Modes of ambiguous communication*

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Abstract

We study cheap talk communication in a simple two actions-two states model featuring an ambiguous state distribution. Equilibrium behavior of both sender (S) and receiver (R) features mixing and we relate each agent's randomization to a specific mode of ambiguous communication. For sufficiently high ambiguity, implementing the S-optimal decision rule with only two messages is impossible if R has aligned preferences. This may in contrast be possible if R has misaligned preferences. Adding a little ambiguity may generate influential communication that is unambiguously advantageous to S.

Keywords: cheap talk, ambiguity.

JEL classification: D81, D83.

"*When I use a word,*" Humpty Dumpty said, in a rather scornful tone, "*it means just what I choose it to mean - nothing more nor less.*". "*The question is,*" said Alice, "*whether you can make words mean so many different things.*". (Lewis Carroll, Through the looking glass)

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1 Introduction

Many situations of advice feature uncertainty about the prior distribution of the state of the world. In medical advice, the distribution of a particular disease across ethnic groups may be unclear. In financial advice, the process governing the value of a given asset may be unknown. We examine a binary cheap talk model featuring Knightean prior uncertainty as well as ambiguity averse agents and we address the following questions. First, how does the addition of ambiguity change the predictions of the classical cheap talk model? Second, does the model generate features that are reminiscent of ambiguous language? We review our findings in what follows.

A preliminary standard result is that agents strictly favour randomization for intermediate (and thereby inconclusive) signal realizations, which allows to hedge in the face of ambiguity. We start by focusing on equilibria that implement the optimal decision rule of S (S-optimal equilibria). Our main objective is to establish the comparative statics of the set of S-optimal equilibria with respect to preference misalignment, message space cardinality and the ambiguity level. In our binary model, the natural measure of preference misalignment between sender (S) and receiver (R) is $\beta = q_S - q_R$, where $q_i \in (0,1)$ describes i's relative sensitivity to type I and II errors ($\beta \geq 0$ as we assume $q_R \leq q_S$). The level of ambiguity is captured by a one dimensional parameter.

Our first main finding is that it is without loss of generality to concentrate on so-called threshold equilibria. In the latter, S sends at most three messages and his communication strategy is described by three thresholds and mixing probabilities computed on the basis of q_S and q_R. In threshold equilibria S occasionally randomizes and his strategy cannot be described as a partitional strategy à la Crawford and Sobel (1982) (CS) but only as mixing over a set of partitional strategies. R also typically randomizes. We interpret randomization by respectively S and R as embodying two different modes of ambiguous communication.

Our next class of findings concerns the impact of the message space cardinality on the existence of S-optimal equilibria. If three messages are available, for any ambiguity level there is a maximal bias $\widetilde{\beta} \in (0,1)$ such that an S-optimal equilibrium exists if and only if $\beta \leq \widetilde{\beta}$. Given high ambiguity, three messages are necessary for the existence of an S-optimal equilibrium independently of bias β. Given intermediate ambiguity there is always an interval of biases $\left[\underline{\beta}, \overline{\beta}\right]$ satisfying $\overline{\beta} < \widetilde{\beta}$ for which two messages suffice. It may furthermore be the case that $\underline{\beta} > 0$, meaning that a low bias renders three messages necessary. Finally, given low ambiguity two messages are always sufficient.

We add four remarks on the above class of findings. First, given intermediate ambiguity perfectly manipulating R thus requires more sophisticated language as β increases from $\beta \in \left[\underline{\beta}, \overline{\beta}\right]$ to $\beta' \in (\overline{\beta}, \widetilde{\beta}]$. In this case, if we pick the S-optimal equilibrium as our prediction for the game, more bias thus implies richer equilibrium language. This reverses the prediction of the CS cheap talk model if we pick the finest equilibrium as the salient prediction for the latter model. A second remark is that under high (resp. intermediate) ambiguity the S-optimal equilibrium for sure (resp. potentially) does not exist if S and R have identical preferences and only two messages are available. This is counterintuitive and we term this the *Doppelgänger Paradox*. A third remark is that under intermediate ambiguity the Doppelgänger Paradox, if arising, is compounded by the existence of an S-optimal equilibrium if R is moderately biased (i.e. $\beta \in \left[\underline{\beta}, \overline{\beta}\right]$). A misaligned R is thus preferable to S than a perfectly aligned R. We term this the *strong Doppelgänger Paradox*. A fourth and final remark is that the above features do not obtain in the absence of ambiguity. In a model featuring two actions, two messages always suffice to implement the (potentially mixed) S-optimal decision rule if S and R have identical preferences.

A third main finding is that there typically now also exist influential communication equilibria that do not implement the S-optimal decision rule, in contrast to the case of no ambiguity. A fourth main finding is that adding a little ambiguity, starting from no ambiguity, can generate the possibility of influential communication and additionally be unambiguously beneficial to S.

Literature review Ambiguous language arguably lacks a theoretical explanation: Existing models that explicitly purport to study ambiguous communication actually generate vagueness (see for example Alesina and Cukierman (1990), Aragonès and Neeman (2000), Callander and Wilson (2008), Tomz and Van Houweling (2009)). In contrast, we find the forms of randomization (by S and R) featured in S-optimal equilibria of our game reminiscent of two common modes of ambiguous communication. In this light, we provide a simple account of ambiguous language as the equilibrium implication of ambiguity in priors.

Our contribution lies at the intersection of the literatures on respectively cheap talk communication and ambiguity. The first was initiated by the seminal model of Crawford and Sobel (1982). The endogenous randomization over messages inducing different beliefs featured in our model bears a relation to the exogenous randomization studied in Board, Blume and Kawamura (2007). In the latter model, an emitted message may be randomly swapped with another during the transmission process. The authors show that this exogenous randomization can be welfare beneficial. Note however that if the sender had access to non-noisy messages he would strictly favour these over noisy messages. Blume and Board (2010, 2012) as well as Gordon and Nöldeke (2015) offer a further exploration of the issues studied in Blume, Board and Kawamura (2007). Finally, Lipman (2009) examines communication with identical player preferences and concludes that vagueness can be efficient only if the informed party exhibits bounded rationality in the form of "vague views of the world". Some of our results are in line with this conjecture.

Our paper also relates to the literature of ambiguity. We model ambiguity based on the Max-Min model (Gilboa (1987), Gilboa and Schmeidler (1989)). It is well-known that no common practice on updating of ambiguity averse preferences has yet emerged. We refer to Siniscalchi (2011) as well as Hanany and Klibanoff (2007, 2009) for a discussion of this issue. Recently, ambiguity has been brought to strategic settings by a number of authors. Bade (2010), Riedel and Sass (2011), Azrieli and Teper (2011) and Hanany, Klibanoff and Mukerji (2015) define general equilibrium concepts under ambiguity. A large array of papers study more specific applications to finance, tournaments or contract theory. Somewhat more related contributions include a number of studies of mechanism design under ambiguity (Bose and Renou (2011), Di Tillio et al. (2011)). The latter contributions, in applying the revelation principle, analyze a messaging game in the presence of ambiguity. Finally, Kellner and Le Quement (2015) analyze ambiguous (Ellsbergian) communication strategies within the CS model and find that for any standard influential equilibrium, there exists an Ellsbergian equilibrium ensuring both S and R a strictly higher ex ante expected payoff. Ambiguity, by triggering Max-Min decision making, acts as a beneficial commitment device for R.

2 The model

There are two agents, a sender S and a receiver R. The state of the world $\omega \in \{A, B\}$ has a subjectively uncertain distribution represented by a set $[P_l(B), P_h(B)]$ of prior probabilities of state B. We assume that $P_h(B) = \frac{1}{2} + e$ and $P_l(B) = \frac{1}{2} - e$, for some $e \in \left(0, \frac{1}{2}\right)$. R can choose among two actions a and b.

Preferences given a unique prior In the absence of ambiguity the preferences of each agent $i \in \{S, R\}$ are described by a parameter $q_i \in (0,1)$ which denotes an agent's relative aversion to type I and type II errors. Payoffs to agent $i \in \{S, R\}$ are given by $\pi_i(b, A) = -q_i$, $\pi_i(a, B) = -(1 - q_i)$ and $\pi_i(a, A) = \pi_i(b, B) = 0$. Define $P_k(B \mid \theta)$ as the posterior probability of B given information event θ and prior $P_k(B) \in \{P_l(B), P_h(B)\}$. Define $E_k(\pi_i(j, \omega) \mid \theta)$ as the expected payoff of action j for agent i given information event θ and prior $P_k(B) \in \{P_l(B), P_h(B)\}$:

$$
\begin{aligned}
E_k(\pi_i(b, \omega) \mid \theta) &= -q_i(1 - P_k(B \mid \theta)) \\
E_k(\pi_i(a, \omega) \mid \theta) &= -(1 - q_i)P_k(B \mid \theta),
\end{aligned}
$$

meaning that for a given prior $P_k(B)$ and a given information event θ, an ambiguity neutral agent i strictly favours action b over a if $P_k(B \mid \theta) > q_i$ and a over b if $P_k(B \mid \theta) < q_i$. An ambiguity neutral agent thus always strictly prefers a pure action except in the knife-edge case where $q_i = P_k(B \mid \theta)$. Preference parameters q_S and q_R are public information and $q_R \leq q_S$.

Preferences given multiple priors Let (α_a, α_b) denote the mixed action assigning probability α_a to a and probability α_b to b. An agent chooses the mixed action (α_a^*, α_b^*) that maximizes the minimal expected payoff across all possible priors given information event θ. Letting Δ_{ab} be the set of all distributions over the action space $\{a, b\}$, α^* satisfies

$$
\alpha^* \in \arg\max_{\alpha \in \Delta_{ab}} \min_{k \in \{l,h\}} \sum_{j \in \{a,b\}} \alpha_j E_k(\pi_i(j, \omega) \mid \theta).
$$

Note that in characterizing the Max-Min action we trivially only need to consider the most extreme (updated) priors $P_l(B \mid \theta)$ and $P_h(B \mid \theta)$ because the expected payoff of action j is increasing in the probability of state J. It is easily seen that the Max-Min action

α^* is characterized as follows. If $P_h(B\,|\theta) < q$ then $\alpha_a^* = 1$. If $P_l(B\,|\theta) < q < P_h(B\,|\theta)$ then $\alpha_a^* = q$. If $q < P_l(B\,|\theta)$ then $\alpha_a^* = 0$. If $P_h(B\,|\theta) = q$ there is a set of Max-Min actions defined by $\alpha_a \in [q, 1]$. If $P_l(B\,|\theta) = q$ there is a set of Max-Min actions defined by $\alpha_a \in [0, q]$.

Information structure S receives a one dimensional signal σ drawn from a state dependent continuous distribution $F_\omega(\sigma)$ with density function $f_\omega(\sigma)$ on a domain $[\underline{\sigma}, \overline{\sigma}]$. We allow both for a bounded domain (for example $[\underline{\sigma}, \overline{\sigma}] = [0, 1]$) or an unbounded domain (for example $\underline{\sigma} = -\infty$ and $\overline{\sigma} = +\infty$). $F_A(\sigma)$ and $F_B(\sigma)$ are such that $\frac{f_B(\sigma)}{f_A(\sigma)}$ is strictly increasing in σ, thus satisfying the Monotone Likelihood Ratio Property (MLRP). We introduce the following useful condition on the distribution of signals:

Assumption 1 *$\forall \sigma \in (\underline{\sigma}, \overline{\sigma})$, it holds true that*

$$\frac{\frac{F_B(\sigma)}{F_A(\sigma)}}{\frac{\partial\left(\frac{F_B(\sigma)}{F_A(\sigma)}\right)}{\partial\sigma}} > \frac{\frac{f_B(\sigma)}{f_A(\sigma)}}{\frac{\partial\left(\frac{f_B(\sigma)}{f_A(\sigma)}\right)}{\partial\sigma}} \text{and } \frac{\frac{1-F_B(\sigma)}{1-F_A(\sigma)}}{\frac{\partial\left(\frac{1-F_B(\sigma)}{1-F_A(\sigma)}\right)}{\partial\sigma}} > \frac{\frac{f_B(\sigma)}{f_A(\sigma)}}{\frac{\partial\left(\frac{f_B(\sigma)}{f_A(\sigma)}\right)}{\partial\sigma}}, \tag{1}$$

For $\sigma \in \{\underline{\sigma}, \overline{\sigma}\}$, the above inequality conditions hold true with weak inequalities.

Assumption 1 is equivalent to log-concavity of the inverse Mills ratios if for all σ, $f_B(\sigma) = f_A(\sigma - c)$ for some positive constant c. An instance of this is when f_B and f_A are two normal distributions with identical variance and means $\mu_A < \mu_B$.

Assumption 2: *It holds true that* $\lim_{\sigma\to\underline{\sigma}} \frac{f_B(\sigma)}{f_A(\sigma)} = 0$ *and* $\lim_{\sigma\to\overline{\sigma}} \frac{f_B(\sigma)}{f_A(\sigma)} = +\infty$.

Assumption 2 is satisfied for a pair of normals as described above. Assumptions 1 and 2 are also satisfied if $f_B(\sigma) = 2\frac{\sigma-\underline{\sigma}}{(\overline{\sigma}-\underline{\sigma})^2}$ and $f_A(\sigma) = 2\frac{\overline{\sigma}-\sigma}{(\overline{\sigma}-\underline{\sigma})^2}$ (which will be assumed together with $[\underline{\sigma}, \overline{\sigma}] = [0, 1]$ in our figures).

Communication protocol and equilibrium The timing of the game G is given as follows. At 0, Nature draws the state ω. At 1, Nature draws a signal according to F_ω. At 2, S issues a message. At 3, R chooses an action. S can communicate costlessly with R by emitting a message $m \in M$, where M is a set of cardinality $n \geq 2$ in which individual messages are numbered $m_1, ..., m_n$. A communication strategy δ of S specifies for each signal σ a distribution $(\delta_1(\sigma), .., \delta_n(\sigma))$ over messages belonging to M, where $\delta_j(\sigma)$ is the probability of sending m_j. A decision strategy ρ of R specifies a distribution $(\rho_a(m), \rho_b(m))$ over $\{a, b\}$ for each possible message m in M. A Weak Perfect Bayesian equilibrium of the game G is given by: 1) a communication strategy δ of S, 2) a decision strategy ρ of R and 3) a system of beliefs satisfying the following two requirements. First, δ and ρ are sequentially rational given the system of beliefs. Second, posterior beliefs for both S and R are generated whenever possible by using prior-by-prior Bayesian updating using equilibrium strategies.

Sequential rationality of ρ implies that R picks an action in the set of Max-Min actions given the posteriors $\{P_l(B\,|m,\delta), P_h(B\,|m,\delta)\}$. Let $E_k(\pi_S\,|m,\rho,\sigma)$ denote the expected payoff of S under the prior $P_k(B)$ at information set σ if R uses the decision strategy ρ. Let Δ_M be the set of all distributions over M. Sequential rationality of δ requires that at any signal σ

$$\delta^* \in \arg\max_{\delta\in\Delta_M} \min_{k\in\{l,h\}} \sum_{r=1}^{n} \delta_r(\sigma) E_k(\pi_S\,|m_r,\rho,\sigma). \tag{2}$$

Whenever a given equilibrium is such that some messages are never sent (i.e. are not sent for any σ), we assume that these out of equilibrium messages all trigger the same beliefs as m_1, which is an equilibrium message in all equilibria that we study. Note that for any equilibrium E featuring a communication strategy δ that makes use of m_1 but leaves a subset of unused messages M_- of cardinality $|M_-|$, one can always trivially construct an alternative equilibrium E' featuring a communication strategy δ' defined as follows: δ' is identical to δ, except that for any σ s.t. $p(m_1\,|\sigma,\delta) > 0$, δ' specifies that $p(m\,|\sigma,\delta') = \frac{p(m_1|\sigma,\delta)}{(1+|M_-|)}$ for any $m \in m_1 + M_-$. Equilibrium E' features no unused messages and trivially exhibits the same mapping as E between $[\underline{\sigma},\overline{\sigma}]$ and beliefs as well as actions of R.

