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Littlest Seesaw model from $S_4 \times U(1)$

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Abstract

We show how a minimal (littlest) seesaw model involving two right-handed neutrinos and a very constrained Dirac mass matrix, with one texture zero and two independent Dirac masses, may arise from $S_4 \times U(1)$ symmetry in a semi-direct supersymmetric model. The resulting CSD3 form of neutrino mass matrix only depends on two real mass parameters plus one undetermined phase. We show how the phase may be fixed to be one of the cube roots of unity by extending the $S_4 \times U(1)$ symmetry to include a product of Z_3 factors together with a CP symmetry, which is spontaneously broken leaving a single residual Z_3 in the charged lepton sector and a residual Z_2 in the neutrino sector, with suppressed higher order corrections. With the phase chosen from the cube roots of unity to be $-2\pi/3$, the model predicts a normal neutrino mass hierarchy with $m_1 = 0$, reactor angle $\theta_{13} = 8.7°$, solar angle $\theta_{12} = 34°$, atmospheric angle $\theta_{23} = 44°$, and CP violating oscillation phase $\delta_{\rm CP} = -93°$, depending on the fit of the model to the neutrino masses.

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1 Introduction

Despite great experimental progress in neutrino physics in the last twenty years [1], the origin of neutrino mass and lepton mixing remains unclear. Although there has been intense theoretical activity in this period, there is still no leading candidate for a theory of neutrino mass and lepton mixing (for reviews see e.g. [2,3]).

From a theoretical point of view the most appealing possibility seems to be the seesaw mechanism in its original formulation involving heavy right-handed Majorana neutrinos [4]. However the seesaw mechanism is very difficult to test experimentally, at least if the right-handed neutrino masses are beyond reach of the LHC, and also introduces many additional parameters. One approach to this problem is to follow the idea of minimality, leading to seesaw theories with smaller numbers of parameters and hence testable predictions [5]. If the predictions are realised experimentally then this may provide indirect experimental support for the seesaw mechanism, and in addition provide insights into the flavour problem. This is the approach we shall follow in this paper.

The most minimal version of the seesaw mechanism involves two right-handed neutrinos [6]. In order to reduce the number of free parameters still further to the smallest number possible, and hence increase predictivity, various approaches to the two right-handed neutrino seesaw model have been suggested, such as postulating one [7] or two [8] texture zeroes, however such two texture zero models are now phenomenologically excluded [9] for the case of a normal neutrino mass hierarchy considered here. The minimal successful scheme with normal hierarchy seems to be a two right-handed model with a Dirac mass matrix (in the diagonal charged lepton mass basis) involving one texture zero and a particular pattern of couplings, together with a diagonal right-handed neutrino mass matrix [10],

$$m^D = \begin{pmatrix} 0 & b \\ a & 3b \\ a & b \end{pmatrix}, \qquad M_R = \begin{pmatrix} M_{\text{atm}} & 0 \\ 0 & M_{\text{sol}} \end{pmatrix}, \tag{1}$$

where a, b are two complex parameters. The seesaw mechanism [4] leads to a light effective Majorana neutrino mass matrix:

$$m^\nu = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + m_b e^{i\eta} \begin{pmatrix} 1 & 3 & 1 \\ 3 & 9 & 3 \\ 1 & 3 & 1 \end{pmatrix}. \tag{2}$$

$m_a = |a|^2/M_{\text{atm}}$ and $m_b = |b|^2/M_{\text{sol}}$ may be taken to be real and positive without loss of generality, the physical predictions only depending on a relative phase whose phenomenologically preferred value is $\eta = 2\pi/3$ [10]. Following the proposed lepton model in [10], this structure has been incorporated into unified models of quarks and leptons in [11]. It has also been shown to lead to successful leptogenesis in which not only the sign of baryon asymmetry is determined by the ordering of the heavy right-handed neutrinos, but also η is identified as the leptogenesis phase, directly linking CP violation in the laboratory with that in the early universe [12].

The implementation of the seesaw mechanism above is an example of sequential dominance (SD) [13] in which the first term in Eq. (2), arising from the first (atmospheric) right-handed neutrino, provides the dominant contribution to the atmospheric neutrino mass, leading to approximately maximal atmospheric mixing, while the second subdominant term from the second (solar) right-handed neutrino gives the solar neutrino mass and controls the solar and reactor mixing and CP violation. If the constrained form of Dirac mass matrix in Eq. (1) is relaxed, but the texture zero is maintained, then SD generally leads to a reactor angle which is bounded by $\theta_{13} \lesssim m_2/m_3$ [7], a prediction that was made a decade before the reactor angle was measured in 2012 [1]. However sharp predictions for the reactor (and solar) angles can only result from applying constraints to the Dirac mass matrix of various types, an approach known as constrained sequential dominance (CSD) [14]. For example, keeping the first column of the Dirac mass matrix fixed $(0, a, a)^T$, a class of CSDn models has emerged [10, 14–17] corresponding to the second column taking the form $(b, nb, (n-2)b)^T$, with a reactor angle approximately given by [18]

$$\theta_{13} \sim (n-1)\frac{\sqrt{2}}{3}\frac{m_2}{m_3}, \tag{3}$$

where CSD1 [14] implies tri-bimaximal (TB) mixing with a zero reactor angle, CSD2 [15] has a reactor angle $\theta_{13} \sim \frac{\sqrt{2}}{3}\frac{m_2}{m_3}$, which is too small, CSD3 [10] in Eq. (1) predicts $\theta_{13} \sim \frac{2\sqrt{2}}{3}\frac{m_2}{m_3}$ which is in good agreement with the experimental value $\theta_{13} \sim 0.15$ [1], and CSD4 [16] predicts $\theta_{13} \sim \sqrt{2}\frac{m_2}{m_3}$, while higher values of $n > 4$ involve increasingly large values of the reactor angle which are disfavoured [17].

The seesaw scheme in Eq. (1) is referred to as either CSD3 or the Littlest Seesaw (LS) [18] since the seesaw mechanism only involves two complex Dirac masses a, b together with two real positive right-handed neutrino masses M_{atm} and M_{sol} (as compared to 18 parameters in the most general three right-handed neutrino seesaw mechanism). The resulting neutrino mass matrix in Eq. (2) involves only three parameters, namely the real positive mass parameters m_a, m_b together with the real phase η. It was realised [10] that if the phase is also fixed to be $\eta = 2\pi/3$ then this leads to a highly predictive and successful scheme, with only two remaining real positive input parameters m_a, m_b which may be determined by the physical neutrino masses m_2, m_3, with $m_1 = 0$ being an automatic prediction of two right-handed neutrinos. The entire PMNS mixing matrix is then uniquely predicted by the model.

Although the Littlest Seesaw is unquestionably minimal and predictive, the *Achilles Heel* of this model has always been its theoretical justification from symmetry. For example, assuming some family symmetry, spontaneously broken by some new Higgs fields (the so-called flavons) in the triplet representation, the structure of the Dirac mass matrix in Eq. (1) may in principle arise from the vacuum alignment of these flavons. However, the desired flavon vacuum alignment $(1, 3, 1)^T$, responsible for the second column of the Dirac mass matrix, does not seem to follow directly from any symmetry, but only indirectly via a sequence of flavon alignments which are mutually orthogonal [10, 17]. However it was recently realised that S_4 might be the best candidate symmetry for producing this alignment [18] since in the real basis it is the minimal symmetry that preserves a U type symmetry capable of equating two of the elements of the alignment, namely the first and third components of $(1,3,1)^T$. However to date it has not proved possible to construct a model in which both the neutrino mass matrix and charged lepton mass matrix structures are enforced by subgroups of the original family symmetry.[1]

In this paper, then, we shall propose a Littlest Seesaw model in which a minimal neutrino mass matrix, simply related to that in Eq. (2), follows from a semi-direct supersymmetric model plus some minimal dynamical constraints. This represents real progress since previously the Littlest Seesaw has only been realised in indirect models not enforced by any (discrete) symmetry considerations. In our semi-direct approach here we shall use $S_4 \times U(1)$ to enforce a version of the Littlest Seesaw which is simply related to that in Eqs. (1,2), by the permutation $L_2 \leftrightarrow L_3$. We shall also show that this new version of CSD3 may also be generalised to CSDn. The starting point for our approach here is the observation that Eq. (2) leads to trimaximal TM_1 mixing [20,21], in which the first column of the tri-bimaximal mixing matrix [22] is preserved. The inspiration for our approach comes from the semi-direct model of trimaximal TM_1 mixing that was developed in [23] in which, denoting the three generators of S_4 as S, U, T, the model preserves a residual Z_3 in the charged lepton sector arising from the T generator, and a Z_2 in the neutrino sector corresponding to the product SU. Following [23], we shall enforce the Littlest Seesaw by similar symmetry arguments, the notable difference being that in our case, instead of having three right-handed neutrinos in a triplet of S_4, the model here involves two right-handed neutrinos which are singlets of S_4.

