Generalised CP and $\Delta(96)$ Family Symmetry

Gui-Jun Ding[1], Stephen F. King[2]

[1]*Department of Modern Physics, University of Science and Technology of China, Hefei, Anhui 230026, China*

[2]*School of Physics and Astronomy, University of Southampton, Southampton, SO17 1BJ, U.K.*

Abstract

We perform a comprehensive study of the $\Delta(96)$ family symmetry combined with the generalised CP symmetry $H_{\rm CP}$. We investigate the lepton mixing parameters which can be obtained from the original symmetry $\Delta(96) \rtimes H_{\rm CP}$ breaking to different remnant symmetries in the neutrino and charged lepton sectors, namely G_{ν} and G_l subgroups in the neutrino and the charged lepton sector respectively, and the remnant CP symmetries from the breaking of $H_{\rm CP}$ are $H^{\nu}_{\rm CP}$ and $H^{l}_{\rm CP}$, respectively, where all cases correspond to a preserved symmetry smaller than the full Klein symmetry, as in the semi-direct approach, leading to predictions which depend on a single undetermined real parameter, which may be fitted to the reactor angle for example. We discuss 26 possible cases, including a global χ^2 determination of the best fit parameters and the correlations between mixing parameters, in each case.

1 Introduction

Following the measurement of the reactor mixing angle θ_{13} by the Daya Bay [1], RENO [2], and Double Chooz [3] reactor neutrino experiments, the three lepton mixing angles θ_{12}, θ_{23}, θ_{13} and both mass-squared differences Δm^2_{sol} and Δm^2_{atm} have been measured to reasonably good accuracy. However the Dirac CP phase and neutrino mass ordering have not been measured so far. If neutrinos are Majorana particles, there are two more Majorana CP phases which play a role in neutrinoless double-beta decay searches. Determining the neutrino mass ordering and measuring the Dirac and Majorana CP violating phases are primary goals of the next generation neutrino oscillation experiments. The CP violation has been firmly established in the quark sector and therefore it is natural to expect that CP violation occurs in the lepton sector as well. Indeed hints of a nonzero $\delta_{\rm CP}$ have begun to show up in global analysis of neutrino oscillation data [4–6].

In recent years, much effort has been devoted to explaining the structure of the lepton mixing angles through the introduction of discrete family symmetries. In this paradigm, one generally assumes a non-abelian discrete flavour group which is broken down to different subgroups in the neutrino and charged lepton sectors. The mismatch between these two subgroups gives rise to particular predictions for the lepton mixing angles. For recent reviews, please see [7] for the model building and relevant group theory aspects. Inspired by the success of discrete family symmetry, it is conceivable to extend the family symmetry to include a generalised CP symmetry $H_{\rm CP}$ [8, 9] which will allow the prediction of both CP phases and mixing angles.

The idea of combining a family symmetry with a generalised CP symmetry has begun to be discussed in the literature. For example, the simple $\mu-\tau$ reflection symmetry, which exchanges a muon (tau) neutrino with a tau (muon) antineutrino in the charged lepton diagonal basis, has been discussed and successfully implemented in a number of models where both atmospheric mixing angle θ_{23} and Dirac CP phase $\delta_{\rm CP}$ were determined to be maximal [10–12]. The phenomenological consequences of imposing both an S_4 flavour symmetry and a generalised CP symmetry have been investigated in a model-independent way [13–15]. All the three lepton mixing angles and CP phases are found to depend on only one free parameter for the symmetry breaking of $S_4 \rtimes H_{\rm CP}$ to $Z_2 \times \rm CP$ in the neutrino sector and to some abelian subgroup of S_4 in the charged lepton sector. Concrete S_4 family models with a generalised CP symmetry have been constructed in Refs. [14–17] where the spontaneous breaking of the $S_4 \rtimes H_{\rm CP}$ down to $Z_2 \times \rm CP$ in the neutrino sector was implemented. A similar generalised analysis has also been considered for A_4 family symmetry [18]. Other models with a family symmetry and a generalised CP symmetry can also be found in Refs. [19–22]. The interplay between flavor symmetries and CP symmetries has been generally discussed in [23,24]. In addition, there are other theoretical approaches involving both family symmetry and CP violation [25–28]. A generalised CP analysis of $\Delta(6n^2)$ has been performed recently [29] based on a direct approach with the full Klein symmetry $Z_2 \times Z_2$ preserved in the neutrino sector and a Z_3 preserved in the charged lepton sector. Here we shall focus on $\Delta(96)$ and relax the requirement of having the full Klein symmetry.