We say that an equilibrium is *influential* if there exist two equilibrium messages m and m' that trigger different beliefs and actions (i.e. $\rho_a(m) \neq \rho_a(m')$). We say that two equilibria are *outcome-equivalent* if they implement the same decision rule, i.e. the same mapping from $[\underline{\sigma},\overline{\sigma}]$ to the set Δ_{ab} of distributions over $\{a,b\}$. Outcome equivalence thus does not take into account how a given distribution over $\{a,b\}$ is attached to a given signal σ in equilibrium, i.e. who randomizes.

The next section provides an analysis of our model. A subsequent section shows that our model is formally equivalent to one involving a continuum of pure actions.

3 Analysis

3.1 Opening remarks

We start with a characterization of agents' optimal decision rules. Given Assumption 2, there exist for each agent i and ambiguity level $e \in \left(0, \frac{1}{2}\right)$ two thresholds $t_i^1(e) \in (\underline{\sigma}, \overline{\sigma})$ and $t_i^2(e) \in (\underline{\sigma}, \overline{\sigma})$ satisfying the following

$$\left(\frac{\frac{1}{2}+e}{\frac{1}{2}-e}\right)\frac{f_B(t_i^1(e))}{f_A(t_i^1(e))} = \left(\frac{\frac{1}{2}-e}{\frac{1}{2}+e}\right)\frac{f_B(t_i^2(e))}{f_A(t_i^2(e))} = \frac{q_i}{1-q_i}.$$

Note that $\lim\limits_{e\to 0} t_i^1(e) = \lim\limits_{e\to 0} t_i^2(e)$ and $t_i^1(e) < t_i^2(e)$ for $e \in \left(0, \frac{1}{2}\right)$.

The thresholds are key to determining agent i's optimal (Max-Min) action as a function of signal σ. For $\sigma < t_i^1(e)$ agent i strictly favours a. For $\sigma \in \left(t_i^1(e), t_i^2(e)\right)$ agent i strictly favours the mixed action $(q_i, 1-q_i)$. For $\sigma > t_i^2(e)$ agent i strictly favours action b. At $\sigma = t_i^1(e)$ there is a set of Max-Min actions given by $(\rho_a, 1-\rho_a)$ s.t. $\rho_a \in [q_i, 1]$. At $\sigma = t_i^2(e)$ there is a set of Max-Min actions given by $(\rho_a, 1-\rho_a)$ s.t. $\rho_a \in [0, q_i]$. Each agent i thus has a (continuum) set of optimal decision rules that differ only regarding the action picked at the measure zero events $\sigma = t_i^1(e)$ and $\sigma = t_i^2(e)$. For all practical purposes (i.e. implementability and payoffs) all rules in this set are identical and slightly abusing vocabulary we shall henceforth define as *the S-optimal decision rule* the rule in this set that specifies action $(q_S, 1-q_S)$ at $\sigma = t_S^1(e)$ and $\sigma = t_S^2(e)$.

On a technical note, it is immediate that for any $e > 0$ we have $t_R^1(e) < t_S^1(e)$ and $t_R^2(e) < t_S^2(e)$ if $q_R < q_S$. $t_S^1(e)$ and $t_S^2(e)$ are respectively strictly decreasing and increasing in e. Also, $\lim\limits_{e\to 0} t_S^1(e) = \lim\limits_{e\to 0} t_S^2(e) \in (\underline{\sigma}, \overline{\sigma})$ while $\lim\limits_{e\to \frac{1}{2}} t_S^1(e) = \underline{\sigma}$ and $\lim\limits_{e\to \frac{1}{2}} t_S^2(e) = \overline{\sigma}$. The latter two limits imply that for $\omega \in \{A, B\}$, $\lim\limits_{e\to \frac{1}{2}} F_\omega(t_S^1(e)) = 0$ and $\lim\limits_{e\to \frac{1}{2}} 1 - F_\omega(t_S^2(e)) = 0$.

Let $t_S = \lim\limits_{e\to 0} t_S^1(e) = \lim\limits_{e\to 0} t_S^2(e)$. Equilibrium behavior in the absence of ambiguity is very simple except in the knife-edge case $P(B\,|\sigma \le t_S)) = q_R$. If there exists an equilibrium featuring influential communication then it implements S's optimal decision rule which is furthermore deterministic. S simply announces truthfully his favoured action by sending m_1 when favouring a and m_2 when favouring b. R takes action a after m_1 and b after m_2.

This equilibrium only exists if R is willing to take action a after m_1, which requires that q_R is not excessively low[1].

In what follows, we perform an analysis of the set of influential equilibria under ambiguity. The set consists of two separate subsets, those equilibria that implement the S-optimal decision rule (S-optimal equilibria) and those that do not.

3.2 S-optimal equilibria

We start with the following two observations.

Lemma 1 *a) If there exists an S-optimal equilibrium, then there exists an S-optimal equilibrium satisfying the following. No more than three messages are sent. Conditional on a given equilibrium message, R either picks pure action a or pure action b or mixed action* $(q_R, 1 - q_R)$.

b) If $q_R < q_S$*, there exists no S-optimal equilibrium in which S never randomizes.*

Point a) states that one can w.l.o.g. focus on S-optimal equilibria in which R only takes three different actions: $(1, 0)$, $(0, 1)$ and $(q_R, 1 - q_R)$. A full proof is given in the Appendix. The argument behind b) is as follows. For intermediate signal realizations S favours the mixed action $(q_S, 1 - q_S)$. In a putative S-optimal equilibrium in which S would never randomize, one message would have to trigger mixed action $(q_S, 1 - q_S)$. Yet we show in the proof of a) that there cannot exist an S-optimal equilibrium in which R chooses this mixed action.

Exhaustively characterizing the set of S-optimal equilibria is both daunting and unnecessary for our purposes. We now introduce a simple subclass of S-optimal equilibria and show that we may w.l.o.g. restrict ourselves to it. In what follows, note that we simply write t_S^i instead of $t_S^i(e)$ whenever no confusion can arise.

Definition 1 ***Threshold equilibrium***

a) A threshold equilibrium with threshold $z \in (t_S^1, t_S^2]$ *(also threshold-z equilibrium) involves the following strategy profile. S emits* m_1 *if* $\sigma < t_S^1$ *and* m_2 *if* $\sigma > t_S^2$*. If* $\sigma \in [t_S^1, z)$*, S emits* m_1

[1] In the knife edge case where $P(B\,|\sigma \leq t_S) = q_R$, there exists for every $x \in [0, 1]$ an equilibrium implementing the following decision rule. Pick b with probability x if $\sigma \leq t_S$ and b for sure if $\sigma > t_S$.

with probability $\frac{q_S - q_R}{1-q_R}$ *and* m_3 *with probability* $\frac{1-q_S}{1-q_R}$. *If* $\sigma \in \left[z, t_S^2\right]$ *, S emits* m_1 *with probability* q_S *and* m_2 *with probability* $1 - q_S$. *R chooses a after* m_1*, b after* m_2 *and mixed action* $(q_R, 1 - q_R)$ *after* m_3.

b) A threshold equilibrium with threshold $z = t_S^1$ *involves the following strategy profile. S emits* m_1 *if* $\sigma < t_S^1$ *and* m_2 *if* $\sigma > t_S^2$. *If* $\sigma \in [t_S^1, t_S^2]$*, S emits* m_1 *with probability* q_S *and* m_2 *with probability* $1 - q_S$. *R chooses a after* m_1*, b after* m_2.

Threshold equilibria feature a communication strategy that is entirely described by the three thresholds $\left\{t_S^1, z, t_S^2\right\}$ and a set of mixing probabilities. We call such a strategy a threshold communication strategy. A threshold equilibrium in which $z = t_S^1$ implies the emission with positive probability of exactly two messages while any remaining threshold equilibrium implies that three messages are emitted with positive probability. Given q_S, q_R, e, let $P_k(B\,|m_i, z)$ denote the conditional probability of B implied by m_i given prior $P_k(B)$ and the threshold strategy z.

All threshold equilibria are outcome equivalent as they all implement S's optimal decision rule. Though not stated explicitly in what follows, note that the set of threshold equilibria typically constitutes a continuum. Many different threshold communication strategies of S allow him to optimally guide R's actions. This multiplicity reflects the fact that there are two ways to trigger the optimal randomization by R in threshold equilibria, a simple and a more sophisticated way. The first is to randomize between m_1 and m_2. The second is to randomize between m_1 and m_3 (where m_3 leads to randomization by R). The two ways are perfect substitutes for S.

We attach three technical remarks on the randomization performed by S in threshold equilibria and S-optimal equilibria in general. First, in the classical CS model, any partitional equilibrium can be reinterpreted as an equilibrium in which S mixes between messages, but such mixing only involves messages that cause identical beliefs and identical actions. The involved mixing is therefore unnecessary as opposed to the mixing that appears in the S-optimal equilibria of our model, which cannot be disposed of as shown in Lemma 1.b). Second, the randomization performed by S in S-optimal equilibria differs from that featured in the *noisy talk* model of Blume, Board and Kawamura (2007) to the extent that randomization in our model is voluntary while it is exogenously generated in the *noisy talk* model. Third, the mixing performed by S in a threshold equilibrium can be reinterpreted as mixing over a set of classical partitional communication strategies upon observation of his private signal.

We wish to characterize the comparative statics of the set of S-optimal equilibria. Key questions are: 1) For a given q_S, what are the values of q_R compatible with the existence of an S-optimal equilibrium and how do these values vary as a function of the ambiguity level e? 2) Does S sometimes need strictly more than two messages to implement his optimal decision rule? Proposition 1 below implies that in seeking to answer the above questions, we may restrict ourselves without loss of generality to the subset of threshold equilibria.

Proposition 1 ***S-optimal equilibria and threshold equilibria***

a) If there exists an S-optimal equilibrium, then there exists a threshold equilibrium.

b) Given only two available messages, any S-optimal equilibrium is a threshold equilibrium with threshold t_S^1.

Point a) is proved in the Appendix. The proof of b) is as follows. In an S-optimal equilibrium featuring only two equilibrium messages, one message (say m_1) must trigger action a for sure and another (say m_2) must trigger b for sure. This is necessary to allow S to induce a for sure below t_S^1 and b for sure above t_S^2. There being no third message available, it follows that for $\sigma \in \left[t_S^1, t_S^2\right]$ S randomizes with probability $(q_S, 1-q_S)$ between m_1 and m_2 . The described strategy profile is a threshold equilibrium profile with threshold t_S^1.

Before stating our characterizations in the next two propositions we introduce constants $e_1, e_2, e_{12}, e_{13} \in \left(0, \frac{1}{2}\right)$ which shall constitute building blocks of our statements. Given q_S, q_R, e, let $P_k(B\,|m_i, z, e)$ denote the conditional probability of B implied by m_i given prior $P_k(B)$ and the threshold strategy z. Given any quintuple q_S, q_R, e, i, k, it can be shown that $\lim\limits_{z \to t_S^1(e)} P_k(B\,|m_i, z, e)$ is independent of q_R. Slightly abusing notation, we let $P_k(B\,|m_i, t_S^1(e), e) = \lim\limits_{z \to t_S^1(e)} P_k(B\,|m_i, z, e)$ given q_S, e, i, k. Constant e_1 is s.t. as e increases, $P_h(B\,|m_1, t_S^1(e), e)$ crosses q_S from below at e_1. e_2 is s.t. as e increases, $P_l(B\,|m_2, t_S^1(e), e)$ crosses q_S from above at e_2. e_{13} is s.t. as e increases $P_h(B\,|m_1, t_S^1(e), e)$ crosses $P_l(B\,|m_3, t_S^1(e), e)$

from below at $e = e_{13}$. Finally, e_{12} is s.t $P_h(B\left|m_1, t_S^1(e), e\right)$ crosses $P_l(B\left|m_2, t_S^1(e), e\right)$ from below at $e = e_{12}$. The constants satisfy the following. First, if $e_1 \neq e_2$ then e_{12} is strictly between e_1 and e_2 while if $e_1 = e_2$ then $e_{12} = e_1$. Second, $e_{13} < \min\{e_{12}, e_1\}$. Note that the constants are constructed for a fixed q_S and fixed distributions F_A and F_B. For more detail, we refer to the Online Appendix.

The following proposition offers a characterization of the comparative statics of the set of S-optimal equilibria in the presence of three messages.

Proposition 2 ***S-optimal equilibria (three messages available)***

Suppose that three messages are available.

i. Fix q_S. For any given e there is a strictly positive threshold $\underline{q}_R(e) < q_S$ such that there exists an S-optimal equilibrium if and only if $q_R \in \left[\underline{q}_R(e), q_S\right]$.

ii. For all $e \leq e_{13}$, $\underline{q}_R(e)$ is continuous and strictly increasing in e.

Proof: see in Appendix.

Point i. implies that an S-optimal equilibrium exists if and only if q_R is not too low relative to q_S. Point ii. shows that when three messages are available, increasing ambiguity is not helpful in so far as S's ability to implement his optimal decision rule is concerned.

In certain situations, communication is restricted to the use binary messages. An expert may for example be allowed only to say "*yes*" or "*no*". We now characterize the set of conflict levels under which S-optimal communication remains possible under this restriction. Note in what follows that $\underline{q}_R(e)$ is the lower bound defined in Proposition 2.

Proposition 3 ***S-optimal equilibria (only two messages available)***

Suppose that exactly two messages are available.

i. Let q_S be such that $e_2 < e_1$. The following holds true.

i.a) If $e > e_{12}$, there exists no S-optimal equilibrium for any q_R while instead if $e \leq e_{12}$, there are strictly positive thresholds $\underline{q}'_R(e)$ and $\overline{q}'_R(e)$ satisfying $\underline{q}'_R(e) \leq \overline{q}'_R(e) \leq q_S$ such that there exists an S-optimal equilibrium if and only if $q_R \in \left[\underline{q}'_R(e), \overline{q}'_R(e)\right]$.

i.b) $\underline{q}'_R(e) = \underline{q}_R(e)$ if $e \leq e_{13}$ while instead $\underline{q}'_R(e) > \underline{q}_R(e)$ if $e \in (e_{13}, e_{12}]$.

i.c) $\overline{q}'_R(e) = q_S$ if $e \leq e_2$ while instead $\overline{q}'_R(e) < q_S$ if $e \in (e_2, e_{12}]$.

i.d) For any $e \leq e_{12}$, $\underline{q}'_R(e)$ is continuous and strictly increasing in e and $\overline{q}'_R(e)$ is continuous and weakly decreasing in e.

ii. Let q_S be such that $e_2 \geq e_1$. Statements of Point i. apply with the following modifications. First, constant e_{12} is everywhere replaced by e_1. Second, Point i.c) is replaced by: $\overline{q}'_R(e) = q_S$ for $e \in (0, e_1]$.

Proof: see in Appendix.