We shall also impose a CP symmetry in the original theory which is spontaneously broken, where unlike [24], there is no residual CP symmetry in either the charged lepton or neutrino sectors. Nevertheless we shall obtain sharp predictions for CP violation by fixing the phase η in the neutrino mass matrix Eq. (2) to be one of the cube roots of unity due to a Z_3 family symmetry, using the mechanism proposed in [25]. In order to achieve this, we suppose that the original $U(1)$ which accompanies S_4 is extended to a product of $U(1)$ factors, where some of these are supposed to be explicity broken to Z_3 subgroups, which are subsequently spontaneously broken along with the S_4. This is perhaps the least appealing feature of our scheme, but it is necessary in order to obtain a sharp input value for the phase η, and hence CP violation, as well as the lepton mixing angles which also depend on η. We shall propose a concrete models along these lines based on S_4 together with one $U(1)$ factor accompanied by five Z_3 symmetries, and show that the desired leading order operator structure in both the Yukawa and vacuum alignment sectors have quite suppressed higher order corrections, leading to reliable predictions for observable neutrino masses as well as lepton mixing and CP parameters.

The layout of the remainder of the paper is as follows. In Section 2 we show how the

[1] In general, neutrino mass models based on discrete family symmetry may be classified into three types [19]: direct, semi-direct and indirect, depending on the residual symmetry preserved in the neutrino and charged lepton sectors. If the full Klein symmetry of the Majorana neutrino mass matrix and the symmetry of the charged lepton mass matrix are identified as subgroups of the original family symmetry, the models are known as direct, while semi-direct (or indirect) models correspond to cases where only a part (or none) of the residual symmetries may be identified as subgroups of the family symmetry.

Littlest Seesaw can arise from S_4 symmetry, avoiding any technical details, making the paper accessible to any casual reader. In Section 3 we show how the necessary vacuum alignments of CSD3 can arise from an F-term mechanism which does not rely on long chains of orthogonality conditions and is simpler than previous attempts. In Section 4 we describe a model of leptons based on $S_4 \times U(1)$ that leads to the Littlest Seesaw, then extend it to $S_4 \times U(1) \times (Z_3)^5$ in order to fix the phase to be a cube root of unity. Finally in Section 5 we briefly comment on charged lepton flavour violation in this model. Section 6 concludes the paper. In addition, Appendix A gives the necessary group theory of S_4, along with the symmetry preserved and broken by various vacuum alignments, and the S_4 Clebsch-Gordan coefficients. Appendix B generalises the version of CSD3 discussed in this paper to a new type of CSDn and presents analytic formulas for neutrino masses and lepton mixing parameters for this case.

2 Littlest Seesaw model from S_4: an overview

Before getting into too many technicalities of symmetry and model building, it is useful to give a sketch of the type of model we will present in this paper. This enables serious readers to have in mind where we are heading before getting immersed in the details, or casual readers to simply read this section of the paper, then jump to the Conclusions. The version of the Littlest Seesaw model in this paper involves lepton doublets which transform under S_4 as $L \sim \mathbf{3'}$, two right-handed neutrinos $N^c_{\rm sol} \sim \mathbf{1}$, $N^c_{\rm atm} \sim \mathbf{1}$ and the up- and down-type Higgs fields $H_{u,d} \sim \mathbf{1}$ with couplings in the superpotential:

$$\frac{\phi'_{\rm atm}}{\Lambda} L H_u N^c_{\rm atm} + \frac{\phi'_{\rm sol}}{\Lambda} L H_u N^c_{\rm sol} \ , \tag{4}$$

where the non-renormalisable terms are suppressed by a dimensionful cut-off Λ and the flavons $\phi'_{\rm atm} \sim \mathbf{3'}$ and $\phi'_{\rm sol} \sim \mathbf{3'}$ are required to have the vacuum alignments[2]

$$\langle \phi'_{\rm atm} \rangle = \varphi'_{\rm atm} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} , \quad \langle \phi'_{\rm sol} \rangle = \varphi'_{\rm sol} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} . \tag{5}$$

An important point we would like to emphasise is that, as discussed in Appendix A, in the S_4 basis employed in this paper the above vacuum alignments preserve the generator product SU, i.e. $SU\langle \phi'_{\rm atm} \rangle = \langle \phi'_{\rm atm} \rangle$ and $SU\langle \phi'_{\rm sol} \rangle = \langle \phi'_{\rm sol} \rangle$, but break T and U separately. Assuming that the charged lepton mass matrix is diagonal, the preserved S_4 subgroup SU is instrumental in enforcing TM$_1$ mixing as in the semi-direct model of [23]. However, unlike [23], this model involves two right-handed neutrinos which are assumed to have a diagonal mass matrix M_R.

The S_4 singlet contraction $\mathbf{3'} \otimes \mathbf{3'} \ \rightarrow \ \mathbf{1}$ implies $(L\phi')_{\mathbf{1}} = L_1 \phi'_1 + L_2 \phi'_3 + L_3 \phi'_2$ (see Appendix A), which leads to the Dirac neutrino mass matrix m^D, together with a diagonal

[2] The minus signs in the third components are related to the S_4 triplet basis as defined in Appendix A.

right-handed neutrino mass matrix M_R,

$$m^D = \begin{pmatrix} 0 & b \\ -a & -b \\ a & 3b \end{pmatrix} \equiv \begin{pmatrix} 0 & b \\ a & b \\ a & 3b \end{pmatrix}, \qquad M_R = \begin{pmatrix} M_{\rm atm} & 0 \\ 0 & M_{\rm sol} \end{pmatrix}, \tag{6}$$

where the equivalence above follows after multiplying L_2 by a minus sign. The seesaw mechanism $m^\nu = -m^D M_R^{-1} m^{D^T}$ implies[3]

$$m^\nu = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + m_b e^{i\eta} \begin{pmatrix} 1 & 1 & 3 \\ 1 & 1 & 3 \\ 3 & 3 & 9 \end{pmatrix}, \tag{7}$$

where without loss of generality, $m_a = |a|^2/M_{\rm atm}$, $m_b = |b|^2/M_{\rm sol}$ may be taken to be real and positive and η is a real phase parameter. Eq. (7) with $\eta = -2\pi/3$ gives a phenomenologically successful and predictive description of neutrino masses and lepton mixing parameters, as first discussed in [10]. In fact the neutrino mass matrix in Eq. (7) with $\eta = \pm 2\pi/3$ is one of the two CSD3 forms first discussed in [10].

The main point we wish to emphasise is that the neutrino mass matrix in Eq. (7), which is related to that in Eq. (2) by the permutation $L_2 \leftrightarrow L_3$, leads to phenomenologically successful predictions for neutrino parameters for a phase $\eta = \pm 2\pi/3$. In Table 1 we compare predictions from the two forms of CSD3 neutrino mass matrix in Eq. (7) and Eq. (2) for some benchmark input parameters m_a, m_b, η. The two types of CSD3 yield identical predictions for the reactor and solar angles as well as the neutrino masses, for the same values of m_a, m_b, while the predictions for the atmospheric angle have the same values of $\sin 2\theta_{23}$ but are in different octants of θ_{23}. It is clear that both types of CSD3 give good predictions for lepton mixing angles, assuming that $\eta = \pm 2\pi/3$. In both examples in Table 1 the CP phase is predicted to be $\delta_{\rm CP} \approx -\pi/2$.[4] For the original CSD3, $\eta = 2\pi/3$ is identified as the leptogenesis phase and the baryon asymmetry of the universe leads to a determination of the lighter atmospheric neutrino mass $M_{\rm atm} = 4 \times 10^{10}$ GeV [12]. For the new type of CSD3 here we expect leptogenesis to fix the lighter solar right-handed neutrino mass to be $M_{\rm sol} = 4 \times 10^{10}$ GeV due to the preferred opposite value of the leptogenesis phase $\eta = -2\pi/3$.

In Appendix B, the mass matrix in Eq. (7) is generalised to a new type of CSDn, and analytic formulas for neutrino masses and lepton mixing parameters are presented for any real value of n (although we are only interested in $n = 3$ here). The results may be compared to the numerical results in [17] and the analytic formulas in [18] for the original version of CSDn based on a generalisation of Eq. (2).

[3] We follow the Majorana mass convention $-\frac{1}{2}\overline{\nu_L} m^\nu \nu_L^c$.