In this paper, then, we study generalised CP symmetry in the context of $\Delta(96)$ where a CP symmetry is assumed to exist at a high energy scale. The generalised CP transformation compatible with an $\Delta(96)$ family symmetry is defined, and a model-independent analysis of the lepton mixing matrix is performed by scanning all the possible remnant subgroups in the neutrino and charged lepton sectors. Relaxing the requirement of having the full Klein symmetry in the neutrino sector given by a subgroup of $\Delta(96)$, as in the semi-direct approach we are led to a large number of possibilities where the results depend on a single parameter, expressed as an angle which determines the reactor angle. We systematically discuss all such possibilities consistent with existing phenomenological data, then analyse in detail the resulting predictions for mixing parameters, including a full discussion of correlations between parameters and a χ^2 determination of the best fit point.

The remainder of this paper is organised as follows. In section 2 we discuss generalised CP with $\Delta(96)$. In section 3 we perform a model independent CP analysis of $\Delta(96)$ subgroups, and categorise all the different possibilities for preserved flavour and CP symmetries in the neutrino and charged lepton sectors. In section 4 we analyse the lepton mixing predictions arising from 26 different possible cases discussed in the previous section. We perform a χ^2 analysis to determine the best fit to current data. Section 5 concludes the paper. The details of the group theory of $\Delta(96)$ are collected in Appendix A.

2 Generalised CP with $\Delta(96)$

It is non-trivial to define a CP transformation consistently in the presence of a family symmetry G_f. Generally the so-called consistency condition must be satisfied [8, 9, 23]:

$$X\rho^*(g)X^{-1} = \rho(g'), \quad g, g' \in G_f\,, \tag{2.1}$$

where $\rho(g)$ denotes the representation matrix for the group element g, X is the generalised CP transformation, which maps a field φ into

$$\varphi(t, \mathbf{x}) \xrightarrow{CP} X\,\varphi^*(t, -\mathbf{x})\,, \tag{2.2}$$

where the obvious action of CP on the spinor indices has been suppressed for the case of φ being spinor. Eq. (2.1) implies that the generalised CP transformation X maps the group element g onto g' and the family group structure is preserved under this mapping. Because X is unitary and therefore invertible, the generalized CP is an automorphism of the family symmetry group, and all the possible unitary matrices of X forms a representation of the automorphism group. Given a solution X to Eq. (2.1), $\rho(h)X$ with any $h \in G_f$ also satisfies the consistency equation Eq. (2.1),

$$\big(\rho(h)X\big)\rho^*(g)\big(\rho(h)X\big)^{-1} = \rho(h)\big[X\rho^*(g)X^{-1}\big]\rho^{-1}(h) = \rho(hg'h^{-1})\,, \tag{2.3}$$

which indicates that the CP transformation $\rho(h)X$ maps the element g into $hg'h^{-1}$, and the corresponding automorphism is the composition of the automorphism of X followed by an inner automorphism $\mathrm{conj}(h) : g' \to hg'h^{-1}$. Therefore, when we investigate the groups of generalised CP transformations consistent with a family symmetry, it is sufficient to only consider the outer automorphism of G_f with the inner automorphism modded out, since the inner automorphism doesn't impose any constraint.

For our family symmetry of interest $G_f = \Delta(96)$ in the present work, only the identity element commutes with all other elements and hence the inner automorphism group is isomorphic to $\Delta(96)$. The group theory of $\Delta(96)$ is discussed in Appendix A. There is only one non-trivial outer automorphism (up to inner automorphism) $\mathfrak{u}$ with

$$S \xrightarrow{\mathfrak{u}} S, \quad T \xrightarrow{\mathfrak{u}} T^2, \quad U \xrightarrow{\mathfrak{u}} U \,. \tag{2.4}$$

