University of Southampton
Faculty of Physical Sciences and Engineering

Non-Polynomial Scalar Field Potentials in the Local Potential Approximation

Ismail Hamzaan Michael Bridle

Submitted for the degree of

DOCTOR OF PHILOSOPHY

April 2017

UNIVERSITY OF SOUTHAMPTON

ABSTRACT

FACULTY OF PHYSICAL SCIENCES AND ENGINEERING

Physics and Astronomy

Doctor of Philosophy

NON-POLYNOMIAL SCALAR FIELD POTENTIALS IN THE LOCAL POTENTIAL APPROXIMATION

by Ismail Hamzaan Michael Bridle

We present the renormalisation group analysis of $O(N)$ invariant scalar field theory in the local potential approximation. Linearising around the Gaussian fixed point, we find the same eigenoperators solutions exist for both the Wilsonian and the Legendre effective actions, given by solutions to Kummer's equations. We find the usual polynomial eigenoperators and the Hilbert space they define are a natural subset of these solutions given by a specific set of quantised eigenvalues. Allowing for continuous eigenvalues, we find non-polynomial eigenoperator solutions, the so called Halpern-Huang directions, that exist outside of the polynomial Hilbert space due to the exponential field dependence.

Carefully analysing the large field behaviour shows that the exponential dependence implies the Legendre effective action does not have a well defined continuum limit. In comparison, flowing towards the infrared we find that the non-polynomial eigenoperators flow into the polynomial Hilbert space. These conclusions are based off RG flow initiated at an arbitrary scale, implying non-polynomial eigenoperators are dependent upon a scale other than k. Therefore, the asymptotic field behaviour forbids self-similar scaling. These results hold when generalised from the Halpern-Huang directions around the Gaussian fixed point to a general fixed point with a general non-polynomial eigenoperator.

Legendre transforming to results of the Polchinski equation, we find the flow of the Wilsonian effective action is much better regulated and always fall into the polynomial Hilbert space. For large Wilsonian effective actions, we find that the non-linear terms of the Polchinski equation forbid any non-polynomial field scaling, regardless of the fixed point.

These observations lead to the conclusion that only polynomial eigenoperators show the correct, self-similar, scaling behaviour to construct a non-perturbatively renormalisable scalar QFT.

Contents

List of Figures

Declaration of Authorship

I declare that this thesis and the work presented in it are my own and has been generated by me as the result of my own original research.

I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University;
- No part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution;
- Where I have consulted the published work of others, this is always clearly attributed;
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
- I have acknowledged all main sources of help;
- The work in this thesis based done mainly by the author are chapters 3 with the exception of sections 3.1, 3.2, and 3.3 where underlying research was done in part by the author's supervisor Prof. Tim Morris. The work of 4 was done in entirely in collaboration with Prof. Tim Morris;
- Results of this thesis has been published in [1];

Signed: ..... Date: .....

Acknowledgements

I would like to thank my wife Lisa, who has supported me throughout the course of this PhD and suffered the stress of completing this thesis with me. Without her, I do not doubt that completing this would have been far more difficult than it was.

I would also offer special thanks to my fellow postgraduate research colleagues over these past four years, especially Andrew M., Andrew L., Anthony, Azaria, Edwin, Fred, Jason, Jürgen, Matt, Marc S., Marc T., Peter, Simon, Tad, Thomas, Tobi, Ronnie, and Zoe, who provided invaluable friendship, stimulating conversation, and a healthy dose of insanity!

Finally, I would like to express my sincerest gratitude to my supervisor Tim Morris, who has contributed enormously to my entire PhD process.

To Lisa

Chapter 1

Introduction

Many physical phenomena are described using microscopic models that derive results through simplifying the situation to the smallest possible groups and interactions possible, for example ferromagnetism described in the Ising Model [2] as two-valued spin operators with nearest-neighbour interactions only or electromagnetic scattering through photon exchange in Quantum Electrodynamics. The simplified models applied in numerous situations has offered insight into the fundamental properties of the physical world that had otherwise eluded physicists. Using our previous example of ferromagnetism, it was only Weiss's addition of describing microscopic phenomena, specifically alignment molecular spins to give an innate magnetic field [3], that could expand Curie's work on paramagnetism [4] to describe ferromagnetism at low temperatures and, rather interestingly, it was where such descriptions failed that lead to further understanding of critical phenomena of phase transitions [5–7].

Consider a system where two or more distinct, but coexisting, states or phases with the same internal symmetries exist in equilibrium. If changes in thermodynamic variables, such as temperature, of the system lead to these phases to become more and more similar then there will exist a point, known as the critical point, where, in terms of thermodynamic quantities, there is no longer any difference between these phases and beyond this point there is only one continuous phase [8]. Critical phenomena is the description of physics near this critical point, where we see a phase transition from distinct coexistent states to a single continuous state. If one or more of the first-order derivatives of thermodynamic potentials are discontinuous then these phase transitions are described as first-order transitions. If these first-order derivatives are continuous,

but higher-order derivatives are discontinuous then the phase transition is described as second-order. However, while applications of classical theories, such mean-field theory as applied by Weiss to magnetism [3], were able to describe these phase transitions qualitatively the numerical results did not match experimental measurements and in many were unable to predict second order phase transitions [5, 6, 8, 9].

In terms of the scientific method, the simplified models are our source of testable hypotheses of the physical world, testing underlying physics such as fundamental interaction restrictions, group symmetries, and particle properties by predicting physical observables or explaining how certain phenomena occur. However, they all have one major barrier in describing the physical world that is intrinsic to this methodology, the physical world is not described by the interaction of the few, but large many body problems with many degrees of freedom. Keeping with our ferromagnet example, applying the Ising model to calculate the critical phenomena at the phase transition in a physical situation will have many orders of magnitude between the length scale of nearest-neighbour spin interactions and a physical ferromagnet sample. As we will show below, it is physics at scales much larger than that of the microscopic model that drives critical phenomena.

There is no case where this problem is more apparent than in the Quantum Field Theory (QFT) applied in high energy particle physics (HEP). Microscopically, in the process of second quantisation we treat each field as an operator on each point in space-time, introducing a degree of freedom for each point in space. The action at this scale, the bare/ultraviolet momentum cutoff scale, sets the allowed interactions between different particles similar to the restriction to nearest-neighbour interactions in Hamiltonians of the Ising model. On the other hand, macroscopically, we find that for any physical observable, fluctuations within the vacuum itself allows for an infinite number of virtual field configurations producing the same transition between initial and final states, restricted only in that they are built up from symmetry conserving interactions and must conserve momentum. While most of these worries are handled effectively in QFT through expansions in small coupling, we cannot solve for all physical theories in this way and strongly interacting theories, quantum chromodynamics for example, are heavily influenced by the vacuum fluctuations. Further to this, comparing the very nature of the calculations done in HEP to complex systems in statistical mechanics (SM) then leads to the idea that a physical QFT, describing particle interactions at

high energy, is in fact heavily influenced by critical phenomena.

Particularly, in regards to the discussions within the thesis, calculations and measurements of critical behaviour throughout the 20th century provided strong evidence that if one compares any two systems that can be broken down to and subsequently described by the same order parameter, or the same gauge group in HEP, then both systems should yield the same critical behaviour. This phenomenon, which is known as the universality of critical behaviour, implies that there must be some fundamental property derived from shared interaction symmetries to give the same critical behaviour, even when systems and macroscopic properties vary significantly. When two or more systems display this universal critical behaviour they are said to belong to a universality class, where each universality class is described by either a different order parameter or a different 'critical point', the point where critical behaviour is observed. For example, in three spatial dimensions and the choice of single component scalar field, φ, invariant under Z_2 discrete symmetry, $\varphi \to -\varphi$ symmetry, we may describe the critical behaviour of fluid-gas phase transition of low temperature atomic fluids, critical binary fluids, or ferromagnetic spontaneous magnetisation at the Curie temperature in the same way [5, 9]. Conversely, given the link between HEP calculations and critical behaviour in SM, the pertinent point to this thesis is that results for critical behaviour found in our HEP calculations should be found in other systems of the same universality class, a fact we will use to help motivate and justify the conclusions of this thesis. Therefore, the concepts derived from universality are pivotal to the arguments we will make in the body of this thesis. As such, exposition about critical behaviour and universality is included in the next section and we clarify the link between HEP and SM critical behaviour in section 1.2.

Outside the region where small couplings expansions are valid it is possible to study a QFT, in particular how the QFT changes with the inclusion of vacuum fluctuations, with the a theoretical tool known as as the Renormalisation Group (RG) or more particularly the continuum version of Wilsonian RG known as the Functional or Exact RG (ERG). Through systematic integration over possible fluctuations RG techniques are able to investigate how an initial QFT changes under the process of renormalising couplings, even when couplings are large. These techniques have then lead to theories of non-perturbative renormalisability, even for theories that show divergent behaviour perturbatively. It was only through non-perturbative RG techniques that led to SM and

HEP results such as the universal equation of state for spontaneous symmetry breaking in Heisenberg models[1] [11] or the classification of fixed points for one component scalar field theories in two dimensions [12]. Particularly, in HEP, the ERG has been the basis of recent work into constructing theories of non-perturbatively renormalisable quantum gravity based upon the idea of asymptotic safety first proposed by Weinberg [13–18]. These theories exploit particular ideas unique to the RG that did not exist in either QFT or SM before Wilson brought the work of many physicists, including Widom, Fisher, Kadanoff, and himself, together to systematically study scale invariant phenomena. Unlike the condensed matter systems that Wilsonian RG has was originally conceived to deal with, calculations for theories such as asymptotically safe gravity do not have easily accessible experimental evidence and so understanding of the RG itself and RG phenomena is pivotal. Particularly, the validity functional truncations applied in recent work on quantum gravity may be assessed exclusively from rigorous understanding of the mathematical structure of the RG [19–38].

Analysis of the RG properties of scalar field theory often proves an excellent testing ground for truncations and assumptions and, in part, is directly relevant to some terms appearing in more complicated theories. Additionally, the RG properties of scalar field theory has been confirmed for a number of years, particularly for polynomial potentials in the Local Potential Approximation (LPA) [39–48], which describes the leading term of the functional truncation known as the derivative expansion, a common approximation where functionals are expanded in terms of derivative operators, or equivalently powers of external momentum [43, 46, 49–52]. This approximation is often applied to asymptotic safety in quantum gravity in the form of functional truncations, particularly in theories that approximate the potential as a polynomial function of the Ricci scalar [19–38]. For the majority of this thesis we specialise to the leading term of the LPA, which differs only in that anomalous dimension vanishes. Further details about the background and application of the LPA can be found in reviews [39, 53–55] and the application in this thesis is provided in chapter 2. Of particular importance to HEP and a key point of motivation is that, within the LPA, corrections from vacuum fluctuations for a quartic scalar field potential, namely the Higgs-like potentials, show that non-trivial interacting terms renormalise to zero in four (wick-rotated) space-time

[1]The Heisenberg model is an expansion of the Ising model, replacing the two-valued spin operators with multivalued quantum operators. [10]

dimensions. In other words, according to RG calculations the Higgs is non-interacting, this is known as the Higgs triviality problem [56, 57]. We will confirm this and other known RG behaviour of scalar field theory in the LPA in chapter 3.

However, work by Halpern and Huang [58–60] has led to debate on the significance of non-polynomial solutions in the LPA, which under certain circumstances give rise to an asymptotically free scalar field theory. Of note, in the parameter space that these solutions solve the RG equations there exist potentials that, if correct, can be chosen such that they demonstrate the required non-trivial minimum for spontaneous symmetry breaking, *i.e.* provide a solution to the Higgs triviality potential. Similar research was subsequently undertaken expanding upon these ideas [61–73] with an aim to investigate or exploit the properties of an asymptotically free scalar field theory.

In section 3.2 we re-derive the original Halpern-Huang non-polynomial potentials and present a very brief discussion of subsequent research based upon them. While this derivation will show that these are naïvely solutions to the LPA RG equations, we show in subsequent sections that these non-polynomial potentials must scale in such a way that they do not produce physical solutions. Specifically, we find that they either belong to a Hilbert space defined by the polynomial potentials or do not show the correct RG behaviour. This first problem would mean they are solely a linear combination of known polynomial solutions, meaning that the correct RG behaviour can only be displayed by the constituent polynomial solutions. Therefore, they do not produce new non-trivial interactions or form a perfect action required for a non-perturbatively renormalisable QFT. The latter problem demonstrates that, unlike the polynomial solutions, these non-polynomial potentials do not preserve the crucial property of scale invariance near a critical point and therefore cannot be considered physically acceptable results. As far as possible these conclusions will be generalised from the Halpern-Huang solutions to non-polynomial potentials in general. Finally we present a full summary of our conclusions in chapter 5.

Many of the key arguments that the subsequent chapters are based on are drawn from key properties of the RG derived from its historical application to critical phenomena. We therefore begin discussions with exposition on critical behaviour in general, utilising a generalised Ising model as a core example.

1.1 Critical Behaviour in d-dimensional Ising Model

To provide an intuitive way to understand the ideas that led to the RG let us consider the Ising model generalised to d dimensions. We choose this model for its simplicity, well known experimental and numerical results when compared to fluid and ferromagnetic systems [5, 8, 9, 74–77], and due to the direct relevance of Ising-type critical behaviour to the conclusions of this thesis. The key linking point is that the universal critical behaviour of many body systems described by an $O(N)$ symmetric scalar field[2], such as the spin operators we will define in this Ising model, should be obtained in our subsequent RG analysis of $O(N)$ scalar field theory. For example, the compressibility of a $d = 3$ fluid in the gas-liquid phase transition and the magnetic susceptibility of a $d = 3$ ferromagnet both are described by an Ising model with a discrete $O(1)$ symmetry [5, 9]. Therefore, any conclusions made in this Thesis about $O(N)$ symmetric scalar field that can be applied to $d = 3,\ N = 1$ are directly testable in the aforementioned systems, a point we will return to in section 3.3.

In detail, our Ising model describes a d dimensional lattice of $\hat{N}$ points with some spacing a between each of them and volume V. Each point has a position described by a d-component vector $\vec{\mathbf{r}}$ at which we describe a molecular or atomic spin by a spin operator $\sigma(\vec{\mathbf{r}})$. The direction of the spins is dictated by a constant magnetic field H applied over the lattice, in what we call the z direction, such that we can treat each spin as a two-valued operator, aligned with or against the magnetic field as given by $\sigma(\vec{\mathbf{r}}) = +1$ and $\sigma(\vec{\mathbf{r}}) = -1$ respectively. As such these spin operators may be treated as a Z_2 ($O(1)$) symmetric scalar field. However, to keep contact with key historic work in SM we will keep to the nomenclature of $\sigma(\vec{\mathbf{r}})$ as spin operators throughout this section. Finally, we choose that the interactions between each spin is limited such that a spin $\sigma(\vec{\mathbf{r}})$ may only interact with nearest-neighbour spins $\sigma(\vec{\mathbf{r}}_2)$.

Then the equilibrium behaviour of the system at a specific temperature is determined by minimising the total energy contributions from interactions between the spins with the overall magnetic field, interactions between spins, and overall thermal fluctu-

[2] A N component scalar field is $O(N)$ symmetric if the physics remains the same under interchange of field indices.

ations. Explicitly, we may write down a Hamiltonian, $\mathcal{H}$, for the system:

$$\mathcal{H} = -\hat{J} \sum_{\substack{\text{nearest} \\ \text{neighbours}}} \sigma(\vec{\mathbf{r}}_1)\sigma(\vec{\mathbf{r}}_2) - H \sum_r \sigma(\vec{\mathbf{r}}) \,, \tag{1.1.1}$$

where we assume $\hat{J} > 0$, giving a total energy decrease or increase if the nearest neighbours align in the same or opposite directions respectively. From this we can define a partition function, $\mathcal{Z}$, that contains all possible state configurations of the entire lattice:

$$\mathcal{Z}[K, h] = \text{Tr}\left[\exp\left(-\frac{\mathcal{H}}{k_b T}\right)\right] , \tag{1.1.2}$$

where k_b is Boltzmann's constant, $K = \hat{J}/k_b T$ and $h = H/k_b T$. The trace in this expression represents integration over the spin degrees of freedom $\sigma(\vec{\mathbf{r}})$ and a sum over possible configurations. More usefully, we can define, in the thermodynamic limit, the free energy density, $\tilde{f}$:

$$\tilde{f}[K, h] = \lim_{N,V\to\infty} \ln\left(\mathcal{Z}[K, h]\right) . \tag{1.1.3}$$

We then obtain the equilibrium behaviour of the system at a given temperature and field by minimising the free energy density. For example, the equation of state for our Ising model could be given by the value of the spontaneous magnetisation density, $m = -\left(\partial f/\partial H\right)|_T$, at equilibrium.

At very low temperatures, specifically far below the value of temperature that describes the critical point, *i.e.* the critical temperature T_c, the probability of thermal fluctuations changing the alignment of spins is quite low. As such, the equilibrium behaviour is dominated by the interaction between spins with the magnetic field, favouring coalignment with the magnetic field, and inter-spin interactions, favouring coalignment of nearest-neighbour spins. This will lead to an ordered system with most, if not all, spins aligning in the direction of magnetic field, *i.e.* $\sigma(\vec{\mathbf{r}}) = 1$ for most $\vec{\mathbf{r}}$, and the overall lattice will be in a net magnetised state. This net magnetisation will then remain with the removal of the external field due to the inter-spin interactions.

Upon removing the external field, an increase in temperature would allow small sets

of the spins to change orientation from $\sigma(\vec{\mathbf{r}}) = 1$ to $\sigma(\vec{\mathbf{r}}) = -1$.[3] Therefore, the thermal fluctuations within the lattice will be seen as continuous groups of spins directly opposing the net magnetisation of the overall system. As the temperature increases the probability of spins flipping will increase leading to larger and more frequent fluctuations within the lattice.

One of the first real qualitative tools to account for these thermal fluctuations was van der Waal's now famous mean-field theory (MFT), which he first used to describe critical fluids before the work was linked to magnetism by Weiss and it was then finally applied in QFT by Landau and Lifshitz with the introduction of order parameters [3,4,6,8,74,78]. In MFT we make the assumption that, compared to the overall system, the size of these thermal fluctuations are small. We can then assume that effect of the thermal fluctuations and the individual degrees of freedom can be safely averaged over and the main driving force in the system is given by the overall expectation value of the spins, $\langle\sigma\rangle$. For the Ising model, the calculations of MFT shows that at zero external field there exists a phase change at the critical temperature, T_c, given by the change of the average spin from zero above T_c to one of two non-zero values, equal in magnitude but opposite in sign, below T_c [5, 6]. Qualitatively, this correctly predicts the first order phase change from a disordered state with no net magnetisation to a net magnetised state aligned either in the positive or negative z direction. Equally, the behaviour of some of the singularities expected from second order phase transitions can be derived. For example, far enough away from the lattice spacing, *i.e.* for some $\vec{\mathbf{r}}_2$ we have $r = |\vec{\mathbf{r}}_2 - \vec{\mathbf{r}}| \gg a$, MFT shows that near the critical point that the connected two-point function tends towards a power law relation given by canonical mass dimension [5, 7]:

$$\begin{aligned} G(\vec{\mathbf{r}}, \vec{\mathbf{r}}_2) =& \langle\sigma(\vec{\mathbf{r}})\sigma(\vec{\mathbf{r}}_2)\rangle - \langle\sigma(\vec{\mathbf{r}})\rangle\langle\sigma(\vec{\mathbf{r}}_2)\rangle\,, \\ \approx& \frac{C}{r^{d-2+\eta}} \qquad \text{as } r \to \infty,\ T \to T_c\,, \end{aligned} \tag{1.1.4}$$

where C is some constant and the exponent giving the anomalous dimension, the difference of scaling from canonical mass dimension, $\eta = 0$.

However, it has been shown to a great deal of accuracy through both numerical

[3] It is important to note that the net magnetisation of the lattice acts like the external field that was originally applied to the lattice, as such we may continue to assume that the spin operators remain two-valued and aligned in the positive or negative z direction.

calculations and experimental evidence that near the critical point the quantitative values of the scaling behaviour, particularly the rate of divergence given by so called critical exponents, are incorrect. For example, one of the key observations made in the 20th century, starting from Onsager's analytic solution for a two-dimensional Ising model applied to fluids [75], is that MFT does not correctly predict $\eta \neq 0$ nor the second order phase transition given by the divergence in the heat capacity in less than four dimensions [5, 8, 79]. This discrepancy is due to the assumption that since only small scale fluctuations are of consequence to the theory they may be safely incorporated into effective parameters that will not change the basic characteristics of the theory, as is done in rewriting the theory in terms of the expectation value $\langle\sigma\rangle$ in MFT. This approach however cannot account for any large scale fluctuations or the possibility for the fluctuation to have dramatic variation in scale.

1.1.1 Droplet Model of Critical Behaviour

To demonstrate this point further, consider each group of spins flipped by thermal fluctuations in a droplet model, similar to the model put forward by Kadanoff in [5] as an application of Widom's model on fluids in [80] to a two dimensional Ising model. We begin with the d dimensional lattice in a net magnetised state with all spins pointing in the positive z direction at a temperature far below the critical temperature and zero external field. Then a thermal fluctuation will take the form of a continuous group of spins flipping to align in the negative z direction, which we will call a droplet. This basic configuration is shown in figure 1.1.1. The inter-spin interactions within the droplet will actually favour these spins to remain aligned against the net magnetisation of the lattice. As there is no external field, the energy density within the group is then equal to that of the rest of the lattice aligned in positive z direction. Therefore, the only increase to the free energy density of the overall system from the droplet will be due to interactions at the interface between the droplet and rest of the lattice. We may consider the free energy increase due to these interactions as an 'energy cost' to flip the group of spins, which will then be proportional to the number of spins external to the droplet that are adjacent to its boundary. Specifically, this cost to form the droplet will be proportional to the area of the boundary surface times the energy per unit area to flip spins just within this boundary.

Intuitively we can see that at temperatures far below the critical temperature there

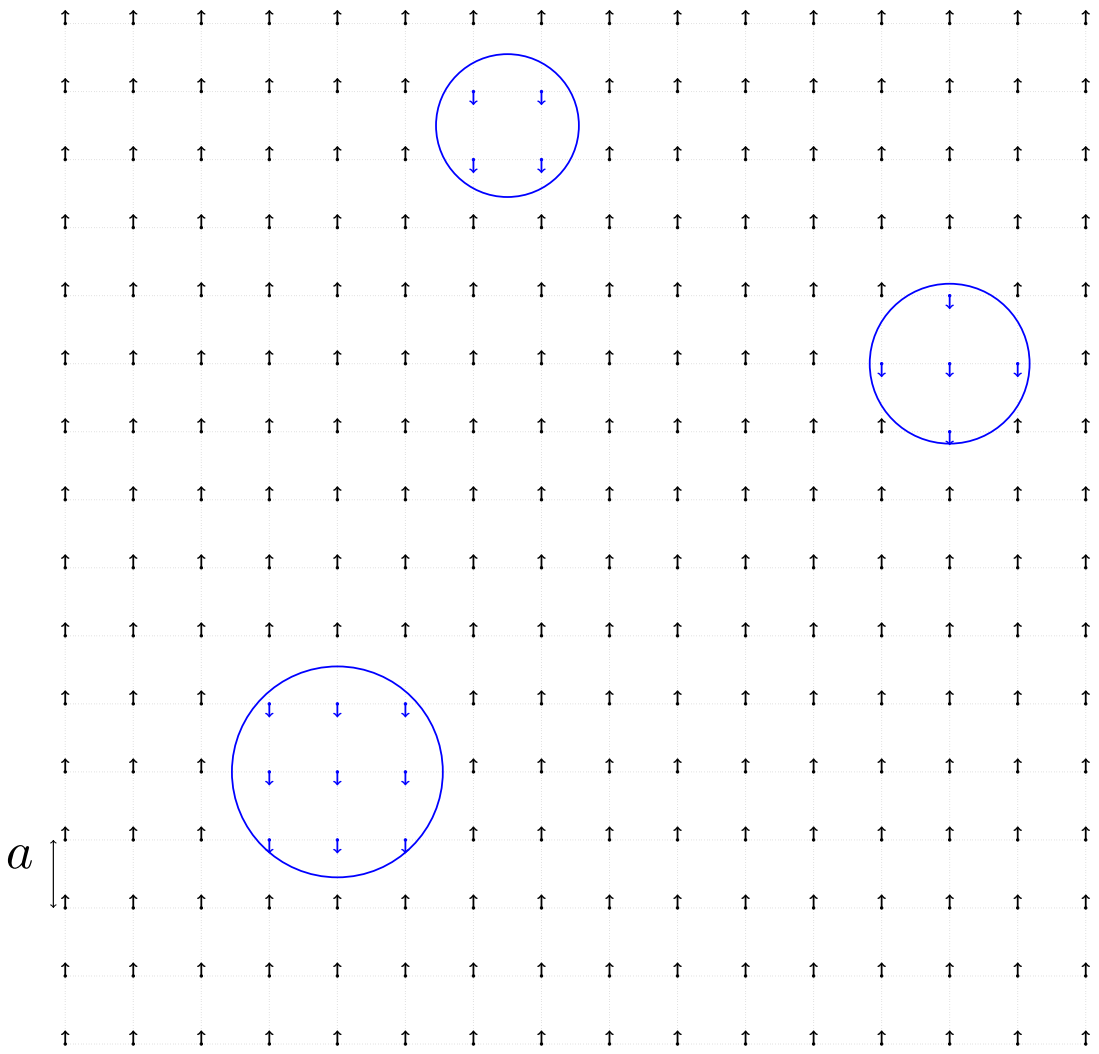

Figure 1.1.1: A section of an infinite two-dimensional lattice of spin operators. We described the thermal fluctuations within the Ising Model as droplet formation, here simplified to be circular in nature for demonstration purposes. When the temperature is much lower than the critical temperature, ($T \ll T_c$), almost all spin operators $\sigma(\vec{\mathbf{r}})$ point in the positive z direction, which we show with the black arrows. Thermal fluctuations, shown in the blue, will then take the form of small, continuous groups of spin operators flipping to the opposing $-z$ direction. These thermal fluctuations may be treated as droplets of opposing spin, where the energy required to create each droplet is proportional to its size. As such, at low temperatures the droplets will be small in size and few in number. [5]

is insufficient thermal energy to support droplets with a large surface area. Additionally, the probability to create each droplet will be small, which will therefore lead to a mostly ordered system with a limited number of small droplets. However, as we approach the critical temperature the increase in available thermal energy within the system will increase the chance to form more droplets of opposing spins. As each spin is locked in the z direction, the increasing number of droplets will lead to a reduction in the magnitude of the lattice's net magnetisation. This drop of net magnetisation will lead to a reduction in the relative free energy increase per unit area to form the boundary of each droplet in the system. The overall effect is that as the critical temperature is approached the thermal fluctuations will lead to more and larger droplets of opposing spins forming. We may consider the maximum size of these droplets as the extent to which spin operators correlate due to the thermal fluctuations, in other words the size of the droplets define the characteristic length at which spins correlate with increasing temperature, *i.e.* the correlation length ξ.

Now, as the temperature increases we will expect the reduction in net magnetisation to lead to large correlation lengths and droplets will tend to become comparable in size to the system itself. If we ignore the boundary and the spins outside of one of these large droplets, it can be treated as its own system with the same the free energy density and an overall net magnetisation density as our low temperature lattice, except this sub-system will be a fraction of the volume of the original lattice and the net magnetisation will be in the $-z$ direction. Treating such a droplet as if it were our lattice, we would therefore expect, with the increase in temperature, that thermal fluctuations within the droplet will lead to smaller droplets that then point in the $+z$ direction. This behaviour is shown in figure 1.1.2. In fact, as we approach the critical temperature, before the phase change from the ordered to disordered system, we will expect that these sub-droplets within this first droplet to become comparable in size to the original droplet. From this we see that as the maximum size of droplets increases, sub-droplets within them will increase in size sufficiently such that they in turn will develop sub-droplets of growing size.

At the critical point then, the net magnetisation decreases sufficiently that the correlation length tends to the size of the lattice itself. As such, according to our droplet model, we will not only begin to see correlations as large as the system itself, but also at every scale below this down to the smallest possible scale, the lattice spacing a. Further,

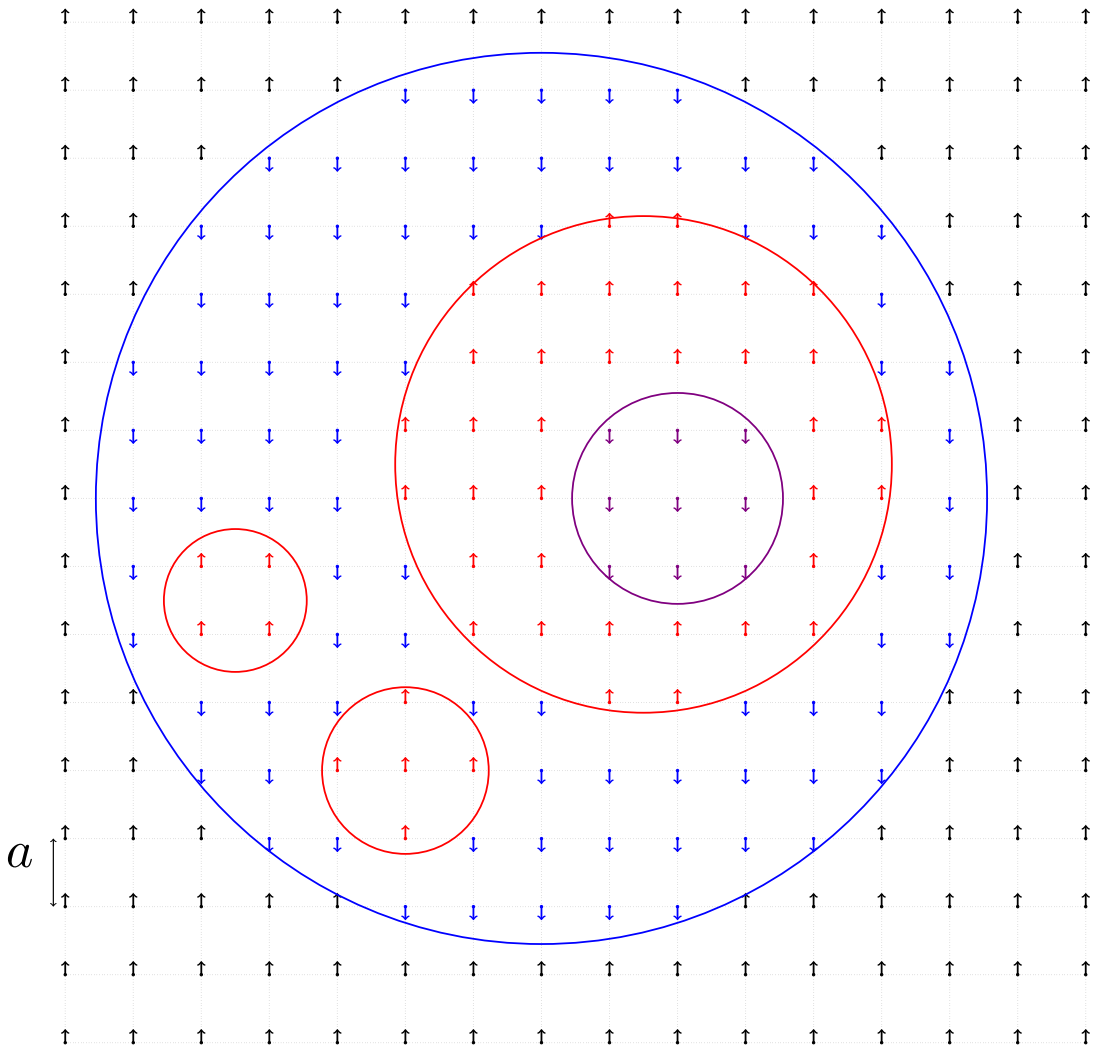

Figure 1.1.2: A section of an infinite two-dimensional lattice of spin operators, near the critical point, *i.e.* when $T \sim T_c$. Again we described the thermal fluctuations within the Ising Model as droplet formation, here simplified to be circular in nature for demonstration purposes. While the droplets can be expected to be near circular in the low temperature regime, this may not be true for larger droplets. The smaller scale thermal fluctuations, shown in blue, grow large as we approach the critical point. These large fluctuations may then be treated similarly to the original Ising model, but with opposite overall magnetisation. It can therefore be expected that the large droplet will form smaller scale droplets within them, shown in red. As more fluctuations occur within the system the overall magnetisation decreases, allowing for more droplets to form. Therefore as internal, red, droplets grow large we can additionally expect that they will form internal droplets, shown in violet. As the critical point is approached the length scale of the largest droplets approaches that of the lattice itself, thus allowing for sub-droplets to form at any scale between the lattice spacing a and the extent of the lattice itself.

from the fact that each of the sub-droplets behave like the next largest droplet we can see that each of smaller scale correlations behave and interact exactly like the larger scale correlations except with an overall smaller characteristic scale. From this, we can see that the discontinuous or singular behaviour exhibited by critical phenomena is in fact driven by not only large scale fluctuations, but also fluctuations that occur at all scales between the lattice spacing and the extent of the lattice itself. In other words, as the system approaches the critical point the physics becomes self-similar, which implies the correlations between the spins have no characteristic scale, they are scale invariant.

With this model in mind, the failure of MFT can easily be understood. The averaging over the spins $\sigma(\vec{r})$ done by rewriting equations of state explicitly in terms of the expectation value $\langle\sigma\rangle$ does not allow for the increasing size of fluctuations nor the dramatic variation in the size of each contributing fluctuation. With the failure of mean-field theory, we see that calculations of critical phenomena cannot be rewritten as effective theories in terms of expectation values and we must consider each individual degree of freedom. This however leads to incredibly complex calculations over each of the individual spin operators $\sigma(\vec{\mathbf{r}})$. Specifically, these calculations must account for correlations with each other spin operator up to and including the distance ξ away from each site $\vec{\mathbf{r}}$, which extends across the entire lattice at the critical point. Further, in the both the thermodynamic limit, $N \to \infty$ & $V \to \infty$, and in HEP we do not have a physical restriction upon the size of the lattice (or equivalently on space-time). Therefore, at the critical point we can expect the correlation length to diverge, implying calculations must be done over fluctuations between our fundamental scale defined by a up to $\xi \to \infty$. Then to attempt calculations of macroscopic properties of the system at equilibrium, such as magnetic susceptibility or other correlation functions, near this critical point we are then required to integrate over an infinite number of microscopic degrees of freedom while taking into account an infinite number of possible fluctuations over all possible length scales.

While it is possible to perform examples of these calculations, either analytically with suitable approximations or numerically with sufficient approximation and computational power, there is merit in investigating the physics between the fundamental scale and the final physical observables, particularly as these critical systems display scale invariant phenomena. If we consider the mesoscopic scale, the scale between the lattice spacing and the macroscopic scale, of a near-critical system then the inclu-

sion of these fluctuations over distances much larger than the lattice spacing a may be considered as correlations between spins beyond the microscopic nearest-neighbour interaction. We observe what may be called 'non-local interactions'. When observing a near-critical system with a mesoscopic correlation length, then there is a sense that the system "loses memory", as Fisher put it [7], of individual microscopic degrees of freedom and instead the system appears as though it was made up of new incredibly complex effective renormalised interactions, that still preserve fundamental symmetries, and parameters defined at the scale of the correlation length.

In particular, one of the key observations made when measuring near critical systems is that just before the system undergoes phase transitions many of the important macroscopic properties of the system can instead be described using power-law scaling relations,[4] where the exponent or part of the exponent is called the critical exponent. It is these scaling relations, or derivatives of them, that display discontinuities or singularities, at some point on the system, when the system reaches the critical point. For example, within the Ising model the connected two-point correlation function, equation (1.1.4), diverges as $r \to 0$ with critical exponent η or the magnetic susceptibility that scales close to the critical point as:

$$\chi = \left.\frac{\partial^2 \tilde{f}}{\partial h^2}\right|_T \propto \left(\frac{T - T_c}{T_c}\right)^{-\gamma}, \tag{1.1.5}$$

where the critical exponent γ is greater than zero in the Ising model, implying χ diverges at the critical point ($T = T_c$). Those exponents that describe a phase transition at a critical point are known as the "relevant" critical exponents.

On the other hand, when comparing the critical behaviour of similar systems, specifically ones that can be described with the same type of order parameters with the same fundamental symmetries, it is possible to describe the phase transitions of these systems with power law behaviour that scales with the same relevant critical exponents. While the overall magnitude and the value of thermodynamic variables at which we observe these scaling relations will differ between systems, the exponents are universal and depend only upon the dimensionality of the system. Using our prior examples, we find that the fluid compressibility at the fluid-gas phase transition in $d = 3$ low temperature

[4]The appearance of power law scaling behaviour is not entirely unexpected if we consider the lack of a well defined length scale for fluctuations near the critical point, as power law behaviour equally implies scale invariance.

atomic fluids and the magnetic susceptibility in $d = 3$ ferromagnet below the Curie temperature, are both described by the same correlation function χ, equation (1.1.5), and both diverge with the same exponent, γ, as $(T - T_c)^{-\gamma}$. That is, for any system described by the same type of order parameter, values such as T_c are not universal but exponents such as η and γ are universal and depend only upon the dimensionality of the system. As eluded to above, this is called the universality of critical behaviour and those systems that share the same critical exponents are said to belong to a universality class.

1.1.2 Scaling Hypothesis of Critical Behaviour

Similar to our consideration of the mesoscopic scale, if we investigate the region nearby, instead of, the critical point we find that the system is continuous and free of any singularities. Further to this, when comparing different systems that belong to the same universality class we may observe, just as importantly, that there exists non-universal behaviour that does not affect the critical behaviour at all. To understand the subtleties that separate systems within the same universality class we need to construct an equation of state for each system near the critical point. The major problem facing physicists in this regard was that many possible equations of state could not reproduce all the correct critical indices. For example, power series expansions using the deviation from the critical point, *e.g.* $(T-T_c)$, as a small parameter were able to construct analytic equations of state, however these expansions in fact enforce the same critical exponents as MFT [7]. The solution to this came from an idea put forward by Widom, whom, while investigating fluids near the fluid-gas critical point, proposed the idea that critical behaviour naturally followed from the fact that the equation of state is homogeneous near the critical point. Specifically, a function between two vector spaces $f : V_1 \to V_2$ over a field $\mathbb{F}$ is considered homogeneous, of degree n, if it satisfies:

$$f(\lambda\vec{\mathbf{x}}) = \lambda^n f(\vec{\mathbf{x}})\,, \tag{1.1.6}$$

for $\vec{\mathbf{x}} \in V$, constant n, and any non-zero $\lambda \in \mathbb{F}$. The homogeneity of the free energy, known as Widom's scaling hypothesis, allows one to write an equation of state in terms of a function that is not only scale-invariant but also analytic everywhere apart from the critical point. Crucially though, it not only allows for the correct critical indices,

but it does so for entire phase boundaries.[5] This last point is best explained through example, for which we adapt Kadanoff's derivation of the scaling hypothesis [81] using what is now known as Kadanoff blocking.