[4] In addition, the CSD3 in Eq. (7) predicts the Majorana phase $\beta = -71.9^\circ$ (as compared to $\beta = 71.9^\circ$ with Eq. (2)) which is not shown in the Table since the neutrinoless double beta decay parameter is $m_{ee} = m_b = 2.684$ meV for the above parameter set which is practically impossible to measure in the foreseeable future.

m_a (meV)	m_b (meV)	η (rad)	θ_{12} (°)	θ_{13} (°)	θ_{23} (°)	$\delta_{\rm CP}$ (°)	m_1 (meV)	m_2 (meV)	m_3 (meV)
26.57	2.684	$-\frac{2\pi}{3}$	34.3	8.67	44.2	-93.3	0	8.59	49.8
26.57	2.684	$\frac{2\pi}{3}$	34.3	8.67	45.8	-86.7	0	8.59	49.8
Value	from	[27]	$33.48^{+0.78}_{-0.75}$	$8.50^{+0.20}_{-0.21}$	$42.3^{+3.0}_{-1.6}$	-54^{+39}_{-70}	0	8.66 ± 0.10	49.57 ± 0.47

Table 1: Benchmark parameters and predictions for CSD3 in Eq. (7) used in this paper (second line) with a fixed phase $\eta = -2\pi/3$, as compared to the version of CSD3 in Eq. (2) (third line) with a fixed phase of $\eta = 2\pi/3$. These predictions, which depend on the theoretical fit [17], as well as possible charged lepton and renormalisation group corrections [26], may be compared to the global best fit values from [27] (for $m_1 = 0$), given in the fourth line (see also [28, 29]).

3 Vacuum alignment for CSD3

In our setup, we rely on the supersymmetric F-term alignment mechanism to generate the appropriate symmetry breaking flavon VEVs. The required driving fields are denoted by X_i, Y_i, Z_i, where the subscript i indicates its S_4 representation. We derive all necessary alignments in a short sequence of steps. Commencing with the primary alignments of triplets flavons, we proceed to generate alignments of doublet flavons. In a final step, the SU preserving CSD3 alignments are obtained from SU symmetric F-term conditions. Our notation is such that the three primary triplet flavons are denoted by $\phi'_{S,U} \sim \mathbf{3'}$, $\phi_T \sim \mathbf{3}$ and $\phi'_t \sim \mathbf{3'}$. The doublet flavons, which are obtained from the primary ones, are $\rho_{S,U} \sim \mathbf{2}$ and $\rho_t \sim \mathbf{2}$. Here, the indices (S, U, T and t) show the symmetry preserving generators, where t corresponds to T multiplied by a Z_3 generator which is not part of S_4. In addition to the triplet and doublet flavons, we also introduce the S_4 singlet flavons $\xi_T \sim \mathbf{1}$ and $\xi_{S,U} \sim \mathbf{1}$.

The primary triplet alignments are derived from simply coupling the square of a flavon triplet to a single driving field X_i. The resulting F-term conditions depend on the S_4 representation of X_i, and the most general solutions of these conditions are given as follows.

$$X_{3'}(\phi'_{S,U})^2 \qquad \longrightarrow \qquad \begin{pmatrix} 1 \\ \omega^n \\ \omega^{2n} \end{pmatrix}, \tag{8}$$

$$X_2(\phi_T)^2 \qquad \longrightarrow \qquad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -2\omega^n \\ -2\omega^{2n} \end{pmatrix}, \tag{9}$$

$$X_1(\phi'_t)^2 \qquad \longrightarrow \qquad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2x \\ -1/x \end{pmatrix}, \tag{10}$$

where the alignments are only fixed up to an integer ($n \in \mathbb{Z}$) or continuous ($x \in \mathbb{R}$) parameter, with $\omega \equiv e^{2\pi i/3}$.

We emphasise that all solutions of the ϕ_T alignments are related by S_4 transformations. It is therefore possible to choose the direction $\langle\phi_T\rangle \propto (1,0,0)^T$ without loss of generality. Moreover, the alignments of $\phi'_{S,U}$ can be brought to the standard $(1,1,1)^T$ form by a T transformation which does not affect the ϕ_T alignment. Finally, the so-selected alignments of ϕ_T and $\phi'_{S,U}$ do not change their form (up to a possible overall sign) under application of a U transformation. This fact allows us to get rid of the ambiguity of the ϕ'_t alignment: the third alignment $(2, 2x, -1/x)^T$ can be removed by requiring orthogonality with $\langle\phi_T\rangle$, which can be enforced in a straightforward way by the term

$$X_{1'}\phi_T\phi'_t \ , \tag{11}$$

in the driving potential. Then, a U transformation can be applied to choose the alignment $\langle\phi'_t\rangle \propto (0,1,0)^T$ without loss of generality. We can thus make use of the following three primary alignments

$$\langle\phi'_{S,U}\rangle \ = \ \varphi'_{S,U}\begin{pmatrix}1\\1\\1\end{pmatrix} , \qquad \langle\phi_T\rangle \ = \ \varphi_T\begin{pmatrix}1\\0\\0\end{pmatrix} , \qquad \langle\phi'_t\rangle \ = \ \varphi'_t\begin{pmatrix}0\\1\\0\end{pmatrix} , \tag{12}$$

to generate new alignments, which together with the primary ones can be used in constructing our CSD3 model of leptons. Note that $\langle\phi'_{S,U}\rangle$ preserves S, U while $\langle\phi_T\rangle$ preserves T as discussed in Appendix A.

The secondary alignments of the doublet flavons $\rho_{S,U}$ and ρ_t originate in the driving terms

$$Y_3\phi'_{S,U}\rho_{S,U} \ , \qquad Y_{3'}\left(\xi_T\phi'_t - \phi_T\rho_t\right) \ , \tag{13}$$

where ξ_T represents an S_4 singlet flavon which does not affect the alignment of ρ_t. We remark that all dimensionless coupling constants of the flavon potential are suppressed for the sake of notational clarity. It is, however, important to keep in mind that such couplings are real in our setup with imposed CP symmetry. A straightforward calculation shows that the F-term conditions resulting from Eq. (13) determine the doublet alignments uniquely to

$$\langle\rho_{S,U}\rangle \ = \ \varrho_{S,U}\begin{pmatrix}1\\1\end{pmatrix} , \qquad \langle\rho_t\rangle \ = \ \varrho_t\begin{pmatrix}0\\1\end{pmatrix} . \tag{14}$$

We point out that the doublet flavon ρ_t is actually not required in constructing the CSD3 alignments. However, it can be used in the charged lepton sector to generate the muon and electron masses.[5]

[5]For the tau mass we can use the triplet flavon ϕ'_t. The product $\phi'_t\rho_t$ yields and effective alignment in the $(0,0,1)^T$ direction and can be used to generate the muon mass. Finally the product $\phi'_t\rho_t\rho_t$ gives rise to an effective vacuum alignment in the $(1,0,0)^T$ direction so that it can be adopted to give mass to the electron. (In principle we could also use the ϕ_T flavon for the electron, but the relative suppression of the electron mass with respect to the tau and muon mass would require an unnatural hierarchy between the VEVs of ϕ_T and ϕ'_t.)

Turning to the derivation of the CSD3 alignments, we first consider the contraction of $\phi'_{S,U}$ and ϕ_T to a $\mathbf{3'}$ of S_4,

$$\left[\langle\phi'_{S,U}\rangle \cdot \langle\phi_T\rangle\right]_{3'} \quad \propto \quad \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} . \tag{15}$$

Although the flavon direction $\langle\phi_T\rangle$ does not respect the SU symmetry, its product with $\langle\phi'_{S,U}\rangle$, contracted to a $\mathbf{3'}$, yields an SU invariant direction. From this result, we immediately see that the driving term

$$Z_{3'}\left(\phi'_{S,U}\phi_T - \xi_{S,U}\phi'_{\mathrm{atm}}\right) \ , \tag{16}$$

with $\xi_{S,U}$ being an S_4 singlet flavon field, generates the alignment

$$\langle\phi'_{\mathrm{atm}}\rangle \ = \ \varphi'_{\mathrm{atm}} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} . \tag{17}$$

Similarly, we can consider the product of ϕ'_{atm} and ϕ'_t to a $\mathbf{3'}$ of S_4,

$$[\langle\phi'_{\mathrm{atm}}\rangle \cdot \langle\phi'_t\rangle]_{3'} \quad \propto \quad \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} . \tag{18}$$

Again, the flavon direction $\langle\phi'_t\rangle$ does not respect SU, yet its product with $\langle\phi'_{\mathrm{atm}}\rangle$ to a $\mathbf{3'}$ does. In order to realise the CSD3 alignment ϕ'_{sol}, we use the particular SU preserving product of Eq. (18) as well as the doublet flavon $\rho_{S,U}$ (whose VEV is invariant under SU) in the driving term

$$\tilde{Z}_{3'}\left(\phi'_{\mathrm{atm}}\phi'_t - \rho_{S,U}\,\phi'_{\mathrm{sol}}\right) \ . \tag{19}$$

To see this, we insert the already aligned flavon directions into Eq. (19). This gives the F-term conditions for $\langle\phi'_{\mathrm{sol}}\rangle = (\beta_1, \beta_2, \beta_3)^T$

$$\varphi'_{\mathrm{atm}}\,\varphi'_t \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} - \varrho_{S,U} \begin{pmatrix} \beta_2 + \beta_3 \\ \beta_3 + \beta_1 \\ \beta_1 + \beta_2 \end{pmatrix} \ = \ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} , \tag{20}$$

which uniquely specify the alignment to the CSD3 one

$$\langle\phi'_{\mathrm{sol}}\rangle \ = \ \varphi'_{\mathrm{sol}} \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix} . \tag{21}$$