Therefore the structure of the automorphism group of $\Delta(96)$ can be summarized as

$$\begin{array}{ll} \mathrm{Z}(\Delta(96)) \cong Z_1, & \mathrm{Aut}(\Delta(96)) \cong \Delta(96) \rtimes Z_2, \\ \mathrm{Inn}(\Delta(96)) \cong \Delta(96), & \mathrm{Out}(\Delta(96)) \cong Z_2 = \{id, \mathfrak{u}\} \,, \end{array} \tag{2.5}$$

where $\mathrm{Z}(\Delta(96))$, $\mathrm{Aut}(\Delta(96))$, $\mathrm{Inn}(\Delta(96))$ and $\mathrm{Out}(\Delta(96))$ denote the center, automorphism group, inner automorphism group and outer automorphism group of $\Delta(96)$ respectively. Under the action of $\mathfrak{u}$, another set of $\Delta(96)$ generators a, b, c and d defined in Eq. (A.3) is mapped into

$$a \xrightarrow{\mathfrak{u}} a^2c^2d^2, \quad b \xrightarrow{\mathfrak{u}} abc^2, \quad c \xrightarrow{\mathfrak{u}} c^3d^3, \quad d \xrightarrow{\mathfrak{u}} d \,. \tag{2.6}$$

Applying this mapping, we see that the three-dimensional representations transform as

$$\mathbf{3} \leftrightarrow \overline{\mathbf{3}}, \quad \mathbf{3}' \leftrightarrow \overline{\mathbf{3}}' \,. \tag{2.7}$$

The other representations are not changed. This transformation is a symmetry of the character table shown in Table 8, if we exchange the conjugacy classes $3C_4 \leftrightarrow 3C_4'$ and $12C_8 \leftrightarrow 12C_8'$. As a result, the non-trivial CP transformation of $\Delta(96)$ has to be a representation of $\mathfrak{u}$ in the sense of Eq. (2.1), i.e.

$$X(\mathfrak{u})\rho^*(g)X^{-1}(\mathfrak{u}) = \rho(\mathfrak{u}(g)) \,. \tag{2.8}$$

Notice that it is sufficient to only impose the consistency equation on the group's generators for discrete family symmetry:

$$\begin{aligned} X(\mathfrak{u})\rho^*(S)X^{-1}(\mathfrak{u}) &= \rho(\mathfrak{u}(S)) = \rho(S), \\ X(\mathfrak{u})\rho^*(T)X^{-1}(\mathfrak{u}) &= \rho(\mathfrak{u}(T)) = \rho(T^2), \\ X(\mathfrak{u})\rho^*(U)X^{-1}(\mathfrak{u}) &= \rho(\mathfrak{u}(U)) = \rho(U) \,, \end{aligned} \tag{2.9}$$

For the two-dimensional representation $\mathbf{2}$, we have

$$\rho^*_{\mathbf{2}}(S) = \rho_{\mathbf{2}}(S), \quad \rho^*_{\mathbf{2}}(T) = \rho_{\mathbf{2}}(T^2), \quad \rho^*_{\mathbf{2}}(U) = \rho_{\mathbf{2}}(U) \,. \tag{2.10}$$

Therefore the corresponding generalized CP transformation is of the form

$$X(\mathfrak{u}) = X_{\mathbf{2}}(\mathfrak{u}) \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,, \tag{2.11}$$

which represents the outer automorphism $\mathfrak{u}$ via $X_{\mathbf{2}}(\mathfrak{u})\rho^*_{\mathbf{2}}(g)X^{-1}_{\mathbf{2}}(\mathfrak{u}) = \rho_{\mathbf{2}}(\mathfrak{u}(g))$. In the same way, for the three-dimensional representations $\mathbf{3}$, $\mathbf{3'}$, $\overline{\mathbf{3}}$, $\overline{\mathbf{3}}'$, $\widetilde{\mathbf{3}}$ and $\widetilde{\mathbf{3}}'$, the CP transformation satisfying the consistency equation Eq.(2.9) is determined to be

$$X_{\mathbf{3}}(\mathfrak{u}) = X_{\mathbf{3'}}(\mathfrak{u}) = X_{\overline{\mathbf{3}}}(\mathfrak{u}) = X_{\overline{\mathbf{3}}'}(\mathfrak{u}) = X_{\widetilde{\mathbf{3}}}(\mathfrak{u}) = X_{\widetilde{\mathbf{3}}'}(\mathfrak{u}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \equiv \mathbb{1}_{3\times 3}\,. \tag{2.12}$$