Note that if assuming $[\underline{\sigma}, \overline{\sigma}] = [0, 1]$, $f_B(\sigma) = 2\sigma$ and $f_A(\sigma) = 2 - 2\sigma$, it holds true that $e_2 < e_1$ iff $q_S > \frac{1}{2}$. Figure 1 below illustrates Propositions 2 and 3 for this signal structure, assuming $q_S = .66$. The horizontally striped area indicates pairs (e, q_R) for which there exists an S-optimal equilibrium if only two messages are available. The diagonally striped area indicates pairs (e, q_R) for which there exists an S-optimal equilibrium if three messages are available.

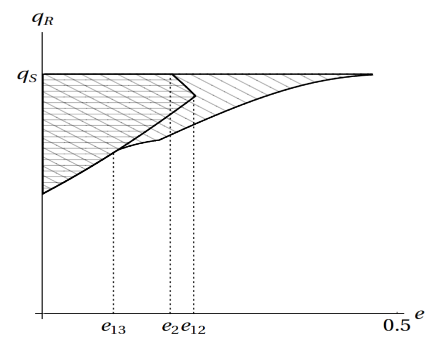

Figure 1: Message number and S-optimal equilibria.

On the usefulness of a third message Proposition 3 shows that the absence of a third message hurts S in terms of his ability to implement his optimal decision rule whenever ambiguity is sufficiently high. We focus on the statements of i., those of ii. being qualitatively virtually identical with one exception which is discussed later. Point i.a) shows that if ambiguity is large ($e > e_{12}$), the S-optimal rule can never be implemented in the absence of a third message. Points i.b) and i.c) show that if e is smaller than e_{12}, there is a closed interval $\left[\underline{q}'_R(e), \overline{q}'_R(e)\right]$ of values of q_R such that an S-optimal equilibrium exists if and only if q_R belongs to this interval. Point i.b) shows that the lower bound $\underline{q}'_R(e)$ is strictly higher than $\underline{q}_R(e)$ if $e \in (e_{13}, e_{12})$ and i.c) states that the upper bound $\overline{q}'_R(e)$ is strictly smaller than q_S if $e \in (e_2, e_{12})$. If $e \in (\max\{e_{13}, e_2\}, e_{12})$, a third message is thus useful if and only if R is either very aligned or very misaligned. For $e \leq \min\{e_{13,}e_2\}$, on the other hand, a third message is always superfluous. Finally, i.c) shows that given only two messages, adding a little ambiguity in the environment is never helpful in so far as S's ability to implement his optimal decision rule is concerned. Indeed, the bounds $\underline{q}'_R(e)$ and $\overline{q}'_R(e)$ are respectively increasing and decreasing in e.

At an abstract level, the usefulness of a third message derives from the following two features of the game. First, given ambiguity aversion agents' optimal decision rules involve three types of behavior, either a or b or mixing. Both agents favour a when the signal is low, b when it is high, and hedging (though with different probabilities) when it is intermediate. If ambiguity is high, there is a common interval of signal realizations where both agents want to randomize. For S, being able to convey whether he wants to randomize can thus naturally be helpful. If ambiguity is instead low, there is no common interval of signals for which both agents want to randomize.

A Doppelgänger Paradox A salient aspect of our characterization is that for e large enough (a sufficient condition being $e \geq \min\{e_1, e_{12}\}$), there exists no S-optimal equilibrium for $q_S = q_R$ when S is restricted to using only two messages. We call this phenomenon *Doppelgänger Paradox* and add some remarks on this in what follows.

First, if $q_R = q_S$, one would expect that there exists a threshold equilibrium with threshold $t^1_S(e)$, thus making the restriction to two messages inconsequential. The intuition for this would be as follows. In such a putative equilibrium, R chooses a after m_1 and b after m_2. S simply randomizes optimally between m_1 and m_2 whenever $\sigma \in \left[t^1_S(e), t^2_S(e)\right]$ and otherwise chooses m_1 or m_2. R, recognizing that his optimal decision rule coincides with that of S, should have no deviation incentive. This intuition is however wrong. Under the assumed updating rule, R simply applies his own optimal (ambiguity averse) Max-Min best response to the received message. The fact that in the postulated equilibrium S has already acted in a way that maximized his own (ambiguity averse) preferences is immaterial. S's ability to successfully hedge against ambiguity does not imply that R is also hedged against ambiguity. The key here is the dynamic inconsistency of R's behavior given the assumed updating rule. Following Hanany and Klibanoff (2007) one could alleviate the paradox by excluding certain priors at certain information sets of R.

For comparison, consider a Bayesian game under expected utility between two players (1 and 2) each endowed with two actions. Assume that in equilibrium, agent 1 randomizes for some types. Suppose that agent 1 cannot act himself but needs to act through a third agent (R) with identical preferences (a Doppelgänger). Communication between agent 1 and R is cheap talk. Under expected utility, a third message would never be necessary to allow agent 1 to implement his desired decision rule through R. Agent 1 would simply optimally randomize between two messages inducing pure actions by R and the latter would have no incentive to deviate.

A second specificity of the model is that given only two messages, an increased preference misalignment can be helpful for S. Given q_S such that $e_2 < e_1$ and $e \in (e_2, e_{12}]$, an S-optimal equilibrium does not exist for $q_R = q_S$ but exists for an interval of values of q_R strictly smaller than q_S (see Figure 1). One might call this the strong Doppelgänger paradox. Note that the latter does not arise if $e_2 \geq e_1$ (corresponding to Point ii. in Proposition 3). Under expected utility, an increase in R's bias would in contrast always hurt S for any fixed message space cardinality.

Interpreting S-optimal equilibria The randomization performed by respectively S or by R in threshold equilibria relates to two common modes of ambiguous communication, each offering an instance of the multiplicity of interpretations that is in our view the essence of ambiguous language. The first mode operates through the ambiguity arising in deriving the implications of the perceived language through a process of introspection (*What shall I do given what I heard?*). The second mode operates through the ambiguity arising in perceiving language through a process of extrospection (*What have I actually heard?*).

We expand on the first mode in what follows. In a threshold equilibrium with $z > t_S^1(e)$, when S sends the partitional message m_3 that gives rise to randomization by R, this is somewhat equivalent to S taking an agnostic stance, stating "*I recommend neither a nor b*." or "*Whether a or b is optimal is a matter of perspective.*". A caveat is that m_3 is admittedly a classical partitional message à la Crawford and Sobel (1982) but the partition is a specific one: It exclusively contains intermediate signals and by definition triggers randomization by R. By sending m_3, S conveys the inconclusiveness of his own information.

We now comment on the second mode. When S mixes between messages triggering different beliefs and responses, this bears some similarity to the choice of versatile formulations that give rise to a distribution over perceived statements. Suppose that besides the classical messages m_1, m_2 and m_3, S also has access to non standard messages which induce a distribution over the observation by R of respectively m_1, m_2 and m_3. Let $\widetilde{m}(x_1, x_2, x_3)$ induce R to see m_i with probability x_i for $i = 1, 2, 3$, with $x_1 + x_2 + x_3 = 1$. Call any such message a *noisy message* and let it be common knowledge that S has access to a rich set of such messages, one for each (x_1, x_2, x_3) s.t. $x_1 + x_2 + x_3 = 1$. Suppose that there exists a simple threshold equilibrium featuring $z = t_S^1$ and thus making use only of standard messages. It follows immediately that there exists an equilibrium in which S is known to use the following communication strategy. He sends m_1 if $\sigma < t_S^1$, m_2 if $\sigma > t_S^2$ and sends the noisy message $\widetilde{m}(q_S, 1 - q_S, 0)$ if $\sigma \in \left[t_S^1, t_S^2\right]$. Similarly, if there exists a threshold equilibrium featuring $z \in (t_S^1, t_S^2)$, then there exists an equilibrium in which S sends m_1 if $\sigma < t_S^1$, m_2 if $\sigma > t_S^2$, the noisy message $\widetilde{m}\left(\frac{q_S - q_R}{1 - q_R}, 0, \frac{1 - q_S}{1 - q_R}\right)$ if $\sigma \in \left[t_S^1, z\right)$ and the noisy message $\widetilde{m}(q_S, 1 - q_S, 0)$ if $\sigma \in \left[z, t_S^2\right]$.

We add two remarks on equilibria featuring noisy messages. First, note that R is aware of the fact that these messages are being used. He recognizes that he sees a standard message m_1, m_2 or m_3 but that S may in fact have sent a noisy message. Equilibria featuring noisy messages thus involve fully rational agents. Second, the use of noisy messages allows to solve the implicit commitment problem inherent to the randomization performed by S. Recall that a Max-Min decision maker is typically not indifferent between the two actions that he randomizes between if randomization is optimal. Noisy messages in essence allow S to delegate the task of randomizing to an outside garbling device.

3.3 Non S-optimal equilibria

Recall that the set of influential and non S-optimal equilibria is generically empty in the absence of ambiguity. This is not the case anymore in the presence of ambiguity. Decision rule $D(q_S, q_R, e)$ is defined as follows. Action a is picked with probability q_R if $\sigma < t_S^2(e)$. b is played with probability one if $\sigma \geq t_S^2(e)$. Consider the following strategy profile which implements D. S emits m_1 if $\sigma < t_S^2(e)$ and m_2 if $\sigma \geq t_S^2(e)$. R chooses mixed action $(q_R, 1-q_R)$ after m_1 and b after m_2. When this profile constitutes an equilibrium we call it the *simple D-equilibrium*. Recall in what follows that $\underline{q}_R(e)$ is the lower bound defined in Proposition 2.

Proposition 4 ***Existence of a simple*** D***-equilibrium***

i. Fix q_S*. For any* $e \in (0, \frac{1}{2})$ *there are strictly positive thresholds* $\underline{q}_R^D(e)$ *and* $\overline{q}_R^D(e)$ *satisfying* $\underline{q}_R^D(e) < \overline{q}_R^D(e) \leq q_S$ *such that the simple D-equilibrium exists if and only if* $q_R \in \left[\underline{q}_R^D(e), \overline{q}_R^D(e)\right]$.

ii. $\lim_{e\to 0}\underline{q}_R^D(e) = \lim_{e\to 0}\underline{q}_R(e)$*. For* $e \in (0, \frac{1}{2})$*,* $\underline{q}_R^D(e)$ *is continuous and strictly decreasing in* e *and* $\overline{q}_R^D(e)$ *is continuous and weakly increasing in* e*.*

Proof: See in Appendix.

A key aspect here is that in equilibrium, S triggers only two actions, one of which is mixed. Though this immediately implies that S will not be able to implement his optimal decision rule, it does not automatically imply the existence of advantageous deviations for S. Though m_1 does not trigger S's optimal action for $\sigma \leq t_S^2(e)$, it triggers a less unattractive action than m_2 does. Point ii. shows that the addition of a little ambiguity may allow for the emergence of the simple D-equilibrium. Figure 2 below considers the same parameter values as Figure 1. The diagonally striped area indicates pairs (e, q_R) for which there exists an S-optimal equilibrium if three messages are available. The vertically striped area indicates pairs (e, q_R) for which the simple D-equilibrium exists.

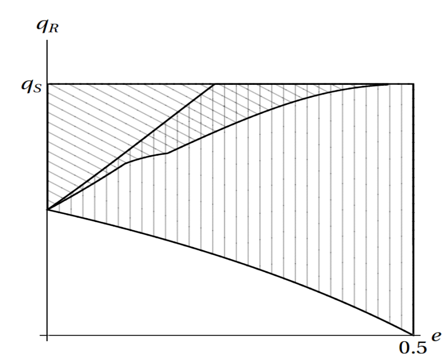

Figure 2: S-optimal equilibrium and simple D-equilibrium.

3.4 The virtues of a little ambiguity

In what follows, we let $\underline{q}_R(0) = \lim_{e\to 0} \underline{q}_R(e)$, where $\underline{q}_R(e)$ is the lower bound defined in Proposition 2. Recall that in the absence of ambiguity ($e = 0$), only the babbling equilibrium exists given $q_R < \underline{q}_R(0)$. For q_R slightly below $\underline{q}_R(0)$, the next proposition evaluates the welfare properties of the influential communication rendered possible by the addition of a little ambiguity. To that end, we compare the expected payoff obtained by agent i in the babbling equilibrium under no ambiguity to that obtained by i in the simple D-equilibrium under ambiguity level e when applying his most adverse prior.

Proposition 5 *Fix $q_S < \frac{1}{2}$. There is an $e^* > 0$ and for any $e \in (0, e^*)$ there is a threshold $\widehat{\underline{q}}_R(e) < \underline{q}_R(0)$ such that if $e \in (0, e^*)$ and $q_R \in \left[\widehat{\underline{q}}_R(e), \underline{q}_R(0)\right)$ then:*

a) The simple D-equilibrium exists.

b) For any prior $P(B) \in \left[P_l^e(B), P_h^e(B)\right]$ the expected utility of S is strictly larger in the simple D-equilibrium than in the babbling equilibrium in the absence of ambiguity.

Proof: See in Appendix.

Our proposition shows that for q_R close enough to $\underline{q}_R(0)$ and e small enough, the addition of a little ambiguity not only generates the possibility of influential communication but also ensures S an increase in expected payoff under any prior in $\left[P_l^e(B), P_h^e(B)\right]$. This conclusion does not apply for R who may well lose under both priors from the transition to positive ambiguity and influential communication.

Figure 3 below illustrates the proposition. We assume the same information structure as in previous figures and set $q_S = .45$. The diagonally striped area indicates parameters for which there exists an equilibrium implementing the S-optimal decision rule. The vertically striped area denotes parameter values for which the simple D-equilibrium exists. The plain grey area denotes parameters for which 1) $q_R < \underline{q}_R(0)$, 2) the simple D-equilibrium exists and 3) the latter improves the expected payoff of S under any prior in $\left[P_l^e(B), P_h^e(B)\right]$ w.r.t. to babbling under no ambiguity.

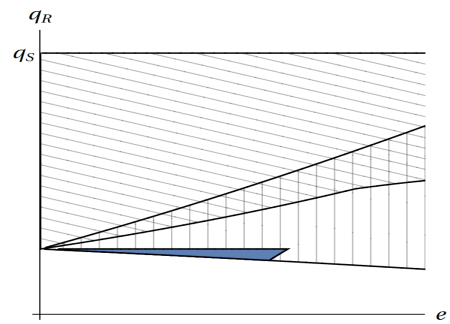

Figure 3: Adding a little ambiguity.

4 Extension to a continuum of actions

We now show that our model is formally equivalent to a model involving a continuum of pure actions. We refer to the jury interpretation of our setup. Let states A and B correspond to the defendant being respectively innocent and guilty. Define furthermore T as the maximal detention time (in years) to which the latter can be sentenced. For every $x \in [0,1]$, let action x consist in detaining the defendant for xT years. Given a state $\omega \in \{A, B\}$ and a given action $x \in [0,1]$, let payoffs to agent $i \in \{S, R\}$ be given by $\pi_i(x, A) = -q_i x$, $\pi_i(x, B) = -(1-q_i)(1-x)$. Defining $E_k(\pi_i(x,\omega) \mid \theta)$ as the expected payoff of action x for agent i given information event θ and prior $P_k(B)$, we now have:

$$E_k(\pi_i(x,\omega) \mid \theta) = -q_i x\,(1 - P_k(B \mid \theta)) - (1-q_i)(1-x)P_k(B \mid \theta). \tag{3}$$

Note two features. Within this modified model, the expected payoff of the pure action x is the same as the expected payoff of the mixed action assigning probability x to pure action 1 and $1-x$ to 0. Secondly, (3) is the expected payoff of the mixed action $(1-x, x)$ in our original model. It follows trivially from this second observation that the optimal decision rule of an agent in this setup mirrors the one featured in the original model given $P_k(B \mid \theta)$. First, choose 0 for sure if $P_h(B\,|\theta) < q$. Second, choose pure action $1-q$ or randomize over $\{0,1\}$ with probability $(q, 1-q)$ if $P_l(B\,|\theta) \leq q \leq P_h(B\,|\theta)$. Finally, choose 1 if $q < P_l(B\,|\theta)$.

