{aligned} \partial_i u_\epsilon &= v_\epsilon^i(0,\cdot) + \int_0^T \partial_i\partial_t v_\epsilon^{n+2} dt && (6.72) \\ &= v_\epsilon^i(0,\cdot) + \int_0^T \partial_t v_\epsilon^i dt && (6.73) \\ &= v_\epsilon^i && (6.74) \end{aligned}$$

Taking now into account that $\mathbf{v}^\epsilon \to \mathbf{v}$ in $L^2([0,T], L^2(\Sigma, \mathbb{R}^{n+2}))$ and that the convolution and derivatives commute we obtain the result that $\mathbf{v}$ is an $L^2([0,T], L^2(\Sigma, \mathbb{R}^{n+2}))$ function of the form $(\partial_1 u, ..., \partial_n u, \partial_t u, u)$.
Collecting the results from this section we have established Theorem 5.

6.3 Applications

Although in the following three examples the spacetimes are not spatially compact we will assume we are working in a local region of the form $\Sigma_{[0,T]} = [0,T] \times \Sigma$ as described in the geometric setting.

Junction Conditions There is a precise mathematical formalism proposed by Israel to describe the junction conditions for two regular spacetimes joined along a non-null singular hypersurface Λ [42]. He noted that if we consider two half-spaces V^+ and V^-, a singular hypersurface, Λ, can be fully characterised by the different extrinsic curvatures (second fundamental forms) associated with its embeddings in V^+ and V^- and a continuous matching condition of the metric through the common boundary. If we use Gaussian coordinates based on Λ, then the normal derivatives of the metric have a jump across Λ with the metric being continuous along Λ. This scenario satisfies the analytic conditions required for the application of Theorem 5 to apply. Notice however that the theorem does not need to make assumptions on the time dependence or matter content of the spacetime.

Impulsive Gravitational Waves A spacetime that contains impulsive gravitational waves described in double null coordinates has line element given by:

$$ds^2 = 2dudv - (1 - u\Theta(u))^2 dx^2 - (1 + u\Theta(u))^2 dx^2 \tag{6.75}$$

where $\Theta(u)$ is the Heaviside step function. The spacetime is vacuum, but has a Weyl tensor with delta function components

$$\begin{aligned} C_{uxux} &= -\delta(u) \\ C_{uyuy} &= \delta(u) \end{aligned}$$

It is important to notice that although the curvature is not bounded this condition is not relevant for Theorem 5 to apply.

Brane-world Cosmologies The Randall–Sundrum Brane-Worlds (RS) are models that explore gravity beyond classical general relativity [17], and also appear in a cosmological context [67]. In the RS model in AdS_5 one has that in Gaussian normal coordinates $X^A = (t, x^i, y)$ based on the brane at $y = 0$, the model has the line element

$$ds^2 = e^{-2|y|/L}(-dt^2 + dx^{i2}) + dy^2 \tag{6.76}$$

This spacetime again satisfies the conditions for the Theorem 5 to be applicable and therefore solutions with finite energy exist.

Notice that the well-posedness result can be extended to other brane world models and even collision of branes [66] as long as the spacetime satisfies the assumptions of Theorem 5. Therefore, one can consider that dynamical models of colliding branes do not produce strong gravitational singularities provided that the spacetime remains $C^{0,1}$ during all the processes.

Chapter 7

Spacetimes with string-like singularities

7.1 The main result

In this chapter, we provide tools to establish the existence and uniqueness of solutions of the wave equation in rough backgrounds with metrics that satisfy certain conditions (see Geometric Conditions 3 in §7.1.1). These are for example satisfied by cosmic string type singularities. The basic proof of the theorem follows the method of Evans [63, §7.2] and uses Galerkin approximation methods together with energy estimates for the wave operator. The method of proof is different from that used in the previous chapters and reflects real differences in the type of singularity under consideration. For the string type singularities under consideration here, the special form of the metric means that we have good energy estimates for the wave operator but not for its adjoint in H^1. On the other hand, the time derivatives of the metric coefficients are well-behaved, which is crucial to the use of a Galerkin approximation. This allows us to prove existence, uniqueness and stability of weak solutions with the required regularity. However, the results differ from those in [63] in that we explicitly lower the differentiability (although see [68]), generalise the results to curved spacetimes with a special emphasis on the $n+1$ decomposition of spacetime and use a different method of proof to establish uniqueness and stability. This allows the result to be generalised if one works with more general gauges. Furthermore, we show that under the Geometric Conditions 3 and the hypothesis of Lemma 5, the energy momentum is not only integrable in spacetime, but can also be defined distributionally on any constant time hypersurface. In §7.3 we discuss how our theorems apply to a class of spacetimes with cosmic string type singularities and show that, from this perspective, cosmic string singularities should not be regarded as strong gravitational singularities, even though the curvature is not in L^∞_{loc} or even in L^1_{loc} in some cases.

7.1.1 The general setting

Let $\Sigma_{(0,T]} = \Sigma \times (0,T]$ be a $(n+1)$- dimensional domain equipped with a Lorentzian metric g_{ab} where Σ is an open bounded region of a n-dimensional manifold. Now using a $n+1$ decomposition of the spacetime, the line element of the metric may be written in the form:

$$ds^2 = +N^2dt^2 - \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{7.1}$$

where N is the lapse function, β^i is the shift and γ_{ij} is the induced metric on Σ. The class of metrics we are going to consider requires that there is a foliation of the domain $\Sigma_{(0,T]}$ and suitable coordinates (t, x^i) such that

Geometric Conditions 3

1. *$\gamma^{ij} \in C^1([0,T], L^\infty(\Sigma))$.*
2. *The scalar coefficient of the volume form given by $\sqrt{\gamma}$ for the induced metric γ_{ij} is bounded from below by a positive real number, i.e., $|\sqrt{\gamma}| > \eta$ for some $\eta \in \mathbb{R}^+$*
3. *The lapse function N can be chosen as $N = \sqrt{\gamma}$.*
4. *The shift can be chosen in such a way that $\beta^i = 0$.*
5. *There exist a constant $\theta > 0$ such that*
$$\sum_{i,j=1}^{n} \gamma^{ij}\gamma\xi_i\xi_j \geqslant \theta|\xi|^2$$
for all $(t,x) \in \Sigma_{(0,T]}, \xi \in \mathbb{R}^n$.

Condition 3 is chosen for simplicity as it allows us to use the dual space in Definition 5 with respect to the Lebesgue measure and also the coefficient of volume form becomes $\sqrt{-g} := N\sqrt{\gamma} = \gamma$. Nevertheless the condition can be weakened to require only that it is a bounded function with a positive lower bound, i.e.

3′. *The lapse function N is $C^1((0,T], L^\infty(\Sigma))$ and $|N| > \omega$ for some $\omega \in \mathbb{R}^+$*

However, this is at the expense of adding linear terms in time and to avoid undue complications in the formulae we do not pursue this here. Note however the method of proof in §7.2.2 has been modified from that in Evans [63] to allow for this possibility. We want to obtain weak solutions to the following initial/boundary problem for the wave equation:

$$\Box_g u = f \text{ in } \Sigma_{(0,T]} \tag{7.2}$$
$$u = 0 \text{ on } \partial\Sigma \times [0,T] \tag{7.3}$$
$$u(0,x) = u_0 \text{ on } \Sigma_0 = \Sigma \times \{t = 0\} \tag{7.4}$$
$$\dot{u}(0,x) = u_1 \text{ on } \Sigma_0 = \Sigma \times \{t = 0\} \tag{7.5}$$

where $f : \Sigma_{(0,T]} \to \mathbb{R}$ is a given source and $u_0 : \Sigma \to \mathbb{R}, u_1 : \Sigma \to \mathbb{R}$ are given initial conditions in suitable function spaces specified below.

7.1.2 Weak solution and the main theorem

For a metric with a general $n+1$ splitting given by (9.1) the wave operator is given by:

$$\begin{aligned}\Box_g u = \frac{1}{N\sqrt{\gamma}}\left(\partial_t\left(N\sqrt{\gamma}\frac{1}{N^2}\partial_t u\right)\right) \\ +\frac{1}{N\sqrt{\gamma}}\left(\partial_t\left(N\sqrt{\gamma}\frac{\beta^i}{N^2}\partial_i u\right) + \partial_j\left(N\sqrt{\gamma}\frac{\beta^j}{N^2}\partial_t u\right)\right) \\ -\frac{1}{N\sqrt{\gamma}}\partial_i\left(N\sqrt{\gamma}(\gamma^{ij} - \frac{\beta^i\beta^j}{N^2})\partial_j u\right)\end{aligned} \tag{7.6}$$

Taking into account the Geometric Conditions 3, we obtain

$$\Box_g u = \frac{\ddot{u}}{\gamma} - \frac{Lu}{\gamma} \tag{7.7}$$

where $-L$ is an elliptic operator in divergence form given by:

$$-Lu = -(\gamma^{ij}\gamma u_j)_i \tag{7.8}$$

Notice that our geometric conditions imply that L is a uniformly elliptic operator. We can associate with the operator $-L$ the following bilinear form given by:

$$B[u,v;t] := \int_{\Sigma} \gamma^{ij}(t,x)\gamma(t,x)u_i v_j dx^n \tag{7.9}$$

Using this bilinear form, we make the following definition of a weak solution.

Definition 5 *We say a function:*

$$u \in L^2(0,T;H_0^1(\Sigma)), \text{ with } \dot{u} \in L^2(0,T;L^2(\Sigma)), \ddot{u} \in L^2(0,T;H^{-1}(\Sigma))$$

is a local weak solution of the hyperbolic initial/boundary problem (7.2) provided that locally:

1. *For each $v \in L^2(0,T;H_0^1(\Sigma))$,*

$$\int_0^T < \ddot{u}, v > dt + \int_0^T B[u,v;t]dt = (f,v)_{L^2(\Sigma_{(0,T]},\nu_g)} \tag{7.10}$$

   where $\nu_g = \sqrt{-g}d^{n+1}x = \gamma d^{n+1}x$ and $< \cdot, \cdot >$ denotes the dual pairing between the $H^{-1}(\Sigma)$ and $H_0^1(\Sigma)$ Sobolev spaces.

2. *$u(0,x) = u_0(x), \dot{u}(0,x) = u_1(x)$ where $u_0 \in H_0^1(\Sigma_0)$ and $u_1 \in L^2(\Sigma_0)$*

We motivate Definition 5 by the following calculation. For the moment assume that the metric and the solution are smooth and that $\Box_g u = f$. Multiplying this by an element $v \in L^2(0,T;H_0^1(\Sigma))$ and integrating we obtain:

$$\begin{aligned}
\int_{\Sigma_{(0,T]}} f v \nu_g &= \int_{\Sigma_{(0,T]}} (\Box_g u) v \nu_g \\
&= \int_{\Sigma_{(0,T]}} \left(\frac{1}{\gamma}\ddot{u} - \frac{1}{\gamma}Lu\right) v\gamma d^{n+1}x \\
&= \int_0^T \int_{\Sigma} (\ddot{u} - Lu)\, v d^n x dt \\
&= \int_0^T \int_{\Sigma} (\ddot{u}v)\, d^n x dt - \int_0^T \int_{\Sigma} (\gamma^{ij}\gamma u_j)_i v d^n x dt \\
&= \int_0^T \int_{\Sigma} (\ddot{u}v)\, d^n x dt + \int_0^T \int_{\Sigma} \gamma^{ij}\gamma u_j v_i d^n x dt \\
&= \int_0^T < \ddot{u}, v > dt + \int_0^T B[u,v;t]dt
\end{aligned}$$

The final equation is the definition of a weak solution provided the regularity of the solution and the metric allows the integral to be well defined. This is indeed the case given the Geometric Conditions 3. The Sobolev embedding theorem in one dimension implies that $u \in C([0,T], L^2(\Sigma)) \cap C^1([0,T], H^{-1}(\Sigma))$ and therefore condition 2 makes sense.
As in the previous chapter we say a weak solution is regular if it satisfies a suitable energy estimate. We therefore make the following definition

Definition 6 **(Regular Weak Solution)** *We say a weak solution* $u \in H^1(\Sigma_{(0,T]})$ *is* **regular** *if* u *satisfies the energy estimate*

$$\begin{aligned} max_{t\in(0,T]} \left(||u(t,\cdot)||_{H_0^1(\Sigma)} + ||\dot{u}(t,\cdot)||_{L^2(\Sigma)} \right) + ||\ddot{u}||_{L^2([0,T];H^{-1}(\Sigma))} \\ \leqslant C \left(||f||_{L^2([0,T];L^2(\Sigma,N\nu_\gamma))} + ||u_0||_{H_0^1(\Sigma)} + ||u_1||_{L^2(\Sigma)} \right) \end{aligned} \tag{7.11}$$

We now state the main result of this chapter.

Theorem 6 **(Well-posedness in H^1)** *Let* $(\Sigma_{(0,T]}, g_{ab})$ *be a region of a spacetime satisfying the Geometric Conditions 3. Then the region* $\Sigma_{(0,T]}$ *is wave-regular. That is the wave equation is well-posed in the following sense: Given* $(u_0, u_1) \in H_0^1(\Sigma_0) \times L^2(\Sigma_0)$ *there exists a unique regular weak solution* u *in* $H^1(\Sigma_{(0,T]})$ *of* $\Box_g u = f$ *in the sense of Definition 5 and Definition 6 with initial conditions*

1. $u(0,\cdot) = u|_{\Sigma_0} = u_0$
2. $\dot{u}(0,\cdot) = \frac{\partial u}{\partial t}|_{\Sigma_0} = u_1$

that is stable with respect to initial data in $H_0^1(\Sigma_0) \times L^2(\Sigma_0)$*. Moreover, the components of the energy momentum tensor associated to the solution satisfy* $T^{ab}[u] \in C^0([0,T], L^1(\Sigma))$*.*

7.2 Proof of the main theorem

7.2.1 Outline of the proof

The proof of Theorem 6 follows Galerkin's method of proving existence of the wave equation by using approximate solutions as shown for example in Evans [63]. Using the energy estimate satisfied by the solution we prove uniqueness and stability. Finally, we show continuity in time. The main steps of the proof are as follows:

1. First, we show that there exist unique m-approximate solutions.
2. Second, we show that the m-approximate solutions satisfy an energy estimate

$$\begin{aligned} max_{t\in(0,T]} \left(||u^m(t,\cdot)||_{H_0^1(\Sigma)} + ||u_t^m(t,\cdot)||_{L^2(\Sigma)} + ||u_{tt}^m||_{L^2([0,T];H^{-1}(\Sigma))} \right) \\ \leqslant C \left(||f||_{L^2([0,T];L^2(\Sigma,N\nu_\gamma))} + ||u_0||_{H_0^1(\Sigma)} + ||u_1||_{L^2(\Sigma)} \right) \end{aligned} \tag{7.12}$$

3. Third, we take the limit $m \to \infty$ and show convergence in an appropriate weak sense to a regular weak solution as defined above.
4. Fourth, using the energy estimate we show uniqueness and stability.
5. Fifth, we give a proof showing that the energy momentum tensor can be defined in each hypersurface as an integrable function.

7.2.2 Existence

To prove existence of solutions we employ *Galerkin's method*. This requires several steps. We begin by showing uniqueness and existence of approximate solutions, then we establish a uniform estimate for the solutions and finally we take a limit in a proper weak topology (see Appendix for the formal definition), which converges to the required weak solution.
We start by choosing smooth functions $w_k(x)$ such that:

$$\{w_k\}_{k=1}^{\infty} \text{ is an orthogonal basis of } H_0^1(\Sigma)$$

$$\{w_k\}_{k=1}^{\infty} \text{ is an orthonormal basis of } L^2(\Sigma)$$

We can form the desired basis by choosing the eigenvectors of the Laplace operator Δ in the given local coordinates [63].
Now fix a positive integer m, write

$$u^m(t,x) := \sum_{k=1}^{m} d_m^k(t) w_k(x) \tag{7.13}$$

and consider for each $k = 1, ..., m$ the equation:

$$(u_{tt}^m, w_k)_{L^2(\Sigma)} + B[u^m, w_k; t] = (f, w_k)_{L^2(\Sigma, N\nu_\gamma)} \tag{7.14}$$

where ν_γ is the volume form associated to the induced metric γ_{ij} on Σ.
The system of equations (7.14) can be arranged as a system of linear ODE's given by

$$\ddot{d}_m^k(t) + \sum_{l=1}^{m} e^{kl}(t) d_m^l(t) = f^k(t) \tag{7.15}$$

where $e^{kl}(t) := B[w_l, w_k; t] = \int_\Sigma \gamma^{ij}(t,x)\gamma(t,x) w_{l_i} w_{k_j} dx^n$ and $f^k(t) := (f, w_k)_{L^2(\Sigma, N\nu_\gamma)}$ for each $k = 1, ..., m$.
We also require that the system satisfies the initial conditions

$$d_m^k(0) = (u_0, w_k)_{L^2(\Sigma_0)}, \quad \dot{d}_m^k(0) = (u_1, w_k)_{L^2(\Sigma_0)} \tag{7.16}$$

for $k = 1, ..., m$.
The functions $e^{kl}(t)$ are continuous in t as $\gamma^{ij}(t,x)\gamma(t,x) \in C^1([0,T], L^\infty(\Sigma))$. Then by standard local existence and uniqueness theorems for linear ordinary differential equations we obtain a unique $d_m^k(t) \in C^0([0,T])$ for every $k = 1, ..., m$.
Therefore we have shown that for each m there is a unique solution, u^m, satisfying (7.14) and (7.16) which we call the *m-approximate solution.*

Energy estimates

In this section, we establish the following energy estimate.

Theorem 7 *There exists a constant C, depending only on Σ, T and the coefficients of L such that*

$$\begin{aligned} max_{t\in(0,T]} \left(||u^m(t,\cdot)||_{H_0^1(\Sigma)} + ||u_t^m(t,\cdot)||_{L^2(\Sigma)} + ||u_{tt}^m||_{L^2([0,T];H^{-1}(\Sigma))} \right) \\ \leqslant C \left(||f||_{L^2([0,T];L^2(\Sigma,N\nu_\gamma))} + ||u_0||_{H_0^1(\Sigma)} + ||u_1||_{L^2(\Sigma)} \right) \end{aligned} \tag{7.17}$$

We start by multiplying equality (7.14) by $\dot{d}^k_m(t)$, sum from $k = 1, ..., m$ and use (7.13) to obtain

$$(u^m_{tt}, u^m_t)_{L^2(\Sigma)} + B[u^m, u^m_t; t] = (f, u^m_t)_{L^2(\Sigma, N\nu_\gamma)} \tag{7.18}$$

Using the fact that that

$$(u^m_{tt}, u^m_t)_{L^2(\Sigma)} = \frac{1}{2}\frac{d}{dt}||u^m_t||^2_{L^2(\Sigma)} \tag{7.19}$$

and that

$$B[u^m, u^m_t; t] = \frac{d}{dt}\left(\frac{1}{2}B[u^m, u^m; t]\right) - \frac{1}{2}\int_\Sigma \left(\gamma^{ij}(t,x)\gamma(t,x)\right)_t u^m_i u^m_j \tag{7.20}$$

we have

$$B[u^m, u^m_t; t] \geqslant \frac{d}{dt}\left(\frac{1}{2}B[u^m, u^m; t]\right) - C_1||u^m||^2_{H^1_0(\Sigma)} \tag{7.21}$$

Combining equations (7.18), (7.19), (7.20) and (7.21) we obtain

$$\frac{d}{dt}\left(||u^m_t||^2_{L^2(\Sigma)} + B[u^m, u^m; t]\right) \tag{7.22}$$

$$\leqslant \quad C_2\left(||u^m_t||^2_{L^2(\Sigma)} + ||u^m||^2_{H^1_0(\Sigma)} + ||f||^2_{L^2(\Sigma, N\nu_\gamma)}\right) \tag{7.23}$$

$$\leqslant \quad C_3\left(||u^m_t||^2_{L^2(\Sigma)} + B[u^m, u^m; t] + ||f||^2_{L^2(\Sigma, N\nu_\gamma)}\right) \tag{7.24}$$

where we have applied the uniform ellipticity condition in order to use the inequality

$$\theta\int_\Sigma |\delta^{ij}u^m_i u^m_j| \leqslant B[u^m, u^m; t] \tag{7.25}$$

If we now define the "energy" $E(t)$ of a solution by:

$$E(t) = ||u^m_t(t,\cdot)||^2_{L^2(\Sigma)} + B[u^m, u^m; t] \tag{7.26}$$

then inequality (7.24) reads

$$\frac{dE(t)}{dt} \leqslant C_3E(t) + C_3||f(t,\cdot)||^2_{L^2(\Sigma, N\nu_\gamma)} \tag{7.27}$$

and an application of Gronwall's inequality gives the estimate

$$E(t) \leqslant e^{C_4 t}\left(E(0) + C_3\int_0^t ||f(\tau,\cdot)||^2_{L^2(\Sigma, N\nu_\gamma)}d\tau\right) \tag{7.28}$$

However, we also have

$$E(0) \leqslant C_5\left(||u_0||^2_{H^1_0(\Sigma)} + ||u_1||^2_{L^2(\Sigma)}\right) \tag{7.29}$$

which follows from the initial conditions for the approximate solutions together with $||u^m(0)||^2_{H^1_0(\Sigma)} \leqslant ||u_0||^2_{H^1_0(\Sigma)}$, $||\dot{u}^m(0)||^2_{L^2(\Sigma)} \leqslant ||u_1||^2_{L^2(\Sigma)}$.
Thus, we obtain

$$\begin{aligned} & max_{t\in(0,T]}\left(||u_t^m||^2_{L^2(\Sigma)} + B[u^m,u^m;t]\right) \\ \leqslant C_6 & \left(||f||^2_{L^2([0,T];L^2(\Sigma,N\nu_\gamma))} + ||u_0||^2_{H_0^1(\Sigma)} + ||u_1||^2_{L^2(\Sigma)}\right) \end{aligned} \tag{7.30}$$

Now we have from equation (7.14) that

$$\begin{aligned} (u_{tt}^m, w_k)_{L^2(\Sigma)} &= -B[u^m, w_k; t] + (f, w_k)_{L^2(\Sigma, N\nu_\gamma)} \\ &\leqslant C_7\left(||u^m||_{H_0^1(\Sigma)} + ||f||^2_{L^2(\Sigma)}\right)||w_k||_{H_0^1(\Sigma)} \end{aligned} \tag{7.31}$$

where we have used the bounds on $N, \sqrt{\gamma}$ given by the geometric condition 3 and the Cauchy-Schwartz inequality.
Since $(u_{tt}^m, w_k) = 0$ for $k > m$ by construction we have proved that

$$||u_{tt}^m||_{H^{-1}(\Sigma)} = \sup_{v\in span\{w_k\}} \frac{(u_{tt}^m, v)_{L^2(\Sigma)}}{||v||_{H_0^1(\Sigma)}} \tag{7.32}$$

$$\leqslant C_8 \sup_{v\in span\{w_k\}} \frac{|(f,v)_{L^2(\Sigma)}| + |B(u^m, v; t)|}{||v||_{H_0^1(\Sigma)}} \tag{7.33}$$

$$\leqslant C_9\left(||f||_{L^2(\Sigma))} + ||u^m||_{H_0^1(\Sigma)}\right) \tag{7.34}$$

Squaring the above inequality, integrating in time and using equation (7.30) we obtain

$$\int_0^T ||\ddot{u}||^2_{H^{-1}(\Sigma)}dt \leqslant C_{12}\left(||f||^2_{L^2([0,T];L^2(\Sigma,N\nu_\gamma))} + ||u_0||^2_{H_0^1(\Sigma)} + ||u_1||^2_{L^2(\Sigma)}\right) \tag{7.35}$$

which concludes the proof. ⊡

Convergence to solutions

We have shown that $\{u^m\}_{m=1}^\infty$ is bounded in $L^2(0,T;H_0^1(\Sigma))$, $\{\dot{u}^m\}_{m=1}^\infty$ is bounded in $L^2(0,T;L^2(\Sigma))$ and $\{u_{tt}^m\}_{m=1}^\infty$ is bounded in $L^2(0,T;H^{-1}(\Sigma))$
We now make use of the following Theorem [63]

Theorem 8 *Let X be a reflexive Banach space and suppose the sequence $\{u_k\}_{k=1}^\infty \subset X$ is bounded. Then there exists a sub-sequence $\{u_{k_j}\}_{j=1}^\infty \subset \{u_k\}_{k=1}^\infty$ and $u \in X$ such that $u_{k_j} \rightharpoonup u$, i.e. $\{u_{k_j}\}$ converges weakly to u.*

Using the theorem there exists a sub-sequence of approximate functions $\{u^{m_l}\}_{l=1}^\infty$ such that

- $u^{m_l} \rightharpoonup L^2(0,T;H_0^1(\Sigma))$
- $\dot{u}^{m_l} \rightharpoonup L^2(0,T;L^2(\Sigma))$
- $\ddot{u}^{m_l} \rightharpoonup L^2(0,T;H^{-1}(\Sigma))$

In order to show that this is a weak solution, we must now verify that the limit of the sequence satisfies conditions 1 and 2 of Definition 5.
To verify condition 1, we multiply (7.14) by a function $\phi(t) \in C^\infty([0,T])$ and integrate with respect to time to give

$$\int_0^T \left((u_{tt}^{m_l}, \phi(t) w_k)_{L^2(\Sigma)} + B[u^{m_l}, \phi(t) w_k; t] \right) dt = \int_0^T (f, \phi(t) w_k)_{L^2(\Sigma, N\nu_\gamma)} \, dt \quad (7.36)$$

Then taking the limit as $m_l \to \infty$, we obtain

$$\int_0^T (< \ddot{u}, \phi(t) w_k > + B[u, \phi(t) w_k; t]) \, dt = \int_0^T (f, \phi(t) w_k)_{L^2(\Sigma, N\nu_\gamma)} \, dt \quad (7.37)$$

Thus for any test function of the form $v = \sum_{k=1}^N \phi^k(t) w_k(x)$ we have that equality (16.41) is satisfied. Moreover, test functions of that form are dense in $L^2(0,T; H_0^1(\Sigma))$. Therefore, we have shown that

$$\int_0^T (< \ddot{u}, v > + B[u, v; t]) \, dt = \int_0^T (f, v)_{L^2(\Sigma, N\nu_\gamma)} \, dt = \int_{\Sigma_{(0,T]}} f v \nu_g \quad (7.38)$$

for any $v \in L^2(0,T; H_0^1(\Sigma))$. ⊡
Finally we need to verify that the solution also satisfies the initial conditions.
We start by choosing any function $v \in C^2([0,T]; H_0^1(\Sigma))$ with $v(T) = \dot{v}(T) = 0$. Using it as a test function and integrating by parts twice with respect to t in (16.42), we find

$$\begin{aligned} &\int_0^T (< u, v_{tt} > + B[u, v; t]) \, dt \\ &= \int_0^T (f, v)_{L^2(\Sigma, N\nu_\gamma)} \, dt - (u(0), \dot{v}(0))_{L^2(\Sigma, \nu_\gamma)} + < \dot{u}(0), v(0) > \end{aligned} \quad (7.39)$$

and we also have that the m-approximate solutions satisfy

$$\begin{aligned} &\int_0^T \left((u^{m_l}, v_{tt})_{L^2(\Sigma)} + B[u^{m_l}, v; t] \right) dt \\ &= \int_0^T (f, v)_{L^2(\Sigma, N\nu_\gamma)} \, dt - (u^{m_l}(0), \dot{v}(0))_{L^2(\Sigma, \nu_\gamma)} + (\dot{u}^{m_l}(0), v(0))_{L^2(\Sigma, \nu_\gamma)} \end{aligned} \quad (7.40)$$

which follows from (7.36) and integrating by parts twice in t again.
Using the initial condition (7.16) and the fact that $\{w_k\}$ is a basis of $L^2(\Sigma_0)$ we obtain

$$u^{m_l}(0, \cdot) \to u_0 \text{ in } L^2(\Sigma_0) \quad (7.41)$$

$$\dot{u}^{m_l}(0, \cdot) \to u_1 \text{ in } L^2(\Sigma_0) \quad (7.42)$$

and therefore again taking the limit $m \to \infty$ we have

$$\begin{aligned} &\int_0^T (< u, v_{tt} > + B[u, v; t]) \, dt \\ &= \int_0^T (f, v)_{L^2(\Sigma, N\nu_\gamma)} \, dt - (u_0, \dot{v}(0))_{L^2(\Sigma, \nu_\gamma)} + (u_1, v(0))_{L^2(\Sigma, \nu_\gamma)} \end{aligned} \quad (7.43)$$

Therefore comparing (7.39) and (7.43) and using that v was arbitrary we conclude that

$$u(0) = u_0 \tag{7.44}$$
$$\dot{u}(0) = u_1 \tag{7.45}$$

Therefore, the limit $u^{m_l} \rightharpoonup u$ gives the desired weak solution. ⊡

7.2.3 Uniqueness and stability with respect to the initial data

The proof of uniqueness and stability relies on the energy estimate (9.106). By letting m tend to infinity and using the fact that the norm is sequentially weakly lower-semicontinuous [63], we obtain the bound that the weak solution satisfies

$$\begin{aligned} &max_{t\in(0,T]}\left(||u(t,\cdot)||_{H_0^1(\Sigma)} + ||\dot{u}(t,\cdot)||_{L^2(\Sigma)}\right) \\ &\leqslant C\left(||f||_{L^2([0,T];L^2(\Sigma,N\nu_\gamma))} + ||u_0||_{H_0^1(\Sigma)} + ||u_1||_{L^2(\Sigma)}\right) \end{aligned} \tag{7.46}$$

Therefore, if $u = w_1 - w_2$ is the difference between two weak solutions satisfying the same initial conditions u_0, u_1 with the same source function f, then u is a weak solution with vanishing initial data $u_0 = u_1 = 0$ and source function $f = 0$.
Hence

$$\left(||u(t,\cdot)||_{H_0^1(\Sigma)} + ||\dot{u}(t,\cdot)||_{L^2(\Sigma)}\right) \leqslant 0 \tag{7.47}$$

for all $0 \leqslant t \leqslant T$ which implies $u = 0$ and therefore $w_1 = w_2$.
We now prove the stability of the solution with respect to the initial data. To make the concept precise, we say that the solution u is continuously stable in $H^1(\Sigma_{(0,T]})$ with respect to initial data in $H_0^1(\Sigma_0) \times L^2(\Sigma_0)$, if given $\epsilon > 0$ there is a δ depending on u_0, u_1 and source function f such that if:

$$\|u_0 - \tilde{u_0}\|_{H_0^1(\Sigma_0)} \leqslant \delta, \text{ and } \|u_1 - \tilde{u_1}\|_{L^2(\Sigma_0)} \leqslant \delta \tag{7.48}$$

for $(u_0, u_1) \in H_0^1(\Sigma_0) \times L^2(\Sigma_0)$ and

$$\|f - \tilde{f}\|_{L^2([0,T];L^2(\Sigma,N\nu_\gamma))} \leqslant \delta \tag{7.49}$$

for $f \in L^2([0,T]; L^2(\Sigma, N\nu_\gamma))$ then

$$\|u - \tilde{u}\|_{H_0^1(\Sigma_{(0,T]})} \leqslant \epsilon \tag{7.50}$$

where $\tilde{u}$ is a solution with initial data given by $\tilde{u}|_{\Sigma_0} = \tilde{u_0}$ and $\tilde{u}_t|_{\Sigma_0} = \tilde{u_1}$ with source function $\tilde{f}$.
Now squaring (7.46) and integrating in time from 0 to $\tau \leqslant T$ we have:

$$\left(\|u - \tilde{u}\|_{\Sigma_{(0,T]}}\right)^2 \leqslant \int_0^\tau \left(||u(t,\cdot)||^2_{H_0^1(\Sigma)} + ||\dot{u}(t,\cdot)||^2_{L^2(\Sigma)}\right) dt \tag{7.51}$$

$$\begin{aligned} &\leqslant K \int_0^\tau \left(||u_0 - \tilde{u_0}||^2_{H_0^1(\Sigma)} + ||u_1 - \tilde{u_1}||^2_{L^2(\Sigma)}\right) \\ &+ \|f - \tilde{f}\|_{L^2([0,T];L^2(\Sigma,N\nu_\gamma))} dt \end{aligned} \tag{7.52}$$

for $\tau \leqslant T$ and with K a suitable constant.
Further choosing $\delta = \frac{\epsilon}{\sqrt{3\tau C}}$ we obtain the inequality:

$$\left(\|u - \tilde{u}\|_{\Sigma_{(0,T]}}\right)^2 \leqslant K \int_0^\tau \frac{2\epsilon^2}{3\tau K} + \frac{\epsilon^2}{3\tau K} dt \leqslant \epsilon^2 \tag{7.53}$$

which establishes stability with respect to the initial data.