For the six-dimensional representation $\mathbf{6}$, the associated CP transformation $X_{\mathbf{6}}(\mathfrak{u})$ is

$$X_{\mathbf{6}}(\mathfrak{u}) = \mathbb{1}_{6\times 6}\,. \tag{2.13}$$

with again $X_{\mathbf{6}}(\mathfrak{u})\rho^*_{\mathbf{6}}(S)X^{-1}_{\mathbf{6}}(\mathfrak{u}) = \rho_{\mathbf{6}}(S)$, $X_{\mathbf{6}}(\mathfrak{u})\rho^*_{\mathbf{6}}(T)X^{-1}_{\mathbf{6}}(\mathfrak{u}) = \rho_{\mathbf{6}}(T^2)$ and $X_{\mathbf{6}}(\mathfrak{u})\rho^*_{\mathbf{6}}(U)X^{-1}_{\mathbf{6}}(\mathfrak{u}) = \rho_{\mathbf{6}}(U)$. The set of consistence equations in Eq. (2.9) are trivially satisfied for the one dimensional representations $\mathbf{1}$ and $\mathbf{1'}$, and we simply take

$$X_{\mathbf{1}}(\mathfrak{u}) = X_{\mathbf{1'}}(\mathfrak{u}) = 1\,. \tag{2.14}$$

Therefore we conclude that the working basis listed in Table 7 is the so-called "CP basis". Including the inner automorphism (the family symmetry transforamtion), the most general CP transformation H_{CP} consistent with $\Delta(96)$ family symmetry is given by

$$\rho_{\mathbf{r}}(h)X_{\mathbf{r}}(\mathfrak{u}) = \rho_{\mathbf{r}}(h), \qquad h \in \Delta(96)\,, \tag{2.15}$$

where h can be any of the 96 group elements of $\Delta(96)$, and $\rho_{\mathbf{r}}(h)$ denotes the representation matrix of h in the irreducible representation $\mathbf{r}$. Hence the generalised CP transformation consistent with the $\Delta(96)$ family symmetry is of the same form as the family group transformation in the chosen basis.

3 Model independent CP analysis of $\Delta(96)$ subgroups

3.1 Subgroups of $\Delta(96)$

In the notation of Appendix A, one finds that $\Delta(96)$ has fifteen Z_2 subgroups, sixteen Z_3 subgroups, seven K_4 subgroups, twelve Z_4 subgroups and six Z_8 subgroups, which in terms of the generators a, b, c and d, can be expressed as follows:

- Z_2 subgroups

$$\begin{array}{llll} Z_2^{(1)} = \{1, c^2\}\,, & Z_2^{(2)} = \{1, d^2\}\,, & Z_2^{(3)} = \{1, c^2d^2\}\,, & Z_2^{(4)} = \{1, ab\}\,, \\ Z_2^{(5)} = \{1, abc\}\,, & Z_2^{(6)} = \{1, abc^2\}\,, & Z_2^{(7)} = \{1, abc^3\}\,, & Z_2^{(8)} = \{1, a^2b\}\,, \\ Z_2^{(9)} = \{1, a^2bd\}\,, & Z_2^{(10)} = \{1, a^2bd^2\}\,, & Z_2^{(11)} = \{1, a^2bd^3\}\,, & Z_2^{(12)} = \{1, b\}\,, \\ Z_2^{(13)} = \{1, bcd\}\,, & Z_2^{(14)} = \{1, bc^2d^2\}\,, & Z_2^{(15)} = \{1, bc^3d^3\}\,, & \end{array}\,.$$

The first three Z_2 subgroups $Z_2^{(1)}$, $Z_2^{(2)}$ and $Z_2^{(3)}$ are related with each under the group conjugation, and the same holds to be true for the remaining Z_2 subgroups $Z_2^{(4)} \ldots Z_2^{(15)}$.

- Z_3 subgroups

$$\begin{array}{lll}
Z_3^{(1)}=\{1,a,a^2\}, & Z_3^{(2)}=\{1,ac,a^2cd\}, & Z_3^{(3)}=\{1,ac^2,a^2c^2d^2\},\\
Z_3^{(4)}=\{1,ac^3,a^2c^3d^3\}, & Z_3^{(5)}=\{1,ad,a^2c^3\}, & Z_3^{(6)}=\{1,ad^2,a^2c^2\},\\
Z_3^{(7)}=\{1,ad^3,a^2c\}, & Z_3^{(8)}=\{1,acd,a^2d\}, & Z_3^{(9)}=\{1,acd^2,a^2c^3d\},\\
Z_3^{(10)}=\{1,acd^3,a^2c^2d\}, & Z_3^{(11)}=\{1,ac^2d,a^2cd^2\}, & Z_3^{(12)}=\{1,ac^2d^2,a^2d^2\},\\
Z_3^{(13)}=\{1,ac^2d^3,a^2c^3d^2\}, & Z_3^{(14)}=\{1,ac^3d,a^2c^2d^3\}, & Z_3^{(15)}=\{1,ac^3d^2,a^2cd^3\},\\
Z_3^{(16)}=\{1,ac^3d^3,a^2d^3\} & &
\end{array},$$

