On Lyapunov functions and data-driven dissipativity

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Abstract: Our contribution in this paper is twofold. In the first part, we study Lyapunov functions when a plant is interconnected with a dissipative stabilizing controller. In the second, we present results on data-driven approach to dissipative systems. In particular, we provide conditions under which an observed trajectory can be used to determine whether a system is dissipative with respect to a given supply rate. Our results are based on linear difference systems for which the use of quadratic difference forms play a central role for dissipativity and Lyapunov theory.

Keywords: Quadratic difference forms, dissipativity, Lyapunov stability, data-driven, discrete-time system, interconnection.

1. INTRODUCTION

The theory of *dissipativity*, first introduced in Willems (1972), plays an important role when dealing with control problems (see Van der Schaft (2012); Hill and Moylan (1980); Ghanbari et al. (2016)). Like most control problems dissipativity theory requires system models (state space, high order differential/difference equations); consequently, when such models are not available it is mandatory to perform system identification before studying dissipative systems. In recent developments of data-based approaches to control (Shi and Skelton (2000); Safonov and Tsao (1997); Maupong and Rapisarda (2016, 2017); Markovsky and Rapisarda (2008)), where the design of a controller is based on system data rather than models, which are in many real-life situations not available, (see Hou and Wang (2013) for a formal definition of *data-driven control* and a summary of approaches in the literature). It has become important to link the theory of dissipativity and system data. For example, to develop tests in which we can directly use system data to determine whether a system is dissipative. In the literature, the concept of dissipativity and data has been explored in Rapisarda and Trentelman (2011). In this case, the authors illustrate how to find a state space representation of a dissipative system using noise-free observed trajectories. Dissipativity using input/output data has also been studied in Montenbruck and Allgöwer (2016), where the authors use finite input/output data to infer a supply rate such that one can apply the small-gain- and feedback- theorems for passive systems to find a controller.

In this paper, given a supply rate as well as an observed trajectory from a system with an unknown model, we show how to determine whether the system is (half-line) dissipative with respect to such a supply rate. We also introduce the notion of $L-$dissipativity for finite observed trajectories. These results could be used in data-driven control and system identification. For example, in the work of Rapisarda and Trentelman (2011) they assume that the given data is generated by a dissipative system with respect to the given supply rate, but with our result it is possible to test if the data is indeed generated by a dissipative system. Our results are based on the concepts of *behavioral system theory* (see Willems (1989, 1991)), hence we use mathematical tools such as *quadratic difference forms* (QdFs)(see Kaneko and Fujii (2000)). Our results differ from other approaches such as that in Montenbruck and Allgöwer (2016), where data is used to infer a supply rate.

In the first part of the paper, we focus on Lyapunov functions and dissipative stabilizing controllers. The relationship between a Lyapunov function and a supply rate of a stabilizing controller has been studied in Rapisarda and Kojima (2010). The authors show that stabilizing using a dissipative controller is equivalent to imposing dissipation on the closed loop system, in the case of *full interconnection* (see Willems (1997)). In this paper, we show the relationship between a Lyapunov function and a storage function of a stabilizing controller interconnected via *partial interconnection* (see Belur (2003)). To show this we assume that the *control variables* are observable from the *to-be-controlled variables.* Then under such assumptions, we show how the Lyapunov function can be expressed using the storage function of the controller and an *observability map.*

Notation. We denote the space of $\mathtt{w}$ dimensional real vectors by $\mathbb{R}^{\mathtt{w}}$ and that of $\mathtt{g} \times \mathtt{w}$ real matrices by $\mathbb{R}^{\mathtt{g}\times\mathtt{w}}$. The space of real matrices with unspecified but finite number of rows and columns is denoted by $\mathbb{R}^{\bullet\times\bullet}$. Let $A \in \mathbb{R}^{\mathtt{g}\times\mathtt{p}}$ then $A^\top$ denotes the transpose of A and colspan(A) denotes the subspace consisting of all linear combination of the columns of A. $\mathrm{tr}(A)$ denotes the trace of A. The ring of polynomials matrices with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$. The ring of two variable polynomial matrices with real coefficients in the indeterminate ζ and η is denoted by $\mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\zeta,\eta]$, and that of symmetric matrices by $\mathbb{R}_s^{\mathtt{g}\times\mathtt{w}}[\zeta,\eta]$. The set of all maps from $\mathbb{Z}$ to $\mathbb{R}$ is denoted by $(\mathbb{R})^{\mathbb{Z}}$. The collection of all linear, closed, shift invariant subspaces of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ equipped with the topology of pointwise convergence is denoted by $\mathscr{L}^{\mathtt{w}}$. We denote by $l_2^{\mathtt{w}}$ the collection of summable time series, i.e. $w \in l_2^{\mathtt{w}}$ means $\sum_{t=-\infty}^{\infty} \|w(t)\|^2 < \infty$ where $\|w\|^2 := w^\top w$. The *Hankel matrix* associated with a trajectory w is defined by

$$\mathfrak{H}(w) := \begin{bmatrix} \dots & w(-1) & w(0) & w(1) & \dots & w(t') & \dots \\ \dots & w(0) & w(1) & w(2) & \dots & w(t'+1) & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots & \\ \dots & w(t-1) & w(t) & w(t+1) & \dots & w(t+t') & \dots \end{bmatrix}.$$

Let $L, J \in \mathbb{Z}_+$, then the Hankel matrix with L block row and J columns is denoted by $\mathfrak{H}_{L,J}(w) \in \mathbb{R}^{L\times J}$.