7.2.4 Integrability of the energy momentum tensor

The regularity of the solutions allows us to interpret the energy momentum tensor of the scalar field u as a tensor with $L^1(\Sigma_{(0,T]})$ components given by

$$T^{ab}[u] = \left(g^{ac}g^{bd} - \frac{1}{2}g^{ab}g^{cd}\right)u_c u_d - \frac{1}{2}g^{ab}u^2 \tag{7.54}$$

We now show that $T^{ab}[u](t,\cdot)$ is in $L^1(\Sigma_t)$ for all $0 \leqslant t \leqslant T$. To prove this, notice that it is enough to show that $u \in C([0,T]; H^1_0(\Sigma)) \cap C^1(0,T; L^2(\Sigma))$. This result also allows us to establish the existence and uniqueness of solutions in $\Sigma_{(0,T]}$ given initial data on any hypersurface Σ_t with $0 \leqslant t \leqslant T$. In this section we closely follow the exposition given in [68].

Proposition 4 *Let u be a weak solution as defined in Definition 7 with $f \in L^2(\Sigma_{(0,T]})$ and initial data $(u_0, u_1) \in H^1_0(\Sigma_0) \times L^2(\Sigma_0)$. Additionally let the metric satisfy the Geometric Conditions 3. Then $u \in C([0,T]; H^1_0(\Sigma)) \cap C^1(0,T; L^2(\Sigma))$.*

To prove this proposition, we use the following Lemma which can be found in [53]

Lemma 4 *Suppose that V, H are Hilbert spaces and $V \hookrightarrow H$ is densely and continuously embedded in H. If*

$$u \in L^\infty(0,T;V), \dot{u} \in L^2(0,T;H), \tag{7.55}$$

then $u \in C_w([0,T]; V)$ is weakly continuous.

Then from the fact that $H^1_0 \hookrightarrow L^2 \hookrightarrow H^{-1}$ and the energy estimate we have that $u \in C_w([0,T]; H^1_0(\Sigma))$ and $\dot{u} \in C_w([0,T]; L^2(\Sigma))$

Lemma 5 *Let u be a weak solution that satisfies $\ddot{u} + Lu \in L^2(0,T; L^2(\Sigma))$ Then*

$$E(t) = \left(||\dot{u}(t,\cdot)||^2_{L^2(\Sigma)} + B[u,u;t]\right) : (0,T] \to \mathbb{R} \tag{7.56}$$

is a continuous function.

Proof. We have, using the equations satisfied by the m-approximate solutions, (7.18) and integrating from 0 to T that

$$\begin{aligned}\sup_{0 \leqslant t \leqslant T} B[u^m(t), u^m(t); t] + ||u^m_t(t,\cdot)||_{L^2(\Sigma)} \\ = B[u^m_0, u^m_0; 0] + ||u^m_t(0,\cdot)||_{L^2(\Sigma)} \\ + \int_0^T \left(B_s[u^m(s), u^m(s); s] + 2(f(s), u^m_t(s))_{L^2(\Sigma, N\nu_\gamma)}\right) ds\end{aligned} \tag{7.57}$$

Now noticing that $\sup_{0\leqslant t\leqslant T}\Big(B[u(t),u(t);t]+||\dot{u}(t,\cdot)||_{L^2(\Sigma)}\Big)$ is a norm equivalent to $L^\infty((0,T),H_0^1(\Sigma))\times L^\infty((0,T),L^2(\Sigma))$, we can use the weak convergence of the subsequence $\{u^{m_l}\}$ to obtain

$$\begin{aligned}\sup_{0\leqslant t\leqslant T} B[u(t),u(t);t]+||\dot{u}(t,\cdot)||_{L^2(\Sigma)}&\\ \leqslant B[u_0,u_0;0]+||u_1||_{L^2(\Sigma)}&\\ +\limsup_{l\to\infty}\int_0^T\Big(B_s[u^{m_l}(s),u^{m_l}(s);s]+2(f(s),u_t^{m_l}(s))_{L^2(\Sigma,N\nu_\gamma)}ds\Big)&\end{aligned}\tag{7.58}$$

where $B_s[u^m(s),u^m(s);s]=\int_\Sigma\left(\gamma^{ij}(s,x)\gamma(s,x)\right)_s u_i^m u_j^m dx^n$ By letting T tend to zero, we find

$$\begin{aligned}\limsup_{T\to 0} B[u(t),u(t);t]+||\dot{u}(t,\cdot)||_{L^2(\Sigma)}&\\ \leqslant B[u_0,u_0;0]+||u_1||_{L^2(\Sigma)}&\end{aligned}\tag{7.59}$$

Since on the other hand, u and $\dot{u}$ are weakly semicontinous, we have

$$\begin{aligned}\liminf_{T\to 0} B[u(t),u(t);t]+||\dot{u}(t,\cdot)||_{L^2(\Sigma)}&\\ \geqslant B[u_0,u_0;0]+||u_1||_{L^2(\Sigma)}&\end{aligned}\tag{7.60}$$

Then (7.59) and (7.60) shows that $B[u(t),u(t);t]+||\dot{u}(t,\cdot)||_{L^2(\Sigma)}$ is continuous at $t=0$. However, we might choose another time as initial time and therefore we have that $E(t)$ is continuous. ⊡

We can now prove Proposition 4.

Proof of Proposition 4. Using the weak continuity of $\dot{u}$, the continuity of E and the continuity of a_t in H_0^1 we find that

$$\begin{aligned}||\dot{u}(t,\cdot)-\dot{u}(t_0,\cdot)||^2_{L^2(\Sigma)}+B[u(t,\cdot)-u(t_0,\cdot),u(t,\cdot)-u(t_0,\cdot);t_0]&\\ =||\dot{u}(t,\cdot)||^2_{L^2(\Sigma)}+||\dot{u}(t_0,\cdot)||^2_{L^2(\Sigma)}&\\ +B[u(t,\cdot),u(t,\cdot);t_0]+B[u(t_0,\cdot),u(t_0,\cdot);t_0]&\\ -2(\dot{u}(t,\cdot),\dot{u}(t_0,\cdot))_{L^2(\Sigma)}-2B[u(t,\cdot),u(t_0,\cdot);t_0]&\\ =||\dot{u}(t,\cdot)||^2_{L^2(\Sigma)}+B[u(t,\cdot),u(t,\cdot);t]&\\ +||\dot{u}(t_0,\cdot)||^2_{L^2(\Sigma)}+B[u(t_0,\cdot),u(t_0,\cdot);t_0]&\\ -2B[u(t,\cdot),u(t_0,\cdot);t_0]-2(\dot{u}(t,\cdot),\dot{u}(t_0,\cdot))_{L^2(\Sigma)}&\\ +B[u(t,\cdot),u(t,\cdot);t_0]-B[u(t,\cdot),u(t,\cdot);t]&\\ =E(t)+E(t_0)+B[u(t,\cdot),u(t,\cdot);t_0]-B[u(t,\cdot),u(t,\cdot);t]&\\ -2\left((\dot{u}(t,\cdot),\dot{u}(t_0,\cdot))_{L^2(\Sigma)}+B[u(t,\cdot),u(t_0,\cdot);t_0]\right)&\end{aligned}$$

Therefore taking the limit $t\to t_0$, we obtain

$$\begin{aligned}
&\lim_{t\to t_0}\left(||\dot u(t,\cdot)-\dot u(t_0,\cdot)||^2_{L^2(\Sigma)}+B[u(t,\cdot)-u(t_0,\cdot),u(t,\cdot)-u(t_0,\cdot);t_0]\right)\\
&=\lim_{t\to t_0} E(t)+E(t_0)+B[u(t,\cdot),u(t,\cdot);t_0]-B[u(t,\cdot),u(t,\cdot);t]\\
&-2\left((\dot u(t,\cdot),\dot u(t_0,\cdot))_{L^2(\Sigma)}+B[u(t,\cdot),u(t_0,\cdot);t_0]\right)\right)\\
&=E(t_0)+E(t_0)-2\left((\dot u(t_0,\cdot),\dot u(t_0,\cdot)+B[u(t_0,\cdot),u(t_0,\cdot);t_0]\right))\\
&=E(t_0)+E(t_0)-2\left(||\dot u(t_0,\cdot)||^2_{L^2(\Sigma)}+B[u(t_0,\cdot),u(t_0,\cdot);t_0]\right)\right)=0
\end{aligned} \tag{7.61}$$

Then using equation (7.25), we have

$$\begin{aligned}
&\lim_{t\to t_0}\left(||\dot u(t,\cdot)-\dot u(t_0,\cdot)||^2_{L^2(\Sigma)}+\theta||u(t,\cdot)-u(t_0,\cdot)||^2_{H^1_0(\Sigma)}\right) \qquad (7.62)\\
&\leqslant \lim_{t\to t_0}\left(||\dot u(t,\cdot)-\dot u(t_0,\cdot)||^2_{L^2(\Sigma)}+B[u(t,\cdot)-u(t_0,\cdot),u(t,\cdot)-u(t_0,\cdot);t_0]\right)
\end{aligned}$$

Hence using (7.61), we conclude

$$\begin{aligned}
&\lim_{t\to t_0}||\dot u(t,\cdot)-\dot u(t_0,\cdot)||_{L^2(\Sigma)}=0\\
&\lim_{t\to t_0}||u(t,\cdot)-u(t_0,\cdot)||_{H^1_0(\Sigma)}=0
\end{aligned} \tag{7.63}$$

so that u is an element of $C([0,T];H^1_0(\Sigma))\cap C^1(0,T;L^2(\Sigma))$ as required. ⊡

7.3 Applications

In this section we discuss how Theorem 6 applies to several physical scenarios. We treat the case of spacetimes with cosmic strings and show that these spacetimes, despite having regions where the curvature behaves as a distribution or in the case of dynamic cosmic strings even develop curvature singularities, the wave equation has well-posed classical dynamics.

Cosmic strings are topological defects that were potentially formed during a phase transition in the early universe. Current observations put tight constraints on the dimensionless string tension $G\mu \leqslant 10^{-8}$ (in Planck units) where $c = 1$, $G = m_{pl}^{-2}$ and μ is the mass per length [70]. Additionally, the effective thickness of a cosmic string is of the order 10^{-29} cm. [16]. This extremely small width justifies what is called "the thin string limit". This is the metric around a static infinitely straight Nambu-Goto string lying along the z-axis satisfying Einstein's equations, which is "conical" in the plane transverse to the string. The line element is given by

$$ds^2 = dt^2 - dz^2 - d\rho^2 - (1-4G\mu)^2\rho^2 d\theta^2 \tag{7.64}$$

where $0 \leqslant \theta < 2\pi$.

By introducing a new angular coordinate $\tilde\theta = (1-4G\mu)\theta$, the spacetime can be seen to be flat everywhere except at $\rho = 0$; where there is an angular deficit of $2\pi(1-A)$ with $A = (1-4G\mu)$. We want to consider the region containing $\rho = 0$, so we transform to Cartesian coordinates $(x, y) = (\rho\cos\theta, \rho\sin\theta)$, which are regular at $\rho = 0$ and rewrite the line element as

$$ds^2 = dt^2 - \frac{x^2 + A^2 y^2}{x^2 + y^2} dx^2 - \frac{2xy(1 - A^2)}{x^2 + y^2} dxdy - \frac{y^2 + A^2 x^2}{x^2 + y^2} dy^2 - dz^2 \tag{7.65}$$

Notice that the metric has a direction dependent limit on the z-axis, so it fails to be C^0 at the axis although it remains bounded.
By direct inspection, one can see that the metric is bounded everywhere, $N = 1$, $\sqrt{\gamma} = A$ and given that $0 < A \leqslant 1$ the uniform ellipticity condition is satisfied. These conditions imply that under a rescaling of the time coordinate the hypothesis of Theorem 6 is satisfied.
In fact, we can consider a time dependent generalisation of this metric given by the line element

$$\begin{aligned} ds^2 = dt^2 - \frac{x^2 + A^2(t) y^2}{x^2 + y^2} dx^2 - \frac{2xy(1 - A^2(t))}{x^2 + y^2} dxdy \\ - \frac{y^2 + A^2(t) x^2}{x^2 + y^2} dy^2 - dz^2 \end{aligned} \tag{7.66}$$

This spacetime represents a dynamical cosmic string with a time dependent deficit angle. The metric satisfies the conditions of the Theorem 6 as long as the angle deficit satisfies $0 < A(t) \leqslant 1$ and the function $A(t)$ is C^1. Moreover, the dynamical cone can develop curvature singularities in contrast with the static cone [71].
Vickers [72, 73] showed that under certain conditions two-dimensional quasi-regular singularities can be seen as generalised strings. Moreover, the strings are totally geodesic and only the normal directions to the string are degenerate. For timelike generalised cosmic strings this guarantees that the time derivatives are not problematic.
Finally, assuming that the generalised cosmic string admits a $3 + 1$ splitting given by a family of L^∞ Riemannian metrics with suitable lapse and shift (see Geometric Conditions 3) and the angle deficit is chosen such that the uniform ellipticity condition is satisfied, we can conclude that generalised cosmic strings are H^1-wave regular.
Notice however that the spinning cosmic string metric given by

$$ds^2 = (dt + 4Jd\theta)^2 - dr^2 - A^2 r^2 d\theta^2 - dz^2 \tag{7.67}$$

does not satisfy the hypothesis of the theorem since $\beta^i \neq 0$ and there are no local coordinates containing a neighbourhood of the z-axis which make β^i vanish.

Part III

The quantum test-field probe.

Chapter 8

Introduction

8.1 Quantum fields on spacetimes with rough metrics

The precise details of the quantisation of the scalar wave equation

$$\Box_g \phi = 0 \tag{8.1}$$

on a general smooth and globally hyperbolic curved spacetime has not a universal well-defined prescription. However, the broad outlines of a suitable mathematical framework for the subject were laid down already in the period 1978-2000 [74, 75]. It is only in the last 10 years (and to a great extent stimulated by work on Einstein's equations) that a renaissance in the pure-mathematical analysis of linear hyperbolic differential equations and related questions in the differential geometry of Lorentzian manifolds has started to make it possible to fill certain fundamental mathematical gaps in this framework which were previously out of reach.

In this chapter we focus on a quantisation scheme where we quantise (8.1) in the case of a rough background and for which finite energy solutions exist (see definition 7 below). The reason why this problem is worth exploring, beyond the mathematical analysis, is grounded in the desire to model quantum matter (our most sophisticated description of matter) in physical scenarios that either requires finite regularity of the metric such as realistic models of stars [15] or perturbation of non-globally hyperbolic spacetimes such as the Kerr and Reissner-Nordström metrics (with their Cauchy Horizons) and also anti-de Sitter spacetime [3].

The *quantisation of the scalar field in a rough* spacetime, or with finite differentiability, requires in the first instance that the classical system is well-posed. Then, the quantisation procedure still needs to take place. This is a delicate issue. However, on the quantum field side, there have been advances regarding the correct notion of how to quantise the field when the regularity of the solution (not of the metric) is finite. In particular, the developments in the characterisation of suitable physical states (which correspond to the notion of Hadamard states [76] in the smooth case), under the name of N-adiabatic states is a key result [77]. In this chapter we do not discuss the Hadamard condition in detail however see the chapter about future outlook and open problems for a comment on this condition.

Chapter 9

Quantisation

In this chapter we describe how to perform the quantisation of a scalar field when one is considering low regularity in both the solutions of the scalar field and the metric tensor. In Section 9.1 we describe the general geometric background and the precise notion of a weak solution that we will use. In Section 9.1.2 we show that the solution space, with finite energy solutions, can be seen as a symplectic vector space and define a sympectomorphism which encodes the dynamics of the system. In Section 9.2.2 we give the algebraic construction of the observables and states of the quantum theory, and describe some key properties and classification of quantum states. In Section 9.4 we state the condition needed to implement the dynamics of the system as a unitary operator. In Section 9.5 we give an application of the framework developed.

9.1 Rough metrics and weak solutions

The purpose of this section is to describe the geometric regions we will consider. Also, we will define a precise notion of weak solutions which be used later in section 9.1.2 to define a finite energy solution.

9.1.1 The general setting

In the section below we describe the geometry of spacetime we will consider.
Let $\mathcal{M} = \Sigma \times [0, T]$ be an $n+1$-dimensional domain equipped with a Lorentzian metric g_{ab} where Σ is an compact closed n-dimensional manifold or a bounded open set of $\mathbb{R}^n$. Now using an $n+1$ decomposition of spacetime the line element of the metric may be written in the form:

$$ds^2 = +N^2 dt^2 - \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{9.1}$$

where N is the lapse function, β^i is the shift and γ_{ij} is the induced metric on Σ. The class of metrics we are going to consider requires that there is a foliation of the domain $\mathcal{M}$ and suitable coordinates (t, x^i) such that the following conditions hold:

Geometric Conditions 4

1. $\gamma^{ij} \in C^{0,1}(\mathcal{M})$
2. *The volume form given by* $\sqrt{\gamma}dx^n$*, for the induced metric* γ_{ij} *is bounded from below by a positive real number, i.e.,* $|\sqrt{\gamma}| > \eta$ *for some* $\eta \in \mathbb{R}^+$

3. The lapse function N can be chosen as $N = \sqrt{\gamma}$.

4. The shift can be chosen in such a way that $\beta^i = 0$

5. There exist a constant $\theta > 0$ such that

$$\sum_{i,j=1}^{n} \gamma^{ij}\gamma\xi_i\xi_j \geqslant \theta|\xi|^2$$

for all $(t,x) \in \mathcal{M}, \xi \in \mathbb{R}^n$.

That means the operator given by $-L(\cdot) := -\partial_j(\gamma^{ij}\gamma\partial_i(\cdot))$ is uniformly elliptic.

These conditions are similar to the Geometric Conditions 3 with one important difference. We are requiring higher regularity of the metric as can be seen in condition 1. This higher regularity requirement is needed because the solutions we will use also need higher regularity (See Definition 9). Our choice of lapse can be changed but the precise notion of weak solution involved also needs to be modified accordingly.

9.1.2 Weak Solutions

We now define the notion of a weak solution to the hyperbolic initial/boundary problem

$$\Box_g u = 0 \text{ in } \mathcal{M} \tag{9.2}$$

$$u(0,x) = u_0 \text{ on } \Sigma_0 = \Sigma \times \{t=0\} \tag{9.3}$$

$$\dot{u}(0,x) = w \text{ on } \Sigma_0 = \Sigma \times \{t=0\} \tag{9.4}$$

If Σ is a bounded open set of $\mathbb{R}^n$ we impose the boundary conditions

$$u = 0 \text{ on } \partial\Sigma \times [0,T] \tag{9.5}$$

First, we notice that the Geometric Conditions 4 allows us to write

$$\Box_g u = \frac{\ddot{u}}{\gamma} - \frac{Lu}{\gamma} \tag{9.6}$$

where $-L$ is an elliptic operator in divergence form given by:

$$-Lu = -(\gamma^{ij}\gamma u_j)_i \tag{9.7}$$

Then, we can associate to the operator $-L$ the bilinear form given by:

$$B[u,v;t] := \int_{\Sigma} \gamma^{ij}(t,x)\gamma(t,x)u_i v_j dx^n \tag{9.8}$$

Now using this bilinear form we make the following definition of a weak solution

Definition 7 *We say a function:*

$$u \in L^2(0,T;H^1(\Sigma)), \text{ with } \dot{u} \in L^2(0,T;L^2(\Sigma)), \ddot{u} \in L^2(0,T;H^{-1}(\Sigma))$$

is a **finite energy solution** *provided that locally:*

1. For each $v \in L^2(0,T;H^1(\Sigma))$,

$$\int_0^T (< \ddot{u}(t,\cdot), v(t,\cdot) > +B[u,v;t])dt = 0 \tag{9.9}$$

where $< \cdot, \cdot >$ *denotes the dual pairing between the* $H^{-1}(\Sigma)$ *and* $H^1(\Sigma)$ *Sobolev spaces*

2.

$$u|_{\Sigma_0} = u(0,x) = u_0(x) \tag{9.10}$$

$$n^a \nabla_a u|_{\Sigma_0} = \frac{\dot{u}(0,x)}{\sqrt{\gamma(0,x)}} = u_1(x) \tag{9.11}$$

where n^a *is the unit normal to* Σ_t, $u_0 \in H^1(\Sigma_0)$ *and* $u_1 \in L^2(\Sigma_0)$

The space of such solutions with the Geometric Conditions 4 will provide a suitable space for the classical theory which then we will quantise.

9.2 Classical structure

The quantisation of a classical theory requires us to change the mathematical structure of the theory. For example, in a Hilbert representation procedure: physical states are represented by vectors in a Hilbert space, $\mathcal{H}$, and field observables are represented by self-adjoint operators [74]. One can also consider other alternative methods of quantisation such as the so called algebraic quantisation. In this framework one construct a unique up to $*$-isomorphism a C^*-algebra, $CCR(V,\Xi)$ (see Appendix for the formal definitions of a C^*-algebra and $CCR(V,\Xi)$) which satisfy the canonical commutation relations and which represent the quantum observables of the theory, then the states of the theory are given by certain positive linear functional on the algebra [74, 75] (see Table 9.1 and 9.2). However, the states of the classical theory are represented by vectors in a symplectic space, (V,Ξ), which is a vector space V over a field F (for example $\mathbb{R}$ or $\mathbb{C}$) equipped with a symplectic structure, Ξ, defined by the following properties

- a bi-linear map $\Xi : V \times V \to F$
- alternating: $\Xi(u,v) = -\Xi(v,u)$ holds for all $u, v \in V$, and
- nondegenerate: $\Xi(u,v) = 0$ for all $v \in V$ implies that $u = 0$.

The classical observables are defined as the smooth functional from V into F. Moreover, one can define the dynamics classically as a symplectomorphism. To be precise, given (V_1,Ξ_1) and (V_2,Ξ_2) be two symplectic vector spaces and $S : V_1 \to V_2$ a linear map. The map S is a *symplectomorphism* if and only if $\Xi_2(Sv,Su) = \Xi_1(v,u)$ for all $v, u \in V_1$. A more detailed description of the symplectomorphism that encodes the dynamics will be given below.

The key point we address in this section is the requirement of constructing a suitable symplectic space, (V,Ξ), that encodes the classical field theory on the rough spacetime. For this purpose we define the space of "symplectic" solutions (see definition 9) and show that there is a symplectic structure associated to it. Moreover, we define the symplectomorphism which encodes the dynamics of the theory.

Table 9.1: Hilbert representation

	Quantum Systems	Classical Systems
States	Hilbert Space $\mathcal{H}$	Symplectic Space (V, Ξ)
Observables	$\{\hat{O}\vert\hat{O} : \mathcal{H} \to \mathcal{H}$ self-adjoint $\}$	$\{f\vert f : V \to \mathbb{R}$ smooth$\}$

Table 9.2: Algebraic quantisation

	Quantum Systems	Classical Systems
States	$\{\omega\vert\omega : CCR(V, \Xi) \to CCR(V, \Xi)$ positive $\}$	Symplectic Space (V, Ξ)
Observables	$CCR(V, \Xi)$	$\{f\vert f : V \to \mathbb{R}$ smooth$\}$

In the smooth case, that is when one is considering a C^∞-globally hyperbolic space-times $\mathcal{M}$, there are at least two options for generating a symplectic form for the classical field theory [74].

9.2.1 Propagators

The first one relies on the existence of advanced and retarded propagators G^+ and G^-. These propagators are defined precisely below

Definition 8 *Let M be a time-oriented connected Lorentzian manifold. A linear map $G^+ : D(M) \to C^\infty(M)$ satisfying*

- $\Box_g G^+ = id_{D(M)}$
- $G^+ \Box_g|_{D(M)} = id_{D(M)}$
- *$supp(G^+\psi) \subset J^+(supp(\psi))$ for all $\psi \in D(M)$*

is called an advanced propagator for $\Box_g$. Similarly, a linear map $G^- : D(M) \to C^\infty(M)$ satisfying

- $\Box_g G^+ = id_{D(M)}$
- $G^+ \Box_g|_{D(M)} = id_{D(M)}$
- *$supp(G^-\psi) \subset J^-(supp(\psi))$ for all $\psi \in D(M)$*

is called a retarded propagator for $\Box_g$.

If these operators exist, we can form the linear map

$$G := G^+ - G^- : D(M) \to C^\infty(M). \tag{9.12}$$

We called this linear map the *causal propagator*. There are other class of propagators such as the Feynman propagator (see Part IV of the thesis).
The causal propagator has some very useful properties that we prove in the lemma below

Lemma 6 *Let G be a causal propagator then*

1. *$G : D(M) \to \mathcal{S}_{sc}$ where $\psi \in \mathcal{S}_{sc}$ if $\Box_g \psi = 0$ and $\psi|_{\Sigma_t}$ and $n^a \nabla_a \psi|_{\Sigma_t}$ are compactly supported for all t.*

2. *The sequence of linear maps*

$$0 \xrightarrow{\Box_g} D(M) \xrightarrow{G} \mathcal{S}_{sc} \xrightarrow{\Box_g} \mathcal{S}_{sc}$$

is a complex, i.e., the composition of any two subsequent maps is zero.