which can be written compactly as

$$Z_3^{(x,y)}=\left\{1,ac^xd^y,a^2c^{x-y}d^x\right\},\quad x,y=0,1,2,3\,. \tag{3.1}$$

All the above Z_3 subgroups are found to be conjugate to each other.

- K_4 subgroups

$$\begin{array}{lll}
K_4^{(1)}=\{1,c^2,d^2,c^2d^2\}, & K_4^{(2)}=\{1,ab,c^2,abc^2\}, & K_4^{(3)}=\{1,abc,c^2,abc^3\},\\
K_4^{(4)}=\{1,a^2b,d^2,a^2bd^2\}, & K_4^{(5)}=\{1,a^2bd,d^2,a^2bd^3\}, & K_4^{(6)}=\{1,b,c^2d^2,bc^2d^2\},\\
K_4^{(7)}=\{1,bcd,c^2d^2,bc^3d^3\}\,. & &
\end{array}$$

Note that $K_4^{(1)}$ is a normal subgroup of $\Delta(96)$, and the remaining K_4 subgroups are conjugate to each other.

- Z_4 subgroups

$$\begin{array}{lll}
Z_4^{(1)}=\{1,cd^2,c^2,c^3d^2\}, & Z_4^{(2)}=\{1,cd^3,c^2d^2,c^3d\}, & Z_4^{(3)}=\{1,c^2d^3,d^2,c^2d\},\\
Z_4^{(4)}=\{1,c,c^2,c^3\}, & Z_4^{(5)}=\{1,d,d^2,d^3\}, & Z_4^{(6)}=\{1,cd,c^2d^2,c^3d^3\},\\
Z_4^{(7)}=\{1,abd^2,c^2,abc^2d^2\}, & Z_4^{(8)}=\{1,abcd^2,c^2,abc^3d^2\}, & Z_4^{(9)}=\{1,a^2bc^2,d^2,a^2bc^2d^2\},\\
Z_4^{(10)}=\{1,a^2bc^2d,d^2,a^2bc^2d^3\}, & Z_4^{(11)}=\{1,bc^2,c^2d^2,bd^2\}, & Z_4^{(12)}=\{1,bc^3d,c^2d^2,bcd^3\}
\end{array}.$$

The twelve Z_4 subgroups fall into three categories applying similarity transformations belonging to $\Delta(96)$: the first contains Z_4^1, $Z_4^{(2)}$ and $Z_4^{(3)}$, the second one $Z_4^{(4)}$, $Z_4^{(5)}$ and $Z_4^{(6)}$ and the third the others $Z_4^{(7)}\ldots Z_4^{(12)}$. The generating elements of the Z_4 subgroups $Z_4^{(1)}$, $Z_4^{(2)}$ and $Z_4^{(3)}$ have two degenerate eigenvalues, the lepton mixing matrix can not be determined uniquely if the flavor symmetry $\Delta(96)$ in broken to $Z_4^{(1)}$, $Z_4^{(2)}$ or $Z_4^{(3)}$ in the charged lepton sector. As a result, we don't consider these cases in the present work.

- Z_8 subgroups

$$\begin{aligned}
Z_8^{(1)}&=\{1,abd,cd^2,abcd^3,c^2,abc^2d,c^3d^2,abc^3d^3\},\\
Z_8^{(2)}&=\{1,abcd,cd^2,abc^2d^3,c^2,abc^3d,c^3d^2,abd^3\},\\
Z_8^{(3)}&=\{1,a^2bc^3,c^2d^3,a^2bcd^3,d^2,a^2bc^3d^2,c^2d,a^2bcd\},\\
Z_8^{(4)}&=\{1,a^2bc^3d,c^2d^3,a^2bc,d^2,a^2bc^3d^3,c^2d,a^2bcd^2\},\\
Z_8^{(5)}&=\{1,bc,cd^3,bc^2d^3,c^2d^2,bc^3d^2,c^3d,bd\},\\
Z_8^{(6)}&=\{1,bc^2d,cd^3,bc^3,c^2d^2,bd^3,c^3d,bcd^2\}\,.
\end{aligned}$$

All the six Z_8 subgroups are conjugate to each other.