Furthermore, the VEVs φ'_{atm} and φ'_{sol} (including the phase) are related via

$$\varphi'_{\mathrm{sol}} \ = \ \frac{\varphi'_t}{2\,\varrho_{S,U}}\,\varphi'_{\mathrm{atm}} \ . \tag{22}$$

Having completed the discussion of the vacuum alignment for supersymmetric CSD3 models, we conclude this section by collecting all terms of the flavon sector in the flavon superpotential. Suppressing all coupling coefficients (which are real in the case of a CP symmetric setup), we have the superpotential

$$\begin{aligned} W_0^{\text{flavon}} \sim\ & X_{3'}(\phi'_{S,U})^2 + X_2(\phi_T)^2 + X_1(\phi'_t)^2 + X_{1'}\phi_T\phi'_t \\ & + Y_3\phi'_{S,U}\rho_{S,U} + Y_{3'}(\xi_T\phi'_t - \phi_T\rho_t) \\ & + Z_{3'}(\phi'_{S,U}\phi_T - \xi_{S,U}\phi'_{\text{atm}}) + \tilde{Z}_{3'}(\phi'_{\text{atm}}\phi'_t - \rho_{S,U}\phi'_{\text{sol}}) \ . \end{aligned} \tag{23}$$

It is important to notice that the flavon potential of Eq. (23) contains only renormalisable terms. As a consequence, the CSD3 alignments derived from the corresponding F-term conditions should be relatively robust when implemented into a concrete model.

4 A concrete model of CSD3

In order to define a model, it is necessary to specify its particle content as well as all symmetries which constrain the couplings of the fields. In Eq. (23), we have already stated the flavon superpotential for generating the CSD3 alignments. By construction, these terms are symmetric under the imposed S_4 family symmetry. Furthermore, it is possible to introduce a $U(1)$ symmetry which allows for all terms of Eq. (23). Such a $U(1)$ must, however, also be consistent with the superpotential terms of the lepton sector. Following the discussion of the Littlest Seesaw model [18], we demand the superpotential terms

$$\begin{aligned} W_0^{\text{lepton}} =\ & \frac{y'_\tau}{\Lambda}LH_dE_3^c\,\phi'_t + \frac{y'_\mu}{\Lambda^2}LH_dE_2^c\,\phi'_t\,\rho_t + \frac{y'_e}{\Lambda^3}LH_dE_1^c\,\phi'_t\,(\rho_t)^2 \\ & + \frac{y_{\text{atm}}}{\Lambda}LH_uN^c_{\text{atm}}\,\phi'_{\text{atm}} + \frac{y_{\text{sol}}}{\Lambda}LH_uN^c_{\text{sol}}\,\phi'_{\text{sol}} + \xi_{\text{atm}}\,N^c_{\text{atm}}N^c_{\text{atm}} + \xi_{\text{sol}}\,N^c_{\text{sol}}N^c_{\text{sol}} \ . \end{aligned} \tag{24}$$

Here we assume the Higgs doublets H_u and H_d to transform trivially under S_4 as well as any additional $U(1)$ symmetry. The neutrino sector of Eq. (24) contains the typical CSD3 Dirac mass terms, while the Majorana mass terms arise from the VEVs of the S_4 singlet flavons ξ_{atm} and ξ_{sol}. We suppress the dimensionless Yukawa couplings in the Majorana sector for brevity.[6] Considering the charged lepton sector, we choose the right-handed electrons as S_4 singlets, while the three generations of left-handed lepton doublets L_i are combined into the S_4 triplet $\mathbf{3}'$. Contracting L with $\langle\phi'_t\rangle$ to an S_4 invariant projects out the third family L_3. Similarly, the S_4 products $L\langle\phi'_t\rangle\langle\rho_t\rangle$ and $L\langle\phi'_t\rangle\langle\rho_t\rangle^2$ project out L_2 and L_1, respectively. We thus obtain a diagonal charged lepton mass matrix in which the hierarchy of masses results from different powers of the suppression factor $1/\Lambda$.

The operators of Eq. (24) put further constraints on a possible $U(1)$ symmetry. Counting the number of fields and comparing this to the number of constraints from Eqs. (23,24), we

[6] Replacing the singlet flavon (Majoron) fields ξ by bare mass parameters M, it is possible to show that the flavon superpotential would include additional renormalisable terms which spoil our successful method of generating the CSD3 alignment.

	fields	S_4	$U(1)$	$U(1)_x$	$Z_3^{(1)}$	$Z_3^{(2)}$	$Z_3^{(3)}$	$Z_3^{(4)}$	$Z_3^{(5)}$
Higgs & leptons	L	$\mathbf{3'}$	$-x_1+z_1$	1	1	0	0	0	0
	E_3^c	$\mathbf{1}$	$x_1-z_1-z_3$	-1	2	0	2	0	0
	E_2^c	$\mathbf{1}$	$x_1-x_2-z_1-2z_3-z_4+z_5$	-4	2	0	1	2	1
	E_1^c	$\mathbf{1}$	$x_1-2x_2-z_1-3z_3-2z_4+2z_5$	-7	2	0	0	1	2
	N_{atm}^c	$\mathbf{1}$	$-z_1$	0	2	0	0	0	0
	N_{sol}^c	$\mathbf{1}$	$-z_2$	0	0	2	0	0	0
	H_d	$\mathbf{1}$	0	0	0	0	0	0	0
	H_u	$\mathbf{1}$	0	0	0	0	0	0	0
flavon fields	$\phi'_{S,U}$	$\mathbf{3'}$	x_1+x_2	2	0	0	0	0	0
	$\rho_{S,U}$	$\mathbf{2}$	$z_1-z_2+z_3$	0	1	2	1	0	0
	$\xi_{S,U}$	$\mathbf{1}$	z_5	0	0	0	0	0	1
	ϕ_T	$\mathbf{3}$	$-x_2+z_5$	-3	0	0	0	0	1
	ξ_T	$\mathbf{1}$	z_4	0	0	0	0	1	0
	ϕ'_t	$\mathbf{3'}$	z_3	0	0	0	1	0	0
	ρ_t	$\mathbf{2}$	$x_2+z_3+z_4-z_5$	3	0	0	1	1	2
	ϕ'_{atm}	$\mathbf{3'}$	x_1	-1	0	0	0	0	0
	ϕ'_{sol}	$\mathbf{3'}$	$x_1-z_1+z_2$	-1	2	1	0	0	0
	ξ_{atm}	$\mathbf{1}$	$2z_1$	0	2	0	0	0	0
	ξ_{sol}	$\mathbf{1}$	$2z_2$	0	0	2	0	0	0
driving fields	$X_{3'}$	$\mathbf{3'}$	$-2x_1-2x_2$	-4	0	0	0	0	0
	X_2	$\mathbf{2}$	$2x_2-2z_5$	6	0	0	0	0	1
	X_1	$\mathbf{1}$	$-2z_3$	0	0	0	1	0	0
	$X_{1'}$	$\mathbf{1'}$	$x_2-z_3-z_5$	3	0	0	2	0	2
	Y_3	$\mathbf{3}$	$-x_1-x_2-z_1+z_2-z_3$	-2	2	1	2	0	0
	$Y_{3'}$	$\mathbf{3'}$	$-z_3-z_4$	0	0	0	2	2	0
	$Z_{3'}$	$\mathbf{3'}$	$-x_1-z_5$	1	0	0	0	0	2
	$\tilde{Z}_{3'}$	$\mathbf{3'}$	$-x_1-z_3$	1	0	0	2	0	0
	X_0	$\mathbf{1}$	0	0	0	0	0	0	0

Table 2: The particle content and symmetries of our CSD3 model. $U(1)$ denotes the most general symmetry consistent with the terms of Eqs. (23,24). $U(1)_x$ is specified by setting $x_1 = -1$, $x_2 = 3$ and $z_i = 0$. The $Z_3^{(i)}$ symmetries are Z_3 subgroups of $U(1)$ with all parameters set to zero except for $z_i = 1$. In addition, we assume a standard $U(1)_R$ symmetry with the charge assignments: +1 for lepton, +2 for driving fields, 0 for Higgs and flavon fields.

can determine the maximal $U(1)$ symmetry which is allowed in our setup. With 25 fields and 18 independent terms, we obtain 7 free parameters which specify the most general $U(1)$ symmetry. Expressing its charges in terms of the parameters $x_{1,2}$ and $z_{1,2,3,4,5}$, we list the complete charge assignments in Table 2.