Proof.
By the definition of the advanced and retarded function we have

$$\begin{aligned} \Box_g G\psi &= \Box_g(G^+\psi - G^-\psi) &\quad (9.13)\\ &= id_{D(M)}\psi - id_{D(M)}\psi = 0 &\quad (9.14) \end{aligned}$$

Now in globally hyperbolic spacetimes we have the following fact. Given a Cauchy surface, Σ, in a globally hyperbolic spacetime M and K a compact subset of $\mathcal{M}$, then $J^{\pm}(K) \cap \Sigma$ is compact [75].Therefore, we know that if we choose $K = supp(\psi)$ we have that $J^{\pm}(supp(\psi)) \cap \Sigma$ is compact. By definition of the advanced and retarded function we have that $supp(G^{\pm}\psi) \subset J^{\pm}(supp(\psi))$ and also by definition of the propagators and the definition of a Cauchy surface, $supp(G^{\pm}\psi)$ and Σ, are closed sets. This implies that $supp(G^{\pm}\psi) \cap \Sigma$ is closed. Moreover, we have the inclusion $supp(G^{\pm}\psi) \cap \Sigma \subset J^{\pm}(supp(\psi)) \cap \Sigma$. Hence, $supp(G^{\pm}\psi) \cap \Sigma$ is a compact set because it is a closed set contained in a compact set. This proves that $G\psi \in \mathcal{S}_{sc}$. To prove point 2 we notice that by definition of the propagators $G\Box_g = \Box_g G = 0$ in $D(M)$. By the result of the first part of the lemma we have that $G(D(M)) \subseteq \mathcal{S}_{sc}$ which concludes the proof. ⊡
We now use the causal propagator to define the symplectic space for any C^{∞} globally hyperbolic Lorentzian manifolds where G^+ and G^- always exist [75].
We first define

$$\tilde{\Omega} : D(M) \times D(M) \to \mathbb{R} \tag{9.15}$$

by

$$\tilde{\Omega}(\psi, \phi) := \int_M G(\psi)\phi\nu_g \tag{9.16}$$

This is not yet the desired structure as it is degenerate due to the fact that if $\phi \in ker(G)$, then $\tilde{\Omega}(\psi, \cdot) = 0$. Nevertheless, we show two important facts about $\tilde{\Omega}$.

Proposition 5 *Let $\tilde{\Omega}$ be defined as in equation (9.16). Then,*

1. *$\tilde{\Omega}$ is bilinear*

2. *$\tilde{\Omega}$ is skew-symmetric*

Proof. Linearity follows directly from the definition of the propagators as linear maps. To show that $\tilde{\Omega}$ is skew-symmetric we notice that

$$\int_M G^{\pm}(\psi)\phi\nu_g = \int_M G^{\pm}(\psi)\Box_g G^{\mp}(\phi)\nu_g \quad (9.17)$$
$$= \int_M \Box_g G^{\pm}(\psi)G^{\mp}(\phi)\nu_g \quad (9.18)$$
$$= \int_M \psi G^{\mp}(\phi)\nu_g \quad (9.19)$$

where we have used the self-adjointness of $\Box_g$, the adjoint operator of $G^{\pm}$ and that $G^{\mp}\phi \cap G^{\pm}\psi$ is compact in a globally hyperbolic spacetime [75].
Therefore we have

$$\tilde{\Omega}(\psi,\phi) = \int_M G(\psi)\phi\nu_g \quad (9.20)$$
$$= -\int_M \psi G(\phi)\nu_g \quad (9.21)$$
$$= -\tilde{\Omega}(\phi,\psi) \quad (9.22)$$

⊡
As already noted, $\tilde{\Omega}$, is not a symplectic structure as it is degenerate. However, if we define the equivalence relation $\psi \sim \phi$ if and only if $\psi - \phi \in ker(G)$; then we have that the quotient space $D(M)/ker(G)$ with the symplectic structure

$$\tilde{\Omega}([\psi],[\phi]) =: \tilde{\Omega}(\psi,\phi),$$

where $[\psi],[\phi] \in D(M)/ker(G)$ is a symplectic space. We area abusing notation and denoting the symplectic structure in the quotient and the degenerate skew-from with the same symbol $\tilde{\Omega}$.
From this point on, one can use the tools in the Appendix to generate the algebra that corresponds to the quantum observables of the theory.

9.2.2 Space of solutions

The second method for constructing the symplectic space is to consider the space of solutions $\mathcal{S}_{sc}$. This is a linear space, by linearity of the wave equation, and define the symplectic structure

$$\Xi(\Psi_1,\Psi_2) =: \int_{\Sigma_t} \pi_1\varphi_2 - \pi_2\varphi_1 \quad (9.23)$$

where $\varphi_{1,2} = \Psi_{1,2}|_{\Sigma_t}$ and $\pi_{1,2} = n^a\nabla_a\Psi_{1,2}\sqrt{h}d^nx|_{\Sigma_t}$.
That Ξ is a symplectic structure and that it is independent of Σ_t will be shown below in the low regularity case, so for the moment, we omit those proofs.
In the C^∞-globally hyperbolic case one can show that the two constructions are equivalent using the causal propagator G. To be precise, we have the following proposition

Proposition 6 *Let $(\mathcal{S}_{sc},\Xi)$ and $(D(M)\backslash ker(G),\tilde{\Omega})$ defined as above. Then we have*

$$\Xi(\Psi_1,\Psi_2) = \tilde{\Omega}([\psi],[\phi]) \quad (9.24)$$

where $\Psi_1 = G(\psi), \Psi_2 = G(\phi)$.

We sketch the proof below. A formal proof of can be found in [78].
Proof.
Let $\Psi_1 \in \mathcal{S}_{sc}$ and $\phi \in D(M)$ such that $supp(\phi) \subset [t_1, t_2] \times \Sigma$. Then, integrating by parts twice we have

$$\int_{\mathcal{M}} \Psi_1 \Box_g G^+ \phi \nu_g = \int_{\mathcal{M}} \Box_g \Psi_1 G^+ \phi \nu_g \\ - \int_{\Sigma_{t_2}} \left(\Psi_1 \frac{\partial G^+ \phi}{\partial t} - G^+ \phi \frac{\partial \Psi_1}{\partial t} \right) \nu_h + \int_{\Sigma_{t_1}} \left(\Psi_1 \frac{\partial G^+ \phi}{\partial t} - G^+ \phi \frac{\partial \Psi_1}{\partial t} \right) \nu_h \tag{9.25}$$

The using the fact that $\Box_g \Psi_1 = 0$ and that $\Sigma_{t_1} \not\subseteq supp(G^+(\phi))$ we obtain

$$\int_{\mathcal{M}} \Psi_1 \Box_g G^+ \phi \nu_g = - \int_{\Sigma_{t_2}} \left(\Psi_1 \frac{\partial G^+ \phi}{\partial t} - G^+ \phi \frac{\partial \Psi_1}{\partial t} \right) \nu_h \tag{9.26}$$

Moreover $\Sigma_{t_2} \not\subseteq supp(G^-(\phi))$ and therefore $G^+\phi = G\phi$ in Σ_{t_2} which implies

$$\int_{\mathcal{M}} \Psi_1 \Box_g G^+ \phi \nu_g = - \int_{\Sigma_{t_2}} \left(\Psi_1 \frac{\partial G \phi}{\partial t} - G \phi \frac{\partial \Psi_1}{\partial t} \right) \nu_h \tag{9.27}$$

If we define $\Psi_2 := G\phi$ and use that $G^+\phi = G\phi$ we have

$$\int_{\mathcal{M}} \Psi_1 \phi \nu_g = -\Xi(\Psi_1, \Psi_2) \tag{9.28}$$

Now we show that there is a $\psi \in D(M)$ such that $G\psi = \Psi_1$.
First, we notice that without loss of generality $\text{supp}(\Psi_1) \subset I^+(K) \cup I^-(K)$ for a compact set K of M. Using a partition of unity subordinated to the open covering $\{I^+(K), I^-(K)\}$ write Ψ_1 as $\Psi_1 = \mathit{\Psi}_1 + \mathit{\Psi}_2$ where $\text{supp}(\mathit{\Psi}_1) \subset I^+(K) \subset J^+(K)$ and $\text{supp}(\mathit{\Psi}_2) \subset I^-(K) \subset J^-(K)$.
Now we define $\psi := -\Box \mathit{\Psi}_2 = \Box \mathit{\Psi}_1$ which implies that $\text{supp}(\psi) \subset J^+(K) \cap J^-(K)$, hence using the globally hyperbolic condition we have that $\psi \in D(M)$.
We check that $G^+\psi = \mathit{\Psi}_1$. Given $\varphi \in D(M)$ we have

$$\int_M \varphi G^+ \Box \mathit{\Psi}_1 = \int_M G^- \varphi \Box \mathit{\Psi}_1 = \int_M \Box G^- \varphi \mathit{\Psi}_1 = \int_M \varphi \mathit{\Psi}_1 \tag{9.29}$$

Similarly, one shows that $G^-\psi = -\mathit{\Psi}_2$.
Hence, $G\psi = \mathit{\Psi}_1 + \mathit{\Psi}_2 = \Psi$.

$$\int_{\mathcal{M}} \psi G(\phi) \nu_g = \Xi(\Psi_1, \Psi_2) \tag{9.30}$$

$\boxdot$.
Therefore for the C^∞-globally hyperbolic case both constructions are equivalent. In our study of quantum field theory with low regularity we choose to work with the second construction for several reasons. First, the definition of G requires the existence of advanced and retarded propagators which work as inverses of the operator $\Box_g : D(M) \to D(M)$. When the metric is no longer C^∞ we no longer have that $\Box_g$ goes from $D(M)$ to $D(M)$. Second, many constructions of $G^\pm$ use the existence of a parametrix.

If one uses for example Hadamard coefficients to construct the parametrix, this requires C^∞ regularity of the metric. Third, the range of G is smooth solutions of compact support, when the metric is no longer smooth, solutions of this regularity are no longer guaranteed. Although these problems may be fixed by a clever choice of function spaces, we do not pursue it here. As mentioned above we choose to work with the second structure which make use of a suitable space of solutions. Because we are going to work below with low regularity metrics satisfying the Geometric Conditions 4, we first do a bookkeeping exercise to find the regularity needed for the solutions to show that Ξ is a symplectic vector space which is independent of the Cauchy surface Σ. This exercise, will allow us to define what we call the symplectic solutions and to define a suitable space of initial data.
We start our bookkepping exercise by noting that in order for $\tilde{\Omega}(\Psi_1, \Psi_2)$ to be well defined as a Lebegue integral we need

$$\Psi \in C^1(\mathbb{R}, L^2(\Sigma)) \cap C^0(\mathbb{R}, H^1(\Sigma)).$$

Also, Ξ should be independent of Σ_t for our solutions. That is we need:

$$\frac{d}{dt}\tilde{\Omega}\left(\Psi_1, \Psi_2\right) = \frac{d}{dt}\int_{\Sigma_t} \pi_1\varphi_2 - \pi_2\varphi_1 = 0 \tag{9.31}$$

We need to use differentiation under the integral sign and the equations of motion or we can also use that $j^\mu = \Psi_1\nabla^\mu\Psi_2 - \Psi_2\nabla^\mu\Psi_1$ is conserved for solutions. These conditions requires extra regularity and the regularity needed is

$$\Psi \in C^1(\mathbb{R}, H^1(\Sigma)) \cap C^0(\mathbb{R}, H^2(\Sigma)) \cap \Psi \in H^2(0,T; L^2(\Sigma)). \tag{9.32}$$

As an additional result, this regularity is also enough to have that

$$\int_M \nabla^a(T_{ab}\xi^b) = 0$$

where ξ^b is a smooth timelike killing vector field.
We now make the following definition, which allow us to define the minimum regularity that our solutions must have to recover the properties of the symplectic structure as in the smooth case.

Definition 9 *A solution u is a symplectic solution if it is a finite energy solution with regularity*

$$u \in C^1([0,T]; H^1(\Sigma)) \cap C^0([0,T]; H^2(\Sigma)), \ddot{u} \in L^2(0,T; L^2(\Sigma)).$$

Remark We call the set of all symplectic solutions $\mathcal{S}$.
We now prove the following proposition that shows the existence of suitable regular initial data that generate weak solutions of the required regularity.

Proposition 7 *Given the pairs $(u_0, w) \in H^2(\Sigma) \times H^1(\Sigma)$ as initial data, then there is a unique symplectic solution.*

We sketch the proof below. A detailed proof can be found in [63].
That finite energy solutions exist has been shown in Part II. Now, the higher regularity condition is shown by showing first higher regularity in time and elliptic regularity to show higher regularity in space.

To show higher regularity in time the method of proof is similar to the calculation done in chapter 7. We fix a positive integer m and differentiate with respect to time the identity (7.14). Writing $\tilde{u}^m := u^m_t$ we obtain

$$(\tilde{u}^m_{tt}, w_k)_{L^2(\Sigma)} + B[\tilde{u}^m, w_k; t] = 0 \tag{9.33}$$

where we using that we are considering the case $f = 0$.
Multiplying by $\ddot{d}^k_m(t)$ and arguing as in the proof of the energy estimates in chapter (7) we observe

$$\begin{aligned} & \frac{d}{dt}\left(||\tilde{u}^m_t||^2_{L^2(\Sigma)} + B[\tilde{u}^m, \tilde{u}^m; t]\right) && (9.34)\\ \leqslant\ & C_2\left(||\tilde{u}^m_t||^2_{L^2(\Sigma)} + ||\tilde{u}^m||^2_{H^1_0(\Sigma)}\right) && (9.35)\\ \leqslant\ & C_3\left(||\tilde{u}^m_t||^2_{L^2(\Sigma)} + B[\tilde{u}^m, \tilde{u}^m; t]\right) && (9.36)\end{aligned}$$

Then we consider the elliptic problem

$$B[u^m, w_k; t] = (-u^m_{tt}, w_k) \tag{9.37}$$

The using elliptic regularity which is Theorem 26 in the Appendix we obtain the estimate

$$||u^m||^2_{H^2(\Sigma)} \leqslant C\left(||u^m_{tt}||^2_{L^2(\Sigma)} + ||u^m||^2_{L^2(\Sigma)}\right) \tag{9.38}$$

Then using the estimate (9.38) in (9.36) and applying Gronwall's inequality, we deduce

$$\begin{aligned} max_{t\in(0,T]} & \left(||u^m(t,\cdot)||_{H^2(\Sigma)} + ||u^m_t(t,\cdot)||_{H^1(\Sigma)} + ||u^m_{tt}||_{L^2(\Sigma))}\right) \\ & \leqslant C\left(||u_0||_{H^2(\Sigma_0)} + ||u_1||_{H^1(\Sigma_0)}\right) \end{aligned} \tag{9.39}$$

Then passing to the limit as $m \to \infty$ we obtain the same estimate for the weak solution u.
We now define the set $\Gamma = \{(u_0, w)|(u_0, w) \in H^2(\Sigma_0) \times H^1(\Sigma_0)\}$ as the space of initial data and the space $\mathcal{C} := \{(\varphi, \pi) = (u_0, \frac{w}{\sqrt{h}}\nu_h)\}$ as the generalised phase space of the theory and recover the classical results of the symplectic structure in the low regularity case.

Proposition 8 *Let $\mathcal{C}$ be as above then $(\mathcal{C}, \Omega)$ is a symplectic space with the symplectic structure*

$$\Omega\left((\varphi_1, \pi_1), (\varphi_2, \pi_2)\right) =: \int_{\Sigma_0} \pi_1\varphi_2 - \pi_2\varphi_1 \tag{9.40}$$

Proof.
$\mathcal{C}$ forms a vector space over $\mathbb{R}$ by using the Banach space structure. Now we show that $\Omega\left((\varphi_1, \pi_1), (\varphi_2, \pi_2)\right)$ is a bilinear, alternating, non-degenerate form. Linearity follows from the following calculation

$$\begin{aligned}\Omega\left((\varphi_1,\pi_1)+(\varphi_3,\pi_3),(\varphi_2,\pi_2)\right) &= \int_{\Sigma_0}(\pi_1+\pi_3)\varphi_2-\pi_2(\varphi_1+\varphi_{3)} \qquad (9.41)\\ &= \int_{\Sigma_0}(\pi_1\varphi_2-\pi_2\varphi_1)+\int_{\Sigma_0}(\pi_3\varphi_2-\pi_2\varphi_3)\\ &= \Omega((\varphi_1,\pi_1),(\varphi_2,\pi_2))+\Omega((\varphi_3,\pi_3),(\varphi_2,\pi_2))\end{aligned}$$

The other part is analogous. Now to prove that the form is alternating we need to show that

$$\Omega(\varphi_1,\pi_1),(\varphi_1,\pi_1))=0$$

which follows from the definition. Finally we show that the form is non-degenerate. Let us assume we have a pair (φ_0,π_0) such that

$$\Omega\left((\varphi_1,\pi_1),(\varphi_0,\pi_0)\right)=:\int_{\Sigma_0}\pi_1\varphi_0-\pi_0\varphi_1=0 \qquad (9.42)$$

for all $(\varphi_1,\pi_1)\in\mathcal{C}$, then taking the elements $(\varphi_1,0),(0,\pi_1)$ with $\varphi_1,w_1\in C_0^\infty(\Sigma_0)$ we obtain

$$\begin{aligned}\Omega\left((\varphi_1,0),(\varphi_0,\pi_0)\right) &= -\int_{\Sigma_0}\frac{1}{\sqrt{h}}w_0\varphi_1\nu_h \qquad (9.43)\\ &= -\int_{\Sigma_0}w_0\varphi_1 d^nx=0 \qquad (9.44)\\ \Omega\left((0,\pi_1),(\varphi_0,\pi_0)\right) &= \int_{\Sigma_0}\frac{1}{\sqrt{h}}w_1\varphi_0\nu_h \qquad (9.45)\\ &= \int_{\Sigma_0}w_1\varphi_0 d^nx=0 \qquad (9.46)\end{aligned}$$

Therefore φ_0,w_0 belong to the orthogonal set of $C_0^\infty(\Sigma_0)$ which is a dense space in $L^2(\Sigma_0)$ and therefore $\varphi_0=\pi_0=0$ which show that Ω is non-degenerate. $\boxdot$.
Now the bijective linear map

$$\tau_0 \;:\; \mathcal{S}\to\mathcal{C} \qquad (9.47)$$

given by

$$u^1\mapsto(u^1(0,\cdot),u_t^1(0,\cdot)dx^n)=(\varphi_1,\pi_1) \qquad (9.48)$$

allow us to define a symplectic structure on $\mathcal{S}$

Proposition 9 *Let $\mathcal{S}$ be a non empty set. Then $(\mathcal{S},\Xi)$ is a symplectic space with the symplectic structure*

$$\Xi\left(u^1,u^2\right)=:\Omega(\tau_0(u^1),\tau_0(u^2)) \qquad (9.49)$$

Proof
Linearity and the fact that the form is alternating follows in the same manner as in the above case. Therefore, we only show the form is non degenerate. Notice that if there is a u^1 such that $\Xi\left(u^1,u^2\right)=0$ for all $u^2\in\mathcal{S}$ there is an element $\tau_0(u^1)=(\varphi_0,\pi_0)$ such that taking the elements $(\varphi_1,0),(0,\pi_1)$ with $\varphi_1,w_1\in C_0^\infty(\Sigma_0)$ we obtain

$$\begin{aligned}
\Omega\left((\varphi_1,0),(\varphi_0,\pi_0)\right) &= -\int_{\Sigma_0}\frac{1}{\sqrt{h}}w_0\varphi_1\nu_h && (9.50)\\
&= -\int_{\Sigma_0} w_0\varphi_1 d^n x = 0 && (9.51)\\
\Omega\left((0,\pi_1),(\varphi_0,\pi_0)\right) &= \int_{\Sigma_0}\frac{1}{\sqrt{h}}w_1\varphi_0\nu_h && (9.52)\\
&= \int_{\Sigma_0} w_1\varphi_0 d^n x = 0 && (9.53)\\
& && (9.54)
\end{aligned}$$

As before this implies φ_0,w_0 belong to the orthogonal set of $C_0^\infty(\Sigma_0)$ which is a dense space in $L^2(\Sigma_0)$ and therefore $\varphi_0=\pi_0=0$. Uniqueness of solutions implies that $u^1=0$. This show that Ξ is non-degenerate. $\boxdot$.
There is no special property of the time $t=0$ and therefore we could have chosen another hypersurface $\Sigma_t=\{t\}\times\Sigma$ and construct weak solutions to the problem

$$\Box_g u = 0 \text{ in } \Sigma_{[0,t)\cup(t,T)} \quad (9.55)$$
$$u(t,x) = u_0 \text{ on } \Sigma_t=\Sigma\times\{t\} \quad (9.56)$$
$$\dot{u}(t,x) = w \text{ on } \Sigma_t=\Sigma\times\{t\} \quad (9.57)$$

and have another identification map

$$\tau_t \; : \; \mathcal{S}\to\mathcal{C} \quad (9.58)$$
$$\tau_t \; : \; u^1\to(u^1(t,\cdot),u^1_t(t,\cdot)dx^n)=(\tilde{\varphi}_1,\tilde{\pi}_1) \quad (9.59)$$

and we would have symplectic structure on $\mathcal{S}$ given by

$$\tilde{\Xi}\left(u^1,u^2\right) =: \Omega(\tau_t(u^1),\tau_t(u^2)) \quad (9.60)$$

Since $(\tilde{\varphi}_1,\tilde{\pi}_1)\neq(\varphi_1,\pi_1)$ it is not clear that $\tilde{\Xi}\left(u^1,u^2\right)=\Xi\left(u^1,u^2\right)$. We show in the next proposition that this is indeed the case.

Proposition 10 *Let $\mathcal{S}$ be a non empty set. Then $(\mathcal{S},\Xi)$ and $(\mathcal{S},\tilde{\Xi})$ define the same symplectic vector space. Moreover, the map $\tau_t\tau_0^{-1}$ is a symplectomorphism.*

Proof.
We have

$$\begin{aligned}\tilde{\Xi}(u^1,u^2)=\Omega(\tau_t(u^1),\tau_t(u^2)) &= \int_\Sigma (u^2(t,\cdot)u^1_t(t,\cdot)-u^1(t,\cdot)u^2_t(t,\cdot))dx^n\\
&(u^2(t,\cdot),u^1_t(t,\cdot))_{L^2(\Sigma)}-(u^1(t,\cdot),u^2_t(t,\cdot))_{L^2(\Sigma)}\end{aligned} \quad (9.61)$$

and

$$\Xi(u^1,u^2) = \Omega(\tau_0(u^1),\tau_0(u^2)) = \int_\Sigma (u^2(0,\cdot)u^1_t(0,\cdot) - u^1(0,\cdot)u^2_t(0,\cdot))dx^n$$
$$(u^2(0,\cdot),u^1_t(0,\cdot))_{L^2(\Sigma)} - (u^1(0,\cdot),u^2_t(0,\cdot))_{L^2(\Sigma)} \tag{9.62}$$

Therefore using Theorem 23 in the Appendix, the definition of a weak solution as in Definition 7 and the regularity of the weak solution given in Definition 9 we have

$$\begin{aligned} &(u^2(t,\cdot),u^1_t(t,\cdot))_{L^2(\Sigma)} - (u^2(0,\cdot),u^1_t(0,\cdot))_{L^2(\Sigma)} \\ &= \textstyle\int_0^T \langle u^1_{tt}(t),u^2(t)\rangle_{H^{-1}(\Sigma),H^1_0(\Sigma)} + \langle \dot u^2(t),u^1_t(t)\rangle_{H^{-1}(\Sigma),H^1_0(\Sigma)}dt \\ &= \textstyle\int_0^T B[u^1(t),u^2(t)]dt + \int_0^T \langle \dot u^2(t),u^1_t(t)\rangle_{H^{-1}(\Sigma),H^1_0(\Sigma)}dt \end{aligned} \tag{9.63}$$

and

$$\begin{aligned} &(u^2_t(0,\cdot),u^1(0,\cdot))_{L^2(\Sigma)} - (u^2_t(t,\cdot),u^1(t,\cdot))_{L^2(\Sigma)} \\ &= -\textstyle\int_0^T \langle u^2_{tt}(t),u^1(t)\rangle_{H^{-1}(\Sigma),H^1_0(\Sigma)}dt - \int_0^T \langle \dot u^1(t),u^2_t(t)\rangle_{H^{-1}(\Sigma),H^1_0(\Sigma)}dt \\ &= -\textstyle\int_0^T B[u^2(t),u^1(t)]dt - \int_0^T \langle \dot u^1(t),u^2_t(t)\rangle_{H^{-1}(\Sigma),H^1_0(\Sigma)}dt \end{aligned} \tag{9.64}$$

Therefore using equation (9.63) and (9.64) we have

$$\tilde\Xi(u^1,u^2) - \Xi(u^1,u^2) = 0 \tag{9.65}$$

In fact using the fact that $\tilde\Xi\left(u^1,u^2\right) = \Xi\left(u^1,u^2\right)$ we have the following

$$\begin{aligned} \Omega((\varphi_1,\pi_1),(\varphi_1,\pi_1)) &= \Omega(\tau_0(u^1),\tau_0(u^2)) && (9.66)\\ &= \Xi(u^1,u^2) && (9.67)\\ &= \tilde\Xi(u^1,u^2) && (9.68)\\ &= \Omega(\tau_t(u^1),\tau_t(u^2)) && (9.69)\\ &= \Omega(\tau_t\tau_0^{-1}(\varphi_1,\pi_1),\tau_t\tau_0^{-1}(\varphi_1,\pi_1)) && (9.70) \end{aligned}$$

Therefore the map $\tau_t\tau_0^{-1} : \mathcal{C} \to \mathcal{C}$ is a symplectomorphism. We call the map $\tau_t\tau_0^{-1}$ the evolution map. Explicitly, we have

$$\tau_t\tau_0^{-1} : (u^1(0,\cdot),\dot u^1(0,\cdot)dx^n) \to (u^1(t,\cdot),\dot u^1(t,\cdot)dx^n) \tag{9.71}$$

9.3 Quantum observables and states

In this section we define the quantum observables and states of the theory in the algebraic approach. Then we describe certain states for which their GNS-representation (see Appendix for a proof of the GNS-representation theorem) contains certain physical properties.

We make the following definitions

Definition 10 *The fundamental observables for the quantum theory in a rough space-time are the elements of the CCR-representation, constructed from the symplectic vector space of weak solutions with finite energy,* $(\mathcal{S}, \Xi)$.

Definition 11 *A state,* ω*, of the quantum field is a linear map*

$$\omega : CCR(\mathcal{S}, \Xi) \to \mathbb{C}$$

satisfying

- $\omega(aa^*) \geqslant 0$ *for all* $a \in \mathcal{A}$.
- $\omega(I) = 1$ *where* I *denotes the identity element of* $\mathcal{A}$.

We now in the next example construct a Weyl system $(\mathcal{A}, W)$ (See the Appendix for a formal definition) for an arbitrary symplectic vector space (V, Ξ).

Example 1

Let $H = L^2(V, \mathbb{C})$ be the Hilbert space of square-integrable complex-valued functions on V with respect to the counting measure, i.e., H consists of those functions $F : V \to \mathbb{C}$ that vanish everywhere except for countably many points and satisfy

$$||F||^2_{L^2} := \sum_{\phi \in V} |F(\phi)|^2 < \infty \tag{9.72}$$

The hermitian product on H is given by

$$(F, G)_{L^2} = \sum_{\phi \in V} \overline{F(\phi)} \cdot G(\phi) \tag{9.73}$$

Let $\mathcal{A} := \mathcal{L}(H)$ be the C^* algebra of bounded linear operators on H as in example (2). We define the map $W : V \to \mathcal{A}$ by

$$(W(\phi)F)(\psi) := e^{\frac{i\Xi(\phi,\psi)}{2}} F(\phi + \psi) \tag{9.74}$$

It is clear that $W(\phi)$ is a bounded linear operator on H for any $\phi \in V$ and $W(0) = id_H = 1$.
To check the rest of the conditions in definition of a Weyl system, notice that making the substitution $\xi = \phi + \psi$ then

$$
\begin{aligned}
((W(\phi)F, G)_{L^2} &= \sum_{\psi \in V} \overline{(W(\phi)F)(\psi)} G(\psi) && (9.75)\\
&= \sum_{\psi \in V} \overline{e^{\frac{i\Xi(\phi,\psi)}{2}} F(\phi+\psi)} G(\psi) && (9.76)\\
&= \sum_{\xi \in V} \overline{e^{\frac{i\Xi(\phi,\xi-\phi)}{2}} F(\xi)} G(\xi-\phi) && (9.77)\\
&= \sum_{\xi \in V} \overline{e^{\frac{i\Xi(\phi,\xi)}{2}}}\, \overline{F(\xi)} G(\xi-\phi) && (9.78)\\
&= \sum_{\xi \in V} \overline{F(\xi)} \overline{e^{\frac{i\Xi(-\phi,\xi)}{2}}} G(\xi-\phi) && (9.79)\\
&= \sum_{\xi \in V} (F, W(-\phi)G)_{L^2} && (9.80)
\end{aligned}
$$

Hence $W(-\phi) = W(\phi)^*$.
Finally we compute

$$
\begin{aligned}
(W(\phi)(W(\psi)F))(\xi) &= e^{\frac{i\Xi(\phi,\xi)}{2}} (W(\psi)F)(\varphi+\xi)) && (9.81)\\
&= e^{\frac{i\Xi(\phi,\xi)}{2}} e^{\frac{i\Xi(\psi,\phi+\xi)}{2}} F(\varphi+\xi+\psi)) && (9.82)\\
&= e^{\frac{i\Xi(\phi,\Xi)}{2}} e^{\frac{i\Xi(\psi+\phi,\xi)}{2}} F(\varphi+\xi+\psi)) && (9.83)\\
&= e^{\frac{i\Xi(\phi,\psi)}{2}} (W(\phi+\psi)F)(\xi) && (9.84)
\end{aligned}
$$

Let $CCR(V,\Xi)$ be the C^*-subalgebra of $\mathcal{L}(H)$ generated by the elements $V(\phi), \phi \in V$. Then the $CCR(V,\Xi)$ together with the map W forms a Weyl-system $(\mathcal{A}, W)$ for (V,Ξ) We have shown that for every symplectic space (V,Ξ) there is a Weyl-system and also a CCR- representation. Uniqueness of the CCR- representation holds up to $*$-isomorphism (This is Theorem 27 in the Appendix).
In definition 11 we have given a broad definition of quantum states. This definition has the minimum requirements such that the GNS-theorem apply. However, the definition is rather formal and the amount of states available is far too big. Therefore, additional conditions need to be imposed in the state space (the space of of all quantum states). In this section we define at the beginning the notion of pure and mixed states which allow us to obtain the statistical description of quantum field theory.
The first states we define are the so called pure and mixed states.