2. BACKGROUND

2.1 Discrete-time systems

A *discrete-time system* is defined by $\Sigma := (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ where $\mathbb{Z}$ is called the *time axis*, $\mathbb{R}^{\mathtt{w}}$ is the *signal space* and $\mathfrak{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ is called the *behavior*. Now, define the *backward shift* operator σ by $(\sigma f)(t) := f(t+1)$. Then Σ is *linear* if $\mathfrak{B}$ is a linear subspace of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$, *time-invariant* if $\sigma\mathfrak{B} \subseteq \mathfrak{B}$ and *complete* if $\mathfrak{B}$ is closed in the topology of pointwise convergence. Consequently, $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}$. It has been shown in Willems (1989, 1991), that $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}$ if and only if there exists $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ such that

$$\mathfrak{B} = \{w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}} | R(\sigma)\, w = 0\}.$$

The *polynomial operator in the shift* $R(\sigma)$ is called a *kernel representation* of $\mathfrak{B}$, therefore, $\mathfrak{B} = \ker(R(\sigma))$. Furthermore, R is said to induce a *minimal kernel representation* if it is full row rank. The number of rows in a minimal representation is called *output cardinality* and is denoted by $\mathtt{p}(\mathfrak{B})$. $\mathtt{p}(\mathfrak{B})$ is equal to the number of outputs of Σ. The number of inputs is called *input cardinality* and is defined by $\mathtt{m}(\mathfrak{B}) := \mathtt{w} - \mathtt{p}(\mathfrak{B})$.

Let $L \in \mathbb{Z}_+$. The restrictions of $\mathfrak{B}$ on the interval $[0, L]$ is defined by

$$\begin{aligned} \mathfrak{B}_{|[0,L]} := \{w : [0, L] \to \mathbb{R}^{\mathtt{w}} \mid \exists\, w' \in \mathfrak{B} \text{ s.t.} \\ w(t) = w'(t) \text{ for } 1 \leqslant t \leqslant L\}. \end{aligned} \quad (1)$$

The integer L appearing in (1) is called the *lag*.

We also deal with discrete-time systems with *latent variables* whose behavior $\mathfrak{B}_{full} \in \mathscr{L}^{\mathtt{w}+1}$, called the *full behavior*, consists of all trajectories (w, ℓ) with w a *manifest variable* trajectory and ℓ a latent variable trajectory. Let $R \in \mathbb{R}^{\mathtt{d}\times\mathtt{w}}[\xi]$ and $M \in \mathbb{R}^{\mathtt{d}\times 1}[\xi]$, then $\mathfrak{B}_{full} \in \mathscr{L}^{\mathtt{w}+1}$ admits a representation of the form $R(\sigma)w = M(\sigma)\ell$, called a *hybrid representation*. $\mathfrak{B}_{full}$ induces the *manifest behavior* defined by $\mathfrak{B} := \{w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} \mid \exists\, \ell \in (\mathbb{R}^{1})^{\mathbb{Z}}\ s.t.\ (w, \ell) \in \mathfrak{B}_{full}\}$. Let the operator $\pi_w : (\mathbb{R}^{\mathtt{w}} \times \mathbb{R}^{1})^{\mathbb{Z}} \to (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ defined by

$$w =: \pi_w(w, \ell),$$

be the *projection onto the w variables* of $\mathfrak{B}_{full}$. Then the manifest behavior is defined by $\mathfrak{B} := \pi_w(\mathfrak{B}_{full})$.

Let $(w_1, w_2) \in \mathfrak{B}$, w_2 is *observable* from w_1 if there exists a map $f : (\mathbb{R}^{\mathtt{w}_1})^{\mathbb{Z}} \to (\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}}$ such that $w_2 = f(w_1)$. Let $w_1, w_2 \in \mathfrak{B}$, then $\mathfrak{B}$ is *controllable* if there exists a $t_1 \geq 0$ and $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t \leq 0$ and $w(t) = w_2(t - t_1)$ for $t \geq t_1$. We denote by $\mathscr{L}^{\mathtt{w}}_{contr}$ the collection of all controllable elements of $\mathscr{L}^{\mathtt{w}}$. It has been proven in Th. 6.6.1, p. 229 of Willems (1997) that $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}_{contr}$ if and only if there exists an integer $\mathtt{l}$ and a matrix $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{l}}[\xi]$ such that $\mathfrak{B} = \mathrm{im}(M)$, i.e. $w = M(\sigma)\ell$ for $\ell \in (\mathbb{R}^{1})^{\mathbb{Z}}$. This is called an *image representation*. M induce an *observable image representation*, i.e. ℓ is observable from w, if and only if $M(\lambda)$ is full column rank for all $\lambda \in \mathbb{C}$. Autonomous behaviors are defined as follows.

Definition 1. $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}$ is *autonomous* if for all $w_1, w_2 \in \mathfrak{B}$, $[w_1(t) = w_2(t) \text{ for } t < 0] \Longrightarrow [w_1 = w_2]$.

Note that for autonomous system $\mathtt{m}(\mathfrak{B}) = 0$.

2.2 Partial Interconnection

We recall some relevant results on *control as interconnection*. Consider a partition of the system variables as follows, w the *to-be-controlled variables*, i.e. system variables to be influenced and c the *control variables*, i.e. system variables chosen to influence the to-be-controlled variables. Let $R \in \mathbb{R}^{\mathtt{g}\times\mathtt{w}}[\xi]$ and $M \in \mathbb{R}^{\mathtt{g}\times\mathtt{c}}[\xi]$ be such that the to-be-controlled system full behavior is defined by

$$\mathcal{P}_{full} := \{(w, c) \mid R(\sigma)w = M(\sigma)c\}$$

and its manifest behavior by

$$\mathcal{P} := \{w \mid \exists\, c \text{ such that } (w, c) \in \mathcal{P}_{full}\} = \pi_w(\mathcal{P}_{full}).$$

Let $C \in \mathbb{R}^{\bullet\times\mathtt{c}}[\xi]$ and define a *controller behaviour* by $\mathcal{C} := \{c \mid C(\sigma)c = 0\}$. Then the behavior

$$\mathcal{K}_{full} := \{(w, c) \mid (w, c) \in \mathcal{P}_{full} \text{ and } c \in \mathcal{C}\},$$

is called the *full controlled behavior*, and is a result of the *partial interconnection* of $\mathcal{P}_{full}$ and $\mathcal{C}$ through the control variables. $\mathcal{K}_{full}$ induces the *manifest controlled behavior* defined by $\mathcal{K} := \pi_w(\mathcal{K}_{full})$. In the this case, $\mathcal{C}$ is said to *implement* $\mathcal{K}$ or $\mathcal{K}$ is *implementable*. Necessary and sufficient conditions for the existence of a controller $\mathcal{C}$ implementing $\mathcal{K}$ are given in Th. 1, p. 55 of Willems and Trentelman (2002).