Near the critical point we can rewrite the dependent variables so that our free energy density is in terms of deviation from the fixed point, *i.e.* $\tilde{f}(K,h) \mapsto \tilde{f}(t,h)$, where $t = \frac{|T-T_c|}{T_c}$ is the reduced temperature deviation and we have also explicitly used the fact that the critical magnetic field, H_c, is zero. For this example, we begin with a similar d-dimensional lattice that is below, but in the region of, the critical temperature and a small overall field H aligned in the positive z direction. As described above, this will lead to an overall net magnetisation that is small but non-zero. To calculate the free energy density we may, using an idea put forward by Kadanoff in 1966 [81], group spins together instead of integrating over each individual spin.

Consider dividing the entire lattice into sub-lattices, or blocks, of length aL so that there are L lattice sites on each side. Each of the blocks will then contain L^d spin operators. In order to see the correct asymptotic behaviour of correlation functions we must assume that the value of $aL \gg a$. However, since the correlation length ξ diverges at the critical point the value of L can in fact be chosen such that $a \ll aL \ll \xi$, which means that relative to the large scale fluctuations in the system each block can be considered small. For the simple case of $d = 2$ dimensions this blocking is shown in figure 1.1.3. The leading contribution to the free energy density from each block can be found by considering each block isolated from the rest of the system and the external field, contributing the free energy density due to interactions between the spin operators at sites within the block, $\tilde{f}_L(t)$. As each cell is small, we expect that it cannot support critical behaviour until the limit $L \to \xi$ is taken. Therefore, $\tilde{f}_L(t)$ is an analytic function of t but not L.

If we now consider the interaction between the spins in the block and the external field, that is for all spin operators $\sigma(\vec{\mathbf{r}})$ in the block b, the interaction with magnetic field will take the form of:

$$-h \sum_{r \in b} \sigma(\vec{\mathbf{r}}) \,. \tag{1.1.7}$$

We can assume that the inter-spin interactions within the block will lead to coalignment

[5] In most critical systems a phase transition between different states occur due to two or more variables. As such, we may define a locus of points where the phase transition occurs with respect to variables, defining a phase boundary for that transition [5,7]. For example, in ferromagnets there exists a phase boundary from an ordered state to the disordered state defined by $T < T_c$, $h = 0$

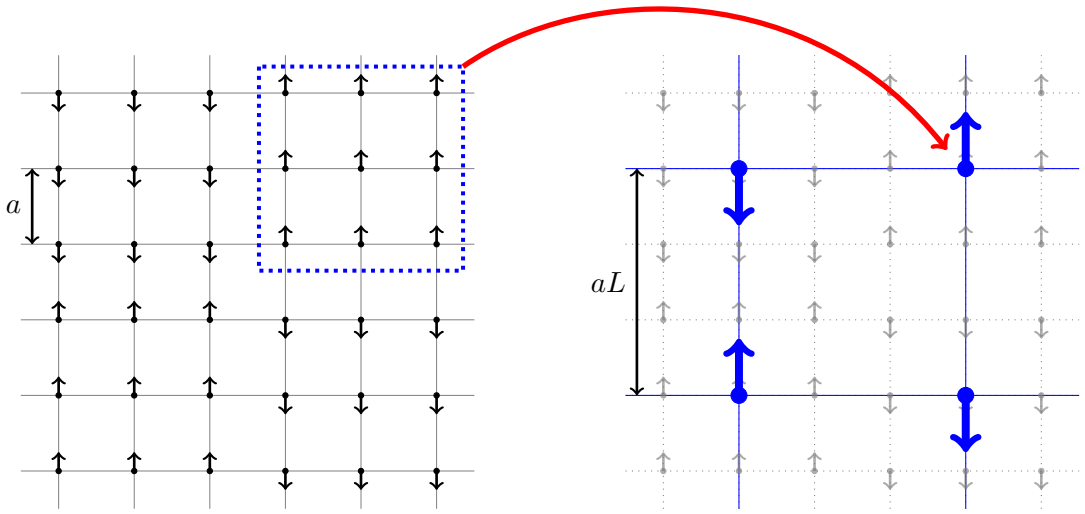

Figure 1.1.3: Example of blocking in a two-dimensional Ising model, with lattice spacing a. A group of spin operators $\sigma(\vec{\mathbf{r}})$, shown on the left, is grouped into blocks of length aL, where here $L = 3$. Each block of spins can then be interpretted as if it were an effective spin operator μ_b at this larger scale aL. *Nota bene* all spin operators within a block point in the same direction, so that each block-spin μ_b is two-valued like the original spin operators $\sigma(\vec{\mathbf{r}})$ [82].

of all the spin operators within the block. As such we may treat the interaction with the external field as if the block was an individual block-spin, which is proportional to the average spin within the block:[6]

$$\sum_{r\in b} \sigma(\vec{\mathbf{r}}) = \mu_b \langle\sigma\rangle_L L^d \,, \tag{1.1.8}$$

where μ_b is the block-spin operator, which we may assume to be double valued like our original spin operators. The expectation value, $\langle\sigma\rangle_L$, is given by:

$$\langle\sigma\rangle_L^2 = \sum_{r\in b}\sum_{r_2\in b} \frac{\langle\sigma(\vec{\mathbf{r}})\sigma(\vec{\mathbf{r_2}})\rangle}{L^{2d}} \,. \tag{1.1.9}$$

Using the fact that the blocks are small compared to large scale fluctuations, we can assert that this expectation value will not display critical phenomena and is essentially a function of L only. Treating each block as if it were an effective spin operator we can then consider nearest-neighbour interactions between the blocks, leading to the

[6]We note here, as was pointed out by Wilson [82], for this to be true all spins within the block must point in the same direction. Otherwise, the block "spin operator" will no longer be two-valued and we cannot treat the blocked lattice as a psuedo-Ising model.

contribution to free energy:

$$-\sum_{\substack{\text{nearest}\\ \text{neighbour}\\ \text{cells}}} \mu_b \mu_b' \tilde{K}(t, L) + \tilde{f}_{int}(t, L)\,, \tag{1.1.10}$$

where $\tilde{f}_{int}$ the free energy density due to direct interactions between blocks and $\tilde{K}$ is the effective coupling of spins that gives the free energy due to block-spins aligning or opposing each other. We can assume the length scale of $\tilde{K}$ and the interactions that make up $\tilde{f}_{int}$ are very small compared to ξ, so both functions can be treated as analytic functions of the t. Then, compared to our original Ising model, the partition function looks the same apart from the addition of the term due to the interactions between blocks.[7]

The correction to free energy compared to that of the isolated blocks is then essentially an Ising model calculation with effective spin parameters $\mu_b(\vec{\mathbf{r}})$, effective spin-spin coupling $\tilde{K}(t, L)$ and a new effective field $\tilde{H}$. One of the points of this effective Ising-model is that we can now calculate the overall system as though it had a fundamental lattice spacing aL and all scales below this have been 'integrated out', we have in essence reduced the number of degrees of freedom in the system while keeping the density of degrees of freedom the same.

This new Ising model will also be described by effective deviation parameters $\tilde{t}$ and $\tilde{h}$. Comparing a sum over block-spin operators defined in equation (1.1.8) to Hamiltonian of the original Ising model, equation (1.1.1), we see that the effective magnetic field variable is given by:

$$\tilde{h}(h, L) = \frac{\tilde{H}}{k_B T} = h\langle\sigma\rangle_L L^d\,. \tag{1.1.11}$$

Then, the free energy density of the blocked system, $\tilde{f}_{block}$, will be given by the sum of the isolated block free energy, the free energy due to the additional interaction term and the original Ising model free energy in terms of the new effective parameters:

$$\tilde{f}_{block}(t, h, L) = \tilde{f}_L(t) + L^{-d}\left(d\tilde{f}_{int}(t, L) + \tilde{f}(\tilde{t}(t, L), \tilde{h}(h, L))\right)\,, \tag{1.1.12}$$

[7] Strictly speaking, the additional term enters partition function within the exponential. As such when we take the logarithm in the definition of the free energy density, equation (1.1.3), the $\tilde{f}_{int}$ term appears additively.

where the factor of L^{-d} is the volume element of each block. In order to recreate the critical behaviour from the original Ising model we need to impose long range correlations of the new spin parameters. This condition on the correlation function $\langle \mu_b \mu_{b'} \rangle$ requires that the new effective parameters $\tilde{t} \ll 1$ and $\tilde{h} \ll 1$. At this point our calculation is entirely a reparametrisation of the original Ising model,[8] as such we can then equate the total free energy density of the original Ising model and that of the block-spin Ising model:

$$\tilde{f}(t,h) = \tilde{f}_L(t) + L^{-d}\left(d\tilde{f}_{int}(t,L) + \tilde{f}(\tilde{t},\tilde{h})\right) . \tag{1.1.13}$$

We can then expect to describe the critical behaviour of the system equivalently with both free energy densities. Consider for example the singularity in the specific heat capacity at zero external field, defined by:

$$C_h \equiv \left.\frac{\partial^2 \tilde{f}(t,h)}{\partial t^2}\right|_h \overset{t\to 0}{\sim} t^{-\alpha} . \tag{1.1.14}$$

Using the fact that the only function on the right hand side of equation (1.1.13) that is not analytic in t is $\tilde{f}$ we find that to reproduce the singularity in the specific heat capacity we must have:

$$t^{2-\alpha} = L^{-d}\tilde{t}^{2-\alpha} . \tag{1.1.15}$$

More generally, the part of the free energy density that contains all singularities and critical behaviour of the blocked system must be related to original system by:

$$\tilde{f}_{sing}(t,h) = L^{-d}\tilde{f}_{sing}(\tilde{t},\tilde{h}) . \tag{1.1.16}$$

Given the relation between t and $\tilde{t}$, for these two functions to be non-zero this is a statement that singular part of the free energy is homogeneous. Further it requires that h is proportional to L, *i.e.* that:

$$\langle \sigma \rangle_L = AL^{-\psi} , \tag{1.1.17}$$

[8] This is only true under the assumption that all spins within a block align, which is not always be true. This assumption is one of the key flaws of Kadanoff's original scaling picture that guided Wilson in his construction of the RG [82].

for some constant A and exponent ψ. This gives us the relation:

$$\tilde{h} = AL^{d-\psi}h\,. \tag{1.1.18}$$

From this we can write equation (1.1.16) in a manifestly homogeneous way:

$$\tilde{f}_{sing}\left(t^{2-\alpha}, \frac{h}{L^{\psi}}\right) = L^{-d}\tilde{f}_{sing}\left(L^{d}t^{2-\alpha}, L^{d}\frac{h}{L^{\psi}}\right)\,. \tag{1.1.19}$$

From this we recognise, setting $\lambda = L^d$, that the singular part of the free energy density is homogeneous to degree $n = 1$. This is essentially a reiteration of the scale invariance of critical phenomena and we may choose any value for λ to obtain the same critical exponents. For example, choosing $\lambda = t^{-(2-\alpha)}$, we may recover Widom's and Kadanoff's original result that allows us to write the singular part of the free energy in terms of a homogeneous scaling function of a single variable [7, 81, 83]:

$$\tilde{f}_{sing}\,(t, h) = t^{2-\alpha}\mathcal{F}\left(\frac{h}{t^{\Delta}}\right)\,, \tag{1.1.20}$$

where $\Delta = \frac{(2-\alpha)}{d}(d - \psi)$. This property of homogeneity then allows for us write an equation of state that is naturally analytic everywhere apart from the critical point, as is required by homogeneity, while still giving the correct critical indices. For example, in the form of written by Widom and Kadanoff we find that by setting $h = 0$ $\tilde{f}_{sing}$ will still be an analytic function of t apart from at the critical point, $t = 0$, and the the scaling of t implies that we obtain the correct critical indices for the entire phase boundary given by $h = 0$.

The process used in this derivation, Kadanoff blocking, is essentially a systematic way to use the self similarity of the system near the critical point. However, it provides no insight into the reason why self-similarity leads to universal critical behaviour. Specifically, one of the key steps in the derivation of the homogeneity of the $\tilde{f}_{sing}$ is the *assumption* that it displays the correct singularities at the critical point. Without this assumption we could not derive the rescaling of $t \to \tilde{t}$. The missing step to answer these questions was provided by Wilson in his development of the Renormalisation Group. That is, if a system is self-similar one can, as before, group degrees of freedom into blocks, rewrite the theory in terms effective parameters defined at the scale given by these blocks and then, pivotally, rescale all parameters of the blocked system

to appear like the original system.[9] This complete rescaling is pivotal in preserving scale-invariance of not only the singular parts of the free energy, but the remaining analytic part of the as well. One of the key findings that came from the RG analysis over Widom's and Kadanoff's scaling hypothesis is the idea that the scaling of variables that do not change the critical behaviour are also universal. These are then described by so called irrelevant critical exponents. The inclusion of them then provided a natural explanation as to why different systems could demonstrate the same critical behaviour, specifically any of the irrelevant scaling variables could take any magnitude away from the critical point without changing critical behaviour as their scaling behaviour vanishes at the critical point. This last step in understanding universality is best understood from a more thorough stand point, as such we will proceed by constructing the RG we will use.

1.2 The Renormalisation Group

The discussion up to this point has been primarily centred around the derivation of critical phenomenon, the observation of universal critical behaviour and particularly about how different systems within the same universality class display the same critical exponents‘. In this current section we will demonstrate that these ideas follow naturally from RG methodology and that the RG encapsulates all the previously covered material on critical phenomena, while providing an effective framework for calculating critical exponents.

The basis of the RG is, like Kadanoff blocking, exploitation of the fact that scale invariance near the critical point implies self-similar behaviour. As such, we do not need to explicitly integrate over each degree of freedom of the system to solve for the effect of large scale fluctuations. Expanding upon the scaling hypothesis of Kadanoff and Widom, similar work on critical behaviour by Fisher and his own work in a way to perform explicit calculations Wilson developed a method to map a Hamiltonian of a near-critical system to one in terms of less degrees of freedom. This method, aptly

[9] We note that many texts written after Kadanoff's original paper [81] often include this rescaling step when referring his derivation. However this step is missing in the original work, which rescales only the two relevant thermodynamic variables of the system. The key difference is that the RG not only rescales all thermodynamic variables but also spatial (or equivalently momentum) variables and the complete Hamiltonian itself such that the partition function not only appears the same but is left invariant [53].

named the Wilsonian Renormalisation Group (WRG) [56], uses the fact that, when microscopic degrees of freedom are represented as field configurations in momentum space, we can describe fluctuations within the system as momentum modes. That is we may explicitly integrate out only the 'high momentum', or equivalently 'small distance', fluctuations relative to a renormalised fundamental scale.

At this point we make an explicit note to avoid confusion in further sections; historically, due to the fact that variables, such as an external magnetic field, that couple to functions of the order parameter can be controlled in a lab, the term field in statistical mechanics literature will often refer to coupling variables where as in QFT the nomenclature of a (quantum) field is derived from their description as the mathematical field object and the coupling variables, which are kept constant during calculations, are referred to as coupling constants. Throughout the rest of this thesis we will keep to QFT nomenclature and refer to order parameters as fields and the equivalent thermodynamic variables as coupling constants instead of fields.

Consider now a general system described by the quantum field φ, that has the Hamiltonian $\mathcal{H}[\varphi]$. The Partition Function is then given by:

$$\mathcal{Z} = \mathrm{Tr}\left[\exp\left(-\frac{\mathcal{H}}{k_B T}\right)\right], \tag{1.2.1}$$

where the trace here now implies integration over spatial, or momentum, coordinates and field configurations. For ease of notation, and as is customary to HEP, we will work in natural units and assume Boltzmann's constant, Planck's constant and the speed of light in a vacuum are all equal to one, *i.e.* $k_B = \hbar = c = 1$, throughout the rest of this thesis. We assume that the Hamiltonian here is defined up to the momentum scale Λ, *i.e.* we assume that if we were working on a lattice the fundamental length scale is given by $a = \Lambda^{-1}$.

We can then define a version of the WRG map: first we coarse-grain the system, integrating out only those momentum degrees of freedom within the shell given by $\Lambda/b < q < \Lambda$, for some real number $b > 1$, then each order parameter and the Hamiltonian can be rescaled and written in terms of renormalised variables such that it looks like the original system. The effect of this map is to change the Hamiltonian $\mathcal{H}$ written at the scale Λ via the map to a new Hamiltonian $\mathcal{H}'$ at the scale $\Lambda' = \Lambda/b$, which describes the same physics but incorporates the effects of all fluctuations within the

momentum shell $\Lambda/b < q < \Lambda$ as a shift of the coupling constants and fields within the Hamiltonian.[10] Crucially, the WRG map is constructed in a way that the partition function of the system, equation (1.2.1), and therefore the physics of the system is left unchanged. It is then be possible, via repeated use of this map, to systematically integrate out the fluctuations from all scales and obtain the macroscopic critical behaviour. There are some subtleties here that should be considered, such as the inclusions of interactions beyond those in the microscopic action, however we defer this discussion until we have covered more technical detail in section 1.2.1, where we will cover each step required to complete the integration over all fluctuations.

The development of Feynman's path integral formalism, which gives the generator of QFT Green's functions in terms of a partition function, allows the use of a continuous form of WRG in QFT applied to high energy particle physics. While the WRG is often applicable where small parameters are available, *e.g.* expanding in space-time dimensions near $d = 4$ using $\varepsilon = d - 4$ or for O(N) symmetric models expanding in $1/N$, many cases in high energy physics do not have small parameters that may be used for series expansions, such as work concerned with possible strong coupling or, in the case of modern applications, non-perturbative functional expansions, such as the derivative expansion. As such an exact formalism is more useful in the context of QFT. We will cover a basic outline of the functional or exact renormalistion group (ERG) for scalar field theory in the next section.

In the context of QFT specifically, the ERG has been used to study the radiative corrections of theories that would otherwise been untenable by the perturbation theory normally used in high energy physics. One of the greatest strengths of the ERG is that we can exploit universal scaling behaviour near critical points to study an arbitrary quantum field theory allowing only for certain assumptions, specifically symmetries of the model and the assumptions required to complete the calculation. Similar to the WRG, we perform only a partial integration over momentum modes, in this case to reparametrise the action and partition function in terms of a floating momentum scale, which we later denote k. It is then possible to examine the space of allowed actions in our QFT in general based upon how they change with this scale. For reviews on the ERG and its applications in QFT see [39, 53–55, 84].

[10] This shift of coupling constants due to fluctuations is very reminiscent of the renormalisation procedure in QFT, which is not entirely unexpected given the name of the RG.

1.2.1 Explicit Setup

As stated above, the ERG can be used to examine an arbitrary QFT, in our case a scalar field theory. We begin with an arbitrary bare action, $\hat{\mathcal{S}}[\varphi]$, for a N component real scalar field φ^a, where lower-case indexes a (and later b and c) are reserved for field indices. Further, we will use Einstein sum notation such that for any field, Lorentz or matrix index a repeated index indicates a sum over this index, *e.g.* $\varphi^a\varphi^a = \sum_{a=1}^{N} \varphi^a\varphi^a$. We impose that this action be symmetric under rotation of field components, $O(N)$ symmetry, and discrete Z_2, $\varphi \to -\varphi$, transformations. We work in, as is typical in the ERG, d dimensional Wick-rotated Euclidean space.[11] For the most part we will avoid using a specific value for our d dimensions and N components such that our results may be compared to numerical and experimental results of the Ising universality class.

We assume the theory is valid up to some overall ultraviolet (UV) momentum cutoff, which add in by hand and denote Λ. As was done by Wegner & Houghton we will introduce a floating momentum cutoff scale, that we shall denote k, using a sharp momentum cutoff. This is done by adding to the bare action $\hat{\mathcal{S}}[\varphi]$ the following cutoff action:

$$\Delta\mathcal{S}[\varphi] = \frac{1}{2}\varphi^a \cdot \mathrm{R}_k \cdot \varphi^a \,, \tag{1.2.2}$$

where R_k is the Fourier transform of some sharp cutoff function, $\hat{\mathrm{R}}_k$, and we have used a dot notation to condense integration of functions over spatial coordinates. That is, for some functions $g(\vec{\mathbf{x}})$ and $h(\vec{\mathbf{x}})$:

$$g \cdot h \equiv \int \mathrm{d}^d\vec{\mathbf{x}}\; g(\vec{\mathbf{x}})h(\vec{\mathbf{x}}) \,, \tag{1.2.3}$$

where we assume integration limits for any variable y is given by $-\infty < y < \infty$ unless otherwise specified. This is not to be confused with vector multiplication, which is given by $\vec{\mathbf{x}}.\vec{\mathbf{x}} \equiv \vec{\mathbf{x}}^\nu\vec{\mathbf{x}}_\nu$. We will differentiate vector quantities, *e.g.* $\vec{\mathbf{x}}$, from their respective magnitudes using italic font, *e.g.* $\vec{\mathbf{x}}.\vec{\mathbf{x}} \equiv x^2$. Additionally, we treat functions of two arguments as a matrix, such that for functions $g(\mathbf{x})$ and $h(\mathbf{y})$ and $\Delta(\vec{\mathbf{x}}, \vec{\mathbf{y}})$:

$$g \cdot \Delta \cdot h \equiv \int \frac{\mathrm{d}^d\vec{\mathbf{q}}}{(2\pi)^d}\;\; \tilde{g}(\vec{\mathbf{q}})\tilde{\Delta}(q)\tilde{h}(-\vec{\mathbf{q}}) \,, \tag{1.2.4}$$

[11] Euclidean space is often used as the Minkowski signature presents problems in splitting high/low momentum modes.

where from now on we reserve tildes to denote the Fourier transform of functions. Here we have implicitly used the translation invariance of the kernel Δ so that we may write its Fourier transform as:

$$\Delta(\vec{\mathbf{x}},\vec{\mathbf{y}}) = \int \frac{\mathrm{d}^d\vec{\mathbf{q}}}{(2\pi)^d}\ \tilde{\Delta}(q)\mathrm{e}^{i\vec{\mathbf{q}}.(\vec{\mathbf{x}}-\vec{\mathbf{y}})}\,. \tag{1.2.5}$$

Finally, operators squared are assumed to contain integration over spatial coordinates, *e.g.* :

$$(\partial_\mu\phi)^2 = \int \mathrm{d}^d\vec{\mathbf{x}}\ \partial_\mu\phi(\vec{\mathbf{x}})\partial^\mu\phi(\vec{\mathbf{x}})\,. \tag{1.2.6}$$

The cutoff function R_k acts as an infrared (IR) cutoff such that momentum modes of the fields φ with momentum p propagate freely for $p > k$ and those modes with $p \leq k$ are suppressed. It is worth noting that so long as the cutoff function R_k does not introduce new divergences then the integration over all momentum modes in the partition function is not altered by the inclusion of the cutoff. Explicitly, this cutoff must be finite at zero momentum so that it adds in no IR divergences for massless fields. Further, to ensure that this cutoff does not change the physics of the system with no IR cutoff, it must vanish when $k \to 0$ [47, 48]:

$$\lim_{q^2/k^2\to 0} \mathrm{R}_k(q^2) > 0\,, \tag{1.2.7}$$

and

$$\lim_{k^2/q^2\to 0} \mathrm{R}_k(q^2) \to 0\,. \tag{1.2.8}$$

Finally, to ensure that we get the correct limits at the overall cutoff scale the cutoff function must diverge:

$$\lim_{k\to\Lambda} \mathrm{R}_k(q^2) \to \infty\,. \tag{1.2.9}$$

At this point we will note that the following derivation follows explicitly from the fact that the cutoff function sharply suppresses momentum modes below the scale k, such that we may treat the field φ as the sum of two separate fields:

$$\varphi = \varphi_> + \varphi_<\,, \tag{1.2.10}$$

where $\varphi_>$ are the momentum modes of φ with $p > k$ and the remaining modes that propagate with $p \leq k$ are denoted $\varphi_<$. By doing so, we may explicitly integrate over only the high momentum modes, $\varphi_>$, in order to coarse-grain the system, similar to the Kadanoff blocking procedure shown above. However, for more general cutoff functions this split of momentum modes *cannot be done exactly*. Instead one must exploit the fact that the cutoff function is an artificial addition and instead renormalise the action so that the physics of the system, or more specifically the full partition function, does not change with the artificial floating scale k. For more detail on the general cutoff derivation please see appendix 5. These considerations are however beyond the scope of this thesis but the arguments made below using a sharp cutoff may be easily adapted for the case of a smooth cutoff.

In similar notation to [42] we can write the overall propagator modified by our IR cutoff function, Δ, as a sum of an ultraviolet (UV) cutoff propagator, Δ_{UV}, and a related infrared cutoff propagator, Δ_{IR}, given by the relations:

$$\Delta = \Delta_{IR} + \Delta_{UV}\,, \tag{1.2.11}$$

where:

$$\Delta_{IR}(q,k) = (Z_k q^2 + \mathrm{R}_k(q^2))^{-1}, \tag{1.2.12}$$

$$\Delta_{UV}(q,k) = \frac{\mathrm{R}_k(q)}{Z_k q^2 (Z_k q^2 + \mathrm{R}_k(q))}\,, \tag{1.2.13}$$

and Z_k is the field renormalisation of the field φ, which we include for completeness. In general we would expect this factor to depend on the external momentum, q, but we have suppressed the dependence at this time.

Each of these two propagators then corresponds to one of our split fields; the momentum modes with $p \leq k$ propagate with an effective ultraviolet cutoff, corresponding to Δ_{UV}, and the momentum modes with $q > k$ propagate with an effective infrared cutoff, corresponding to Δ_{IR}.

Within the scope of this thesis we may, without loss of generality, choose to use

Litim's 'optimised cutoff' [47]:[12]

$$\mathrm{R}_k(q) = Z_k(k^2 - q^2)\Theta(k^2 - q^2)\,, \tag{1.2.14}$$

where Θ is the Heaviside step function. The addition of the cutoff action allows us to define the infrared and ultraviolet cutoff propagators explicitly, which in momentum space are given by:

$$\Delta_{IR}(q,k) = (Z_k q^2 + \mathrm{R}_k)^{-1} = \begin{cases} (Z_k q^2)^{-1} & \text{for } q > k\,, \\ (Z_k k^2)^{-1} & \text{for } q \leq k\,, \end{cases} \tag{1.2.15}$$

and

$$\Delta_{UV}(q,k) = \frac{1}{Z_k q^2} - \Delta_{IR}(q,k) = \begin{cases} 0 & \text{for } q > k \\ (Z_k q^2)^{-1}(1 - \frac{Z_k q^2}{k^2}) & \text{for } q \leq k \end{cases} \tag{1.2.16}$$

If we now split off the kinetic terms of the bare action:

$$\hat{\mathcal{S}}[\varphi] + \Delta\mathcal{S}[\varphi] = \frac{1}{2}\varphi^a_> \cdot \Delta^{-1}_{IR} \cdot \varphi^a_> + \frac{1}{2}\varphi^a_< \cdot \Delta^{-1}_{UV} \cdot \varphi^a_< + \mathcal{S}[\varphi_> + \varphi_<]\,, \tag{1.2.17}$$

where $\mathcal{S}[\varphi]$ is the interaction part of the bare action, then the inclusion of our sharp cutoff action allows us to write our partition function explicitly as:

$$\begin{aligned} \mathcal{Z}[\mathrm{J}] =& \int \mathcal{D}\varphi \ \exp\left(-\hat{\mathcal{S}}[\varphi] - \Delta\mathcal{S}[\varphi] + \mathrm{J}^a \cdot \varphi^a\right) \\ =& \mathcal{N} \int \mathcal{D}\varphi_< \exp\left(-\frac{1}{2}\varphi^a_< \cdot \Delta^{-1}_{UV} \cdot \varphi^a_<\right) \mathcal{Z}_k[J, \varphi_<]\,, \end{aligned} \tag{1.2.18}$$

where $\mathcal{N}$ is some overall constant of proportionality that does not contribute to correlation functions, which we will ignore from now on, and the functional $\mathcal{Z}_k$ is given by:

$$\mathcal{Z}_k[J, \varphi_<] = \int \mathcal{D}\varphi_> \exp\left(-\frac{1}{2}\varphi^a_> \cdot \Delta^{-1}_{IR} \cdot \varphi^a_> - \mathcal{S}[\varphi_> + \varphi_<] + \mathrm{J}^a \cdot (\varphi^a_> + \varphi^a_<)\right)\,. \tag{1.2.19}$$

The overall partition function of our QFT can then be calculated piecewise by per-

[12] The explicit choice of cutoff may change the results within the ERG, but up to the order of the functional expansion we employ this choice of cutoff does not effect our results [42, 48].

forming the integration within the partition function $\mathcal{Z}_k$, which physically represents integrating over all possible vacuum fluctuations of momentum modes with $k < p < \Lambda$. The remaining integration over momentum modes between 0 and k in the overall partition function $\mathcal{Z}$ can then be rewritten as if our QFT had the overall cutoff replaced by our scale k and with a new more complicated effective action known as the Wilsonian effective action $\mathcal{S}_k^{eff}$ [39, 52]. As with our discussion about the mesoscopic scale above, this step may introduce interactions beyond those within bare action $\hat{\mathcal{S}}$. In fact, in order to keep the partition function invariant with changes in k we must in general treat the bare action as if it contained any possible interaction that does not violate the fundamental symmetries of the field φ, albeit with most interactions having vanishing coupling at the bare scale.

It is important to note that for the integration over high momentum modes, $\varphi_>$, in $\mathcal{Z}_k$ the low-momentum modes, $\varphi_<$, act as a background field, as such we may treat them as a constant within $\mathcal{Z}_k$. This can be exploited by considering the shift in field variable $\varphi_> = \varphi - \varphi_<$ and then rewriting the interactions in $\mathcal{S}$ as functions of partial derivatives of our source terms, *i.e.* as a function of $\delta/\delta \mathrm{J}$, to give $\mathcal{Z}_k$, as:

$$\begin{aligned} \mathcal{Z}_k[\mathrm{J}, \varphi_<] = &\exp\left(\frac{1}{2}\varphi_<^a \cdot \Delta_{IR}^{-1} \cdot \varphi_<^a - \mathcal{S}\left[\delta/\delta \mathrm{J}\right]\right) \times \cdots \\ &\cdots \times \int \mathcal{D}\varphi \exp\left(-\frac{1}{2}\varphi^a \cdot \Delta_{IR}^{-1} \cdot \varphi^a + \varphi^a \cdot (\mathrm{J}^a + \Delta_{IR}^{-1} \cdot \varphi_<^a)\right) . \end{aligned} \tag{1.2.20}$$

If we now apply the change of variables $\varphi' = \varphi - \Delta_{IR} \cdot \mathrm{J} - \varphi_<$ then the path integral over φ' is a decoupled Gaussian integral that can be solved exactly. Integrating over φ' and then applying all derivatives within $\mathcal{S}[\delta/\delta \mathrm{J}]$, noting that the functional derivatives are replaced with $\Phi = \Delta_{IR} \cdot \mathrm{J} + \varphi_<$ or Δ_{IR}, we find that [52, 85]:

$$\mathcal{Z}_k[\mathrm{J}, \varphi_<] = \exp\left(\frac{1}{2}\mathrm{J}^a \cdot \Delta_{IR} \cdot \mathrm{J}^a + \mathrm{J}^a \cdot \varphi_<^a - \mathcal{S}_k[\Phi]\right) , \tag{1.2.21}$$

where we have removed a trivial multiplicative constant that does contribute the correlation functions. The functional $\mathcal{S}_k$, which we call the Wilsonian (effective) interactions,

is the interaction part of the Wilsonian effective action:[13]

$$\mathcal{S}_k^{eff}[\Phi] = \frac{1}{2}\Phi^a \cdot \Delta_{UV}^{-1} \cdot \Phi^a + \mathcal{S}_k[\Phi]\,. \tag{1.2.22}$$

An important point to make here is that the integration carried out in $\mathcal{Z}_k$ to remove those fields in our partition function with momentum greater than the scale k is simply a reparametrisation of the original partition function. Therefore the Green's functions of the QFT defined by $\mathcal{Z}$ do not depend on the scale k. Specifically, taking the limit of $\varphi_< \to 0$ in the partition function $\mathcal{Z}_k$ in equation (1.2.18) we find that:

$$\begin{aligned}\lim_{\varphi_< \to 0}(\mathcal{Z}_k[\mathrm{J}, \varphi_<]) =& \mathcal{Z}_k[\mathrm{J}] \\ =& \int \mathcal{D}\varphi_> \ \exp\left(-\frac{1}{2}\varphi_>^a \cdot \Delta_{IR}^{-1} \cdot \varphi_>^a - \mathcal{S}[\varphi_>] + \mathrm{J}^a \cdot \varphi_>^a\right),\end{aligned} \tag{1.2.23}$$

which is the exact partition function for a QFT of the N scalar fields $\varphi_>$ with an infrared cutoff at the scale k. Furthermore, the limit $\varphi_< \to 0$ coincides with the limit $k \to 0$, where the full integration of the partition function is obtained, *i.e.* the physics of the final result does not depend upon the choice to consider $\varphi_<$ as a background field in our calculations and we obtain the correct physical results in the limit that all fluctuations are integrated over.

From this we can see that changes in the scale k do not affect the physics of the system. Instead, we find that we may change the scale k to examine how integrating over greater number of momentum modes changes how the partition function $\mathcal{Z}$ is parametrised by changes in the action $\mathcal{S}_k$. That is, the changes in the action $\mathcal{S}_k$ with the scale k can be used to examine how including increasing numbers of fluctuations renormalises our QFT, with the limit $k \to 0$ giving the inclusion of all possible fluctuations, *i.e.* the original full partition function. In the opposite limit, $k \to \Lambda$, the infrared cutoff propagator $\Delta_{IR} \to 0$, or equivalently $\Delta_{UV} \to \Delta$, and there is no functional integration in $\mathcal{Z}_k$ as $\varphi_>$ vanishes, *i.e.* $\Phi = \varphi_<$ and $\varphi = \varphi_<$. The result of this is that

[13] We note that the dependence of the Wilsonian interaction on the cutoff function R_k is non-trivial. However, apart from the addition of the cutoff action in equation (1.2.18) none of the steps taken have broken a symmetry of the model. Therefore, the Wilsonian effective action and the Wilsonian interactions both have the same symmetries as the bare action apart from those symmetries broken by the cutoff action.

in the limit $k \to \Lambda$ our partition function in equation (1.2.18) takes the form:

$$\mathcal{Z}[\mathrm{J}] = \mathcal{N} \int \mathcal{D}\varphi_< \ \exp\left(-\frac{1}{2}\varphi^a_< \cdot \Delta^{-1}_{UV} \cdot \varphi^a_< - \mathcal{S}_k[\varphi_<] + \mathrm{J}^a \cdot \varphi^a_<\right), \tag{1.2.24}$$

that is, in the limit $k \to \Lambda$ the Wilson interactions are approximately the bare interactions:

$$\lim_{k\to\Lambda}(\mathcal{S}_k[\Phi]) \approx \mathcal{S}[\varphi]\,. \tag{1.2.25}$$

To find the scale dependence of $\mathcal{S}_k$ we first exploit the fact that the scale dependence of both the partition function $\mathcal{Z}_k$ and $\mathcal{S}_k$ is independent of J. Taking the scale derivative of equation (1.2.19) at constant J we find that:

$$\left.\frac{\partial \mathcal{Z}_k[\mathrm{J},\varphi_<]}{\partial k}\right|_{\mathrm{J}} = -\frac{1}{2}\left(\frac{\delta}{\delta \mathrm{J}^a} - \varphi^a_<\right) \cdot \partial_k \Delta^{-1}_{IR} \cdot \left(\frac{\delta}{\delta \mathrm{J}^a} - \varphi^a_<\right) \mathcal{Z}_k[\mathrm{J},\varphi_<]\,. \tag{1.2.26}$$

Inserting the integrated form of $\mathcal{Z}_k$ from equation (1.2.21) into this we then find the flow equation of the Wilsonian interactions $\mathcal{S}_k$, which is given by Polchinski's equation [52, 86]:

$$\partial_k \mathcal{S}_k[\Phi] = \frac{1}{2}\frac{\delta \mathcal{S}_k}{\delta \Phi^a} \cdot \frac{\partial \Delta_{UV}}{\partial k} \cdot \frac{\delta \mathcal{S}_k}{\delta \Phi^a} - \frac{1}{2}\mathrm{Tr}\left[\frac{\partial \Delta_{UV}}{\partial k} \cdot \frac{\delta^2 \mathcal{S}_k}{\delta\Phi\delta\Phi}\right], \tag{1.2.27}$$

where the trace is taken over spatial or momentum coordinates and field indices.

As the integration in $\mathcal{Z}_k$ gives the Wilsonian interactions $\mathcal{S}_k$, the Polchinski equation can be thought of as physically describing the change in the effective couplings of the low momentum fields with integrations over different numbers of quantum and classical field excitations [52] with momentum $p^2 > k^2$. Alternatively, we may extract the same physics by examining the flow fully renormalised vertices, *i.e.* one-particle irreducible (1PI) vertices generated by the Legendre effective action, which will then physically represent integration of only the quantum field excitations with momentum $p^2 > k^2$.