3.2 Leptonic Mixing from Remnant Symmetries

Lepton mixing can be derived from a flavor symmetry G_f and the generalised CP symmetry $H_{\rm CP}$ breaking to remnant symmetries in the charged lepton and neutrino sectors respectively. In concrete models, this is usually achieved via a spontaneous breaking using some scalar fields charged under this symmetry into different subgroups of the full symmetry group. The charge assignments are chosen such that there are different residual symmetries in the charged lepton and neutrino sectors. The misalignment between the two residual symmetries leads to particular predictions for the PMNS matrix. In this method, only the structure of full symmetry group and its remnant symmetries are assumed and we do not need to consider the breaking mechanism, i.e. how the required vacuum alignment achieving the remnant symmetries is dynamically realized. In the following, we assume that the family symmetry G_f is spontaneously broken to the G_ν and G_l subgroups in the neutrino and the charged lepton sector respectively, and the remnant CP symmetries from the breaking of H_{CP} are H^ν_{CP} and H^l_{CP}, respectively. The mismatch between the remnant symmetry groups $G_\nu \rtimes H^\nu_{CP}$ and $G_l \rtimes H^l_{CP}$ gives rise to particular values for both mixing angles and CP phases. As usual, the three generations of the left-handed (LH) lepton doublets are unified into a three-dimensional representation $\mathbf{3}$ of G_f. The same results would be obtained if the lepton doublets were assigned to $\mathbf{3}'$ of $\Delta(96)$, since the representation $\mathbf{3}'$ differs from $\mathbf{3}$ only in the overall sign of the generator U. Furthermore, if the LH lepton doublets are embedded into the $\Delta(96)$ triplets $\overline{\mathbf{3}}$ or $\overline{\mathbf{3}}'$, the following predictions for the lepton mass matrices and the diagonalization matrices would become their complex conjugate. The invariance under the residual family symmetries G_ν and G_l implies that the neutrino mass matrix m_ν and the charged lepton mass matrix m_l satisfy

$$\begin{aligned} &\rho^T_{\mathbf{3}}(g_{\nu_i}) m_\nu \rho_{\mathbf{3}}(g_{\nu_i}) = m_\nu, \quad g_{\nu_i} \in G_\nu, \\ &\rho^\dagger_{\mathbf{3}}(g_{l_i}) m^\dagger_l m_l \rho_{\mathbf{3}}(g_{l_i}) = m^\dagger_l m_l, \quad g_{l_i} \in G_l\,, \end{aligned} \tag{3.2}$$

where the charged lepton mass matrix m_l is given in the so-called right-left convention, $l^c m_l l$, and $\rho_{\mathbf{3}}(g)$ denotes the representation matrix of the element g in the irreducible representation $\mathbf{3}$. Furthermore, both mass matrices m_ν and m_l are constrained by the residual CP symmetry as follows:

$$\begin{aligned} &X^T_{\nu\mathbf{3}} m_\nu X_{\nu\mathbf{3}} = m^*_\nu, \qquad\quad X_{\nu\mathbf{3}} \in H^\nu_{CP}, \\ &X^\dagger_{l\mathbf{3}} m^\dagger_l m_l X_{l\mathbf{3}} = (m^\dagger_l m_l)^*, \quad X_{l\mathbf{3}} \in H^l_{CP}\,. \end{aligned} \tag{3.3}$$