For every threshold equilibrium with threshold z in the original model, there is an outcome equivalent equilibrium in this model that differs from the former only to the extent that R now picks the pure action $1 - q_R$ instead of randomizing, thus avoiding the implicit commitment problem associated with randomization in the original model. In contrast, S can still not dispense of mixing between equilibrium messages, which shows that mixing is not an artefact of the binary action space assumed in the original setup.

Lemma 2 *There exists no S-optimal equilibrium in which S never randomizes between messages.*

Proof: Identical to that of Lemma 1.b) and therefore omitted.

5 Conclusion

We have established the basic properties of a simple binary cheap talk model within an ambiguous environment. From a formal perspective, the main novel feature of equilibria is that S often randomizes between messages that trigger different (pure or mixed) actions by R. The communication strategy of S is thus not reducible to a simple partitional strategy. Other key properties relate to the comparative statics effect of interest misalignment, language richness and exogenous ambiguity.

6 Appendix

6.1 Proof of Lemma 1.a)

Outline Step 1 shows that we can restrict ourselves to equilibria in which any given action of R is triggered only by one equilibrium message. Step 2 shows that there cannot exist an S-Optimal equilibrium in which R after some message chooses a mixed action $(\alpha, 1-\alpha)$ s.t. $\alpha \in (q_R, 1)$. Step 3 shows that it is w.l.o.g. to assume that no message triggers a mixed action $(\alpha, 1-\alpha)$ s.t. $\alpha \in (0, q_R)$ and concludes.

We recall the following notation. Given communication strategy δ we denote by $\delta_i(\sigma)$ the likelihood that message m_i is sent given signal σ. Assuming that communication strategy δ uses finitely many messages given by $m_1, .., m_n$, it thus holds true that $\sum_{r=1}^{n} \delta_r(\sigma) = 1$ for any σ. We denote a given communication strategy by $\delta = (\delta_1, \delta_2, .., \delta_n)$.

Step 1 *Let δ be featured in an S-optimal equilibrium E. Assume that given δ, m_r and $m_{r'}$ are both sent with positive probability and trigger the same mixed action $(\alpha, 1-\alpha)$. Consider the strategy $\widetilde{\delta}$ constructed as follows. Set $\widetilde{\delta}_r(\sigma) = \delta_r(\sigma) + \delta_{r'}(\sigma)$ and $\widetilde{\delta}_{r'}(\sigma) = 0$. Set $\widetilde{\delta}_{r''}(\sigma) = \delta_{r''}(\sigma)$ for any $r'' \neq r, r'$. There exists an S-optimal equilibrium featuring $\widetilde{\delta}$.*

Proof: Note that given ω, r, r' it holds true that

$$P(m_r \left| \omega, \widetilde{\delta} \right.) = P(m_r \left| \omega, \delta \right.) + P(m_{r'} \left| \omega, \delta \right.). \tag{4}$$

Suppose that m_r and $m_{r'}$ trigger pure action a given δ. Then it must be that for $s = r, r'$

$$\frac{P(m_s \left| B, \delta \right.)}{P(m_s \left| A, \delta \right.)} \leq \frac{1-P_h(B)}{P_h(B)} \frac{q_R}{1-q_R}.$$

It follows from the above inequality, (4) and Lemma R (Point a)) that

$$\frac{P(m_r \left| B, \widetilde{\delta} \right.)}{P(m_r \left| A, \widetilde{\delta} \right.)} \leq \frac{1-P_h(B)}{P_h(B)} \frac{q_R}{1-q_R}.$$

We may thus conclude that given $\widetilde{\delta}$, action a is optimal after m_r. Suppose that m_r and $m_{r'}$ trigger a mixed action $(\alpha, 1-\alpha)$ s.t. $\alpha \in (q_R, 1)$ given δ. Then it must be that for $s = r, r'$

$$\frac{1-P_h(B)}{P_h(B)} \frac{q_R}{1-q_R} = \frac{P(m_s \left| B, \delta \right.)}{P(m_s \left| A, \delta \right.)}.$$

It follows from the above equality, (4) and Lemma R (Point a)) that

$$\frac{1-P_h(B)}{P_h(B)} \frac{q_R}{1-q_R} = \frac{P(m_r \left| B, \widetilde{\delta} \right.)}{P(m_r \left| A, \widetilde{\delta} \right.)}.$$

We may thus conclude that given $\widetilde{\delta}$, the mixed action $(\alpha, 1-\alpha)$ is optimal after m_r. Suppose that m_r and $m_{r'}$ trigger mixed action $(q_R, 1-q_R)$ given δ. Then it must be that for $s = r, r'$

$$\frac{1-P_h(B)}{P_h(B)} \frac{q_R}{1-q_R} \leq \frac{P(m_s \left| B, \delta \right.)}{P(m_s \left| A, \delta \right.)} \leq \frac{q_R}{1-q_R} \frac{1-P_l(B)}{P_l(B)}.$$

It follows from the above double inequality, (4) and Lemma R (Point a)) that

$$\frac{1-P_h(B)}{P_h(B)}\frac{q_R}{1-q_R} \leq \frac{P(m_r \left| B, \widetilde{\delta} \right.)}{P(m_r \left| A, \widetilde{\delta} \right.)} \leq \frac{q_R}{1-q_R}\frac{1-P_l(B)}{P_l(B)}.$$

We may thus conclude that given $\widetilde{\delta}$, the mixed action $(q_R, 1-q_R)$ is optimal after m_r. The same argument can be used to analyze the remaining possible equilibrium actions.

Step 2 *There exists no S-optimal equilibrium featuring an equilibrium message m_r that satisfies the following. First, it is not only sent for $\sigma = t_R^1$. Second, it triggers a mixed action of R specifying that a is picked with a probability belonging to $(q_R, 1)$.*

Proof: Assume an S-optimal equilibrium E featuring a message m_r *not only sent for* $\sigma = t_R^1$ that triggers a mixed action $(\alpha, 1-\alpha)$ of R s.t. $\alpha \in (q_R, 1)$. For such a mixed action to be part of the set of Max-Min actions, m_r must trigger belief q_R when applying prior $P_h(B)$. Given that m_r triggers this belief and that m_r is not sent only if $\sigma = t_R^1$, it follows that m_r must be sent with strictly positive probability for some $\sigma < t_R^1$ as well as for some $\sigma > t_R^1$. If this is true, then the S-optimal decision rule is however not implemented in E given that b is taken with positive probability for some $\sigma < t_R^1 \leq t_S^1$.

Step 3 *If there exists an S-optimal equilibrium, then there exists an S-optimal equilibrium in which no equilibrium message triggers a mixed action $(\alpha, 1-\alpha)$ satisfying $\alpha \in (0, q_R)$.*

Proof: Assume that there exists an S-optimal equilibrium E that features δ. By steps 1 and 2, it is without loss of generality to assume that m_1 (m_2) is the unique message that triggers a (b) for sure, m_3 is the unique message triggering $(q_R, 1-q_R)$ while for any $r \geq 4$, m_r is the unique message triggering $(\alpha_r, 1-\alpha_r)$ satisfying $\alpha_r \in (0, q_R)$. Denote the strategy profile featured in E by φ.

We now construct a strategy profile φ' that constitutes an S-optimal equilibrium E' that is such that that for some particular $r \geq 4$ (call it $\widetilde{r}$), the mixed action $(\alpha_{\widetilde{r}}, 1-\alpha_{\widetilde{r}})$ is never triggered in equilibrium. Profile φ' features the communication strategy δ' defined as follows. Strategy δ' is identical to δ except that if $\sigma \in \left[t_S^1, t_S^2\right]$,

$$\delta'_{\widetilde{r}}(\sigma) = \delta_{\widetilde{r}}(\sigma)\frac{\alpha_{\widetilde{r}}}{q_R}$$

and

$$\delta'_2(\sigma) = \delta_2(\sigma) + \left(1 - \frac{\alpha_{\widetilde{r}}}{q_R}\right)\delta_{\widetilde{r}}(\sigma).$$

Note that it follows that $\delta'_{\widetilde{r}}(\sigma) \geq 0$ and $\delta'_2(\sigma) \geq \delta_2(\sigma)$.

Profile φ' assigns the following strategy to R. Pick the mixed action $(q_R, 1-q_R)$ after $m_{\widetilde{r}}$. After any other message, pick the same action as in the S-optimal equilibrium E.

We briefly recall some properties of the equilibrium E before checking incentives of S and R in the putative equilibrium E'. First, the probabilities $\delta_3(\sigma), \delta_4(\sigma), ..$ can only be strictly positive if $\sigma \in [t_S^1, t_S^2]$. Also, $\delta_1(\sigma) = 1$ below t_S^1 and $\delta_2(\sigma) = 1$ above t_S^2. Moreover, in E message m_r triggers belief q_R when applying the prior $P_l(B)$, for any $r \geq 4$.

We now verify that the constructed strategy δ' defines a probability distribution over messages for any σ. We only need to consider $\sigma \in [t_S^1, t_S^2]$. Note that

$$\delta'_{\widetilde{r}}(\sigma) - \delta_{\widetilde{r}}(\sigma) = -(\delta'_2(\sigma) - \delta_2(\sigma))$$

while for any remaining $r \geq 1, \delta'_r(\sigma) = \delta_r(\sigma)$. It thus follows that given $\sum_{r\geq 1} \delta_r(\sigma) = 1$ it also holds true that $\sum_{r\geq 1} \delta'_r(\sigma) = 1$. Moreover, the equalities defining $\delta'_{\widetilde{r}}(\sigma)$ and $\delta'_2(\sigma)$ imply that $\delta'_r(\sigma) \geq 0$ for any r.

The putative equilibrium E' implements the S-optimal decision rule if for $\sigma \in [t_S^1, t_S^2]$

$$\delta'_2(\sigma) + \sum_{r\geq 3} \delta'_r(\sigma)(1-\alpha_r) = 1 - q_S.$$

The above equality is true since

$$\begin{aligned}
& \delta'_2(\sigma) + \delta'_{\widetilde{r}}(\sigma)(1-q_R) + \sum_{r\geq 3, r\neq \widetilde{r}} \delta'_r(\sigma)(1-\alpha_r) \\
= \ & \delta_2(\sigma) + \left(1 - \frac{\alpha_{\widetilde{r}}}{q_R}\right)\delta_{\widetilde{r}}(\sigma) + \delta_{\widetilde{r}}(\sigma)\frac{\alpha_{\widetilde{r}}}{q_R}(1-q_R) + \sum_{r\geq 3, r\neq \widetilde{r}} \delta'_r(\sigma)(1-\alpha_r) \\
= \ & \delta_2(\sigma) + \sum_{r\geq 3} \delta_r(\sigma)(1-\alpha_r) = 1 - q_S.
\end{aligned}$$

The last equality follows because δ is S-optimal.

We now check incentives of R in the putative equilibrium E'. Recall that in the S-optimal equilibrium E,

$$\frac{P(m_{\widetilde{r}}|B,\delta)}{P(m_{\widetilde{r}}|A,\delta)} = \frac{\int_{\underline{\sigma}}^{\overline{\sigma}} \delta_{\widetilde{r}}(\sigma) f_b(\sigma) d\sigma}{\int_{\underline{\sigma}}^{\overline{\sigma}} \delta_{\widetilde{r}}(\sigma) f_a(\sigma) d\sigma} = \frac{q_R}{1-q_R} \frac{1 - P_l(B)}{P_l(B)}. \tag{5}$$

Given that for any σ $\delta'_{\widetilde{r}}(\sigma)$ simply equals $\delta_{\widetilde{r}}(\sigma)$ multiplied by a constant, it follows that in equilibrium E' message $m_{\widetilde{r}}$ triggers belief q_R when applying the prior $P_l(B)$. R is thus willing to play a with probability q_R after $m_{\widetilde{r}}$ in E'. Regarding m_2,note that by definition

$$\frac{P(m_2|B,\delta')}{P(m_2|A,\delta')} = \frac{P(m_2|B,\delta) + \int_{\underline{\sigma}}^{\overline{\sigma}} \left(\delta'_2(\sigma) - \delta_2(\sigma)\right) f_b(\sigma) d\sigma}{P(m_2|B,\delta) + \int_{\underline{\sigma}}^{\overline{\sigma}} \left(\delta'_2(\sigma) - \delta_2(\sigma)\right) f_a(\sigma) d\sigma}.$$

By construction of $\delta'_2(\sigma)$

$$\frac{P(m_2|B,\delta')}{P(m_2|A,\delta')} = \frac{P(m_2|B,\delta) + \left(1 - \frac{\alpha_{\widetilde{r}}}{q_R}\right) \int_{\underline{\sigma}}^{\overline{\sigma}} \delta_{\widetilde{r}}(\sigma) f_b(\sigma) d\sigma}{P(m_2|B,\delta) + \left(1 - \frac{\alpha_{\widetilde{r}}}{q_R}\right) \int_{\underline{\sigma}}^{\overline{\sigma}} \delta_{\widetilde{r}}(\sigma) f_a(\sigma) d\sigma}.$$

Since $1 - \frac{\alpha_r}{q_R} \geq 0$, inequality $\frac{P(m_2|B,\delta)}{P(m_2|A,\delta)} \geq \frac{q_R}{1-q_R} \frac{1-P_l(B)}{P_l(B)}$ and equality (5) imply by Lemma R (Point a)) that $\frac{P(m_2|B,\delta')}{P(m_2|A,\delta')} \geq \frac{q_R}{1-q_R} \frac{1-P_l(B)}{P_l(B)}$. Hence it is optimal for R to play b with probability one after m_2 in equilibrium E'.

We have now constructed an equilibrium E' in which exactly two messages m_3 and $m_{\widetilde{r}}$ trigger the mixed action $(q_R, 1 - q_R)$. It follows by the argument given in step 1 that there exists an equilibrium E'' featuring the following strategy profile. S uses δ'' such that $\delta''_3(\sigma) = \delta'_3(\sigma) + \delta'_{\widetilde{r}}(\sigma)$ and $\delta''_{\widetilde{r}}(\sigma) = 0$ and $\delta''_s(\sigma) = \delta'(\sigma)$ for any $s \neq 3, \widetilde{r}$. On the other hand, R's strategy is the same in E'' as in E'.

Starting from the S-optimal equilibrium E, we have now constructed an S-optimal equilibrium E'' featuring one message less than E and which satisfies the same core properties as E: Message m_1 (m_2) is the unique message that triggers a (b) for sure, m_3 is the unique message triggering $(q_R, 1 - q_R)$ while for any $r \geq 4$, m_r (if still used) is the unique message triggering mixed action $(\alpha_r, 1 - \alpha_r)$ satisfying $\alpha_r \in (0, q_R)$. One can iterate this procedure until obtaining an equilibrium in which only the three messages m_1, m_2, m_3 are used and trigger respectively a, b or $(q_R, 1 - q_R)$. Note that in order to simplify exposition, we are assuming a finite number of messages. The arguments carry over to a continuum of messages.■

6.2 Proof of Proposition 1

Outline We here prove Point a). Given Lemma 1.a), we may restrict ourselves to equilibria in which only m_1, m_2 and m_3 might be sent with positive probability and trigger respectively a for sure, b for sure and $(q_R, 1-q_R)$. The proof is organized as follows. Step 1 describes the constraints that must be satisfied in an S-optimal equilibrium of the type described above. Step 2 states additional properties that S's strategy must satisfy. We then consider two cases corresponding to respectively $t_R^2 \leq t_S^1$ and $t_R^2 \in (t_S^1, t_S^2]$. In the case of $t_R^2 \leq t_S^1$, we show that if there exists an S-optimal equilibrium, then there exists an S-optimal equilibrium that features the threshold strategy $z = t_S^1$ (step 3). In the case of $t_R^2 \in (t_S^1, t_S^2]$, we proceed through two steps. We first show that if there exists any S-optimal equilibrium, then one can construct an S-optimal equilibrium in which m_3 is used with a certain constant probability if $\sigma \in (t_S^1, t_R^2]$ (step 4). We then show that if there exists an S-optimal equilibrium of the latter type, one can construct an S-optimal equilibrium which is a threshold equilibrium (step 5).