The first step to defining the Legendre effective action is to write the generator of connected diagrams in the presence of our IR cutoff as:

$$\begin{aligned} \mathrm{W}_k[\mathrm{J},\varphi_<] &\equiv \ln\left(\mathcal{Z}_k[\mathrm{J},\varphi_<]\right) \\ &\equiv \ln\left(\int \mathcal{D}\varphi_> \exp\left(-\frac{1}{2}\varphi^a_> \cdot \Delta^{-1}_{IR} \cdot \varphi^a_> - \mathcal{S}[\varphi_> + \varphi_<] + \mathrm{J}^a \cdot (\varphi^a_> + \varphi^a_<)\right)\right), \end{aligned} \tag{1.2.28}$$

from which we can define a 'classical field', ϕ, as the expectation value of the quantum

field in the presence of our IR cutoff scale k:

$$\phi^a(\vec{\mathbf{x}}) \equiv \frac{\delta \mathrm{W}_k[\mathrm{J},\varphi_<]}{\delta \mathrm{J}^a(\vec{\mathbf{x}})} = \langle \varphi^a \rangle \,, \tag{1.2.29}$$

where the angle brackets denoting an expectation value of some operator ζ is now reserved for the interpretation:

$$\langle \zeta \rangle = \frac{\int \mathcal{D}\varphi_> \; \zeta \; \exp\left(-\frac{1}{2}\varphi^a_> \cdot \Delta^{-1}_{IR} \cdot \varphi^a_> - \mathcal{S}[\varphi_> + \varphi_<] + \mathrm{J}^a \cdot (\varphi^a_> + \varphi^a_<)\right)}{\mathcal{Z}_k[\mathrm{J},\varphi_<]} \,. \tag{1.2.30}$$

With this we can Legendre transform W_k to define the Legendre Effective Action with an overall ultraviolet cutoff Λ and effective infrared cutoff k:[14]

$$\Gamma^{eff}_k[\phi,\varphi_<] = -\mathrm{W}_k[J,\varphi_<] + \mathrm{J}^a \cdot \phi^a \,. \tag{1.2.31}$$

We note here that while the quantum fields $\varphi_<$ and $\varphi_>$ do not carry any scale dependence in k the definition of classical field caries implicit scale dependence. Therefore, if we wish to observe the change of effective couplings of the 'field' ϕ in Γ^{eff}_k we must reparametrise the source function as a scale dependent function of the classical field. That is:

$$\mathrm{J} \to \mathrm{J}(\phi, k) \,. \tag{1.2.32}$$

Taking the scale derivative we find that:

$$\begin{aligned} \left.\frac{\partial \Gamma^{eff}_k[\phi,\varphi_<]}{\partial k}\right|_\phi &= -\left.\frac{\partial \mathrm{W}_k[\mathrm{J},\varphi_<]}{\partial k}\right|_\mathrm{J} - \frac{\delta \mathrm{W}_k[\mathrm{J},\varphi_<]}{\delta \mathrm{J}^a} \cdot \left.\frac{\partial \mathrm{J}^a}{\partial k}\right|_\phi + \left.\frac{\partial \mathrm{J}^a}{\partial k}\right|_\phi \cdot \phi^a \,, \\ &= \frac{1}{2}\left\langle \varphi^a_> \cdot \partial_k \Delta^{-1}_{IR} \cdot \varphi^a_> \right\rangle \,. \end{aligned} \tag{1.2.33}$$

Denoting the connected two-point function as G and decomposing this into expectation values:

$$\mathrm{G}_{ab}(\vec{\mathbf{x}},\vec{\mathbf{y}}) = \frac{\delta^2 \mathrm{W}_k[\mathrm{J},\varphi_<]}{\delta \mathrm{J}^a(\vec{\mathbf{x}})\delta \mathrm{J}^b(\vec{\mathbf{y}})} = \langle \varphi^a(\vec{\mathbf{x}})\varphi^b(\vec{\mathbf{y}})\rangle - \langle \varphi^a(\vec{\mathbf{x}})\rangle\langle \varphi^b(\vec{\mathbf{y}})\rangle \,, \tag{1.2.34}$$

[14]The definition of Γ^{eff}_k in terms of W_k gives the Legendre effective action the same symmetries as Wilsonian effective action, *i.e.* the same symmetries as the bare action apart from those explicitly broken by the cutoff action. We also note that due to the implicit addition of the IR cutoff action in W_k this Legendre transform need only be convex for in the limit $k \to 0$ [84].

we can rewrite equation (1.2.33) as:

$$\partial_k \Gamma_k^{eff}[\phi, \varphi_<]\Big|_\phi = \frac{1}{2}\mathrm{Tr}\left[\mathrm{G}\cdot\partial_k\Delta_{IR}^{-1}\right] + \frac{1}{2}(\phi^a - \varphi_<^a)\cdot\partial_k\Delta_{IR}^{-1}\cdot(\phi^a - \varphi_<^a)\,. \tag{1.2.35}$$

From the definition of G we find that:

$$\int \mathrm{d}^d\vec{\mathbf{z}}\ \mathrm{G}_{ac}(\vec{\mathbf{x}},\vec{\mathbf{z}})\frac{\delta^2\Gamma_k^{eff}}{\delta\phi^c(\vec{\mathbf{z}})\delta\phi^b(\vec{\mathbf{y}})} = \hat{\delta}_{ab}\delta^d(\vec{\mathbf{x}}-\vec{\mathbf{y}})\,, \tag{1.2.36}$$

where we use the notation $\hat{\delta}$ for the Kronecker delta function and δ^d for the d-dimensional Dirac delta function. This means we may replace the factor G with the inverse of the full propagator, the second functional derivative of Γ_k^{eff} with respect to ϕ.

Using the integrated form of $\mathcal{Z}_k$, equation (1.2.21), in the definition of W_k, equation (1.2.28), we can say that:

$$\frac{\delta \mathrm{W}_k[\mathrm{J},\varphi_<]}{\delta \mathrm{J}} = \Delta_{IR}\cdot\mathrm{J} + \varphi_< - \frac{\delta\mathcal{S}_k}{\delta\Phi}\cdot\Delta_{IR}\,, \tag{1.2.37}$$

which equally implies that:

$$\frac{\delta \mathrm{W}_k}{\delta\varphi_<} = \Delta_{IR}^{-1}\cdot(\phi-\varphi_<)\,. \tag{1.2.38}$$

Taking a similar functional derivative of the Legendre transform, equation (1.2.31), with respect to $\varphi_<$ we find that:

$$\frac{\delta\Gamma_k^{eff}[\phi,\varphi_<]}{\delta\varphi_<} = -\frac{\delta \mathrm{W}_k}{\delta\varphi_<}\,. \tag{1.2.39}$$

Therefore, Γ_k^{eff} must have a massless kinetic term of the form $(\phi-\varphi_<)\cdot\Delta_{IR}^{-1}\cdot(\phi-\varphi_<)$ and a remaining interaction functional that is independent of the low momentum fields $\varphi_<$. We can therefore extract this massless kinetic term of the Legendre effective action to define the interaction part, Γ_k, which we call the Legendre (effective) interactions [42, 52, 85]:

$$\Gamma_k^{eff}[\phi,\varphi_<] = \frac{1}{2}(\phi-\varphi_<)^a\cdot\Delta_{IR}^{-1}\cdot(\phi-\varphi_<)^a + \Gamma_k[\phi]\,. \tag{1.2.40}$$

The additional $\varphi_<$ terms in the kinetic term above comes from the fact that our definition of W_k in terms of $\mathcal{Z}_k$, equation (1.2.28), does not integrate over $\varphi_<$. However, the fact that the low momentum fields do not appear in the Legendre interactions implies

Γ_k gives the same interaction terms as if our definition of W_k was defined using the full partition function $\mathcal{Z}$, equation (1.2.18), instead of $\mathcal{Z}_k$. Additionally, using integrated form $\mathcal{Z}_k$, equation (1.2.21), in our Legendre transformation, equation (1.2.31), we find that after extracting the kinetic terms of the Legendre effective action and rearranging to get $\Phi = \Delta_{IR}\cdot \mathrm{J} + \varphi_<$ that we can generalise the Legendre transformation to transform Legendre interactions and Wilsonian interactions [42]:

$$\mathcal{S}_k[\Phi] = \Gamma_k[\phi] + \frac{1}{2}(\phi^a - \Phi^a)\cdot\Delta_{IR}^{-1}\cdot(\phi_a - \Phi_a)\,. \tag{1.2.41}$$

Like with the Wilsonian interactions, the limit of $k \to 0$ returns the full functional integration over vacuum fluctuations, at which point the Legendre effective action equals the full one-particle irreducible generating functional, $\Gamma_k^{eff} = \Gamma$. In the limit $k \to \Lambda$ we can use the fact that $\varphi_> \to 0$ in the definition of W_k, equation (1.2.28), to find that $\phi^a \to \Phi^a$, which implies that in the UV both Wilsonian and Legendre interactions are approximately the bare interactions:

$$\lim_{k\to\Lambda}(\Gamma_k[\phi]) = \lim_{k\to\Lambda}(\mathcal{S}_k[\Phi]) \approx \mathcal{S}[\varphi]\,. \tag{1.2.42}$$

Finally, using the decomposition in equation (1.2.40) we can remove the kinetic like term in the flow equation of Γ_k^{eff}, equation (1.2.35), to obtain the flow equation of the Legendre interactions, which we shall call the Legendre flow equation [50, 52, 87]:

$$\partial_k\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\left[\hat{\delta}_{ab} + \Delta_{IR}\cdot\frac{\delta^2\Gamma_k[\phi]}{\delta\phi^a\delta\phi^b}\right]^{-1}\cdot\frac{\partial_k\Delta_{IR}^{-1}}{\Delta_{IR}}\right]\,. \tag{1.2.43}$$

We include the derivation of both the Polchinski equation and the Legendre flow equation as arguments in section 4.2 utilise the fact that both of these equations contain the same information, which allows us to exploit the behaviour of the Legendre interactions in the flow of the Wilsonian interactions. In fact, the choice of cutoff operator, equation (1.2.14), was made so that within the approximations we will use the duality between the Wilsonian interactions and the Legendre interactions is unchanged [42], at least for polynomial potentials as will be shown in section 4.2.

1.2.2 Renormalisation Group Flows

With the flow equations, we can solve for the effective interactions at a particular scale k. Taking one such solution, Γ_k, we may consider the interactions as a series of linearly independent operator couplings, *i.e.* :

$$\Gamma_k[\phi] = \sum_{n=1}^{\infty} g_n(k)\psi_n[\phi]\,, \tag{1.2.44}$$

where each coupling is weighted by a 'coupling constant' g_n. Then the change of Γ_k with scale k can be thought of as a change of the coupling constants. If we consider the space spanned by coupling constants corresponding to any of all possible operators $\psi_n[\phi]$ constrained only by the our $O(N)$ & Z_2 symmetries, then any possible action Γ_k may be represented as a point in this space, which we call theory space (TS). Changing the scale k would shift the point Γ_k, representing a shift to a new action with different coupling constants. Multiple changes in our scale would then map out a curve, or ERG flow, in TS with direction of the flows dictated by decreasing k.

This picture of RG flows leads to intuitive understanding of the RG behaviour for a theory. For example one can readily see that the two ends to each flow is given by the limit where all fluctuations are integrated over, $k \to 0$, and the microscopic limit, $k \to \Lambda$. However, recalling that each starting action Γ_k was defined by integrating the momentum modes between k and Λ we can see that any flow of this kind is implicitly dependent on Λ, *i.e.* any change to the cutoff we introduced by hand fundamentally changes the flow. Such theories are ill defined. Further, in HEP the existence of some overall UV momentum scale, or equivalent lattice spacing, can break the gauge symmetries required in most models and complicates fundamental symmetries such as Lorentz invariance [88].

Particularly, scale invariance of correlation functions within QFT, and of near critical phenomena in SM, links all length scales. Therefore, unless we can explicitly remove the overall cutoff scale then any physical observable will carry non-trivial dependence upon it. Examining the flow equations carefully, we can observe this dependence within correlation functions, defined by functional derivatives of the Wilsonian or Legendre interactions. For example, we can easily imagine a case where a correlation function that depends upon some external momentum $\vec{\mathbf{p}}$ includes an unconstrained loop integral over some virtual momentum $\vec{\mathbf{q}}$, then the overall cutoff enters non-trivially by restricting the

sum of these two momenta $|\vec{\mathbf{p}} + \vec{\mathbf{q}}|$ within the loop integral [85]. This is a well known artefact of what can be considered "lattice regularisation", where we will see that the scales defined in the QFT will become proportional to Λ, leading to divergent physical quantities when if we attempt to remove the cutoff, *i.e.* we send $\Lambda \to \infty$ [88]. The root of this problem is the fact that up to this point we have not rescaled the our action at all, we have simply shifted the IR momentum scale. Without this step we cannot make use of scale invariance and therefore cannot guarantee that the addition of the UV or IR cutoff scale does not add additional scale dependence to the correlation functions. In fact, so long as the action depends upon some overall momentum cutoff scale then we can show that the physics of the system will gain additional scale dependence for each momentum scale we introduce [85]. If we consider now that the Legendre interactions could be written to depend upon two different UV cutoffs were denoted Γ_k^{Λ} and $\Gamma_k^{\dot{\Lambda}}$, then the two would be related by:

$$\Gamma_k^{\Lambda}[\phi] = \Gamma_k^{\dot{\Lambda}}[\dot{\phi}] + \frac{1}{2}(\phi - \dot{\phi}) \cdot (\Delta_{\Lambda}^{\dot{\Lambda}})^{-1} \cdot (\phi - \dot{\phi})\,, \tag{1.2.45}$$

where $\Delta_{\Lambda}^{\dot{\Lambda}}(p) = C_{\Lambda}^{\dot{\Lambda}}(p)/p^2$, with $C_{\Lambda}^{\dot{\Lambda}}(p)$ giving the difference in the overall UV cutoffs [85]. Following from the point above, we can see that if this second UV cutoff exists then from this relation we expect Γ_k to contain non-trivial dependence on both overall cutoff scales. We therefore require a solution that allows us to remove any UV cutoff scales.

The nature of the flow equations then imply that, after rescaling, RG flows corresponding to physical theories are those flows that do not depend upon the overall cutoff Λ, *i.e.* those flows where this artificial cutoff has been removed, $\Lambda \to \infty$, which is known as taking the continuum limit.[15] However, as pointed by Morris [39, 52], if the cutoff operator R_k varies sufficiently quickly then the momentum integrals in flow equations are sensitive only to momentum of order k and therefore do not depend on the overall cutoff Λ. That is, if R_k varies sufficiently fast we can directly incorporate the rescaling of the system into the RG flow itself and we may solve for flows directly from continuum limit in terms of renormalised variables. Doing so, we may rewrite our effective interactions as functions of running couplings, e.g. $g_i(k)$, and anomalous dimension, the scale derivative of the logarithmic field renormalisation $\eta(k) = -k\partial_k \ln(Z_k)$, that

[15] The continuum limit is defined in SM as taking the lattice spacing $a \to 0$, which in natural units is equivalent to taking the cutoff $\Lambda = a^{-1} \to \infty$.

depend only upon IR cutoff scale k and not the ratio k/Λ [39]:

$$\mathcal{S}_k[\Phi] = \mathcal{S}[\Phi]\,(g_1(k), \cdots, g_n(k), \eta(k)) \quad \text{and} \quad \Gamma_k[\phi] = \Gamma[\phi](g_1(k), \cdots, g_n(k), \eta(k))\,. \tag{1.2.46}$$

This defines self-similar RG flow that then depends only upon the IR cutoff scale k.

We can then consider the special case of an action, $\Gamma_k = \Gamma_*$, that does not flow with k, a fixed point (FP) action. Such actions represents the case where the action does not change with the addition of fluctuations at any scale. There is however one caveat we must consider about this. Unless we work in dimensionless variables then it is possible for certain actions to appear similar to a fixed point after field and operator redefinitions. This problem is essentially an artefact of improper rescaling of the system to appear like the original one due to inherent scale dependence of physical variables. That is, we *must* use dimensionless variables to preserve self-similarity of RG flows in order to find a fixed point.

As such, we now introduce a dimensionless scale, known as RG time so that we may work in dimensionless variables later:[16]

$$t = \ln(\mu/k)\,, \tag{1.2.47}$$

for some arbitrary momentum scale μ that, as we will see below, links the dimensionless RG flow to the physical scale of the real world. This will also allow us to change the description of an action changing with the inclusion of more vacuum fluctuations, decreasing the scale k, to be RG flow of an action with increasing "time". For all other quantities we define dimensionless quantities by scaling with powers of the IR cutoff scale k and we introduce a bar notation to denote that they are dimensionless, e.g the dimensionless form of our scalar field is given by:

$$[\varphi] = k^{\frac{1}{2}(d-2)} \quad \Longrightarrow \quad \bar{\varphi} = k^{-\frac{1}{2}(d-2)}\varphi\,, \tag{1.2.48}$$

where we denote the dimension of an operator χ with square brackets $[\chi]$.

Redefining all variables to be dimensionless using the IR cutoff scale k we can now say that the *only* scale within an action is k. Therefore, an action that is invariant

[16] The author postulates that this nomenclature likely derives from the comparison of RG flows with evolution equations.

under a change of the scale k, a FP action, is scale invariant. The scale invariance of the FP implies that it is independent of the overall cutoff Λ, which means we may trivially take the continuum limit. On the other hand, if we consider introducing a mass of the particle we also introduce a mass scale to theory, therefore a FP action must represent a massless continuum limit. However, from the Wilsonian RG point of view, each step along a RG flow represents an integration over some momentum shell to rewrite the theory and steadily reduce all scales in theory from $\Lambda \to 0$. Therefore, characteristic length scales of the theory, specifically the correlation length ξ, must be rescaled with each RG step. However, applying this to a FP action we find that means this length scale cannot change. This can only be true if $\xi = 0$, which will lead to strictly non-interacting physics, or $\xi = \infty$ [88]. As such, we immediately identify that these fixed point actions are not only massless continuum limits but are in fact critical points of the system where the correlation length diverges.

However, the physical world is not described by massless theories. Therefore, a physical QFT that describes particle physics cannot be given by a FP action. Thus we consider the theory space near a fixed point action Γ_*, *i.e.* an action Γ_k given by:

$$\Gamma_k[\phi, t] = \Gamma_*[\phi] + \varepsilon\delta\Gamma[\phi, t]\,, \tag{1.2.49}$$

where ε is dimensionless and strictly infinitesimal. We can exploit the small parameter ε and linearise the Legendre flow equation (or equivalently the Polchinski equation with an appropriate perturbation of the Wilsonian interactions). Now, we want to construct actions that can use the scale invariance of the fixed point. Therefore at some value of RG time, particularly the IR and UV, or in the limit $\varepsilon \to 0$ the solution must fall back into fixed point for all values of the field. This can only be done if the action $\delta\Gamma_k$ is separable:

$$\delta\Gamma[\phi, t] = \mathcal{F}[\phi]f(t)\,, \tag{1.2.50}$$

for some functional that depends only on our field $\mathcal{F}$ and some RG time dependent function $f(t)$. We can guess the scale dependence of the function $f(t)$ by again using the fact that our solution must fall back into fixed point. If we consider an infinitesimal RG transformation, *i.e.* a change of the scale t, denoted by $\mathcal{R}$ applied to our perturbation around the fixed point, equation (1.2.49), then the fact that the perturbation linearises

the flow equation implies that:

$$\mathcal{R}(\Gamma_*[\phi] + \varepsilon\delta\Gamma_k[\phi,t]) = \Gamma_* + \varepsilon\left(\delta\Gamma_k[\phi,t] + \delta t\ \frac{\partial}{\partial t}\delta\Gamma_k[\phi,t]\right)\,, \tag{1.2.51}$$

where we have used the fact that the transformation is infinitesimal to linearly interpolate from $t \to t + \delta t$. If we now impose the condition that this result must fall back into the fixed point when the perturbation goes to zero, then the scale invariance of the fixed point implies that [7]:

$$\lim_{\delta\Gamma_k\to 0}\left(\delta\Gamma_k[\phi,t] + \delta t\ \frac{\partial}{\partial t}\delta\Gamma_k[\phi,t]\right) = 0\,. \tag{1.2.52}$$

This implies that $\partial_t\delta\Gamma_k[\phi,t]$ must be at least proportional to $\delta\Gamma_k$. More specifically, we find that the perturbations satisfy an eigenvalue equation:

$$\partial_t\delta\Gamma_k[\phi,t] = \lambda\delta\Gamma_k[\phi,t]\,, \tag{1.2.53}$$

with some eigenvalue λ. Given the fact that the perturbation is separable we then find that:

$$\delta\Gamma_k[\phi,t] = \Gamma_\lambda[\phi;t] = \mathcal{F}_\lambda[\phi]\exp(\lambda t)\,, \tag{1.2.54}$$

which leads the interpretation that the RG flows that are controlled by the fixed point, those that fall into the fixed point as the perturbation vanishes, are given by the eigenoperators $\mathcal{F}_\lambda$ with corresponding "couplings" given by $\varepsilon\exp(\lambda t)$ [7, 39, 56, 84].

From the scaling, $\exp(\lambda t)$, we can see that if $\lambda < 0$ then the perturbation Γ_λ flows towards the fixed point as we include more fluctuations, we decrease $k \to 0$ ($t \to \infty$). These describe the irrelevant scaling operators or, in terms of flows, irrelevant directions. Equally if $\lambda > 0$ then Γ_λ grows away from the fixed point as $k \to 0$, which describe the relevant scaling operators or relevant directions[17]. An example of such flows are shown in figure 1.2.1. If $\lambda = 0$ then the operators are called marginal and we need to go beyond the linearised solution, using ε^2 corrections to the flow equation, to see if the corresponding operators are (marginally) relevant or (marginally) irrelevant. There is one more possible flow near the FP, so called limit cycles. These operators are cyclical and neither flow into or out of the FP, however these are beyond the scope of this thesis

[17] These are in fact the same relevant/irrelevant scaling variables discussed above, however we will make the link between RG relevant/irrelevant directions and the corresponding critical behaviour below.

and we will not cover them any further.

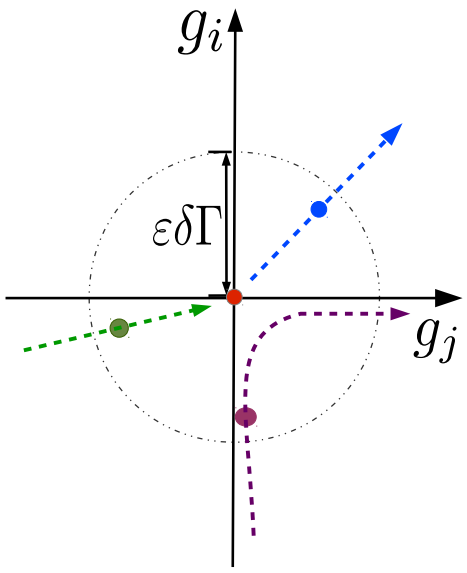

Figure 1.2.1: Qualitative example of RG flow in a two dimensional plane in theory space close to the fixed point, shown by the red circle, parametrised by the two couplings corresponding to the eigenoperators $\mathcal{F}_i$ and $\mathcal{F}_j$. The dashed circle represents the region near the fixed point given by any small perturbation, $\varepsilon\delta\Gamma_k \ll 1$. Initial RG flow points are shown in by bold circles and the RG flow direction, given by decreasing k (or increasing t), are shown by the direction of arrows. In the green is shown the case where both g_i and g_j are irrelevant, therefore a sum of the two will flow into the fixed point. In the blue is shown the case where they are both relevant, therefore the sum of the two will flow away from the fixed point in both the g_i and the g_j direction. In the violet is shown the case where g_i is irrelevant and g_j is relevant. In this final case the flow will begin going towards the fixed point, following the irrelevant variable, before rapidly shooting away from the fixed point towards large g_j.

Using the fact that these eigenoperators are linearly independent solutions to our flow equation around the FP one can construct an action as a sum of these eigenoperators:

$$\Gamma[\phi;t] = \Gamma_* + \sum_\lambda C_\lambda \Gamma_\lambda[\phi;t], \tag{1.2.55}$$

for some constants c_λ. If this action is constructed of only the relevant eigenoperators then in the UV, specifically at the continuum limit, $k \to \infty$ ($t \to -\infty$) the action will flow towards the original FP, *cf.* the flow shown in blue in figure 1.2.1. As such, the action is uniquely defined by the FP and the corresponding coupling constants, weightings of each linearly independent eigenoperators. Thus, such an action is not dependent upon our artificial cutoff Λ even though the action caries scale dependence

in k.

In general, however, the action describing some system will also include irrelevant operators, a key point that historic SM work did not take into account before the introduction of RG. If we assume that eigenoperators remain small enough that we remain in the linear regime then from the scaling above, we can intuitively see that by renormalising a general action, near the fixed point, made up of all relevant and irrelevant eigenoperators it will tend towards:

$$\lim_{t\to\infty}(\Gamma[\phi;t]) = \Gamma_* + \sum_{\text{relevant}} C_\lambda \Gamma_\lambda[\phi; t=\infty]\,, \tag{1.2.56}$$

cf. the violet flow shown in figure 1.2.1. If we again identify the FP action as a critical point, this is a statement of universal critical behaviour.

To make this more clear, consider the link with SM. Then the couplings $C_\lambda \varepsilon \exp(\lambda t)$ would be identified as dimensionless or reduced thermodynamic variables measuring the deviation from the critical point. Then we can think of shift in theory space around a FP as a change of the thermodynamic variables. While in HEP the couplings will be kept constant, in SM they are variables that can be modified in the lab. If we consider then those thermodynamic variables that do not effect critical behaviour, given by the irrelevant directions, are changed to be non-zero then this is equivalent to shifting the microscopic action in theory space in those irrelevant directions. Then the RG behaviour shows that the action will flow into the fixed point, *i.e.* the deviation from irrelevant variables from the fixed point value is *suppressed*, showing that for any value of the irrelevant thermodynamic variables we get the same critical behaviour. On the other hand, we find that for any non-zero value of a relevant thermodynamic variable, the RG flow *enhances* the deviation from the critical point. This property of the RG then naturally shows how the enhancement or suppression particular thermodynamic variables leads to universal critical behaviour, with each universality class governed by an RG fixed point and its corresponding eigenoperators.

This can be simplified by a diagrammatic approach to these flows. Consider the vector space of all actions defined by any linear sum of irrelevant directions, restricted only by the requirement that they solve the linearised flow regime. Now, the flow of any these actions will be represented by a continuous line pointed directly towards the critical point, which we use to define the origin. These flows can then be considered

to define a continuous manifold, specifically part of a manifold known as the critical manifold[18], that can interpreted as, similar to how Fisher describes it [7], the "domain of attraction" of this critical point, this is shown in figure 1.2.2. Now, if we consider each point on the critical manifold displaced in the direction of one relevant direction by a small amount, then following the flow of each of these points will now no longer lead directly into the critical point, instead we see that with each step the flow enhances the the deviation in the relevant direction. If we follow the lines of the flow, this will appear as a distortion of once flat domain of attraction manifold away from the fixed point in the form similar to the addition of curvature to the manifold. Following the flows further we see this distortion towards the axis defined by the relevant direction is enhanced to the point that the relevant direction defines a singularity in "domain of attraction" manifold at the origin. That is, the enhancement of relevant thermodynamic variables by RG flow implies that as soon as a relevant direction is "switched on" it destroys the attraction towards fixed points, it is only these variables that can obstruct or destroy universal critical behaviour.

The picture of RG flows then gives a natural explanation to universal critical behaviour and why only a subset of variables can lead to a critical point. With this link, we may use the RG to find the FP that defines a universality class for the appropriate order parameter in the appropriate number of dimensions. Then, the corresponding relevant/irrelevant critical exponents are related to the relevant/irrelevant eigenvalues and eigenoperators corresponding to the FP.

One final piece of SM behaviour that can be naturally explained with RG flow is phase changes and the phase interface of a system. Consider the case where more than one relevant direction exists, *e.g.* $\hat{t}$ and $\hat{h}$, and one relevant direction, $\hat{t}$, can flow in opposite directions from the physical critical point. Physically, this is the case where in physical dimensions the coupling can be tuned to either a positive or negative scale, *i.e.* the constant multiplicative constant C_λ in equation (1.2.56) can be either postive or negative. Then the critical manifold of irrelevant directions defines a manifold in theory space dividing physical phases of the system. Consider starting on the critical surface, this action will have no effect on critical behaviour. Moving the starting action off this surface in either the positive or negative $\hat{t}$ direction, then the flow will be dominated by

[18]Strictly speaking the critical manifold is given by all actions that flow into the fixed point, including non-eigenoperator solutions outside the linear regime.

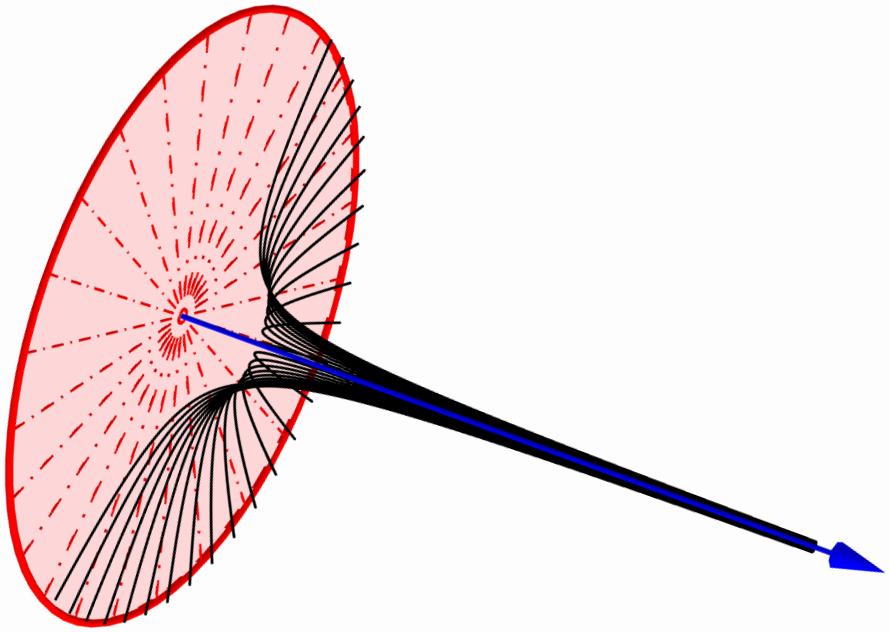

Figure 1.2.2: A qualitative example RG flows near the critical manifold. The critical manifold in this example is shown by the red circle, with a few examples of flows of irrelevant variables shown in the dashed red lines. This manifold can be interpreted as the "domain of attraction" of a FP, any action that begins on the critical manifold will flow towards the fixed point. Examples of the flow for actions that start just off the critical manifold in the direction of a relevant direction, shown as the blue arrow, are shown in black. We see the suppression of irrelevant variables as flow directing towards the fixed point, parallel to the critical manifold. Then, the relevant direction introduces what can be interpreted as an instability in the attraction towards the fixed point that in turn causes any subsequent flow to tend asymptotically far away from the FP in the relevant direction.

this one relevant direction. Specifically, the RG flow will magnify the distance from the critical surface in either the positive $\hat{t}$ direction *or* the negative $\hat{t}$ direction only. This in essence describes how the enhancement of a small deviation from the critical point can lead two completely distinct final states. An example of this RG flow behaviour is shown in figure 1.2.3. Then, if the RG flow leads to the same distinct phases described above for a small range of the second relevant direction, $0 < h < \hat{h}$, then we continue to describe a two phase system depending on whether the initial value of $t < 0$ or $t > 0$. We then find the limiting value of the second relevant variable, $\hat{h}$, will then define the phase boundary . Generalised to any number of relevant variables we can say that different phases of the system are divided in theory space by the critical manifold and

each phase has a boundary defined by the limiting values of the other relevant variables, beyond which the RG flow leads final states not controlled by the critical behaviour of that FP.

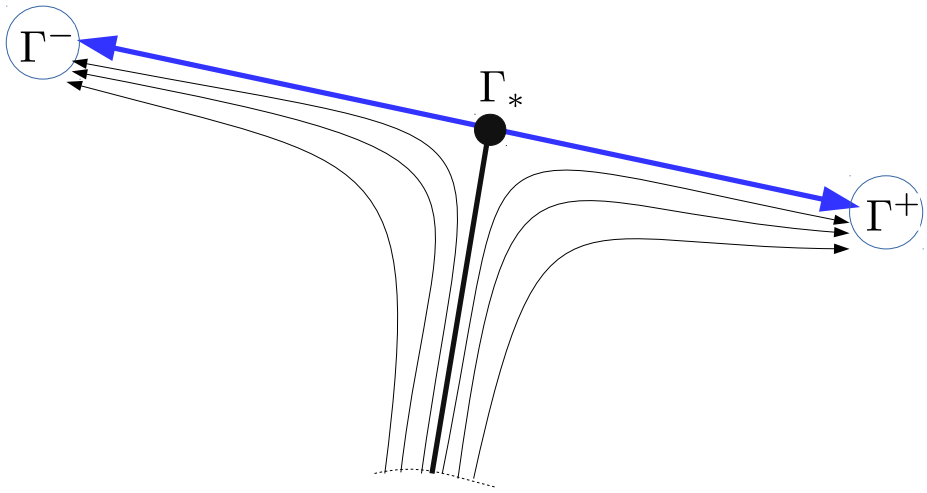

Figure 1.2.3: Schematic example of RG flow picture of phase changes. Here the theory space near the fixed point is shown as the critical manifold, shown as the black line, leading to fixed point, labelled as Γ_*, with one relevant direction, shown in the blue, that can take positive or negative values. Any action on the critical manifold will flow towards the fixed point Γ_*. However, if we consider an action shifted in either the positive or negative direction of one relevant variable away from the critical manifold, equivalently shifting a relevant thermodynamic variable off the critical value, the RG flow leads to one of two distinct final states, *i.e.* different physical phases, shown in blue [7]. In terms of a quantum field theory, this is the equivalent of tuning the dimensionless coupling $g_\lambda = \varepsilon \left(\frac{\mu}{k}\right)^\lambda$ to either a positive or negative physical scale μ, where *either scale defines a different physical theory.*

1.2.3 Renormalisation in Quantum Field Theory

Up to this point we have qualitatively derived many aspects of critical phenomena in SM and linked them to RG behaviour and RG flows. However, calculations in the

realm of HEP differ from those in SM by two key points. Firstly, we assume in general that we are doing zero temperature calculations in the vacuum. More specifically, we treat particles as as quantised vacuum excitations at zero temperature that may interact for only a very small period of time. Secondly, as a consequence of the first point, physical quantities are measured at fixed bare coupling, that is the quantities g_k at the UV scale $k = \Lambda$ are not variables of a universe sized laboratory and are instead set by fundamental properties of the universe or physical observables, *e.g.* they represent particle masses or fundamental gauge group coupling constants. As such, tuning values towards or away from criticality no longer has any physical meaning.

Equally, we are not at liberty to choose some starting action in the theory space unless we wish to concede the fact that there exists an infinite number of possible couplings as free parameters that must then be constrained via experimental measurements. It is this point in particular that leads to the conditions for non-perturbative renormalisability of a QFT derived from RG behaviour. In HEP, we consider a QFT renormalisable if and only if a finite number of parameters must be tuned to obtain finite particle masses and couplings. A trivial solution to this condition is given by FP actions, where each parameter in the system is fixed but all masses are zero.

However, recalling the added complication of non-trivial cutoff dependence for any non-fixed RG flow and the scale dependence of physical, instead of dimensionless, parameters means that any non-fixed action described in theory space will pick up dependence on the overall cutoff scale Λ. This will, as we mentioned above, not only breaks many of the symmetries we wish to preserve in a physical QFT but will lead to infinite masses if we are not careful in taking the continuum limit. As such, we may conclude that a physical, or renormalisable, theory must be controlled by, but not equal to, a FP action, such that the we may remove the artificial scale dependence introduced by Λ and we can take the required continuum limit. Therefore, if we then wish to construct a massive QFT with a finite set of parameters still controlled by the FP we require an action based upon eigenoperators of the FP, specifically the relevant and marginally relevant eigenoperators.

Consider again the critical manifold of fixed point, this represents all actions that give a massless continuum limit, as each action flows into a FP. If we shift a bare action from this manifold a small distance off this manifold then the RG flow of this action will, to begin with, move directly towards the FP parallel to the critical manifold, but

then "shoot away", as Morris puts it [39], along that relevant direction towards an infinitely massive QFT (in the limit $\Lambda \to \infty$),*cf.* the flows off the critical manifold in figure 1.2.2.[19] In order to obtain a finite mass in the continuum limit we must then tune the starting bare action back onto the critical manifold and at the same time, renormalise physical quantities appropriately for the diverging correlation length [39] and tune it to a physical scale, such as a particle mass, found by experiment [88]. In the limit that the bare action returns to the critical manifold the flow will split into two parts, the first will flow directly into the fixed point and the second part, known as the renormalised trajectory, will emanate straight out of the fixed point following the relevant direction. Particularly, since we know that this renormalised trajectory is linked to the FP through successive RG steps, the continuum limit must be controlled by the FP. Therefore, as a result, cannot have artefacts from the overall cutoff. Thus, a properly tuned renormalised trajectories is strictly finite and the number of free parameters is restricted by the FP and the tuned couplings of the renormalised trajectory.

At this point we must consider an important subtlety of the full RG flow of eigenoperators. Near the FP, specifically in the region where we may linearise the flow equations, the scaling behaviour is strictly given by the eigenvalues of the eigenoperators. Therefore, near the FP we can construct our renormalised QFT on the renormalised trajectory in terms of the n relevant and marginally-relevant eigenoperators tuned by some rates α, *e.g.* [39]:

$$\Gamma_k[\phi] = \Gamma_* + \sum_{i=1}^{n} \alpha_i \left(\frac{\mu}{k}\right)^{\lambda_i} \mathcal{F}_\lambda[\phi]\,. \tag{1.2.57}$$

However, the full flow equation is non-linear. Thus, in general the renormalised trajectory cannot be made of sum of all the relevant eigenoperators. Particularly, for any eigenoperator that is not constant in our fields there exists a value at for the field at which our perturbation around the fixed point is no longer small, *i.e.* $\delta\Gamma_k \sim \mathcal{O}(1)$, for any value of k and the expansion of the flow equations in powers of ε no longer converges. Therefore, to construct a true renormalised trajectory we must consider the matching of linearised solutions to the full non-linear solution for each relevant eigenoperator. Specifically, the RG flow of relevant eigenoperator must correctly to the

[19] We know that the QFT must have infinite mass as no part of the flow controls the overall cutoff dependence, which introduces the infinite mass as we take the continuum limit $\Lambda \to \infty$.

FP UV for all values of the field, *i.e.* in both the linear and non-linear regime, in order to protect the continuum limit of the renormalised trajectory from artefacts introduced by our regularisation. Then, the true renormalised trajectory will be given by tuning only those relevant solutions that we can construct the RG flow from UV to the IR.