Definition 12 *A state ω is said to be mixed if it can be expressed in the form*

$$\omega = c_1\omega_1 + c_2\omega_2 \tag{9.85}$$

where $c_1, c_2 > 0$ and ω_1, ω_2 are states.
Otherwise ω is said to be pure.

The mixed states are states such that the GNS-representation of $\mathcal{A}$ is reducible (there are closed proper subrepresentations under the action of $\{\pi_\omega(a) : a \in \mathcal{A}\}$), while for

pure states it is irreducible [81]. In terms of physical content statistical ensembles of pure states are mixed states.
The Fock representation of the scalar field in the Minkowski vacuum [74] may be mimiced by the so called quasi-free states in curved spacetime.
First, we let $\mu : \mathcal{S} \times \mathcal{S} \to \mathbb{R}$ be an arbitrary (real) inner product on $\mathcal{S}$. That satisfies

$$\mu(u_1, u_1)\mu(u_2, u_2) \geqslant \frac{1}{4}\left(\Xi(u_1, u_2)\right)^2 \tag{9.86}$$

Then, we have the following definition

Definition 13 *A state ω such that*

$$\omega[W(u)] = e^{-\frac{\mu(u,u)}{2}} \tag{9.87}$$

for all $u \in \mathcal{S}$ is called a quasi-free state.

The state extends to all $\mathcal{A}$ by linearity and continuity. In addition, for every element of the form $W(u)$, ω satisfy the positivity condition. However, it does not necessarily satisfy it for linear combinations, that is the reason for (9.86) which is a sufficient and necessary condition to guarantee positivity [74].
If ω fails to satisfy

$$\mu(u_1, u_1) = \text{ l.u.b}_{u_2 \neq 0} \frac{\left(\Xi(u_1, u_2)\right)^2}{4\mu(u_2, u_2)} \tag{9.88}$$

then the quasi-free state fails to be a pure state and is a mixed state. In stationary spacetimes, thermal equilibrium states (or KMS states defined below) at finite temperature are represented by quasi-free states which fail to satisfy (9.88) [74].
The fact that these states give Fock representations in curved spacetime is given by the following Proposition

Proposition 11 *Given the symplectic vector space $(\mathcal{S}, \Xi)$ and a inner product that satisfies (9.86) then there exist a pair (K, H) called the one-particle structure associated to $(\mathcal{S}, \Xi, \mu)$ where H is a complex Hilbert space and $K : \mathcal{S} \to H$ is a map satisfying*

1. *K is $\mathbb{R}$-linear and the map $K + iK : \mathcal{S} \to H$ is dense in H.*
2. *$(Ku, Kv)_H = \mu(u, v) + \frac{i}{2}\Xi(u, v)$ for all $u, v \in \mathcal{S}$*

Proof. The proof can be found in [74],
This one-particle structure would form the basis of our discussion about quantum dynamics in section 9.4.
Now, we will see the relationship between the symmetries of spacetime and the algebraic framework so far developed.
Given a one parameter group of isometries $\{\xi_s\}_{s\in\mathbb{R}}$ generated by a killing vector χ it is possible to define a one parameter group of $*$-algebra automorphism (see formal definition in the Appendix) $\alpha_s^\chi : \mathcal{A} \to \mathcal{A}$ by the condition

$$\alpha_s^\chi(1) = 1 \tag{9.89}$$
$$\alpha_s^\chi(W(u)) = W(u(\xi_s(\cdot))) \tag{9.90}$$

Then we have the following definition

Definition 14 *A state ω_χ is χ-invariant if*

$$\omega_\chi(\cdot) = \omega_\chi(\alpha_s^\chi(\cdot)) \tag{9.91}$$

for all $s \in \mathbb{R}$.

When passing to the GNS-representation. One can see [79] that there is a one-parameter family of unitary operators such that

$$\begin{aligned} U_s^\chi|\psi\rangle &= |\psi\rangle && (9.92)\\ U_s^\chi \pi_{\omega_\chi}(a) U_s^{\chi *} &= \pi_{\omega_\chi}(\alpha_s^{(\chi)}(a)) && (9.93) \end{aligned}$$

for all $s \in \mathbb{R}$ and $a \in \mathcal{A}$
If $\{U_s^\chi\}$ are strongly continuous there is a unique self-adjoint operator H, called the such that $U_s^\chi = e^{-ish}$ for every $s \in \mathbb{R}$ and $H|\psi\rangle = 0$. If χ is a timelike vector, the spectrum of H is in the interval $[0, \infty)$ and $|\psi\rangle$ is the unique vector with eigenvalue 0, then ω_χ is called a ground state.
Another important kind of invariant states are the KMS states. We say ω_β is a KMS state for α_s^χ of inverse temperature $\beta > 0$ if for each $a, b \in \mathcal{A}$ there exists an analytic function F which is holomorphic for $0 < lm(z) < \beta$ and continuous for $0 \leqslant Im(z) \leqslant \beta$ so that $\omega(a\alpha_s^\chi(b)) = F(s)$ and $\omega(\alpha_s^\chi(a)b) = F(s + i\beta)$ for all $s \in \mathbb{R}$. KMS states describe physical systems in thermal equilibrium where the dynamics is given by the $*$-algebra isomorphism $\{\alpha_s^\chi\}_{s\in\mathbb{R}}$ [80].

9.4 Unitary equivalence and quantum evolution

9.4.1 Unitary equivalence

In this subsection we will give sufficient and necessary condition for two quantum field theories defined by quasi free states ω_1, ω_2 with inner products μ_1 and μ_2 to be unitarily equivalent i.e. there is an unitary operator $\mathcal{U}$ such that $\mathcal{U}\pi_{\omega_1}(a)\mathcal{U}^{-1} = \pi_{\omega_2}$. The physical content of this assertion is that both theories give the same predictions such as the expectation value of the quantum observables with respect the cyclic vectors $|\psi\rangle_1, |\psi\rangle_2$. However notice that in general one will have $\mathcal{U}|\psi\rangle_1 \neq |\psi\rangle_2$ which corresponds to the creation of particles in quantum field theory.
The basic analysis of the issue of unitary equivalence can be found in [82].
Let μ_1 and μ_2 be two inner products in $\mathcal{S}$ satisfying (9.88). Then a necessary condition for unitary equivalence is that μ_1 and μ_2 are equivalent norms in $\mathcal{S}$. That is for all $u \in \mathcal{S}$ there are two positive constants M, N such that

$$M\mu_1(u, u) \leqslant \mu_2(u, u) \leqslant N\mu_1(u, u) \tag{9.94}$$

If this condition is not satisfied is easy to proof that a contradiction must arise if the theories are unitary equivalent (see [74] for the construction of such a contradiction). Notice that if μ_1 and μ_2 satisfy (9.94), then a Cauchy sequence in μ_1 is a Cauchy sequence if and only if it is a Cauchy sequence in μ_2. Therefore, the completion of $\mathcal{S}$ with respect μ_1 or μ_2 is equivalent. We call the (complexification) of this complete space ${\mathcal{S}_\mu}^{\mathbb{C}}$.
Condition (9.94) is necessary for the unitary equivalence of the theory, but it is not sufficient. Now, we will state the additional conditions to guarantee the unitary

equivalence. Let K_i where $i = 1,2$ be the orthogonal projection $K_i : \mathcal{S}^{\mathbb{C}}_\mu \to H_i$, and $\overline{K_i}$ the orthogonal projection $\overline{K_i} : \mathcal{S}^{\mathbb{C}}_\mu \to \overline{H_i}$ with respect the inner product $2\mu_i(\,\bar{u}\,,\,v\,) = (\,u\,,\,v\,)_{H_i}$.
We define $\alpha^2_1 := K_{1\upharpoonright H_2}$ and $\beta^2_1 := \overline{K_1}_{\upharpoonright H_2}$, and in an analogous way α^1_2 and β^1_2.
Then given $\chi, \zeta \in H_2$, we have that

$$
\begin{aligned}
(\chi,\zeta)_{H_2} &= \mu_2(\,\bar{\chi}\,,\,\zeta\,) \\
&= -i\Xi(\overline{\chi},\zeta) \qquad (9.95)\\
&= -i\Xi(\overline{K_1\chi + \overline{K_1}\chi}, K_1\zeta + \overline{K_1}\zeta) \\
&= (\alpha^2_1\chi, \alpha^2_1\zeta)_{H_1} - (\beta^2_1\chi, \beta^2_1\zeta)_{H_1}, \qquad (9.96)
\end{aligned}
$$

where we used the identity $K_1 + \overline{K_1} = I$, and the orthogonality of H_1 and $\overline{H_1}$ with respect to the inner product $2\mu_1$.
Moreover,

$$
\alpha^{2\dagger}_1\alpha^2_1 - \beta^{2\dagger}_1\beta^2_1 = I. \qquad (9.97)
$$

where $\alpha^\dagger$ denotes the hermitian adjoint operator of α.
In a similar way we obtain

$$
\begin{aligned}
\alpha^{2\dagger}_1\overline{\beta}^2_1 = \beta^{2\dagger}_1\overline{\alpha}^2_1 \quad &, \qquad \alpha^{2\dagger}_1 = \alpha^1_2, \\
\alpha^{1\dagger}_2\alpha^1_2 - \beta^{1\dagger}_2\beta^1_2 = I \quad &, \qquad \alpha^{1\dagger}_2\overline{\beta}^1_2 = \beta^{1\dagger}_2\overline{\alpha}^1_2, \quad \overline{\beta}^{2\dagger}_1 = -\beta^1_2 \qquad (9.98)
\end{aligned}
$$

The maps $\{\alpha^i_j, \beta^i_j \,|\, i,j = 1,2\,; i \neq j\}$ satisfying properties (9.97)-(9.98) are called *Bogoliubov transformations.*
The next theorem establish the sufficient and necessary conditions for the unitary equivalence of the theories.

Theorem 9 *The necessary and sufficient conditions for the quantum field theories defined by the norms μ_1 and μ_2 to be unitary equivalent are that*

1. *μ_1 and μ_2 satisfy (9.94).*
2. *The map β^1_2 (equivalently β^2_1) satisfy $\mathrm{Tr}(\beta^{1\dagger}_2\beta^1_2) < \infty$ (equivalently $\mathrm{Tr}(\beta^{2\dagger}_1\beta^2_1) < \infty$).*

Proof. For a proof of this theorem see [82].

9.4.2 Quantum evolution

In this subsection we focus in the quantum dynamics of the scalar field. As we have shown in (9.71) the evolution map between initial data encode the dynamics of the field. Now, we give a description of the dynamics instead in the space of Cauchy data, $\mathcal{C}$ in the space of solutions $\mathcal{S}$. We give first a brief description of the dynamics in the space of Cauchy data.

1. Consider Cauchy data $(\varphi,\pi)_{t_0} \in \mathcal{C}$ on the initial surface given by $t = t_0$. By well-posedness there is a unique solution $u = \tau^{-1}_{t_0}(\varphi,\pi)_{t_0}$ for such initial data ($\tau^{-1}_{t_0} : \mathcal{C} \to \mathcal{S}$ is the map between initial data an solutions).

2. Then we determine the Cauchy data $(\varphi,\pi)_{t_f}$ for the solution u *on the initial surface given by* $t=t_f$, where $t_f>t_0$. The relationship between u and $(\varphi,\pi)_{t_f}$ is through the map $\tau_{t_f}^{-1}:\mathcal{C}\to\mathcal{S}$ associated to the initial surface $t=t_f$; $u=\tau_{t_f}^{-1}(\varphi,\pi)_{t_f}$.

3. The map $t_{(t_f,t_0)}:=\tau_{t_f}\circ\tau_{t_0}^{-1}:\mathcal{C}\to\mathcal{C}$, $(\varphi,\pi)_{t_0}\mapsto(\varphi,\pi)_{t_f}$, is the finite temporal evolution in $\mathcal{C}$. As shown in Section 9.1.2, this is a symplectomorphism. The set of all such finite transformations with parameter t, $\{t_{(t,t_0)}\}$, will be denoted by $SP_t(\mathcal{C})$.

In the space of solutions the map

$$T_{(t_f,t_0)}=\tau_{t_0}^{-1}\circ t_{(t_f,t_0)}\circ\tau_{t_0}=\tau_{t_0}^{-1}\circ\tau_{t_f}:\mathcal{S}\to\mathcal{S}$$

encodes the dynamics.
If $\tilde{u}$ is the evolution of u (i.e., $\tilde{u}=T_{(t_f,t_0)}u$), notice that $I_{t_0}^{-1}\tilde{u}=I_{t_f}^{-1}u$; that is the evolved Cauchy data at $t=t_f$ of the *initial solution* u is equal to the Cauchy data at $t=t_0$ of the “evolved" solution $\tilde{u}$. The set of such map, $\{T_{(t,t_0)}\}$, forms a group and would be denoted by $SP_t(\mathcal{S})$.
While in linear theories of finite dimension each element of $SP_t(\mathcal{S})$ can always be represented as a unitary operator, in the infinite dimensional case and in particular in the scalar field case the elements of $\{T_{(t,t_0)}\}$ can not always be represented by an unitary operator. The key point is to notice that given an inner product μ for each element $T_{(t,t_0)}$ one has an inner product $\mu_{T_{(t,t_0)}}(\cdot,\cdot)=\mu(T_{(t,t_0)}(\cdot),T_{(t,t_0)}(\cdot))$. Notice that by construction the inner product $\mu_{T_{(t,t_0)}}$ satisfies (9.88) if μ satisfies (9.88). Therefore, an element of $T_{(t,t_0)}$ would be implemented as a unitary operator if and only if $\mu_{T_{(t,t_0)}}$ and μ define unitary equivalent theories [82]. We state this fact as a theorem

Theorem 10 *Given a GNS-representation of a pure quasi free state given by an inner product μ, the finite temporal evolution (in this representation) would be given by unitary operators if and only if the inner product μ and the inner product induced by evolution, μ_T, are unitarily equivalent $\forall T\in SP_T(\mathcal{S})$.*

Notice that the implementation of finite time translations as unitary operators is a different question to that of having a ground state and implementing time translations as unitary operators. The implementation of a symmetry as described above in section (11) is equivalent to having a strongly continuous unitary group generated by a self-adjoint operator which is the Hamiltonian. It can be seen that if this family of strongly operators exists then the finite time transformations described in this section can also be implemented as unitary operators. However, it is not the case that implementing the finite time evolution as unitary operators implies the existence of a self-adjoint operator which correspond to the Hamiltonian [83].

9.5 Applications

9.5.1 Static $C^k(k\geqslant 2)$ spacetimes

In this section we are going to explicitly quantise and implement the dynamics of the field for a particular $1+1$ metric. We discuss the differences between this result and previous results in the literature.

Consider a metric g_{ab} in a region of the form $I^2 = [0,T] \times (0,\pi)$ such that the line element is of the form

$$ds^2 = (1 + |x|^k)dt^2 - (1 + |x|^k)dx^2 \tag{9.99}$$

where $k \geqslant 2$ and even. Then the metric g_{ab} is C^k.
Notice that for this metric we have the following lapse, shift, volume form and induced volume form in $[0,\pi]$

$$\begin{aligned} N &= \sqrt{(1 + |x|^k)} && (9.100)\\ \beta^i &= 0 && (9.101)\\ \sqrt{-g}(\gamma) &= (1 + |x|^k) && (9.102)\\ \sqrt{\gamma} &= \sqrt{(1 + |x|^k)} && (9.103) \end{aligned}$$

which satisfies the geometric conditions 4.
Now using the Galerkin's approximation methods as in chapter 7 and the higher regularity estimates, we have that there is a unique weak solution u as in definition 9. Therefore, the solution satisfies that for each $v \in L^2(0,T; H^1(\Sigma))$,

$$\int_0^T (< \ddot{u}(t,\cdot), v(t,\cdot) > + B[u,v;t])dt = 0 \tag{9.104}$$

where $< \cdot,\cdot >$ denotes the dual pairing between the $H^{-1}(\Sigma)$ and $H^1(\Sigma)$ Sobolev spaces and where

$$B[u,v;t] := \int_0^\pi g^{11}\gamma u_x v_x dx = \int_a^b u_x v_x dx \tag{9.105}$$

Moreover, the m- approximate solutions (See definition 18.21) satisfy the energy estimate

$$||u^m(t,\cdot)||_{H^1(I^2)} \leqslant C\left(||u_0||_{H^1([0,\pi])} + ||h||_{L^2([0,\pi])}\right) \tag{9.106}$$

and the sequence of $\{u^m\}$ converges weakly to u in $H^1(I^2)$ and therefore converges strongly to u in $L^2(I^2)$ i.e,

$$||u^m - u||_{L^2(I^2)} \to 0 \tag{9.107}$$

as $m \to \infty$.
For concreteness, we use as the orthonormal basis $\{w_k(x) = \sin(k\pi x)\}_k$ which are eigenfunctions with eigenvalue k^2 where $k \in \mathbb{N}$. Notice that in this case for each m- approximate solution $u^m := \sum_{k=1}^{m} d_m^k(t) w_k(x)$ then each $d_m^k(t)$ using equation 7.14 satisfies

$$\ddot{d}_m^k(t) + C^2k^2 d_m^k(t) = 0 \tag{9.108}$$

for $k = 1, ..m$.

The general solution of (9.108) is

$$d_m^k(t) = M_k e^{iCkt} + N_k e^{-iCkt} \tag{9.109}$$

for constants M_k, N_k that depends on the initial conditions.
Then, inserting this solution in the general form of the m-approximate solution, reordering and using the fact that u is a real solution the u^m approximate solutions take the form

$$u^m = \sum_{k=1}^{m} A_k w_k(x) + \overline{A_k} w_k(x) = \sum_{k=1}^{m} a_k e^{iCkt} w_k(x) + \overline{a_k} e^{-iCkt} w_k(x) \tag{9.110}$$

where C, a_k, a_k^* are constants.
Moreover, using the strong convergence of the solution u in $L^2(I^2)$ we obtain

$$u = \sum_{k\in\mathbb{N}} A_k w_k(x) + \overline{A_k} w_k(x) = \sum_{k\in\mathbb{N}} a_k e^{iCkt} w_k(x) + \overline{a_k} e^{-iCkt} w_k(x) \tag{9.111}$$

Notice that this equality only holds in a L^2 sense.
With this decomposition of the solution we can define an inner product in $\mathcal{S}$ by restricting to the initial surface

$$\mu(u,v) =: Re \sum_{k\in\mathbb{N}} \overline{A_k(0)} B_k(0) =: Re \sum_{k\in\mathbb{N}} \overline{a_k} b_k \tag{9.112}$$

where a_k are the coefficients of u and b_k are the coefficients of v using the decomposition as in (9.111). Notice that the series converge by Parseval's theorem which is Theorem 24 in the Appendix. Moreover, we have that the inner product in independent of time or in other words independent of the choice of surface $t = cte$ used. Thus,

$$\mu_T(u,v) = \mu(u,v) \tag{9.113}$$

for all $T \in SP_t(\mathcal{S})$.
Therefore, the dynamics of the field can be implemented by unitary operators.
We would like to point out that in this case it was fundamental the choice of basis $w_k(x) = \sin(k\pi x)$. A different choice would make the inner product time dependent and therefore a more delicate analysis have to be made to see that Theorem 10 is satisfied.
We would like to remark that the scalar field quantisation on a static spacetime is well understood in the smooth case (where the metric and solutions are smooth) [84]. Kay showed that, in fact, one can promote classical dynamics to unitary dynamics in the globally hyperbolic case with smooth metrics in the Hamiltonian sense. That is there is a self-adjoint operator that generate the dynamics.
One of the key theorems used in the proof is Chernoff's Lemma.

Theorem 11 Chernoff's lemma *Let T be a symmetric operator with dense domain $\mathcal{D} \subset \mathcal{H}$ a (complex) Hilbert space. Suppose T maps $\mathcal{D}$ into itself. Suppose in addition that there is a one-parameter group $V(t)$ of unitary operators on $\mathcal{H}$ such that $V(t)\mathcal{D} \subset \mathcal{D}$, $V(t)T = TV(t)$ on $\mathcal{D}$ and*

$$\frac{d}{dt} V(t)u = iTV(t)u; \quad u \in \mathcal{D} \tag{9.114}$$

Then T^n is essentially self-adjoint for all n.

Kay used the theorem on the operator $h := gA$ where

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and

$$A = \begin{pmatrix} m^2 - \Delta_\gamma & 0 \\ 0 & 1 \end{pmatrix}$$

with dense domain $\mathcal{D} = C_0^\infty(\Sigma) \times C_0^\infty(\Sigma)$ and Δ_γ is the Laplace-Beltrami operator of the induced spatial metric γ_{ij}. It is straightforward to see that the theorem breakdowns if the operator Δ_γ does not have smooth coefficients and therefore this previous result does not include our current example.

Part IV

Future outlook and open problems

Our inquiry has reached a point where we can step back and look at what we have done so far. This exercise should be seen as a road intersection where one needs to take out the map again in order to continue the journey rather than a mountain climber who sees the vast valley from the top of the mountain. The rest of the journey depends on where one wants to go. In this spirit, we will let the reader arrive at their own conclusions. Nevertheless, it is true that this is not a completed work, there are still many interesting paths that need to be explored and analysed. We would like to show some of these directions below.

The propagation of singularities

Our work in the second part of the thesis has focused only on the fact that the classical test-field has H^1 regularity. However, one can be more precise and give a characterisation about where and in what direction the solutions are no longer C^∞.

To begin, let u be a distribution. We say u is smooth at x' if there exists an open neighbourhood $U \subset \mathbb{R}^n$ of x' and a smooth function $\varphi \in C^\infty(U)$ such that $u(\phi) = \int_{\mathbb{R}^n} \varphi\phi dx^n$ for all test functions $\phi \in C_0^\infty(U)$.

The singular support, *singsupp*, of a distribution u is the complement in $\mathbb{R}^n$ of the set of all points at which u is smooth. The singular support tells us "where" a distribution fails to be smooth, while the concept of a wavefront set allow us to define also "in which direction" the distribution is singular.

To define the wavefront set, we first define a co-vector $\xi \in \mathbb{R}^n \backslash \{0\}$ as a direction of rapid decay for u at x if there exists a conic neighbourhood Γ (a subset of $\mathbb{R}^n \backslash \{0\}$ such that if $v \in \Gamma$ then $\rho v \in \Gamma$ for $\rho > 0$) of ξ and a smooth function $\chi \in C_0^\infty(\mathbb{R}^n)$ which does not vanish at x such that

$$(1 + |\xi|)^N |\widehat{\chi u(\xi)}| \to 0$$

as $\xi \to 0$ in Γ for all $N \in \mathbb{N}$ where $\widehat{\chi u(\xi)}$ is the Fourier transform of $\chi u(\zeta)$. The set of singular directions of u at x, $\Sigma_x(u)$ is the complement in $\mathbb{R}^n \backslash \{0\}$ of the set of directions of rapid decay of u at x.

Then we say $WF(u)$ of u is

$$WF(u) := \{(x, \chi) \in \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\}) | \chi \in \Sigma_x(u)\}.$$

Accordingly, the definition of a wavefront set may be extended to manifolds M in the following way: We say $(x, \chi) \in WF(u) \subset T^*M \backslash \{0\}$ if and only if there exists a chart neighbourhood $(\phi_\alpha, \mathcal{U}_\alpha)$ of x such that the corresponding coordinate expression of (x, χ) belongs to $WF(u(\phi_\alpha)) \subset \mathbb{R}^n \times (\mathbb{R}^n \backslash \{0\})$ where n is the dimension of M.

The phenomena of propagation of singularities correspond to the answer to the following question. What is $WF(u)$? For concreteness, we focus on the wave equation $\Box_g u = f$ where $g_{ab} \in C^\infty$ in the sense of distributions i.e. $\int_M u \Box_g \phi \mu_g = \int_M f \phi \mu_g$ for all test functions ϕ.

The key concept to answer this question is the characteristic set. To define this set, we first define the principal symbol of the operator $\Box_g$ as the homogeneous polynomial $p(x, \eta) = g^{ab}(x)\eta_a\eta_b$ where η_a, η_b are the components of a co-vector η. In the case of Minkowski spacetime, we have $p_M(x, \eta) = -\eta_0^2 + \eta_1^2 + \eta_2^2 + \eta_3^2$ where $\eta = (\eta_0, \eta_1, \eta_2, \eta_3)$. The characteristic set of $\Box_g$, $char(\Box_g)$, is the set of points in T^*M such that $p(x, \eta) = 0$ which is the light-cone bundle

$$C = \{(x, \eta) | g_x(\eta, \eta) = 0\}$$

which is ruled by null geodesics called bicharacteristics ([85], Chapter 3).
Now we can state the result of the propagation of singularities theorem

Theorem 12 *Let $\Box_g u = f$ where $g_{ab} \in C^\infty$. Furthermore, let $\gamma : [a,b] \to T^*M$ be a bicharacteristic. If u satisfies $\gamma([a,b]) \cap WF(f) = \emptyset$, then either $\gamma([a,b]) \subset WF(u)$ or $\gamma([a,b]) \cap Wf(u) = \emptyset$. Moreover, independent of the condition $\gamma([a,b]) \cap WF(f) = \emptyset$, we have $WF(u) \subset WF(f) \cup char(\Box_g)$.*

A proof of the Theorem can be found in [86, 79]. There is also a refinement of the notion of wavefront set in terms of Sobolev spaces.
We say a distribution u is microlocally H^s at $(x,\chi) \in (\mathbb{R}^n \backslash \{0\})$ if there exists a conic neighbourhood Γ of χ and a smooth function $\varphi \in C_0^\infty(\mathbb{R}^n), \varphi(x) \neq 0$ such that

$$\int_{\mathbb{R}^n} (1+|\xi|^2)^s |\widehat{\varphi u}(\xi)|^2 d^n\xi < \infty.$$

The Sobolev wavefront set $WF^s(u)$ of a distribution u is the complement, in $T^*(\mathbb{R}^n)\backslash\{0\}$, of the set of all pairs (x,χ) at which u is microlocally in H^s; a similar construction as in the smooth case generalises to manifolds.
With the notion of the Sobolev wavefront set one can thus prove similar theorems as the one above but in the low regularity setting. Results in that directions consider regularity where the second derivative of the metric is in $L^1([0,T], L^\infty(\Sigma))$ [87] or manifolds with corners [88].