We introduce the following notation. We denote by $P_k(\omega\,|m_i, \delta)$ the probability of state ω conditional on receiving m_i when applying prior $P_k(B)$, for $k \in \{l, h\}$. Slightly abusing notation, we denote by $P(m_i|\omega, z)$ and $P_k(\omega\,|m_i, z)$ the counterparts of $P(m_i|\omega, \delta)$ and $P_k(\omega\,|m_i, \delta)$ for the case where S uses a threshold strategy featuring threshold z.

Step 1 *a) An S-optimal equilibrium featuring a communication strategy δ specifying that only m_1 and m_2 are emitted with positive probability exists if and only if*

$$P_h(B\,|m_1, \delta) \leq q_R \leq P_l(B\,|m_2, \delta).$$

b) An S-optimal equilibrium featuring a strategy δ in which messages m_1, m_2 and m_3 are emitted with positive probability exists if and only if the above double inequality holds and furthermore $P_l(B\,|m_3, \delta) \leq q_R$.

Proof: We first consider Point a). An S-optimal equilibrium featuring a strategy δ in which only messages m_1, m_2 are emitted with positive probability exists if and only if

$$\max\{P_l(B\,|m_1, \delta), P_h(B\,|m_1, \delta)\} \leq q_R \leq \min\{P_l(B\,|m_2, \delta), P_h(B\,|m_2, \delta)\}.$$

The LHS inequality ensures that R chooses a with probability one after m_1 while the

RHS inequality ensures that R picks b for sure after m_2. Note that by definition it is always true that $P_l(B\,|m_1,\delta\,) \leq P_h(B\,|m_1,\delta\,)$ and that $P_l(B\,|m_2,\delta\,) \leq P_h(B\,|m_2,\delta\,)$.

We now consider Point b). Consider an equilibrium in which m_1, m_2 and m_3 are emitted with positive probability. Besides the previously imposed conditions, it must also be that $P_l(B\,|m_3,\delta\,) \leq q_R \leq P_h(B\,|m_3,\delta\,)$ so as to ensure that R randomizes after m_3. Given that in an S-optimal equilibrium, m_3 is only sent for $\sigma \geq t_S^1$, it follows by Lemma F that

$$\frac{P_h(B)}{1-P_h(B)} \frac{\int_{t_S^1}^{\overline{\sigma}} \delta_3(\sigma) f_B(\sigma) d\sigma}{\int_{t_S^1}^{\overline{\sigma}} \delta_3(\sigma) f_A(\sigma) d} \geq \frac{P_h(B)}{1-P_h(B)} \frac{f_B(t_S^1)}{f_A(t_S^1)} = \frac{q_S}{1-q_S},$$

i.e. $q_S \leq P_h(B|m_3,\delta)$. Hence $q_R \leq P_h(B\,|m_3,\delta\,)$.

Step 2 *a) In any S-optimal equilibrium featuring a communication strategy* δ, $\delta_1(\sigma) = 1$ *for* $\sigma < t_S^1$ *and* $\delta_2(\sigma) = 1$ *for* $\sigma > t_S^2$. *For* $\sigma \in \left[t_S^1, t_S^2\right]$ *,*

$$\delta_1(\sigma) + \delta_3(\sigma) q_R = q_S. \tag{6}$$

and $\delta_1(\sigma) \geq \frac{q_S - q_R}{1-q_R}$ *and* $\delta_3(\sigma) \leq \frac{1-q_S}{1-q_R}$.

b) It holds true that for $\sigma \in \left[t_S^1, t_S^2\right]$ *,*

$$\delta_3(\sigma) - \frac{1-q_S}{1-q_R} = -\frac{1}{q_R}\left(\delta_1(\sigma) - \frac{q_S - q_R}{1-q_R}\right) \tag{7}$$

Proof: The equality (6) ensures a is played with probability q_S for any $\sigma \in \left[t_S^1, t_S^2\right]$ as required by the S-optimal decision rule. Equality (6) implies $\delta_3(\sigma) = \frac{q_S - \delta_1(\sigma)}{q_R}$. From this latter equality and the fact that $\delta_1(\sigma) + \delta_3(\sigma) \leq 1$ it follows that $\delta_1(\sigma) \geq \frac{q_S - q_R}{1-q_R}$ or equivalently $\delta_3(\sigma) \leq \frac{1-q_S}{1-q_R}$ for all $\sigma \in \left[t_S^1, t_S^2\right]$. Equation (7) directly follows from using $\delta_3(\sigma) = \frac{q_S - \delta_1(\sigma)}{q_R}$.

Step 3 *Suppose that* $t_R^2 \leq t_S^1$. *If there exists an S-optimal equilibrium, then there exists an S-optimal equilibrium that features the threshold strategy* $z = t_S^1$.

Proof: Suppose first that $t_R^2 < t_S^1$. Assume that there exists an S-optimal equilibrium featuring a strategy δ that assigns positive probability to m_3 which triggers mixed action $(q_R, 1-q_R)$. We know from step 2 that it can only be true that $\delta_3(\sigma) > 0$ if $\sigma \in \left[t_S^1, t_S^2\right]$. Given this and the fact that $t_R^2 < t_S^1$, it must however then be the case by Lemma F that

$\frac{P_l(B|m_3,\delta)}{P_l(A|m_3,\delta)} > \frac{q_R}{1-q_R}$, implying that R responds to m_3 by choosing action b for sure. It follows that δ cannot constitute an equilibrium. Note that if an S-optimal equilibrium is such that only m_1 and m_2 are used and trigger respectively a for sure and b for sure, then this equilibrium features the threshold strategy $z = t_S^1$.

Suppose now that $t_R^2 = t_S^1$. Assume that there exists an S-optimal equilibrium featuring a strategy δ that assigns positive probability conditional on some signal to m_3 which triggers mixed action $(q_R, 1-q_R)$. We know from step 2 that it can only be true that $\delta_3(\sigma) > 0$ if $\sigma \in \left[t_S^1, t_S^2\right]$. In order to have $\frac{P_l(B|m_3,\delta)}{P_l(A|m_3,\delta)} \leq \frac{q_R}{1-q_R}$, it must be that m_3 is only sent for $\sigma = t_S^1$. However, if this is the case then there exists an S-optimal equilibrium where δ is replaced by a δ' that is identical to δ except that for $\sigma = t_S^1$, S randomizes between m_1 and m_2 with probabilities $(q_S, 1-q_S)$. This equilibrium is a threshold equilibrium with threshold $z = t_S^1$.

Step 4 *Suppose that $t_R^2 \in (t_S^1, t_S^2]$. If there exists an S-optimal equilibrium featuring a communication strategy δ, then there exists an S-optimal equilibrium featuring δ' defined as follows:*

$$\delta'(\sigma) = \left(\frac{q_S - q_R}{1-q_R}, 0, \frac{1-q_S}{1-q_R}\right) \text{ if } \sigma \in \left[t_S^1, t_R^2\right], \tag{8}$$

$$\delta'(\sigma) = \delta(\sigma) \text{ if } \sigma \notin \left[t_S^1, t_R^2\right]. \tag{9}$$

Proof: Since δ is featured in an S-optimal equilibrium, it holds true that $\frac{P(m_1|B,\delta)}{P(m_1|A,\delta)} \leq \frac{q_R}{1-q_R}\frac{1-P_h(B)}{P_h(B)}$, $\frac{P(m_3|B,\delta)}{P(m_3|A,\delta)} \leq \frac{q_R}{1-q_R}\frac{1-P_l(B)}{P_l(B)}$ and $\frac{P(m_2|B,\delta)}{P(m_2|A,\delta)} \geq \frac{q_R}{1-q_R}\frac{1-P_l(B)}{P_l(B)}$. Consider the strategy δ' defined above. We shall prove that all message constraints are satisfied given δ'.

Consider first the m_1-constraint. Note that

$$\frac{P(m_1|B,\delta')}{P(m_1|A,\delta')} = \frac{P(m_1|B,\delta) - \int_{t_S^1}^{t_R^2}\left(\delta_1(\sigma) - \frac{q_S-q_R}{1-q_R}\right) f_B(\sigma)d\sigma}{P(m_1|A,\delta) - \int_{t_S^1}^{t_R^2}\left(\delta_1(\sigma) - \frac{q_S-q_R}{1-q_R}\right) f_A(\sigma)d\sigma}.$$

Recall that $\delta_1(\sigma) \geq \frac{q_S-q_R}{1-q_R}$ for any $\sigma \in \left[t_S^1, t_R^2\right]$ as proved in the preceding step. Given $t_R^1 \leq t_S^1$ and Lemma F, it holds true that

$$\frac{\int_{t_S^1}^{t_R^2}\left(\delta_1(\sigma) - \frac{q_S-q_R}{1-q_R}\right) f_B(\sigma)d\sigma}{\int_{t_S^1}^{t_R^2}\left(\delta_1(\sigma) - \frac{q_S-q_R}{1-q_R}\right) f_A(\sigma)d\sigma} \geq \frac{q_R}{1-q_R}\frac{1-P_h(B)}{P_h(B)}. \tag{10}$$

Using together inequality $\frac{P(m_1|B,\delta)}{P(m_1|A,\delta)} \leq \frac{q_R}{1-q_R}\frac{1-P_h(B)}{P_h(B)}$, inequality (10) and the fact that $\delta_1(\sigma) \geq \frac{q_S-q_R}{1-q_R}$ for $\sigma \in \left[t_R^2, t_S^2\right]$, Lemma R (Point b)) implies that $\frac{P(m_1|B,\delta')}{P(m_1|A,\delta')} \leq \frac{q_R}{1-q_R}\frac{1-P_h(B)}{P_h(B)}$.

Consider now the m_3-constraint. Note that

$$\begin{aligned}\frac{P(m_3|B,\delta')}{P(m_3|A,\delta')} &= \frac{P(m_3|B,\delta) - \int_{t_S^1}^{t_R^2}\left(\delta_3(\sigma) - \frac{1-q_S}{1-q_R}\right) f_B(\sigma)d\sigma}{P(m_3|A,\delta) - \int_{t_S^1}^{t_R^2}\left(\delta_3(\sigma) - \frac{1-q_S}{1-q_R}\right) f_A(\sigma)d\sigma} \\ &= \frac{P(m_3|B,\delta) + \frac{1}{q_R}\int_{t_S^1}^{t_R^2}\left(\delta_1(\sigma) - \frac{q_S-q_R}{1-q_R}\right) f_B(\sigma)d\sigma}{P(m_3|A,\delta) + \frac{1}{q_R}\int_{t_S^1}^{t_R^2}\left(\delta_1(\sigma) - \frac{q_S-q_R}{1-q_R}\right) f_A(\sigma)d\sigma}.\end{aligned}$$

The second above equality uses equation (7). Recall also that $\delta_1(\sigma) \geq \frac{q_S-q_R}{1-q_R}$ for any $\sigma \in \left[t_S^1, t_R^2\right]$. By Lemma F it holds true that

$$\frac{\int_{t_S^1}^{t_R^2}\left(\delta_1(\sigma) - \frac{q_S-q_R}{1-q_R}\right) f_B(\sigma)d\sigma}{\int_{t_S^1}^{t_R^2}\left(\delta_1(\sigma) - \frac{q_S-q_R}{1-q_R}\right) f_A(\sigma)d\sigma} \leq \frac{q_R}{1-q_R}\frac{1-P_l(B)}{P_l(B)}. \tag{11}$$

Using together inequality $\frac{P(m_3|B,\delta)}{P(m_3|A,\delta)} \leq \frac{q_R}{1-q_R}\frac{1-P_l(B)}{P_l(B)}$, inequality (11) and the fact that $\delta_1(\sigma) \geq \frac{q_S-q_R}{1-q_R}$ for any $\sigma \in \left[t_S^1, t_R^2\right]$, Lemma R (Point a)) implies that $\frac{P(m_3|B,\delta')}{P(m_3|A,\delta')} \leq \frac{q_R}{1-q_R}\frac{1-P_l(B)}{P_l(B)}$.

Consider finally the m_2-constraint. Since $\delta_2'(\sigma) = 0$ if $\sigma \leq t_2^R$, Lemma F implies that $\frac{P(m_2|B,\delta')}{P(m_2|A,\delta')} \geq \frac{q_R}{1-q_R}\frac{1-P_l(B)}{P_l(B)}$. Hence, δ' satisfies all message constraints and constitutes an S-optimal equilibrium.

Step 5 *Suppose that $t_R^2 \in (t_S^1, t_S^2]$. Suppose there exists an S-optimal equilibrium featuring a communication strategy δ satisfying $\delta(\sigma) = \left(\frac{q_S-q_R}{1-q_R}, 0, \frac{1-q_S}{1-q_R}\right)$ if $\sigma \in \left[t_S^1, t_R^2\right]$. Then there exists a threshold equilibrium featuring $z \geq t_R^2$.*

Proof: Suppose that there exists an S-optimal equilibrium featuring δ with $\delta(\sigma) = \left(\frac{q_S-q_R}{1-q_R}, 0, \frac{1-q_S}{1-q_R}\right)$ whenever $\sigma \in \left[t_S^1, t_R^2\right]$. We distinguish two cases. In Case 1, $\frac{P(m_3|B,t_S^2)}{P(m_3|A,t_S^2)} \leq \frac{P(m_3|B,\delta)}{P(m_3|A,\delta)}$. In Case 2, $\frac{P(m_3|B,t_S^2)}{P(m_3|A,t_S^2)} > \frac{P(m_3|B,\delta)}{P(m_3|A,\delta)}$. We begin with Case 1 and prove that in this case there exists a threshold equilibrium with $z = t_S^2$.

Consider the m_3-constraint. Since δ is used in an S-optimal equilibrium, it holds true that $\frac{P(m_3|B,\delta)}{P(m_3|A,\delta)} \leq \frac{q_R}{1-q_R}\frac{1-P_l(B)}{P_l(B)}$. It follows immediately that $\frac{P(m_3|B,t_S^2)}{P(m_3|A,t_S^2)} \leq \frac{q_R}{1-q_R}\frac{1-P_l(B)}{P_l(B)}$.