Let $\mathcal{C}$ be a controller that implements $\mathcal{K}$ and assume that the c are observable from w then the following result holds true.

Proposition 1. Let $\mathcal{P}_{full} \in \mathscr{L}^{\mathtt{w}+\mathtt{c}}, \mathcal{K} \in \mathscr{L}^{\mathtt{w}}$ and $\mathcal{C} \in \mathscr{L}^{\mathtt{c}}$. Assume that $R, D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi], M \in \mathbb{R}^{\bullet\times\mathtt{c}}[\xi]$ and $C \in \mathbb{R}^{\bullet\times\mathtt{c}}[\xi]$ are such that $\mathcal{P}_{full} = \ker([R(\sigma)\ -M(\sigma)])$, $\mathcal{K} = \ker(D(\sigma))$ and $\mathcal{C} = \ker(C(\sigma))$. Furthermore, assume that $\mathcal{C}$ implements $\mathcal{K}$ via partial interconnection through c with respect to $\mathcal{P}_{full}$ and that c is observable from w. Then following statements are equivalent:

(1) $c = P(\sigma)w$ where P is defined by $P := NR + GD$ with $G \in \mathbb{R}^{\mathtt{c}\times\bullet}[\xi]$ such that $G(\lambda)$ is full column rank for all $\lambda \in \mathbb{C}$ and $N \in \mathbb{R}^{\mathtt{c}\times\bullet}[\xi]$,
(2) there exists $F \in \mathbb{R}^{\bullet\times\bullet}[\xi]$ such that $NM = I_{\mathtt{c}} + FC$.

Proof. See Prop. 1, p. 40 of Maupong and Rapisarda (2017).

2.3 Quadratic difference form

Let $\Phi \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}[\zeta, \eta]$ defined by $\Phi(\zeta, \eta) := \sum_{k,j=0}^{N} \Phi_{k,j} \zeta^k \eta^j$ where $N \in \mathbb{N}$ and $\Phi_{k,j} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}$. Then for all $w_i \in (\mathbb{R}^{\mathtt{w}_i})^{\mathbb{Z}}$, with $i = 1, 2$, $\Phi(\zeta, \eta)$ induces a *bilinear difference form* defined by

$$L_\Phi(w_1, w_2)(t) := \sum_{k,j=0}^{N} w_1(t+k)^\top \Phi_{kj} w_2(t+j).$$

Assume that $\mathtt{w}_1 = \mathtt{w}_2 = \mathtt{w}$ and $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$. Then for all $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$, $\Phi(\zeta, \eta)$ induces a *quadratic difference form* (QdF)

$$Q_\Phi(w)(t) := \sum_{k,j=0}^{N} w(t+k)^\top \Phi_{kj} w(t+j).$$

$\Phi(\zeta, \eta)$ is *symmetric* if $\Phi(\zeta, \eta) = \Phi(\zeta, \eta)^\top$. $\Phi(\zeta, \eta)$ is closely associated with the *coefficient matrix* defined by $\tilde{\Phi} := [\Phi_{i,l}]_{i,l=0,1,2,\dots}$. $\tilde{\Phi}$ is an infinite matrix with only a finite block $N \times N$ nonzero entries, see pp. 1708-1709 of Willems and Trentelman (1998). Even though $\tilde{\Phi}$ is infinite, the largest power of ζ and η in $\Phi(\zeta, \eta)$ is finite. Therefore, we define the *effective size* of $\tilde{\Phi}$ by

$$N := \min \{N' \mid \Phi_{k,j} = 0 \text{ for all } k, j > N'\}.$$

We shall denote by $\tilde{\Phi}_N$ the coefficient matrix with effective size N.

Define the map

$$\nabla : \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta] \to \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$$

by

$$\nabla \Phi(\zeta, \eta) := (\zeta\eta - 1)\Phi(\zeta, \eta).$$

$\nabla\Phi$ induces a QdF $Q_{\nabla\Phi}$ which defines the *rate of change* of Q_Φ defined by

$$Q_{\nabla\Phi} := Q_\Phi(w)(t+1) - Q_\Phi(w)(t),$$

for $t \in \mathbb{Z}$ and $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$.

Definition 2. Let $\Phi \in \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$, a QdF Q_Φ is called *non-negative* if $Q_\Phi(w)(t) \geq 0$ for all $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ and $t \in \mathbb{Z}$. If $Q_\Phi(w)(t) \geq 0$ and if $Q_\Phi(w)(t) = 0$ implies that $w = 0$ for all $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ then we call Q_Φ *positive.*

Non-negativity and positivity are denoted by $Q_\Phi \geq 0$ and $Q_\Phi > 0$, respectively. Moreover, $Q_\Phi \geq 0$ if and only if $\tilde{\Phi} \geq 0$, see Prop. 2.1, p. 33 of Kaneko and Fujii (2000). Now we define non-negative and positive along $\mathfrak{B}$.

Definition 3. Let $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}$ then Q_Φ is *non-negative along* $\mathfrak{B}$ if $Q_\Phi(w)(t) \geq 0$ for all $w \in \mathfrak{B}$. Furthermore, if $Q_\Phi \geq 0$ and $Q_\Phi(w)(t) = 0$ implies that $w = 0$ for all $w \in \mathfrak{B}$ then Q_Φ is *positive along* $\mathfrak{B}$.