This subtlety to renormalisation is pivotal to the conclusions of this thesis, particularly that the scaling behaviour of non-polynomial potentials do not have the RG behaviour required to take the continuum limit, which we will demonstrate in section 3.3 and 3.4, and that as we decrease the floating cutoff k the non-polynomial directions collapse into a polynomial renormalised trajectory, which we will demonstrate in chapter 4. These two conclusions are drawn upon t behaviour relative to a specific initial value, implying that *the full RG behaviour of the non-polynomial directions is not self-similar* and is, in fact, dependent upon a scale other than k. Therefore, they cannot be used to form a QFT with a finite number of free parameters and can only give finite masses and observables with an infinite number of counter-terms.

However, before discussing eigenoperators, relevance of the eigenoperators and their completeness we must first overcome the fact that the flow equations are non-linear second order functional equations to find the FP Γ_*. We therefore move to motivate the approximation we use to solve the flow equations in the next chapter before we discussing the solutions in the linear regime in chapter 3.

Chapter 2

Derivative Expansion for $O(N)$ Scalar Field Theory

In the previous chapter we have seen how the picture of RG flows in a theory space of possible actions leads intuitively to universal critical phenomenon and how the eigenoperators of a fixed point link both critical behaviour and renormalisability in QFT. Of particular importance to HEP, according the RG a physical QFT is directly related to the relevant eigenoperator perturbations of a fixed point action. With this introduction in mind, we now move on to the considerations of finding such fixed point solutions and their corresponding eigenoperators. However, to do so we must find fixed solutions to RG flow equations and we quickly come across problems in almost all of the applicable cases in HEP we wish to study using the ERG, specifically cases regarding strong couplings or those where the form of the bare action is not known, thereby immediately forbidding the use of traditional perturbation theory.

Particularly, the full non-perturbative flow equations, equations (1.2.27) and (1.2.43), are second order non-linear functional differential equations. From a technical point of view, we may interpret the integrals within the functionals as infinite sums over the field configurations and instead consider the flow equations as closed sets containing an infinite number of coupled non-linear partial differential equations. Therefore, solutions to these equations require either a very particular ansatz or some level of approximation. Aside from the case of the effective actions being equal to an overall vacuum constant, *i.e.* a trivial theory with no interactions, there are no known analytical solutions to the flow equations without prior approximation.

Of particular note, for an $O(N)$ symmetric scalar field the $\epsilon = 4-d$ expansion [56,89] leads to fairly accurate numerical results and the $N \to \infty$ limit of $1/N$ expansions [90–92] leads to exact solutions of the flow equation. Taking functional derivatives of the flow equations to construct flow equations of effective vertices, in the case of $\frac{\delta^n \mathcal{S}_k[\Phi]}{\delta \Phi^n}$, or fully renormalised vertices, in the case of $\frac{\delta^n \Gamma_k[\phi]}{\delta \phi^n}$, we see from the flow equations that the flow of each vertex is then related to each other m-point vertex up to $m = n + 2$. This is simply a restatement of the fact that each differential equation encoded by the flow equations is coupled to an infinite number of other differential equations. Using a small parameter such as $1/N$ as a suitable control parameters one may rewrite these flow equations in a 'perturbative' series so that the number coupled equations, the or equivalently the number of coupled terms, is reduced by powers of the control parameter. As such, to each order of the control parameter we need only solve a finite number of coupled equations as long as the flow equation converges with each order the control parameter. In the extreme limit where the control parameter tends to zero we would then expect the number of coupled terms to reach a minimum and in the case of $1/N$ expansions we obtain directly soluble system [92]. Additionally, unlike the traditional perturbation theory in QFT in terms of small coupling constants, these expansions can be done in terms of exact flows directly and therefore still study strong coupling theories inaccesible by traditional perturbation theory, albeit restricted such that large couplings do not break the convergence of the series expansion with respect to the control parameter.

Aside from these direct expansions in small quantities the only other way to control the complexity of the flow equations is to explicitly expand the functional form of either the effective actions or the flow equations in terms of operators invariant under the symmetries of the effective actions and then truncate this expansion to some order. Specifically, we assume either the flow equation or the effective action takes the form of a sum of these possible invariants times a scale and momentum dependent coefficient function and expand the flows in terms of these coefficient functions. Due to the nature of these expansions we will refer to them as functional truncations, rather than series expansions. In comparison to the small parameter expansions, many of these functional truncations do not converge to the known results from either the large N expansion or from experimental measurements of critical exponents. For example, the truncation of effective actions in operator invariants as originally proposed by Golner and Riedel

in their scaling field method [93–95], or equivalently expansion in polynomial powers of the field [50, 52, 96, 97], does not converge beyond a certain order of the truncation. The nature of such expansions, truncations in powers of fields around some classical field value, implies that the fields do not fluctuation greatly. As such, these examples of functional truncation will lead to results similar to mean field theory, which implies they cannot recover the correct critical behaviour. Particularly, it can be shown that truncations in powers of the field lead to additional spurious fixed points that should not be found [50, 52].

Of particular note, functional truncations are not expansions in a small parameter, the reduction in the number of coupled differential equations for the coefficient functions is an artificial imposition. Therefore, the convergence of the truncation to flow equation is not guaranteed and rigorous testing must be done with each additional invariant included to avoid inaccurate or spurious results. It is this point that motivates the following study based upon the functional truncation in terms of derivative operators, the derivative expansion of the ERG. This method has been the centre point for recent research regarding scalar field theory, specifically for possible application to asymptotic safety and quantum gravity [19–38].

The derivative expansion uses an anstatz for the effective actions as a series of all linearly independent differential operators, specifically under integration by parts, that do not break the symmetries of our field or the effective actions $\mathcal{S}_k$ and Γ_k [42,43,46,50,98]. It is particularly important that this approximation and any others applied afterwards preserve *all symmetries* of the effective actions, including accidental ones, or spurious RG behaviour and unphysical results will be introduced, as we found in collaboration with Dietz and Morris with regards to the so called single field approximation [99]. For an $O(N)$ scalar field this restricts each term to include only even powers of the field, with each field index summed over, and due to global Lorentz symmetry each term must equally contain an even number of derivatives with no hanging lorentz indices. This expansion was first derived from the case of vanishing external momenta in SM systems, *e.g.* [44], as in that case the derivative operators would allow the series expansion to be written as a ratio of vanishingly small external momenta divided by some characteristic momentum scale to give a convergent series with a well defined expansion parameter. However, this is not always the case and in many cases beyond scalar field theory it is impossible to impose that external momentum is small and one would

expect that the characteristic momenta within loop corrections to grow large enough that higher order terms of the expansion should become relevant if they exist. It is for these reasons that functional truncations must be verified at each order. However, these concerns are beyond the scope of this thesis and we continue by concentrating solely on the leading terms of the derivative expansion.

In general, the derivative expansion gives the Legendre effective action as:

$$\begin{aligned}\Gamma_k^{eff}[\phi,\varphi_<]-&\frac{1}{2}(\phi-\varphi_<)^a\cdot \mathrm{R}_k\cdot(\phi-\varphi_<)^a \\ &=\int \mathrm{d}^d\vec{\mathbf{x}}\ \mathrm{V}(\rho,t)+\frac{1}{2}\mathrm{Z}_k(\rho)(\partial_\mu(\phi^a-\varphi_<^a))^2+\frac{1}{4}\mathrm{Y}_k(\rho)(\partial_\mu\rho)^2+\mathcal{O}\left(\partial^4\right),\end{aligned} \tag{2.0.1}$$

where $\rho=\frac{\phi^a\phi^a}{2}$, V is the effective average potential, or Legendre (effective) potential, for $N>1$ field components we get a second field renormalisation constant Y_k, and we use $\mathcal{O}\left(\partial^4\right)$ as short hand to denote all terms with four or more derivates. The appearance of the low-momentum fields in this equation is due to the definition of the kinetic term of Γ_k^{eff} in equation (1.2.40), however we note that the higher order derivative terms and terms with powers of φ^4 or higher must be independent on the low-momentum fields due to the fact that the effective interactions are independent of them, $\frac{\delta\Gamma_k[\phi]}{\delta\varphi_<}=0$. The Wilsonian effective action then takes a similar form given by:

$$\mathcal{S}_k^{eff}[\Phi]=\frac{1}{2}\Phi^a\cdot\Delta_{UV}^{-1}\cdot\Phi^a+\int \mathrm{d}^d\vec{\mathbf{x}}\ \mathrm{U}(\sigma)+\frac{1}{4}\mathrm{Y}_k(\sigma)(\partial_\mu\sigma)^2+\mathcal{O}\left(\partial^4\right), \tag{2.0.2}$$

where $\sigma=\frac{\Phi^a\Phi^a}{2}$, U is the Wilsonian (effective) potential, the field renormalisation factors that depend on Z_k are within the propagator Δ_{UV}, and again we use $\mathcal{O}\left(\partial^4\right)$ as short hand to denote all terms with four or more derivatives.

Finally to find physical fixed point solution we require the RG flow to be analytic away from the fixed point, as discussed in chapter 1. Therefore the potentials, U and V, each of the field renormalisation functions, Z_k and Y_k, are required to be analytic for all σ and ρ and the field renormalisation $\mathrm{Z}_k\neq 0$ so that we do not introduce an IR singularity into renormalised propagators [50].

As discussed above, the convergence of expansions of this kind is not guaranteed, even perturbatively. [43, 50, 51]. However, the order of the truncation applied here has been shown to reproduce the exact results from the large N expansion [51, 92] and has been used in recent and historical work in calculating critical exponents in

SM [43–45, 50]. We can therefore adequately avoid the issue of convergence and apply the following in terms of field theoretic results and understanding of the RG itself as applied to scalar field theory. Of more importance to us is that the particular form of solutions becomes heavily dependent upon the cutoff used [46]. However, the leading order term for the flow equations, the so called local potential approximation (LPA) that the work of this thesis is based on, is independent of the cutoff. Further, our specific choice of cutoff operator ensures that the Legendre transform between the Wilson effective action and the Legendre effective action, equation (1.2.41), remains exact [42].

Before deriving the LPA we first remind the reader that in order find a sensible fixed point solution we must first shift to dimensionless variables so that the scale dependence of the flow equations is in terms of k only. This can be done explicitly by absorbing the scaling dimension of each of our variables in terms of k to some power [40, 43, 50][1]:

$$\begin{aligned} \mathrm{V} \to \bar{\mathrm{V}} = k^{-d}\mathrm{V}\,, \qquad & \phi \to \bar{\phi} = k^{-\frac{1}{2}(d-2+\eta)}\phi\,, \qquad & \rho \to \bar{\rho} = k^{-(d-2+\eta)}\rho\,, \\ \mathrm{U} \to \bar{\mathrm{U}} = k^{-d}\mathrm{U}\,, \qquad & \Phi \to \bar{\Phi} = k^{-\frac{1}{2}(d-2+\eta)}\Phi\,, \qquad & \sigma \to \bar{\sigma} = k^{-(d-2+\eta)}\sigma\,, \\ \mathrm{Z}_k \to \bar{\mathrm{Z}}_k = k^{+\eta}\mathrm{Z}_k\,, \qquad & \mathrm{Y}_k \to \bar{\mathrm{Y}}_k = k^{d-2+\eta}\mathrm{Y}_k\,, & \end{aligned} \tag{2.0.3a}$$

and finally we also need to rescale all momentum variables:

$$\vec{\mathbf{q}} \to \mathbf{q} = k^{-1}\vec{\mathbf{q}}\,. \tag{2.0.3b}$$

If we change the scale derivative in our flow equations to be dimensionless through the change $\ln(\mu/k) \to t$, we can rewrite our flow equations as:

$$(\,\partial_t|_\phi + \frac{1}{2}(d-2-\eta)\Delta_\phi + \Delta_\partial|_\phi)\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\left[\hat{\delta}_{ab} + \Delta_{IR}\cdot\frac{\delta^2\Gamma_k[\phi]}{\delta\phi^a\delta\phi^b}\right]^{-1}\cdot\frac{\partial_k\Delta_{IR}^{-1}}{\Delta_{IR}}\right], \tag{2.0.4}$$

where $\Delta_\phi = \phi.\partial_\phi$ is the field counting operator, which counts the number of fields in at any given vertex, and Δ_∂ is the derivative counting operator, which can be represented

[1] We remind the reader that we denote dimensionless quantities with a bar, *i.e.* $\bar{\sigma}$ is the dimensionless form of σ. This is not done for vectors to avoid confusion with vector notation.

by:

$$\Delta_{\partial}|_{\phi} = \int \frac{\mathrm{d}^d \vec{\mathbf{q}}}{(2\pi)^d}\ \tilde{\phi}(\vec{\mathbf{q}}) q^{\mu} \frac{\partial}{\partial q^{\mu}} \frac{\delta}{\delta \tilde{\phi}(\vec{\mathbf{q}})}. \tag{2.0.5}$$

Equally the Polchinski equation can be written as:

$$(\partial_t|_{\Phi} + \frac{1}{2}(d-2-\eta)\Delta_{\Phi} + \Delta_{\partial}|_{\Phi})\mathcal{S}_k[\Phi] = \frac{1}{2}\frac{\delta \mathcal{S}_k}{\delta \Phi^a} \cdot \frac{\partial \Delta_{UV}}{\partial k} \cdot \frac{\delta \mathcal{S}_k}{\delta \Phi^a} - \frac{1}{2}\mathrm{Tr}\left[\frac{\partial \Delta_{UV}}{\partial k} \cdot \frac{\delta^2 \mathcal{S}_k}{\delta \Phi \delta \Phi}\right], \tag{2.0.6}$$

with equivalent field and derivative counting operators now in terms of Φ.

2.1 Local Potential Approximation

The leading term for the derivative expansion will then be given by expanding the effective actions in derivative operators, equations (2.0.1) and (2.0.2), and then removing all terms with derivatives. While the LPA can be derived by removing the derivative terms in our ansatz, equations (2.0.1) and (2.0.2), the same result can be obtained directly from the flow equations, equations (2.0.4) and (2.0.6), by removing the non-linear derivative terms on the right hand side [50]. Expanding both sides of the flow equations in derivative operators and truncating the right hand side to order ∂^0, we find that the right hand side of the flow equations become independent of Z_k and Y_k. The left hand side of the flow equation then states that the entirety of the flow of these quantities comes from their scaling dimension, that is in physical variables they do not scale with k and in fact follow mean field evolution. From this we find:

$$\partial_t \bar{\mathrm{Z}}_k(\bar{\rho}, t) + (d-2+\eta)\bar{\rho}\partial_{\bar{\rho}}\bar{\mathrm{Z}}_k(\bar{\rho}, t) + \eta \bar{\mathrm{Z}}_k(\bar{\rho}, t) = 0\,, \tag{2.1.1}$$

and

$$\partial_t \bar{\mathrm{Y}}_k(\bar{\rho}, t) + (d-2+\eta)\bar{\rho}\partial_{\bar{\rho}}\bar{\mathrm{Y}}_k(\bar{\rho}, t) + (d-2-\eta)\bar{\mathrm{Y}}_k(\bar{\rho}, t) = 0\,. \tag{2.1.2}$$

At a fixed point, setting $\partial_t \bar{\mathrm{Z}}_k(\bar{\rho}, t) = \partial_t \bar{\mathrm{Y}}_k(\bar{\rho}, t) = 0$, we find that the canonical scaling of Z_k implies that:

$$\mathrm{Z}_k(\rho) \propto \rho^{\frac{-\eta}{d-2+\eta}}\,, \tag{2.1.3}$$

which implies that for Z_k to be non-zero and analytic at $\bar{\rho} = 0$, to avoid IR divergences, we must set $\eta = 0$. To preserve canonical normalisation this implies that we must set the constant of proportionality to unity, *i.e.* $\mathrm{Z}_k = 1$. This condition then implies that:

$$\mathrm{Y}_k(\rho) \propto \rho^{-1} \,. \tag{2.1.4}$$

Therefore, to obtain an analytic result for all ρ we must set the constant of proportionality to zero, *i.e.* $\mathrm{Y}_k = 0$. We then find the same can be said for any operator of higher powers of derivatives in the potentials. This is essentially the statement that when we exclude the derivative terms from the right hand side of the flow equations the interacting part of the effective actions is given by a local potential function, U or V, and all higher order derivative terms do not follow canonical mass scaling, which is to be expected as they then have negative mass dimension. This is the local potential approximation (LPA):

$$\mathcal{S}_k[\Phi] \overset{LPA}{=} \int \frac{\mathrm{d}^d x}{(2\pi)^d} \;\; \mathrm{U}(\sigma, t) \,, \tag{2.1.5}$$

$$\Gamma_k[\varphi] \overset{LPA}{=} \int \frac{\mathrm{d}^d x}{(2\pi)^d} \;\; \mathrm{V}(\rho, t) \,. \tag{2.1.6}$$

2.2 Flow Equations of the Effective Potentials

With this we have now simplified the problem from non-linear functional differential equations to non-linear second order partial differential equations for the effective potentials. The left hand side of both flow equations are determined by the scaling of our dimensionless variables, equation (2.0.3), which gives us:

$$\partial_t \bar{\mathrm{V}}(\bar{\rho}, t)\Big|_\rho + (d-2)\bar{\rho}\partial_{\bar{\rho}}\bar{\mathrm{V}}(\bar{\rho}, t) - d\bar{\mathrm{V}}(\bar{\rho}, t) = k^{-d}\partial_t \Gamma_k[\phi] \,, \tag{2.2.1}$$

and

$$\partial_t \bar{\mathrm{U}}(\bar{\sigma}, t)\Big|_\sigma + (d-2)\bar{\sigma}\partial_{\bar{\sigma}}\bar{\mathrm{U}}(\bar{\sigma}, t) - d\bar{\mathrm{U}}(\bar{\sigma}, t) = k^{-d}\partial_t \mathcal{S}_k[\Phi] \,, \tag{2.2.2}$$

respectively. For the Polchinski equation, equation (2.0.6), there is no matrix inversion to complicate the calculation of the remaining side of the flow equation. In fact, when we insert our LPA ansatz, equation (2.1.5), and perform the momentum integration

and sum over field indices we find in dimensionless variables, regardless of the cutoff used:

$$\partial_t \bar{\mathrm{U}} + (d-2)\bar{\sigma}\partial_{\bar{\sigma}}\bar{\mathrm{U}} - d\bar{\mathrm{U}} = -2\bar{\sigma}(\partial_{\bar{\sigma}}\bar{\mathrm{U}})^2 + N\partial_{\bar{\sigma}}\bar{\mathrm{U}} + 2\bar{\sigma}\partial_{\bar{\sigma}}^2\bar{\mathrm{U}}\,, \tag{2.2.3}$$

which we will call the LPA Polchinski equation, where we have suppressed the field and RG time dependence of $\bar{\mathrm{U}}$ for brevity and we have absorbed the $d-1$ dimensional solid angle Ω and factors of $(2\pi)^d$ into the potential by shifting [43, 50]:

$$\mathcal{S}_k \to \frac{\Omega}{(2\pi)^d}\mathcal{S}_k \tag{2.2.4}$$

and

$$\Phi^a \to \sqrt{\frac{\Omega}{(2\pi)^d}}\Phi^a.$$

An important note to make here is that cutoff independence of the LPA Polchinski equation is a property of the LPA. Specifically, when working in dimensionless variables we find that under the global rescaling applied above the cutoff dependence apparent in the Polchinski equation, equation (1.2.27), appears in sub-leading terms of the derivative expansion [98]. Therefore, in the LPA we find that the flow of the Wilsonian interactions is cutoff independent. The only exception to this is when the derivation of the Polchinski equation, and therefore the LPA Polchinski equation, breaks down as is the case when $\mathcal{S}_k = \Gamma_k$. An example of this is shown in section 2.2.1, where we find a different flow equation for $\mathcal{S}_k$ using the Wegner and Houghton method. However, in the general case where the derivation of the Polchinski and LPA Polchinski equation is valid we find that the flow of $\bar{\mathrm{U}}$ is independent of the cutoff in the LPA.

The equivalent calculation to find the flow of the Legendre potential is complicated by the matrix inversion of the full propagator, divided by the IR cutoff propagator, on the right hand side of the Legendre flow equation, equation (2.0.4), that has explicit dependence on the cutoff. This equivalently means that the choice of cutoff explicitly changes the form of the flow for the Legendre potential, V. However, our particular choice of cutoff operator allows for the Legendre transform between the Wilsonian and Legendre effective interactions, equation (1.2.41), to hold in the LPA [42]. Written in

terms of our dimensionless quantities we then find that:

$$\bar{\mathrm{U}}(\sigma,t) = \bar{\mathrm{V}}(\bar{\rho},t) + \frac{1}{2}(\bar{\phi}-\Phi)^2\,, \tag{2.2.5}$$

where there is no implied integration over the term $(\bar{\phi}-\Phi)^2$. This implies, as is expected from a Legendre transform, that the scale derivative of both actions are equal:

$$\partial_t \bar{\mathrm{U}}(\bar{\sigma},t)\Big|_{\bar{\sigma}} = \partial_t \bar{\mathrm{V}}(\bar{\rho},t)\Big|_{\bar{\rho}}\,. \tag{2.2.6}$$

By taking field derivatives of this Legendre transform we can then find a relation between the IR cutoff classical field, ϕ, and the effective UV cutoff quantum fields Φ:

$$\Phi^a - \phi^a = \Phi^a \partial_{\bar{\sigma}} \bar{\mathrm{U}}(\bar{\sigma},t) = \phi^a \partial_{\bar{\rho}} \bar{\mathrm{V}}(\bar{\rho},t)\,. \tag{2.2.7}$$

As this implies that the two fields point in the same direction [42, 92], we can rewrite this in terms of $\bar{\rho}$ and $\bar{\sigma}$:

$$\sqrt{\bar{\sigma}} - \sqrt{\bar{\rho}} = \sqrt{\bar{\sigma}}\partial_{\bar{\sigma}} \bar{\mathrm{U}}(\bar{\sigma},t) = \sqrt{\bar{\rho}}\partial_{\bar{\rho}} \bar{\mathrm{V}}(\bar{\rho},t)\,, \tag{2.2.8a}$$

or equivalently:

$$\sqrt{\frac{\bar{\rho}}{\bar{\sigma}}} = 1 - \partial_{\bar{\sigma}} \bar{\mathrm{U}}(\bar{\sigma},t) = (1 + \partial_{\bar{\rho}} \bar{\mathrm{V}}(\bar{\rho},t))^{-1} \tag{2.2.8b}$$

or

$$\bar{\sigma} = \bar{\rho}(1 + \partial_{\bar{\rho}} \bar{\mathrm{V}}(\bar{\rho},t))^2\,. \tag{2.2.8c}$$

As such, we may transform the results from the flow equation for Legendre potential $\bar{\mathrm{V}}$ to an equivalent result for the Wilsonian potential $\bar{\mathrm{U}}$ that, as we have shown above, is independent of the cutoff. Therefore, even if the form of the flow equation does depend on the cutoff used, so long as the Legendre transform holds our results do not.

Absorbing a factor of Δ_{IR}^{-1} into the inverse matrix, to get the inverse exact propagator, on the right hand side of the Legendre flow equation, equation (2.0.4). Using the fact we have discarded all derivatives we can write the exact propagator in momentum

space as:

$$\hat{\delta}^{ab}\Delta_{IR}^{-1}+\cdot\frac{\delta^2\Gamma_k[\phi]}{\delta\phi^a(-\mathbf{q})\delta\phi^b(\mathbf{q})}=\hat{\delta}^{ab}(p^2+\mathrm{R}_k(p^2))+(\hat{\delta}^{ab}\partial_\rho V(\rho,t)+\phi^a\phi^b\partial_\rho^2 V(\rho,t))\,. \quad (2.2.9)$$

Therefore, in the LPA the full propagator takes the form $A\hat{\delta}^{ab}+B\phi^a\phi^b$. We may then calculate the inverse by noting a matrix of this form has $N-1$ eigenvalues A, corresponding to the Goldstone modes, and the one eigenvalue $A+\phi^2 B$, corresponding to the radial mode [43]. Performing the inversion and noting that the momentum integral within the trace can be averaged over all directions of the momentum, *i.e.* shifting to d dimensional polar coordinates we perform the angular integration, we find that in dimensionless variables we get [1, 43, 47]:

$$\partial_t\bar{\mathrm{V}}(\bar{\rho},t)\Big|_{\bar{\rho}}+(d-2)\bar{\rho}\partial_{\bar{\rho}}\bar{\mathrm{V}}(\bar{\rho},t)-d\bar{\mathrm{V}}(\bar{\rho},t)=(N-1)\mathfrak{L}(\partial_{\bar{\rho}}\bar{V}(\bar{\rho},t))+\mathfrak{L}(\partial_{\bar{\rho}}\bar{V}+2\bar{\rho}\partial_{\bar{\rho}}^2\bar{V}(\bar{\rho},t))+N\,, \quad (2.2.10)$$

where we have again absorbed the $d-1$ dimensional solid angle Ω and factors of $(2\pi)^d$ into the potential with equivalent shifts:

$$\Gamma_k\to\frac{\Omega}{(2\pi)^d}\Gamma_k \quad (2.2.11)$$

and

$$\phi^a\to\sqrt{\frac{\Omega}{(2\pi)^d}}\phi^a\,.$$

The last term in equation (2.2.10) is given by a shift in the vacuum energy $\bar{\mathrm{V}}\mapsto\bar{\mathrm{V}}+N/d$, which is done to simplify the transformation between $\bar{\mathrm{V}}$ and $\bar{\mathrm{U}}$ but has no physical consequences [1]. Finally, the function $\mathfrak{L}(\zeta)$ on the right hand side gives a so called threshold function that depends on the cutoff used [47]:

$$\mathfrak{L}(\zeta)=\frac{d}{2}\int_0^\infty \mathrm{d}y\; y^{d/2}\frac{\partial_t r(y)}{y(1+r)+\zeta}\,, \quad (2.2.12)$$

where we use $y=\mathbf{q}^2=q^2k^{-2}$ and we define a dimensionless version of our cutoff function, equation (1.2.14), given by:

$$r(y)=q^{-2}\mathrm{R}_k(q^2)=\left(\frac{1}{y}-1\right)\Theta(1-y)\,. \quad (2.2.13)$$

Our choice of a sharp cutoff allows us to evaluate these threshold functions analytically. Specifically, Litim's optimised cutoff gives us:

$$\partial_t r(y) = -\frac{2}{y}\Theta(1-y) + (1-y)\delta(1-y)\,, \tag{2.2.14}$$

$$\implies \mathfrak{L}(\zeta) = \frac{d}{2}\int_0^\infty \mathrm{d}y\ \frac{y^{d/2}}{y(1+r(y))+\zeta}\left(-\frac{2}{y}\Theta(1-y) + (1-y)\delta(1-y)\right)\,. \tag{2.2.15}$$

The delta function in second term and the change of integration limits implied by the step function in the first term then trivially leads to the result:

$$\mathfrak{L}(\zeta) = -\,(1+\zeta)^{-1}\,, \tag{2.2.16}$$

which gives us the LPA Legendre flow equation:

$$\partial_t \bar{\mathrm{V}} + (d-2)\bar{\rho}\partial_{\bar{\rho}}\bar{\mathrm{V}} - d\bar{\mathrm{V}} = -\frac{N-1}{1+\partial_{\bar{\rho}}\bar{\mathrm{V}}} - \frac{1}{1+\partial_{\bar{\rho}}\bar{\mathrm{V}} + 2\bar{\rho}\partial_{\bar{\rho}}^2\bar{\mathrm{V}}} + N\,, \tag{2.2.17}$$

where we have suppressed the field and time dependence of $\bar{\mathrm{V}}$ for brevity.

2.2.1 Cutoff Dependence of the LPA Legendre flow equation

Before continuing, we first make a comparison the historic work of Wegner and Houghton to demonstrate the cutoff dependency of the LPA Legendre equation. The first real result for studying the RG flow of the effective potential was method developed by Wegner and Houghton, where the blocking of momentum modes was done by integrating only over those modes within an infinitesimal momentum shell defined by $k - \mathrm{d}k \leq |p| \leq k$ [90]. This equivalently means that the cutoff function only supports momentum modes within this infinitesimal momentum shell. Therefore, the integration over the high momentum modes, $\varphi_>$, can be done by using a saddle-point approximation and in the limit of $\mathrm{d}k \to 0$ this simplifies to an exact Gaussian integral. The exact relation in the limit of $\mathrm{d}k \to 0$ is then:

$$\partial_t \mathcal{S}_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\ln(\mathcal{S}_k^{(2)}[\phi])\right]\,. \tag{2.2.18}$$

If we now take apply the LPA, shift to dimensionless variables, and apply the global shifts of the effective potentials and momentum above we then find that in the LPA

the flow of the effective potential becomes:

$$\partial_t \bar{\mathrm{V}} + (d-2)\bar{\rho}\partial_{\bar{\rho}}\bar{\mathrm{V}} - d\bar{\mathrm{V}} = (N-1)\ln(1+\partial_{\bar{\rho}}\bar{\mathrm{V}}) + \ln(1+\partial_{\bar{\rho}}\bar{\mathrm{V}} + 2\bar{\rho}\partial_{\bar{\rho}}^2\bar{\mathrm{V}}) + N\,, \quad (2.2.19)$$

where we have used the fact that the propagator has $N-1$ eigenvalues $(1+\partial_{\bar{\rho}}\bar{\mathrm{V}})$ and one eigenvalue $(1+\partial_{\bar{\rho}}\bar{\mathrm{V}}+2\bar{\rho}\partial_{\bar{\rho}}^2\bar{\mathrm{V}})$ when applying the trace. A keen reader will note that this equation is terms of the Legendre potential and not the Wilsonian potential as one would expect from taking the LPA of $\mathcal{S}_k$. However, a special property of this method is that $\mathcal{S}_k = \Gamma_k$, as can be seen from the Legendre transform relation in equation (1.2.41) after taking into account the fact that cutoff here implies that $\Phi = \phi$. As the two pictures are the same using the Wegner Houghton method we chose to write the equation in terms of the Legendre effective potential, $\bar{\mathrm{V}}$, as this result represents the limit where our threshold functions, equation (2.2.12), in the LPA Legendre equation, equation (2.2.10), are evaluated with an explicit sharp cutoff [46, 84]:

$$\mathrm{r}_{sharp}(y) = \frac{1}{\theta\,(y-1)} - 1 = \begin{cases} 0 & \text{for } q^2 \geq k^2\,, \\ \infty & \text{for } q^2 < k^2\,. \end{cases} \quad (2.2.20)$$

That is, when the cutoff function supports RG flow in only an infinitesimal momentum shell we obtain an equivalent result as that of the Wegner and Houghton method.

This flow equation differs from the one first derived by Wegner and Houghton [90] due to their use of physical, instead of dimensionless, variables and a single-component scalar field. The Wegner Houghton result has since been used in various different forms, including as one of the first applications of the LPA by Nicoll et al. [44], which the above equation is much closer to. In terms of our scalar field theory, this comparison allows us to see explicitly that the non-linear flow of the Legendre potential $\bar{\mathrm{V}}$ is heavily dependent upon the cutoff used. This fact will become more relevant in discussions about renormalised trajectories and completeness of a QFT in chapter 4, where comparisons between the cutoff dependent results of the LPA Legendre flow equation and the cutoff independent results of the LPA Polchinski equation are compared, however it does not directly impact the final conclusions of this thesis.

2.3 Fixed Point Potentials and Eigenoperators

Having simplified the functional flow equations to partial differential equations we may now attempt to solve for fixed point effective potentials, $\bar{\mathrm{U}}_*$ and $\bar{\mathrm{V}}_*$. These potentials are scale independent, therefore we set the time derivatives to zero in the the flow of the potentials, equations (2.2.3) and (2.2.17), to zero. It is simple to see that both equations have the trivial solution where we set the potentials to a constant, the non-interacting or Gaussian fixed point (GFP), which is zero in the LPA[2]:

$$\bar{\mathrm{U}}_*(\bar{\sigma}) \overset{GFP}{=} 0 \qquad \bar{\mathrm{V}}_*(\bar{\rho}) \overset{GFP}{=} 0\,. \tag{2.3.1}$$

In fact, for $d = 4$, the most interesting case in HEP, we find that there are no interacting fixed point potentials and the Gaussian FP is the only known solution to these flow equations [56]. In lower dimensions, e.g. for $d = 3$, we can find the equivalent of the Wilson-Fisher fixed point potential [40, 43, 56, 89], but the exact structure of the FP potential does not change the arguments we will present. Therefore, for the majority of the remaining work and to make contact with HEP models such as the Higgs mechanism we will concentrate upon potentials based upon the only $d = 4$ fixed point, the Gaussian fixed point. Non-perturbatively renormalisable QFTs will therefore, as stated above, then be based upon linearised solutions around the GFP and it is a subset of these solutions that led Halpern and Huang to derive their non-polynomial potentials [58–60].

To find the eigenoperators we linearise the flow equations using a similar ansatz to equation (1.2.49) for each of the effective potentials:

$$\begin{aligned} \bar{\mathrm{U}}(\bar{\sigma}, t) =& \bar{\mathrm{U}}_* + \varepsilon \delta \mathrm{U}(\bar{\sigma}, t) && \text{(2.3.2a)} \\ =& \varepsilon \exp(\lambda_\sigma t) u_\lambda(\bar{\sigma})\,, \end{aligned}$$

[2]We find that the $\bar{\mathrm{V}}_*(\bar{\rho}) = 0$ from the LPA Legendre flow equation, equation (2.2.17), only after including the shift $\bar{\mathrm{V}} \mapsto \bar{\mathrm{V}} + N/d$. For the sharp cutoff flow equation, equation (2.2.19), this shift will in fact make $V_*(\bar{\rho}) = -N/d$. This just shows that the exact form of the solution to the LPA Legendre flow equation is cutoff dependent. However, as most of our results regarding the Legendre potential will be derived from using the Litim optimised cutoff, we assume that GFP is given by $\bar{\mathrm{V}}_*(\bar{\rho}) = 0$ throughout the rest of this thesis.

and

$$\begin{aligned}\bar{\mathrm{V}}(\bar{\rho},t) =&\bar{\mathrm{V}}_* + \varepsilon\delta\mathrm{V}(\bar{\rho},t) \\ =&\varepsilon\exp(\lambda_\rho t)v_\lambda(\bar{\rho})\,,\end{aligned} \tag{2.3.2b}$$

where we assume ε is strictly infinitesimal, that is we will accept the validity of the linearisation step for any operator $\delta\mathrm{U}$ or $\delta\mathrm{V}$. In the second line of each of these equations we have used the fact that the GFP potentials $\bar{\mathrm{U}}_* = \bar{\mathrm{V}}_* = 0$. The functions of fields, u_λ and v_λ are then the eigenoperators, or "scaling fields", with corresponding eigenvalues λ_σ and λ_ρ respectively. These eigenvalues then determine the scaling dimension of the eigenoperators through their couplings:

$$g(t) = \varepsilon\exp(\lambda t)\,, \tag{2.3.3}$$

which will then give, in terms of physical variables, corresponding physical couplings:

$$g_\lambda(t) = \varepsilon\mu^\lambda = k^\lambda g(t)\,, \tag{2.3.4}$$

of mass dimension λ. We can see that while the dimensionless couplings may scale to large numbers, these physical couplings cannot. If we recall the discussion in section 1.2.3 on tuning relevant eigenoperators to construct a renormalised QFT, that process is equivalent to tuning the two parameters ε and μ that define these physical couplings.

As the left hand side of each of the flow equations is given by the dimensions of the potentials, we can see that, apart from interchange of the potentials and eigenvalues, the perturbation around the GFP gives the same result on the left hand side for either potential. Expanding the right hand side of the three flow equations for the effective potentials, the LPA Polchinski equation (2.2.3), the LPA Legendre flow equation (2.2.17), and the sharp cutoff flow equation (2.2.19), in powers of ε we find that the flows of the resulting eigenoperators differ only at $\mathcal{O}(\varepsilon^2)$ and higher order. This is a special property of the GFP in the LPA. Specifically, in the LPA the differences of each linearised flow equation is characterised by the value of the fixed point at $\mathcal{O}(\varepsilon)$. Therefore, at the GFP these differences vanish due to the special property $\bar{\mathrm{U}}_* = \bar{\mathrm{V}}_* = 0$. Therefore, in

the linear regime each flow equation reduces to the same linearised flow equation:

$$\lambda w(z) + (d-2)z\partial_z w(z) - dw(z) = N\partial_z w(z) + 2z\partial_z^2 w(z) + \mathcal{O}(\varepsilon)\,, \tag{2.3.5}$$

up to replacing z with $\bar{\rho}$ or $\bar{\sigma}$ and we find that up to corrections of order ε:

$$\lambda_\sigma = \lambda_\rho = \lambda$$

and

$$u_\lambda(\bar{\sigma}) = v_\lambda(\bar{\rho}) = w(z)\,,$$

i.e. the eigenoperators u_λ and v_λ are the same. This result is a property unique to the Gaussian fixed point. From our Legendre transform relations between the fields, *cf.* equation (2.2.8c) we find that:

$$\begin{aligned}\bar{\sigma} =& \bar{\rho} + \partial_{\bar{\rho}}\bar{\mathrm{V}}_*(\bar{\rho}) + (\partial_{\bar{\rho}}\bar{\mathrm{V}}_*(\bar{\rho}))^2 + \mathcal{O}(\varepsilon) \\ =& \bar{\rho} + \mathcal{O}(\varepsilon)\,,\end{aligned} \tag{2.3.6}$$

which implies that:

$$\bar{\mathrm{U}}(\bar{\sigma}, t) = \bar{\mathrm{V}}(\bar{\rho}, t) + \mathcal{O}(\varepsilon)\,. \tag{2.3.7}$$

From this we can conclude that, in the linear regime, we can concentrate upon only the solutions to the Legendre potential, $\bar{\mathrm{V}}$. We now dedicate the next chapter to solving for the eigenoperator solutions of the LPA Legendre flow equation around the GFP before then confirming if they have the correct RG behaviour near the fixed point required to take the continuum limit. Particularly, we see that there exists two different classes of solutions given by the quantised polynomial eigenoperators and non-quantised non-polynomial eigenoperators, the latter of which can be shown to not have a suitable RG behaviour for large fields.