Since the theory still preserves both remnant family symmetry and remnant CP symmetries after symmetry breaking, they have to be compatible with each other, and the corresponding consistency equation should be fulfilled

$$\begin{aligned} &X_{\nu\mathbf{r}} \rho^*_{\mathbf{r}}(g_{\nu_i}) X^{-1}_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(g_{\nu_j}), \qquad g_{\nu_i}, g_{\nu_j} \in G_\nu, \\ &X_{l\mathbf{r}} \rho^*_{\mathbf{r}}(g_{l_i}) X^{-1}_{l\mathbf{r}} = \rho_{\mathbf{r}}(g_{l_j}), \qquad g_{l_i}, g_{l_j} \in G_l\,, \end{aligned} \tag{3.4}$$

where $X_{\nu\mathbf{r}}$ and $X_{l\mathbf{r}}$ are the elements of H^ν_{CP} and H^l_{CP}, respectively. Given a set of solutions $X_{\nu\mathbf{r}}$ and $X_{l\mathbf{r}}$, we can straightforwardly see that $\rho_{\mathbf{r}}(g_{\nu_i}) X_{\nu\mathbf{r}}$ and $\rho_{\mathbf{r}}(g_{l_i}) X_{l\mathbf{r}}$ are also solutions to the above consistency equations in Eq. (3.4). The invariance conditions of Eqs. (3.2)-(3.3) allow us to reconstruct the lepton mass matrices m_ν and $m_l^\dagger m_l$, and ultimately we can determine the lepton mixing matrix U_{PMNS}. Furthermore, if the residual family symmetries are another pair of subgroup G'_ν and G'_l which are conjugate to G_ν and G_l under the action of the group element $h \in G_f$, i.e.

$$G'_\nu = hG_\nu h^{-1}, \qquad G'_l = hG_l h^{-1}\,. \tag{3.5}$$

Then the consistent residual CP symmetries $H_{CP}^{\nu'}$ and $H_{CP}^{l'}$ are related to H_{CP}^{ν} and H_{CP}^{l} by

$$H_{CP}^{\nu'} = \rho_{\mathbf{r}}(h)H_{CP}^{\nu}\rho_{\mathbf{r}}^T(h), \qquad H_{CP}^{l'} = \rho_{\mathbf{r}}(h)H_{CP}^{l}\rho_{\mathbf{r}}^T(h)\,, \tag{3.6}$$

and the resulting neutrino and charged lepton mass matrices are of the form

$$m'_\nu = \rho_{\mathbf{3}}^*(h)m_\nu\rho_{\mathbf{3}}^\dagger(h), \qquad m_l'^\dagger m'_l = \rho_{\mathbf{3}}(h)m_l^\dagger m_l\rho_{\mathbf{3}}^\dagger(h). \tag{3.7}$$

Hence the remnant subgroups G'_ν and G'_l lead to the same predictions for the lepton mixing matrix U_{PMNS} as G_ν, G_l case [18].

Having completed a general discussion of the implementation of a generalised CP symmetry with a family symmetry, we now concentrate on the case of interest in which the family symmetry $G_f = \Delta(96)$ and a generalised CP symmetry $H_{\rm CP}$ consistent with $\Delta(96)$ is imposed. Thus, the theory respects the full symmetry $\Delta(96)\rtimes H_{\rm CP}$. In the following, we shall perform a model independent study of the constraints that these symmetries impose on the neutrino mass matrix, the charged lepton mass matrix and the PMNS matrix by scanning all the possible remnant symmetries $G_{\rm CP}^{\nu}\cong G_\nu\rtimes H_{\rm CP}^{\nu}$ and $G_{\rm CP}^{l}\cong G_l\rtimes H_{\rm CP}^{l}$. Note that all the possible lepton mixing patterns derived from $\Delta(96)$ family symmetry breaking has been completed by one of us in Ref. [30], where the generalised CP symmetry is not imposed. Other related work on $\Delta(96)$ flavor symmetry can be found in Refs. [31,32]. It is sufficient to consider only a small number of representative cases which leads to different results for mixing angles and CP phases, since different choices of G_ν and G_l related by group conjugation generate the same result. We further restrict ourselves to the case of Majorana neutrinos, which implies that the remnant family symmetry G_ν can only be $K_4\cong Z_2\times Z_2$ or Z_2 subgroups. For the case of $G_\nu = K_4$, the lepton flavor mixing is completely fixed as shown in Ref. [30], and seven mass independent textures including the well-known tri-bimaximal, bimaximal and Toorop-Feruglio-Hagedorn (TFH) mixing patterns can be produced. In the following, we shall concentrate on the case of $G_\nu = Z_2$ and generalised CP symmetry is imposed. By considering all the possible family symmetry breaking, we find only six viable cases listed below.