Non-negativity and positivity along $\mathfrak{B}$ are denoted by $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ and $Q_\Phi \overset{\mathfrak{B}}{>} 0$, respectively.

2.4 Dissipative systems

Let $\Phi \in \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ and $\mathfrak{B} \in \mathscr{L}_{contr}^{\mathtt{w}}$. Then $\mathfrak{B}$ is $\Phi-$*dissipative* if $\sum_{t=-\infty}^{\infty} Q_\Phi(w)(t) \geq 0$ for all $w \in l_2^{\mathtt{w}} \cap \mathfrak{B}$; $\Phi-$ *strictly dissipative* if there exits $\epsilon > 0$ such that $\sum_{t=\infty}^{\infty} Q_\Phi(w)(t) \geq \epsilon \sum_{t=0}^{\infty} \|w(t)\|^2$ for all $w \in l_2^{\mathtt{w}} \cap \mathfrak{B}$; and $\Phi-$*half-line dissipative* if $\sum_{t=0}^{\infty} Q_\Phi(w)(t) \geq 0$ for all $w \in (l_2^{\mathtt{w}} \cap \mathfrak{B})_{|[0,\infty]}$. Q_Φ is called the *supply rate.*

The properties of *storage* and *dissipation* function associated with the supply rate are defined as follows.

Definition 4. Let $\Phi \in \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ and $\mathfrak{B} \in \mathscr{L}_{contr}^{\mathtt{w}}$.

(1) The QdF Q_Ψ induced by $\Psi \in \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ is called a *storage function* for $\mathfrak{B}$ with respect to Φ if $Q_{\nabla\Psi}(w)(t) \leqslant Q_\Phi(w)(t)$ for all $w \in \mathfrak{B}$ and $t \in \mathbb{Z}$,
(2) The QdF Q_Δ induced by $\Delta \in \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ is called a *dissipation function* for $\mathfrak{B}$ with respect to Φ if $Q_\Delta(w)(t) \geq 0$ for all $w \in \mathfrak{B}$, $t \in \mathbb{Z}$ and if $\sum_{t=-\infty}^{\infty} Q_\Phi(w)(t) = \sum_{t=-\infty}^{\infty} Q_\Delta(w)(t)$ for all $w \in l_2^{\mathtt{w}} \cap \mathfrak{B}$.

Supply rates, storage functions and dissipation functions are associated as follows.

Proposition 2. Let $\mathfrak{B} \in \mathscr{L}_{contr}^{\mathtt{w}}$ and let $\Phi, \Psi, \Delta \in \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$. Assume that $\mathfrak{B} = \operatorname{im}(M(\sigma))$, where $M \in \mathbb{R}^{\mathtt{w} \times 1}[\xi]$ is full column rank for all nonzero $\lambda \in \mathbb{C}$. Then the following statements are equivalent:

1. For all $w \in l_2^{\mathtt{w}} \cap \mathfrak{B}$, $\sum_{t=-\infty}^{\infty} Q_\Phi(w)(t) \geq 0$;
2. Q_Φ admits a storage function for $\mathfrak{B}$;
3. Q_Φ admits a dissipation function for $\mathfrak{B}$;
4. $M(e^{-j\omega})^\top \Phi(e^{-j\omega}, e^{j\omega}) M(e^{j\omega}) \geq 0$ for all $\omega \in [0, 2\pi]$.

Moreover, there is a one-one relationship between $\Psi(\zeta, \eta)$ and $\Delta(\zeta, \eta)$ through $\Phi(\zeta, \eta)$, described by

$$(\zeta\eta - 1)\Psi(\zeta, \eta) = \Phi(\zeta, \eta) - \Delta(\zeta, \eta).$$

Proof. See Prop. 3.3, p. 39 of Kaneko and Fujii (2000).

Now let Φ be a constant matrix, i.e. $\Phi \in \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}$ and denote by $\sigma_+(\Phi)$ and $\sigma_-(\Phi)$ the number of positive and negative eigenvalues of Φ. A case of interest in this work is when $\mathfrak{B}$ is $\Phi-$dissipative and $\sigma_+ = \mathtt{m}(\mathfrak{B})$, i.e. when the number of positive eigenvalues of Φ is equal to the number of inputs of $\mathfrak{B}$. This is referred to as *liveness condition* (see Willems and Trentelman (2002) section VI-B).

2.5 Lyapunov stability

Definition 5. Let $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}$. $\mathfrak{B}$ is *asymptotically stable* if $[w \in \mathfrak{B}] \Rightarrow [\lim_{t\to\infty} w(t) = 0]$.

Note that a necessary condition for asymptotic stability is that $\mathfrak{B}$ has to be autonomous. In the following result we give sufficient condition for asymptotic stability using QdFs.

Theorem 3. $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}$ is asymptotically stable if there exist $\Psi \in \mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ such that $Q_\Psi \overset{\mathfrak{B}}{\geq} 0$ and $Q_{\nabla\Psi} \overset{\mathfrak{B}}{<} 0$.

Proof. See Lemma 3, p. 2913 of Kojima and Takaba (2005)

The QdF Q_Ψ satisfying conditions of Th. 3 is called *Lyapunov function* for $\mathfrak{B}$. Necessary conditions, which a

Lyapunov function is a solution of *two-variable polynomial Lyapunov equations* are given in Lemma 4 and Th. 1 p.2913 of Kojima and Takaba (2005).

3. MAIN RESULTS

In the first subsection, we study Lyapunov functions of systems stabilized by a dissipative controller. The second subsection is dedicated to dissipativity using data.

3.1 Dissipative controllers

We start with the following result, where we show how the supply rate of a controller is associated with the supply rate of the controlled system.