Chapter 3

Eigenoperators of $O(N)$ Scalar Field Theory

Having applied the leading approximation of the Derivative expansion to find a fixed point potential, the Gaussian fixed point, the next step to construct a non-perturbatively renormalisable quantum field theory is to construct the renormalised trajectory, *cf.* equation (1.2.57). The renormalised trajectory, given by a tuned sum of relevant eigenoperators, is required to emanate directly from the fixed point so that in the ultraviolet, which is given by $t \to -\infty$,[1] the perturbation falls into the FP. In other words, we may use the scale invariance of the FP to remove any dependence upon the overall ultraviolet momentum cutoff. However, if we consider the fact that the renormalised trajectory defined at some RG scale k may be considered as a bare action renormalised by a partial integration of the partition function, *cf.* equation (1.2.18), we must equally show that the renormalised trajectory has a finite IR limit, given by $t \to \infty$, so that the functional integration within the partition function can be completed. This must then be true for the total sum of relevant eigenoperators as well as each individual eigenoperator when the linearisation of the effective action is applicable.

To then reiterate from our discussion of critical phenomena *and* renormalisation in QFT we can say that an eigenoperator is considered relevant not because it displays a positive eigenvalue in the linearised regime, but because it strictly flows directly from the RG fixed point, the critical point, to a finite infrared value away from the fixed point as we integrate over fluctuations, *i.e.* reduce k, for all values of the field. Therefore,

[1] As we solve for flows directly from the continuum limit ($\Lambda \to \infty$) the UV limit, $k \to \Lambda$, gives $t = \ln(\mu/k) \to -\infty$.

while it is necessary for the eigenoperators to show positive scaling dimension from the linearised analysis of perturbations around the fixed point, this is not sufficient.

Particularly, from the linear analysis alone the authors Halpern and Huang proposed a new class of eigenoperators given by non-polynomial functions with a spectrum of possible scaling dimensions, eigenvalues, that therefore show, for positive eigenvalues, the necessary condition required that they flow into the FP in the UV. Non-polynomial potentials based upon these eigenoperators could then be considered as "asymptotically free" as RG flow into the GFP would imply the "turning off" of interactions. However, upon moving away from the linear analysis we show that full RG flow of the non-polynomial direction does not flow back into the FP in the UV and they therefore cannot be considered truly "relevant" eigenoperators.

The first step to show this is to first solve for the dimensionless eigenoperators. From the linear flow equation, equation (2.3.5), we can see that for both the Polchinksi equation and the Legendre flow equation the eigenoperator perturbations around the Gaussian FP can be shown to be identical within the LPA. As such, we will concentrate upon only the Legendre effective potential V to find the eigenoperators of $O(N)$ scalar field theory around the GFP.

To solve the eigenoperator equation, equation (2.3.5), we first collect all terms on one side of and perform the change of variable $\bar{\rho} \to \kappa\omega$, where $\kappa = \frac{2}{(d-2)}$. This change will rewrite the flow of our eigenoperators, $v_\lambda(\bar{\rho})$, manifestly as a confluent hypergeometric equation, also known as Kummer's equation [100–102]:

$$\omega\partial_\omega^2 v_\lambda(\omega) + \left(\frac{N}{2} - \omega\right)\partial_\omega v_\lambda(\omega) - \frac{(\lambda - d)}{(d-2)} v_\lambda(\omega) = 0\,. \tag{3.0.1}$$

Mathematically, as this equation is second order linear ordinary differential equation the solution will have two free parameters, giving two linearly independent solutions for each value of the constants N, d, and λ. The linearly independent solutions to this equation correspond to the functions constructed at one of the two singularities of the equation, the Kummer M function at the regular singularity $\omega = 0$ and the Tricomi U function at the irregular singularity $\omega = \infty$ respectively.

However, physically a general eigenoperator is required to be finite in the limit $\omega \to 0$ and therefore cannot be constructed using Tricomi's function. Thus the eigenoperators are given by Kummer's M function. On the other hand, as these eigenoperators

span all field values $0 \leq \omega < \infty$ we find that above a certain field value that depends upon the parameters of theory the Kummer M function takes an asymptotic form that depends upon the Tricomi U function. An interested reader is encouraged to explore these considerations and consult literature on special functions [100–102], but for the scope of this thesis we find that eigenoperators are generally of the form of Kummer's M function that has a finite series expansion at the origin but transitions at large fields to what we find to be exponential field dependence. Particularly, the Kummer M function can subsequently be written as two different classes of solutions depending on if the coefficient of $v_\lambda(\omega)$ in Kummer's equation, equation (3.0.1), is a negative integer. That is we obtain a subset of polynomial solutions if the eigenvalues belong to a set of quantised values and a continuous spectrum of solutions for any other value. It is these other non-quantised eigenvalue solutions that Halpern and Huang proposed as interacting potentials around the GFP [58,59]. However, before we rederive their result we will first review the quantised eigenvalue solutions, that have been known for many years [39–45].

3.1 Eigenoperators for Quantised Eigenvalues

From a historical point of view, the investigations on eigenoperators have always been based upon polynomial functions of the order parameter. While it is possible to mathematically solve flow equations beyond this restriction, removing the polynomial restriction allows for a continuous infinity of eigenvalues. However, recalling the link between the eigenoperators and critical behaviour in thermodynamics, we find that for each eigenoperator perturbation around a fixed point the corresponding coupling, with mass dimension given by the eigenvalue, is in fact some measurable thermodynamic quantity. Therefore, a continuum of eigenoperators would imply a continuum of thermodynamic variables with a continuum of scaling dimensions that should enter into thermodynamic potentials! This is obviously not the case in any measured condensed matter system, instead we find these systems are described by only a finite set of thermodynamic parameters with quantised mass dimension. Universality of critical phenomena then suggests that if there is a continuum of non-polynomial eigenoperators in HEP we *must* find a physical reason why the number of relevant thermodynamic variables that drives critical phenomena is finite, even for any of the dramatically different

systems that exist within one universality class.

The general solution to the eigenoperator equation, equation (3.0.1), was first found by the mathematician Kummer in 1837 [103] and is given by the confluent hypergeometric function of the first kind ${}_1\mathrm{F}_1$, or Kummer's function M, *i.e.* we can say that for any eigenoperator:

$$v_\lambda(\omega) \propto \mathrm{M}\left(\frac{\lambda - d}{d-2}, \frac{N}{2}, \omega\right) . \tag{3.1.1}$$

The Kummer function is defined by the hypergeometric series given by:

$$\mathrm{M}(c_1, c_2, \omega) = \sum_{m=0}^{\infty} \frac{(c_1)^{\overline{m}}}{(c_2)^{\overline{m}}} \frac{\omega^m}{m!} , \tag{3.1.2}$$

where $(c_1)^{\overline{m}}$ is the (Pochhammer) rising factorial of some number c_1 defined for any non-negative m as:

$$(c_1)^{\overline{0}} = 1 \tag{3.1.3a}$$

$$(c_1)^{\overline{m}} = c_1(c_1+1)(c_1+2)\cdots(c_1+m-1) = \frac{\hat{\Gamma}(c_1+m)}{\hat{\Gamma}(c_1)} . \tag{3.1.3b}$$

There is a unique property given to the Kummer function from these rising factorials. Consider the case that c_1 is some negative integer $-n$ for some non-negative integer n, $n \in \mathbb{Z}^*$, then one can see that for all $m > n$ the rising factorial $(c_1)^{\overline{m}} = 0$. Therefore, for any $c_2 \neq -j$ and $c_1 = -n$, $\{n, j\} \in \mathbb{Z}^*$, the Kummer function collapses to a polynomial of degree n, which are in fact proportional to generalised or associated Laguerre polynomials [100–102]. We therefore find that the solutions to the eigenoperator equation is naturally quantised by a non-negative integer n that solves $c_1 = -n$, which gives the relation:

$$c_1 = \frac{\lambda - d}{d-2} = -n . \tag{3.1.4}$$

These quantised solutions are then rank n polynomials of ω. Specifically, inverting the relation between n and λ we find that for:

$$\lambda = \lambda_n = d - n(d-2) , \tag{3.1.5}$$

we get eigenoperator solutions in the form associated Laguerre polynomials:

$$v_\lambda(\omega) \overset{\lambda=\lambda_n}{=} v_n(\omega) = (-2\kappa)^n \mathrm{L}_n^{\frac{N}{2}-1}(\omega) , \tag{3.1.6}$$

where $\mathrm{L}_n^{\alpha-1}$ is the rank n associated Laguerre polynomial or Sonine polynomial. These polynomials can be written in a sum form as [100–102]:

$$\mathrm{L}_n^{\alpha-1}(\omega) = \sum_{m=0}^{n} (-1)^m \frac{\hat{\Gamma}\left(n+\alpha\right)}{\hat{\Gamma}\left(\alpha+m\right)(n-i)!} \frac{\omega^m}{m!}\,, \tag{3.1.7}$$

where $\hat{\Gamma}$ is the gamma function, which is needed to generalise factorials for non-integer values of α. We note that this definition is valid for any real value of α that is not a negative integer, therefore this solution covers any number of components N. The factor of $(-2\kappa)^n$ multiplying the Laguerre polynomials in equation (3.1.6) is there so that the highest power of $\phi^a\phi^a$ in v_n is canonically normalised to

$$\frac{(\phi^a\phi^a)^n}{n!}\,,$$

and each lower power of $\phi^a\phi^a$ is then generated by successive tadpole corrections. In the linear regime, we know that associated coupling, g_{2n}, of this canonically normalised term is of order ε and is therefore small. As such, we can equate these quantised directions with the usual interactions in perturbative scalar QFT, where the $\frac{(\phi^a\phi^a)^n}{n!}$ terms representing the bare interaction of the QFT, with scaling dimension λ_n, and tadpole corrections giving effecive vertices for interactions with $n-2$ or fewer fields.

However, we note that the Laguerre polynomials form a complete set of orthogonal functions under a generalised Laguerre weight [100–102], which can be written in terms of our polynomial directions so that:

$$\int_0^\infty \mathrm{d}\omega\ \omega^{\frac{N}{2}-1}\mathrm{e}^{-\omega} v_n(\omega)v_m(\omega) = (-2\kappa)^{n+m}\frac{\hat{\Gamma}\left(n+\frac{N}{2}\right)}{\hat{\Gamma}(n+1)}\hat{\delta}_{nm}\,, \tag{3.1.8}$$

where here $\{m,n\} \in \mathbb{Z}^*$. This can easily be generalised to allow any potential $\bar{\mathrm{V}}(\bar{\rho},t)$ to be expanded in terms of our polynomial eigenoperators using:

$$\bar{\mathrm{V}}(\bar{\rho},t) = \sum_{n=0}^{\infty} g_{2n}(t) v_n\left(\kappa^{-1}\bar{\rho}\right)\,, \tag{3.1.9}$$

so long as the total potential is square-integrable with respect to the generalised Laguerre weight, *i.e.* :

$$\int_0^\infty \mathrm{d}\omega\ \omega^{\frac{N}{2}-1}\mathrm{e}^{-\omega}\left|\bar{\mathrm{V}}\left(\kappa\omega,t\right)\right|^2<\infty\,. \tag{3.1.10}$$

The associated couplings, g_{2n}, are then defined by:

$$g_{2n}(t)=(2\kappa)^{-2n}\,\frac{\hat{\Gamma}(n+1)}{\hat{\Gamma}\left(n+\frac{N}{2}\right)}\int_0^\infty \mathrm{d}\omega\ \omega^{\frac{N}{2}-1}\mathrm{e}^{-\omega}v_n(\omega)\bar{\mathrm{V}}\left(\kappa\omega,t\right)\,, \tag{3.1.11}$$

where the convergence of this integral is guaranteed for any analytic potential $\bar{\mathrm{V}}$ that satisfies equation (3.1.10).

The exponential decay in the integral weight implies that the convergence of the expansion in equation (3.1.9) is almost always true. As such, we can define a Hilbert space structure formed by the quantised eigenoperators, where the convergence and the norm is defined by the generalised Laguerre weight. That is, not only can we expand a potential as a finite series of polynomial directions, by which we mean each associated coupling in the series is finite, but we may also represent the potential in a basis of these polynomial directions so long as:

$$\lim_{n_{max}\to\infty}\left\{\int_0^\infty \mathrm{d}\omega\ \omega^{\frac{N}{2}-1}\mathrm{e}^{-\omega}\left|\bar{\mathrm{V}}\left(\frac{2}{(d-2)}\omega,t\right)-\sum_{n=0}^{n_{max}}g_{2n}(t)v_n(\omega)\right|^2\right\}=0\,. \tag{3.1.12}$$

The difference here is subtle. Having a finite series expansion in polynomial eigenoperators is a necessary condition to show that the potential is given by a sum of polynomial eigenoperators, however it is possible to exist outside of the Hilbert space even if the series expansion is finite, *e.g.* the case where each coefficient goes to zero. However, if the limit in equation (3.1.12) is true, then we can show that the series expansion and the potential are exactly equal, proving that the potential is in fact given by a sum of polynomial directions.

However, the flow equations near the FP are linear. As such, each eigenoperator in any given linear combination of eigenoperators solves the linearised flow equation individually. Recalling that the continuum limit of any physical theory is connected to, via RG flow, and controlled by the linear regime of the FP this implies near the FP, *i.e.* for $t\to-\infty$, any scale dependent potential $\bar{\mathrm{V}}_Q(\bar{\rho},t)$ that fits into the Hilbert space

defined by these quantised directions, may be solved for directly from the quantised directions rather than the flow of some bare action. Explicitly, we may then say that:

$$\bar{\mathrm{V}}_Q(\bar{\rho},t)=\sum_{\text{relevant}} c_n \exp(\lambda_n t) v_n\left(\kappa^{-1}\bar{\rho}\right)+\sum_{\text{irrelevant}} d_n \exp(\lambda_n t) v_n\left(\kappa^{-1}\bar{\rho}\right)\,, \tag{3.1.13}$$

for some coefficients c_n and d_n. Therefore, with increasing RG time $\bar{\mathrm{V}}_Q$ will flow towards a potential where the irrelevant eigenoperators are suppressed and only the relevant eigenoperators remain in the expansion given by equation (3.1.13). This is exactly the behaviour one would expect within linearised regime; so long as we are within the linearised regime, *i.e.* the domain of attraction of a FP, any potential will flow to suppress the irrelevant directions and magnify the relevant variables. However, recalling the conditions required for a renormalisable QFT discussed in section 1.2.3, this is equivalent to saying that any bare potential belonging to the Hilbert space of quantised directions merely represents a possible action tuned off the critical manifold for any $c_n \neq 0$. These therefore provide no further information on the critical structure of the theory $O(N)$ scalar field theory and can only provide a physical QFT after they are carefully tuned back onto the critical manifold to give the renormalised trajectory.

3.1.1 Quantised Interactions in $d = 4$ Dimensions

If we now specialise to the dimensionality of interest in HEP, $d = 4$ dimensions, so that we may apply our findings to models like the Higgs, we find $\omega = \bar{\rho}$ and that the quantised eigenoperators correspond to eigenvalues:

$$\lambda_n = 4 - 2n\,. \tag{3.1.14}$$

The $n = 0$ case has positive scaling dimension, $\lambda_n > 0$, but v_0 corresponds to a trivial the vacuum energy term that is completely unconstrained by the eigenoperator equation, equation (3.0.1). It does not however lead to any interesting physics and we therefore ignore the case of $n = 0$. This means that the equivalent perturbative couplings, regardless of the number of components, are relevant if and only if $n = 1$. This sole relevant coupling, $v_1(\omega)$, has scaling dimension $\lambda_1 = 2$ and can therefore be identified as, up to an additive constant, the mass term associated to $g_2 = m^2/k^2$ and physical 'coupling' m^2. The next lowest value of n corresponds to the ϕ^4 coupling,

the Higgs coupling for $N = 4$, with marginal scaling dimension $\lambda_2 = 0$. Taking into account ε^2 corrections we find that associated coupling g_4 is a marginally irrelevant, that is it vanishes in the limit of $t \to +\infty$. All higher power couplings, corresponding to the $n \geq 3$, all have negative scaling dimension, $\lambda_n < 0$ for $n \geq 3$, and are therefore irrelevant. From this, we can conclude the Higgs theory, or any other $O(N)$ scalar field theory, has no non-trivial interactions in $d = 4$ dimensions. This is the Higgs triviality problem and we confirm the usual picture that the in $d = 4$ the continuum limit of a free massive scalar field.

3.2 Non-Polynomial Eigenoperators of the Gaussian Fixed Point

One of the key points in Halpern and Huang's derivation of the non-polynomial eigenoperator is that there is no physical reason in HEP why we should expect a quantised set of coupling parameters. In HEP we are not working with thermodynamic variables, instead the couplings represent fundamental physical parameters that we are not free to change. As such, we might expect that if an infinite number of possible parameters existed, the physical world would only permit a small subset that could be constrained via measurements of physical observables. While this is at odds with the expectations from the universality of critical phenomena, the possibility to have a continuum of eigenvalues is not physically restricted *a priori*.

Allowing for non-polynomial eigenoperators and non-quantised eigenvalues we must then consider the remaining cases given by the Kummer function solutions for any real $c_1 \neq -n$, *i.e.* $\lambda \neq d - n(d-2)$ ($n \in \mathbb{Z}^*$), where the hypergeometric series given by equation (3.1.2) is not truncated at some power of the field. Explicitly we find the non-polynomial eigenoperators take the form:

$$v_\lambda(\omega) \overset{\lambda \neq \lambda_n}{=} w_\lambda(\omega) = s_\lambda \mathrm{M}\left(\frac{\lambda - d}{d-2}, \frac{N}{2}, \omega\right) , \tag{3.2.1}$$

where the normalisation constant s_λ is the sign of $(\lambda - d)(\lambda - 2)$. We include this normalisation constant so that the these non-quantised directions are positive for large ω, *i.e.* they provide a perturbation that is bounded from below for a positive associated coupling. Therefore, w_λ is normalised to s_λ (± 1) at $\bar{\rho} = 0$. At large ω the Kummer M

tends to the asymptotic form [101]:

$$w_\lambda(\omega) = C \exp(\omega)\, \omega^{-p} \left(1 + O\left(\frac{1}{\omega}\right)\right), \tag{3.2.2}$$

where:

$$p = \frac{N}{2} - \frac{\lambda - d}{d-2} \qquad \text{and} \qquad C = \frac{\hat{\Gamma}\left(N/2\right)}{\left|\hat{\Gamma}\left(\frac{\lambda-d}{d-2}\right)\right|}. \tag{3.2.3}$$

Since these non-polynomial directions solve the eigenoperator ODE, equation (3.0.1), they must each be linearly independent solutions, for each real $\lambda \neq d - n(d-2)$. Therefore, they cannot be expanded in the quantised directions as in equation (3.1.9). This can be shown trivially by substituting the asymptotic form of the non-polynomial direction, equation (3.2.2), into the coefficient equation for the expansion, equation (3.1.11), and noting that for sufficiently large n, specifically:

$$n > -\frac{\lambda - d}{d-2}, \tag{3.2.4}$$

the integrals do not converge. These non-polynomial directions therefore live outside the Hilbert space spanned by the quantised directions, given by equation (3.1.6), *i.e.* we therefore find that there exists an infinite number of possible independent eigenoperator solutions characterised by λ.

However, we stress that this is a *mathematical* property of these non-polynomial directions that is only true within linearised regime, *i.e.* very close to the Gaussian fixed point and for finite fields only. We will show in section 3.3 that once we consider the RG flow for all fields they do not show the correct RG behaviour to obtain a continuum limit and in chapter 4 we find that when we attempt to solve for the IR (t increasing) evolution, the full RG flow drives the non-polynomial directions back into the Hilbert space of polynomial potentials.

3.2.1 Halpern-Huang Potentials

Based upon the flow exclusively in the linearised regime the non-polynomial eigenoperators *are linearly independent from the polynomial directions* and therefore present a unique solution to the Higgs triviality problem. If we again specialise to $d = 4$ and $N = 4$, then we find that $\kappa = 1 \implies \omega = \bar{\rho}$ and the series expansion of the

non-polynomial directions can be written as:

$$w_\lambda(\bar{\rho}) = s_\lambda \sum_{m=0}^{\infty} \frac{\hat{\Gamma}\left(\frac{1}{2}(\lambda-4)+m\right)}{\hat{\Gamma}\left(m+2\right)\hat{\Gamma}\left(\frac{1}{2}(\lambda-4)\right)} \frac{\bar{\rho}^m}{m!}\,, \tag{3.2.5}$$

where we note for $\lambda \neq \lambda_n = 4-2n$, *i.e.* we don't get the polynomial solutions, each of the gamma functions that appear here are finite. Therefore, the coefficients of each term of these hypergeometric series are finite.

We can therefore say that in the linear regime these non-polynomial solutions provide a different class of eigenoperator compared to the quantised directions. Additionally, as λ can take any real value $\lambda \neq \lambda_n$, there must exist some of these eigenoperators that are relevant, *i.e.* they flow into fixed point in the UV, given by $\lambda > 0$. This led to the authors Halpern and Huang to conclude that these non-polynomial eigenoperators represent "asymptotically free" interactions, as they vanish into the non-interacting GFP in the UV. Of particular note is that these potentials display a non-trivial minimum for $0 < \lambda < 2$, which potentially yield symmetry-breaking potentials that are directly applicable to the Higgs sector of the standard model and may solve the Higgs triviality problem. Examples of these types of potentials are shown in figure 3.2.1. From these we can see the non-trivial minimum is shifting to larger values of the field ρ as the eigenvalue increases, which is to be expected given the two limiting value of $\lambda = 0$, where the Kummer M functions is quadratic in $\bar{\rho}$, and $\lambda = 2$, where the Kummer M function is linear in $\bar{\rho}$.

The possibility of "asymptotically free" scalar field theory defined by these Halpern-Huang directions has lead to numerous research publications. These publications are primarily about possible cosmological implications [61–63,65] and non-polynomial gravity [68–72]. Additional studies have looked to confirm these results, such as Periwal's derivation in terms of the Polchinski equation [27], Halpern's analysis of scattering amplitudes [64], and Gies' large N study [66]. We particularly emphasise the results of Gies' large N study that showed two key points. Firstly, that for any of the possibly symmetry breaking potentials, *i.e.* $0 < \lambda < 2$, the non-trivial minimum disappears when the full partition function is integrated over. Secondly, the nature of these solutions naturally allows for the mass scale of in the infrared to differ from the ultraviolet cutoff by many orders of magnitude, softening but not solving the "hierarchy problem". Another, more recent paper, that is particularly interesting is the paper by Gies and

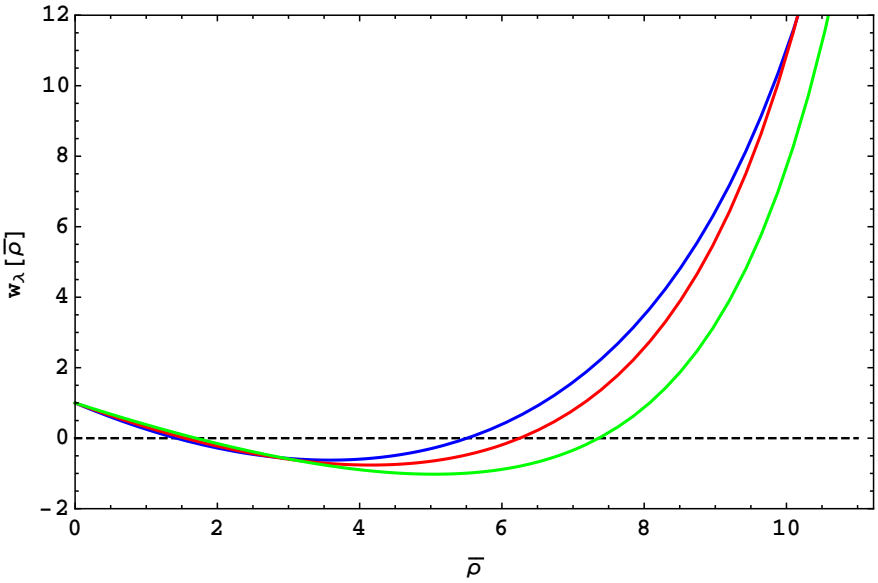

Figure 3.2.1: Three examples of possible relevant symmetry-breaking non-polynomial eigenoperator potentials, $w_\lambda(\bar{\rho})$ for $N = 4$ and $d = 4$. In the blue we show $\lambda = 0.6$, in the red $\lambda = 1$, and in the green $\lambda = 1.4$.

Scherer [67], trying to construct Yukawa interactions using the renormalisation group. In this paper, they find that the inclusion of fermionic degrees of freedom implies that the scalar GFP, no scalar-scalar interactions in the action, no longer exists. Specifically, a scalar field theory with no self-interactions represents a RG fixed point if and only if the Yukawa interaction is zero throughout the RG flow. Physically, this may be interpreted as the fact that even without scalar-scalar interactions there can exist additional vacuum fluctuations due to the Yukawa interaction. Therefore, the Halpern-Huang directions cannot be used in place of the standard model Higgs interaction as the Yukawa terms forbid the GFP! While this only forbids Halpern-Huang relevant directions in a Yukawa theory, it does not forbid non-polynomial scalar field potentials in general. However, any further analysis along these lines is beyond the scope of this thesis.

3.3 Ultraviolet properties: (Non)Existence of a Physical Continuum Limit

Up to this point we have been following an almost entirely mathematical analysis of the eigenoperators based upon the assumption that the RG flow near the fixed point is given by linearised limit, given by equation (3.0.1). If however one wants to

maintain that the non-polynomial directions, given by equation (3.2.1), actually do correspond to a continuous infinity of possible physical couplings $g_\lambda = \varepsilon(\mu/t)^\lambda$, then, due to the universality of critical phenomena, one must find a way to exclude the equivalent perturbations in all the so far experimentally realisable continuous phase transitions described by scalar field theory, including in numerical simulations [5, 8, 9, 74–77]. Particularly, from our discussion regarding RG flow at the fixed point in section 1.2.2 and the connection to SM we can see that a continuous spectrum of couplings g_λ, or equivalently relevant couplings, would imply a continuous spectrum of thermodynamic variables that destroy critical behaviour, which is in stark contrast to the two parameter set[2] of Ising-like models! Furthermore, the fact that these couplings represent physical thermodynamic quantities in SM implies that continuous values of λ should lead an infinite set of thermodynamic variables with mass dimension k^λ.

Even if we avoid this issue by decoupling SM and HEP and consider that only in HEP can we find non-quantised eigenvalues, we must still ensure that these non-polynomial solutions to the linearised flow equations do give suitable physical results. Specifically, in order for a renormalised trajectory to emanate from a FP it must flow into the FP in the UV, for any value of the field. It is considering this point that the above analysis starts to fail. That is the linearisation of the LPA flow equations around the fixed point hinges upon the assumption that $\bar{\mathrm{U}} \ll 1$ and $\bar{\mathrm{V}} \ll 1$ so that we may discard any non-linear terms. However, for any constant ε, no matter how small, we can expect that the field $\bar{\rho}$ will grow sufficiently that the convergence of the expansion in ε is no longer true, even when including corrections of $\mathcal{O}(\varepsilon^2)$ and higher powers in ε, since we find that the (non-constant) quantised solutions, equation (3.1.6), grow like $(\kappa^{-1}\bar{\rho})^n$, and the non-polynomial solutions, equation (3.2.1), grow like $\exp(\kappa^{-1}\bar{\rho})$, *cf.* (3.2.2). Mathematically the issue is the difference between uniform and point-wise convergence: it is not possible to find a value of ε such that the perturbation around the fixed point is uniformly bounded by some finite value $\mathcal{M}$. Equivalently, this implies that there always exists some value of ε small enough that we may linearise the flow equations over some finite domain of $\bar{\rho}$ and t, but there is *no value of* ε *small enough* that ensures the linearisation of the flow equations is a valid procedure over all possible field values, $0 \leq \bar{\rho} < \infty$, for any t.

It is possible to avoid this large field problem, for example by including a field

[2]Namely the deviation from critical temperature and the reduced magnetic field. [5, 8, 9, 74–77]

cutoff operator so that the potential is small or zero beyond some very large field $\bar{\rho}$ or by simply asserting that the fields $\bar{\rho}$ can be assumed to be sufficiently small in physical observables, as is effectively done in traditional perturbation theory or more generally for scaling field methods. However, as noted by Morris [40, 41] and implicitly shown above, the large field $\bar{\rho}$ dependence is pivotal to the linear independence of the non-polynomial directions. It is this property that makes the non-polynomial solutions a new interaction[3], since it is only the large field behaviour, equation (3.2.2), that prevents the convergence of expansion in terms of quantised eigenoperators, equation (3.1.9). Specifically, from the definition of the Hilbert space spanned by the quantised directions, the norm defined with respect to the generalised Laguerre weight implies that this expansion is valid if for any general potential:

$$\left|\bar{\mathrm{V}}(\bar{\rho},t)\right| < \left|(\bar{\rho})^{\frac{N}{4}} \exp\left(\frac{\bar{\rho}}{2\kappa}\right)\right| . \tag{3.3.1}$$

Therefore, in the case that the non-polynomial directions are *mathematically* linearly-independent solutions, they are not *linearised* solutions, *i.e.* it is incorrect to use the linearised flow equation to solve for their t dependence. As such we conclude that analysis beyond the linear regime is therefore necessary before we can draw any conclusions as to the physical relevance of these non-quantised perturbations. Equally, to be rigorous, similar non-linear flow analysis is also necessary to classify the quantised perturbations.[4]

The fact that the flow equations are first order differential equations in the RG time t implies we may, if we wish, specify a single 'boudary' condition, *i.e.* choose the $\bar{\rho}$ dependence of $\bar{\mathrm{V}}(\bar{\rho},t)$ at some 'initial point' $t = t_0$.[5] The RG flow of this potential can then in principle be solved for uniquely by the given flow equation. We can then explore the non-perturbative RG flow outside of the linear regime, *i.e.* for the asymptotically large values.

In order to do this for the either the polynomial or the non-polynomial directions we must assume that the potential is equal to one such direction, *i.e.* any given v_n,

[3] By new interaction we explicitly mean not studied in statistical mechanics or renormalisation group literature prior to the paper by Halpern and Huang.

[4] Excluding any vacuum energy eigenoperator, such as the $\lambda = 4$ eigenoperator in $d = 4$ dimensions, as they carry no field dependence and are therefore small for any sufficiently small ε.

[5] Given the definition of $t = \ln(\mu/k)$, with arbitrary μ, this is in fact some arbitrary point on the flow.

equation (3.1.6), or w_λ, equation (3.2.1), at some initial $t = t_0$. Without any loss of generality we can take this initial value of $t_0 = 0$. At this point we reiterate that a true continuum limit is required not only to allow for the limit $\Lambda \to \infty$, but to also ensure that the RG flow does not depend upon any scale other than k. Therefore, given the arbitrary nature of the definition of t_0, we must impose that for any true relevant eigenoperator the RG flow should not carry dependence upon this initial value of t.

Choosing some positive eigenvalue λ so that in the linear regime, with small field values, the potential flows into the GFP for decreasing t we can then set the potential at some initial RG time:

$$\bar{\mathrm{V}}(\bar{\rho}, 0) = \bar{\mathrm{V}}_*(\bar{\rho}) + \varepsilon\, v_\lambda(\kappa^{-1}\bar{\rho})\,, \tag{3.3.2}$$

agreeing with (2.3.2b) at $t = 0$. Here v_λ will be any one v_n or w_λ. Obviously similar comments apply to $\bar{\mathrm{U}}(\bar{\sigma}, t)$, however, as was shown in section 2.2, the full non-perturbative flow equations of $\bar{\mathrm{U}}$ and $\bar{\mathrm{V}}$ differ. Particularly, the flow of the Legendre effective potential $\bar{\mathrm{V}}$ is cutoff dependent and therefore we must consider the non-perturbative flow of each potential separately. We will exploit this fact later on using the generalised Legendre transform, equation (2.2.5). For now, we concentrate on the behaviour of $\bar{\mathrm{V}}(\bar{\rho}, t)$.

We consider first the quantised perturbations, *i.e.* setting $\lambda = \lambda_n$ and $v_\lambda = v_n$ in equation (3.3.2). The number of relevant quantised eigenoperators is dependent upon the dimension of the system. Particularly, we find that relevant and marginal eigenoperators, given by $\lambda_n \geq 0$, correspond to $n \leq d/(d-2)$, *i.e.* for $d = 4$ the only relevant and marginal eigenoperators correspond to $n = 1, 2$ respectively and for $d = 3$ the and relevant eigenoperators correspond to $n = 1, 2$ and the marginal eigenoperator is given by $n = 3$. We have ignored the $n = 0$ case as this eigenoperator is field independent and can always be treated with the linearised analysis.

For $n = 1$ we get the linearised mass term perturbation, with scaling dimension $\lambda_1 = 2$ in any dimension, given by:

$$\bar{\mathrm{V}}_1(\bar{\rho}, t) = \varepsilon\, v_1(\kappa^{-1}\bar{\rho}) \exp(2t)\,, \tag{3.3.3}$$

where from the equation (3.1.6)

$$v_1(\kappa^{-1}\bar{\rho}) = 2\bar{\rho} - N\,. \tag{3.3.4}$$

Due to the constant term, $v_1(\kappa^{-1}\bar{\rho})$ is linear inhomogeneous in $\bar{\rho}$. Substituting this potential into our flow equation, equation (2.2.17), (or equivalently (2.2.19)), we see that the right hand side of the flow equations are actually independent of $\bar{\rho}$. Since the left hand side of the flow equations are linear, the linearised solution for the $\bar{\rho}$ piece is therefore correct whatever its magnitude. Therefore, we can conclude that non-perturbatively

$$\bar{\mathrm{V}}_1(\bar{\rho}, t) = 2\varepsilon \exp(2t)\bar{\rho} + c(t)\,, \tag{3.3.5}$$

where $c(t)$ represents the non-perturbative field independent term. The non-linear terms on the right hand side of the flow equation are then known explicitly, therefore the constant term $c(t)$ may be solved for exactly and $c(t)$ is therefore not given by the linearised solution $-N\epsilon \exp 2t$. However, since there is no $\bar{\rho}$ dependence in the non-linear terms on the right hand side of the flow equations, the linear $\bar{\rho}$ dependence of the linearised solution is valid over the entire flow. Furthermore, $c(t) = -N\epsilon \exp 2t$ is a good approximation for all $t \ll t_c$ where $t_c = -\frac{1}{2}\ln(N\epsilon)$ is the large positive RG time such that $c(t) \sim 1$. We therefore conclude that the linearised mass term appropriately tends towards the GFP in the UV and we can confirm that it has an appropriate continuum limit required to be a relevant direction as the linearised analysis suggested.

It is important to note that in physical variables, *cf.* equations (2.0.3), the mass term perturbation can be written as:

$$\begin{aligned} \mathrm{V}(\rho, t) = k^d \bar{\mathrm{V}}(\bar{\rho}, t) =& 2\varepsilon k^d \bar{\rho} \exp(2t) - N\varepsilon k^d \mathrm{e}^{-2t} \\ =& 2\varepsilon\mu^2\rho - N\varepsilon\mu^2 k^{(d-2)}\,, \end{aligned} \tag{3.3.6}$$

for $t \ll t_c$. Therefore in physical variables the field dependent piece is actually independent of t (*i.e.* has mean field evolution) while the constant term *actually diverges*, for all $d > 2$, in the continuum limit as $\Lambda \to \infty$. This is an example of how it is important that we use dimensionless variables, otherwise we may not be able to determine whether or not $\mathrm{V}(\rho, t)$ correctly flows into the Gaussian fixed point in the UV, $t \to -\infty$.

For the quantised perturbations, the (marginally) irrelevant eigenoperators at the

linearised level corresponding to integer $n \geq d/(d-2)$,[6] we are already certain that they do not have the correct behaviour to fall into the Gaussian fixed point since even when $V \ll 1$, the RG time dependence is such that the perturbation grows as away from the fixed point as t becomes increasingly negative.[7]

In less than four dimensions, the quartic and higher polynomial eigenoperators may become relevant or marginally relevant according to $\lambda_n = d - n(d-2)$ from the linearised analysis. For example, the quartic potential ($\bar{\rho}^2$) becomes relevant for $d < 4$. The full non-perturbative treatment of these polynomials becomes increasingly complicated due to the non-linear flow terms. However, as Morris has pointed out [40, 43, 50], for $n \geq 2$ the large field behaviour of the polynomials eigenoperators implies that the non-linear right hand side of the flow of the Legendre effective potential, equations (2.2.17) or (2.2.19), is negligible compared to the at least linear left hand side. Therefore, we may solve for the asymptotic field flow of the polynomial eigenoperators as though they followed mean field evolution, by which we mean the RG flow is stationary *in physical units*. Starting from our intial perturbation, equation (3.3.2), with $v_\lambda = v_n$ for $n \geq 2$ we get from mean field evolution that in the large field regime:

$$\bar{\mathrm{V}}(\bar{\rho}, t) \sim \mathrm{e}^{dt} \bar{\mathrm{V}}\left(\bar{\rho}\,\mathrm{e}^{-(d-2)t}, 0\right) = \bar{\mathrm{V}}_*(\bar{\rho}) + \varepsilon \mathrm{e}^{dt} v_n\left(\kappa^{-1}\bar{\rho}\,\mathrm{e}^{-(d-2)t}\right), \tag{3.3.7}$$

as $\bar{\rho} \to \infty$. This implies that the perturbations take the form:

$$v_n(\kappa\bar{\rho}) \sim c\bar{\rho}^{(d-\lambda_n)/(d-2)}, \quad \text{as} \quad \bar{\rho} \to \infty, \tag{3.3.8}$$

where c is some unknown constant that must be determined from the flow equations [40, 43, 50]. However, we find for these polynomial perturbations we may always absorb the time evolution into a running coupling, $g(t)$, as is required to define self-similar RG flow and preserve scale-invariance, *cf.* section 1.2.2. Specifically we find then find that:

$$\bar{\mathrm{V}}(\bar{\rho}, t) \sim \bar{\mathrm{V}}_*(\bar{\rho}) + \varepsilon \mathrm{e}^{\lambda_n t} c\bar{\rho}^{(d-\lambda_n)/(d-2)}, \tag{3.3.9}$$

regardless of the fixed point we expand around. From this we conclude that any

[6] In $d = 4$ the marginal operator is irrelevant, for dimensions less than 4 analysis of each marginal operator will have to be done separately.

[7] Further comments on quantised (ir)relevant perturbations about both the Gaussian fixed point and general fixed points can be found in refs. [39–41]

(marginally) relevant polynomial eigenoperator, *for any fixed point*, shows the correct scaling in terms of a physical coupling g_λ of mass dimension λ_n that falls into the FP in the UV, given by the scaling $\mathrm{e}^{\lambda_n t}$, even for large $\bar{\rho}$. They can therefore be used to form a renormalised trajectory required for a physical QFT.