Consider the m_1-constraint. Since δ is used in an S-optimal equilibrium, it is true that $\frac{P(m_1|B,\delta)}{P(m_1|A,\delta)} \leq \frac{q_R}{1-q_R}\frac{1-P_h(B)}{P_h(B)}$. Note that $\frac{P(m_1|B,t_s^2)}{P(m_1|A,t_s^2)} = \frac{P(m_1|B,\delta)-\int_{t_R^2}^{t_S^2}\left(\delta_1(\sigma)-\frac{q_S-q_R}{1-q_R}\right)f_B(\sigma)d\sigma}{P(m_1|A,\delta)-\int_{t_R^2}^{t_S^2}\left(\delta_1(\sigma)-\frac{q_S-q_R}{1-q_R}\right)f_A(\sigma)d\sigma}$. We know furthermore that $\delta_1(\sigma) \geq \frac{q_S-q_R}{1-q_R}$ for $\sigma \in \left[t_R^2, t_S^2\right]$. Using Lemma F, it thus holds true that

$$\frac{\int_{t_R^2}^{t_S^2}\left(\delta_1(\sigma) - \frac{q_S-q_R}{1-q_R}\right) f_B(\sigma)d\sigma}{\int_{t_R^2}^{t_S^2}\left(\delta_1(\sigma) - \frac{q_S-q_R}{1-q_R}\right) f_A(\sigma)d\sigma} \geq \frac{q_R}{1-q_R}\frac{1-P_l(B)}{P_l(B)} \geq \frac{q_R}{1-q_R}\frac{1-P_h(B)}{P_h(B)}. \tag{12}$$

Using together inequality $\frac{P(m_1|B,\delta)}{P(m_1|A,\delta)} \leq \frac{q_R}{1-q_R}\frac{1-P_h(B)}{P_h(B)}$, inequality (12) and the fact that $\delta_1(\sigma) \geq \frac{q_S-q_R}{1-q_R}$ for $\sigma \in \left[t_R^2, t_S^2\right]$, Lemma R (Point b)) implies that $\frac{P(m_1|B,t_S^2)}{P(m_1|A,t_S^2)} \leq \frac{q_R}{1-q_R}\frac{1-P_h(B)}{P_h(B)}$.

Consider finally the m_2-constraint. In a threshold equilibrium featuring $z = t_S^2$, m_2 is sent if and only if $\sigma \geq t_S^2 \geq t_R^2$. It follows by Lemma F that $P_l(B|m_2, t_S^2) \geq q_S \geq q_R$. Hence, the threshold strategy featuring $z = t_S^2$ satisfies all equilibrium constraints corresponding to m_1, m_2 and m_3.

We now examine Case 2, thus assuming $\frac{P(m_3|B,t_S^2)}{P(m_3|A,t_S^2)} > \frac{P(m_3|B,\delta)}{P(m_3|A,\delta)}$. We shall show that we can find a threshold strategy featuring $z \in [t_R^2, t_S^2]$ such that $\frac{P(m_3|B,z)}{P(m_3|A,z)} = \frac{P(m_3|B,\delta)}{P(m_3|A,\delta)}$, $\frac{P(m_1|B,z)}{P(m_1|A,z)} \leq \frac{P(m_1|B,\delta)}{P(m_1|A,\delta)}$ and $P_l(B|m_2, z) \geq q_R$, implying that the identified threshold strategy constitutes an equilibrium.

Consider first the m_3-constraint. Note that $\frac{P(m_3|B,t_R^2)}{P(m_3|A,t_R^2)} \leq \frac{P(m_3|B,\delta)}{P(m_3|A,\delta)}$. To see this note the following. First, under δ and the threshold strategy $z = t_R^2$, m_3 is not sent for $\sigma < t_S^1$ and sent with probability $\frac{1-q_S}{1-q_R}$ for $\sigma \in (t_S^1, t_R^2)$. Second $\delta_3(\sigma) \geq 0$ for $\sigma \in (t_R^2, t_S^2]$ while under the threshold equilibrium $z = t_R^2$, m_3 is sent with probability zero for $\sigma > t_R^2$. The stated inequality then follows by Lemma F (Part b)) combined with Lemma R (Part a)). Using the fact that $\frac{P(m_3|B,z)}{P(m_3|A,z)}$ is continuous and increasing in z as well as the double inequality

$$\frac{P(m_3|B, t_R^2)}{P(m_3|A, t_R^2)} \leq \frac{P(m_3|B, \delta)}{P(m_3|A, \delta)} < \frac{P(m_3|B, t_S^2)}{P(m_3|A, t_S^2)},$$

it follows that there is a $z^* \in [t_R^2, t_S^2]$ such that

$$\frac{P(m_3|B, z^*)}{P(m_3|A, z^*)} = \frac{P(m_3|B, \delta)}{P(m_3|A, \delta)}. \tag{13}$$

Recall that given the threshold strategy $z > t_S^1$, m_3 is sent with probability $\frac{1-q_S}{1-q_R}$ for $\sigma \in [t_S^1, z)$ while it is sent with probability zero otherwise. We thus have

$$\frac{P(m_3|B,z^*)}{P(m_3|A,z^*)} = \frac{P(m_3|B,\delta) + \int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_B(\sigma)d\sigma - \int_{z^*}^{t_S^2}\delta_3(\sigma)f_B(\sigma)d\sigma}{P(m_3|A,\delta) + \int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_A(\sigma)d\sigma - \int_{z^*}^{t_S^2}\delta_3(\sigma)f_A(\sigma)d\sigma}. \tag{14}$$

Equalities (13) and (14) are compatible only in two scenarios, which we call I and II. Scenario I is that the terms

$$\int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_B(\sigma)d\sigma - \int_{z^*}^{t_S^2}\delta_3(\sigma)f_B(\sigma)d\sigma \tag{15}$$

and

$$\int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_A(\sigma)d\sigma - \int_{z^*}^{t_S^2}\delta_3(\sigma)f_A(\sigma)d\sigma \tag{16}$$

are both zero. Scenario II is that either both of the above terms are strictly positive or both of these are strictly negative and additionally

$$\frac{\int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_B(\sigma) - \int_{z^*}^{t_S^2}\delta_3(\sigma)f_B(\sigma)}{\int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_A(\sigma) - \int_{z^*}^{t_S^2}\delta_3(\sigma)f_A(\sigma)} = \frac{P(m_3|B,\delta)}{P(m_3|A,\delta)}. \tag{17}$$

For scenario II, we will now show that the two terms must actually both be strictly positive. Recall first that by definition $\frac{P(m_3|B,\delta)}{P(m_3|A,\delta)} \leq \frac{q_R}{1-q_R}\frac{1-P_l(B)}{P_l(B)}$. Second, Lemma F implies that.

$$\frac{q_R}{1-q_R}\frac{1-P_l(B)}{P_l(B)} \leq \frac{\int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_B(\sigma)d\sigma}{\int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_A(\sigma)d\sigma} < \frac{\int_{z^*}^{t_S^2}\delta_3(\sigma)f_B(\sigma)d\sigma}{\int_{z^*}^{t_S^2}\delta_3(\sigma)f_A(\sigma)d\sigma}.$$

Combining these inequalities implies

$$\begin{aligned}\frac{\int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_B(\sigma)d\sigma - \int_{z^*}^{t_S^2}\delta_3(\sigma)f_B(\sigma)d\sigma}{\int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_A(\sigma)d\sigma - \int_{t_R^2}^{z^*}\delta_3(\sigma)f_A(\sigma)d\sigma} &\leq \frac{\int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_B(\sigma)d\sigma}{\int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_A(\sigma)d\sigma} \\ &< \frac{\int_{z^*}^{t_S^2}\delta_3(\sigma)f_B(\sigma)d\sigma}{\int_{z^*}^{t_S^2}\delta_3(\sigma)f_A(\sigma)d\sigma}.\end{aligned}$$

Using the above double-inequality and the assumption that (15) and (16) have the same sign, Ratio Lemma R (Point d)) implies that (15) and (16) are both strictly positive.

Consider now the m_1-constraint. We compare $\frac{P(m_1|B,z^*)}{P(m_1|A,z^*)}$ with $\frac{P(m_1|B,\delta)}{P(m_1|A,\delta)}$. Using equation (7), one can write

$$\frac{P(m_1|B,z^*)}{P(m_1|A,z^*)} = \frac{P(m_1|B,\delta) - q_R\left(\int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_B(\sigma)d\sigma - \int_{z^*}^{t_S^2} \delta_3(\sigma) f_B(\sigma)d\sigma\right)}{P(m_1|A,\delta) - q_R\left(\int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_A(\sigma)d\sigma - \int_{z^*}^{t_S^2} \delta_3(\sigma) f_A(\sigma)d\sigma\right)}. \tag{18}$$

Note that the numerator in the LHS ratio equals the numerator in the RHS ratio. The same holds true for denominators. It follows that the numerator and the denominator in the RHS ratio are both strictly positive.

In scenario I, the terms multiplying q_R vanish and hence $\frac{P(m_1|B,z^*)}{P(m_1|A,z^*)} = \frac{P(m_1|B,\delta)}{P(m_1|A,\delta)}$. It follows that the m_1-constraint is satisfied for the threshold strategy given by z^*, as it was satisfied for δ. In scenario II, as in any S-optimal equilibrium, δ is s.t. $P_h(B\,|m_1,\delta) \leq q_R \leq P_h(B\,|m_3,\delta)$ and thus $P(B\,|m_1,\delta) \leq P(B\,|m_3,\delta)$. Given equation (17), it thus follows that

$$\frac{P(m_1|B,\delta)}{P(m_1|A,\delta)} \leq \frac{\int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_B(\sigma)d\sigma - \int_{z^*}^{t_S^2} \delta_3(\sigma) f_B(\sigma)d\sigma}{\int_{t_R^2}^{z^*}\left(\frac{1-q_S}{1-q_R} - \delta_3(\sigma)\right) f_A(\sigma)d\sigma - \int_{z^*}^{t_S^2} \delta_3(\sigma) f_A(\sigma)d\sigma}.$$

Given this inequality and the equality (18), then Lemma R (Point b)) implies that $\frac{P(m_1|B,z^*)}{P(m_1|A,z^*)} \leq \frac{P(m_1|B,\delta)}{P(m_1|A,\delta)}$, so the m_1-constraint is satisfied also in scenario II.

Consider finally the m_2-constraint. In the constructed threshold equilibrium featuring a threshold $z^* \geq t_R^2$, m_2 is sent if and only if $\sigma \geq t_R^2$. It follows by Lemma F that $P_l(B|m_2,z^*) \geq q_R$. Hence, we have identified a $z^* \in [t_R^2, t_S^2]$ such that the threshold strategy z^* satisfies all three message constraints and thus constitutes an equilibrium.■

6.3 Preliminary analysis of constraints

In what follows, we shall study the incentive compatibility constraints of S and call them respectively m_1-, m_2- and m_3-constraint. We first study each separately, examining comparative statics w.r.t. z, e and q_R, and then study their pairwise relations. The lemmas appearing below are proved in the Online Appendix. We introduce some notation in what follows. In a threshold equilibrium featuring threshold z, we denote by $P_k(B\,|m_i, z, q_S, e)$ the conditional probability of state B given message m_i when applying prior $P_k^e(B)$, for $i \in \{2, 3\}$. The counterpart for m_1 is denoted by $P_k(B\,|m_1, z, q_R, q_S, e)$. Note that the latter expression is a function of q_R in contrast to the other two, except in the special case of $z = t_S^1(e)$. To stress the fact that $P_k(B|m_1, t_S^1(e), q_R, q_S, e)$ is independent of q_R, we shall write $P_k(B|m_1, t_S^1(e), 0, q_S, e)$. Message m_3 is never sent in a threshold equilibrium with threshold $t_S^1(e)$ so that $P_l(B\,|m_3, t_S^1(e), q_S, e)$ is not well defined. We slightly abuse notation and set $P_l(B|m_3, t_S^1(e), q_S, e) = \lim\limits_{z \to t_S^1(e)} P_l(B\,|m_3, z, q_S, e)$. Given $i \in \{2, 3\}$ and $\omega \in \{A, B\}$, we let $P(m_i|z, q_S, e, \omega)$ denote the probability that m_i is sent conditional on the state being ω. Let $P(m_1|z, q_R, q_S, e, \omega)$ denote the equivalent for message m_1. Given $i \in \{2, 3\}$, $k \in \{h, l\}$ and $\omega \in \{A, B\}$, we let $P_k(\omega, m_i, z, q_S, e)$ denote the probability that the state is ω and that m_i is sent when using prior $P_k(B)$. Let $P_k(\omega, m_1, z, q_R, q_S, e)$ denote the equivalent for message m_1.

Lemma M1 *i.a) Given e and q_S, there is a threshold $\widehat{z}(e) \in \left[t_S^1(e), t_S^2(e)\right]$ such that the following is true. If $z < \widehat{z}(e)$, there is no q_R such that $P_h(B\,|m_1, z, q_R, q_S, e) \leq q_R$. If $z \geq \widehat{z}(e)$, there is a unique value $\Psi(z, q_S, e) \in (0, q_S]$ such that $P_h(B\,|m_1, z, q_R, q_S, e) \leq q_R$ iff $q_R \geq \Psi(z, q_S, e)$. For $z \geq \widehat{z}(e)$ the expression $\Psi(z, q_S, e)$ satisfies $P_h(B|m_1, z, \Psi(z, q_S, e), q_S, e) = \Psi(z, q_S, e)$ and is continuous as well as strictly decreasing in z.*

i.b) If $\widehat{z}(e) > t_S^1(e)$ then $\Psi(\widehat{z}(e), q_S, e) = q_S$. If $\widehat{z}(e) = t_S^1(e)$ then $\Psi(\widehat{z}(e), q_S, e) = P_h(B|m_1, t_S^1(e), 0, q_S, e) \leq q_S$.

ii.a) $P_h(B|m_1, t_S^1(e), 0, q_S, e)$ is continuous and strictly increasing in e.

ii.b) There is a constant $e_1 \in \left(0, \frac{1}{2}\right)$ such that $P_h(B|m_1, t_S^1(e), 0, q_S, e) < q_S$ iff $e < e_1$.

ii.c) If $e \leq e_1$ then $\widehat{z}(e) = t_S^1(e)$. If $e > e_1$ then $\widehat{z}(e) > t_S^1(e)$.

Lemma M2 *i.a) $P_l(B\,|m_2, z, q_S, e)$ is continuous and strictly increasing in z*

i.b) $P_l(B|m_2, t_S^2(e), q_S, e) > q_S$.

ii.a) $P_l(B|m_2, t_S^1(e), q_S, e)$ is continuous and strictly decreasing in e.

ii.b) There is a constant $e_2 \in \left(0, \frac{1}{2}\right)$ such that $P_l(B|m_2, t_S^1(e), q_S, e) < q_S$ iff $e > e_2$.

Lemma M3 *i.a) $P_l(B\,|m_3, z, q_S, e) < q_S\ \forall e > 0$ and z.*

i.b) $P_l(B\,|m_3, z, q_S, e)$ is continuous and strictly increasing in z.

ii.a) $P_l(B\,|m_3, t_S^1(e), q_S, e)$ *is continuous and strictly decreasing in* e.
ii.b) $\lim_{e\to 0} P_l(B\,|m_3, t_S^1(e), q_S, e) = q_S$.
ii.c) $P_l(B\,|m_3, t_S^1(e), q_S, e)$ *approaches* 0 *for* $e \to \frac{1}{2}$.