Proposition 4. Let $\mathcal{P}_{full} \in \mathscr{L}^{\mathtt{w}+\mathtt{c}}, \mathcal{K} \in \mathscr{L}^{\mathtt{w}}$ and $\mathcal{C} \in \mathscr{L}^{\mathtt{c}}$. Assume that $\mathcal{C}$ implement $\mathcal{K}$ via partial interconnection with $\mathcal{P}_{full}$, and that c is observable from the w. Let $P \in \mathbb{R}^{\mathtt{c}\times\mathtt{w}}[\xi]$ satisfying conditions of Prop. 1 and $\Phi \in \mathbb{R}_s^{\mathtt{c}\times\mathtt{c}}[\zeta,\eta]$. Then the following statements are equivalent:

(1) $\mathcal{C}$ is Φ-dissipative,
(2) $\mathcal{K}$ is Φ_1-dissipative where $\Phi_1 \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is defined by $\Phi_1(\zeta,\eta) := P(\zeta)^\top\Phi(\zeta,\eta)P(\eta)$.

Proof. By the assumption that $\mathcal{C}$ implement $\mathcal{K}$ and that c is observable from the w it follows from Prop. 1 that $c = P(\sigma)w$ for all $w \in \mathcal{K}$. To show (1) $\Rightarrow$ (2) Assume that $\mathcal{C}$ is Φ-dissipative, then $\sum\limits_{t=-\infty}^{\infty} Q_\Phi(c)(t) \geq 0$ for all $c \in l_2^{\mathtt{c}} \cap \mathcal{C}$. Now substitute $c = P(\sigma)w$ into $\sum\limits_{t=-\infty}^{\infty} Q_\Phi(c)(t) \geq 0$ then $\sum\limits_{t=-\infty}^{\infty} Q_\Phi(P(\sigma)w)(t) \geq 0$. Define $\Phi_1(\zeta,\eta) := P^\top(\zeta)\Phi(\zeta,\eta)P(\eta)$. Then $\sum\limits_{t=-\infty}^{\infty} Q_{\Phi_1}(w)(t) \geq 0$ for all $w \in l_2^{\mathtt{w}} \cap \mathcal{K}$ such that $c = P(\sigma)w \in l_2^{\mathtt{c}} \cap \mathcal{C}$. Therefore, $\mathcal{K}$ is Φ_1-dissipative.
(2) $\Rightarrow$ (1) Assume that $\mathcal{K}$ is Φ_1-dissipative where $\Phi_1(\zeta,\eta) := P(\zeta)^\top\Phi(\zeta,\eta)P(\eta)$. Then $\sum\limits_{t=-\infty}^{\infty} Q_{\Phi_1}(w)(t) \geq 0$ for all $w \in l_2^{\mathtt{w}} \cap \mathcal{K}$. Since $c = P(\sigma)w$ for all $w \in \mathcal{K}$ then applying $w' \in l_2^{\mathtt{w}} \cap \mathcal{K}$ such that $c' = P(\sigma)w' \in l_2^{\mathtt{c}} \cap \mathcal{C}$, to $\sum\limits_{t=-\infty}^{\infty} Q_{\Phi_1}(w')(t)$ implies applying $c' = P(\sigma)w' \in l_2^{\mathtt{c}} \cap \mathcal{C}$ to $\sum\limits_{t=-\infty}^{\infty} Q_\Phi(c')(t)$. Now since $\sum\limits_{t=-\infty}^{\infty} Q_{\Phi_1}(w)(t) \geq 0$ then $\sum\limits_{t=-\infty}^{\infty} Q_\Phi(c)(t) \geq 0$, hence, $\mathcal{C}$ is Φ-dissipative.

In Prop. 4 above, we have shown that interconnecting a plant with a dissipative controller results in the controlled behavior being dissipative with supply rate parametrized by the supply rate of the controller and the observability map used to construct the control variables from the to-be-controlled variables. Now, consider the following definition.

Definition 6. Let $\mathcal{K}$ be implementable via partial interconnection with $\mathcal{P}_{full}$. A controller $\mathcal{C}$ is called *stabilizing through c* if

i. $\mathcal{C}$ implements $\mathcal{K}$ and
ii. $[w \in \mathcal{K}] \Rightarrow [\lim_{t\to\infty} w(t) = 0]$.

In the sequel we shall refer to $\mathcal{C}$ satisfying conditions of Def. 6 as a *stabilizing controller*.

Theorem 5. Under the assumptions of Prop. 1, assume that $\mathcal{C} \in \mathscr{L}_{contr}^{\mathtt{c}}$ is a stabilizing controller that implements $\mathcal{K}$. Let $\Phi \in \mathbb{R}_s^{\mathtt{c}\times\mathtt{c}}$, assume that $\mathcal{C}$ is strictly dissipative with respect to Φ and that $\sigma_+(\Phi) = \mathtt{m}(\mathcal{C})$. Then there exists $\Psi, \Delta \in \mathbb{R}_s^{\mathtt{c}\times\mathtt{c}}[\zeta,\eta]$ such that $Q_\Psi \geq 0$, $Q_\Delta \geq 0$ induce storage and dissipation functions of $\mathcal{C}$, respectively. Moreover, define $\Psi' := P(\zeta)^\top\Psi(\zeta,\eta)P(\eta)$ then $Q_{\Psi'}$ is a Lyapunov function for $\mathcal{K}$.