The renormalised energy momentum tensor (REMT)

The main motivation for the regularity chosen for our classical weak solutions u was that they have a well-defined distributional energy-momentum tensor $T_{ab}[u]$. However, when we move to the third part of the thesis this important observable was not mentioned. We want to come back to this issue now.
A natural requirement for the quantisation of the scalar field is the existence of states ω such that the quantum observable $\langle T_{ab}\rangle_\omega$ is well-defined. In the case of a C^∞ globally hyperbolic spacetime, the existence of states that satisfy the microlocal spectrum condition is a sufficient condition [76]. The microlocal condition is a condition on the two-point function of the state ω, which in the case of quasi-free states determine the state completely [79].
The two-point function of a state ω in the case of a C^∞-globally hyperbolic spacetime is defined for any $\phi, \psi \in D(M)$ as

$$\omega_2(\phi,\psi) =: \omega(G(\phi)G(\psi)) = -\frac{\partial^2}{\partial t \partial s}|_{t,s=0} \quad \omega\left(W(G(t\phi))W(G(t\psi))\right)$$

where G is the causal propagator as defined in Equation (9.12). We further notice that by using the Schwarz kernel theorem, G, G^+ and G^- are also a bidistributions which we denote by $\tilde{G}$, $\tilde{G}^+$ and $\tilde{G}^-$. Then $\omega_2 : D(M \times M) \to \mathbb{R}$ is a bidistribution which is also a bisolution i.e.

$$\omega_2(\Box_g \phi, \psi) = \omega_2(\phi, \Box_g \psi) = 0$$

We mention that the two point function allows us to also define the Feynman propagator ω_F of a state ω to be $\omega_F = i\omega_2 + \tilde{G}^+$.
To define the microlocal spectrum condition precisely, we define the sets

$$\begin{gathered}C = \{(x_1,\eta,x_2,\tilde{\eta}) \in T^*(M\times M)\backslash 0;\\ g^{ab}(x_1)\eta_a\tilde{\eta}_b = 0, g^{ab}(x_2)\eta_a\tilde{\eta}_b = 0, (x_1,\eta) \sim (x_2\tilde{\eta})\}\\ C^+ = \{(x_1,\eta,x_2,\tilde{\eta}) \in C; \eta_0 \geqslant 0 \tilde{\eta}_0 \geqslant 0\}\\ C^- = \{(x_1,\eta,x_2,\tilde{\eta}) \in C; \eta_0 \leqslant 0 \tilde{\eta}_0 \leqslant 0\}\end{gathered}$$

where $(x_1,\eta) \sim (x_2,\tilde{\eta})$ means that $\eta,\tilde{\eta}$ are cotangent to the null geodesic γ at x_1 resp. x_2 and parallel transports of each other along γ.
We now define the microlocal spectrum condition.

Definition 15 *A state ω_H on the algebra $CCR(\mathcal{S},\Xi)$ satisfies the microlocal spectrum condition if its two point function ω_2 is a distribution $D'(M\times M)$ and satisfies the following wavefront set condition*

$$WF(\omega_{2H}) = C^+ \subset WF(\tilde{G})$$

The states that satisfy the microlocal spectrum condition are called Hadamard states.

A characterisation in terms of the Sobolev-wavefront set has been obtained by Junker and Schrohe which was used to study the larger class of adiabatic states [77].

Definition 16 *A quasifree state ω_N on the algebra $CCR(\mathcal{S},\Xi)$ is called an adiabatic state of order $N\in\mathbb{R}$ if its two-point function ω_{2N} is a distribution that satisfies the following H^s-wavefront set condition for all $s \leqslant N+\frac{3}{2}$*

$$WF^s(\omega_{2N}) \subset C^+$$

Hadamard states are adiabatic states of any order.

The analysis done by Junker and Schrohe was done in the C^∞-globally hyperbolic spacetime. However, these states seems to be the adequate states for the analysis of $\langle T_{ab}\rangle_\omega$ in rough spacetimes.
We also point out that one further issue that has to be addressed is that the causal propagator $\tilde{G}$ is used extensively to define the correct singular structure of the Hadamard states. Therefore, it might be interesting to look for a rough version of this propagator as well. We mention below some probable conditions that such a propagator must satisfy:
A rough propagator $G_r : V \to \mathcal{S}$ is a linear map from a function space V that satisfies the following properties:

1. G_r is onto.
2. $G_r\phi = 0$ if and only if $\phi = (\Box_g + m^2)\psi$ (in some weak sense).
3. For all $u\in\mathcal{S}$ and all $\phi\in V$, we have $\int_M u\phi\nu_g = \tilde{\Xi}(G_r\phi,u)$.

Moreover, the sequence of linear maps

$$0 \xrightarrow{\Box_g} V \xrightarrow{G_r} \mathcal{S} \xrightarrow{\Box_g} \mathcal{S}$$

is a complex, i.e. the composition of any two subsequent maps is zero.
These conditions are satisfied in the C^∞ globally hyperbolic spacetime by choosing $V = D(M), \mathcal{S} = \mathcal{S}_{sc}$ as mentioned in Part III.

Black holes and the early universe

The physical motivation for studying singularities comes from two scenarios: Black holes and the early universe. In this section, we address several directions of research and some results about the description of singularities in these physical scenarios.

As mentioned in the introduction of the thesis, the singularity theorems only establish geodesic incompleteness and not the precise nature of the incompleteness. In examples such as Friedmann Robertson Walker or Schwarzschild the metric suffers a loss of differentiability and eventually fails to be extended as a C^2 Lorentzian manifold [89, 90, 91]. However in the Kerr and Reissner–Nordström spacetimes, there is a loss of global hyperbolicity instead. In these cases, while the maximal globally hyperbolic development is unique [3], one may extend the spacetime (even in a C^∞ manner) in non-unique ways. The boundary of the original manifold in the extension is known as the Cauchy horizon. It is expected that under small perturbations the extensions are unstable and some loss of regularity will occur, preventing the extension. Moreover, a detailed analysis of Einstein's equations shows that in general one should expect the singularity in the conformal compactification to be of null type (rather than spacelike).[92]. The analysis of such situations is part of the content of The Strong Cosmic Censorship Conjecture. In fact, Dafermos has shown that the outcome of the conjecture depends critically on the differentiability of the metric allowed in the extensions of the maximal Cauchy development [40].

In Part I of this thesis, we analysed the singularity in the $1 + 1$ case of a Friedmann-Robertson-Walker metric and found that the singularity is a point. We have also mentioned that in the 4-dimensional Friedmann-Robertson-Walker and Schwarzschild case the complete space $\overline{M}$ is no longer a Hausdorff space. This raises the question about the physicality of the construction, although some alternatives have been proposed to fix the situation [26]. In cases like Kerr and Reissner-Nordström, one would expect that the b-boundary coincides with the picture given by the conformal compactification in which the boundary is the Cauchy horizon which is a null hypersurface. However, a detailed calculation of this spacetimes has not been done.

In Part II of this thesis, we used as a necessary condition for a singularity to be regarded as a strong gravitational singularity that in any neighbourhood of it the evolution of the wave equation is not locally well-posed in H^1. Therefore, we expect that around the singularity of Schwarzschild any rough extensions will satisfy this. In the case of Kerr and Reissner-Nordström, the behaviour of test fields at the interior of black holes has been studied from the point of view of global well-posedness as the local well-posedness does not present any difficulties. In those cases, it has been shown that the decay rates of the field at the event horizon are crucial to the existence of an H^1_{loc} solution at the interior up to the Cauchy Horizon [51, 52]. Also, in the Reissner-Nordström case bounded solutions exist in the extension through the Cauchy Horizon. More strikingly, is the fact that in the case of extremal Reissner-Nordström or Reissner-Nordström-De-Sitter (RN-DS) one can prove that such finite energy solutions exist [93]. In fact, these works support recent evidence that the Strong Cosmic Censorship may not hold in these scenarios under the criteria of the regularity of C^0 metrics and L^2-Christoffel symbols. If one holds the view that generically under gravitational collapse, there must be a loss of regularity, then one would need to analyse global well-posedness of wave equations in rough spacetimes.

In the case of the early universe, the loss of regularity at the initial singularity would

require analysing spacetime with rough metrics if one insists on extending the manifold as a Lorentzian manifold [94]. However, there is the possibility to extend the metric as a degenerate metric, where the signature is no longer two and one admits zero eigenvalues in the "metric" [89, 95]. If one follows this ideology, then one needs to study a more complex setting of a degenerate hyperbolic equation. In this direction, there have been interesting partial results. For example, even if all coefficients are smooth, a smooth solution at the point where the metric vanishes may not exist [96]. However, also analytical conditions have been found for the well-posedness of such problems [97] but the physicality of such conditions remains to be analysed.
In Part III, we probed spacetime with quantum fields. In this case, it is shown that $\langle T_{ab}\rangle_\omega$ for certain states ω diverges in general when one approaches Schwarzschild type singularities [99], the cosmological singularity [100] or a Cauchy Horizon [98]. Moreover, because the quantisation procedure relies on the classical theory it is only recently that a more precise mathematical approach has been developed that allows one to understand these scenarios. We point out the result by Marković and Poisson [101] which shows that in the RN-DS the Cauchy Horizon is stable classically but probing it with quantum matter one obtains that states are ill-behaved as they approach the horizon. This is puzzling, as the (in)-stability of the Cauchy Horizon is a classical question, and nevertheless, the instability can only be seen through quantum considerations. Moreover, the environment changes drastically when one considers quantum fields in curved spacetime as there is particle creation in black holes [102] and in cosmological models [103]. All these phenomena have to be included to give a complete picture of the final fate of black hole and cosmological singularities in quantum field theory in curved spacetime.

Semiclassical relativity and quantum gravity

As mentioned in the introduction, a complete consistent description of space and time in the physical sciences is still missing. Therefore, one must go beyond General Relativity. A conservative way to do this is what is called the semiclassical approximation of General Relativity in which one considers the equation

$$G_{ab} = \langle T_{ab}\rangle_\omega$$

where $\langle T_{ab}\rangle_\omega$ is the renormalised energy-momentum tensor.
The equations are now fourth order in the metric g_{ab} which makes their analysis more convoluted than the second order equations of General Relativity. A more worrisome issue is the fact that the theory may not be reasonable from a physical perspective because stable classical solutions may not exist. For example the stabilty of Minkowski spacetime requires additional ad-hoc constraints [104]. Another aspect that needs to be addressed is the fact that one needs an additional criteria to determine which state ω is being used. Also, a critical issue is the tension that arises between the conservation of energy-momentum built into General Relativity and the measurement process in quantum theory which could allow the local violation of this conservation law.
Regarding the main topic of this thesis, the semiclassical approximation will likely allow singularities as predicted by the singularity theorems. Notice that the theorems cannot be straightforwardly applied as the energy conditions are not true for quantum matter fields [105]. However, if these negative energy violations are small enough, geodesic incompleteness must follow. Initial steps towards a full quantum versions of

the singularity theorems have benefited from the quantum energy inequalities [106] which have allowed partial version of quantum singularity theorems [107]. One can even take the point of view of a characterisation of singularities along the lines of the strong cosmic censorship in which one regards a singularity as the region where the semiclassical equations break down as a well-posed problem. As mentioned above, this will be a lot harder than the already difficult second order non-linear classical version. A more radical step requires one to consider versions that quantise gravity or make gravity emerge from more fundamental structures. A complete and fair description of these approaches is beyond the scope of this thesis. However, we would like to discuss some general expectations some approaches such as Loop Quantum Gravity, Asymptotic Safety, Euclidean quantum gravity, Discrete Quantum Gravity and String theory have in common. A comprehensive overview of the current approaches to quantum gravity can be found in [108].

One condition that is expected from a consistent picture of quantum gravity is that singularities are resolved and therefore the problem of singularities disappears. This is similar to the ultraviolet catastrophe where singularities or divergences in the energy appeared in the black body case and only when the quantum picture emerges the singularity disappears. In the quantum gravity community, research has been done showing how under the perspective of quantum gravity cosmological singularities [109] or black hole singularities are resolved [110] in certain particular scenarios. While this is a promising step forward, there are theoretical arguments showing that quantum gravity cannot solve all singularities of GR. Otherwise the theory becomes inconsistent [111]. At the end, the main problem of our current understanding of quantum gravity is not only the inherent complexity of quantum gravity, but also the lack of precise criteria for the classical structure of singularities which the quantum regime needs to look at.

From a physical perspective, the only judge to solve this dispute can be an experiment. Hence, until we have experimental data and a theory which is able to model a singularity-free picture of gravitational physics, these approaches remain speculative. Since their conception, semiclassical gravity and quantum gravity have stubbornly refused experimental falsification. This tells us just how frustrating the search for a gravitational singularity-free description can be. Nevertheless, while there is time and will, the human spirit may be up for the task.

Part V

Appendix

Chapter 10

Differential Geometry

10.1 Manifolds

A function f on an open set $\mathcal{U}$ of $\mathbb{R}^n$ is said to be C^k if f is k-times continuously differentiable. A function f on an open set $\mathcal{U}$ of $\mathbb{R}^n$ is said to be Lipschitz if there is some constant K such that for each pair of points $p, q \in \mathcal{U}$, $|f(p) - f(q)| \leqslant K|p-q|$, where $|p|$ denotes the usual Euclidean distance. We denote by $C^{k,1}$ those C^{k-1} functions where the k-derivative is a Lipschitz function.
A C^k *n-manifold* $\mathcal{M}$ is a set $\mathcal{M}$ together with a C^k structure. A C^k *structure* consists on an *atlas* $\{\mathcal{U}_\alpha, \phi_\alpha\}$ where the $\mathcal{U}_\alpha$ are subsets of $\mathcal{M}$ and the ϕ_α are one-one maps of the corresponding $\mathcal{U}_\alpha$ to open sets in $\mathbb{R}^n$ such that

1. the $\mathcal{U}_\alpha$ cover $\mathcal{M}$, i.e. $\mathcal{M} = \bigcup_\alpha \mathcal{U}_\alpha$,

2. if $\mathcal{U}_\alpha \bigcap \mathcal{U}_\beta$ is non-empty, then the map

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(\mathcal{U}_\alpha \bigcap \mathcal{U}_\beta) \to \phi_\alpha(\mathcal{U}_\alpha \bigcap \mathcal{U}_\beta)$$

   is a C^k map of an open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^n$.

Each $\mathcal{U}$ is a *local coordinate neighbourhood* where a point $p \in \mathcal{U}_\alpha$ has the coordinates of $\phi_\alpha(p)$ in $\mathbb{R}^n$. An atlas is said to be compatible with a given C^k atlas if their union is a C^k atlas for all $\mathcal{M}$. The atlas consisting of all atlases compatible with the given atlas is called the *complete atlas* of the manifold. A C^k manifold with boundary is defined in the same way replacing $\mathbb{R}^n$ by $\mathbb{R}^n_+$ (the upper half plane). A $C^r(r > 0)$ manifold has a unique C^∞ structure ([3]). A l-dimensional manifold $\mathcal{N}$ is an *embedded sub-manifold* of the n- dimensional manifold $\mathcal{M}$ provided $\mathcal{N}$ is locally described as the common locus:

$$F^1(x^1, .., x^n) = 0, .., F^{n-l}(x^1, .., x^n) = 0$$

of $n - l$ C^1 functions that are independent in the sense that the Jacobian matrix $[\frac{\partial F^\alpha}{\partial x^i}]$ has rank $(n-l)$ at each point of the locus.
If γ is a 1-embedded (possibly piecewise) sub-manifold on $\mathcal{M}$, we call it a *curve* on $\mathcal{M}$. If γ is a curve such that there is not another curve γ' with the property that $\gamma \subset \gamma'$ then γ is an *inextendible curve*. If a curve γ defined in a open subset (a, b) satisfies that $\lim_{t\to a} \gamma(t) = x$, $\lim_{t\to b} \gamma(t) = y$ and x, y are in $\mathcal{M}$, we call x, y the *endpoints* of the curve γ. Notice coordinates can be seen asa family of curves on $\mathcal{M}$ which we call *coordinate curves*. An arbitrary coordinate curve denoted γ_y can be written as

the common locus $x^a - C_a = 0$ with C_a constants and for all $x^a \neq y$. For example, in cartesian coordinates a curve γ_z correspond to the geometric place $x - C_1 = y - C_2 = 0$ for arbitrary constants C_1 and C_2.
The *topology* τ *of* $\mathcal{M}$ is the topology induced by the maps $\{\phi_\alpha\}$ on $\mathcal{M}$. The open sets are unions of sets of the form $\{\phi_\alpha^{-1}(U)|U \in \tau_{\mathbb{R}^n}\}$. If $\mathcal{M}$ is a manifold with boundary, then the *boundary of* $\mathcal{M}$, denoted by $\partial\mathcal{M}$, is defined to be the set of all points of $\mathcal{M}$ whose image under a map ϕ_α lies on the boundary of $\mathbb{R}^n_+$.
A topological space $\mathcal{M}$ is said to be a *Hausdorff space* if whenever p, q are two distinct points in $\mathcal{M}$, there exist disjoint open sets $\mathcal{U}, \mathcal{V}$ in $\mathcal{M}$ such that $p \in \mathcal{U}, q \in \mathcal{V}$. An atlas $\{\mathcal{U}_\alpha, \phi_\alpha\}$ is said to be *locally finite*, if every point $p \in \mathcal{M}$ has an open neighbourhood which intersects only a finite number of the sets $\mathcal{U}_\alpha$. $\mathcal{M}$ is *paracompact* if for every atlas $\{\mathcal{U}_\alpha, \phi_\alpha\}$ there exists a locally finite atlas $\{\mathcal{V}_\beta, \psi_\beta\}$ with each $\mathcal{V}_\beta$ contained in some $\mathcal{U}_\alpha$. We require the topology τ to be Hausdorff and paracompact. This will assure good local behaviour and the existence of partitions of unity which are useful to go from local properties to global ones.
Given a covering $\{\mathcal{U}_\alpha\}, \alpha \in J$ of $\mathcal{M}$ by $\{\mathcal{U}_\alpha\}$, a *partition of unity* subordinate to this cover is a family of real-valued functions $f_\alpha : \mathcal{M} \to \mathbb{R}$ with the following properties:

1. $f_\alpha \geqslant 0$.
2. The collection $\{\mathrm{supp}(f_\alpha)\}$ is locally finite (at each point $x \in \mathcal{M}$ there are only a finite number of functions $f_\alpha \neq 0$
3. The support of f_α is a closed subset of the patch $\mathcal{U}_\alpha$.
4. $\sum_\alpha f_\alpha(p) = 1$ for all $x \in \mathcal{M}$.

Then that partition is said to be of class C^k if the f_α are of class C^k. Unless otherwise stated, all manifolds will be paracompact C^∞ connected Hausdorff manifolds.

10.2 Vectors, One forms, and Tensors

The differential structure of $\mathcal{M}$ is enough to define *vectors* X^a on each point p of $\mathcal{M}$. Given a C^1 curve γ such that $\gamma(t_0) = p$ with tangent vector X^a at p is the operator which maps each C^1 function f into the number

$$X^a(f)|_{t_0} = \lim_{h\to 0} \frac{f(\gamma(t_0 + h)) - f(\gamma(t_0))}{h};$$

that is, $X^a(f)$ is the derivative of f in the direction of $\gamma(t)$ with respect to the parameter t. We define the *tangent space*, $T_p(\mathcal{M})$, of a point p in $\mathcal{M}$ as the space of all tangent vectors to $\mathcal{M}$ at p. This is a n-dimensional vector space where a basis is given by the tangent vectors of the coordinate curves that satisfies $p \in \gamma_a$ with $a \in \{1, 2, ..., n\}$. We write this particular basis as $\{\frac{\partial}{\partial x^a}\}$. Note that this implies that $T_p(\mathcal{M}) \cong \mathbb{R}^n$ The tangent space allows us to define a differential function φ from a manifold $\mathcal{M}$ to $\mathcal{N}$. We say that the map φ is differentiable with differential $D\varphi$ if for all p in $\mathcal{M}$ the map $D\varphi_p : T_p\mathcal{M} \to T_{\varphi(p)}\mathcal{N}$ is a linear map, explicitly it takes the form $D\varphi_p(X^a)(f) = X^a(f \circ \varphi)|_p := Y^a|_{\varphi(p)}$ where f is smooth real function on $\mathcal{N}$.
If the map f is differentiable with a differentiable inverse we call f a diffeomorphism. The *dual space*, $T_p^*(\mathcal{M})$, is the space of all linear functions from T_p to $\mathbb{R}$. The elements, V^a, of this space are called *co-vectors* and span a n-dimensional vector

space. A basis $\{E^a\}$ of $T_p^*(\mathcal{M})$ is called dual to a basis $\{E_a\}$ if $E^a(E_b) = \delta_a^b$. A *tensor type (s,r)*, $T^{a_1...a_s}_{b_1...b_r}$ at p is a linear function from the Cartesian product form by r copies of T_p^* and s copies of T_p to $\mathbb{R}$. The space of all such tensors is called the *tensor product* $T^r_s(\mathcal{M})$. The *components of a tensor* $T^{a_1...a_r}_{b_1...b_s}$ with respect to the co-vectors and vectors $\{E_{a_1}...E_{a_r}, E^{b_1}...E^{b_s}\}$ is just the evaluation of the tensor $T^{a_1...a_r}_{b_1...b_s}(E_{a_1}, ..., E_{a_r}, E^{b_1}..., E^{b_s})$. Given two tensor $T^{a_1...a_r}_{b_1...b_s}$,$S^{a_1...a_q}_{b_1...b_t}$ we can form the tensor $T^{a_1...a_r}_{b_1...b_s} \otimes S^{a_1...a_q}_{b_1...b_t}$ by taking vectors and covectors
$\{E_1, .., E_r, E^1, .., E^s, \tilde{E}_1, .., \tilde{E}_q, \tilde{E}^1, .., \tilde{E}^t\}$ and sending it to the number
$T^{a_1...a_r}_{b_1...b_s}(E_1, .., E_r, E^1, .., E^s)S^{a_1...a_q}_{b_1...b_t}(\tilde{E}_1, .., \tilde{E}^q, \tilde{E}^1, .., \tilde{E}^t)$.
A $(0, r)$ tensor $T_{a_1...a_r}$ which components are of the form

$$\frac{1}{p!}\{\text{ alternating sum over all permutation of indices} a_1 \text{ to } a_r\}$$

is called a *r-form.* We denote this type of components by $T_{[a_1...a_r]}$. The space of all r- form at a point p is denoted by $\Lambda^r_p(\mathcal{M})$. If A is a r-form and B is a q-form we defined the *wedge product* $A \wedge B$ as the $(r+q)$-form given by a tensor with components $A^{[a_1...a_r}B^{\alpha_1...\alpha_q]}$.
There is a special operation d on r-forms called *exterior derivative* defined by the following properties:

- d is linear;
- $d(\Lambda^r) \subset \Lambda^{r+1}$;
- If f is a function df is the differential defined above;
- $d^2 = 0$;
- $d(A \wedge B) = dA \wedge B + (-1)^r \wedge dB$ with $A \in \Lambda^r$.

10.3 Metrics

A *metric* is a symmetric tensor g of type $(0, 2)$ at each point p. Given a basis $\{E^a\}$ of T_p the ab-component of g is $= g(E^a, E^b)$. The signature of g at p is the number of positive eigenvalues minus the number of negatives ones. If g has signature $|n-2|$ we called g a *Lorentzian metric.* With a Lorentzian metric on $\mathcal{M}$, the non zero vectors at p can be divided into three classes: a vector $X \in T_p$ being said to be *timelike*,*null* or *spacelike* according to whether $g(X, X)$ is negative, zero or positive. A curve γ is called *timelike*,*null* or *spacelike* according to whether $g(\dot{\gamma}, \dot{\gamma})$ is negative, zero or positive in all the domain of definition of γ.
If we are in a Riemann manifold then the metric define a *distance function*

$$d(x, y) : (x, y) \in \mathcal{M} \times \mathcal{M} \to \inf_\gamma \left\{ \int_x^y \|\dot{\gamma}\| \right\} \in \mathbb{R}$$

where the infimum is taken over all piecewise C^1 curves γ from x to y. Moreover, the distance function allows us to define a topology. A basis of that topology is given by the set $\{B(x, r) : y \in \mathcal{M} | d(x, y) \leqslant r \forall x \in \mathcal{M}\}$. The topology naturally induce a notion of convergence. We say the sequence $\{x_n\}$ converges to y if for $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that for any $n \geqslant N$ $d(x_n, y) \leqslant \epsilon$. A sequence that satisfies this conditions is called a *Cauchy sequence.* If every Cauchy sequence converges we say that $\mathcal{M}$ is *metrically complete.*

10.4 Hypersurfaces and volume forms

If S is an $n-1$-dimensional orientable manifold and $\Theta : S \to \mathcal{M}$, satisfies that $D\Theta : TS \mapsto T\mathcal{M}$ is injective, then the image $\Theta(S)$ is said to be a *hypersurface* in $\mathcal{M}$. If g_{ab} is a metric on $\mathcal{M}$, the imbedding induces a metric h called the *induced metric* given by $h_p(\cdot,\cdot) = g_{\Theta(p)}(D\Theta(\cdot), D\Theta(\cdot))$ for any vector in T_pS. If g is positive definitive the metric h would be positive definite. If g is Lorentzian metric we use the one form n_a defined by the normal bundle (see below in subsection 10.6) and notice that h will be

- Lorentz if $g^{ab}n_an_b > 0$ and we call the hypersurface a timelike hypersurface,
- degenerate if $g^{ab}n_an_b = 0$ and we call the hypersurface a null hypersurface,
- positive definite if $g^{ab}n_an_b < 0$ and we call the hypersurface a spacelike hypersurface.

Given a metric g in a n-dimensional manifold $\mathcal{M}$ we called the n-form

$$\nu_g := \sqrt{|det(g)|}dx^1 \wedge ... \wedge dx^n$$

the *volume form* associated to g. Given a vector X^a, the form $j = \nu_g(X^a, ..., \cdot)$ defines a $n-1$-form. We define the *divergence* of X^a, $div(X^a)$ as the unique scalar such that $dj =: div(X^a)\nu_g$.

10.5 Divergence Theorem

The Stoke's theorem can be stated as

Theorem 13 • *Let $\mathcal{M}$ denote a bounded, (n)-dimensional, orientable C^1 manifold in $\Omega \subset \mathbb{R}^n$ with regular boundary $\partial\mathcal{M}$ that is a bounded, $n-1$-dimensional, orientable C^1 hypersurface and such that $\overline{\partial\mathcal{M}}$ is a lipschitz domain.*

- *Let w denote a n-dimensional differential form of the class $C^1(\mathcal{M})$.*

Then,

$$\int_{\mathcal{M}} dw = \int_{\partial\mathcal{M}} w \tag{10.1}$$

In the case when we choose $dw = div(X^a)\nu_g$ for some vector X^a we called the result the *divergence theorem*.
There is a version of the Theorem in low regularity that we state below.

Theorem 14 *Let Ω be a compact set with compact closure with Lipschitz boundary and let Z^a be a vector field on an $(n+1)$-dimensional manifold M with continuous metric g_{ab} and metric volume element ν_g. If*

$$Z^a \in W^{1,1}_{loc}(\Omega), g_{ab} \in W^{1,n}_{loc}(\Omega),$$

then the Stokes identity holds:

$$\int_{\partial\Omega} j = \int_{\Omega} dj$$

for $j = i_{Z^a}\nu_g$ and where $dj = div(Z^a)\nu_g$

A proof of this can be found in [112].

10.6 Fibre Bundles

A *Fibre Bundle* consists of manifolds E, B, and F with a C^∞ surjection $\pi : E \to B$ satisfying that for every e in E, there is an open neighbourhood $U \subset B$ of $\pi(e)$ such that $\pi^{-1}(U)$ is diffeomorphic to the product space $U \times F$, in the sense that for each point $p \in U$ there is a diffeomorphism ϕ_p of $\pi^{-1}(p)$ onto F such that the map Ψ defined by $\Psi(e) = (\pi(e), \phi_{\pi(e)})$ is a diffeomorphism. B is called the base space of the bundle, E the total space, and F the fibre. When the fibre F is a vector space the fibre bundle is called a *vector bundle*.
The *trivial bundle* is the fibre bundle given by the Cartesian product of the base space and the fibre to generate the total space, $E = B \times F$. The map π is given by projection on the first factor.
The *tangent bundle*, $T\mathcal{M}$, is the vector bundle with base space $B = \mathcal{M}$, total space $E = \cup_{p\in\mathcal{M}} T_p$ and fibre $F = \mathbb{R}^n$. The manifold structure on E is define by local coordinates $\{z^A\} = \{(x^a, V^{\alpha_a})\}$ where $\{x^a\}$ are local coordinates on $\mathcal{M}$ and V^{α_a} are the components of the vector V^a at T_p with respect to $\{\frac{\partial}{\partial x^a}\}$.
The *cotangent bundle*, $T^*\mathcal{M}$, is the vector bundle with dual to the tangent budnle in the sense that the fibre $\tilde{F}$ are one-forms. The manifold structure on E is define by local coordinates $\{z^A\} = \{(x^a, V_{\alpha_a})\}$ where $\{x^a\}$ are local coordinates on $\mathcal{M}$ and V_{α_a} are the components of the one-form V^a at T_p with respect to $\{dx^a\}$.
The *normal bundle*, $N(\mathcal{N}, \mathcal{M})$, of a l-sub-manifold $\mathcal{N}$ embedded in a n-manifold $\mathcal{M}$ is the vector bundle with base space $B = \mathcal{N}$, total space $E = \cup_{p\in\mathcal{M}} N_p$ where N_p is the vector space $T_p(\mathcal{M}) \backslash T_p(\mathcal{N})$ of and fibre $F = \mathbb{R}^{n-l}$. The manifold structure on E is define by local coordinates $\{\tilde{z}^A\} = \{(\tilde{x}^a, V^a + [W^a])\}$ where $\{\tilde{x}^i\}$ are local coordinates on $\mathcal{N}$ and $V^a + [W^a]$ are the components of the vector V^a at $T_p\mathcal{M}$ with respect to the coordinate basis $\{x^a\}$ on $\mathcal{M}$. The map π projects each point of N_p into p. The normal bundle of a hypersurface S defines a unique one-form field denoted by n_a.
Let $\mathcal{N}$ be a closed oriented sub-manifold of dimension k in an oriented manifold $\mathcal{M}$ of dimension n. A *tubular neighbourhood* of $\mathcal{N}$ is by definition an open neighbourhood of $\mathcal{N}$ in $\mathcal{M}$ diffeomorphic to a vector bundle of rank $n-k$ over $\mathcal{N}$. The tubular neighbourhood theorem states that every sub-manifold $\mathcal{N}$ in $\mathcal{M}$ has a tubular neighbourhood $\mathcal{T}$, and that in fact $\mathcal{T}$ is diffeomorphic to the normal bundle $N(\mathcal{N}, \mathcal{M})$.