Lemma M4 *a) There is an* $e_{13} \in (0, e_1)$ *such that*

$$P_l(B|m_3, t_S^1(e), q_S, e) > P_h(B|m_1, t_S^1(e), 0, q_S, e)$$

iff $e < e_{13}$.
b) Given $e \leq e_{13}$ *it holds true that* $\widehat{z}(e) = t_S^1(e)$ *and that*

$$\Psi(z, q_S, e) < P_l(B\,|m_3, z, q_S, e),\ \forall z \in \left(t_S^1(e), t_S^2(e)\right].$$

c) If $e > e_{13}$ *and* $P_l(B|m_3, t_S^2(e), q_S, e) > \Psi(t_S^2(e), q_S, e)$ *then there exists some*

$$z^* \in \left[\widehat{z}(e), t_S^2(e)\right] \setminus \left\{t_S^1(e)\right\}$$

such that $\Psi(z, q_S, e) > P_l(B\,|m_3, z, q_S, e)$ *iff* $z \in [\widehat{z}(e), z^*)$.
d) If $e > e_{13}$ *and* $P_l(B|m_3, t_S^2(e), q_S, e) \leq \Psi(t_S^2(e), q_S, e)$ *then* $\Psi(z, q_S, e) \geq P_l(B\,|m_3, z, q_S, e)$ *for* $z \in \left[\widehat{z}(e), t_S^2(e)\right]$.

Lemma M5 *It holds true that* $P_l(B\,|m_2, z, q_S, e) > P_l(B\,|m_3, z, q_S, e), \forall z \in \left[t_S^1(e), t_S^2(e)\right]$.

Lemma M6 *a) There is an* $e_{12} \in \left(e_{13}, \frac{1}{2}\right)$ *such that*

$$P_h(B|m_1, t_S^1(e), 0, q_S, e) > P_l(B|m_2, t_S^1(e), q_S, e)$$

iff $e > e_{12}$.
b) If $e_2 < e_1$ *then* $e_2 < e_{12} < e_1$. *If* $e_2 > e_1$ *then* $e_2 > e_{12} > e_1$. *If* $e_2 = e_1$ *then* $e_2 = e_{12} = e_1$.

6.4 Proof of Proposition 2

Outline The proof of Point i. is given in steps 1-4 and is organized as follows. We first study the case of $e \leq e_{13}$ and then examine the case of $e > e_{13}$. The latter case subdivides into two subcases I and II. In subcase I, $P_l(B|m_3, t_S^2(e), q_S, e) > \Psi(t_S^2(e), q_S, e)$ whereas in subcase II the latter inequality is reversed. The proof of Point ii. appears in step 5.

Step 1 *a) A threshold equilibrium featuring $z = t_S^1(e)$ exists if and only if*

$$P_h(B\,|m_1, z, q_R, q_S, e) \leq q_R \leq P_l(B\,|m_2, z, q_S, e).$$

b) A threshold equilibrium featuring $z > t_S^1(e)$ exists if and only if the above inequalities hold and in addition $P_l(B\,|m_3, z, q_S, e) \leq q_R$.

Proof: This was proved in step 1 of the proof of Proposition 1.

Step 2 *Assume that $e \leq e_{13}$. There exists some threshold equilibrium if and only if*

$$q_R \geq P_h(B|m_1, t_S^1(e), 0, q_S, e).$$

This implies that $\underline{q}_R(e) = P_h(B|m_1, t_S^1(e), 0, q_S, e)$.

Proof: Let $e \leq e_{13}$ so by definition (see Lemma M4) it holds true that

$$P_h(B|m_1, t_S^1(e), 0, q_S, e) \leq P_l(B|m_3, t_S^1(e), q_S, e).$$

From Lemma M3, we know that $P_l(B|m_3, t_S^1(e), q_S, e) \leq q_S$, $\forall z$. From Lemma M5, we know that $P_l(B\,|m_3, z, q_S, e) < P_l(B\,|m_2, z, q_S, e)$, $\forall z$.

Consider first $q_R < P_h(B|m_1, t_S^1(e), 0, q_S, e)$. There exists no $t_S^1(e)$-equilibrium as the m_1-constraint is violated. We know from Lemma M3 that $P_l(B\,|m_3, z, q_S, e)$ is increasing in z so that for any $z > t_S^1(e)$, $P_l(B\,|m_3, z, q_S, e) > P_h(B|m_1, t_S^1(e), 0, q_S, e)$. Hence, for $z > t_S^1(e)$, $q_R < P_l(B\,|m_3, z, q_S, e)$ which means that the m_3-constraint is violated. There thus exists no threshold equilibrium with $z > t_S^1(e)$.

Consider $q_R \in \left[P_h(B|m_1, t_S^1(e), 0, q_S, e), P_l(B|m_3, t_S^1(e), q_S, e)\right]$. Given that it holds true that $P_l(B|m_3, t_S^1(e), q_S, e) < P_l(B|m_2, t_S^1(e), q_S, e)$, it follows that $q_R \leq P_l(B|m_2, t_S^1(e), q_S, e)$. For such values of q_R, the m_1- and the m_2-constraints are thus satisfied in a $t_S^1(e)$-equilibrium. Hence, the $t_S^1(e)$-equilibrium (which does not involve m_3) exists.

Consider $q_R \in \left(P_l(B|m_3, t_S^1(e), q_S, e), P_l(B|m_3, t_S^2(e), q_S, e)\right]$. For any such q_R, there exists a threshold equilibrium for the unique value of z at which $q_R = P_l(B\,|m_3, z, q_S, e)$. For such a pair q_R, z, the m_3-constraint is satisfied because $q_R = P_l(B\,|m_3, z, q_S, e)$. The m_1-constraint is satisfied because we know from Lemma M4 that for any z, $P_l(B\,|m_3, z, q_S, e) \geq \Psi(z, q_S, e)$. The m_2-constraint is satisfied because $q_R = P_l(B\,|m_3, z, q_S, e)$ while we know from Lemma M5 that $P_l(B\,|m_3, z, q_S, e) < P_l(B\,|m_2, z, q_S, e)$.

Consider finally $q_R \in \left(P_l(B|m_3, t_S^2(e), q_S, e), q_S\right]$. In a putative $t_S^2(e)$-equilibrium, the m_1-constraint is satisfied because we know from Lemma M5 that $P_l(B|m_3, t_S^2(e), q_S, e) > \Psi(t_S^2(e), q_S, e)$. The m_3-constraint is satisfied by assumption. The m_2-constraint is satisfied because we know from Lemma M2 that $q_S < P_l(B|m_2, t_S^2(e), q_S, e)$. Hence the $t_S^2(e)$-equilibrium exists. Figure A below illustrates the case of $e \leq e_{13}$. We assume the same information structure as in previous figures, setting $q_S = .85$ and $e = .12$. The m_2-constraint does not appear because it is satisfied for all considered pairs q_R, z.

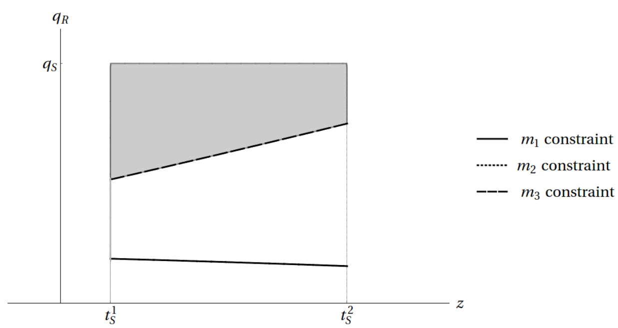

Figure A: Illustration of constraints ($e \leq e_{13}$)

Step 3 *(Subcase I) Assume $e > e_{13}$ and $P_l(B|m_3, t_S^2(e), q_S, e) > \Psi(t_S^2(e), q_S, e)$. There exists some threshold equilibrium if and only $q_R \in [P_l(B\,|m_3, z^*, q_S, e), q_S]$, where z^* is the unique value in the set $\left[\widehat{z}(e), t_S^2(e)\right] \setminus \{t_S^1(e)\}$ such that $\Psi(z^*, q_S, e) = P_l(B\,|m_3, z^*, q_S, e)$. This implies that $\underline{q}_R(e) = P_l(B\,|m_3, z^*, q_S, e)$, for the above defined z^*.*

Proof: We know from Lemma M4 that there exists some $z^* \in \left[\widehat{z}(e), t_S^2(e)\right] \setminus \{t_S^1(e)\}$ such that $\Psi(z^*, q_S, e) = P_l(B\,|m_3, z^*, q_S, e)$ while in contrast $\Psi(z, q_S, e) > P_l(B\,|m_3, z, q_S, e)$ for $z \in [\widehat{z}(e), z^*)$ and $\Psi(z, q_S, e) < P_l(B\,|m_3, z, q_S, e)$ for $z \in (z^*, t_S^2(e)]$.

Consider first $q_R < P_l(B\,|m_3, z^*, q_S, e)$. For such values of q_R, there exists no z such that $q_R \geq \max\{P_l(B\,|m_3, z, q_S, e), \Psi(z, q_S, e)\}$. It follows that for such values of q_R, there exists no threshold equilibrium.

Consider $q_R \in \left[P_l(B|m_3, z^*, q_S, e), P_l(B|m_3, t_S^2(e), q_S, e)\right]$. For any such q_R, there exists a threshold-z equilibrium for the value of z defined by the intersection of the horizontal line q_R with $P_l(B|m_3, z, q_S, e)$. For any such pair q_R, z, we know that the m_1- and m_3-constraints are satisfied because it holds true that $q_R \geq P_l(B|m_3, z, q_S, e)$ and $q_R \geq \Psi(z, q_S, e)$. For any such pair q_R, z, the m_2-constraint is satisfied because for any z, we know from Lemma M5 that $P_l(B|m_3, z, q_S, e) < P_l(B|m_2, z, q_S, e)$.

Consider finally $q_R \in \left(P_l(B|m_3, t_S^2(e), q_S, e), q_S\right]$. For these values of q_R, the $t_S^2(e)$-equilibrium exists. Indeed, the m_1- and m_3-constraints are satisfied because

$$q_R \geq \max\left\{P_l(B|m_3, t_S^2(e), q_S, e), \Psi(t_S^2(e), q_S, e)\right\}.$$

Furthermore, the m_2-constraint is satisfied because $q_S < P_l(B|m_2, t_S^2(e), q_S, e)$. Subcase I is illustrated in Figure B below. We assume the same information structure as in previous figures, setting $q_S = .85$ and $e = .19$.

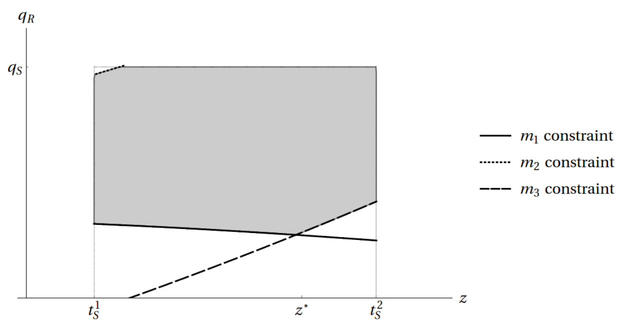

Figure B: Illustration of constraints ($e > e_{13,}$ subcase I)

Step 4 *(Subcase II) Assume $e > e_{13}$ and $P_l(B|m_3, t_S^2(e), q_R, q_S, e) \leq \Psi(t_S^2(e), q_S, e)$. There exists a threshold equilibrium if and only if $q_R \geq \Psi(t_S^2(e), q_S, e)$. This implies that $\underline{q}_R(e) = \Psi(t_S^2(e), q_S, e)$.*

Proof: Consider first $q_R < \Psi(t_S^2(e), q_S, e)$. Since $\Psi(z, q_S, e)$ is decreasing in z on $\left[\widehat{z}(e), t_S^2(e)\right]$, there exists no $z \in \left[\widehat{z}(e), t_S^2(e)\right]$ such that $q_R > \Psi(z, q_S, e)$.

Consider instead $q_R \geq \Psi(t_S^2(e), q_S, e)$. For any such q_R, the $t_S^2(e)$-equilibrium exists. The m_1- and m_3-constraints are satisfied because

$$q_R \geq \Psi(t_S^2(e), q_S, e) \geq P_l(B|m_3, t_S^2(e), q_S, e).$$

The m_2-constraint is satisfied because $q_S < P_l(B|m_2, t_S^2(e), q_S, e)$ (see Lemma M2). Subcase II is illustrated in the figure below. We assume the same information structure as in previous figures, setting $q_S = .85$ and $e = .24$.

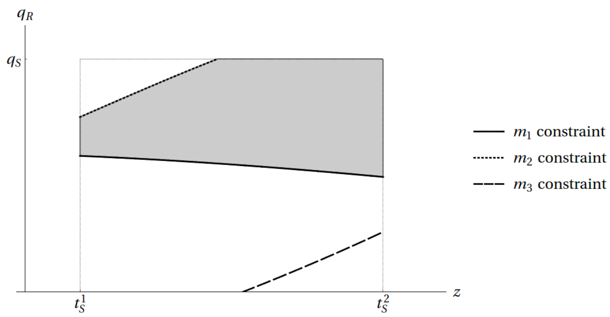

Figure C: Illustration of constraints ($e > e_{13,}$ subcase II)

Step 5 This proves Point ii. Note that for $e \leq e_{13}$, we have $\underline{q}_R(e) = P_h(B|m_1, t_S^1(e), 0, q_S, e)$. We know from Lemma M1 that $P_h(B|m_1, t_S^1(e), 0, q_S, e)$ is continuous and strictly increasing in e.■

6.5 Proof of Proposition 3

Outline Point i.a) is proved in step 1. Point i.b) is proved in steps 2-3. Point i.c) is proved in step 4. Point i.d) is proved in step 5. The proof of Part ii is omitted, as it is very similar to the proof of Part i. Recall in what follows the following inequalities. First, if $e_1 \neq e_2$ then e_{12} is strictly between e_1 and e_2 while if $e_1 = e_2$ then $e_{12} = e_1$. Second, $e_{13} < \min\{e_{12}, e_1\}$.

Step 1 *Let $e_2 < e_1$. An S-optimal equilibrium using only two messages exists for some q_R if and only if $e \leq e_{12}$.*

Proof: Given that only two messages are allowed, there exists an S-optimal equilibrum if and only if there exists a threshold equilibrium with threshold $t_S^1(e)$ (see Proposition 1.b)). A $t_S^1(e)$-equilibrium exists for some $q_R \leq q_S$ if and only if $P_h(B|m_1, t_S^1(e), 0, q_S, e) \leq \min\left\{q_S, P_l(B|m_2, t_S^1(e), q_S, e)\right\}$. To see this, recall that given $q_R \leq q_S$, a $t_S^1(e)$-equilibrium exists if and only if q_R satisfies

$$P_h(B|m_1, t_S^1(e), 0, q_S, e) \leq q_R \leq \min\left\{q_S, P_l(B|m_2, t_S^1(e), q_S, e)\right\}.$$

From Lemma M1 we know that $P_h(B|m_1, t_S^1(e), 0, q_S, e) \leq q_s$ iff $e \leq e_1$. From Lemma M6, we know that $P_h(B|m_1, t_S^1(e), 0, q_S, e) \leq P_l(B|m_2, t_S^1(e), q_S, e)$ iff $e \leq e_{12}$. Now, note that

$e_{12} < e_1$. Consequently, if $e \in (0, e_{12}]$ there exists a non empty interval of values of q_R for which there exists a $t_S^1(e)$-equilibrium while if $e > e_{12}$ then no value of q_R is compatible with a $t_S^1(e)$-equilibrium.

Step 2 *(Lower bound with two messages) Let $e_2 < e_1$. If $e \le e_{12}$ then*

$$\underline{q}'_R(e) = P_h(B|m_1, t_S^1(e), 0, q_S, e).$$

Proof: The fact that $\underline{q}'_R(e) = P_h(B|m_1, t_S^1(e), 0, q_S, e)$ follows immediately from the necessary and sufficient conditions stated in step 1. Recall that $P_h(B|m_1, t_S^1(e), 0, q_S, e)$ and $P_l(B|m_2, t_S^1(e), q_S, e)$ are both independent of q_R.