Proof. The existence of Ψ and Δ follows from Prop. 2 and the fact that $Q_\Psi \geq 0$ follows from $\sigma_+(\Phi) = \mathtt{m}(\mathcal{C})$ and Th. 6.4, p. 1726 of Willems and Trentelman (1998). Now we show that $Q_{\Psi'} \overset{\mathcal{K}}{\geq} 0$ and $Q_{\nabla\Psi'} \overset{\mathcal{K}}{<} 0$. Since $Q_\Psi \geq 0$ then $Q_\Psi \overset{\mathcal{C}}{\geq} 0$, from Prop. 1 $c = P(\sigma)w$ for all $w \in \mathcal{K}$. Hence $Q_\Psi(c)(t) \geq 0$ for all $c \in \mathcal{C}$ implies that $Q_\Psi(P(\sigma)w)(t) \geq 0$ for all $w \in \mathcal{K}$, therefore, $Q_{\Psi'} \overset{\mathcal{K}}{\geq} 0$. Now since $\mathcal{C}$ is strictly Φ-dissipative then there exists ϵ such that $Q_{\nabla\Psi}(c)(t) = Q_\Phi(c)(t) - Q_\Delta(c)(t) - \epsilon\|c(t)\|^2$, hence, $Q_{\nabla\Psi}(P(\sigma)w)(t) = Q_\Phi(P(\sigma)w)(t) - Q_\Delta(P(\sigma)w)(t) - \epsilon\|P(\sigma)w(t)\|^2$ for $w \in \mathcal{K}$. Since by definition $Q_\Delta \geq 0$ then $Q_\Phi(P(\sigma)w)(t) - Q_\Delta(P(\sigma))(t) - \epsilon\|P(\sigma)w(t)\|^2 < 0$, consequently, $Q_{\nabla\Psi}(P(\sigma)w)(t) < 0$ which implies $Q_{\nabla\Psi'} \overset{\mathcal{K}}{<} 0$.

It follows from Th. 5 that interconnecting a system with a strictly dissipative controller under the liveness conditions, one can construct a Lyapnov function of the closed loop system using the storage function of the controller and the observability map in Prop. 1. This is illustrated with an example below.

Example 1. Consider the system in Fig. 1 corresponding to a *resonant* two-port electrical circuit. If we want to drive the voltage V_1 to zero asymptotically, we can consider its interconnection with a controller, connected to the right-hand side port to impose constraints on the port variables V_2 and I_2. Consequently, the circuit is considered to be the to-be-controlled system $\mathcal{P}_{full}$, with $w = V_1$ as the to-be-controlled variable and $c = \mathrm{col}(V_2, I_2)$ as the control variables.

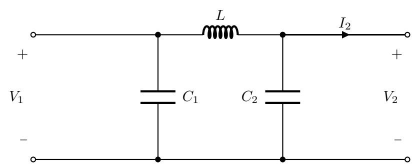

Fig. 1. To-be-controlled system.

The controller, $\mathcal{C}$, connected to the port with voltage V_2 is depicted in Fig. 2. It can be easily verified that $\mathcal{C}$ stabilizes $\mathcal{P}$. For simplicity of computations we consider $L = 1$H,

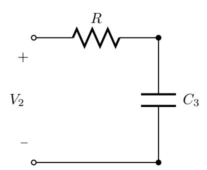

Fig. 2. Stabilizing controller.

$C_1 = C_2 = C_3 = 1$F and $R = 1\Omega$. Hence, $\mathcal{P}_{full}$ and

$\mathcal{C}$ admit hybrid and kernel representations respectively induced by

$$R(\xi) = \begin{bmatrix} 1 \\ \xi \end{bmatrix},\ M(\xi) = \begin{bmatrix} \xi^2+1 & 0 \\ -\xi & -1 \end{bmatrix};\ \text{and}\ C(\xi) = [\xi\ \ -\xi-1]\ .$$

The interconnection of $\mathcal{P}_{full}$ and $\mathcal{C}$ via c results in the controlled system behaviour, $\mathcal{K}$, whose kernel representation is induced by

$$D(\xi) = \left[\xi^4 + 2\xi^3 + 2\xi^2 + 3\xi\right].$$

By the assumption that c are observable w then the observability is induced by

$$P(\xi) = \begin{bmatrix} \xi^2+1 \\ \xi^3+2\xi \end{bmatrix}.$$

Note that since $\mathcal{C}$ is $\Phi-$dissipative, with $\Phi = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and a storage function induced by $\Psi = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{bmatrix}$. From Th. 5 we can compute a functional for $\mathcal{K}$ induced by

$$\begin{aligned}\Psi' &= P(\zeta)^\top \Psi(\zeta,\eta) P(\eta) \\ &= \zeta^3\eta^3 - \frac{1}{2}\zeta^2\eta^3 - \frac{1}{2}\zeta^3\eta^2 + 2\zeta\eta^3 + \frac{1}{2}\zeta^2\eta^2 + 2\zeta^3\eta - \frac{1}{2}\eta^3 \\ &\quad - \zeta\eta^2 - \zeta^2\eta - 1/2\zeta^3 + \frac{1}{2}\eta^2 + 4\zeta\eta + 1/2\zeta^2 - \eta - \zeta + \frac{1}{2}.\end{aligned}$$

It is a matter of straightforward verification that $Q_{\Psi'}$ is a Lyapunov function for $\mathcal{K}$.

3.2 Dissipativity using data

We introduce concepts of dissipativity based on data.

Lemma 6. Let $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}$, $\tilde{w} \in \mathfrak{B}$, and the Hankel matrix associated with $\tilde{w}$ by $\mathfrak{H}(\tilde{w})$. Then col span$(\mathfrak{H}(\tilde{w})) = \mathfrak{B}$.

Proof. Recall that the columns of $\mathfrak{H}(\tilde{w})$ consists of the shifts of $\tilde{w}$, now since $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}$ then $\mathfrak{B}$ is linear and shift invariant. Therefore, all linear combinations of the shifts of $\tilde{w}$ are elements of $\mathfrak{B}$.

Note that the above results hold in the case $\tilde{w} \in \mathfrak{B}_{|[0,\infty]}$. Moreover, col span$(\mathfrak{H}_L(\tilde{w})) = \mathfrak{B}_{|[0,L]}$, where $L \in \mathbb{Z}_+$.