10.7 Lie Groups

A *Lie group*, G, is a group that is also a manifold such that the group operation

$$(a, b) \in G \times G \to ab^{-1} \in G$$

is a C^∞ function. The group structure of a Lie group allows us to define a left action, L_g given by:

$$L_g : h \in G \to gh \in G$$

for all g in G. The manifold structure defines a differential $dL_g : TG \to TG$. A *left invariant vector field* is a map $V : G \to TG$ that is fixed under the differentials of left translations, this means:

$$DL_g V(h) = V(L_g h) = V(gh), \forall h \in G.$$

The left invariant vector fields of a Lie group form a vector space and an algebra under a bilinear operation called the Lie bracket. This algebra is the *Lie algebra* $\mathfrak{g}$ of the Lie group G. As a vector space $\mathfrak{g}$ is isomorphic to the tangent space T_eG where e is the identity element of the group G. For every $\vec{g}$ in $\mathfrak{g}$ there is a one parameter subgroup

$$\varphi : t \in \mathbb{R} \to g \in G$$

such that $\varphi(0) = e$ and $\varphi(t + \tau) = \varphi(t)\varphi(\tau)$ with the property that φ is a curve with tangent vector $\vec{g}$ at $\dot{\varphi}(0)$. We define the *exponential map*:

$$\exp : \vec{g} \in \mathfrak{g} \to \varphi(1) \in G.$$

We have then that $\varphi(t) = \exp t\vec{g}$.
There is an important representation of a Lie group, G, in its Lie algebra $\mathfrak{g}$ named the *adjoint representation* given by the automorphisms

$$ad(g) : h \in G \to ghg^{-1} \in G, \forall g \in G$$

which induces an automorphism

$$ad(g)_* : \vec{g} \in \mathfrak{g} \to d(ad(g))\vec{g} \in \mathfrak{g}, \forall g \in G$$

Classical examples of real Lie groups are the matrix groups: $GL(n, \mathbb{R})$, $O(n, \mathbb{R})$, $SO(n, \mathbb{R})$, $SO(1, n, \mathbb{R})$ and $SU(n, \mathbb{R})$ with Lie algebras: $\mathfrak{gl}(n, \mathbb{R})$, $\mathfrak{o}(n, \mathbb{R})$, $\mathfrak{so}(n, \mathbb{R})$, $\mathfrak{so}(1, n, \mathbb{R})$ and $\mathfrak{su}(n, \mathbb{R})$.

10.8 G-Principal Bundles

A *principal G-bundle* P, is a fibre bundle with a right action, R_g that acts freely

$$(p, g) \in P \times G \to R_g(p) = pg \in P;$$

where the base space B is the quotient space of P by the equivalence relation $(p, g) \sim (q, g')$ if $p = q$ and there is h in G such that $g = hg'$ and there is a mapping φ of $\pi^{-1}(\mathcal{U})$ into G satisfying $\varphi(\overline{p}g) = \varphi(\overline{p})g$ for all $\overline{p}$ in $\pi^{-1}(\mathcal{U})$ and g in G.
Let $\mathcal{M}$ be a n-dimensional manifold. A *frame*, $\{E^a\}_p$ at p is an ordered basis of T_p. Let $L(\mathcal{M})$ be the set of all frames $\{E^a\}$ at all points on $\mathcal{M}$ with the projection π sending a frame at p to p. Then the *general linear group* $GL(n, \mathbb{R})$ has a natural right action on $\{E^a\}$ by right multiplication, i.e., given $(\{E^a\}, A^b_a)$ the action of $A^b_a \in GL(n, \mathbb{R})$ on $\{E^a\}$ is $\{E^b = E^a A^b_a\}$ If $\{x^a\}$ are coordinates on $\mathcal{M}$ and we choose the frame $\{\frac{\partial}{\partial x^a}\}$ then it can be shown that the coordinates $(x^a, X^{\alpha_a}_{\alpha_b})$ are a local coordinate system of $\mathcal{LM}$ where $X^{\alpha_a}_{\alpha_b}$ represent the ab matrix element of the change of basis matrix between $\{\frac{\partial}{\partial x^a}\}$ and any other frame $\{E^b\}$. In fact this choice makes $L(\mathcal{M})$ a G-principal bundle called the *frame bundle*. Moreover, if we have a metric in $\mathcal{M}$ and we restrict the frames to just orthonormal frames we obtain another G-principal bundle called the *orthonormal frame bundle*, $O(\mathcal{M})$ with Lie Group the *orthonormal group*, $SO(n, \mathbb{R})$ or $SO^+(1, n, \mathbb{R})$ depending on the signature.
Given a principal G-bundle P and a fibre bundle E with fibre F where P and E have the same base space B then E is called an *associated bundle* to P, if the quotient space of $P \times F$ by the equivalence relation given by the action

$$(\overline{p}, f) \in P \times F \to (\overline{p}g, fg^{-1}) \in P \times F$$

and denoted as $P \times_G F$ is E. As an example mention that the tangent bundle, $T(\mathcal{M})$, of a n- dimensional manifold $\mathcal{M}$ is isomorphic to an associated vector bundle of $\mathcal{LM}$ with fibre $\mathbb{R}^n$.
Every tangent space $T_{\overline{p}}P$ of a G-principal bundle P has a subspace called the *vertical subspace* $V_{\overline{p}}$ given by the kernel of the differential $D\pi$ restricted at $\overline{p}$. Explicitly,

$$V_{\overline{p}} = \{E^a \in T_{\overline{p}}P | D\pi_{\overline{p}}(E^a) = 0 \in T_{\pi(p)}B\}$$

10.9 Canonical one forms and Connections

The *solder form* of a frame bundle $L(\mathcal{M})$ is the map:

$$\theta : TL(\mathcal{M}) \to \mathbb{R}^n : (\overline{p}, Q) \to \pi_p(D\pi(\overline{p}, Q))$$

where $\overline{p}$ is a point in $L(\mathcal{M})$ and Q is an element of $T_{\overline{p}}L\mathcal{M}$. The solder form for the orthonormal bundle $O(\mathcal{M})$ is defined similarly. Notice that $V_{\overline{p}} \subset ker(\theta)$.
A *connection* $\overline{\nabla}$ in a G-principal bundle is an assignment of a subspace $H_{\overline{p}}$ called the *horizontal subspace* of $T_{\overline{p}}(P)$ for all $\overline{p}$ in P such that:

- $T_{\overline{p}}P = H_{\overline{p}} \oplus V_{\overline{p}}$
- For any $\overline{p}, \overline{q} \in P$ there is a C^∞ curve γ such that $T_{\gamma(t)}P = H_{\gamma(t)} \oplus V_{\gamma(t)}$.
- $H_{\overline{p}g} = D_{\overline{p}}R_g(H_{\overline{p}})$ for every $\overline{p} \in P$ and $g \in G$.

A *connection form* ϖ of a connection $\overline{\nabla}$ in a G-principal bundle is C^∞ map:

$$\varpi : TP \to \mathfrak{g}$$

with the following properties:

- If $\varpi(X) = 0$ then $X \in H_{\overline{p}}$ for some $\overline{p}$ in P
- For all g in G and all C^∞ maps $X : P \to TP$

$$\varpi(DR_g(X)) = ad_*(g^{-1})\varpi(X)$$

- For all $\vec{g} \in \mathfrak{g}, \varpi(X^*_{\overline{p}}) = \vec{g}$ where X^*_p is the tangent vector at $t = 0$ of a curve given by $\gamma(t) = R_{\exp t\vec{g}}(\overline{p})$

Connections and connection forms determine one another uniquely.
We shall now express the connection ϖ by a family of forms each defined in an open subset of the base manifold $\mathcal{M}$. Let $\{\mathcal{U}_\alpha\}$ be an open covering of $\mathcal{M}$ with a family of diffeomorphism $\psi_\alpha : \pi^{-1}(\mathcal{U}_\alpha) \to U_\alpha \times G$ and the corresponding family of transition functions $\psi_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \to G$. For each α, let $\sigma_\alpha : \mathcal{U}_\alpha \to P$ be defined by $\sigma_\alpha =: \psi_\alpha^{-1}(x, e), x \in \mathcal{U}_\alpha$, where e is the identity in G.
For each α, define on $\mathcal{U}_\alpha$

$$\varpi_\alpha(v) := \varpi(D\sigma_\alpha(v))$$

for all $v \in TU_\alpha$. Then ϖ_α is a $\mathfrak{g}$-valued one-form. We also define on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, a $\mathfrak{g}$-valued one-form by

$$\theta_\alpha(v) := \theta(D\psi_{\alpha\beta}(v))$$

The family of one forms $\{\varpi_\alpha, \theta_\alpha\}$ define uniquely ϖ [28].
If we pick a particular chart $\mathcal{U}_\alpha$ and a basis $\{E^a_b\}$ of $\mathfrak{g}$. It is possible to define n^3 functions Γ^a_{bc}, on $\mathcal{U}_\alpha$ such that $\varpi_\alpha(\cdot) = \Sigma(\Gamma^a_{bc} dx^b(\cdot))E^c_a$, on $\mathcal{U}_\alpha$. The functions Γ^a_{bc} are called the *Christoffel symbols*.
As in the case of the principle bundle, the connection forms ϖ_α define a connection ∇. The connection ∇ satisfies that:

- If $T^{a_1\ldots a_r}_{b_1\ldots b_s}$ is a C^r tensor of type (r,s), then $\nabla T^{a_1\ldots a_r}_{b_1\ldots b_s}$ us a C^{r-1} tensor field of type $(r, s+1)$,
- ∇ is linear,
- for arbitrary tensor fields $T^{a_1\ldots a_r}_{b_1\ldots b_s}$, $S^{a_1\ldots a_q}_{b_1\ldots b_t}$, we have

$$\nabla T^{a_1\ldots a_r}_{b_1\ldots b_s} \otimes S^{a_1\ldots a_q}_{b_1\ldots b_t} = \left(\nabla T^{a_1\ldots a_r}_{b_1\ldots b_s}\right) \otimes S^{a_1\ldots a_q}_{b_1\ldots b_t} + T^{a_1\ldots a_r}_{b_1\ldots b_s} \otimes \nabla S^{a_1\ldots a_q}_{b_1\ldots b_t}$$

  .

- $\nabla f = df$ for any function f.

If the principal G-bundle is $\mathcal{LM}$ we define a *covariant derivative* of a vector Y^a along X^a on the base manifold B by the vector $\nabla_{X^a} Y^a := \nabla^{\mathcal{M}} Y^a(X^a)$. In general we define The covariant derivative of a tensor $T^{a_1\ldots a_r}_{b_1\ldots b_s}$ along X^a to be the tensor $\nabla_{X^a} T^{a_1\ldots a_r}_{b_1\ldots b_s} := \nabla T^{a_1\ldots a_r}_{b_1\ldots b_s}(X^a)$ and denote its components by $T^{\alpha_{a_1}\ldots\alpha_{a_r}}_{\alpha_{b_1}\ldots\alpha_{b_s};\alpha_l}$.
This derivative provides a notion of parallel propagation on the manifold $\mathcal{M}$ along a curve γ. Let $\gamma(t)$ be a C^1. A vector, V^a, that satisfies

$$\nabla_{\dot\gamma} V^a = 0$$

is said to be parallel along γ. If a curve γ with tangent vector $\dot\gamma$ satisfies the equation

$$\nabla_{\dot\gamma}\dot\gamma = 0 \tag{10.2}$$

then γ is called a *geodesic* which is the analogue of straight lines in curved manifolds. The parameter of γ is called an *affine parameter* and the length with respect to this parameter *affine length.*
An important tensor defined by the connection is the $(3,1)$ *Riemann tensor* given by:

$$\mathcal{R}(X^a, Y^b, Z^c) := \nabla_{X^a}\nabla_{Y^b} Z^c - \nabla_{Y^b}\nabla_{X^a} Z^c - \nabla_{[X^a, Y^b]} Z_c$$

for any three vectors X^a, Y^b, Z^c and where $[X^a, Y^b]$ is the Lie Bracket of vector fields [3]. This tensor defines the curvature of the manifold.
If we have a metric g_{ab} on $\mathcal{M}$ we say the connection is *metric compatible* if $\nabla_{\frac{\partial}{\partial x^c}} g_{ab} = 0$ for any choice of basis $\{\frac{\partial}{\partial x^c}\}$. If in addition it is satisfied that $\nabla_{Y^b} X^a - \nabla_{X^a} Y^b = [X^a, Y^b]$ for any vectors X^a, Y^a we called the connection the *Levi-Civita connection* of the metric g_{ab}. This connection always exists and is unique.
In addition, there is also a connection $\hat\nabla$ induced in any associated bundle of $\mathcal{LM}$. In particular the associated vector bundle isomorphic V to $T\mathcal{M}$ allows us to describe the geometry in what is called Cartan's method. Given $\{E^a\}$ an orthonormal basis of vectors we have that $\hat\nabla_{E^c} E_a = \omega^c_{ab} E^b$ where $\omega^b_a = \omega^b_{ca} E^c$ are called the connection one-forms on V.

Then we can express the Levi-Civita conditions, where the line element of the metric takes the form $ds^2 = \mu_{ab} E_a \otimes E_b$ with μ_{ab} a matrix of constants, as:

$$\omega_b^{(a} \mu^{c)b} = 0$$

and

$$dE_a = -\omega_a^b \wedge E_b. \tag{10.3}$$

The curvature 2-form is defined by the equations:

$$\Omega_b^a = d\omega_b^a + \omega_c^a \wedge \omega_b^c \tag{10.4}$$

which is related to the Riemann tensor through

$$\Omega_b^a = \frac{1}{2} R_{cdb}^a E^c \wedge E^d \tag{10.5}$$

where R_{cdb}^a are the components of the Riemann tensor in the frame $\{E_a\}$.

10.10 Conjugate points

Let γ belong to a smooth 1-parameter system of affinely parametrised geodesics in a manifold with a Lorentzian metric. This system of geodesics can be described by a smooth map μ from a strip $\{(t,v)|t_0 < t < t_1, -\epsilon < v < \epsilon\}$ into $\mathcal{M}$, where each path defined by v equal to a constant is an affinely parametrised geodesic, parametrised by t, and $\gamma(t) = \mu(t,0)$. Now we denoted by $T^a = \mu^*(\frac{\partial}{\partial t})$ and $V^a = \mu^*(\frac{\partial}{\partial v})$ the vectors given by the standard embedding in $\mathcal{M}$ of the coordinate vectors $\frac{\partial}{\partial t}, \frac{\partial}{\partial v}$. These vectors satisfy the condition that $[T^a, V^a] = 0$ as they are "coordinate vectors" and V^a satisfies the geodesic equation $\nabla_{V^a} V^a = 0$. This conditions can be simultaneously stated as the condition that V^a satisfies the *geodesic deviation equation*

$$\nabla_{\dot{\gamma}} \nabla_{\dot{\gamma}} V^a = R_{bcd}^a T^b T^c T^d \tag{10.6}$$

A *Jacobi field* is any vector V^a defined along γ that satisfies the geodesic deviation equation. A *conjugate point* is a vanishing point of the Jacobi field.
Let γ be null (timelike) geodesic meeting a smooth two-surface $\mathcal{T}$ (three dimensional hypersurface S at the point p). Then a point $q \in \Gamma$ is said to be *conjugate to* $\mathcal{T}(S)$ *on* γ if and only if a non trivial Jacobi field exist on γ which vanishes at p but not everywhere along γ, and which arises from a one-paramter system of affinely parametrised null geodesics which are all orthogonal to $\mathcal{T}(S)$ at their intersection.
Often to guarantee the existence of conjugate points it is not needed to solve the full geodesic deviation equation. Lets consider a congruence of timelike geodesics with future-pointing unit tangent field u^a in a Lorentzian manifold with metric g_{ab}. The *expansion*, θ; *vorticity*, $\mathcal{W}^{ab}$; and *shear*, σ^{ab} are defined uniquely by the expresion

$$u_{\alpha_a;\alpha_b} = \frac{1}{3}\theta \left(u_{\alpha_a} u_{\alpha_b} - g_{\alpha_a \alpha_b}\right) + \sigma_{\alpha_a \alpha_b} + \mathcal{W}_{\alpha_a \alpha_b}$$

and the requirments that $\mathcal{W}^{ab}$ is antisymmetric and σ^{ab} is symmetric. We are using the notation α_a to denote the components of the corrsponding tensor. In particular we have that θ defined as

$$\theta := \mathrm{div}(u^a)$$

satisfies the *Raychaudhuri equation*

$$\frac{d\theta}{dt} = -R_{\alpha_a\alpha_b}u^{\alpha_a}u^{\alpha_b} - \frac{1}{3}\theta^2 - \sigma_{\alpha_a\alpha_b}\sigma^{\alpha_a\alpha_b} + \mathcal{W}_{\alpha_a\alpha_b}\mathcal{W}^{\alpha_a\alpha_b}.$$

where $\sigma_{\alpha_a\alpha_b}\sigma^{\alpha_a\alpha_b}$ is a positive and we have assumed without lose of generality that the *vorticity*, ϖ^{ab}, of the geodesics vanishes. A detailed explanation of this tensors and the derivation of the Raychaudhuri equation can be found in [3].
Notice that if $R_{ab}T^aT^b \geqslant 0$ then we have an inequality of the form

$$\frac{d\theta}{dt} \leqslant -\frac{1}{3}\theta^2 \leqslant 0.$$

Therefore if the initial value of $\theta(0) = \theta_0$ is negative we have the inequality

$$\frac{1}{\theta} \geqslant \frac{1}{\theta_0} + \frac{t}{3}$$

This implies that θ must go to negative infinity in an interval $[0, -\frac{3}{\theta_0}]$. Otherwise, we will have a negative decreasing function which is bigger than zero. This blow up of θ signals the formation of a conjugate point in the interval $[0, -\frac{3}{\theta_0}]$. A precise statement of this effect can be found in ([3], Chapter 4). We stated below for reference.

Theorem 15 *Let S be a spacelike hypersurface and γ a curve orthogonal to S. If $R_{ab}T^aT^b \geqslant 0$ and $\theta(S \cap \gamma) = \theta_0 < 0$ then there will be a point conjugate to S along the curve γ within a distance $\frac{3}{-\theta_0}$ from S, provided that $\gamma(s)$ can be extended that far.*

A critical result of geodesics that contain conjugate points is that they can not be of maximal affine length. This is the content of the following Theorem:

Theorem 16 *A timelike geodesic γ from S to q is maximal if and only if there is no point conjugate to S along γ.*

The proof can be found in [3, 13].

Chapter 11

Causality theory

A Lorentzian manifold $\mathcal{M}$ is called *time orientable* if it is possible to define continuously a division of non-spacelike vector into two classes, arbitrarily called future directed and past-directed. These kind of spacetimes allow us to define a non-spacelike curve γ as *future directed* (*past directed*) if the tangent vectors of γ are future directed (past directed). We will assume time orientability through all the discussion.

To define the causal structure we will require that the metric on $\mathcal{M}$ is C^2. In fact it was proven that one can relax this condition to $C^{1,1}$ or C^0(imposing certain conditions) and still obtain all the results valid for C^2[11]. We demand the C^2 condition for simplicity.

We define for sets S,$\mathcal{U}$ the *chronological future* $I^+(S,\mathcal{U})$ of S relative to $\mathcal{U}$ as the set of all points in $\mathcal{U}$ which can be reached from S by a future-directed timelike curve in $\mathcal{U}$. This set is open in the manifold topology. For lower differentiability than $C^{1,1}$ it is not known if this condition still holds [11]. This set can be seen as all the events a massive particle can reach if it is initially on S.

The *causal future* of S relative $\mathcal{U}$ is denoted by $J^+(S,\mathcal{U})$ and it is defined as the union $S \bigcap U$ with the set of all points in $\mathcal{U}$ which can be reached from S by a future-directed non-spacelike curve in $\mathcal{U}$. It is straightforward to see that $I^+(S,\mathcal{U}) \subset \mathcal{J}^+(\mathcal{S},\mathcal{U})$. This set can be seen as all the events any known particle can reach if it is initially on S.

The *future horismos* of S relative to $\mathcal{U}$, denoted by $E^+(S,\mathcal{U})$ is defined as

$J^+(S,\mathcal{U})\backslash\mathcal{I}^+(\mathcal{S},\mathcal{U})$. If $\mathcal{U}$ is a convex normal neighbourhood around p then $E^+(p,\mathcal{U})$ is generated by the future-directed null geodesics in $\mathcal{U}$ from p, and form the boundary in $\mathcal{U}$ of both $I^+(p,\mathcal{U})$ and $J^+(p,\mathcal{U})$ (proposition 4.5.1,[3]). All definitions are dual for the past just changing future for past in the previous definitions and if $U = \mathcal{M}$ then we just write $I^+(S), J^+(S), E^+(S)$.

If $(\mathcal{M}, g)$ and $(\mathcal{M}', g')$ are spacetimes, a bijection $f : \mathcal{M} \to \mathcal{M}$ between spacetimes such that given p, q the statement $p \in J^+(q)$ if and only if $f(p) \in J^+(f(q))$ is a *causal isomorphism.*

The *chronology condition* is the requirement that spacetime does not have closed time-like curves. The *causality condition* holds if there are no closed non-spacelike curves. This conditions are not apriori ruled out by Einstein's equation. Nevertheless, it is consideration leads to difficult interpretations about free will and the nature of the scientific method. Moreover, in our daily experience we do not experience situations where we return to previous events. This is why we would believe (until experiments tell other wise) that our spacetime must avoid closed causal curves. The following definitions are a refinement of the causal conditions needed to achieve a spacetime as close as possible to the spacetime of our know experience.

The spacetime is called *future distinguishing* if $I^+(p) = I^+(q)$ implies that $p = q$. This condition states that for every p in $\mathcal{M}$ there is a neighbourhood $\mathcal{U}$ of p such that no future non-spacelike curve from p intersects more than once. This formalises the idea that we ca not come arbitrarily close to any point p in $\mathcal{M}$ through a future non-spacelike curves that passes through p . An exact analogue defines *past distinguishing* spacetimes and a future and past distinguishing spacetime is called *distinguishing.* We also mention the fact that if spacetimes $(\mathcal{M}, g)$ and $(\mathcal{M}', g')$ are distinguishing and f is a causal isomorphism then f is a conformal isometry.[113]

The *strong causality condition* holds if for every p in $\mathcal{M}$ there is a neighbourhood $\mathcal{U}$ of$p$ such that no non-spacelike curve from p intersects more than once. Trivially strong causality implies future distinguishing and past distinguishing. Closely related to strong causality is that of *imprisonment.* A non-spacelike curve γ that is future-inextendible is called:*totally imprisoned* if it enters and remain within a compact set S; *partially imprisoned* if it continually re-enters a compact set S. If a compact set S is strongly causal there are not totally or partially imprisoned curves in S.

The conditions so far only apply for a given spacetime $\mathcal{M}$. However, spacetimes which can be achieved by infinitesimals perturbations to $\mathcal{M}$ might have for example closed causal curves. The macroscopic nature of our experience require us to define stability conditions as we are not able to measure any property of spacetime to arbitrary accuracy. To do this we define a strict partial order $<$ defined on the set of all Lorentzian metrics on $\mathcal{M}$ denoted by $Lor(\mathcal{M})$. We say $g < g'$if and only if given $X^a \in T_p$ such that $g(X^a, X^a) \leqslant 0$ implies $g'(X^a, X^a) < 0$ for all $p \in \mathcal{M}$.

A spacetime $(\mathcal{M}, g)$ is *stably causal* if there exist $g' \in Lor(\mathcal{M})$ such that $g < g'$ and g' satisfies the causality condition. This conditions establish that the spacetime $(\mathcal{M}, g)$ and any spacetime reached through a small perturbation in the sense that opens slightly its lightcones would remain causal.

The definition of *causal continuity* requires the outer continuity of the sets I^+, I^-, plus to be future and past distinguishing. The set $I^+(p)$ is outer continuous at some $p \in \mathcal{M}$ if, for any compact subset $K \subset \mathcal{M} \backslash I^+(p)$ there exists an open neighbourhood $\mathcal{U}$ around p such that $K \subset \mathcal{M} \backslash I^+(q)$ for all $q \in \mathcal{U}$. There is a conjecture by Borde and Sorkin that states that causal discontinuous spacetimes and infinite burst of energy for a scalar field propagating in such backgrounds are related.[114] This conjecture might lead to a better understanding of topology change in GR and hence in quantum gravity.

Let S be a closed set then we define the region $D^+(S)$ to the future of S called the *future domain of dependence* of S as the set of all points $p \in \mathcal{M}$ such that every inextendible non-spacelike curve intersects S. The future boundary of $D^+(S) = \bar{D^+}(S) \backslash I^-(D(S))$ is called the *future Cauchy horizon* is S and is denoted by $H^+(S)$. This set is a closed *achronal* (every two points in $H^+(S)$ ca not be joined by timelike curves) .In a similar way we can define the past domain of dependence and the past Cauchy horizon. A *partial Cauchy surface* S is a spacelike hypersurface which no non-spacelike curve intersect more than once. A partial Cauchy surface S that satisfies $D(S) = D^+(S) \bigcup D^-(S) = \mathcal{M}$ is a *Cauchy surface* and we say that $\mathcal{M}$ is globally hyperbolic.

Leray [3] introduce this concept originally by defining the space $C(p, q)$ of all C^0 non-spacelike curves from p to q (a C^0 curve γ is spacelike if there is a piecewise C^∞ spacelike curve joining any two points in γ), regarding two curves to be the same if one is a reparametrization of the other. The topology of $C(p, q)$ is defined by saying that a neighbourhood of γ in $C(p,q)$ consists of all the curves in $C(p,q)$ whose points in $\mathcal{M}$ lie in a neighbourhood $\mathcal{W}$ of the points of γ in $\mathcal{M}$. There are several equivalence of global hyperbolicity where the proofs can be found in [3].

Theorem 17 *A spacetime $\mathcal{M}$ is globally hyperbolic if one of the following equivalent statements hold:*

- *There is a hypersurface S such that S is a Cauchy surface.*
- *$\mathcal{M}$ is strongly causal and given any p,q on $\mathcal{M}$ $J^+(p) \cap J^-(q)$ is compact in the manifold topology*
- *$C(p,q)$ is compact for all p,q in $\mathcal{M}$.*

The importance of globally hyperbolic spacetimes is that the condition is sufficient to have *globally well-posededness* for the wave equation.
In the proof of the singularity theorems one uses critically the existence of maximal geodesics in globally hyperbolic regions. We state the precise theorem which proof can be found in [44].

Theorem 18 *If A and B are closed subsets of a compact set S, thought which strong causality holds, then there is a curve γ from a point of A to a point B which maximizes the lengths of the curve.*

In particular we can choose $A = \{p\}$ and $B = S$ where S is a compact Cauchy surface as we do in the outline of the proof of the Singularity Theorems.
The existence of certain kind of functions is sufficient or necessary for certain causal conditions to hold. The main type of functions consider are *time functions* and *temporal functions.* A function $t : \mathcal{M} \to \mathbb{R}$ is a time function if it is continuous and strictly increasing on any future-directed causal curve. If, additionally, each level hypersurface $S_a = t^{-1}(a)$ is a Cauchy hypersurface (for all a in the image of t), then t is a *Cauchy time function.* A smooth function $T : \mathcal{M} \to \mathbb{R}$ is a temporal function if its gradient is everywhere timelike and past-pointing. If, additionally, each (spacelike) level hypersurface $S_a = T^{-1}(a)$ is a Cauchy hypersurface (for all a in the image of T), then T is a *Cauchy temporal function.* A complete discussion about the causal conditions, time functions and temporal functions can be found on [115].
We now show the logical implications of the causal conditions described above [115].

$$\begin{gathered}\text{Globally Hyperbolic}\\ \Downarrow \\ \text{Causally Continuous}\\ \Downarrow \\ \text{Stably Causal}\\ \Downarrow \\ \text{Strongly Causal}\\ \Downarrow \\ \text{Distinguishing}\end{gathered}$$

$$\Downarrow$$

Causal

$$\Downarrow$$

Chronological

Chapter 12

Banach and Hilbert spaces

Let X denote a real linear space. A mapping $||\cdot|| : X \to [0, \infty)$ is called a *norm* if

- $||u + v|| \leqslant ||u|| + ||v||$ for all $u, v \in X$.
- $||\lambda u|| = |\lambda|||u||$.
- $||u|| = 0$ if and only if $u = 0$.