Now we must apply the same analysis to the non-polynomial eigenoperators w_λ, equation (3.2.1), to prove that they not only qualify as true relevant perturbations when $\lambda > 0$, as suggested by the linearised analysis, but also provide linearly independent solutions compared to the polynomial eigenoperators. If we now choose the potential to be equal to one such eigenoperator at an arbitrary initial time $t_0 = 0$:

$$\bar{\mathrm{V}}(\bar{\rho}, 0) = \varepsilon w_\lambda\Big(\kappa^{-1}\bar{\rho}\Big) \ , \tag{3.3.10}$$

we cannot now write down an analytic form for the exact solution $\bar{\mathrm{V}}(\bar{\rho}, t)$ in general for large fields. However the right hand side of either the LPA Legendre flow equation or the sharp cutoff flow equation, equations (2.2.17) and equation (2.2.19) respectively, cannot contribute to the t dependence of the asymptotic behaviour, *cf.* equation (3.2.2). In fact, the exponential dependence of the Kummer function for large fields implies that for large $\bar{\rho}$ the right hand side the sharp cutoff equation, equation (2.2.19), can contribute at most a term of $\mathcal{O}(\bar{\rho})$ and the right hand side of the LPA Legendre flow equation, equation (2.2.17), is actually exponentially suppressed. Therefore, like in with the $n \geq 2$ polynomial potentials, the leading t dependence of the asymptotic expansion is found by requiring that the left hand side of the flow equations vanish, *i.e.* it follows for large $\bar{\rho}$ that in physical units non-polynomial potentials do not evolve at all and are, again, given by the mean field evolution result. Therefore, in physical units we find that:

$$\mathrm{V}(\rho, t) = \mathrm{V}(\rho, 0) = \varepsilon\mu^d w_\lambda\Big(\mu^{-(d-2)}\kappa^{-1}\rho\Big) \ . \tag{3.3.11}$$

In dimensionless variables appropriate to determining the RG flow, *cf.* equation (2.0.3), near the Gaussian fixed point and again using $\bar{\rho} = \kappa\omega$, the mean field evolution implies:

$$\bar{\mathrm{V}}\left(\kappa\omega, t\right) = \mathrm{e}^{dt}\bar{\mathrm{V}}\left(\kappa\omega\mathrm{e}^{-(d-2)t}, 0\right) = \varepsilon\mathrm{e}^{dt} w_\lambda\Big(\omega\mathrm{e}^{-(d-2)t}\Big) \ , \tag{3.3.12}$$

i.e. the asymptotic non-linear flow of the non-polynomial potentials takes the form:

$$\bar{\mathrm{V}}\left(\kappa\omega, t\right) = \varepsilon\, \hat{C}\, \frac{\mathrm{e}^{\left(2d+N\kappa^{-1}-\lambda\right)t}}{\omega^{p}}\, \exp\left(\omega \mathrm{e}^{-(d-2)t}\right)\left[1 + O\left(\frac{\mathrm{e}^{(d-2)t}}{\omega}\right)\right]. \tag{3.3.13}$$

We see that far from falling in to the GFP in the UV, $t \to -\infty$, in the large $\bar{\rho}$ regime the perturbation actually diverges rapidly, dominated by the $\exp\left(\kappa^{-1}\bar{\rho}\mathrm{e}^{-(d-2)t}\right)$ term. That is, in the large field regime, the ultraviolet behaviour of the non-quantised perturbations fail to behave correctly as relevant perturbations about the Gaussian fixed point: they do not generate an RG flow that emanates from the GFP for all values of $\bar{\rho}$ and thus cannot be used to form a continuum limit governed by the GFP. Additionally, we cannot avoid this problem by limiting the the field values to the linear regime as it is these large field values that support the linear independence of the non-polynomial directions from polynomial directions, *cf.* equations (3.1.9) and (3.1.12).

We have seen that (3.3.13) can be deduced from neglecting the right hand sides of the flow equations. This neglect is justified providing $\bar{\mathrm{V}}(\bar{\rho}, t) \gg 1$. Despite its simplicity, this argument is therefore inherently non-perturbative. In fact, as mentioned above, for large $\bar{\rho}$ the right hand side of (2.2.19) can contribute at most a term of $\mathcal{O}(\bar{\rho})$, while the the left hand side is at least $\mathcal{O}(\mathrm{e}^{\bar{\rho}})$, and the right hand side of (2.2.17) is actually exponentially suppressed. Therefore, we recognise that contributions from the right hand side of the flow equation are infinitely suppressed in the large $\bar{\rho}$ regime, in the sense that they make no contribution to the t dependence of the asymptotic expansion, equation (3.3.13), even if we had carried the multiplicative series corrections in $1/\bar{\rho}$ to infinite order.

We can generalise this to a potential constructed out of any sum of these non-polynomial eigenoperators. Specifically, if we replace our initial potential $V(\bar{\rho}, 0)$, equation (3.3.10), by an initial perturbation that is a general linear combination of the non-polynomial eigenoperators:

$$\bar{\mathrm{V}}(\bar{\rho}, 0) = \varepsilon \int \mathrm{d}\lambda\, \varsigma(\lambda)\, w_{\lambda}(\kappa^{-1}\bar{\rho}), \tag{3.3.14}$$

for some sufficiently well-behaved density factor $\varsigma(\lambda)$, then either this perturbation can already be re-expressed in terms of the polynomial eigenoperators, as in equation (3.1.9), and thus will have RG evolution determined by them, or it lies outside the

Hilbert space defined by the polynomial eigenoperators in which case, according to equation (3.1.12), the total potential must grow for some finite large value of $\bar{\rho}$ at least as fast as $\bar{\rho}^{-N/4}\exp(\bar{\rho}/2\kappa)$. In this latter case we see again, by trivially adapting the argument below equation (3.3.13), that in the large $\bar{\rho}$ regime the potential will diverge away from the GFP as $t \to -\infty$. Therefore, any general non-polynomial potential cannot be formed using a renormalised trajectory from the GFP, *i.e.* it is not possible to construct a non-polynomial potential using GFP to control the continuum limit. Applying this to $d = 4$ we can see that the so called Halpern-Huang directions are, in fact, not asymptotically free as we can apply the above logic to show that they do not display the correct UV behaviour required for a continuum limit.

3.4 Generalising to Any Fixed Point

We have already seen that the polynomial eigenoperators in LPA can be shown to have the correct ultraviolet behaviour, even for non-trivial fixed points [40,43,50]. If we wish to extend the arguments regarding non-polynomial eigenoperators above to any fixed point potential we must first linearise the flow equations around a general fixed point potential. In particular, if we consider $d < 4$ dimensions, the flow equation for the effective potentials, equations (2.2.3), (2.2.17), and (2.2.19), have non-perturbative Wilson-Fisher fixed points [56, 89] which famously are known to describe the universal properties of a wide variety of statistical physics and condensed matter systems, such as the gas-liquid phase transition in $d = 3$ with $N = 1$ [5, 9]. Assuming there is a non-trivial fixed point potential $\bar{\mathrm{V}}_*(\bar{\rho}) \neq 0$, then applying the separation of variables ansatz, *cf.* equation (2.3.2), to a small perturbation around the fixed point gives us:

$$\bar{\mathrm{V}}(\bar{\rho}, t) = \bar{\mathrm{V}}_*(\bar{\rho}) + \varepsilon \exp(\lambda t) v_\lambda(\bar{\rho}). \tag{3.4.1}$$

If we now linearise the LPA Legendre flow equation, equation (2.2.17), around a general fixed point we get:

$$(\lambda - d)v_\lambda + (d-2)\bar{\rho}\partial_{\bar{\rho}}v_\lambda = \frac{(N-1)\partial_{\bar{\rho}}v_\lambda}{(1+\partial_{\bar{\rho}}\bar{\mathrm{V}}_*)^2} + \frac{\partial_{\bar{\rho}}v_\lambda + 2\bar{\rho}\partial_{\bar{\rho}}^2 v_\lambda}{(1+\partial_{\bar{\rho}}\bar{\mathrm{V}}_* + 2\bar{\rho}\partial_{\bar{\rho}}^2\bar{\mathrm{V}}_*)^2}\,, \tag{3.4.2}$$

where we have suppressed the $\bar{\rho}$ dependence of v_λ and $\bar{\mathrm{V}}_*$. For any given fixed point potential $\bar{\mathrm{V}}_*$, we may consider the terms dependent on the fixed point only as independent

functions of $\bar{\rho}$ compared to the eigenoperators v_λ. Particularly, writing:

$$f1(\bar{\rho}) = (1 + \partial_{\bar{\rho}} \bar{\mathrm{V}}_*(\bar{\rho}))^{-1} \tag{3.4.3a}$$

and

$$f2(\bar{\rho}) = (1 + \partial_{\bar{\rho}} \bar{\mathrm{V}}_*(\bar{\rho}) + 2\bar{\rho}\partial_{\bar{\rho}}^2 \bar{\mathrm{V}}_*(\bar{\rho}))^{-1}, \tag{3.4.3b}$$

then we may rewrite the flow of the eigenoperators as:

$$(\lambda - d)v_\lambda + (d-2)\bar{\rho}\partial_{\bar{\rho}} v_\lambda = (N-1)(f1(\bar{\rho}))^2 \partial_{\bar{\rho}} v_\lambda + (f2(\bar{\rho}))^2 (\partial_{\bar{\rho}} v_\lambda + 2\bar{\rho}\partial_{\bar{\rho}}^2 v_\lambda)\,. \tag{3.4.4}$$

then we can see that the eigenoperator equation is a linear ODE. If we consider the form of the two functions $f1$ and $f2$, then the fact that the fixed point potentials are smooth [40, 43, 50] implies that the functions $f1$ and $f2$ must remain finite for finite values of $\bar{\rho}$. This can be seen by noting that the right hand side of the flow equations for $\bar{\mathrm{V}}$, equations (2.2.17) and (2.2.19), can be written in terms of $f1$ and $f2$, therefore if either diverges for any $\bar{\rho}$ then the left hand side of the flow equations implies that either the fixed point or the derivative of the fixed point diverges, *i.e.* $\bar{\mathrm{V}}$ *is not smooth*, which is contradictory. From this we can say that the two function $f1$ and $f2$ do not introduce additional singularities to eigenoperator equation, equation (3.4.4). Then, there is a mathematical theorem that says that we can transform this ODE into a hypergeometric equation, similar to the Kummer equation given in equation (3.0.1), which has non-polynomial solutions given by a confluent hypergeometric equation for all λ apart from a discrete set λ_n where these solutions collapse to polynomial solutions [100–102].[8] These solutions then define a one-parameter set for each λ, with this parameter being the normalisation which we are free to choose since the equation is linear.

We therefore conclude that for any fixed point, including the Wilson-Fisher fixed point in $d = 3$ that describes physical critical phenomena in SM, there exists a continuous spectrum of non-polynomial eigenoperators. However, if we examine the Hilbert space defined by the polynomial eigenoperators around the GFP we find that *the same*

[8] The theorem states that any linear second order differential equation with at most three regular singular points can always be transformed into a hypergeometric equation. In the case that there are only two singularities, then we have a special limit where two regular singularities have merged into an irregular singularity, giving a confluent hypergeometric equation. Kummer's equation, equation (3.0.1) is then the the simplest confluent hypergeometric equation [100–102].

Hilbert space of polynomial directions applies to any fixed point. From this we can immediately conclude that for non-polynomial eigenoperators to be relevant around any fixed point, not just the GFP, and independent from the polynomial eigenoperators then they must scale at least as fast $\bar{\rho}^{N/4}\mathrm{e}^{\bar{\rho}/2\kappa}$ for large $\bar{\rho}$. One can then trivially adapt the arguments following equation (3.3.13) to show that even perturbing around non-trivial fixed points non-polynomial potentials do not show correct ultraviolet behaviour required for a physical continuum limit.

From this we can see that, in general, non-polynomial potentials do not connect to the FP at large field values due the exponential scaling and therefore cannot be used to construct a renormalised trajectory required for a finite mass QFT. There is one other major problem that plague these non-polynomial potentials, which one can intuit from the lack a well defined continuum limit, namely that the RG time evolution cannot be separated out of these non-polynomial directions. They therefore do not define self-similar RG flow, implying that they depend upon more than the RG scale k. We now proceed to move away from UV analysis in order to demonstrate this, by showing that as we flow towards a physical theory in the infrared, relative to an arbitrary $t = 0$, that these non-polynomial eigenoperators flow back into Hilbert space of polynomial eigenoperators, *i.e.* the linear independence of the non-polynomial directions is in fact dependent upon the scale that defines $t = 0$.

Chapter 4

Infrared Properties: Scale Dependence of Renormalisation Group Flows.

We have shown that non-polynomial eigenoperators do not show the correct RG behaviour, relative to an arbitrary initial RG time $t_0 = 0$, to flow into back into a FP potential in UV, *i.e.* for $t \to -\infty$. This therefore means that it is not possible to take the continuum limit with a non-polynomial potential. This however covers only half of the possible RG flow from the initial RG time t_0. We now proceed to investigate the effect of integrating over more momentum modes, including additional vacuum fluctuations, by considering the flow in the remaining $t > 0$ domain. We find that the non-polynomial directions of the Gaussian fixed point in fact flow back into the Hilbert space of polynomial directions beyond a set small positive value of t. This implies that, upon complete integration of the path integral only the polynomial interactions remain.

Further, by comparing the RG flow at finite positive t, $t \to 0+$, and $t < 0$ we can see that the RG behaviour of these non-polynomial directions are in fact dependent upon an arbitrary additional scale through the definition of t_0. This implies that the non-polynomial potentials based upon the eigenoperators of the Gaussian fixed point cannot show self-similar flow, instead the flow implicitly introduces dependence upon the mass scale μ through the definition $t = \ln(\mu/k)$. More specifically, dependence upon an additional mass scale would implicitly introduce dependence upon the overall cutoff scale [85] that will lead to divergences in the continuum limit. This final conclusion is

sufficient to show that any non-polynomial potential based upon the non-polynomial eigenoperators w_λ cannot lead to a finite mass QFT in the limit $\Lambda \to \infty$. This conclusion is then generalised to non-trivial fixed points at which point we find that only polynomial potentials can show self-similar flow, *i.e.* we prove that non-polynomial potentials within the LPA are non-perturbatively non-renormalisable.

Expanding on this by explicitly investigating the infrared behaviour of the LPA Polchinski equation we find that the flow of the Wilsonian interactions is much better controlled. Due to the nature of the LPA Polchinski equation we find that the RG flow for large potentials is in fact suppressed towards a finite polynomial limit controlled by what is known in SM literature as the high temperature fixed point, a massive non-interacting fixed point given by a quadratic potential. This suppression then implies that the expansion of non-polynomial potentials in the basis given by polynomial eigenoperators will always converge, *i.e.* the RG behaviour is strictly given by the polynomial eigenoperators only. As these findings are a direct consequence of non-perturbative RG flow of the Wilsonian potential, which we found above follows directly from the condition required for non-polynomial interactions to be independent interactions, we conclude that the non-perturbative RG flow of the Wilsonian interactions forbids independent non-polynomial interactions around the Gaussian fixed point. Finally these conclusions are expanded to include non-trivial fixed points, completing our proof that only polynomial interactions show the correct RG behaviour for $O(N)$ scalar field theory in the LPA.

4.1 Couplings in the Legendre effection action

From our analysis, it can be shown that there is always a field value $\bar{\rho}$ large enough that the RG flow of one of the non-polynomial eigenoperators w_λ is determined by its asymptotic behaviour, leading to the mean-field result that we repeat here to refer to throughout this chapter:

$$\bar{\mathrm{V}}\left(\kappa\omega, t\right) = \varepsilon\, \hat{C}\, \frac{\mathrm{e}^{\left(2d+N\kappa^{-1}-\lambda\right)t}}{\omega^{p}}\, \exp\left(\omega \mathrm{e}^{-(d-2)t}\right)\left[1 + O\left(\frac{\mathrm{e}^{(d-2)t}}{\omega}\right)\right], \tag{4.1.1}$$

where we again use $\bar{\rho} = \kappa\omega$, *i.e.* such that neglect of the right hand side of the flow equations continues to be justified:

$$\partial_t \bar{\mathrm{V}}(\bar{\rho}, t) + (d-2)\bar{\rho}\partial_{\bar{\rho}}\bar{\mathrm{V}}(\bar{\rho}, t) - d\bar{\mathrm{V}}(\bar{\rho}, t) \approx 0\,. \tag{4.1.2}$$

To be precise, let $\omega = \omega_{asy}(t)$ be such that $\bar{\mathrm{V}}(\rho, t) \gg 1$ according to (4.1.1). Inverting, we find

$$\omega_{asy} \sim \mathrm{e}^{(d-2)t}\ln(1/\epsilon) + A\,, \tag{4.1.3}$$

for some large constant A.[1] From the time dependence of this field value we can say that for a given fixed RG time t_f, we can approximate the potential given by the initial condition $\bar{\mathrm{V}}(\bar{\rho}, 0) = \varepsilon w_\lambda(\bar{\rho})$ for all $\omega > \omega_{asy}(t_f)$ by the asymptotic form given in equation (4.1.1) for all $t < t_f$. The same cannot be said for any value of $t > t_f$ as the value of $\omega_{asy}(t) > \omega_{asy}(t_f)$.

We can then investigate the linear independence of a non-polynomial potential from the polynomial directions, using equation (3.1.9), for $t > 0$. Expanding the potential in terms of polynomial directions:

$$\bar{\mathrm{V}}(\bar{\rho}, t) = \sum_{n=0}^{\infty} g_{2n}(t) v_n(\kappa^{-1}\bar{\rho})\,, \tag{4.1.4}$$

cf. equation (3.1.9) and equation (3.1.11). Then approximating the polynomial eigenoperators by their leading power, $v_n(\omega) \sim 2^n \frac{\omega^n}{n!}$, implies that the integrand in the definition of g_{2n}, equation (3.1.11), has leading ω dependence for $\omega > \omega_{asy}$:

$$\omega^{n+\frac{(\lambda-d)}{(d-2)}-1}\mathrm{e}^{-a\omega}\,, \quad \text{where} \quad a = 1 - \mathrm{e}^{-(d-2)t} > 0\,. \tag{4.1.5}$$

Thus the integrals defining $g_{2n}(t)$ converge due to the exponential factor within a that behaves as $0 \leq \mathrm{e}^{-(d-2)t} < 1$ for all $t > 0$. Recalling that it is specifically the asymptotic, $\omega > \omega_{asy}$, behaviour that forbid the expansion of the non-polynomial directions, $w_\lambda(\bar{\rho})$, in terms of the polynomial directions, $v_n(\bar{\rho})$, we see that *as soon as $t > 0$, no matter how small, the couplings in the expansion* (3.1.9) *are well defined* [41].

In more detail, we can say from the asymptotic behaviour of the integrand within the couplings g_{2n}, equation (4.1.5), that for $n < \frac{(d-\lambda)}{(d-2)}$, the integrals defining g_{2n} are

[1] See sec. 4.2 for the higher order terms.

controlled by the fact $\omega^{n+\frac{(\lambda-d)}{(d-2)}-1} < \omega^{-1}$ for $t = 0$. Therefore, for $n < \frac{(d-\lambda)}{(d-2)}$ the integrals converge even for $a = 0$ and thus are of $\mathcal{O}(\varepsilon)$, for $t \geq 0$. Comparing this to the $d = 4$ Halpern-Huang potentials, *cf.* section 3.2.1, we find that for the most interesting potentials, the potentially symmetry-breaking potentials with $\lambda < 2$, this corresponds to $\mathcal{O}(\varepsilon)$ values for the running mass $g_2(t)$ and vacuum energy $g_0(t)$ for all finite $t \geq 0$.

Since we cannot have $n = \frac{(d-\lambda)}{(d-2)}$, the remaining possibility is $n > \frac{(d-\lambda)}{(d-2)}$.[2] For small positive t, we can taylor expand the factor a to find that the leading time dependence of the integrand is given by $a \approx (d-2)t$. Therefore the integrals only just converge due to the factor of $\mathrm{e}^{-a\omega}$ and are therefore dominated by the behaviour at very large ω. We can then compute the leading t dependence of the coefficients g_{2n} from the large ω behaviour. The coefficients corresponding to polynomial directions with $n > \frac{(d-\lambda)}{(d-2)}$, for $0 < t \ll 1$, are then given by:[3]

$$\begin{aligned} g_{2n}(t) \approx & \frac{\varepsilon(d-2)^{2n}}{2^{3n}} \frac{C}{\hat{\Gamma}\left(n+\frac{N}{2}\right)} \int\limits_{\omega_{asy}}^{\infty} \mathrm{d}\omega \; \omega^{n+\frac{(\lambda-d)}{(d-2)}-1} \, \mathrm{e}^{-(d-2)t\omega} \\ \approx & \frac{\varepsilon(d-2)^{2n}}{2^{2n}} \, 2^{\frac{(\lambda-d)}{(d-2)}} \frac{\hat{\Gamma}\left(\frac{N}{2}\right)}{\hat{\Gamma}\left(n+\frac{N}{2}\right)} \frac{\hat{\Gamma}\left(n+\frac{(\lambda-d)}{(d-2)}\right)}{\left|\hat{\Gamma}\left(\frac{\lambda-d}{d-2}\right)\right|} \left(\frac{1}{2(d-2)t}\right)^{n+\frac{(\lambda-d)}{(d-2)}} . \end{aligned} \tag{4.1.6}$$

Therefore, we prove that a potential based upon the non-polynomial eigenoperators of the GFP does have an expansion in the quantised directions for any $t > 0$ such that the corresponding couplings are infinitesimal. However, we see that while all the above couplings $g_{2n}(t)$ are finite for $t > 0$, they all diverge as $t \to 0^+$. In particular, we find that they diverge faster for increasingly irrelevant couplings, *i.e.* for larger n, *cf.* section (3.1). In this sense we see that non-linear flow of the non-polynomial directions imply that the $t = 0$ point is in fact already infinitely far from the GFP. However, as t increases to positive values, these couplings rapidly shrink as expected for the irrelevant couplings.

Indeed, the leading time behaviour for $t > 0$, equation (4.1.6), shows that for sufficiently small ε all couplings g_{2n} of increasing irrelevancy up to some maximum $n < n_{max}(\varepsilon)$ already shrink to the regime $g_{2n}(t) \ll 1$ before even for $t \ll 1$. We

[2] $n = \frac{(d-\lambda)}{(d-2)}$ implies that $\lambda = \lambda_n$, *cf.* equation (3.1.5), *i.e.* the eigenoperators are polynomial.

[3] In the first line of equation 4.1.6, we drop the $\mathcal{O}(1)$ integral over $0 < \omega < \omega_{asy}$ and insert the leading term from $w_n(\omega)$, *i.e.* $w_n(z) \approx 2^n \frac{\omega^n}{n!}$. Thus we can ignore an additive $\mathcal{O}(\varepsilon)$ correction. Additionally, we can ignore the multiplicative $\mathcal{O}(t)$ correction from the Taylor expansion of the ω independent exponential in t. In the second line we use the the fact that $0 < t \ll 1$ to drop multiplicative corrections of $\mathcal{O}(t\omega_{asy})$ coming from the lower limit of the integral.

might expect to have to take $\varepsilon \to 0$ to return to the GFP and form a continuum limit, particularly due to the poor behaviour in the ultraviolet for finite ε found in the previous chapter. If so, since $n_{max} \propto \ln(1/\varepsilon)$ we can say that the maximum n for which $g_{2n}(t) \ll 1$ increases without bound, implying that eventually, in the limit $\varepsilon \to 0$, all couplings shrink to infinitesimal with increasing t already for $t \ll 1$, *i.e.* if we attempt to construct a continuum limit from the $t > 0$ flow we find that the non-polynomial eigenoperators can be given by a sum of polynomial directions with infinitesimal couplings.

Furthermore, the asymptotic form of the integrand, equation (4.1.5), has a maximum at

$$\omega_{max} = \frac{n + \frac{(\lambda - d)}{(d-2)} - 1}{\left(1 - \mathrm{e}^{-(d-2)t}\right)}, \tag{4.1.7}$$

which is an $\mathcal{O}(1)$ value for any t larger than infinitesimal, after which the exponential decay of $\mathrm{e}^{-a\omega}$ takes over. Therefore, for $t > 0$ the large ω values actually make a negligible contribution to expansion coefficients and the integral is dominated by the bounded region $0 < \omega \lesssim O(1)$. Recalling the overall ε multiplier, we have thus established that *for all finite* $t > 0$, any potential that starts as a non-polynomial eigenoperator is given by a linear sum of polynomial eigenoperators whose couplings $g_{2n}(t)$ are not only finite, but infinitesimal ($\mathcal{O}(\varepsilon)$).

To be precise we prove that the couplings g_{2n} converge to a finite infinitesimal value. To complete the proof we must also consider the convergence of the full potential to full series expansion, *i.e.* the convergence with respect to the generalised Laguerre weight, *cf.* equation (3.1.12). The series convergence to the exact solution $\bar{\mathrm{V}}(\bar{\rho}, t)$ is given once the integral in (3.1.10) converges. From the asymptotic form, equation (4.1.1), we see that this happens as soon as $1 - 2\mathrm{e}^{-(d-2)t} > 0$. Therefore, we find that the coefficients g_{2n} become finite and infinitesimal for all $t > 0$ and that *the complete expansion in polynomial directions converges for all* $t > \frac{\ln(2)}{(d-2)}$, which implies that any potential $\bar{\mathrm{V}}(\bar{\rho}, t)$ based upon the non-polynomial eigenoperators $w_\lambda(\bar{\rho})$ flows back into the Hilbert space spanned by the quantised directions $v_n(\bar{\rho})$.

It is then reasonable to expect that the infrared ($t \to \infty$ or equivalently $k \to 0$) fate of the non-polynomial directions is just governed by the perturbative and non-linear evolution of the polynomial directions, given that the couplings g_{2n} are $\mathcal{O}(\varepsilon)$, which as we have reviewed, includes only a finite number of relevant directions. Particularly,

following the analysis in section 3.3 up to equation (3.3.9), this leads to a perturbative potential formed by only the relevant polynomial directions in the infrared. In $d = 4$ this then leads to the high temperature fixed point with all interactions decayed away leaving only the diverging $v_1(\bar{\rho})\,\mathrm{e}^{2t}$ mass term, recovering the Higgs triviality problem.

At first sight this picture seems deeply contradictory to the fact that the non-linear flow of the eigenoperators, *i.e.* the large field behaviour, is fixed to be mean-field, *viz.* equation (3.3.12). This mean-field evolution would imply that in physical variables, *cf.* equation (3.3.11), the potential remains that of the original non-polynomial perturbation, equation (3.3.10), and should not actually depend on t at all. While it appears as though these two pictures differ greatly, in actuality these two pictures are consistent with each other.

We have already seen from our introduction on the RG, *cf.* section 1.2.2, and analysis of the linear mass term perturbation, *cf.* equation (3.3.6), that true RG properties are only manifest in properly scaled dimensionless variables. This is why we do not see the flow of the non-polynomial potential in physical variables falling back into the Hilbert space spanned by the quantised directions. However, we find instead that in physical variables the Hilbert space is growing to accommodate the non-polynomial interaction. This follows from the dimensional scaling, equation (2.0.3), which maps the explicit exponential in the generalised Laguerre weight, *cf.* equation (3.1.8), from $\exp(-\omega) \to \exp\left(-\omega/k^{(d-2)}\right)$, which thus overcomes the $\exp\left(\omega/\mu^{(d-2)}\right)$ present in physical form of the non-polynomial directions, equation (3.3.11), as soon as $k < \mu$.

4.2 Couplings in the Wilsonian effective action

To further address these issues, we now use the fact that the flows of the Legendre interactions and the Wilsonian interactions are equivalent under an exact duality, specifically in the LPA the flows of the Legendre potential $\bar{\mathrm{V}}$ can be transformed into the flow for the Wilsonian potential $\bar{\mathrm{U}}$ using a cutoff dependent generalised Legendre transformation [42]. Using this transformation we can turn the exact solution $\bar{\mathrm{V}}(\bar{\rho}, t)$ of the LPA Legendre flow equation, equation (2.2.17), with the non-polynomial boundary condition, equation (3.3.10), into an exact solution $\bar{\mathrm{U}}(\bar{\sigma}, t)$ of the LPA Polchinski flow equation, equation (2.2.3). This $\bar{\mathrm{U}}(\bar{\sigma}, t)$ representation, which effectively just builds back in the one-particle reducible contributions missing from the one-particle

irreducible $\bar{\mathrm{V}}(\bar{\rho}, t)$ [42, 85], has improved properties. The equivalent expansion in the polynomial perturbations for the Wilsonian potential is given by:

$$\bar{\mathrm{U}}(\bar{\sigma}, t) = \sum_{n=0}^{\infty} h_{2n}(t)\, v_n(\bar{\sigma})\,, \tag{4.2.1}$$

where we have the same polynomial directions due to the fact we are solving near the Gaussian fixed point: we have already seen that the Gaussian fixed point for the Polchinski flow, equation (2.2.3), is equal to GFP in the Legendre flow and that in the linearised regime, *i.e.* infinitesimally close to the FP, we obtain the same Kummer's equation, equation (3.0.1), and thus the same spectrum of eigenoperator solutions. We can therefore treat the series expansion in equation (4.2.1) with the same convergence criteria as that for the Legendre potential, equation (3.1.9), with the interchange $\bar{\rho} \to \bar{\sigma}$ and $\bar{\mathrm{V}} \to \bar{\mathrm{U}}$. However, as we will show, due to the Legendre transform duality, the couplings $h_{2n}(t)$ in fact converge and remain infinitesimal, $\mathcal{O}(\varepsilon)$, for all finite t including for $t \leq 0$. Furthermore, the equivalent convergence of the series expansion to the exact potential $\bar{\mathrm{U}}(\bar{\sigma}, t)$ with respect to the generalised Laguerre norm can be proven for all finite t.

Before proceeding we review the non-linear evolution of the mass perturbation as this can be shown to flow towards a fixed point in the infrared, *i.e.* when $t \to \infty$. This special case is due to the fact that for polynomials of rank 1 the right hand side of the LPA Polchinski equation, equation (2.2.3), has no field dependence from $\bar{\mathrm{U}}$. Starting with a potential defined by:

$$\bar{\mathrm{U}}(\bar{\sigma}, 0) = \varepsilon\, w_1(\bar{\sigma})\,, \tag{4.2.2}$$

we can examine the flow outside the linear regime using the LPA Polchinski equation. Its evolution follows a similar analysis to that of the quantised eigenoperators in section 3.3 above with the addition that we use the exact map given by generalised Legendre transform, equation (2.2.5). However, we note that we can just as easily solve for it directly from the LPA Polchinski equation. Neglecting the uninteresting vacuum energy term, we find from the rank one polynomial eigenoperator, *cf.* equation (3.1.6), gives us a potential of the form: that the solution takes the form:

$$\bar{\mathrm{U}}(\bar{\sigma}, t) = 2h_2(t)\bar{\sigma}\,. \tag{4.2.3}$$

Substituting this into the LPA Polchinski equation, equation (2.2.3), we find that:

$$\partial_t h_2(t) = 2h_2(t) - 4h_2^2(t)\,. \tag{4.2.4}$$

Then, from the fact that this must give us $h_2(0) = \varepsilon$ to agree with our initial potential we find that:

$$h_2(t) = \frac{\varepsilon \mathrm{e}^{2t}}{1 + 2\varepsilon\,(\mathrm{e}^{2t} - 1)}\,. \tag{4.2.5}$$

We therefore see that the non-linear term on the right hand side of the flow equation, namely the $2\bar{\sigma}(\partial_{\bar{\sigma}}\bar{\mathrm{U}})^2$ term, moderates the growth of this relevant coupling in the infrared and instead of diverging as e^{2t}, like the corresponding coupling in the Legendre potential, $g_2(t) = \varepsilon \mathrm{e}^{2t}$, as shown in equation (3.3.5), it obtains a limiting value given by:

$$\lim_{t\to\infty}(h_2(t)) = \frac{1}{2}\,. \tag{4.2.6}$$

Therefore, the mass term perturbation tends to a fixed point value:

$$\bar{\mathrm{U}}(\bar{\sigma}, t) \to \bar{\mathrm{U}}_*(\bar{\sigma}) = \bar{\sigma} - \frac{N}{d} \quad \text{as } t \to \infty\,, \tag{4.2.7}$$

which is in fact a fixed point of the flow, known in the literature as the high temperature fixed point.[4] It is this 'compactification' of the flow of the relevant coupling which will make it easier to see what is going on with the infrared evolution of the non-polynomial directions[5]. Specifically, we find that as we flow towards the IR the growth of the Wilsonian interactions is suppressed to be at most linear in $\bar{\sigma}$ and all flow becomes controlled by the high temperature fixed point.

Starting, as before, with some initial Legendre potential at $t = 0$ made of one of the polynomial or non-polynomial directions, *e.g.* equation (3.3.2), we can use the LPA form of the generalised Legendre transfomation, equation (2.2.5), to get an initial

[4]The factor of N/d can is found from the LPA Polchinski equation by assuming a fixed point potential of the form $\bar{\mathrm{U}}_*(\bar{\sigma}) = \bar{\sigma} + c$.

[5]The nature of the LPA Polchinski equation makes it appear as if the RG flow towards the IR reduces the number of possible interactions. If we consider this graphically then it would appear as it the space of allowed interactions is compacted as we flow towards the IR, which we describe as 'compactification' of the flow.

condition for $\bar{\mathrm{U}}(\bar{\sigma}, t)$:

$$\begin{aligned}\bar{\mathrm{U}}(\bar{\sigma}, 0) =& \bar{\mathrm{V}}(\bar{\rho}, 0) + \frac{1}{2}(\phi - \Phi)^2 \\ =& \bar{\mathrm{V}}_*(\bar{\rho}) + \varepsilon v_\lambda(\bar{\rho}) + \frac{1}{2}(\phi - \Phi)^2 .\end{aligned} \tag{4.2.8}$$

We have already shown that in the LPA the same the value of the GFP is the same for both potentials, $\bar{\mathrm{U}}_* = \bar{\mathrm{V}}_* = 0$ and that small perturbations around the GFP gives the same eigenvalue equation, Kummer's equation (3.0.1). We can see this directly from the Legendre transform, equation (2.2.5). Using the fact that the transformation implies that the two fields $\bar{\rho}$ and $\bar{\sigma}$ point in the same direction we can show that:[6]

$$\bar{\sigma} = \bar{\rho}(1 + \partial_{\bar{\rho}} \bar{\mathrm{V}}(\bar{\rho}, t))^2 . \tag{4.2.9}$$

Then for an infinitesimal perturbation around the fixed point we get:

$$\begin{aligned}\bar{\sigma} =& \bar{\rho}(1 + \partial_{\bar{\rho}} \bar{\mathrm{V}}_* + \varepsilon \partial_{\bar{\rho}} v_\lambda(\bar{\rho}))^2 \\ =& \bar{\rho}(1 + \partial_{\bar{\rho}} \bar{\mathrm{V}}_*(\bar{\rho}))^2 + \mathcal{O}(\varepsilon v_\lambda(\bar{\rho})) ,\end{aligned} \tag{4.2.10}$$

i.e. for infinitesimal perturbations around the Gaussian fixed point, $\bar{\mathrm{V}}_* = 0$, $\bar{\sigma} = \bar{\rho}$ up to corrections of order ε. Setting $\bar{\mathrm{V}}_*(\bar{\rho}) = 0$ in our initial condition based on the Legendre transform, equation (4.2.8), we then have:

$$\begin{aligned}\bar{\mathrm{U}}(\bar{\sigma}, 0) =& \varepsilon v_\lambda(\bar{\rho}) + (\sqrt{\bar{\rho}} - \sqrt{\bar{\sigma}})^2 \\ =& \varepsilon v_\lambda(\bar{\rho}) + \varepsilon^2 \bar{\rho} (\partial_{\bar{\rho}} v_\lambda(\bar{\rho}))^2 ,\end{aligned} \tag{4.2.11}$$

where we have used equation (2.2.8a) to rewrite the fields in terms of the Legendre potential. We know that for finite $\bar{\rho}$, *i.e.* $\bar{\rho} \sim \mathcal{O}(1)$, that the linearised solution is valid for all t, *i.e.* $\varepsilon v_\lambda \ll 1$ and we can ignore the second term. Thus, we can conclude, up to $\mathcal{O}(\varepsilon)$ corrections that all conclusions made in the linear regime for $\bar{\mathrm{V}}$ are equally valid when applied to perturbations of $\bar{\mathrm{U}}$ around the GFP. For non-trivial fixed points the Legendre transformation is non-trivial, even in terms of linearised solutions, due

[6] Evidently $\bar{\sigma}(\bar{\rho})$ is then monotonic increasing function, justifying the use of the Legendre transform, despite the fact for some potentials there exists non-trivial minima, which imply that $\bar{\mathrm{V}}$ may not increase monotonically, *e.g.* the possible spontaneous symmetry breaking potentials pointed out by Halpern and Huang [58, 59] *cf.* section 3.2.1.

to the fact that $\bar{\sigma}$ becomes a function of $\bar{\rho}$ and $\bar{\mathrm{V}}_*(\bar{\rho})$. However, it can be shown that the Legendre transform relations can be used to transform the LPA Legendre flow equation into the LPA Polchinski equation [42]. Therefore, while the eigenoperators are manifestly the same around the GFP, we still find the perturbations around any fixed point will give the same eigenvalues, as expected from universality, and after appropriately transforming the fields, equivalent eigenoperators.