Step 3 *(Lower bound with three messages, in a comparative perspective) Let $e_2 < e_1$. If $e \le e_{13}$ then*

$$\underline{q}_R(e) = P_h(B|m_1, t_S^1(e), 0, q_S, e).$$

If $e \in (e_{13}, e_{12}]$ then $\underline{q}_R(e) = P_l(B|m_3, z, q_R, q_S, e)$ for some $z > t_S^1(e)$ that satisfies

$$P_l(B|m_3, z, q_R, q_S, e) < P_h(B|m_1, t_S^1(e), 0, q_S, e).$$

Proof: The characterization of $\underline{q}_R(e)$ is provided in the proof of Proposition 2. We know from Lemma M4 that $e_{13} \in (0, e_{12})$ and that if $e > e_{13}$ then

$$P_h(B|m_1, t_S^1(e), 0, q_S, e) > P_l(B|m_3, t_S^1(e), q_S, e).$$

We also know from Lemma M3 that $P_l(B|m_3, z, q_R, q_S, e)$ is decreasing in z for $z \in \left[t_S^1(e), t_S^2(e)\right]$.

Step 4 *(Upper bound with two messages) Let $e_2 < e_1$. If $e \le e_2$ then $\overline{q}'_R(e) = q_S$. If $e \in (e_2, e_{12}]$ then $\overline{q}'_R(e) = P_l(B|m_2, t_S^1(e), q_S, e) < q_S$.*

Proof: The fact that $\overline{q}'_R(e) = \min\left\{q_S, P_l(B|m_2, t_S^1(e), q_S, e)\right\}$ follows immediately from the necessary and sufficient conditions stated in step 1. The constant e_2 is such that $P_l(B|m_2, t_S^1(e), q_S, e) < q_S$ iff $e > e_2$. We know from Lemma M6 that $e_2 < e_{12}$ given our assumption that $e_2 < e_1$. It follows that if $e \le e_2$ then we have $\min\left\{q_S, P_l(B|m_2, t_S^1(e), q_S, e)\right\} = q_S$ while if $e > e_2$ we instead have $\min\left\{q_S, P_l(B|m_2, t_S^1(e), q_S, e)\right\} = P_l(B|m_2, t_S^1(e), q_S, e)$.

Step 5 *The lower bound $\underline{q}'_R(e) = P_h(B|m_1, t_S^1(e), 0, q_S, e)$ is continuous and strictly increasing in e* for *$e \in (0, e_{12}]$. The upper bound $\overline{q}'_R(e) = q_S$ for $e \in (0, e_2]$. The upper bound $\overline{q}'_R(e) = P_l(B|m_2, t_S^1(e), q_S, e)$ is continuous and strictly decreasing in e for $e \in (e_2, e_{12}]$. Also, $P_l(B|m_2, t_S^1(e_2), q_S, e_2) = q_S$.*

Proof: This was proved in Lemmas M1 and M2.■

6.6 Proof of Proposition 4

Outline Steps 1-2 prove Point i. whereas steps 3-4 prove Point ii.

Step 1 *Given q_S, q_R, e, the simple D-equilibrium exists iff*

$$\frac{P_l^e(B)}{1-P_l^e(B)}\frac{F_B(t_S^2(e))}{F_A(t_S^2(e))} \leq q_R \leq \frac{P_h^e(B)}{1-P_h^e(B)}\frac{F_B(t_S^2(e))}{F_A(t_S^2(e))}. \tag{19}$$

Proof: R is willing to randomize after m_1 iff (19) holds true. On the other hand, R is willing to take action b for sure after message m_2 iff $\frac{q_R}{1-q_R} \leq \frac{P_l^e(B)}{1-P_l^e(B)}\frac{1-F_B(t_S^2(e))}{1-F_A(t_S^2(e))}$. The latter inequality is always satisfied given that by definition $\frac{q_S}{1-q_S} \leq \frac{P_l^e(B)}{1-P_l^e(B)}\frac{1-F_B(t_S^2(e))}{1-F_A(t_S^2(e))}$. To see this, note that $t_S^2(e)$ is by definition such that $\frac{q_S}{1-q_S} = \frac{P_l^e(B)}{1-P_l^e(B)}\frac{f_B(t_S^2(e))}{f_A(t_S^2(e))}$ and apply Lemma F. Regarding S, note that for $\sigma \geq t_S^2(e)$, the sender prefers b to be played for sure as discussed in the opening remarks (section 3.1). Given R's equilibrium strategy, she can achieve this by sending m_2 for sure. For $\sigma \in [t_S^1(e), t_S^2(e))$, the mixed action assigning probablity q_S to a is optimal for S. Sending m_1 for sure (resulting in a being played with probability $q_R \leq q_S$) allows her to come as close as possible to her optimal action. Similarly, for $\sigma < t_S^1(e)$, S's optimal action is the pure action a, and sending m_1 for sure results in a being played with the highest possible probability.

Step 2 *Fix q_S. For any $e > 0$* and given *$q_R \leq q_S$, there exist thresholds $\underline{q}_R^D(e)$ and $\overline{q}_R^D(e)$ satisfying $\underline{q}_R^D(e) < \overline{q}_R^D(e) \leq q_S$ such that the simple D-equilibrium exists iff $q_R \in \left[\underline{q}_R^D(e), \overline{q}_R^D(e)\right]$.*

Proof: It follows immediately from step 1 that $\frac{\underline{q}_R^D(e)}{1-\underline{q}_R^D(e)} = \frac{P_l^e(B)}{1-P_l^e(B)}\frac{F_B(t_S^2(e))}{F_A(t_S^2(e))}$ and $\frac{\overline{q}_R^D(e)}{1-\overline{q}_R^D(e)} = \frac{P_h^e(B)}{1-P_h^e(B)}\frac{F_B(t_S^2(e))}{F_A(t_S^2(e))}$. Note that for any $e \in (0, \frac{1}{2})$, $\frac{P_l^e(B)}{1-P_l^e(B)}\frac{F_B(t_S^2(e))}{F_A(t_S^2(e))} < \frac{q_S}{1-q_S}$ (by the definition of $t_S^2(e)$ and applying Lemma F) so that $\underline{q}_R^D(e) < q_S$. Note furthermore that for $e > 0$, it is trivially true that

$$\frac{P_l^e(B)}{1-P_l^e(B)}\frac{F_B(t_S^2(e))}{F_A(t_S^2(e))} < \frac{P_h^e(B)}{1-P_h^e(B)}\frac{F_B(t_S^2(e))}{F_A(t_S^2(e))}.$$

Step 3 *It holds true that $\lim_{e\to 0}\underline{q}_R^D(e) = \lim_{e\to 0}\underline{q}_R(e)$.*

Proof: Note that $\lim_{e\to 0}\frac{\underline{q}_R(e)}{1-\underline{q}_R(e)} = \lim_{e\to 0}\frac{P_h^e(B)}{1-P_h^e(B)}\frac{F_B(t_S^1(e))}{F_A(t_S^1(e))}$ which is equal to

$$\lim_{e\to 0}\frac{\underline{q}_R^D(e)}{1-\underline{q}_R^D(e)} = \lim_{e\to 0}\frac{P_l^e(B)}{1-P_l^e(B)}\frac{F_B(t_S^2(e))}{F_A(t_S^2(e))}$$

given that $\lim_{e\to 0} t_S^1(e) = \lim_{e\to 0} t_S^2(e)$ and $\lim_{e\to 0} P_l^e(B) = \lim_{e\to 0} P_h^e(B)$.

Step 4 *For any e, $\underline{q}_R^D(e)$ is strictly decreasing in e and $\overline{q}_R^D(e)$ is weakly increasing in e.*

Proof: Regarding $\underline{q}_R^D(e)$, note first that $\frac{P_l^e(B)}{1-P_l^e(B)}\frac{F_B(t_S^2(e))}{F_A(t_S^2(e))}$ is strictly decreasing in e, as shown in Lemma E2 which is stated and proved in section 3 of the Online Appendix. Note furthermore that $\lim_{e\to\frac{1}{2}}\frac{P_l^e(B)}{1-P_l^e(B)}\frac{F_B(t_S^2(e))}{F_A(t_S^2(e))} = 0$. Regarding $\overline{q}_R^D(e)$, note that $\frac{P_h^e(B)}{1-P_h^e(B)}\frac{F_B(t_S^2(e))}{F_A(t_S^2(e))}$ is strictly increasing in e. To see this, note first $t_S^2(e)$ increases in e so that $\frac{F_B(t_S^2(e))}{F_A(t_S^2(e))}$ increases in e. Note also that $\frac{P_h^e(B)}{1-P_h^e(B)}$ increases in e. It follows that $\overline{q}_R^D(e)$ is strictly increasing in e until it reaches value q_S, after which it remains equal to q_S.■

6.7 Proof of Proposition 5

Outline Step 1 analyzes equilibrium and payoffs in the absence of ambiguity. Steps 2 and 3 analyze equilibrium and payoffs in the presence of ambiguity with a focus on the simple D-equilibrium prediction. Step 3 shows the continuity in q_R and e of the expected payoff of S in the simple D-equilibrium under ambiguity $e > 0$ and prior $P_k^e(B)$. Step 4 is about comparing expected payoffs under the no ambiguity and the ambiguity scenario. Step 5 uses the previously established facts in order to conclude. In what follows, we slightly abuse notation and let $\underline{q}_R(0) = \lim_{e\to 0}\underline{q}_R(e)$. The threshold $\underline{q}_R(0)$ thus denotes the lowest value of q_R for which influential communication is possible in the absence of ambiguity. We similarly let $t_S^i(0) = \lim_{e\to 0} t_S^i(e)$, for $i \in \{1, 2\}$.

Step 1 *Given $e = 0$ and $q_R < \underline{q}_R(0)$, the unique equilibrium outcome is such that R always picks action b. S obtains expected payoff $\Pi_0(q_S) = -\frac{1}{2}q_S$.*

Proof: Recall that under $e = 0$, if $q_R \neq \underline{q}_R(0)$ the S-optimal equilibrium is the only equilibrium with influential communication. For $e = 0$ and $q_R < \underline{q}_R(0)$ there exists no S-optimal equilibrium. It follows that given $e = 0$ and $q_R < \underline{q}_R(0)$, there is no influential communication. In the absence of any influential communication and given a unique prior $P(B) = \frac{1}{2}$, R simply picks the ex ante optimal action which is b given $q_R \leq q_S < \frac{1}{2}$.

Step 2 *a) For any $e \in \left(0, \frac{1}{2}\right)$, $\underline{q}_R^D(e)$ is continuous and strictly decreasing in e whereas $\overline{q}_R^D(e)$ is continuous and weakly increasing in e. b) $\underline{q}_R(0) = \lim_{e \to 0} \underline{q}_R^D(0)$. c) For any $e \in \left(0, \frac{1}{2}\right)$ and $q_R \in \left[\underline{q}_R^D(e), \underline{q}_R(0)\right]$, there exists an equilibrium implementing decision rule $D(q_S, q_R, e)$.*

Proof: Given e, recall that threshold $\underline{q}_R^D(e)$ (resp. $\underline{q}_R^D(e)$) is the lowest (resp. highest) q_R compatible with the existence of the simple D-equilibrium. Points a) and b) were proved in Proposition 4. It follows from Points a) and b) that for any $e \in \left(0, \frac{1}{2}\right)$, $\underline{q}_R^D(e) < \underline{q}_R(0)$. Point c) follows immediately from the facts stated in Points a) and b).

Step 3 *Denote by $\Pi_D(e, q_S, q_R, k)$ the expected payoff of S given decision rule $D(q_S, q_R, e)$ under prior $P_k^e(B), k \in \{l, h\}$. $\Pi_D(e, q_S, q_R, k)$ is continuous in q_R and e.*

Proof: The expected payoff of S given decision rule $D(q_S, q_R, e)$ is given by

$$
\begin{aligned}
& \Pi_D(e, q_S, q_R, k) \\
= \quad & -(1 - P_k^e(B)) \left[F_A(t_S^2(e))(1 - q_R) + 1 - F_A(t_S^2(e))\right] q_S - P_k^e(B) F_B(t_S^2(e)) q_R (1 - q_S).
\end{aligned}
$$

To see this, recall that under decision rule $D(q_S, q_R, e)$, R plays b with probability $1 - q_R$ for $\sigma < t_S^2(e)$ and plays b with probability one for $\sigma \geq t_S^2(e)$. It follows that given state $\omega \in \{A, B\}$, under decision rule $D(q_S, q_R, e)$ R chooses b with probability $F_\omega(t_S^2(e))(1 - q_R) + 1 - F_\omega(t_S^2(e))$. R instead chooses a with probability $F_\omega(t_S^2(e) q_R$. The expression $\Pi_D(e, q_S, q_R, k)$ is trivially continuous in q_R. It is continuous in e because $t_S^2(e)$ and $P_k^e(B)$ are continuous in e.

Step 4 *There exists an $e^* > 0$ and a function $\underline{q}_R^*(e)$ defined on $[0, e^*)$ and satisfying $\underline{q}_R^*(e) < \underline{q}_R(0)$ $\forall e \in [0, e^*)$ such that the following is true. If the pair e, q_R satisfies $e \in [0, e^*)$ and $q_R \in [\underline{q}_R^*(e), \underline{q}_R(0)]$, then $\Pi_D(e, q_S, q_R, k) > \Pi_0(q_S)$, $\forall k \in \{l, h\}$.*

Proof: For fixed e, q_S, q_R, k, we compare the welfare of S in an equilibrium implementing $D(q_S, q_R, e)$ to S's welfare in an equilibrium in which b is always chosen. Note first that

$$\Pi_D(0, q_S, \underline{q}_R(0), l) = \Pi_D(0, q_S, \underline{q}_R(0), h).$$

This is true because $t_S^1(0) = t_S^2(0)$. Second, note that for any $q_R \leq q_S$

$$\Pi_D(0, q_S, q_R, h) - \Pi_0(q_S) > 0. \tag{20}$$

To see this note that given $e = 0$, S favours a scenario in which $D(q_S, q_R, 0)$ is implemented to one in which R always chooses action b. Recall that $t_S^1(0) = t_S^2(0)$ and note that for $e = 0$, the S-optimal decision rule is to choose a for sure below $t_S^1(0)$ and b for sure above $t_S^1(0)$. The decision rule $D(q_S, q_R, 0)$ admittedly differs from the S-optimal decision rule but it dominates the "always pick b" rule. It yields a strictly higher payoff than the "always pick b" rule for $\sigma < t_S^1(0)$ by ensuring that a is picked with positive probability. It yields the same payoff as the latter for $\sigma \geq t_S^1(0)$. A special case of the inequality (20) is

$$\Pi_D(0, q_S, \underline{q}_R(0), h) - \Pi_0(q_S) > 0.$$

Note that $\Pi_D(e, q_S, q_R, l)$ and $\Pi_D(e, q_S, q_R, h)$ are continuous in e and q_R. The statement follows.

Step 5 *Given a pair* e, q_R *satisfying* $e \in (0, e^*)$ *and* $q_R \in \left[\max\left\{\underline{q}_R^D(e), q_R^*(e)\right\}, \underline{q}_R(0)\right]$ *there exists an equilibrium implementing decision rule* $D(q_S, q_R, e)$ *and furthermore*

$$\min\{\Pi_D(e, q_S, q_R, l), \Pi_D(e, q_S, q_R, h)\} > \Pi_0(q_S).$$

Proof: Applying simultaneously the statements of steps 2 and 4, the above statement follows.■

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