We say X is normed if there is norm on X. A sequence $\{u^m\} \subset X$ is called a *Cauchy sequence* provided for each $\epsilon > 0$ there exist $N > 0$ such that

$$||u^k - u^l|| < \epsilon$$

for all $k, l > N$. If each Cauchy sequence in X converges, that is, whenever $\{u^m\}$ is a Cauchy sequence, there exists $u \in X$ such that

$$\lim_{m\to\infty} ||u - u^m|| = 0$$

then X is *complete*. A *Banach space* X is a complete, normed linear space.
We say that a linear operator $A : X \to Y$ between Banach spaces is *bounded* if and only if

$$||A|| := \sup_{||x||\leqslant 1} \{||Ax||\} \tag{12.1}$$

The set of all bounded linear functionals on X is the *dual space* of X. We denote it by X^*. A Banach space is *reflexive* if $(X^*)^* = X$.
We say a sequence $\{u^m\} \subset X$ *converges weakly* to $u \in X$ written $u^m \rightharpoonup u$ if

$$u^*(u^m) \to u^*(u) \tag{12.2}$$

for each bounded linear functional $u^* \in X^*$.
In addition, we have

$$||u|| \leqslant \liminf_{m\to\infty} ||u^m|| \tag{12.3}$$

A mapping $(\cdot, \cdot) : X \times X \to \mathbb{R}$ is called an *inner product* if

- $(u, v) = (v, u)$.
- The mapping $u \to (u, v)$ is linear for each $v \in X$.

- $(u, u) \geqslant 0$ for all $u \in X$.
- $(u, u) = 0$ if and only if $u = 0$.

If $(\cdot, \cdot)$ is an inner product, the associated norm is

$$||u|| = (u, u)^{\frac{1}{2}}$$

A *Hilbert space* H is a Banach space X endowed with an inner product that generates a norm.

Chapter 13

Measure Theory

A family $\mathcal{X}$ of subsets of a set X is said to be a *σ-algebra* in case:

- $\emptyset, X \in \mathcal{X}$.
- If $A \in \mathcal{X}$ then $X \backslash A \in \mathcal{X}$.
- If $A_n \in \mathcal{X}$ for $n \in \mathbb{N}$ then $\bigcup_{n \in \mathbb{N}} \{A_n\} \in \mathcal{X}$.

An ordered pair $(X, \mathcal{X})$ is called a *measurable space.*
A function f on X to $\mathbb{R}$ is said to be *measurable* if for every real number r the set

$$\{x \in X : f(x) > r\} \tag{13.1}$$

belongs to $\mathcal{X}$.
An *extended function* is a function where we allow $\infty, -\infty$ to be part of the range and define an extended measurable function as above including now $\infty, -\infty$.
Unless otherwise stated, we will used the σ-algebra called the *Borel algebra* generated by all the open intervals in $\mathcal{M}$.

13.1 Measures

A *measure* is an extended real-valued function μ defined on a σ-algebra $\mathcal{X}$ of subsets of X such that:

- $\mu(\emptyset) = 0$.
- $\mu(E) \geqslant 0$ for all $E \in \mathcal{X}$.
- μ is *countably additive* in the sense that if $\{E_n\}$ is any disjoint sequence of sets in $\mathcal{X}$, then

$$\mu(\bigcup \{E_n\}) = \sum_n \mu(E_n) \tag{13.2}$$

A *measure space* is a triple $(X, \mathcal{X}, \mu)$. We say that a proposition holds *μ-almost everywhere* if there exists a subset $N \in \mathcal{X}$ with $\mu(N) = 0$ such that the proposition holds in the complement of N.

13.2 Integrals

Let $M^+(X,\mathcal{X})$ be the collection of all non negative measurable functions. A function ψ is *simple* if it has only a finite number of values and we can always express it as $\psi = \sum_j a_j \chi_{E_j}$, where $a_j \in \mathbb{R}$ and χ_{E_j} is the characteristic function of a set E_j in $\mathcal{X}$. We define the integral of ψ with respect to μ to be:

$$\int \psi d\mu = \sum_j a_j \mu(E_j) \tag{13.3}$$

If $f \in M^+(X,\mathcal{X})$ we define the *integral of f with respect to* μ to be the extended real number

$$\int f d\mu = sup \int \psi d\mu \tag{13.4}$$

where the supremum is taken over all simple functions satisfying $0 \leqslant \psi \leqslant f$.
The collection $L^1(X,\mathcal{X},\mu)$ of *integrable functions* consists of all real-valued $\mathcal{X}$-measurable functions such that$f = f_1 - f_2$ where $f_1, f_2 \in M^+(X,\mathcal{X})$ and have finite integrals with respect to μ. In that case, we define the *integral* of f with respect to μ to be:

$$\int f d\mu = \int f_1 d\mu - \int f_2 d\mu \tag{13.5}$$

13.3 Banach space-valued functions

We extend the notions of differentiability, measurability and integrability to mappings

$$\mathbf{f} : [0,T] \to X \tag{13.6}$$

where X is Banach space, with norm $||\cdot||$.
The definition of continuity and pointwise differentiability of Banach-valued functions are the same as in the scalar case. A function $\mathbf{f}$ is *strongly continuous* at $t \in (0,T)$ if $||f(s) - f(t)|| \to 0$ as $s \to t$. A function $\mathbf{f}$ is *strongly differentiable* at $t \in (0,T)$, with strong pointwise derivative $\mathbf{f_t(t)}$, if

$$\mathbf{f}_t(t) = \lim_{h \to 0} \frac{||\mathbf{f}(t+h) - \mathbf{f}(t)||}{h} \tag{13.7}$$

where the limit exists strongly in X. A function $\mathbf{s} : [0,T] \to X$ is called *simple* if it has the form

$$\mathbf{s}(t) = \sum_{i=1}^{m} \chi_{E_i} u_i \tag{13.8}$$

where each E_i is a Lebesgue measurable subset of $[0,T]$ and $u_i \in X$. A function $\mathbf{f}$ is *strongly measurable* if there exist simple functions s_k such that

$$\lim_{k \to \infty} ||s_k(t) - f(t)|| = 0$$

for a.e. $0 \leqslant t \leqslant T$.
We define the integral of $\mathbf{s} = \sum_{i=1}^{m} E_i u_i$ as

$$\int_0^T \mathbf{s} dt := \sum_{i=1}^{m} |E_i| u_i \tag{13.9}$$

If $\mathbf{f}$ is a strongly measurable function and there exists a sequence $\mathbf{s_k}$ of simple functions such that

$$\int_0^T ||\mathbf{s_k} - \mathbf{f}||dt \to 0 \tag{13.10}$$

as $k \to \infty$, then we say $\mathbf{f}$ is *summable.* For $\mathbf{f}$ summable, we define the *integral* as

$$\int_0^T \mathbf{f}dt := \lim_{k\to\infty} \int_0^T \mathbf{s_k}dt \tag{13.11}$$

Chapter 14

Functional Analysis

14.1 Regularity theory

A function f defined on a nonempty open set $\Omega \subset \mathbb{R}^n$ is called a *test function* if $f \in C^\infty(\Omega)$ and there is a compact set $K \subset \Omega$ such that the support of f lies in K. The set of all test functions is denoted by $C_0^\infty(\Omega)$.
The space of function $C_0^\infty(\Omega)$ admits a topology. To be precise, let $\{\phi_n\}$, $n \in \mathbb{N}$ and ϕ elements of $C_0^\infty(\Omega)$. We define a topology on $C_0^\infty(\Omega)$ by saying that ϕ_n converges to ϕ in $D(\Omega)$, if there is a compact subset K of Ω such that the support of all the ϕ_n (and of ϕ) lie in K and, moreover, for all $\epsilon > 0$ there is N such that for all $x \in K$ if $n > N$ then $\mid D_\alpha \phi_n - D_\alpha \phi \mid < \epsilon$ for all α. We denote this topological space as $D(\Omega)$.
A *distribution* is a linear mapping $\alpha : D(\Omega) \to \mathbb{R}$ which is continuous in the following sense: If $\phi_n \to \phi$ in $D(\Omega)$, then $\alpha(\phi_n) \to \alpha(\phi)$. The set of all distributions is denoted by $D'(\Omega)$.
Let $\tilde{L}^p(\Omega, \mu)$ be the set of all C^0 $g : \bar{\Omega} \to \mathbb{R}$ for which

$$\|g\|_p = \left(\int_\Omega |g|^p \right)^{\frac{1}{p}} d\mu \tag{14.1}$$

is finite. Then the completion of $\tilde{L}^p(\Omega, \mu)$ with respect to $\|\cdot\|_p$ is the space called $L^p(\Omega, \mu)$. We say f is in $L^p_{loc}(\Omega, \mu)$ if f is in $L^p(K, \mu)$ for all compact sets K properly contained in Ω . The space $L^\infty(\Omega, \mu)$ is defined as $\{f \text{ s.t } |f(x)| \leqslant C \text{ for } \mu\text{-almost everywhere}\}$
Let u in $L^1_{loc}(\Omega, \mu)$. The weak derivative $v = \frac{\partial^\alpha u}{\partial x^\alpha}$ of u, if it exists, is a $v \in L^1_{loc}(\Omega, \mu)$ that satisfies:

$$\int_\Omega u \frac{\partial^\alpha \phi}{\partial x^\alpha} d\mu = (-1)^{|\alpha|} \int_\Omega v \phi d\mu$$

for all $\phi \in D(\Omega)$.
Let k be a non- negative integer and let $1 \leqslant p \leqslant \infty$. Then we define the *Sobolev space* $W^{k,p}(\Omega, \mu)$ to be the set of all distributions $u \in L^p(\Omega, \mu)$ such that the weak derivatives $\frac{\partial^\alpha u}{\partial x^\alpha} \in L^p(\Omega, \mu)$ for $|\alpha| \leqslant k$.
When $p = 2$ we can define an inner product:

$$(u, v)_k = \Sigma_{|\alpha| \leqslant k} \int_\Omega \frac{\partial^\alpha u}{\partial x^\alpha} \frac{\partial^\alpha v}{\partial x^\alpha} d\mu \tag{14.2}$$

that defines a norm

$$(u, u)_k = (\|u\|^k_\Omega)^2$$

and we denote this Sobolev space as $H^k(\Omega,\mu)$.
We define also the *local Sobolev spaces* $W^{k,p}_{loc}(\Omega,\mu)$. We say f is in $W^{k,p}_{loc}(\Omega,\mu)$ if given any compact set K properly contained in Ω, f is in $W^{k,p}(K,\mu)$
There is a series of embeddings of L^p and Sobolev spaces which we state below:

1. If $k \leqslant l$ then $W^{k,p}(\Omega,\mu) \to W^{l,p}(\Omega,\mu)$.
2. If Ω has a finite measure, $\mu(\Omega) < \infty$, then
$$L^q(\Omega,\mu) \subset L^p(\Omega,\mu)$$
for any $1 \leqslant q \leqslant p \leqslant \infty$.
3. If either $kp \geqslant n$ or $m = n$ and $p = 1$ then
$$W^{k,p}(\Omega,\mu) \to C^j_B(\Omega)$$
where $C^j_B(\Omega)$ are the C^j functions with bounded derivatives in Ω up to order j.
4. If $kp = n$ then
$$W^{k,p}(\Omega,\mu) \to L^q(\Omega,\mu)$$
for $p \leqslant q \leqslant \infty$.
5. If $kp \leqslant n$ then
$$W^{k,p}(\Omega,\mu) \to L^q(\Omega,\mu)$$
for $p \leqslant q \leqslant \frac{kp}{n-kp}$.

The embeddings should be understand in the sense of representatives. For example, the statement
$$W^{k,p}(\Omega,\mu) \to C^j_B(\Omega)$$
means that for each element $\phi \in W^{k,p}(\Omega,\mu)$ there is a $C^j_B(\Omega)$ function which is equal to ϕ μ-almost everywhere.
Sometimes it is necessary to restrict a Sobolev function to another subspace. To do so, we will state the following result whose proof can be found in [116].
Let k be a positive integer. Assume that Ω is a C^k n-manifold with bounded boundary. Then if $kp \leqslant n$ and $p \leqslant q \leqslant \frac{(n-1)p}{n-kp}$ there is a bounded *trace operator*
$$T : W^{k,p}(\Omega,\mu) \to L^q(\partial\Omega,\nu).$$
where μ,ν are the Lebesgue measures in the respective spaces. If $kp = n$, then the embedding holds for $p \leqslant q \leqslant \infty$.
We also might want to approximate functions in a Sobolev space through C^∞ functions. The way to do this is by smoothing with *strict delta nets*. A net $\{(\rho_n)\} \in (0,1]$ of smooth functions on $\mathbb{R}^n$ is called a strict delta net, if

- $supp(\rho_n) = \{p \in \mathbb{R}^n | \rho_n \neq 0\} \to 0$ for $n \to 0$.
- $\int \rho_n d\mu \to 1$ for $n \to 0$.
- ρ_n is uniformly bounded in $L^1(\Omega,\mu)$.

For any strict delta net $\{(\rho_n)\}$ we denote by $d\rho_n$ the *diameter of the support* of ρ_n , i.e. $d\rho_n := sup\{|x| : x \in supp(\rho_n)\}$. For any $f \in L^1_{loc}(\Omega, \mu)$ we call the *convolution* f_n of f with a strict delta net $\{(\rho_n)\}$ a smoothing of f , given by

$$f_n(x) := \rho_n * f(x) = \int_{B(x,d\rho_n)} f(x-y)\rho_n(y)dy \tag{14.3}$$

where $B(x, r)$ denotes the open ball of radius r around x.
The following theorem states the wanted result. The proof can be found in [116].

Theorem 19 *The smoothing of any $f \in L^1_{loc}(\Omega)$ has the following properties:*

- *$f_n \in C^\infty(\Omega_{d\rho_n})$ and $f_n \to f$ almost everywhere. The convergence here is pointwise.*
- *If f is continuous the convergence is actually uniform on compact subsets of Ω.*
- *If $f \in W^{m,p}(\Omega) for 1 \leqslant p < \infty$ then $f_n \to f \in W^{m,p}(\Omega)$. The convergence here is in the norm given by $W^{m,p}(\Omega)$.*
- *The derivative operator commute with smoothing, i.e. $\partial_x f_n(x) = \rho_n * \partial_x f$*

This result implies that the C^∞ functions are *dense* in any Sobolev space, that means we can always find sequences of C^∞ functions that converge to any desired function in a given Sobolev space. In fact, in the case of L^p spaces the functions $C^\infty_0$are dense. Sometimes it is convenient to parametrized the strict delta net from $[1, \infty)$ instead of the interval $(0, 1]$. In this thesis, we will use both, butit will be made from the context which convention we are using.
When the measure is the one associated to the volume form dx^n, (or dx^{n+1} when the domain is the whole spacetime), we will omit the measure and just write $L^2(\Sigma)$. We will also use this convention when considering the details of the function spaces $L^2(\Sigma, \nu_h)$. However, this will be mentioned explicitly in the text when it occurs.
We define $H^{-k}(\Sigma)$, the set of all linear functionals on $H^k_0(\Sigma)$. Moreover, if Σ is $\mathbb{R}^n$ or a compact manifold $H^{-k}(\Sigma)$ denotes the dual space of $H^k(\Sigma)$. Given an element $v \in H^{-k}(\Sigma)$ and $u \in H^k(\Sigma)$ we denote the action or *dual pairing* of v to u as the real number $< u, v >$.
We now extend the notion of L^p space and $W^{k,p}$ to Banach-valued functions.
We say $u \in L^p(0, T; X)$ if and only if

$$\left(\int_0^T ||u||^p_X \right)^{\frac{1}{p}} < \infty \tag{14.4}$$

and we define the *Sobolev space* $W^{k,p}(0, T; X)$ as the space of all functions $u \in L^p(0, T; X)$ such that $\frac{d^s}{dt^s}u \in L^p(0, T; X)$ for $s = 1, 2, ..., k$.
We say $u \in C^k([0, T]; X)$ if and only if

$$max_{t\in[0,T]} \left(\sum_{\alpha \leqslant k} \left\| \frac{d^k}{dt} u(t) \right\|_X \right) < \infty \tag{14.5}$$

14.2 Important Theorems

Theorem 20 Hahn-Banach Theorem *Let X be a Banach space and let M be a linear subspace of X. Let l_M be a bounded linear mapping from M to $\mathbb{R}$. Then there is a linear functional $l \in X^*$ such that $l|_M = l_M$ and $||l|| = ||l_M||$.*

Theorem 21 Riesz Representation Theorem *Let H be a Hilbert space, with inner product $(,)$. Then, H^*, the dual space of H can be canonically identified with H. More precisely, for each $u^* \in H^*$ there exists a unique element $u \in H$ such that*

$$u^*(v) = (u, v) \tag{14.6}$$

for all $v \in H$.

Theorem 22 Banach-Alaoglu Theorem *Let X be a reflexive Banach space and suppose the sequence $\{u^m\} \subset X$ is bounded. Then there exists a subsequence $\{u^{k_l}\} \subset \{u^k\}$ and $u \in X$ such that $u^{k_l} \rightharpoonup u$.*

Theorem 23 Integration by parts in Bochner spaces *Let $V \subset H \subset V^*$ where $V = H^1(\Sigma), H = L^2(\Sigma)$ and $V^* = H^{-1}(\Sigma)$ and u, v elements of the set $W = \{w \in L^2(0,T;V) \mid w' \in L^2(0,T;V^*)\}$, then*

$$(u(T), v(T))_H - (u(0), v(0))_H = \int_0^T \langle u'(t), v(t)\rangle_{V^*,V} + \langle v'(t), u(t)\rangle_{V^*,V}$$

Theorem 24 Parseval's identity *Suppose that H is an inner-product space. Let B be an orthonormal basis of H; i.e., an orthonormal set which is total in the sense that the linear span of B is dense in H. Then*

$$\|x\|^2 = (x, x) = \sum_{v \in B} |(x, v)|^2 .$$

Chapter 15

Elliptic Theory

Theorem 25 Elliptic Regularity *Let Σ be a compact manifold and Δ_h the Laplace-Beltrami operator associated to a smooth Riemannian metric h_{ij} and $f \in L^2(\Sigma)$. Then there exists a*

$$u \in H^2(\Sigma)$$

that satisfies the estimate

$$||u||_{H^2(\Sigma)} \leqslant C\left(||f||_{H^2(\Sigma)} + ||u||_{L^2(\Sigma)}\right) \tag{15.1}$$

and

$$\Delta_h u = f \tag{15.2}$$

a.e. in Σ.

The proof can be found in [63].

Theorem 26 Elliptic Regularity version two. *Assume $L = \partial_j(a^{ij}\partial_i(\cdot))$ is an elliptic operator with $a^{ij} \in C^1(\Sigma)$ and $f \in L^2(\Sigma)$. Suppose that u satisfies*

$$B(u, v) = (f, v)_{L^2(\Sigma)}$$

for $v \in H^1(\Sigma)$. Then

$$u \in H^2(\Sigma)$$

and satisfies the estimate

$$||u||_{H^2(\Sigma)} \leqslant C\left(||f||_{H^2(\Sigma)} + ||u||_{L^2(\Sigma)}\right) \tag{15.3}$$

The proof can be found in [117] and in [63] for the case when Σ is an open bounded set in $\mathbb{R}^n$.

Chapter 16

Parabolic Theory

In this section we give the main results concerning the energy estimates and regularity of solutions to the parabolic problem

$$v_{tt} - \epsilon\Delta_h v \quad = \quad f \tag{16.1}$$

$$v(0,\cdot) = g \tag{16.2}$$

where $f \in L^2(\Sigma_{[0,T]})$, $g \in H^1(\Sigma)$, Δ_h is the Laplace-Beltrami operator associated to a smooth Riemannian metric h_{ij} and ϵ is a positive real number.
The results of this section are stated in three propositions. The first proposition states the existence of suitable m-approximate solutions. The second proposition allows us to obtain suitable energy estimates that are uniform in m. The third proposition shows the existence of the solutions and their regularity.

Proposition 12 *For each $m \in \mathbb{N}$ there is a unique function $v^{\epsilon,m}$ such that*

$$(\partial_t v^{\epsilon,m}, w_k)_{L^2(\Sigma)} + B[v^{\epsilon,m}, w_k; t] = (f, w_k)_{L^2(\Sigma)} \tag{16.3}$$

and

$$d_m^k(0) = (g, w_k)_{L^2(\Sigma)} \tag{16.4}$$

for $k = 1, ..., m$ and a.e. in $0 \leqslant t \leqslant T$ where

$$B[v^{\epsilon,m}, w_k] = \epsilon \int_\Sigma h^{ij}\partial_i(v^{\epsilon,m})\partial_j(w_k)\nu_h d^n x, \tag{16.5}$$

Therefore $B[u,v]$ is the bilinear form in $H^1(\Sigma)\times H^1(\Sigma)$ associated to Δ_h and w_k are a joint orthogonal basis for $H^1(\Sigma)$ and $L^2(\Sigma)$. If $\Sigma \subset \mathbb{R}^n$ then we need to change $H^1(\Sigma)$ for $H_0^1(\Sigma)$.

Proof. We start by choosing smooth functions $w_k(x)$ such that:

$$\{w_k\}_{k=1}^{\infty} \text{ is an orthogonal set in } H^1(\Sigma)$$

$$\{w_k\}_{k=1}^{\infty} \text{ is an orthonormal basis of } L^2(\Sigma)$$

We can form the desired basis by choosing the eigenfunctions of the Laplace-Beltrami operator Δ_h [118]. If $\Sigma \subset \mathbb{R}^n$ then we can choose the eigenfunctions of the Laplace operator Δ. However we need to change $H^1(\Sigma)$ for $H_0^1(\Sigma)$ [63].

Now fix a positive integer m, write

$$\upsilon^m(t,x) := \sum_{k=1}^{m} d_m^k(t) w_k(x) \tag{16.6}$$

and consider for each $k = 1, ..., m$ the equation:

$$(\upsilon_t^m, w_k)_{L^2(\Sigma)} + B[\upsilon^m, w_k; t] = (f, w_k)_{L^2(\Sigma)} \tag{16.7}$$

The system of equations (16.3) can be arranged as a system of linear ODEs for each $k = 1, ..., m$ given by

$$\dot{d}_m^k(t) + \lambda^k d_m^k(t) = f^k(t) \tag{16.8}$$

where $e^{kl}(t) := B[w_l, w_k; t] = \epsilon \int_\Sigma h^{ij}(t,x) w_{l_i} w_{k_j} \nu_h$, $f^k(t) := (f, w_k)_{L^2(\Sigma)}$ and λ^k is the k eigenvalue of Δ_h.
We also require that the system satisfies the initial condition

$$d_m^k(0) = (g, w_k)_{L^2(\Sigma)}, \tag{16.9}$$

for $k = 1, ..., m$.
Then by standard local existence and uniqueness theorems for linear ordinary differential equations, we obtain a unique $d_m^k(t) \in C^1([0,T])$ for every $k = 1, ..., m$.
Therefore we have shown that for each m there is a unique solution, υ^m, satisfying (16.3) and (16.4) which we call the *m-approximate solution*. Therefore $\upsilon^m = \upsilon^{\epsilon,m}$. ⊡

Proposition 13 *There exist a constant C, depending only on Σ, T, ϵ such that*

$$\begin{aligned} &\left(sup_{t\in[0,T]}||\upsilon^m(t)||^2_{H^1(\Sigma)} + ||\upsilon_t^m||^2_{L^2(0,T;L^2(\Sigma))} + \right) \\ &\qquad \leqslant C\left(||f||^2_{L^2([0,T];L^2(\Sigma))} + ||g||^2_{H^1(\Sigma)}\right) \end{aligned} \tag{16.10}$$

Proof. Fixing m we multiply (16.3) by $d_m^k(t)$ and sum from $k = 1, ..., m$ to obtain

$$(\upsilon_t^m, \upsilon^m)_{L^2(\Sigma)} + B[\upsilon^m, \upsilon^m] = (f, \upsilon^m)_{L^2(\Sigma)} \tag{16.11}$$

Noticing the following inequalities

$$||\upsilon^m||^2_{H^1(\Sigma)} \leqslant KB[\upsilon^m, \upsilon^m] \tag{16.12}$$

$$(\upsilon_t^m, \upsilon^m)_{L^2(\Sigma)} = \frac{d}{dt}(||\upsilon^m||)^2_{L^2(\Sigma)} \tag{16.13}$$

$$(f, \upsilon^m)_{L^2(\Sigma)} \leqslant \frac{1}{2}\left(||f||^2_{L^2(\Sigma)} + ||\upsilon^m||^2_{L^2(\Sigma)}\right) \tag{16.14}$$

Therefore we have

$$\frac{d}{dt}(||\upsilon^m||)^2_{L^2(\Sigma)} \leqslant \frac{d}{dt}(||\upsilon^m||)^2_{L^2(\Sigma)} + ||\upsilon^m||^2_{H^1(\Sigma)} \tag{16.15}$$

$$\leqslant C_1 \frac{d}{dt}(||\upsilon^m||)^2_{L^2(\Sigma)} + B[\upsilon^m, \upsilon^m] \tag{16.16}$$

$$\leqslant C_2\left(||f||^2_{L^2(\Sigma)} + ||\upsilon^m||^2_{L^2(\Sigma)}\right) \tag{16.17}$$

Taking the notation $\eta(t) := ||\upsilon^m(t)||^2_{L^2(\Sigma)}$ and $\xi(t) := ||f(t)||^2_{L^2(\Sigma)}$ the inequalities (16.15) and (16.17) give

$$\dot{\eta}(t) \leqslant C_2(\eta(t) + \xi(t)) \tag{16.18}$$

Thus applying Gronwall's inequality and returning to the original variables yields the estimate

$$sup_{t\in[0,T]}||\upsilon^m||^2_{L^2(\Sigma)} \leqslant C_2\left(||f||^2_{L^2(\Sigma)} + ||\upsilon^m(0)||^2_{L^2(\Sigma)}\right) \tag{16.19}$$

Noticing that $||\upsilon^m(0)||^2_{L^2(\Sigma)} \leqslant ||g||^2_{L^2(\Sigma)}$ we obtain the estimate

$$sup_{t\in[0,T]}||\upsilon^m||^2_{L^2(\Sigma)} \leqslant C_2\left(||f||^2_{L^2(\Sigma)} + ||g||^2_{L^2(\Sigma)}\right) \tag{16.20}$$

Returning to the inequalities (16.12), (16.13) and (16.14) and using (16.11) and the estimate (16.20) we have

$$\begin{aligned} ||\upsilon^m||^2_{H^1(\Sigma)} &\leqslant C_2\left(||f||^2_{L^2(\Sigma)} + ||\upsilon^m||^2_{L^2(\Sigma)}\right) && (16.21) \\ &\leqslant C_3\left(||f||^2_{L^2(\Sigma)} + ||g||^2_{L^2(\Sigma)}\right) && (16.22) \end{aligned}$$

Integrating from 0 to T we obtain

$$||\upsilon^m||^2_{L^2(0,T;H^1(\Sigma))} \leqslant C_4\left(||f||^2_{L^2(0,T;L^2(\Sigma))} + ||g||^2_{L^2(\Sigma)})\right) \tag{16.23}$$

Now multiply (16.3) by $\dot{d^k_m}(t)$ and sum from $k = 1, .., m$ to discover

$$(\upsilon^m_t, \upsilon^m_t)_{L^2(\Sigma)} + B[\upsilon^m, \upsilon^m_t; t] = (f, \upsilon^m_t)_{L^2(\Sigma)} \tag{16.24}$$

Noticing the following inequalities

$$\begin{aligned} \frac{d}{dt}\left(\frac{1}{2}B[\upsilon^m, \upsilon^m]\right) &= B[\upsilon^m, \upsilon^m_t] && (16.25) \\ (f, \upsilon^m_t)_{L^2(\Sigma)} &\leqslant \frac{1}{2}\left(||f||^2_{L^2(\Sigma)} + ||\upsilon^m_t||^2_{L^2(\Sigma)}\right) && (16.26) \end{aligned}$$

Therefore we have

$$||\upsilon^m_t||^2_{L^2(\Sigma)} + \frac{d}{dt}\left(\frac{1}{2}B[\upsilon^m, \upsilon^m]\right) \leqslant \frac{1}{2}\left(||f||^2_{L^2(\Sigma)} + ||\upsilon^m_t||^2_{L^2(\Sigma)}\right) \tag{16.27}$$

Integrating in time, we find

$$\begin{aligned} &||\upsilon^m_t||^2_{L^2(0,T;L^2(\Sigma))} + \tfrac{1}{2}B[\upsilon^m(t), \upsilon^m(t)] \\ &\leqslant \left(\tfrac{1}{2}B[\upsilon^m(0), \upsilon^m(0) + ||f||^2_{L^2(0,T;L^2(\Sigma))}\right) + \tfrac{1}{2}||\upsilon^m_t||^2_{L^2(0,T;L^2(\Sigma))} && (16.28) \end{aligned}$$

Again noticing that

$$||\upsilon^m(0)||_{H^1(\Sigma)} \leqslant ||g||_{H^1(\Sigma)} \quad (16.29)$$
$$B[\upsilon^m(t),\upsilon^m(t)] = ||\upsilon^m(t)||_{H^1(\Sigma)} \quad (16.30)$$

the expressions follows from compact embeddings of Sobolev spaces in compact manifolds or bounded domains in $\mathbb{R}^n$ where one has to take instead of the H^1 norm the H^1_0 norm [119].
Therefore using (16.30) and (16.29) in (16.28) we obtain

$$||\upsilon^m_t||^2_{L^2(0,T;L^2(\Sigma))} \leqslant C_5 \left(||g||_{H^1(\Sigma)} + ||f||^2_{L^2(0,T;L^2(\Sigma))}\right) \quad (16.31)$$

$$sup_{t\in[0,T]}||\upsilon^m(t)||^2_{H^1(\Sigma)} \leqslant C_6 \left(||g||_{H^1(\Sigma)} + ||f||^2_{L^2(0,T;L^2(\Sigma))}\right) \quad (16.32)$$

$\boxdot$

Notice that the estimates bound the sequence $\{\upsilon^m\}$ in $L^2(0,T;H^1(\Sigma))$ and $\{\upsilon^m_t\}$ in $L^2(0,T;L^2(\Sigma))$. Therefore there is a subsequence that converges weakly to an element υ^ϵ in $L^2(0,T;H^1(\Sigma))$ with $\{\upsilon^\epsilon_t\}$ in $L^2(0,T;L^2(\Sigma))$. We have add the ϵ to remark that at this point there is still an ϵ dependence.
Moreover, norms are lower semicontinuous [120], so we have

$$||\upsilon^\epsilon(t)|| \leqslant \liminf_{m\to\infty} ||\upsilon^{\epsilon,m}(t)|| \quad (16.33)$$

a.e. for $0 \leqslant t \leqslant T$. [120]
Therefore using (16.32) we have

$$\begin{aligned} ||\upsilon^\epsilon(t)||_{H^1(\Sigma)} &\leqslant sup_{t\in[0,T]}||\upsilon^\epsilon||_{H^1(\Sigma)} && (16.34)\\ &\leqslant sup_{t\in[0,T]} \liminf_{m\to\infty} ||\upsilon^{\epsilon,m}||_{H^1(\Sigma)} && (16.35)\\ &\leqslant sup_{t\in[0,T]} sup_{m\in\mathbb{N}}||\upsilon^{\epsilon,m}||_{H^1(\Sigma)} && (16.36)\\ &\leqslant C_8\left(||g||_{H^1(\Sigma)} + ||f||^2_{L^2(0,T;L^2(\Sigma))}\right) && (16.37)\end{aligned}$$

Therefore υ^ϵ is in $L^\infty(0,T;H^1(\Sigma))$.