However, when we consider the full *non-perturbative* flow, where the potentials are $\mathcal{O}(1)$ or larger, then the $\mathcal{O}(\varepsilon v_\lambda(\bar{\rho}))$ corrections become large and the exact solutions of $\bar{\mathrm{U}}(\bar{\sigma}, t)$ must be found from the LPA Polchinski equation, equation (2.2.3), with the boundary condition given by (4.2.8). From the Legendre transform, the fact that Φ_a and ϕ_a point in same direction, implies that:

$$\sqrt{\frac{\bar{\rho}}{\bar{\sigma}}} = 1 - \partial_{\bar{\sigma}}\bar{\mathrm{U}}(\bar{\sigma}, t) = \frac{1}{1 + \partial_{\bar{\rho}}\bar{\mathrm{V}}(\bar{\rho}, t)} \,. \tag{4.2.12}$$

Setting $\bar{\mathrm{V}}(\bar{\rho}, 0) = \varepsilon\, v_\lambda(\bar{\rho})$, we see that when $\bar{\rho} \to \infty$, the Legendre transform implies that $\partial_{\bar{\sigma}}\bar{\mathrm{U}} \to 1$ as $\bar{\sigma} \to \infty$. This implies that the Wilsonian potential $\bar{\mathrm{U}}(\bar{\sigma}, t)$ cannot grow faster than $\bar{\sigma}$ as $\bar{\sigma} \to \infty$, for all finite RG time, and thus in fact for both the infrared and ultraviolet behaviour. This is true for any $\bar{\mathrm{V}}$ growing faster than $\bar{\rho}$ and thus includes all polynomial interactions v_n with $n > 1$ and all the non-polynomial directions. If we attempt to generalise this to non-polynomial potentials in general, then from the fact they must be grow faster than $\bar{\rho}^{N/4}\mathrm{e}^{\bar{\rho}/2\kappa}$ for large $\bar{\rho}$, *cf.* equation (3.1.12) and section 3.4, implies that any non-polynomial potential found in terms of the Legendre effective potential leads to $\partial_{\bar{\sigma}}\bar{\mathrm{U}} \to 1$ as $\bar{\sigma} \to \infty$. We see therefore that the large field behaviour is generally under much better control in the Wilsonian picture.

Attempting to expand the Wilsonian effective potential in terms of the polynomial eigenoperators we can compute the couplings, *cf.* equation (3.1.11), for the expansion in equation (4.2.1) as:

$$h_{2n}(t) = (2\kappa)^{-2n}\, \frac{\hat{\Gamma}(n+1)}{\hat{\Gamma}\left(n + \frac{N}{2}\right)} \int_0^\infty \mathrm{d}\hat{\omega}\, \hat{\omega}^{\frac{N}{2}-1}\, \mathrm{e}^{-\hat{\omega}}\, w_n(\hat{\omega})\, \bar{\mathrm{U}}\left(\kappa\hat{\omega}, t\right) \,, \tag{4.2.13}$$

where we now explicitly differentiate $\hat{\omega} = \kappa^{-1}\bar{\sigma}$ from the equivalent field $\omega = \kappa\bar{\rho}$ defined in terms of $\bar{\rho}$. Since $\bar{\mathrm{U}}(\bar{\sigma}, t)$ grows no faster than $\bar{\sigma}$ for any Legendre potential that grows faster than $\bar{\rho}$, which includes all eigenoperators apart from v_1 and v_0, these integrals

always converge. Furthermore, this behaviour implies that the Wilsonian potential expanded in polynomial directions is square integrable, *cf.* equation (3.1.10), and the series expansion will always converge to the potential, *cf.* equation (3.1.12). In other words we have proven that *the dual potential* $\bar{\mathrm{U}}(\bar{\sigma}, t)$ *of the non-perturbative evolution of a Legendre potential constructed from non-polynomial directions can always (viz. for all finite* t*) be expanded in terms the polynomial directions such that the series expansion converges to right limit, and such that the couplings* $h_{2n}(t)$ *remain finite for all finite* t*.*

We have shown that for large $\bar{\sigma}$, $\bar{\mathrm{U}}(\bar{\sigma}, t) \to \bar{\sigma}$ up to corrections that grow slower than $\bar{\sigma}$. We see therefore that *the large field behaviour is governed by the high temperature fixed point* (4.2.7) *for all RG time* including the initial $t = 0$.

It is then possible to compute the leading corrections to this. Again using the fact that ϕ^a and Φ^a point in the same direction, the Legendre transformation and equation (4.2.12) implies:

$$\begin{aligned} \bar{\mathrm{U}}(\bar{\sigma}, t) = (\sqrt{\bar{\rho}} - \sqrt{\bar{\sigma}})^2 + \bar{\mathrm{V}}(\bar{\rho}, t) &= (\sqrt{\bar{\rho}} - \sqrt{\bar{\sigma}})^2 + \int \mathrm{d}\bar{\rho} \left(\sqrt{\frac{\bar{\sigma}}{\bar{\rho}}} - 1 \right) \qquad (4.2.14) \\ &= \bar{\sigma} - 2\sqrt{\bar{\rho}\bar{\sigma}} + \int \mathrm{d}\bar{\sigma} \, \frac{\mathrm{d}\bar{\rho}}{\mathrm{d}\bar{\sigma}} \sqrt{\frac{\bar{\sigma}}{\bar{\rho}}} \, . \qquad (4.2.15) \end{aligned}$$

The second line is in a useful form for inverting the transformation to obtain a series expansion for $\bar{\rho}(\bar{\sigma}, t)$ so that we can solve for the leading corrections iteratively. Let us mention however that the first line gives, by integration by parts, the most compact expression:

$$\begin{aligned} \bar{\mathrm{U}}(\bar{\sigma}, t) &= (\sqrt{\bar{\rho}} - \sqrt{\bar{\sigma}})^2 - \bar{\rho} + 2\sqrt{\bar{\rho}\bar{\sigma}} - \int \frac{d\bar{\rho}}{\sqrt{\bar{\sigma}}} \frac{d\bar{\sigma}}{d\bar{\rho}} \sqrt{\bar{\rho}} \\ &= \bar{\sigma} - \int d\bar{\sigma} \, \sqrt{\frac{\bar{\rho}}{\bar{\sigma}}} \, . \qquad (4.2.16) \end{aligned}$$

Assuming again that $\bar{\mathrm{V}}(\bar{\rho}, 0) = \varepsilon w_\lambda(\kappa^{-1}\bar{\rho})$, we can invert the Legendre transform of the fields, equation (2.2.8c), for large $\bar{\rho}$ using equation (4.1.1). As the square of the derivative of the Legendre effective potential, the $(\partial_{\bar{\rho}}\bar{\mathrm{V}})^2$, term dominates, one finds:

$$\bar{\rho} = \mathrm{e}^{(d-2)t} \ln(\sqrt{\bar{\sigma}}/\varepsilon) + \cdots \qquad (4.2.17)$$

Therefore for both $\bar{\sigma}$ and $\bar{\rho}$ of order of magnitude $\mathrm{e}^{(d-2)t} \ln(1/\varepsilon)$ or larger, *i.e.* when

either effective potential is large enough to leave the linearised solution, one can show that

$$\bar{\rho}(\bar{\sigma},t) = \kappa^{-1}\mathrm{e}^{(d-2)t}\left\{\ln\left(\frac{\sqrt{\bar{\sigma}}}{\varepsilon}\right) + \left(p-\frac{1}{2}\right)\ln\left(\ln\left(\frac{\sqrt{\bar{\sigma}}}{\varepsilon}\right)\right) \right. \\ \left. -(\kappa+1)t - \ln\hat{C} + O\left(\frac{\ln\ln\left(\sqrt{\bar{\sigma}}/\varepsilon\right)}{\ln(\sqrt{\bar{\sigma}}/\varepsilon)}\right)\right\} . \tag{4.2.18}$$

Thus we find that

$$\bar{\mathrm{U}}(\bar{\sigma},t) = \bar{\sigma} - 2\mathrm{e}^{\kappa^{-1}t}\sqrt{\bar{\sigma}\,\ln\left(\sqrt{\bar{\sigma}}/\varepsilon\right)} + O\left(\sqrt{\bar{\sigma}}\,\frac{\ln\ln\left(\sqrt{\bar{\sigma}}/\varepsilon\right)}{\sqrt{\ln(\sqrt{\bar{\sigma}}/\varepsilon)}}\right) . \tag{4.2.19}$$

Despite appearances the subleading term remains of the same size as the leading term as t increases, since the formula is only valid for $\bar{\sigma} \gtrsim \mathrm{e}^{(d-2)t}\ln(1/\varepsilon)$. Carefully discarding the terms that are the same size or smaller than the neglected corrections $O(\cdots)$, one can verify directly that (4.2.19) solves the Wilson/Polchinski LPA flow equation, equation (2.2.3).

We can then compute estimates for the couplings $h_{2n}(t)$ in the expansion, equation (4.2.1). From the equation for the h_{2n} (4.2.13), using the leading power for $v_n(\hat{\omega})$, we see they are bounded by an integral expression of form:

$$h_{2n}(t) \lesssim \hat{A}\int_0^\infty \mathrm{d}\hat{\omega}\;\hat{\omega}^{\frac{N}{2}+n-1}\,\mathrm{e}^{-\hat{\omega}}\,\bar{\mathrm{U}}\left(\kappa\hat{\omega},t\right) , \tag{4.2.20}$$

where:

$$\hat{A} = \frac{(d-2)^{2n}}{2^{3n}}\frac{\hat{\Gamma}(n+1)}{\hat{\Gamma}\left(n+\frac{N}{2}\right)} \tag{4.2.21}$$

Using equations (2.2.8c) and (4.2.14) with (2.2.8a), we can write the integral in terms of $\bar{\rho}$ and $\bar{\mathrm{V}}$:

$$h_{2n}(t) \lesssim \\ \hat{A}\int_0^\infty \mathrm{d}\omega\;\omega^{\frac{N}{2}+n-1}\left(1+\bar{\mathrm{V}}'+\kappa\omega\bar{\mathrm{V}}''\right)\left(\bar{\mathrm{V}}+\kappa\omega(\bar{\mathrm{V}}')^2\right)\left(1+\bar{\mathrm{V}}'\right)^{N+2n-1}\mathrm{e}^{-\omega\left(1+\bar{\mathrm{V}}'\right)^2}, \tag{4.2.22}$$

where, for brevity, we indicate a derivative with respect to $\bar{\rho}$ as:

$$\partial_{\bar{\rho}}\bar{\mathrm{V}}(\bar{\rho},t) = \kappa^{-1}\partial_{\omega}\bar{\mathrm{V}}(\kappa\omega,t) = \bar{\mathrm{V}}' \quad \text{and} \quad \partial^2_{\bar{\rho}}\bar{\mathrm{V}}(\bar{\rho},t) = \kappa^{-2}\partial^2_{\omega}\bar{\mathrm{V}}(\kappa\omega,t) = \bar{\mathrm{V}}''\,. \tag{4.2.23}$$

The integral is cut-off sharply for $\omega > \omega_1(t)$, where $\omega_1(t)$ satisfies $\omega(\partial_{\bar{\rho}}\bar{\mathrm{V}})^2 = 1$. This is a consequence of the exponential growth of the potential for large fields, *cf.* equation (4.1.1), therefore the factor of $\omega(1+\partial_{\bar{\rho}}\bar{\mathrm{V}})^2 \approx \omega((\partial_{\bar{\rho}}\bar{\mathrm{V}})^2)$ grows as an exponential in the field. In other words, the last term in the integral vanishes as the negative exponential of an exponential for $\omega > \omega_1(t)$. Since equation (4.2.18) in fact solves $\bar{\rho}(\partial_{\bar{\rho}}\bar{\mathrm{V}})^2 = \bar{\sigma}$, it is defined by inverting $\bar{\sigma} = \bar{\rho}(1+\partial_{\bar{\rho}}\bar{\mathrm{V}})^2$ for $\bar{\mathrm{V}} \gg 1$, we already know that $\omega_1(t) = \kappa^{-1}\,\bar{\rho}(\bar{\sigma},t)|_{\bar{\sigma}=1}$.

For $\bar{\rho} \sim \mathcal{O}(1)$, the integrand is the same as in equation (3.1.11), more precisely it collapses to the approximation

$$\omega^{\frac{N}{2}+n-1}\mathrm{e}^{-\omega}\bar{\mathrm{V}}\left(\kappa\omega,t\right)\,, \tag{4.2.24}$$

since in this case $\bar{\mathrm{V}} \sim \mathcal{O}(\varepsilon)$ allowing one to drop all the 'correction' terms in (4.2.22). The balance of terms changes as ω approaches $\omega_1(t)$. From (4.2.18), we see that $\omega_1(t) = \kappa^{-1}\,\bar{\rho}(\bar{\sigma},t)|_{\bar{\sigma}=1}$ diverges as $\omega_1 \sim \mathrm{e}^{(d-2)t}\ln(1/\varepsilon)$. Thus, from equation (2.2.8c) $\partial_{\bar{\rho}}\bar{\mathrm{V}} = 1/\sqrt{\omega_1} \sim \mathrm{e}^{-\kappa^{-1}t}/\sqrt{\ln(1/\varepsilon)}$ and thus from (4.1.1) $\bar{\mathrm{V}} \sim \mathrm{e}^{(1-\kappa^{-1})t}/\sqrt{\ln(1/\varepsilon)}$ and $\partial^2_{\bar{\rho}}\bar{\mathrm{V}} \sim \mathrm{e}^{-(\kappa^{-1}+1)t}/\sqrt{\ln(1/\varepsilon)}$. Applying these estimates to equation (4.2.22), we see that in this regime the first two brackets in the integrand are now dominated by the $\kappa\omega\partial^2_{\bar{\rho}}\bar{\mathrm{V}}$ and $\kappa\omega(\partial_{\bar{\rho}}\bar{\mathrm{V}})^2$ terms respectively. However, it is the $\mathrm{e}^{-\omega(1+\partial_{\bar{\rho}}\bar{\mathrm{V}})^2}$ term that makes the most significant contribution as, from $\omega_1(\partial_{\bar{\rho}}\bar{\mathrm{V}})^2 = 1$, we find that $\mathrm{e}^{-\omega(1+\partial_{\bar{\rho}}\bar{\mathrm{V}})^2} \approx \mathrm{e}^{-\omega} \sim \varepsilon^{\exp((d-2)t)}$.

The situation is illustrated in figure 4.2.1. Putting all these observations together we see that for finite $t > 0$ only the $\bar{\rho} \sim \mathcal{O}(1)$ regime contributes, due to the compactification of the flow at leading order and thus the leading estimates for the couplings in the two pictures agree $h_{2n}(t) = g_{2n}(t) \sim \mathcal{O}(\varepsilon)$, which implies that the convergence of the series expansion to polynomial potentials is guaranteed. For finite $t < 0$, we do not see this compactification and instead find that the $\omega \sim \omega_1$ regime dominates and thus one finds $h_{2n}(t) \sim \varepsilon^{\exp(d-2)t}$ to leading order. Further, the suppression from the $\mathrm{e}^{-\omega}$ factor can be applied to the series convergence, *cf.* equation (3.1.12), and we find that the series expansion converges to $\bar{\mathrm{U}}$ for all finite t. These estimates for the

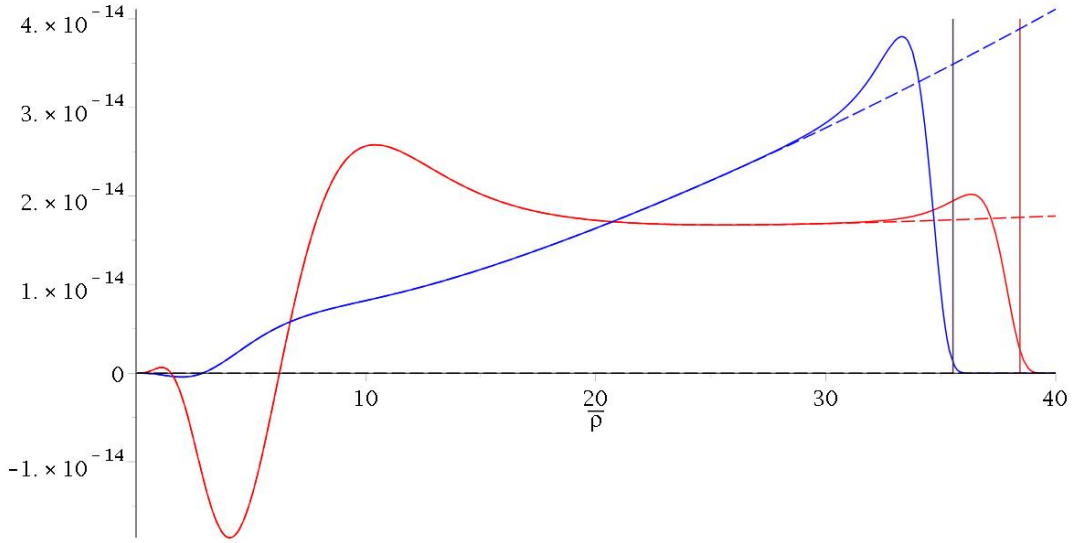

Figure 4.2.1: The exact integrand in equation (4.2.22) for $h_6(t)$ is plotted (together with the pre-multiplier) for the case $t = 0$, as follows from (3.3.10). We have set $d = 4$ and $N = 4$, $n = 3$ and $\varepsilon = 10^{-12}$. In red is shown the result for $\lambda = 1$, *i.e.* for a possible spontaneous symmetry-breaking Halpern-Huang potential (non-polynomial Higgs candidate), and in blue is plotted the integrand/10 for the case $\lambda = 3$, *i.e.* for a symmetry preserving Halpern-Huang potential (a more general case non-polynomial potential). The small minimum in the latter case appears because the potential starts out negative, but does not signify a non-trivial minimum. The dotted line curves show the equivalent integrands for $g_6(t)$, as it appears in equation (4.2.24), the equivalent to equation (3.1.11) taking only the leading power of v_n. The integrands only deviate once ω approaches ω_1. Shown in the black and red verticle lines are the values $\omega = \omega_1(0)$ for the cases $\lambda = 1, 3$ respectively. The figure thus verifies that the values for ω_1 closely approximate the effective cutoff points for the h_6 integrands.

couplings receive multiplicative corrections of form $(\ln(1/\varepsilon))^q$, for some finite power q, whose precise value would require a more in-depth analysis. However already we confirm that $\bar{\mathrm{U}}(\bar{\sigma}, t)$ *can be expanded as a convergent series in the quantised direction*, as in equation (3.1.9), for all t, and such that the couplings $h_{2n}(t)$ are infinitesimal for infinitesimal ε and all finite t.

On the other hand, let us remark that the solution we are analysing is very different from the solution of the Wilson/Polchinski flow equation that one would obtain from the boundary condition $\bar{\mathrm{U}}(\bar{\sigma}, 0) = \varepsilon\, v_\lambda(\bar{\sigma})$ by setting $v_\lambda = v_n(\bar{\sigma})$ with $n > 1$ or $v_\lambda = w_\lambda(\bar{\sigma})$. Indeed, in this case we find that, in the large $\bar{\sigma}$ region where $\bar{\mathrm{U}}(\bar{\sigma}, t) \gg 1$, the right hand side of the LPA Polchinski equation cannot be neglected and thus mean field evolution like in equation (4.1.1) does not take place. In particular the classical part itself (the first term on the right hand side of the LPA Polchinski equation, equation

(2.2.3)) continues to contribute to the evolution of $\bar{\mathrm{U}}(\bar{\sigma}, t)$. Furthermore, we see from our Legendre transform relation in equation (2.2.8b) that $\bar{\rho}$ must vanish at the point $\bar{\sigma}_0$ where $\partial_{\bar{\sigma}}\bar{\mathrm{U}}(y_0, t) = 1$. Therefore $\bar{\rho}(\bar{\sigma})$ is no longer monotonic increasing in these cases, causing the Legendre transform relation to break down.

Additionally, when generalising to non-trivial fixed point potentials, including the Wilson-Fisher fixed point in d=3, the Legendre transform relations and the conclusions we derive from them above become more complicated due to the fact that $\bar{\mathrm{V}}_* \neq 0$. While conclusions based upon setting our the effective potential to one of the polynomial eigenoperators $\bar{\mathrm{U}}(y, 0) = \varepsilon v_n(\bar{\sigma})$ in these cases would require rigorous in-depth analysis due to the growth of the $(\partial_{\bar{\sigma}}\bar{\mathrm{U}}(\bar{\sigma}, t)^2)$ term, the form of the flow equation, equation (2.2.3), implies that we can immediately forbid non-polynomial potentials.

Consider linearising the LPA Polchinski equation around an arbitrary fixed point, using a small perturbation of the form:

$$\bar{\mathrm{U}}(\bar{\sigma}, t) = \bar{\mathrm{U}}_*(\bar{\sigma}) + \varepsilon \exp(\lambda t) v_\lambda(\bar{\sigma})\,, \tag{4.2.25}$$

we get

$$(\lambda - d)v_\lambda + (d-2)\bar{\sigma}\partial_{\bar{\sigma}}v_\lambda = (N - 4\bar{\sigma}\partial_{\bar{\sigma}}\bar{\mathrm{U}}_*)\partial_{\bar{\sigma}}\bar{\mathrm{U}} + 2\bar{\sigma}\partial_{\bar{\sigma}}^2 v_\lambda\,, \tag{4.2.26}$$

where we have supressed the $\bar{\sigma}$ dependence of v_λ and $\bar{\mathrm{U}}_*$. This equation may easily be identified as a confluent hypergeometric equation for smooth $\bar{\mathrm{U}}_*(\bar{\sigma})$, as is required for any physical fixed point potential [40, 43, 50]. Therefore, regardless of the fixed point, we may consider an initial non-polynomial potential. As before, in order for these to non-polynomial directions to not only prove relevant, but linearly independent to the polynomial direction for the entire flow we must show they lie outside of the Hilbert space of polynomial interactions. From the convergence criteria we can then imply that these non-polynomial directions must grow at least exponentially, *cf.* equation (3.1.12). Specifically, they must scale at least for large fields as fast as:

$$\bar{\sigma}^{\frac{N}{4}} \exp\left(\frac{\bar{\sigma}}{2\kappa}\right)\,, \tag{4.2.27}$$

and we can adapt arguments from above to show that such large field behaviour *must be treated with the full flow equation*, in this case the LPA Polchinski equation.

From the stability conditions of the flow equation we can restrict the parameter space for the effective potential so that it must be smooth and have no divergences apart from $\bar{\sigma} \to \infty$ [40, 43, 50]. We can therefore reasonably expect a non-polynomial potential to be of the form, for large $\bar{\sigma}$:

$$\bar{\mathrm{U}}(\bar{\sigma}, t) \sim f(\bar{\sigma}, t) \mathrm{e}^{\frac{1}{2\kappa} g(\bar{\sigma}, t)} , \tag{4.2.28}$$

where the functions f and g must not introduce singularities for finite $\bar{\sigma}$, we impose that f grows less than exponentially, *i.e.* the leading behaviour is in terms of $\mathrm{e}^{g(\bar{\sigma}, t)}$, and we must have g be at least $\mathcal{O}(\bar{\sigma})$. For large $\bar{\sigma}$, this then implies that the derivative of the potential is then approximately given by:

$$\partial_{\bar{\sigma}} \bar{\mathrm{U}}(\bar{\sigma}, t) \sim f(\bar{\sigma}, t) \mathrm{e}^{\frac{1}{2\kappa} g(\bar{\sigma}, t)} \left(\frac{1}{2\kappa} \partial_{\bar{\sigma}} g(\bar{\sigma}, t) + \mathcal{O}\left(\frac{\partial_{\bar{\sigma}} f(\bar{\sigma}, t)}{f(\bar{\sigma}, t)} \right) \right) . \tag{4.2.29}$$

However, if we now divide the LPA Polchinski equation, equation (2.2.3), by a factor of $\partial_{\bar{\sigma}} \bar{\mathrm{U}}$ then for a general non-polynomial potential of the form given in equation (4.2.28) we find that each term apart from the original $(\partial_{\bar{\sigma}} \bar{\mathrm{U}})^2$ term can contribute at most $g(\bar{\sigma}, t)/\partial_{\bar{\sigma}} g(\bar{\sigma}, t)$, which we can reasonably expect to be $\mathcal{O}(\bar{\sigma})$. That is, for large $\bar{\sigma}$ we find that $\partial_{\bar{\sigma}} \bar{\mathrm{U}} \sim \mathcal{O}(\bar{\sigma})$ for all finite t, *i.e.* the non-linear flow of the effective potential forbids exponential growth of the potential. We again find that due to the non-linear terms 'compactification' the flow leads to a finite polynomial result in the IR, in this case to be at most $\mathcal{O}(\bar{\sigma}^2)$. Therefore we can always expand an effective potential $\bar{\mathrm{U}}$ in terms of polynomial potentials, which means that the effective potential $\bar{\mathrm{U}}$ is in fact always polynomial in the LPA. The general conclusion for polynomial potentials then applies and we find that for the Wilsonian potential, the RG flow will always tend towards the high temperature fixed point in the infrared.

Combining all the above observations for the Wilsonian potential $\bar{\mathrm{U}}(\bar{\sigma}, t)$: we find that for any non-polynomial one-particle irreducible potential, Legendre effective potential, that the 'compactification' of the RG flow, or equivalent growth of the Hilbert space of polynomial directions, leads to the dual one-particle reducible potential, the Wilsonian effective potential, to be polynomial. These polynomial potentials then lead to an infinite mass, non-interacting continuum limit defined by the high temperature fixed point, even for non-trivial fixed points. Finally, the non-linear flow of the Wilsonian effective potential, defined by the LPA Polchinski equation, implies that it is not

possible for a generic potential to exists outside of the Hilbert space of polynomial directions. That is, the Wilsonian effective potential is always polynomial.

Chapter 5

Concluding Remarks

The exact renormalisation group has historically been used to study both the critical behaviour and the possibility for non-perturbative renormalisability of systems described by $O(N)$ invariant scalar field theory. These studies are founded upon the scale invariance of near-critical systems and exploit the physical consequence of the scaling phenomena drawn from them. In the first chapter of this thesis we schematically detailed the roots of the renormalisation group, namely the scaling phenomena found in systems near phase transitions that lead to Widom's and Kadanoff's scaling picture [81, 83] and then expanded by Wilson's development of the renormalisation group [6, 7, 56]. The aim of this introduction was to both set the scene and bring out key ideas of the renormalisation group. Firstly, from the statistical physics point of view, the study of critical systems led to the idea of the universality of critical phenomena, implying that the solutions found within our study should be directly applicable to well documented cases of the Ising universality class amongst others by specialising to the relevant number of scalar field components, N, and space-time dimensionality d. Examples in $d = 3$ include entangled polymers for $N = 0$, atomic/molecular fluid-gas transitions described the Ising model for $N = 1$, superfluidic Helium described the XY model for $N = 2$, and the Heisenberg model of ferromagnetism for $N = 3$ [8, 9]. Secondly, from derivation of the renormalisation group as a partial integration over vacuum fluctuations, we found that the non-perturbative renormalisability of a quantum field theory is a direct consequence of the same critical behaviour studied in statistical mechanics. Thirdly, there exists a class of actions known as fixed point actions that do not change with the inclusion of vacuum fluctuations and therefore

do not depend on any scale. Fourthly, by taking a small perturbation around these fixed points one can find the "eigenoperators" or "scaling fields" of a fixed point, from which we may define all critical exponents, that scale directly into or away from the fixed point, given by the irrelevant and relevant eigenoperators respectively. Finally, by constructing a properly tuned set of the relevant eigenoperators one can construct an action that flows directly out of the fixed point, allowing for the removal of unphysical scales, such as momentum regularisation, without introducing divergences, thereby ensuring non-perturbative renormalisability of the theory. These ideas could then be applied to our study of non-polynomial scalar field potentials.

The study of these non-polynomial potentials is motivated from two stand points. Firstly, they may occur in the functional methods applied to modern gravitational theories, specifically with regards to asymptotically safe gravity and modified gravity models. Many of these models apply, amongst other approximations, an expansion in derivative operators and apply an ansatz of the action that includes a function of Ricci scalar to describe the derivative independent parts of the action [13–38]. The leading term of this derivative expansion, known as the local potential approximation, is found by dropping all derivative terms giving an action given by only this (polynomial) function of the Ricci scalar, *i.e.* $f(R)$ gravity. Due to the nature of these studies the only true way to test approximations and the assumptions applied is through rigorous understanding of the renormalisation group equations and renormalisability of quantum field theories in general. Particularly, most modified gravity models assume the function of the Ricci scalar must be polynomial and to understand the consequence of dropping this assumption can only be done by understanding the influence of the non-linear renormalisation group equations and expansions around possible fixed points. We simplify this problem however, by comparing these theories to the much simpler case of a d-dimensional $O(N)$ scalar field theory, which, unlike modified gravity theories, has accessible experimental counterparts for certain dimensionality and number of components N. Secondly, within the renormalisation group study of $O(N)$ scalar field theory two authors, Kenneth Halpern and Kerson Huang [58–60], proposed that by removing the assumption that the potentials must be polynomial in the field one can find non-polynomial solutions to the renormalisation group equations around the Gaussian fixed point, an action given by a field independent constant. Up to the point of this proposal the Gaussian fixed point represented only a non-interacting limit for

scalar field theories. However, by introducing a new parameter space of relevant eigenoperators Halpern and Huang then proposed, through non-polynomial eigenoperators, the possibility that the Gaussian fixed point provided, instead of a trivial physical result, a non-interacting ultraviolet completion of a quantum field theory, *i.e.* the theory was asymptotically free and therefore free of ultraviolet divergences. These theories then gained effective self interactions through the inclusion of vacuum fluctuations. The study of scalar field theory is therefore motivated by considerations of quantum gravity and the motivation of studying non-polynomial potentials is then derived from the proposal of Halpern and Huang.

Other research based upon this proposal was then undertaken by various other authors [61–73]. Of these studies, there are two interesting points regarding the application of "asymptotically free" scalar field theory that are worth repeating. Firstly, as shown by Gies [66], in the large N limit it can be shown that the RG flow of Halpern-Huang directions will always lead to $O(N)$ symmetry preserving potentials when all vacuum fluctuations are included. That is, even if such non-polynomial potentials could be used to define a renormalisable theory, this theory could not show spontaneous symmetry breaking. Therefore, the large N analysis of the Halpern-Huang directions forbids a Higgs like mechanism. Additionally to this, the second result of interest, derived by Gies and Scherer, is that when adding in fermion interactions to construct Yukawa interactions, as is required by the Higgs mechanism to give fermionic mass terms, the RG flow then shows that the Gaussian fixed point disappears altogether, *i.e. Halpern-Huang directions cannot exist* [67]. While this final point does not rule out non-polynomial scalar field potentials with Yukawa interactions in general, it implies that it cannot be done using the Gaussian fixed point to control the continuum limit and a separate fixed point solution must be found.

In terms of our study of these non-polynomial potentials, the key technical detail that was found in the background of the renormalisation group regards the scale dependence of a quantum field theory. Following the derivation of the exact renormalisation group, the effective action of a QFT defined at the floating RG scale are from incomplete functional integration inside of the partition function of momentum modes between some overall cutoff scale Λ, that we add in by hand, and the RG scale k. This integration is then implicitly dependent upon this overall momentum cutoff. While it is possible to remove the scale dependence of variables within the partition function

using powers of the RG scale, we cannot remove all scale dependence so long as the overall cutoff exists. Further, it was proven by Morris and Slade, by comparing the Legendre effective action defined at one overall cutoff to the effective action found with a different overall cutoff, that the effective action of the QFT defined at any RG scale will gain implicit dependence on any momentum scales within the theory [85]. Dependence upon the overall momentum cutoff not only breaks fundamental symmetries but causes, without possible fine tuning of an infinite number of interactions, all possible scales defined within the QFT will become dependent on the overall cutoff, leading to divergent behaviour in the limit $\Lambda \to \infty$. Therefore, in order to control the number of free parameters and avoid divergent behaviour we then find that a renormalisable QFT is controlled by RG fixed points.

However, while the fixed points represent scale independent actions that can be used to circumvent this issue by removing the cutoff, they represent massless theories. Perturbing around the RG fixed points, one then finds the eigenoperators of the fixed point, solutions that scale into the fixed or directly away from the fixed point as we complete the functional integration of the partition function. For the relevant eigenoperators, those that emanate from the fixed point, we find that the RG flow *must link them to fixed point* in the ultraviolet. That is, for any such eigenoperator that flows directly out of the fixed point we know that $\Lambda \to \infty$ limit is defined by a scale invariant action, therefore this scale cannot affect the scales of eigenoperator. A sum of these relevant eigenoperators may then be used to construct a potential, known at the renormalised trajectory, that has flows back into the fixed point in the ultravioulet. As such, the UV limit of the renormalised trajectory is controlled by the fixed point and we may safely take the continuum limit, $\Lambda \to \infty$. A QFT based upon this will then be finite but have no intrinsic scale. Therefore, a renormalisable QFT must then be based upon such a renormalised trajectory where the couplings of the various relevant eigenoperators are tuned to physical scale of the theory. It is the construction of the renormalised trajectory, or more specifically obtaining the continuum limit, that proves if an eigenoperator can truly lead to a renormalisable QFT.

Such issues are less important from the point of view of critical behaviour; the relevance of an eigenoperator is less about the construction of a finite field theory and more about which eigenoperators may or may not destroy critical behaviour with the change of their corresponding coupling. Particularly, we see in the condensed matter

laboratory that the scale dependent couplings of an eigenoperator in fact represent the deviation of physical thermodynamic variables from their critical value. Therefore, in this case, the new parameter space of eigenoperators introduced by Halpern and Huang should equally lead to a new parameter space of the scaling field operators that should, in theory, be coupled to thermodynamic variables. The lack of a continuum of thermodynamic variables coupled with discrete set of universal critical exponents then leasds one to question to validity of the proposed non-polynomial eigenoperator solutions.

Having motivated the work, we then proceeded to setup the functional renormalisation group equations for $O(N)$ scalar field theory in section 1.2.1. Within this we found that under a generalised Legendre transform one could transform the Wilsonian effective action to the Legendre effective action. The former physically represents the one-particle-reducible effective interactions obtained by integrating out the classical and quantum fields between the overall cutoff and the RG scale k, while the latter physically represents the one-particle-irreducible effective interactions obtained by only integrating over the quantum fields. As the two effective actions could be shown to give equivalent physics the relation between them was exploited in chapter 4 to extend the conclusions regarding the Legendre effective potential found in section 3.3 and 3.4 to the Wilsonian effective potential. Further, from this relation we found that while the Legendre potential may appear non-polynomial, the dual Wilsonian potential from the Legendre transform is always polynomial. Therefore, to be rigorous our study included non-polynomial potentials for both the Wilsonian effective action and the Legendre effective action.

We then proceeded in the second chapter to obtain the leading order flow equations in the derivative expansion for $O(N)$ scalar field theory. Expanding each effective action as series of linearly independent, specifically under integration by parts, derivative operators invariant under the $O(N)$ and Lorentz symmetry and any of the symmetries of our field or the effective actions $\mathcal{S}_k$ and Γ_k, including accidental symmetries [99], we then find, as shown in section 2.1, that by dropping all terms on the right hand side of the flow equations of order $\mathcal{O}(\partial^2)$ and above that the effective actions must be given by a local action. In this way we see that the local potential approximation can in fact be derived from both a restriction of the form of the effective actions and from a direct truncation of the full flow equations, which then leads to $\mathcal{O}(\partial^2)$ and higher

terms flowing only due to their canonical dimension, *i.e.* in physical variables they are fixed values that do not depend upon the renormalisation group scale that are set by physical considerations. This is gives the same local potential approximation first used to solve the Wegner-Hougton equation, derived from renormalisation group for scalar fields, at vanishing external momenta [44, 45]. Converting the flow equations to be in terms of the effective potentials in section 2.2 we then found in section 2.3 that for $d = 4$ dimensions there is only one fixed point potential, the Gaussian fixed point given where each of the effective potential vanishes. Linearising around this fixed point we then found that the LPA version of both the Polchinski flow equation and the Legendre flow equation lead to, for any sensible cutoff profile, the same eigenoperators.

The third chapter was dedicated to solving the flow equations near the Gaussian fixed point, particularly for the cases where eigenoperators could be used generate the continuum limit of a renormalised trajectory. The eigenoperator equation found by linearising any of the flow equations around the Gaussian fixed point was the same second order linear ordinary differential equation, Kummer's equation, given in equation (3.0.1). We briefly discussed the mathematical nature of the equation and found that only two types of solutions could represent a physical potential, given by the two classes of solutions derived from Kummer's confluent hypergeometric function $\mathrm{M}(\hat{a}, \hat{b}, z)$. In section 3.1 we discuss the first class of solutions, given where the first variable of $\mathrm{M}(\hat{a}, \hat{b}, z)$ is quantised as a negative integer including zero, $\hat{a} = -n$ for n being a non-negative integer, $n \in \mathbb{Z}^*$. In this case the Kummer function collapses to a generalised Laguerre polynomial of rank n, equation (3.1.7), which give the well known and documented polynomial eigenoperators of the $O(N)$ scalar field theory [39–48].[1] Particularly, these polynomial eigenoperators define a Hilbert space of polynomial interactions with respect to the generalised Laguerre weight that can be used to represent any possible local potential so long as the expansion coefficients, *cf.* equation (3.1.11), are finite and the overall series expansion converges back to the potential, *cf.* equation (3.3.1). A true non-polynomial potential must then exist outside of the Hilbert space of polynomial interactions. That is if we attempt the series expansion given in equation (3.1.9), then either the coefficients must diverge or the whole series expansion should not converge to the original potential.

[1] In some literature these polynomials are written as the entirely equivalent Hermite polynomials. However, the use of generalised Laguerre polynomials in this thesis is due to the fact that they are a direct subset of Kummer's M function.

Examining the eigenvalues of the these quantised solutions we find that only a small number of these solutions represent relevant eigenoperators. Particularly, in $d = 4$ dimensions we find that the only two relevant directions are given by an uninteresting vacuum constant and the linearised mass term. In this way we recover the usual Higgs triviality problem that the proposal of Halpern and Huang could resolve. Of particular importance regarding these polynomial directions is that even for large fields we show that they flow correctly into/out of the fixed point, even for non-trivial fixed points, as shown in section 3.3. As such we find that these polynomial solutions solve the renormalisation group equations for all values of the RG time t and may be used to construct a proper renormalised trajectory.

In section 3.2 we found that the non-polynomial directions were given by Kummer's M function for any eigenvalue $\lambda \neq d - n(d-2)$ with $n \in \mathbb{Z}^*$, implying that there exists a continuum of relevant eigenoperators! These solutions had an infinite series Taylor expansion for small fields but could be approximated by exponential field dependence for larger field values. Particularly, we find it is this asymptotic field behaviour that prevents the convergence of the polynomial direction expansion of the non-polynomial directions. This can easily be seen from the fact that the Kummer equation must have each polynomial direction as well as each non-polynomial direction be linearly independent solutions, as each solution is unique for each given λ. However, we stress that this is solely a mathematical property of the linearised equation that cannot be carried over to full non-linear RG flow. Further, in section 3.3, we find unless the potential is given by a constant, which is only the case for the vacuum constant eigenoperator, we can always find, for any solution to the linearised flow equations, a field value at which point the convergence of linearisation is no longer true. We note that this is one of the greatest distinctions between high energy physics and critical phenomena in condensed matter, the fact that we must consider the field (or equivalently the order parameter) is unbounded. Therefore, true physical eigenoperators solutions must solve the linearised flow equations as well the full non-linear flow equation.