Let $\tilde{w} \in \mathfrak{B}_{|[0,\infty]}$ then from the above Lemma for all $\tilde{w}' \in \mathfrak{B}$ there exists $\alpha \in \mathbb{R}^\infty$ such that $\tilde{w}' = \mathfrak{H}(\tilde{w})\alpha$. Therefore, the Hankel matrix associated with an arbitrary $\tilde{w}' \in (l_2^{\mathtt{w}} \cap \mathfrak{B})_{|[0,\infty]}$ is given by

$$\mathfrak{H}(\tilde{w}') = \underbrace{\begin{bmatrix} \tilde{w}(0) & \tilde{w}(1) & \tilde{w}(2) & \dots \\ \tilde{w}(1) & \tilde{w}(2) & \tilde{w}(3) & \dots \\ \tilde{w}(2) & \tilde{w}(3) & \tilde{w}(4) & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}}_{\mathfrak{H}(\tilde{w})} \underbrace{\begin{bmatrix} \alpha(0) & 0 & 0 & 0 & \dots \\ \alpha(1) & \alpha(0) & 0 & 0 & \dots \\ \alpha(2) & \alpha(1) & \alpha(0) & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}}_{\mathcal{A}(\alpha)}. \tag{2}$$

Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ with the coefficient matrix $\tilde{\Phi}$ and compute

$$\mathfrak{H}^\top(w')\tilde{\Phi}\mathfrak{H}(w') = \begin{bmatrix} Q_\Phi(\tilde{w}')(0) & L_\Phi(\tilde{w}',\sigma\tilde{w}')(0) & L_\Phi(\tilde{w}',\sigma^2 w')(0) & \dots \\ L_\Phi(\sigma\tilde{w}',\tilde{w}')(0) & Q_\Phi(\sigma\tilde{w}',\sigma\tilde{w}')(0) & L_\Phi(\sigma\tilde{w}',\sigma^2\tilde{w}')(0) & \dots \\ L_\Phi(\sigma^2\tilde{w}',\tilde{w}')(0) & L_\Phi(\sigma^2\tilde{w}',\sigma\tilde{w}')(0) & Q_\Phi(\sigma^2\tilde{w}',\sigma^2\tilde{w}')(0) & \dots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

Notice that

$$\operatorname{tr}(\mathfrak{H}(w')^\top\tilde{\Phi}\mathfrak{H}(w')) = \sum_{t=0}^{\infty} Q_\Phi(\tilde{w}')(t),$$

moreover, since $\mathfrak{H}(\tilde{w}') = \mathfrak{H}(\tilde{w})\mathcal{A}(\alpha)$ then

$$\operatorname{tr}(\mathfrak{H}(w')^\top\tilde{\Phi}\mathfrak{H}(w')) = \operatorname{tr}(\mathcal{A}(\alpha)^\top\mathfrak{H}(\tilde{w})^\top\tilde{\Phi}\mathfrak{H}(\tilde{w})\mathcal{A}(\alpha)).$$

This leads us to the following result.

Theorem 7. Let $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}_{contr}$ and $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ with the coefficient matrix denoted by $\tilde{\Phi}$, and $\tilde{w} \in \mathfrak{B}_{|[0,\infty]}$. Then the following statements are equivalent,

(1) $\mathfrak{B}$ is half-line dissipative with respect to Φ,
(2) for all $\mathcal{A}(\alpha)$, such that $\tilde{w}'$ whose Hankel matrix is defined by (2) is such that $\tilde{w}' \in (l_2^{\mathtt{w}} \cap \mathfrak{B})_{|[0,\infty]}$ then $\operatorname{tr}(\mathcal{A}(\alpha)^\top\mathfrak{H}(\tilde{w})^\top\tilde{\Phi}\mathfrak{H}(\tilde{w})\mathcal{A}(\alpha)) \geq 0$, where $\alpha^\top \in \mathbb{R}^{1\times\infty}$.

Proof. From Lemma 6, it follows that for all $\mathcal{A}(\alpha)$ then $\tilde{w}'$ whose Hankel matrix is defined by (2) belongs to $\mathfrak{B}_{|[0,\infty]}$. We proceed to prove (1) $\implies$ (2) as follows. Assume that $\mathfrak{B}$ is $\Phi-$half-line dissipative and recall that $\operatorname{tr}(\mathcal{A}(\alpha))^\top\mathfrak{H}(\tilde{w})^\top\tilde{\Phi}\mathfrak{H}(\tilde{w})\mathcal{A}(\alpha)) = Q_\Phi(\tilde{w}')(0) + Q_\Phi(\tilde{w}')(1) + \dots = \sum_{t=0}^{\infty} Q_\Phi(\tilde{w}')(t)$. Now by the assumption that $\mathfrak{B}$ is $\Phi-$half-line dissipative it follows that $\sum_{t=0}^{\infty} Q_\Phi(\tilde{w}')(t) \geq 0$ for all $\tilde{w}' \in (l_2^{\mathtt{w}} \cap \mathfrak{B})_{|[0,\infty]}$ which implies for all $\mathcal{A}(\alpha)$, such that $\tilde{w}' \in (l_2^{\mathtt{w}} \cap \mathfrak{B})_{|[0,\infty]}$ then
$\operatorname{tr}(\mathcal{A}(\alpha))^\top\mathfrak{H}(\tilde{w})^\top\tilde{\Phi}\mathfrak{H}(\tilde{w})\mathcal{A}(\alpha))) \geq 0$.
(2) $\implies$ (1) Assume $\operatorname{tr}(\mathcal{A}(\alpha)^\top\mathfrak{H}(\tilde{w})^\top\tilde{\Phi}\mathfrak{H}(\tilde{w})\mathcal{A}(\alpha)) \geq 0$ for all $\mathcal{A}(\alpha)$ such that $\tilde{w}' \in (l_2^{\mathtt{w}} \cap \mathfrak{B})_{|[0,\infty]}$. Since $\operatorname{tr}(\mathcal{A}(\alpha))^\top\mathfrak{H}(\tilde{w})^\top\tilde{\Phi}\mathfrak{H}(\tilde{w})\mathcal{A}(\alpha)) = Q_\Phi(\tilde{w}')(0) + Q_\Phi(\tilde{w}')(1) + \dots = \sum_{t=0}^{\infty} Q_\Phi(\tilde{w}')(t)$ then it follows that $\sum_{t=0}^{\infty} Q_\Phi(\tilde{w}')(t) \geq 0$, therefore, $\sum_{t=0}^{\infty} Q_\Phi(\tilde{w}')(t) \geq 0$ for all $\mathcal{A}(\alpha)$ such that $\tilde{w}' \in (l_2^{\mathtt{w}} \cap \mathfrak{B})_{|[0,\infty]}$ which implies that $\mathfrak{B}$ is half-line dissipative with respect to the supply rate Q_Φ.