Proposition 14 *There exist a constant C, depending only on Σ, T, ϵ such that*

$$\begin{aligned} &\left(||\upsilon^\epsilon||^2_{L^2(0,T;H^2(\Sigma))}+\right)\\ \leqslant\, &C\left(||f||^2_{L^2([0,T];L^2(\Sigma))} + ||g||^2_{H^1(\Sigma)}\right)\end{aligned} \quad (16.38)$$

Moreover, υ^ϵ is the unique function that satisfies

$$(\upsilon^\epsilon_t - \epsilon\Delta_h\upsilon^\epsilon)_{L^2(\Sigma)} = (f,v)_{L^2(\Sigma)} \quad (16.39)$$

with $\upsilon^\epsilon(0) = g$ a.e. for $0 \leqslant t \leqslant T$.

Proof
First multiply (16.3) by a function $m(t) \in C^{\infty}([0,T])$ and integrate with respect to time to give

$$\int_0^T \Big((v_t^m, m(t)w_k)_{L^2(\Sigma)} + B[v^m, m(t)w_k; t] \Big)\, dt = \int_0^T (f, m(t)w_k)_{L^2(\Sigma)}\, dt \qquad (16.40)$$

Then taking $m = m_l$ which is the subsequence that converges weakly and taking the limit as $m_l \to \infty$ we obtain

$$\int_0^T \Big((v_t^\epsilon, m(t)w_k)_{L^2(\Sigma)} + B[v^\epsilon, m(t)w_k] \Big)\, dt = \int_0^T (f, m(t)w_k)_{L^2(\Sigma)}\, dt \qquad (16.41)$$

Thus for any test function of the form $V = \sum_{k=1}^{N} m^k(t)w_k(x)$ we have that equality (16.41) is satisfied. Moreover, test functions of that form are dense in $L^2(0,T;H_0^1(\Sigma))$. Therefore, we have shown that

$$\int_0^T \Big((v_t^\epsilon, V)_{L^2(\Sigma)} + B[v^\epsilon, V] \Big)\, dt = \int_0^T (f, V)_{L^2(\Sigma)}\, dt \qquad (16.42)$$

for any $V \in L^2(0,T;H^1(\Sigma))$. Hence in particular

$$(v_t^\epsilon, \tilde{V})_{L^2(\Sigma)} + B[v^\epsilon, \tilde{V}] = \Big(f, \tilde{V}\Big)_{L^2(\Sigma)} \qquad (16.43)$$

for any $\tilde{V} \in H^1(\Sigma)$.
Moreover, using (16.43) and integrating $B[v^\epsilon, \bar{V}]$ by parts once, we can rewrite the equality to read

$$(v^\epsilon, \epsilon\Delta_h \bar{V})_{L^2(\Sigma)} = \Big(f - v_t^\epsilon, \bar{V}\Big)_{L^2(\Sigma)} =: \Big(\tilde{f}, \tilde{V}\Big)_{L^2(\Sigma)} \qquad (16.44)$$

for any $\bar{V} \in C_0^\infty(\Sigma)$ a.e. for $0 \leqslant t \leqslant T$.
Therefore v^ϵ is a weak solution to the elliptic problem $\epsilon\Delta_h v^\epsilon = \tilde{f}$. Using elliptic regularity, we obtain the following estimates

$$||v^\epsilon||_{H^2(\Sigma)} \leqslant C\left(||\tilde{f}||_{L^2(\Sigma)} + ||v^\epsilon||_{L^2(\Sigma)}\right) \qquad (16.45)$$

A proof for the case of compact manifolds can be found in [121]. For bounded domains in $\mathbb{R}^n$ the result can be found in [63]. Therefore we have

$$||v^\epsilon||_{H^2(\Sigma)} \leqslant C\left(||f||_{L^2(\Sigma)} + ||v_t^\epsilon||_{L^2(\Sigma)} + ||v^\epsilon||_{L^2(\Sigma)}\right) \qquad (16.46)$$

Then integrating in time and using the estimate (16.10) we find

$$\left(||v^\epsilon||^2_{L^2(0,T;H^2(\Sigma))} + \right) \leqslant C\left(||f||^2_{L^2([0,T];L^2(\Sigma))} + ||g||^2_{H^1(\Sigma)}\right) \qquad (16.47)$$

Now using (16.43) and integrating by parts once we obtain

$$(v_t^\epsilon - \epsilon\Delta_h v^\epsilon, \bar{V})_{L^2(\Sigma)} = \Big(f, \bar{V}\Big)_{L^2(\Sigma)} \qquad (16.48)$$

for any $\tilde{V} \in H^1(\Sigma)$ a.e. for $0 \leqslant t \leqslant T$. Note that $\Delta_h \upsilon^\epsilon$ make sense because (16.47) guarantees that $\upsilon^\epsilon \in L^2(0,T;H^2(\Sigma))$.
To check for the initial conditions, we integrate (16.42) by parts to obtain

$$\int_0^T \Big(-(\upsilon^\epsilon, V_t)_{L^2(\Sigma)} + B[\upsilon^\epsilon, V]\Big)\,dt = \int_0^T (f,V)_{L^2(\Sigma)}\,dt + (u(0), V(0))_{L^2(\Sigma)} \tag{16.49}$$

for any $V \in C^1(0,T;H^1(\Sigma))$ with $V(T) = 0$. Similarly, from (16.40) and noticing that the equation is valid for all $V \in C^1(0,T;H^1(\Sigma))$ and integrating by parts, we obtain

$$\int_0^T \Big(-(\upsilon^m, V_t)_{L^2(\Sigma)} + B[\upsilon^m, V]\Big)\,dt = \int_0^T (f,V)_{L^2(\Sigma)}\,dt + (\upsilon^m(0), V(0))_{L^2(\Sigma)} \tag{16.50}$$

Then taking the limit $m \to \infty$ and taking into account (16.4) we know that $\upsilon^m(0) \to g$ in $L^2(\Sigma)$ and therefore we find

$$\int_0^T \Big(-(\upsilon^\epsilon, V_t)_{L^2(\Sigma)} + B[\upsilon^\epsilon, V]\Big)\,dt = \int_0^T (f,V)_{L^2(\Sigma)}\,dt + (g, V(0))_{L^2(\Sigma)} \tag{16.51}$$

As V was arbitrary and comparing (16.49) and (16.51), we conclude $u(0) = g$.
That υ^ϵ is the unique solution follows from (16.47). If $\bar{\upsilon}^\epsilon$ was another solution with the same initial data g and source function f, then the difference $\tilde{\upsilon}^\epsilon = \upsilon^\epsilon - \bar{\upsilon}^\epsilon$ is a solution to the problem with vanishing initial data and source function. Then the estimate gives the inequality

$$||\tilde{\upsilon}^\epsilon||^2_{L^2(0,T;H^2(\Sigma))} \leqslant 0 \tag{16.52}$$

which gives $\tilde{\upsilon}^\epsilon = 0$ and therefore $\upsilon^\epsilon = \bar{\upsilon}^\epsilon$. ⊡

Collecting all the results so far, we have shown that there is a unique solution υ^ϵ in $L^2(0,T;H^2(\Sigma)) \bigcap L^\infty(0,T;H^1(\Sigma))$ with υ^ϵ_t in $L^2(0,T;L^2(\Sigma))$ that satisfies

$$(\upsilon^\epsilon_t - \epsilon\Delta_h \upsilon^\epsilon, \bar{V})_{L^2(\Sigma)} = \Big(f, \bar{V}\Big)_{L^2(\Sigma)} \tag{16.53}$$

for any $\tilde{V} \in H^1(\Sigma)$, a.e. for $0 \leqslant t \leqslant T$ with initial data $u(0) = g$ where $g \in H^1(\Sigma)$ and satisfies the estimates (16.31), (16.32) and (16.47).

Chapter 17

C^*-algebra

Let $\mathcal{A}$ be an associative $\mathbb{C}$-algebra, let $||\cdot||$ be an norm on the $\mathbb{C}$-vector space $\mathcal{A}$, and let $*:\mathcal{A}\to\mathcal{A}, a\to a^*$ be a $\mathbb{C}$- antilinear map. Then $(\mathcal{A},||\cdot||,*)$ is called a *C^*-algebra*. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. An *algebra homomorphism*

$$\pi:\mathcal{A}\to\mathcal{B}$$

is called a $*$-morphism if for all $a\in\mathcal{A}$ we have

$$\pi(a^*)=\pi(a)^*.$$

A map $\pi:\mathcal{A}\to\mathcal{A}$ is called a *$*$-automorphism* if it is an invertible $*$-morphism.

Example 2

Let $(\mathcal{F},(\cdot,\cdot))$ be a complex Hilbert space, let $\mathcal{A}=\mathcal{L}(\mathcal{F})$ be the algebra of bounded operators on H. Let $||\cdot||$ be the operator norm, i.e.,

$$||a|| = \sup_{x\in H \quad ||x||=1} ||ax|| \tag{17.1}$$

Let a^* be the adjoint to a, i.e.,

$$(ax,y)=(x,a^*y) \tag{17.2}$$

for all $x,y\in H$.

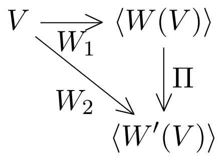
We now introduce the definition of a Weyl system and a CCR- representation of (V,Ξ) A *Weyl system* of the symplectic vector space (V,Ξ) consist of a C^*-algebra $\mathcal{A}$ with unit and a map $W:V\to\mathcal{A}$ such that for all φ,ψ

1. $W(0)=1,$
2. $W(-\varphi)=W(\varphi)^*$
3. $W(\varphi)\cdot W(\psi)=e^{-i\Xi(\varphi,\psi)/2}W(\varphi+\psi)$

A Weyl system $(\mathcal{A},W)$ of a symplectic vector space (V,Ξ) is called a *CCR- representation* of (V,Ξ) if $\mathcal{A}$ is generated as a C^*- algebra by the elements $W(\varphi)$, $\varphi\in V$. In this case we call $\mathcal{A}$ a CCR- algebra of (V,Ξ) and denoted by $CCR(V,\Xi)$.

Notice that following the propagator approach (9.1.2) the generators are labelled by elements $\varphi\in D(M)$ while in the solution approach the label are solutions $\varphi\in\mathcal{S}$. To prove that we need the following preliminary lemmas

Lemma 7 *Given a Weyl-stem $(\mathcal{A}, W)$ of the symplectic structure (V, Ξ) Then the linear span of $W(\phi), \phi \in V$ is closed under multiplication and under $*$. Moreover, if $(\mathcal{A}', W')$ is another Weyl system of the same symplectic vector space (V, Ξ), then there is a unique $*$-isomorphism, Π, such that the following diagram commutes*



where $\langle W(V)\rangle$ and $\langle W'(V)\rangle$ are the linear span of $W(\phi), W'(\phi)$ where $\phi \in V$.

Lemma 8 *Let $(\mathcal{A}, W)$ be a Weyl system of a symplectic vector space (V, Ξ). Then*

$$||a||_{max} := \sup\{||a||_0 ||a||_0 \text{ is a } C^* - \text{norm on } \langle W(V)\rangle\} \tag{17.3}$$

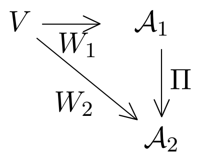
Lemma 9 *Let $(\mathcal{A}, W)$ be a Weyl-system of a symplectic vector space (V, Ξ). Then the completion $\overline{\langle W(V)\rangle}$ of $W(V)$ with respect to $||\cdot||_{max}$ is simple, i.e., it has no nontrivial close two sided $*$-ideals.*

Lemma 10 *Let $\mathcal{A}$ and B be C^*- algebras with unit. Each injective unit preserving $*$-morphism $\Pi : \mathcal{A} \to B$ satisfies*

$$||\Pi(a)|| = ||a|| \tag{17.4}$$

The proof of this lemma can be found in [75]. Now we show the uniqueness of the CCR- representation. The theorem can also be found in [75], but we add it for clarity.

Theorem 27 *Let (V, Ξ) be a symplectic vector space and let $(\mathcal{A}_1, W_1)$ and $(\mathcal{A}_2, W_2)$ be two CCR- representations of (V, Ξ). Then there exists a unique $*$- isomorphism, $\Pi : \mathcal{A}_1 \to \mathcal{A}_2$ such that the diagram*



Proof.
What needs to be show is that the $*$-isomorphism $\Pi : \langle W_1(V)\rangle \to \langle W_2(V)\rangle$ as constructed in lemma 7 extends to an isometry $(\mathcal{A}_1, ||\cdot||_1) \to (\mathcal{A}_2, ||\cdot||_2)$. Since the pullback of the norm $||\cdot||_2$ in $\mathcal{A}_2$ to $\langle W(V)\rangle$ via Π is a C^*-norm we have that $||\Pi(a)||_2 \leqslant ||a||_{max}$ for all $a \in \langle W_1(V)\rangle$. Hence Π extends to an $*$-morphism $\overline{\langle W_1(V)\rangle} \to \mathcal{A}_2$. By lemma 9 the kernel of Π is trivial, hence Π is injective. Lemma 10 implies that $\Pi : (\overline{\langle W_1(V)\rangle}, ||\cdot||_{max}) \to (\mathcal{A}_2, ||\cdot||_2)$ is an isometry.
$\boxdot$
As a corollary we obtain the following result

Proposition 15 *Let* (V_1, Ξ_1) *and* (V_2, Ξ_2) *be two symplecto vector spaces and let* $S : V_1 \to V_2$ *be a symplectic linear map. Then there exist a unique injective* $*$*-morphism*

$$CCR(S) : CCR(V_1, \Xi_1) \to CCR(V_2, \Xi_2) \tag{17.5}$$

such that the diagram

$$\begin{array}{ccc} V_1 & \xrightarrow[S]{\qquad\qquad} & v_2 \\ \downarrow W_1 & & \downarrow W_2 \\ CCR(V_1, \Xi_1) & \xrightarrow[CCR(S)]{} & CCR(V_2, \Xi_2) \end{array}$$

commutes.

Chapter 18

Gelfand-Naimark-Segal (GNS) representation theorem

In the previous section we have defined the quantum observables and states of the theory. Now, we show how from the algebraic formulation one can construct the Quantum field theory of a scalar field in a Hilbert representation. This is the content of the GNS Theorem.

Theorem 28 GNS Theorem
Let $\mathcal{A}$ be a C^*- algebra with identity element and $\omega : \mathcal{A} \to \mathbb{C}$ be a linear map satisfying $\omega(a^*a) \geqslant 0 \forall a \in \mathcal{A}, \omega(I) = 1$. Then there exist a Hilbert space $\mathcal{F}$, a representation $\pi : \mathcal{A} \to L(\mathcal{F})$ and a vector $|\psi >\in \mathcal{F}$ such that:

$$\omega(\mathcal{A}) = \langle\psi|\pi(\mathcal{A})|\psi\rangle$$

satisfying the property that $|\psi\rangle$ is cyclic.
Furthermore, the triple $(\mathcal{F}, \pi, |\psi\rangle)$ is uniquely determined up to unitary equivalence.

Before the proof of the theorem we show the following lemmas

Lemma 11 *The scalar product on the algebra given by*

$$(a, b) = \omega(a^*b) \tag{18.1}$$

satisfies

- $(\lambda_1 a + d, \lambda_2 b + c) = \lambda_1\overline{\lambda_2}(a, b) + \lambda_1(a, c) + \overline{\lambda_2}(d, b) + (d, c)$
- $(a, b) = \overline{(b, a)}$
- $(a, a) \geqslant 0$

for all $a, b \in \mathcal{A}$

Proof. That this scalar product is linear from the right and antilinear from the left follows directly from the linearity of ω and the definition.
Now notice that

$$\omega((\lambda a + b)^*(\lambda a + b)) = |\lambda|^2\omega(a^*a) + \overline{\lambda}\omega(a^*b) + \lambda\omega(b^*a) + \omega(b^*b) \tag{18.2}$$

Now taking $b = 1$ and $\lambda = i$ gives

$$\omega((ia+1)^*(ia+1)) = -\omega(a^*a) - i\omega(a^*) + i\omega(a) + 1 \tag{18.3}$$

and taking $b = 1$ and $\lambda = 1$ gives

$$\omega((a+1)^*(a+1)) = \omega(a^*a) + \omega(a^*) + \omega(a) + 1 \tag{18.4}$$

Therefore using that $\omega(c^*c) \in \mathbb{R}^+$ we have from (18.3)

$$\Im(-i\omega(a^*) + i\omega(a)) = 0 \tag{18.5}$$

and from (18.4)

$$\Im(\omega(a^*) + \omega(a)) = 0 \tag{18.6}$$

writing $\omega(a) = z_1 + iw_1$, $\omega(a^*) = z_2 + iw_2$ we obtain from (18.5) and (18.6) that $z_1 = z_2$ and that $w_1 = -w_2$. Therefore we have proved that $\omega(a^*) = \overline{\omega(a)}$.
This allows us to conclude that

$$\overline{(a,b)} = \overline{\omega(a^*b)} = \omega((a^*b)^*) = \omega(b^*a) = (b,a) \tag{18.7}$$

Moreover, we have by positivity of the state that

$$(a,a) = \omega(a^*a) \geqslant 0 \tag{18.8}$$

⊡
The scalar product so far is not as inner product because of the elements $a \in \mathcal{A}$ such that $\omega(a^*a) = 0$. The next lemma will allow us to fix this situation.

Lemma 12 *The set*

$$N_\omega = \{a \in \mathcal{A} | \omega(a^*a) = 0\} \tag{18.9}$$

is a closed left ideal in $\mathcal{A}$.

Proof.
That N_ω is closed follows from continuity of ω and using that N_ω is the kernel of the map $\omega(a^*a)$.
Now we show that if $a \in \mathcal{A}$ and $c \in N_\omega$ the $ac \in N_\omega$
First notice that

$$\begin{aligned} 0 &\leqslant (a - \lambda b, a - \lambda b) &(18.10)\\ &= (a,a) - \overline{\lambda}(a,b) - \lambda(a,b) + |\lambda|^2(b,b) &(18.11)\end{aligned}$$

Now if $(b,b) \neq 0$, make $\lambda = \frac{(a,b)}{(b,b)}$ to obtain

$$\begin{aligned} 0 &\leqslant (a - \lambda b, a - \lambda b) &(18.12)\\ &= (a,a) - \frac{|(a,b)|^2}{(b,b)} - \frac{|(a,b)|^2}{(b,b)} + \frac{|(a,b)|^2}{(b,b)} &(18.13)\\ &= (a,a) - \frac{|(a,b)|^2}{(b,b)} &(18.14)\end{aligned}$$

which reordering gives

$$|(a,b)|^2 \leqslant (a,a)(b,b) \tag{18.15}$$

If $(b,b)=0$, make $\lambda = n(a,b)$ where n is a natural number, to obtain

$$\begin{aligned} 0 &\leqslant (a-\lambda b, a-\lambda b) && (18.16)\\ &= (a,a) - n\overline{(a,b)}(a,b) - n\overline{(a,b)}(a,b) && (18.17)\\ &= (a,a) - 2n|(a,b)|^2 && (18.18) \end{aligned}$$

which reordering gives

$$2n|(a,b)|^2 \leqslant (a,a) \tag{18.19}$$

As this inequality holds for all n, it follows that $(a,b)=0$.
Then

$$0 = |(a,b)|^2 \leqslant (a,a)(b,b) = 0 \tag{18.20}$$

which can be expressed as

$$|\omega(a^*b)|^2 \leqslant \omega(a^*a)\omega(b^*b) \tag{18.21}$$

Notice then that if $a \in \mathcal{A}$ and $c \in N_\omega$

$$\begin{aligned} \omega((ac)^*ac) &= \omega(c^*a^*ac) && (18.22)\\ &= (c, a^*ac) && (18.23)\\ &\leqslant \sqrt{(c,c)}\sqrt{(a^*ac, a^*ac)} && (18.24)\\ &= \sqrt{\omega(c^*c)}\sqrt{(a^*ac, a^*ac)} && (18.25)\\ &= 0 && (18.26) \end{aligned}$$

Therefore $ac \in N_\omega$ ⊡

Notice also that by (18.21) if a or b are in N_ω then $\omega(a^*b) = 0$
Now we prove Theorem 28.
Proof of Th. 28.
Define the bi-linear form

$$([a],[b])_{\mathcal{F}} = \omega(a^*b) \tag{18.27}$$

on $\mathcal{A}/N_\omega$ which is the quotient algebra generated by the equivalence relationship $a \sim c \iff a - c \in N_\omega$ and $[a]$ denotes the equivalence class $a + N_\omega$.
Note that the bilinear form is well-defined because given two element $a \sim a'$ we have

$$\begin{aligned} ([a],[b])_{\mathcal{F}} &= \omega(a^*b) && (18.28)\\ &= \omega((a+N_\omega)^*b) && (18.29)\\ &= \omega((a'+N_\omega)^*b) && (18.30)\\ &= \omega(a'^*b) + \omega(N_\omega^*b) && (18.31)\\ &= \omega(a'^*b) + \omega(N_\omega b) && (18.32)\\ &= \omega(a'^*b) = ([a'],[b])_{\mathcal{F}} && (18.33) \end{aligned}$$

and therefore the result is independent of the choice of representative. The proof for the other side is analogous and we omit it to avoid repetition.
Also we know by lemma 18.1 that the bilinear form defines a semi-inner product. We show that the bi-linear form in fact defines an inner product by showing that the form is non-degenerate.
Let $[c] \in \mathcal{A}/N_\omega$ such that $([c],[b])_\mathcal{F} = [0]$ for all $[b] \in \mathcal{A}/N_\omega$ then we have

$$0 \leqslant ([c],[c])_\mathcal{F} = \omega(c^*c) = 0 \tag{18.34}$$

Therefore the $c \in N_\omega$ which implies $[c] = [0]$.
We define the Hilbert space $\mathcal{F}$ as the completion of $\mathcal{A}/N_\omega$ under the distance function given by the inner product $(\cdot,\cdot)_\mathcal{F}$.
We also, define the map π_ω by

$$\pi_\omega : \quad \mathcal{A} \quad \to L(\mathcal{F}) \tag{18.35}$$
$$a \quad \to \pi_\omega(a)[b] = [ab] \tag{18.36}$$

We show now that the map π_ω is a representation.
Notice that $||\pi_\omega(a)[b]||_\mathcal{F} = ||[ab]||_\mathcal{F} \leqslant ||[a]||_\mathcal{F}||[b]||_\mathcal{F}$
Therefore $\pi_\omega(a)$ is a bounded operator with domain $\mathcal{A}/N_\omega$. Then, we can extended $\pi_\omega(a)$ to a unique continuous operator $\pi_\omega(a)$ with domain $\mathcal{F}$. In fact we define $\pi_\omega(a)F = \lim_{n\to\infty} \pi_\omega(a)F_n$ where $F \in \mathcal{F}$ and $F_n \in \mathcal{A}/N_\omega$.
That π_ω is linear and multiplicative, i.e $\pi_\omega(a+b) = \pi_\omega(a) + \pi_\omega(b)$ and $\pi_\omega(ab) = \pi_\omega(a)\pi_\omega(b)$ follows from the definition of the quotient algebra and the definition of π_ω.
To check that the map preserves adjoints notice that if $a, b, c \in \mathcal{A}$, we have

$$(\pi_\omega(a^*)[b],[c])_\mathcal{F} = ([a^*b],[c])_\mathcal{F} \tag{18.37}$$
$$= \omega((a^*b)^*c) \tag{18.38}$$
$$= \omega(b^*ac) \tag{18.39}$$
$$= ([b],[ac])_\mathcal{F} \tag{18.40}$$
$$= ([b],\pi(a)[c])_\mathcal{F} \tag{18.41}$$
$$= (\pi_\omega^*(a)[b],[c])_\mathcal{F} \tag{18.42}$$

Therefore π_ω is a representation of $\mathcal{A}$ in $L(\mathcal{F})$.
Next, we show that [1] is a cyclic vector. This follows directly from the fact that $\pi_\omega(a)[1] = [a]$ and therefore $\pi_\omega(\mathcal{A})[1] = \mathcal{A}/N_\omega$ which is dense in $\mathcal{F}$.
Finally , notice that

$$\omega(a) = \omega(1^*a1) = ([1],[a])_\mathcal{F} = ([1],\pi_\omega(a)[1])_\mathcal{F} \tag{18.43}$$

Therefore we have shown the existence of a Hilbert space $\mathcal{F}$, a representation π_ω and a cyclic vector [1] given any C^* algebra with unit element and a state ω. We call the triplet $(\mathcal{F}, \pi_\omega, [1])$ a GNS representation.
We now show that this choice is unique up to unitary equivalence. Let hilbert space $\mathcal{F}'$, a representation π'_ω and a cyclic vector ξ be another GNS representation.
Then we define the operator U as

$$
\begin{aligned}
U: \quad \mathcal{A}/N_\omega & \to \mathcal{D}' & (18.44)\\
\pi(a)[1] & \to \pi'_\omega(a)\xi & (18.45)
\end{aligned}
$$

where $\mathcal{D}'$ is a dense subspace of $\mathcal{F}'$.
Notice that the map is well defined as

$$
\begin{aligned}
(\pi'_\omega(a)\xi, \pi'_\omega(a)\xi)_{\mathcal{F}'} &= (\xi, \pi'_\omega(a^*a)\xi)_{\mathcal{F}'} & (18.46)\\
&= \omega(a^*a) & (18.47)\\
&= ([1], \pi_\omega(a)[1])_{\mathcal{F}} & (18.48)\\
&= (\pi_\omega(a)[1], \pi_\omega(a)[1])_{\mathcal{F}} & (18.49)
\end{aligned}
$$

so if $\pi'_\omega(a)\xi = 0$ that implies $\pi_\omega(a)[1] = 0$.
Moreover, (18.46) show that the operator U is an isometry.
In fact we extend U to the whole $\mathcal{F}$ and $\mathcal{F}'$ by continuity and using the density of $\mathcal{A}/N_\omega$ and $\mathcal{D}'$ the extension of U is a unitary map.
Finally we show the unitary equivalence of the representation by showing that $U\pi_\omega(a) = \pi'_\omega(a)U$ for all $a \in \mathcal{A}$.
This follows from

$$
\begin{aligned}
U\pi_\omega(a)(\pi_\omega(b)[1]) &= U(\pi_\omega(ab)[1]) & (18.50)\\
&= \pi'_\omega(ab)\xi & (18.51)\\
&= \pi'_\omega(a)\pi'_\omega(b)\xi & (18.52)\\
&= \pi'_\omega(a)U(\pi_\omega(b)[1]) & (18.53)
\end{aligned}
$$

$\boxdot$

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