Imposing this general $U(1)$ for arbitrary parameters $x_{1,2}$ and $z_{1,2,3,4,5}$ is tantamount to imposing seven independent $U(1)$ symmetries. It is straightforward to show that such a powerful symmetry, while being consistent with all term of Eqs. (23,24), does not allow for any other relevant term. We have checked this result explicitly for terms with up to five flavon fields, finding no extra term at all.[7]

As discussed in [18], the Littlest Seesaw requires the relative phase factor $\omega = e^{2\pi i/3}$ between the two contributions to the effective light neutrino mass matrix. In a CP conserving setup, such a phase factor can only originate in complex flavon VEVs. In order to predict phases, it is necessary to find a way of driving flavon VEVs to certain values with given phases. An obvious option is to introduce a completely neutral driving field X_0 which couples to both, some power of a flavon field ϕ as well as a bare mass parameter. For instance, $X_0(\phi^2 - M^2)$ entails a real VEV for the flavon ϕ provided that M is real. Such a method has been applied previously, e.g. in [25]. In order to drive a flavon VEV to a complex value whose phase factor is ω^k, it is suggestive to make use of couplings such as $X_0(\phi^3/\Lambda - M^2)$, see e.g. [23]. Clearly, this structure is forbidden if the flavon ϕ carries a non-trivial $U(1)$ charge. However, a non-trivial Z_3 charge is possible; in fact, it is even necessary in order to forbid the quadratic term $X_0\phi^2$.

On the right-hand side of Table 2, we have defined particular subgroups of the general $U(1)$ which, as mentioned earlier, can be understood as seven independent $U(1)$ symmetries. The $U(1)_x$ symmetry is defined by choosing $x_1 = -1$, $x_2 = 3$ and $z_i = 0$. The $Z_3^{(i)}$ symmetries are obtained as discrete subgroups of the general $U(1)$ with all parameters set to zero except for $z_i = 1$. Imposing only $U(1)_x$ and the five $Z_3^{(i)}$ symmetries, it is possible to drive the VEVs of the flavons with zero $U(1)_x$ charge to values with a phase factor ω^k. As the so-reduced symmetry could, in principle, allow for other new terms in the superpotential, we have to check for such unwanted operators. In addition to the terms of Eq. (23), we find the following cubic terms in the flavon potential,

$$W_1^{\text{flavon}} \sim X_0 \left[\frac{(\xi_{\text{atm}})^3 + (\xi_{\text{sol}})^3 + (\xi_T)^3 + (\xi_{S,U})^3 + (\phi'_t)^3 + (\rho_{S,U})^3 + \phi'_{S,U}(\phi'_{\text{atm}})^2}{\Lambda} - M^2 \right]. \tag{25}$$

All terms with one driving field coupling to four flavons are forbidden, while there exist many allowed, though strongly suppressed, terms with five flavons. In the Dirac-type terms of the lepton sector, the first new terms involve four flavon fields and are therefore highly suppressed. Finally, we find extra contributions to the mass terms of the right-handed neutrinos with four or more flavons. The complete model based on the $U(1)_x \times Z_3^{(1)} \times Z_3^{(2)} \times$

[7] Imposing only one particular $U(1)$ symmetry rather than seven independent $U(1)$s, it is also possible to forbid all relevant unwanted operators. For instance, with the somewhat arbitrary choice $(x_1, x_2, z_1, z_2, z_3, z_4, z_5) = (4, 16, -61, 88, 53, -61, 7)$, the driving potential does not have any non-renormalisable operator with three flavon fields. Likewise, the first new Dirac-type terms of the lepton sector involve four flavons and are therefore highly suppressed. For the right-handed neutrinos, we encounter a new contribution to $N^c_{\text{atm}} N^c_{\text{atm}}$ with two flavons, however, the first off-diagonal term $N^c_{\text{atm}} N^c_{\text{sol}}$ already requires five flavons.

$Z_3^{(3)} \times Z_3^{(4)} \times Z_3^{(5)}$ symmetry is therefore given by the superpotentials

$$W^{\text{flavon}} = W_0^{\text{flavon}} + W_1^{\text{flavon}} + \left(\frac{1}{\Lambda^3} X \phi^5 + \cdots \right) , \tag{26}$$

$$W^{\text{lepton}} = W_0^{\text{lepton}} + \left(\frac{1}{\Lambda^4} L H_d E_i^c \phi^4 + \frac{1}{\Lambda^4} L H_u N_i^c \phi^4 + \frac{1}{\Lambda^3} N_i^c N_j^c \phi^4 + \cdots \right) , \tag{27}$$

where the higher order terms in brackets are only written schematically with X or ϕ representing any of the driving or flavon fields of the model.

These observations show that the reduced symmetry on the right-hand side of Table 2 is sufficient to control the coupling of driving, flavon and lepton fields. Moreover, Eq. (25) allows us to constrain the VEVs of the flavons. The existence of the mixed term $\phi'_{S,U}(\phi'_{\text{atm}})^2$ in Eq. (25) follows from the particular charge assignment under the $U(1)_x$ symmetry. However, inserting the vacuum alignment, this term vanishes identically. Hence we can ignore it in the following. Adding six copies of the driving field X_0, we obtain six independent F-term equations which decouple if linearly combined. Then, the VEVs of the flavons ξ_{atm}, ξ_{sol}, ξ_T, $\xi_{S,U}$, ϕ'_t and $\rho_{S,U}$ are driven to values where the phase factor is some power of ω.[8] Due to the symmetries, many of these phase factors can however be removed. For instance, if ξ_{atm} has a phase factor ω^k, this can be modified to ω^0 by a $Z_3^{(1)}$ transformation. Since ξ_{atm} is uncharged under any of the other symmetries, a $Z_3^{(2)}$ transformation can be applied without modifying the trivial phase of ξ_{atm}. On the other hand, such a $Z_3^{(2)}$ transformation can remove the phase of ξ_{sol}. This procedure can be applied further to render real the VEVs of ξ_{atm}, ξ_{sol}, ξ_T, $\xi_{S,U}$ as well as ϕ'_t. Having exhausted all $Z_3^{(i)}$ symmetries, the phase factor of $\rho_{S,U}$ cannot be removed. Similarly to the $Z_3^{(i)}$ transformations, the $U(1)_x$ symmetry can be used to remove the phase of $\langle \phi'_{\text{atm}} \rangle$. Defining the phase factor of $\langle \phi'_{S,U} \rangle$ to be $e^{i\alpha}$, the phases of the VEVs of the remaining flavons ϕ_T, ρ_t and ϕ'_{sol} are fixed by Eqs. (16,13,19), respectively. In summary, we can work in a basis where the VEVs of all flavons are real except for the following list

$$\frac{\varrho_{S,U}}{|\varrho_{S,U}|} = \omega^k , \qquad \frac{\varphi'_{\text{sol}}}{|\varphi'_{\text{sol}}|} = \omega^{-k} , \tag{28}$$

$$\frac{\varphi'_{S,U}}{|\varphi'_{S,U}|} = \frac{\varrho_t}{|\varrho_t|} = e^{i\alpha} , \qquad \frac{\varphi_T}{|\varphi_T|} = e^{-i\alpha} . \tag{29}$$

Adopting this phase convention together with the alignments derived in Section 3, we can deduce the mass matrices from the terms of the lepton superpotential W_0^{flavon} of Eq. (24). Mindful of the Clebsch-Gordan coefficients of S_4 in the T-diagonal basis, stated explicitly in Appendix A, we obtain the Dirac neutrino mass matrix

$$m^D = \frac{v_u}{\Lambda} \begin{pmatrix} 0 & y_{\text{sol}} |\varphi_{\text{sol}}| \, \omega^{-k} \\ -y_{\text{atm}} \varphi_{\text{atm}} & -y_{\text{sol}} |\varphi_{\text{sol}}| \, \omega^{-k} \\ y_{\text{atm}} \varphi_{\text{atm}} & 3 \, y_{\text{sol}} |\varphi_{\text{sol}}| \, \omega^{-k} \end{pmatrix} , \tag{30}$$

[8]Notice that – with the respective alignments – $\langle \phi'_t \rangle^3$ as well as $\langle \rho_{S,U} \rangle^3$ have non-vanishing contractions to an S_4 singlet. This is not the case for $\langle \phi'_{S,U} \rangle^3$.

where the only complex quantity is given explicitly by the factor ω^{-k}. Absorbing the minus signs into the second lepton doublet field, which ultimately gets absorbed into the right-handed muon field when the charged lepton masses are made real and positive, we obtain a Dirac mass matrix with the sign conventions of Eq. (6) [also see Appendix B, Eq. (42)]. This is our preferred convention which we will adopt in the following. The 2×2 right-handed Majorana mass matrix takes the real and diagonal form $M_R = \mathrm{diag}(\xi_{\mathrm{atm}}, \xi_{\mathrm{sol}})$, continuing to suppress the dimensionless Yukawa couplings in the Majorana sector for brevity. Applying the seesaw formula results in the effective light neutrino mass matrix

$$m^\nu = \frac{v_u^2}{\Lambda^2}\left[\frac{(y_{\mathrm{atm}}\varphi'_{\mathrm{atm}})^2}{\xi_{\mathrm{atm}}}\begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix} + \frac{(y_{\mathrm{sol}}|\varphi'_{\mathrm{sol}}|)^2}{\xi_{\mathrm{sol}}}\,\omega^{-2k}\begin{pmatrix}1&1&3\\1&1&3\\3&3&9\end{pmatrix}\right]. \tag{31}$$