From these considerations we examined the large field behaviour of both the quantised solutions and the non-polynomial solutions by assuming the potential was equal to a small perturbation from the fixed point in the direction of only one eigenoperator at $t = 0$, $\bar{\mathrm{V}}(\bar{\rho}, 0) = \bar{\mathrm{V}}_*(\bar{\rho}) + \varepsilon v_\lambda(\bar{\rho})$. Examining the large field behaviour using the LPA form of the Legendre flow equation we found that the right hand side of the equation

is suppressed and the flow of the potential is given solely by the dimensionality of the potential, *i.e.* it follows mean field evolution. Examining the polynomial directions we find that the asymptotic solution of the polynomial eigenoperators allows for the time evolution to always to separated into an infinitesimal coupling, as shown in equation (3.3.9), and therefore always shows the correct scaling for all field values. Particularly, the relevant polynomial directions flow back into the fixed point in the UV and therefore have a well defined continuum limit. On the other hand, when the mean field evolution takes over the flow of the non-polynomial potentials we find that, instead of flowing directly to or from the fixed point in the ultraviolet, the perturbation diverges away from the fixed point, as shown by equation (3.3.13) for GFP non-polynomial directions. In this way we proved that in the large field regime, the ultraviolet behaviour of the non-quantised perturbations fail to behave correctly as relevant eigenoperators and thus cannot generate the required Renormalised Trajectory emanating from a fixed point. Therefore, they cannot be used to form a continuum limit required to remove the artificial momentum cutoff and we cannot construct a renormalisable QFT using them. Further, we also saw that this conclusion could be generalised to any linear combination of these non-quantised directions that cannot be expanded in terms of the quantised perturbations, implying that non-polynomial eigenoperators of the GFP fail to behave correctly to obtain a continuum limit. In section 3.4 these conclusions were expanded for non-trivial fixed points, showing that mathematically non-polynomial eigenoperators appear around any fixed point. We could then show for non-polynomial directions in general, the asymptotic field behaviour lead to mean-field behaviour, which in turn leads to the non-polynomial direction diverging from the fixed point in the UV. From this, we conclude that the Legendre effective potential must be polynomial in order for us to take the continuum limit.

Despite the simplicity of the arguments, the asymptotic field analysis is inherently non-perturbative. The same asymptotic mean-field behaviour can be established for all field values larger than some $\bar{\rho}_{asy}(t) = \kappa\omega_{asy}(t)$ in the IR ($t > 0$) domain. However, we found, contrary to the UV behaviour, that the RG time evolution of the asymptotic mean-field behaviour suppressed the exponential field dependence in the IR. In this way we proved in section 4.1 that as soon as $t > 0$, no matter how small, that the couplings in the expansion of the non-polynomial directions in polynomial directions are well defined, while for $t > \ln(2)/(d-2)$ the Legendre effective potential based

on any non-polynomial eigenoperator falls fully back into the Hilbert space spanned by the polynomial interactions, specifically the convergence of the series expansion to the full potential is guaranteed. Furthermore we proved that for all *finite* $t > 0$, the corresponding couplings $g_{2n}(t)$ are infinitesimal, $\mathcal{O}(\varepsilon)$. Particularly we find that for $n < (d-\lambda)/(d-2)$ the couplings are $\mathcal{O}(\varepsilon)$ even at $t = 0$, while for $n > (d-\lambda)/(d-2)$, they diverge as a power of t as $t \to 0^+$. In this sense we see that the $t = 0$ point of the initial non-polynomial potential is in fact already infinitely far from the Gaussian fixed point. With all the above properties in mind, we see why at finite ε, the non-quantised solutions should more properly be viewed as no more special than any other finite perturbation added to the fixed point. However from the fact that the expansion coefficients are infinitesimal in the IR, as in equation (4.1.6), we also establish that in the limit $\varepsilon \to 0$ all couplings shrink to infinitesimal already at arbitrarily small positive $t > 0$. We therefore conclude that, at least for sufficiently small ε, that the infrared ($t \to +\infty$) fate of the non-polynomial directions is just governed by the perturbative evolution of these quantised direction. In four space-time dimensions with marginally irrelevant quartic coupling this leads in the infrared to the non-interacting but massive high temperature fixed point with all interactions decayed away leaving only a diverging $v_1(z)\mathrm{e}^{2t}$ mass term.

We note that these conclusions are drawn from the asymptotic mean-field behaviour of the potentials. However, given the fact that the this behaviour is governed by the canonical dimensionality of the potential the asymptotic flow is in fact linear. As a consequence of this, we may expand the above conclusions beyond simply expanding in terms of polynomial directions. Specifically, because the behaviour outside of the linearised regime, the flow of $\mathcal{O}(\varepsilon)$ perturbation to the fixed point, is equally linear, we may say explicitly that non-polynomial eigenoperators cannot be considered strictly relevant or irrelevant. Instead, we may only see the correct renormalisation group flow by considering each of the linearly independent polynomial constituents separately. By doing so we find, by mean field evolution, that the RG scale dependence may always be absorbed into a running coupling factor, as shown in equation (3.3.9). While this factoring is trivial within the linearised regime, the fact that it cannot be done simply for the non-polynomial directions, as can be seen in equation (4.1.1), implies that if they were independent solutions then they do not represent properly constructed renormalisation group flow. Specifically, as pointed out by Morris [39], unless the

action can be written in terms of running of couplings we cannot define self-similar renormalisation group flow. That is, *the RG flow is not solely dependent upon our floating integration scale k.*[2] This can be demonstrated simply by consideration of the starting potentials used in our studies in sections 3.3, 4.1, and 4.2. Given the definition of the renormalisation group time, $t = \ln(\mu/k)$, we then find that to set an initial condition $\bar{\mathrm{V}}(\bar{\rho}, t_0) = \varepsilon v_\lambda(\bar{\rho})$ for any t_0 (where we choose $t_0 = 0$) now includes an implicit value of μ. Particularly, the fact that the convergence of the series expansion is given by *the strict value* $t > \ln(2)/(d-2)$ implies that the convergence of non-polynomial directions to a sum of polynomial directions is dependent upon a physical momentum scale! It is therefore unsurprising that the non-polynomial directions cannot form a proper continuum limit as the property of self-similarity, which allows us to write the flow in the single scale k, is crucial for an action to be fixed under the renormalisation group, which we discuss in section 1.2.2, or to flow into/out of the fixed point, as we show in our discussion of physical variables with regard to the linearised mass term around equation (3.3.6). On the other hand, for the polynomial directions, the fact that flow is self similar implies that a physical scale is introduced solely when, in physical variables, a polynomial direction is tuned such that the renormalised trajectory matches the physical scale found in experiment [39, 88]!

Finally in section 4.2 we used the generalised Legendre transform relation, (2.2.5), to study the RG evolution of the corresponding exact solution $\bar{\mathrm{U}}(\bar{\sigma}, t)$ to the LPA Polchinski equation, equation (2.2.3). Here the high temperature fixed point is given by a genuine fixed point of the flow, and the 'compactification' of RG flow implies that the expansion of non-polynomial directions is better controlled. Particularly, we find that the corresponding quantised couplings $h_{2n}(t)$ are finite for all t. Additionally, we saw that the Legendre transform relation remains valid and the large field behaviour is governed by the high temperature fixed point (4.2.7), for all finite RG time, both positive and negative. We also proved that this expansion in polynomial direction, equation (4.2.1), is well defined and convergent always, *i.e.* for all t. Additionally we found that the leading dependence for the couplings vanishes in the limit $\varepsilon \to 0$, as is to be expected from an infinitesimal perturbation.

Expanding upon these ideas to generalise to non-trivial fixed points we find, unlike with the LPA Legendre equation, the LPA Polchinski equation does not lead to mean

[2] text

field evolution for large fields, instead that the large field behaviour is entirely controlled by the high temperature fixed point. By linearising around a non-trivial fixed point we show that there should, again, exist non-polynomial mathematical solutions. However, when attempting to force a general asymptotic form of this solution through the LPA Polchinski equation we find that non-linear terms control the flow sufficiently that they cannot grow fast enough to leave the Hilbert space of polynomial interactions. From this, we can conclude that it impossible for a physical Wilsonian effective potential to be non-polynomial. Combined with the fact that non-polynomial Legendre effective potentials do not display the required self-similar flow to form a continuum limit, *we conclude in general that, according to the exact renormalisation group, there are no renormalisable non-polynomial potentials in the local potential approximation.*

Appendix A

Renormalisation Group Equations from General Cutoff Functions

If we consider blocking the degrees of freedom within a scalar field theory using a general cutoff function, such that it smoothly suppress momentum modes of the scalar field φ above the momentum scale k, then we cannot follow the derivation shown in section 1.2.1. This is due to the fact that split of momentum modes *cannot be done exactly*. Instead we block degrees of freedom by smoothly suppressing momentum modes above the floating scale k and allow momentum modes below k to propagate freely. This case is better described by using a multiplicative cutoff, $\mathcal{C}_{UV}$, instead of the additive cutoff function, R_k, used above. Specifically, we choose $\mathcal{C}_{UV}$ such the low momentum modes propagate with the UV cutoff propagator defined as:

$$\Delta_{UV}(q^2) = \frac{\mathcal{C}_{UV}}{q^2} \tag{A.0.1}$$

and we must have $\mathcal{C}_U V = 1$ for $q^2 \leq k^2$ and $\mathcal{C}_U V \to 0$ sufficiently quickly for $q^2 > k^2$ such that the high momentum modes remain mostly unaffected by the cutoff function. This will then describe a situation where high momentum modes may be integrated out (almost) freely, with $\mathcal{C}_{UV}$ acting as an infrared cutoff in the path integral that suppresses the integration over the low momentum modes. Conversely, with the high momentum modes integrated out the cutoff function then describes an ultraviolet cutoff

propagator for a QFT of the low momentum modes. This description allows us to write the partition function as:

$$\begin{aligned}\mathcal{Z}[\mathrm{J}] &= \int \mathcal{D}\varphi \ \exp\left(-\hat{\mathcal{S}}[\varphi] + \mathrm{J}^a \cdot \varphi^a\right), \\ &= \int \mathcal{D}\varphi \ \exp\left(-\mathcal{S}_k[\varphi] - \frac{1}{2}\varphi^a \cdot \Delta_{UV}^{-1} \cdot \varphi^a + \mathrm{J}^a \cdot \varphi^a\right),\end{aligned} \tag{A.0.2}$$

where this description reparametrises the partition function in terms of the bare action $\hat{\mathcal{S}}$ to be in terms of a scale dependent UV cutoff propagator and an effective action that can be identified as the Wilsonian interactions, $\mathcal{S}_k$, of a UV cutoff effective QFT. As an additional condition we assume that the source term does not couple with the high momentum modes. This condition implies that:

$$\mathrm{J} \cdot \partial_k \Delta_{UV} = 0 \tag{A.0.3}$$

The physics of the system must not change with our cutoff scale k, therefore we may impose:

$$\partial_k \mathcal{Z}[\mathrm{J}] = 0 = -\int \mathcal{D}\varphi \ \left(\partial_k \mathcal{S}_k[\varphi] + \frac{1}{2}\varphi^a \cdot \partial_k \Delta_{UV}^{-1} \cdot \varphi^a\right) \exp\left(-\Sigma_k[\varphi, \mathrm{J}]\right), \tag{A.0.4}$$

where we define:

$$\Sigma_k[\varphi, \mathrm{J}] = \mathcal{S}_k[\varphi] + \frac{1}{2}\varphi^a \cdot \Delta_{UV}^{-1} \cdot \varphi^a - \mathrm{J}^a \cdot \varphi^a. \tag{A.0.5}$$

In order for the statement $\partial_k \mathcal{Z}[\mathrm{J}] = 0$ to be true the Wilsonian interactions must change with the scale k. This is simply a statement of renormalising couplings as we integrate out degrees of freedom. To find this scale dependence we can exploit the fact that the total functional derivative [104]:

$$\partial_k \Delta_{UV} \left[\frac{\delta^2}{\delta\varphi(\vec{\mathbf{p}})\delta\varphi(-\vec{\mathbf{p}})} \mathrm{e}^{-\Sigma_k} + \Delta_{UV} \frac{\delta}{\delta\varphi(\vec{\mathbf{p}})}(\varphi(\vec{\mathbf{p}})\mathrm{e}^{-\Sigma_k}) + \Delta_{UV} \frac{\delta}{\delta\varphi(-\vec{\mathbf{p}})}(\varphi(-\vec{\mathbf{p}})\mathrm{e}^{-\Sigma_k})\right] \tag{A.0.6}$$

$$= \Delta_{UV}^2 \left[\mathcal{N} - \varphi(\vec{\mathbf{p}})\partial_k \Delta_{UV} \varphi(-\vec{\mathbf{p}}) - \frac{\delta \mathcal{S}_k}{\delta\varphi(\vec{\mathbf{p}})} \partial_k \Delta_{UV} \frac{\delta \mathcal{S}_k}{\delta\varphi(-\vec{\mathbf{p}})} + \partial_k \Delta_{UV} \frac{\delta^2 \mathcal{S}_k}{\delta\varphi(\vec{\mathbf{p}})\delta\varphi(-\vec{\mathbf{p}})}\right],$$

where $\mathcal{N}$ is some field independent constant, we have used the fact that J only couples to momentum modes below k, and we have suppressed field indices and the field dependence of $\mathcal{S}_k$ and Σ for brevity. This therefore implies that the flow of the Wilsonian

interactions is given by [52, 86]:

$$\partial_k \mathcal{S}_k[\varphi] = \frac{1}{2}\frac{\delta \mathcal{S}_k}{\delta \varphi^a} \cdot \frac{\partial \Delta_{UV}}{\partial k} \cdot \frac{\delta \mathcal{S}_k}{\delta \varphi^a} - \frac{1}{2}\mathrm{Tr}\left[\frac{\partial \Delta_{UV}}{\partial k} \cdot \frac{\delta^2 \mathcal{S}_k}{\delta \varphi \delta \varphi}\right], \tag{A.0.7}$$

which we recognise as the Polchinski equation. The difference between this version and the one found above, *cf.* equation (1.2.27), is then from the difference in fields, φ compared to Φ, and the cutoff dependence introduced by the propagator Δ_{UV}. However, the fact that the Polchinksi equation has the same form in both general and sharp cutoff formalism implies that the LPA Polchinski equation will take the same form. If we now consider the work by Ball et. al. [98], this is sufficient to show that the LPA Polchinski equation does not depend on the form of the smooth *or* sharp cutoff used, excluding the case of Wegner and Houghton type cutoff.

If we now consider the derivation of the Legendre flow equation for a general cutoff, we must first define the Legendre transform of the connected correlation function generator, W_k, which now cannot depend on low and high momentum modes separately. Explicitly we start with:

$$\tilde{\Gamma}_k[\phi] = -\mathrm{W}_k[\mathrm{J}] + \mathrm{J}^a \cdot \phi^a\,, \tag{A.0.8}$$

similar to the Legendre transform with a sharp cutoff, *cf.* equation 1.2.31, except W_k is defined directly from the partition function in equation (A.0.2):

$$\mathrm{W}_k = \ln(\mathcal{Z}_k)\,, \tag{A.0.9}$$

and now the classical field is given by:

$$\phi(x) = \frac{\delta \mathrm{W}_k[\mathrm{J}]}{\delta \mathrm{J}(x)} = \langle \varphi^a \rangle\,. \tag{A.0.10}$$

The angle brackets notation here denotes an expectation value of some operator, *e.g.* ζ, given by:

$$\big\langle \zeta \big\rangle = \frac{\int \mathcal{D}\varphi\ \zeta\ \exp\left(-\Sigma_k[\varphi, \mathrm{J}]\right)}{\mathcal{Z}[\mathrm{J}]}\,. \tag{A.0.11}$$

If we now take the scale derivative we find that:

$$\begin{aligned}\left.\frac{\partial\tilde{\Gamma}_k[\phi]}{\partial k}\right|_\phi =& -\left.\frac{\partial \mathrm{W}_k[\mathrm{J}]}{\partial k}\right|_\mathrm{J} - \frac{\delta \mathrm{W}_k[\mathrm{J},\varphi_<]}{\delta \mathrm{J}^a}\cdot\left.\frac{\partial \mathrm{J}^a}{\partial k}\right|_\phi + \left.\frac{\partial \mathrm{J}^a}{\partial k}\right|_\phi\cdot\phi^a\,, \\ =&\frac{1}{2}\left\langle\varphi^a\cdot\partial_k\mathrm{R}_k\cdot\varphi^a\right\rangle\,.\end{aligned} \tag{A.0.12}$$

$$\mathrm{G}_{ab}(\vec{\mathbf{x}},\vec{\mathbf{y}}) = \frac{\delta^2\mathrm{W}_k[\mathrm{J}]}{\delta\mathrm{J}^a(\vec{\mathbf{x}})\delta\mathrm{J}^b(\vec{\mathbf{y}})} = \langle\varphi^a(\vec{\mathbf{x}})\varphi^b(\vec{\mathbf{y}})\rangle - \langle\varphi^a(\vec{\mathbf{x}})\rangle\langle\varphi^b(\vec{\mathbf{y}})\rangle\,, \tag{A.0.13}$$

we can rewrite equation (A.0.12) as:

$$\partial_k\tilde{\Gamma}_k[\phi]\Big|_\phi = \frac{1}{2}\mathrm{Tr}\left[\mathrm{G}\cdot\partial_k\mathrm{R}_k\right] + \frac{1}{2}\phi^a\cdot\partial_k\mathrm{R}_k\cdot\phi^a\,. \tag{A.0.14}$$

From the definition of G we find that:

$$\int \mathrm{d}^d\vec{\mathbf{z}}\;\mathrm{G}_{ac}(\vec{\mathbf{x}},\vec{\mathbf{z}})\frac{\delta^2\tilde{\Gamma}_k}{\delta\phi^c(\vec{\mathbf{z}})\delta\phi^b(\vec{\mathbf{y}})} = \hat{\delta}_{ab}\delta^d(\vec{\mathbf{x}}-\vec{\mathbf{y}})\,, \tag{A.0.15}$$

which can be used to replace G with the inverse of the full propagator, given by the second functional derivative of $\tilde{\Gamma}_k$. Defining the bare propagator in the presence of our additive infrared cutoff function:

$$\Delta_{IR}(p^2) = (p^2+\mathrm{R}_k)^{-1}\,, \tag{A.0.16}$$

we can extract the kinetic term of the Legendre effective action to define the equivalent to the Legendre interactions defined in equation (1.2.40):

$$\tilde{\Gamma}_k[\phi] = \frac{1}{2}\phi^a\cdot\Delta_{IR}^{-1}\cdot\phi^a + \Gamma_k[\phi]\,. \tag{A.0.17}$$

Noting the fact that $\partial_k\mathrm{R}_k = \partial_k\Delta_{IR}^{-1}$ we then find that the flow of the Legendre interactions for a general cutoff function is given by: [50, 52, 87]:

$$\partial_k\Gamma_k[\phi] = \frac{1}{2}\mathrm{Tr}\left[\left[\hat{\delta}_{ab} + \Delta_{IR}\cdot\frac{\delta^2\Gamma_k[\phi]}{\delta\phi^a\delta\phi^b}\right]^{-1}\cdot\frac{\partial_k\Delta_{IR}^{-1}}{\Delta_{IR}}\right]\,. \tag{A.0.18}$$

We therefore find that the renormalisation group flow of the Legendre interactions for a general cutoff appears the same as the flow given a sharp cutoff function, *cf.* equation (1.2.43). One of the key points to take from this similarity is the fact that when we shift

to dimensionless variables and apply the LPA we will find the same flow equation as given by equation (2.2.10). As with the Polchinksi equation we therefore find that the conclusions we derive with a sharp cutoff scheme may be easily generalised to a smooth cutoff scheme, however in this case careful considerations of the cutoff dependence of the threshold functions in equation (2.2.10) must be made.

Bibliography

[1] I. Hamzaan Bridle and Tim R. Morris. Fate of nonpolynomial interactions in scalar field theory. *Phys. Rev.*, D94:065040, 2016, arXiv:1605.06075.

[2] E. Ising. Beitrag zur Theorie des Ferromagnetismus. *Zeitschrift für Physik*, 31:253–258, February 1925.

[3] Pierre Weiss. L'hypothèse du champ moléculaire et la propriété ferromagnétique. *J. Phys. Theor. Appl.*, 6(1):661–690, 1907.

[4] P. Curie. Sur la symétrie dans les phénomènes physiques, symétrie d'un champ électrique et d'un champ magnétique. *J. Phys. Theor. Appl.*, 3(1):393–415, 1894.

[5] L.P. Kadanoff. Scaling, Universality and Operator Algebras. In *Phase Transitions and Critical Phenomena*, volume 5A, chapter 1, pages 2–34. Academic Press, London, 1976.

[6] Leo P. Kadanoff. More is the Same: Phase Transitions and Mean Field Theories. *J. Statist. Phys.*, 137:777, 2009, arXiv:0906.0653.

[7] Michael E. Fisher. Renormalization group theory: Its basis and formulation in statistical physics. *Rev. Mod. Phys.*, 70:653–681, Apr 1998.

[8] M E Fisher. The theory of equilibrium critical phenomena. *Reports on Progress in Physics*, 30(2):615–731, 1967.

[9] J. Zinn-Justin. *Quantum Field Theory and Critical Phenomena*. International series of monographs on physics. Clarendon Press, 2002.

[10] W. Heisenberg. Über quantentheoretische Umdeutung kinematischer und mechanischer Beziehungen. *Zeitschrift für Physik*, 33(1):879–893, 1925.

[11] J. Berges, N. Tetradis, and C. Wetterich. Critical equation of state from the average action. *Phys. Rev. Lett.*, 77:873–876, Jul 1996.

[12] Tim R. Morris. The Renormalization group and two-dimensional multicritical effective scalar field theory. *Phys. Lett.*, B345:139–148, 1995, hep-th/9410141.

[13] Steven Weinberg. Ultraviolet Divergences in Quantum Theories of Gravitation. In *General Relativity: An Einstein Centenary Survey*, pages 790–831. CUP Archive, 1979.

[14] M. Reuter. Nonperturbative evolution equation for quantum gravity. *Phys. Rev.*, D57:971–985, 1998, hep-th/9605030.

[15] Max Niedermaier and Martin Reuter. The Asymptotic Safety Scenario in Quantum Gravity. *Living Rev. Rel.*, 9:5–173, 2006.

[16] Roberto Percacci. Asymptotic Safety. In D Oriti, editor, *Approaches to Quantum Gravity: Toward a New Understanding of Space, Time and Matter*, pages 111–128. Cambridge University Press, Cambridge, 2009, arXiv:0709.3851.

[17] Daniel F. Litim. Fixed Points of Quantum Gravity and the Renormalisation Group. In *Proceedings, Workshop on From quantum to emergent gravity: Theory and phenomenology (QG-Ph): Trieste, Italy, June 11-15, 2007*. 2008, arXiv:0810.3675. [PoSQG-Ph,024(2007)].

[18] Martin Reuter and Frank Saueressig. Quantum Einstein Gravity. *New J. Phys.*, 14:055022, 2012, arXiv:1202.2274.

[19] Juergen A. Dietz, Tim R. Morris, and Zoë H. Slade. Fixed point structure of the conformal factor field in quantum gravity. 2016, arXiv:1605.07636.

[20] Labus, Peter and Morris, Tim R. and Slade, Zoë H. Background independence in a background dependent renormalization group. *Phys. Rev.*, D94(2):024007, 2016, arXiv:1603.04772.

[21] Nobuyoshi Ohta, Roberto Percacci, and Gian Paolo Vacca. Renormalization Group Equation and scaling solutions for f(R) gravity in exponential parametrization. *Eur. Phys. J.*, C76(2):46, 2016, arXiv:1511.09393.

[22] Maximilian Demmel, Frank Saueressig, and Omar Zanusso. RG flows of Quantum Einstein Gravity in the linear-geometric approximation. *Annals Phys.*, 359:141–165, 2015, arXiv:1412.7207.

[23] Juergen A. Dietz and Tim R. Morris. Background independent exact renormalization group for conformally reduced gravity. *JHEP*, 04:118, 2015, arXiv1502.07396.

[24] Maximilian Demmel, Frank Saueressig, and Omar Zanusso. Fixed Functionals in Asymptotically Safe Gravity. In *Proceedings, 13th Marcel Grossmann Meeting on Recent Developments in Theoretical and Experimental General Relativity, Astrophysics, and Relativistic Field Theories (MG13): Stockholm, Sweden, July 1-7, 2012*, pages 2227–2229. 2015, arXiv:1302.1312.

[25] Maximilian Demmel, Frank Saueressig, and Omar Zanusso. A proper fixed functional for four-dimensional Quantum Einstein Gravity. *JHEP*, 08:113, 2015, arXiv:1504.07656.

[26] Nobuyoshi Ohta, Roberto Percacci, and Gian Paolo Vacca. Flow equation for $f(R)$ gravity and some of its exact solutions. *Phys. Rev.*, D92(6):061501, 2015, arXiv:1507.00968.

[27] Roberto Percacci and Gian Paolo Vacca. Search of scaling solutions in scalar-tensor gravity. *Eur. Phys. J.*, C75(5):188, 2015, arXiv:1501.00888.

[28] Astrid Eichhorn. The Renormalization Group flow of unimodular f(R) gravity. *JHEP*, 04:096, 2015, 1501.05848.

[29] Elisa Manrique and Martin Reuter. Bimetric Truncations for Quantum Einstein Gravity and Asymptotic Safety. *Annals Phys.*, 325:785–815, 2010, arXiv:0907.2617.

[30] Dario Benedetti and Francesco Caravelli. The Local potential approximation in quantum gravity. *JHEP*, 06:017, 2012, arXiv:1204.3541. [Erratum: JHEP10,157(2012)].

[31] Maximilian Demmel, Frank Saueressig, and Omar Zanusso. Fixed-Functionals of three-dimensional Quantum Einstein Gravity. *JHEP*, 11:131, 2012, arXiv:1208.2038.

[32] Juergen A. Dietz and Tim R. Morris. Asymptotic safety in the f(R) approximation. *JHEP*, 01:108, 2013, arXiv:1211.0955.

[33] Juergen A. Dietz and Tim R. Morris. Redundant operators in the exact renormalisation group and in the f(R) approximation to asymptotic safety. *JHEP*, 07:064, 2013, arXiv:1306.1223.

[34] Maximilian Demmel, Frank Saueressig, and Omar Zanusso. RG flows of Quantum Einstein Gravity on maximally symmetric spaces. *JHEP*, 06:026, 2014, arXiv:1401.5495.

[35] Kevin Falls, Daniel F. Litim, Konstantinos Nikolakopoulos, and Christoph Rahmede. Further evidence for asymptotic safety of quantum gravity. *Phys. Rev.*, D93(10):104022, 2016, arXiv:1410.4815.

[36] Pedro F. Machado and Frank Saueressig. On the renormalization group flow of f(R)-gravity. *Phys. Rev.*, D77:124045, 2008, arXiv:0712.0445.

[37] Martin Reuter and Holger Weyer. Conformal sector of Quantum Einstein Gravity in the local potential approximation: Non-Gaussian fixed point and a phase of unbroken diffeomorphism invariance. *Phys. Rev.*, D80:025001, 2009, arXiv:0804.1475.

[38] Alessandro Codello, Roberto Percacci, and Christoph Rahmede. Investigating the Ultraviolet Properties of Gravity with a Wilsonian Renormalization Group Equation. *Annals Phys.*, 324:414–469, 2009, arXiv:0805.2909.

[39] Tim R. Morris. Elements of the continuous renormalization group. *Prog. Theor. Phys. Suppl.*, 131:395–414, 1998, hep-th/9802039.

[40] Tim R. Morris. Three-dimensional massive scalar field theory and the derivative expansion of the renormalization group. *Nucl. Phys.*, B495:477–504, 1997, hep-th/9612117.

[41] Tim R. Morris. On the fixed point structure of scalar fields. *Phys. Rev. Lett.*, 77:1658, 1996, hep-th/9601128.

[42] Tim R. Morris. Equivalence of local potential approximations. *JHEP*, 07:027, 2005, hep-th/0503161.

[43] Tim R. Morris and Michael D. Turner. Derivative expansion of the renormalization group in O(N) scalar field theory. *Nucl. Phys.*, B509:637–661, 1998, hep-th/9704202.

[44] J. F. Nicoll and T. S. Chang. An Exact One Particle Irreducible Renormalization Group Generator for Critical Phenomena. *Phys. Lett.*, A62:287–289, 1977.

[45] Anna Hasenfratz and Peter Hasenfratz. Renormalization Group Study of Scalar Field Theories. *Nucl. Phys.*, B270:687–701, 1986.

[46] Daniel F. Litim. Derivative expansion and renormalization group flows. *JHEP*, 11:059, 2001, hep-th/0111159.

[47] Daniel F. Litim. Optimized renormalization group flows. *Phys. Rev.*, D64:105007, 2001, hep-th/0103195.

[48] Daniel F. Litim. Universality and the renormalisation group. *JHEP*, 07:005, 2005, hep-th/0503096.

[49] Geoffrey R. Golner. Nonperturbative renormalization-group calculations for continuum spin systems. *Phys. Rev. B*, 33:7863–7866, Jun 1986.

[50] Tim R. Morris. Derivative expansion of the exact renormalization group. *Phys. Lett.*, B329:241–248, 1994, hep-ph/9403340.

[51] Tim R. Morris and John F. Tighe. Convergence of derivative expansions of the renormalization group. *JHEP*, 08:007, 1999, hep-th/9906166.

[52] Tim R. Morris. The Exact renormalization group and approximate solutions. *Int. J. Mod. Phys.*, A9:2411–2450, 1994, hep-ph/9308265.

[53] Oliver J. Rosten. Fundamentals of the Exact Renormalization Group. *Phys. Rept.*, 511:177–272, 2012, arXiv:1003.1366.

[54] C. Bagnuls and C. Bervillier. Exact renormalization group equations. An Introductory review. *Phys. Rept.*, 348:91, 2001, hep-th/0002034.

[55] Bertrand Delamotte. An Introduction to the nonperturbative renormalization group. *Lect. Notes Phys.*, 852:49–132, 2012, cond-mat/0702365.

[56] K. G. Wilson and John B. Kogut. The Renormalization group and the epsilon expansion. *Phys. Rept.*, 12:75–200, 1974.

[57] M. LÃijscher and P. Weisz. Scaling laws and triviality bounds in the lattice φ^4 theory. *Nuclear Physics B*, 290:25 – 60, 1987.

[58] Kenneth Halpern and Kerson Huang. Fixed point structure of scalar fields. *Phys. Rev. Lett.*, 74:3526–3529, 1995, hep-th/9406199.

[59] Kenneth Halpern and Kerson Huang. Nontrivial directions for scalar fields. *Phys. Rev.*, D53:3252–3259, 1996, hep-th/9510240.

[60] K. Halpern and Kerson Huang. Reply to: Comment on 'Fixed point structure of scalar fields'. *Phys. Rev. Lett.*, 77:1659, 1996.

[61] Kerson Huang, Hwee-Boon Low, and Roh-Suan Tung. Scalar Field Cosmology I: Asymptotic Freedom and the Initial-Value Problem. *Class. Quant. Grav.*, 29:155014, 2012, arXiv:1106.5282.

[62] Kerson Huang, Hwee-Boon Low, and Roh-Suan Tung. Scalar Field Cosmology II: Superfluidity and Quantum Turbulence. *Int. J. Mod. Phys.*, A27:1250154, 2012, arXiv:1106.5283.

[63] Kerson Huang, Hwee-Boon Low, and Roh-Suan Tung. Cosmology of an asymptotically free scalar field with spontaneous symmetry breaking. 2010, arXiv:1011.4012.

[64] Kenneth Halpern. Cross-section and effective potential in asymptotically free scalar field theories. *Phys. Rev.*, D57:6337–6341, 1998, hep-th/9708124.

[65] Vincenzo Branchina. Nonperturbative renormalization group potentials and quintessence. *Phys. Rev.*, D64:043513, 2001, hep-ph/0002013.

[66] Holger Gies. Flow equation for Halpern-Huang directions of scalar O(N) models. *Phys. Rev.*, D63:065011, 2001, hep-th/0009041.

[67] Holger Gies and Michael M. Scherer. Asymptotic safety of simple Yukawa systems. *Eur. Phys. J.*, C66:387–402, 2010, arXiv:0901.2459.

[68] Artur R. Pietrykowski. Interacting scalar fields in the context of effective quantum gravity. *Phys. Rev.*, D87(2):024026, 2013, arXiv:1210.0507.

[69] B. Altschul. Nonpolynomial normal modes of the renormalization group in the presence of a constant vector potential background. *Nucl. Phys.*, B705:593–604, 2005, hep-th/0403093.

[70] B. Altschul. Asymptotically free Lorentz and CPT-violating scalar field theories. 2004, hep-th/0407173.

[71] B. Altschul and V. Alan Kostelecky. Spontaneous Lorentz violation and nonpolynomial interactions. *Phys. Lett.*, B628:106–112, 2005, hep-th/0509068.

[72] Brett Altschul, Quentin G. Bailey, and V. Alan Kostelecky. Lorentz violation with an antisymmetric tensor. *Phys. Rev.*, D81:065028, 2010, arXiv:0912.4852.

[73] Vipul Periwal. Halpern-Huang directions in effective scalar field theory. *Mod. Phys. Lett.*, A11:2915–2920, 1996, hep-th/9512108.

[74] J.D. van der Waals. *Over de Continuïteit van den Gas-en Vloeistoftoestand*. Leiden, 1873.

[75] Lars Onsager. Crystal statistics. i. a two-dimensional model with an order-disorder transition. *Phys. Rev.*, 65:117–149, Feb 1944.

[76] Franz J. Wegner. Critical Exponents in Isotropic Spin Systems. *Phys. Rev. B*, 6:1891–1893, Sep 1972.

[77] Michael E. Fisher and P. Pfeuty. Critical Behavior of the Anisotropic n-Vector Model. *Phys. Rev. B*, 6:1889–1891, Sep 1972.

[78] L.D. Landau and E.M. Lifshitz. *Statistical Physics*, volume 5 of *Course of Theoretical Physics*. Elsevier Science, 2013.

[79] P Heller. Experimental investigations of critical phenomena. *Reports on Progress in Physics*, 30(2):731, 1967.

[80] B. Widom. Surface Tension and Molecular Correlations near the Critical Point. *The Journal of Chemical Physics*, 43(11):3892–3897, 1965.

[81] L. P. Kadanoff. Scaling laws for Ising models near T(c). *Physics*, 2:263–272, 1966.

[82] Kenneth G. Wilson. Renormalization Group and Critical Phenomena. I. Renormalization Group and the Kadanoff Scaling Picture. *Phys. Rev. B*, 4:3174–3183, Nov 1971.

[83] B. Widom. Equation of State in the Neighborhood of the Critical Point. *J.Chem.Phys.*, 43(11):3898–3905, 1965.

[84] Juergen Berges, Nikolaos Tetradis, and Christof Wetterich. Nonperturbative renormalization flow in quantum field theory and statistical physics. *Phys. Rept.*, 363:223–386, 2002, hep-ph/0005122.

[85] Tim R. Morris and Zoë H. Slade. Solutions to the reconstruction problem in asymptotic safety. *JHEP*, 11:094, 2015, arXiv:1507.08657.

[86] Joseph Polchinski. Renormalization and Effective Lagrangians. *Nucl. Phys.*, B231:269–295, 1984.

[87] Christof Wetterich. Exact evolution equation for the effective potential. *Phys. Lett.*, B301:90–94, 1993.

[88] P. Hasenfratz. Perfect actions: From the theoretical background to recent developments. *Prog. Theor. Phys. Suppl.*, 131:189–231, 1998.

[89] Kenneth G. Wilson and Michael E. Fisher. Critical Exponents in 3.99 Dimensions. *Phys. Rev. Lett.*, 28:240–243, Jan 1972.

[90] Franz J. Wegner and Anthony Houghton. Renormalization Group Equation for Critical Phenomena. *Phys. Rev. A*, 8:401–412, Jul 1973.

[91] Jordi Comellas and Alex Travesset. O (N) models within the local potential approximation. *Nucl. Phys.*, B498:539–564, 1997, hep-th/9701028.

[92] Marco D'Attanasio and Tim R. Morris. Large N and the renormalization group. *Phys. Lett.*, B409:363–370, 1997, hep-th/9704094.

[93] E. Riedel, G. R. Golner, and K. E. Newman. Scaling Field Representation of Wilson's Exact Renormalization Group Equation. *Annals Phys.*, 161:178–238, 1985.

[94] K. E. Newman, E. K. Riedel, and S. Muto. Q State Potts Model BY Wilson's Exact Renormalization Group Equation. *Phys. Rev.*, B29:302–313, 1984.

[95] Kathie E. Newman and Eberhard K. Riedel. Critical exponents by the scaling-field method: The isotropic N-vector model in three dimensions. *Phys. Rev.*, B30:6615–6638, 1984.

[96] Tim R. Morris. Momentum scale expansion of sharp cutoff flow equations. *Nucl. Phys.*, B458:477–503, 1996, hep-th/9508017.

[97] A. Margaritis, G. Odor, and A. Patkos. Series Expansion Solution of the Wegner-houghton Renormalization Group Equation. *Z. Phys.*, C39:109, 1988.

[98] Richard D. Ball, Peter E. Haagensen, I. Latorre, Jose, and Enrique Moreno. Scheme independence and the exact renormalization group. *Phys. Lett.*, B347:80–88, 1995, hep-th/9411122.

[99] I. Hamzaan Bridle, Juergen A. Dietz, and Tim R. Morris. The local potential approximation in the background field formalism. *JHEP*, 03:093, 2014, arXiv:1312.2846.

[100] M. Abramowitz and I.A. Stegun. *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables.* Applied mathematics series. Dover Publications, 1964.

[101] N.N. Lebedev and R.A. Silverman. *Special Functions and Their Applications.* Dover Books on Mathematics. Dover Publications, 1972.

[102] F.W.J. Olver. *Asymptotics and special functions.* Computer science and applied mathematics. Academic Press, 1974.

[103] E. E. Kummer. De integralibus quibusdam definitis et seriebus infinitis. *Journal für die reine und angewandte Mathematik*, 1837:228–242, Jan 1837.

[104] J. Alexandre. BUSSTEPP 2016 lecture notes: Exact Wilsonian Renormalization. 2015, 1508.07763.