Since we are dealing with observed trajectories it is necessary to consider the fact that it is not possible to have infinite observed trajectories in reality. Therefore, conditions on finite trajectories are needed. For finite observed trajectories consider the following definition.

Definition 7. Let $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}_{contr}$, $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ and $L \in \mathbb{Z}_+$. $\mathfrak{B}$ is called $\Phi-L$*-dissipative* if $\sum_{t=0}^{L} Q_\Phi(w)(t) \geq 0$ for all $w \in \mathfrak{B}_{|[0,L]}$.

Let $\alpha^\top \in \mathbb{R}^{1\times L}$ and define the matrix $\mathcal{A}_{2L}(\alpha) \in \mathbb{R}^{2L\times L}$ by

$$\mathcal{A}_{2L}(\alpha) := \begin{bmatrix} \alpha(0) & 0 & 0 & \dots & 0 \\ \alpha(1) & \alpha(0) & 0 & \dots & 0 \\ \alpha(2) & \alpha(1) & \alpha(0) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha(L) & \alpha(L-1) & \alpha(L-2) & \dots & \alpha(0) \\ 0 & \alpha(L) & \alpha(L-1) & \dots & \alpha(1) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \alpha(L) \end{bmatrix},$$

Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, and assume that the effective size of $\tilde{\Phi}$ is $N \in Z_+$. Choose $L \in \mathbb{Z}_+$ such that $L \geq N$ denote by $\tilde{\Phi}_L$ the $L \times L$ matrix

$$\begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \dots & \Phi_{0,N} & 0_{\mathtt{w}\times(L-N)\mathtt{w}} \\ \Phi_{1,0} & \Phi_{1,1} & \dots & \Phi_{1,N} & 0_{\mathtt{w}\times(L-N)\mathtt{w}} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \Phi_{N,0} & \Phi_{N,1} & \dots & \Phi_{N,N} & 0_{\mathtt{w}\times(L-N)\mathtt{w}} \\ 0_{(L-N)\mathtt{w}\times\mathtt{w}} & 0_{(L-N)\mathtt{w}\times\mathtt{w}} & 0_{(L-N)\mathtt{w}\times\mathtt{w}} & 0_{(L-N)\mathtt{w}\times\mathtt{w}} & 0_{(L-N)\mathtt{w}\times\mathtt{w}} \end{bmatrix}.$$

Finally, let $T \in \mathbb{N}$, $\tilde{w} \in \mathfrak{B}_{|[0,T]}$ and denote the Hankel matrix of depth L and finite number $2L$ of columns by $\mathfrak{H}_{L,2L}(\tilde{w})$.

Theorem 8. Let $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}_{contr}$. Furthermore, let $L \in \mathbb{Z}_+$ $\tilde{w} \in \mathfrak{B}_{|[0,T]}$, $\mathcal{A}_{2L}(\alpha)$ and $\tilde{\Phi}_L$ as above. $\mathfrak{B}$ is $\Phi - L$-dissipative if and only if for all $\mathcal{A}_{2L}(\alpha)$,
$\mathrm{tr}(\mathcal{A}_{2L}(\alpha)^\top \mathfrak{H}_{L\times 2L}(\tilde{w})^\top \tilde{\Phi}_L \mathfrak{H}_{L\times 2L}(\tilde{w})\mathcal{A}_{2L}(\alpha)) \geq 0.$

Proof. Follows the same argument in the proof of Th. 7.

In the following Proposition, we give the relationship between $\Phi - L$-dissipative and Φ-dissipative.

Proposition 9. Let $\mathfrak{B} \in \mathscr{L}^{\mathtt{w}}_{contr}$ and $L \in \mathbb{Z}_+$, $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. If for all $w \in l_2^{\mathtt{w}} \cap \mathfrak{B}$ and $t_0 \in \mathbb{Z}$ there exist $w' \in \mathfrak{B}_{|[t_0,t_0+L]}$ satisfying $w(t) = w'(t)$ for $t_0 \leqslant t \leqslant t_0 + L$ such that $\sum_{t=t_0}^{t_0+L} Q_\Phi(w')(t) \geq 0$ then $\mathfrak{B}$ is $\Phi-$dissipative.

Proof. Since for all L long samples of w it holds that $\sum_{t=t_0}^{t_0+L} Q_\Phi(w)(t) \geq 0$ then $\sum_{-\infty}^{\infty} Q_\Phi(w)(t) \geq 0$, hence, $\mathfrak{B}$ is $\Phi-$dissipative.

Remark 1. In a future research direction, it would be interesting to investigate if $\Phi - L-$dissipativity can prove $\Phi-$dissipativity. A possible starting point would be to consider how to infer the storage function from data.

4. CONCLUSION

In Th. 5, we have shown that the partial interconnection of a dissipative controller with a system results in the Lyapunov function of the controlled system been expressed using the storage function of the controller and observability map used to construct control variable trajectories from to-be-controlled variable trajectories. In subsection 3.2 we have proved necessary and sufficient condition to determine whether a system is dissipative with respect to given supply rate using observed trajectories. In Th. 7 we proved conditions for half-line dissipativity and Th. 8 in the case of finite observed trajectory. For the case of finite trajectories, we introduced the notion of L dissipative, in Def. 7. Current ongoing research effort aims at finding how the results of subsection 3.2 can be applied to design a controller using data-driven control methods.

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