Choosing $k = 2$, which is one of the three physically distinct possible choices $k = 0, 1, 2$, the neutrino mass matrix is of the form of Eq. (45) but with fixed values of $n = 3$ and $\eta = -2\pi/3$,

$$m^\nu = m_a \begin{pmatrix}0&0&0\\0&1&1\\0&1&1\end{pmatrix} + m_b e^{-i2\pi/3}\begin{pmatrix}1&1&3\\1&1&3\\3&3&9\end{pmatrix}, \tag{32}$$

as in Eq. (7) but with fixed phase $\eta = -2\pi/3$ leading to leptonic CP violation with Dirac phase $\delta \sim -\pi/2$, and good values of lepton mixing angles as discussed in Section 2. As the VEVs of all flavons which appear in the neutrino sector of W_0^{lepton}, see Eq. (24), respect the SU symmetry, this neutrino matrix satisfies that symmetry as well and is therefore of the trimaximal TM_1 form.

Considering the charged lepton sector, we find the diagonal mass matrix

$$m^\ell = \frac{v_d\,\varphi'_t}{\Lambda}\begin{pmatrix}y'_e\frac{\varrho_t^2}{\Lambda^2}&0&0\\0&y'_\mu\frac{\varrho_t}{\Lambda}&0\\0&0&y'_\tau\end{pmatrix} \equiv v_d \begin{pmatrix}y_e&0&0\\0&y_\mu&0\\0&0&y_\tau\end{pmatrix}, \tag{33}$$

where the only complex quantity is given by the value of the doublet VEV ϱ_t. Its phase factor $e^{i\alpha}$ can, however, be absorbed into a field redefinition of the right-handed electrons E_2^c and E_1^c. As such, the phase α, as introduced in Eq. (29), does not contribute to the phase structure of the PMNS mixing matrix, and the effective Yukawa couplings y_e, y_μ, y_τ defined above may be taken to be real without loss of generality. We emphasise that the hierarchy of physical effective Yukawa couplings $y_e \ll y_\mu \ll y_\tau \ll 1$ has a natural explanation in this model, arising from the smallness of flavon VEVs compared to the cut-off scale Λ, assuming the primordial Yukawa couplings $y'_e, y'_\mu, y'_\tau \sim O(1)$.

As for the neutrinos, the structure of the charged lepton mass matrix m^ℓ is also related to symmetry, although in a slightly more intricate way. While the alignments of ϕ'_t and ρ_t do not change their direction under a T transformation, both pick up the phase factor ω^2.[9] A subsequent $Z_3^{(3)}$ transformation $c^{(3)}$ can undo this change of the phase so that the

[9]For the triplet ϕ'_t this can be seen in Eq. (36). For the doublet ρ_t we note, that the corresponding T generator is also diagonal with $T = \mathrm{diag}\,(\omega, \omega^2)$, see e.g. [23].

combined $c^{(3)}T$ transformation can be identified as the symmetry of the charged lepton sector which is responsible for guaranteeing a diagonal mass matrix m^ℓ. In this sense the model of leptons presented here is a semi-direct model, since the residual symmetry of the lepton mass matrices of both neutrino and charged lepton sectors may be identified as different subgroups of S_4, namely SU in the neutrino sector and T (combined with a $Z_3^{(3)}$ transformation) in the charged lepton sector.

5 Charged lepton flavour violation

Since the model is supersymmetric we can expect charged lepton flavour violation in this model, due to one-loop diagrams involving sleptons, neutralinos and charginos [30–32]. In the mass insertion approximation, the processes arise from having off-diagonal slepton mass squared and trilinear matrices at low energies in the super-CKM basis in which the charged lepton masses are diagonal. With flavour symmetry present, the high energy slepton mass squared and trilinear matrices are controlled by the flavour symmetry and generally yield only small off-diagonal entries. Unfortunately this is a rather delicate and complex issue, with precise estimates depending on an expansion in flavon fields, canonical normalisation and rotations to the super-CKM basis in which the charged lepton masses are diagonal, followed by renormalisation group running to low energies, along the lines of a recent analysis based on an $SU(5) \times S_4 \times U(1)$ Grand Unified Theory (GUT) of flavour [33].

Ignoring the effects of the operator expansion, canonical normalisation and super-CKM rotations (which are anyway highly suppressed in this model where the charged lepton mass matrix is diagonal), the slepton mass squared and trilinear matrices do not violate flavour at high energies, and the only remaining effect arises from renormalisation group running. Then, using the analytic results in [34], we may make a simple estimate for the branching ratio of $\mu \to e\gamma$ as follows. At leading order in a mass insertion approximation [30–32] the branching fraction of $\mu \to e\gamma$ is given by [34]:

$$\mathrm{BR}(\mu \to e\gamma) \approx \frac{\alpha^3}{G_F^2} f(M_2, \mu, m_{\tilde{\nu}}) |m^2_{\tilde{L}_{21}}|^2 \tan^2\beta \ , \tag{34}$$

where the off-diagonal slepton doublet mass squared is given in the leading log approximation (LLA) by

$$m^{2(\mathrm{LLA})}_{\tilde{L}_{21}} \approx -\frac{(3m_0^2 + A_0^2)}{8\pi^2} |b|^2 \ln \frac{M_{\mathrm{GUT}}}{M_{\mathrm{sol}}} \ , \tag{35}$$

and the remainder of the notation is fairly standard and given in [30–32]. In the present model leptogenesis fixes $M_{\mathrm{sol}} = 4 \times 10^{10}$ GeV and the neutrino fit fixes $m_b = v_u^2 |b|^2 / M_{\mathrm{sol}} \sim$ 2.7 meV, which implies $|b| \sim 10^{-3}$. The smallness of the Yukawa coupling b is due to its non-renormalisable origin $b \sim \frac{\varphi'_{\mathrm{sol}}}{\Lambda}$. This contrasts with other semi-direct models such as those in [33] where the neutrino Yukawa couplings are $O(1)$, and implies that in this model, charged lepton flavour violation such as $\mu \to e\gamma$ will be relatively highly suppressed, at least according to our very simple estimate based on the assumptions above.

6 Conclusions

In this paper, guided by the principles of minimality and symmetry, we have been led to a highly predictive theory of neutrino mass and lepton mixing in which all CP phases are fixed and the neutrino masses and the entire lepton mixing matrix are determined by only two real input mass parameters. Starting from the most elegant mechanism for the origin of neutrino mass, namely the seesaw mechanism, we have focused on the most minimal version involving two right-handed neutrinos. Pursuing minimality, we were then led to consider a two right-handed neutrino seesaw model with one texture zero and a constrained form of Dirac mass matrix involving only two independent Dirac masses with the structure of Eq. (6), simply related to the CSD3 structure in Eq. (1) by $L_2 \leftrightarrow L_3$. Our main achievement is to show that the new version of CSD3 can be obtained from symmetry arguments based on S_4, working in the basis where the diagonal T generator can enforce the diagonality of the charged lepton mass matrix due to a residual Z_3 symmetry, while the preserved S_4 subgroup SU in the neutrino sector with a residual Z_2 symmetry is instrumental in enforcing TM_1 mixing. The resulting scheme combines minimality with symmetry, leading to a high degree of predictivity, where the predictions are protected from higher order corrections by the full symmetry of the model.

We then proposed a realistic model of leptons, based on $S_4 \times U(1)$ symmetry, with two right-handed neutrinos, where a straightforward F-term vacuum alignment results in a neutrino mass matrix with the form of Eq. (7). The relatively simple model corresponds to the left half of Table 2 (to the left of the double vertical lines) in which the symmetry is only $S_4 \times U(1)$. However in order to achieve the phenomenologically desired phase of $\eta = -2\pi/3$ we were forced to extend the symmetries of the model (but not the particle content) to include a $(Z_3)^5$ symmetry in the right half of Table 2 (to the right of the double vertical lines). This enabled us to impose a CP symmetry, then spontaneously break it in a controlled way, such that the phase is constrained to be one of the cube roots of unity, however leaving no residual CP symmetry in the charged lepton or neutrino sectors. With the phase chosen from the cube roots of unity to be $\eta = -2\pi/3$, all CP phases are fixed and the baryon asymmetry of the universe then will determine the lighter solar right-handed neutrino mass to be $M_{\mathrm{sol}} = 4 \times 10^{10}$ GeV. The model predicts a normal neutrino mass hierarchy with $m_1 = 0$, reactor angle $\theta_{13} = 8.7^o$, solar angle $\theta_{12} = 34^o$, atmospheric angle $\theta_{23} = 44^o$, and CP violating oscillation phase $\delta_{\mathrm{CP}} = -93^o$, depending on the fit of the model to the neutrino masses and possible renormalisation group corrections. These predictions will be tested soon.

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Appendix

A S_4 group theory

Throughout this paper we work in the T diagonal basis of S_4, as in [2]:

$$S = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \quad \text{for } \mathbf{3} \text{ or } \mathbf{3}', \tag{36}$$

and

$$U = \mp\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad SU = US = \mp\frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix}, \quad \text{for } \mathbf{3}, \mathbf{3}' \text{ respectively.} \tag{37}$$

In this basis the symmetry preserving vacuum alignments are as follows:

$$\phi_T \sim \mathbf{3} \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \text{ preserves } T, \text{ breaks } S, U,$$

$$\phi'_T \sim \mathbf{3}' \sim \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \text{ preserves } T, U \text{ breaks } S,$$

$$\phi_S \sim \mathbf{3} \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \text{ preserves } S \text{ breaks } T, U,$$

$$\phi'_S \sim \mathbf{3}' \sim \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \text{ preserves } S, U \text{ breaks } T,$$

$$\phi_{SU} \sim \mathbf{3} \sim \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \text{ preserves } SU \text{ breaks } T, U,$$

and the two important SU preserving alignments for $\mathbf{3}'$ flavons,

$$\phi'_{\text{atm}} \sim \mathbf{3}' \sim \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \text{ preserves } SU \text{ breaks } T, U, \tag{38}$$

$$\phi'_{\rm sol} \sim \mathbf{3}' \sim \begin{pmatrix} 1 \\ n \\ 2-n \end{pmatrix}, \text{ preserves } SU \text{ breaks } T, U, \tag{39}$$

where we fix $n = 3$ such that

$$\phi'_{\rm sol} \sim \mathbf{3}' \sim \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}, \text{ preserves } SU \text{ breaks } T, U. \tag{40}$$

In the following we summarise the Kronecker products and Clebsch-Gordan coefficients. The non-trivial S_4 product rules are listed below, where we use the number of primes within the expression

$$\boldsymbol{\alpha}^{(\prime)} \otimes \boldsymbol{\beta}^{(\prime)} \ \to \ \boldsymbol{\gamma}^{(\prime)} \ , \tag{41}$$

to classify the results. We denote this number by p, e.g. in $\mathbf{3} \otimes \mathbf{3}' \to \mathbf{3}'$ we get $p = 2$. Then the Clebsch-Gordan coefficients are given as follows [35]:

$$\mathbf{1}^{(\prime)} \otimes \mathbf{1}^{(\prime)} \ \to \ \mathbf{1}^{(\prime)} \quad \left\{ \begin{array}{ll} & \mathbf{1} \otimes \mathbf{1} \ \to \ \mathbf{1} \\ p = \text{even} & \mathbf{1}' \otimes \mathbf{1}' \ \to \ \mathbf{1} \\ & \mathbf{1} \otimes \mathbf{1}' \ \to \ \mathbf{1}' \end{array} \right\} \quad \alpha\beta \ ,$$

$$\mathbf{1}^{(\prime)} \otimes \mathbf{2} \ \to \ \mathbf{2} \quad \left\{ \begin{array}{ll} p = \text{even} & \mathbf{1} \otimes \mathbf{2} \ \to \ \mathbf{2} \\ p = \text{odd} & \mathbf{1}' \otimes \mathbf{2} \ \to \ \mathbf{2} \end{array} \right\} \quad \alpha \begin{pmatrix} \beta_1 \\ (-1)^p \beta_2 \end{pmatrix} ,$$

$$\mathbf{1}^{(\prime)} \otimes \mathbf{3}^{(\prime)} \ \to \ \mathbf{3}^{(\prime)} \quad \left\{ \begin{array}{ll} & \mathbf{1} \otimes \mathbf{3} \ \to \ \mathbf{3} \\ & \mathbf{1}' \otimes \mathbf{3}' \ \to \ \mathbf{3} \\ p = \text{even} & \mathbf{1} \otimes \mathbf{3}' \ \to \ \mathbf{3}' \\ & \mathbf{1}' \otimes \mathbf{3} \ \to \ \mathbf{3}' \end{array} \right\} \quad \alpha \begin{pmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{pmatrix} ,$$

$$\mathbf{2} \ \otimes \ \mathbf{2} \ \to \ \mathbf{1}^{(\prime)} \quad \left\{ \begin{array}{ll} p = \text{even} & \mathbf{2} \otimes \mathbf{2} \ \to \ \mathbf{1} \\ p = \text{odd} & \mathbf{2} \otimes \mathbf{2} \ \to \ \mathbf{1}' \end{array} \right\} \quad \alpha_1\beta_2 + (-1)^p \alpha_2\beta_1 \ ,$$

$$\mathbf{2} \ \otimes \ \mathbf{2} \ \to \ \mathbf{2} \quad \left\{ \begin{array}{ll} p = \text{even} & \mathbf{2} \otimes \mathbf{2} \ \to \ \mathbf{2} \end{array} \right\} \quad \begin{pmatrix} \alpha_2\beta_2 \\ \alpha_1\beta_1 \end{pmatrix} ,$$

$$\mathbf{2} \ \otimes \ \mathbf{3}^{(\prime)} \ \to \ \mathbf{3}^{(\prime)} \quad \left\{ \begin{array}{ll} p = \text{even} & \begin{array}{l} \mathbf{2} \otimes \mathbf{3} \ \to \ \mathbf{3} \\ \mathbf{2} \otimes \mathbf{3}' \ \to \ \mathbf{3}' \end{array} \\ & \\ p = \text{odd} & \begin{array}{l} \mathbf{2} \otimes \mathbf{3} \ \to \ \mathbf{3}' \\ \mathbf{2} \otimes \mathbf{3}' \ \to \ \mathbf{3} \end{array} \end{array} \right\} \quad \alpha_1 \begin{pmatrix} \beta_2 \\ \beta_3 \\ \beta_1 \end{pmatrix} + (-1)^p \alpha_2 \begin{pmatrix} \beta_3 \\ \beta_1 \\ \beta_2 \end{pmatrix} ,$$

$$\mathbf{3}^{(\prime)} \otimes \mathbf{3}^{(\prime)} \ \to \ \mathbf{1}^{(\prime)} \quad \left\{ \begin{array}{ll} & \mathbf{3} \otimes \mathbf{3} \ \to \ \mathbf{1} \\ p = \text{even} & \mathbf{3}' \otimes \mathbf{3}' \ \to \ \mathbf{1} \\ & \mathbf{3} \otimes \mathbf{3}' \ \to \ \mathbf{1}' \end{array} \right\} \quad \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 \ ,$$

$$\mathbf{3}^{(\prime)} \otimes \mathbf{3}^{(\prime)} \;\to\; \mathbf{2} \quad \left\{ \begin{array}{ll} p = \text{even} & \begin{array}{l} \mathbf{3} \otimes \mathbf{3} \;\to\; \mathbf{2} \\ \mathbf{3}' \otimes \mathbf{3}' \;\to\; \mathbf{2} \end{array} \\ \\ p = \text{odd} & \mathbf{3} \otimes \mathbf{3}' \;\to\; \mathbf{2} \end{array} \right\} \quad \begin{pmatrix} \alpha_2\beta_2 + \alpha_3\beta_1 + \alpha_1\beta_3 \\ (-1)^p(\alpha_3\beta_3 + \alpha_1\beta_2 + \alpha_2\beta_1) \end{pmatrix},$$

$$\mathbf{3}^{(\prime)} \otimes \mathbf{3}^{(\prime)} \;\to\; \mathbf{3}^{(\prime)} \quad \left\{ p = \text{odd} \quad \begin{array}{l} \mathbf{3} \otimes \mathbf{3} \;\to\; \mathbf{3}' \\ \mathbf{3} \otimes \mathbf{3}' \;\to\; \mathbf{3} \\ \mathbf{3}' \otimes \mathbf{3}' \;\to\; \mathbf{3}' \end{array} \right\} \quad \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \\ 2\alpha_2\beta_2 - \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix},$$

$$\mathbf{3}^{(\prime)} \otimes \mathbf{3}^{(\prime)} \;\to\; \mathbf{3}^{(\prime)} \quad \left\{ p = \text{even} \quad \begin{array}{l} \mathbf{3} \otimes \mathbf{3} \;\to\; \mathbf{3} \\ \mathbf{3}' \otimes \mathbf{3}' \;\to\; \mathbf{3} \\ \mathbf{3} \otimes \mathbf{3}' \;\to\; \mathbf{3}' \end{array} \right\} \quad \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \end{pmatrix}.$$

B A new type of CSDn

We may define a new general class of CSDn models as follows. In the diagonal charged lepton and two right-handed neutrino mass basis, CSDn is defined in this paper, up to phase choices, by the Dirac mass matrix in LR convention:[10]

$$m^D = \begin{pmatrix} 0 & b \\ a & (n-2)b \\ a & nb \end{pmatrix}. \tag{42}$$

The (diagonal) right-handed neutrino mass matrix M_R with rows $(\overline{N^c}_{\rm atm}, \overline{N^c}_{\rm sol})^T$ and columns $(N_{\rm atm}, N_{\rm sol})$ is,

$$M_R = \begin{pmatrix} M_{\rm atm} & 0 \\ 0 & M_{\rm sol} \end{pmatrix}, \quad M_R^{-1} = \begin{pmatrix} M_{\rm atm}^{-1} & 0 \\ 0 & M_{\rm sol}^{-1} \end{pmatrix}. \tag{43}$$

The low energy effective Majorana neutrino mass matrix is given by the seesaw formula

$$m^\nu = -m^D M_R^{-1} m^{D^T}, \tag{44}$$

which, after multiplying the matrices in Eqs. (42,43), for a suitable choice of physically irrelevant overall phase, gives

$$m^\nu = m_a \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} + m_b e^{i\eta} \begin{pmatrix} 1 & n-2 & n \\ n-2 & (n-2)^2 & n(n-2) \\ n & n(n-2) & n^2 \end{pmatrix}, \tag{45}$$

[10] Note that this version of CSDn differs from that considered in [18], where the second column of the Dirac mass matrix was $(b, nb, (n-2)b)^T$. For this reason we consider the TB mixing matrix in a different convention. Compared to the analytic formulas in [18], the new version of CSDn leads to a change in sign in the parameters y and hence t and ϵ^ν, with x, z, A, B unchanged, compared to the original version. This implies that the reactor and solar mixing angle formulas are unchanged, but the atmospheric angle formula changes due to the sign change in ϵ^ν, which has the effect of reversing the octant for the atmospheric angle. The formula for $\sin\delta$ also involves a change in sign.

where η is the only physically important phase, which depends on the relative phase between the second and first column of the Dirac mass matrix, $\arg(b/a)$, as well as $m_a = \frac{|a|^2}{M_{\rm atm}}$ and $m_b = \frac{|b|^2}{M_{\rm sol}}$. This can be thought of as the minimal (two right-handed neutrino) predictive seesaw model since only four real parameters m_a, m_b, n, η describe the entire neutrino sector (three neutrino masses as well as the PMNS matrix, in the diagonal charged lepton mass basis). η is identified with the leptogenesis phase, while m_b is identified with the neutrinoless double beta decay parameter m_{ee}.

Consider the tri-bimaximal TB mixing matrix [22] in the following sign convention:

$$U_{\rm TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} . \tag{46}$$

We then observe from Eq. (45) that

$$m^{\nu} \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} . \tag{47}$$

In other words the column vector $(2,1,-1)^T$ is an eigenvector of m^{ν} with a zero eigenvalue, i.e. it is the first column of the TB mixing matrix, corresponding to $m_1 = 0$. We conclude that the neutrino mass matrix leads to so-called TM_1 mixing [20, 21], in which the first column of the mixing matrix is fixed to be that of the TB mixing matrix, but the other two columns are not uniquely determined,

$$U_{\rm TM1} = \begin{pmatrix} \sqrt{\frac{2}{3}} & - & - \\ \frac{1}{\sqrt{6}} & - & - \\ -\frac{1}{\sqrt{6}} & - & - \end{pmatrix} . \tag{48}$$

Since the neutrino mass matrix yields TM_1 mixing as discussed above, it can be block diagonalised by the TB mixing matrix,

$$m^{\nu}_{\rm block} = U^T_{\rm TB} m^{\nu} U_{\rm TB} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & y \\ 0 & y & z \end{pmatrix} , \tag{49}$$

where we find,

$$x = 3 m_b e^{i\eta}, \quad y = -\sqrt{6} m_b e^{i\eta}(n-1), \quad z = |z| e^{i\phi_z} = 2[m_a + m_b e^{i\eta}(n-1)^2] \, . \tag{50}$$

It only remains to put $m^{\nu}_{\rm block}$ into diagonal form, with real positive masses, which can be done exactly analytically of course, since this is just effectively a two by two complex symmetric matrix which may be diagonalised with a rotation angle θ^{ν}_{23}. This procedure leads to the following exact analytic results for neutrino masses and lepton mixing parameters [18].

Taking the Trace (T) and Determinant (D) of the non-trivial 2×2 neutrino mass matrix times its Hermitian conjugate we find

$$m_2^2 + m_3^2 = T \equiv |x|^2 + 2|y|^2 + |z|^2 \ , \tag{51}$$
$$m_2^2 m_3^2 = D \equiv |x|^2|z|^2 + |y|^4 - 2|x||y|^2|z| \cos A \ , \tag{52}$$

from which we extract the exact results for the neutrino masses,

$$m_3^2 = \frac{1}{2}T + \frac{1}{2}\sqrt{T^2 - 4D} \ , \tag{53}$$
$$m_2^2 = D/m_3^2 \ , \tag{54}$$
$$m_1^2 = 0 \ . \tag{55}$$

The exact expression for the reactor angle is given below,

$$\sin\theta_{13} = \frac{1}{\sqrt{6}}\left(1 - \sqrt{\frac{1}{1+t^2}}\right)^{1/2} , \tag{56}$$

where

$$t = \frac{-2\sqrt{6}m_b(n-1)}{2|m_a + m_b e^{i\eta}(n-1)^2| \cos(A-B) - 3m_b \cos B} \ , \tag{57}$$

with

$$\tan B = \frac{2|m_a + m_b e^{i\eta}(n-1)^2| \sin A}{3m_b + 2|m_a + m_b e^{i\eta}(n-1)^2| \cos A} \ , \tag{58}$$

and

$$A = \arg[m_a + m_b e^{i\eta}(n-1)^2] - \eta. \tag{59}$$

The solar angle is given in terms of the reactor angle by the TM_1 mixing sum rule in three equivalent exact forms,

$$\tan\theta_{12} = \frac{1}{\sqrt{2}}\sqrt{1 - 3s_{13}^2} \quad \text{or} \quad \sin\theta_{12} = \frac{1}{\sqrt{3}}\frac{\sqrt{1-3s_{13}^2}}{c_{13}} \quad \text{or} \quad \cos\theta_{12} = \sqrt{\frac{2}{3}}\frac{1}{c_{13}} \ , \tag{60}$$

where we have defined $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$. To first order in s_{13}, The solar angle $\tan\theta_{12}$ approximately takes the TB value of $1/\sqrt{2}$.

The exact expression for the atmospheric angle is given by

$$\tan\theta_{23} = \frac{|1 + \epsilon_{23}^\nu|}{|1 - \epsilon_{23}^\nu|} \ , \tag{61}$$

where

$$\epsilon_{23}^\nu = \sqrt{\frac{2}{3}} t^{-1}\left[\sqrt{1+t^2} - 1\right] e^{-iB} \ , \tag{62}$$

and t and B are given in Eqs. (57,58,59). The atmospheric angle $\tan\theta_{23}$ is maximal when $B = \pm\pi/2$ since then $|1+\epsilon^{\nu}_{23}|$ is equal to $|1-\epsilon^{\nu}_{23}|$.

Mixing sum rules for TM_1 mixing can be expressed as an exact relation for $\cos\delta$ in terms of the other lepton mixing angles [21],

$$\cos\delta = -\frac{\cot 2\theta_{23}(1-5s^2_{13})}{2\sqrt{2}s_{13}\sqrt{1-3s^2_{13}}} \ . \tag{63}$$

Note that, for maximal atmospheric mixing, $\theta_{23} = \pi/4$, we see that $\cot 2\theta_{23} = 0$ and therefore this sum rule predicts $\cos\delta = 0$, corresponding to maximal CP violation $\delta = \pm\pi/2$. The prospects for testing the TM_1 atmospheric sum rules Eqs. (60,63) in future neutrino facilities was discussed in [36].

Using the Jarlskog invariant [37] we find the exact relation [18]:

$$\sin\delta = \frac{24m_a^3m_b^3(n-1)\sin\eta}{m_3^2m_2^2\Delta m^2_{32}s_{12}c_{12}s_{13}c^2_{13}s_{23}c_{23}} \ . \tag{64}$$

Note the positive sign in Eq. (64), which means that, for $n>1$, the sign of $\sin\delta$ takes the same value as the sign of $\sin\eta$, in the convention we use to write our neutrino mass matrix, namely $-\frac{1}{2}\overline{\nu_L}m^{\nu}\nu^c_L$. The above exact results for $\cos\delta$ and $\sin\delta$ completely fix the value of the Dirac oscillation phase δ.

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