- $G_l = Z_3^{(2)}$, $G_\nu = Z_2^{(2)}$
- $G_l = Z_3^{(2)}$, $G_\nu = Z_2^{(9)}$
- $G_l = Z_3^{(2)}$, $G_\nu = Z_2^{(10)}$
- $G_l = K_4^{(3)}$, $G_\nu = Z_2^{(9)}$

- $G_l = Z_4^{(7)}$, $G_\nu = Z_2^{(9)}$
- $G_l = Z_8^{(1)}$, $G_\nu = Z_2^{(9)}$.

We begin this study with an analysis of the charged lepton sector.

3.3 Charged lepton sector

3.3.1 $G_l = Z_3^{(2)} = \{1, ac, a^2cd\}$

Now the full symmetry $\Delta(96) \rtimes H_{CP}$ is broken down to $G_{CP}^l \cong Z_3^{(2)} \rtimes H_{CP}^l$ in the charged lepton sector. The element $X_{l\mathbf{r}}$ of H_{CP}^l should satisfy the consistency equation

$$X_{l\mathbf{r}}\rho_{\mathbf{r}}^*(ac)X_{l\mathbf{r}}^{-1} = \rho_{\mathbf{r}}(g'), \quad g' = ac, a^2cd\,. \tag{3.8}$$

It is found that the remnant CP transformation H_{CP}^l can be

$$H_{CP}^l = \left\{\rho_{\mathbf{r}}(1), \rho_{\mathbf{r}}(ac), \rho_{\mathbf{r}}(a^2cd), \rho_{\mathbf{r}}(a^2b), \rho_{\mathbf{r}}(bcd), \rho_{\mathbf{r}}(abc)\right\}\,. \tag{3.9}$$

The charged lepton mass matrix m_l must respect both the residual family symmetry $Z_3^{(2)}$ and the generalised CP symmetry H_{CP}^l, i.e.

$$\rho_{\mathbf{3}}^\dagger(ac)m_l^\dagger m_l\rho_{\mathbf{3}}(ac) = m_l^\dagger m_l \tag{3.10a}$$

$$X_{l\mathbf{3}}^\dagger m_l^\dagger m_l X_{l\mathbf{3}} = \left(m_l^\dagger m_l\right)^*\,. \tag{3.10b}$$

Eq. (3.10a) implies that the $m_l^\dagger m_l$ is diagonal, i.e.,

$$m_l^\dagger m_l = \mathrm{diag}(m_e^2, m_\mu^2, m_\tau^2)\,, \tag{3.11}$$

up to permutation of diagonal entries. For $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(1), \rho_{\mathbf{r}}(ac), \rho_{\mathbf{r}}(a^2cd)$, the invariance condition of Eq. (3.10b) is automatically satisfied, and no additional constraints are imposed. For the remaining values of $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(a^2b), \rho_{\mathbf{r}}(bcd), \rho_{\mathbf{r}}(abc)$, the residual CP invariant condition of Eq. (3.10b) implies $m_e = m_\mu$. Hence, this case is not viable phenomenologically.

3.3.2 $G_l = K_4^{(3)} = \{1, abc, c^2, abc^3\}$

The hermitian combination $m_l^\dagger m_l$ is constrained by the remnant family symmetry $K_4^{(3)}$ as

$$\begin{aligned}&\rho_{\mathbf{3}}^\dagger(abc)m_l^\dagger m_l\rho_{\mathbf{3}}(abc) = m_l^\dagger m_l\,,\\ &\rho_{\mathbf{3}}^\dagger(c^2)m_l^\dagger m_l\rho_{\mathbf{3}}(c^2) = m_l^\dagger m_l\,.\end{aligned} \tag{3.12}$$

Then the most general charged lepton mass matrix satisfying these equations is of the form

$$m_l^\dagger m_l = \begin{pmatrix} R_{11} & (1+i\sqrt{3})R_{12} & (1-i\sqrt{3})R_{13} \\ (1-i\sqrt{3})R_{12} & R_{11} & (1+i\sqrt{3})R_{13} \\ (1+i\sqrt{3})R_{13} & (1-i\sqrt{3})R_{13} & R_{11}-2R_{12}+2R_{13} \end{pmatrix}\,, \tag{3.13}$$

where R_{11}, R_{12} and R_{13} are real parameters. It is diagonalized by the unitary matrix

$$U_l = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\pi i/3} & \frac{1}{\sqrt{3}}e^{2\pi i/3} & -\frac{1}{\sqrt{6}}e^{2\pi i/3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}}e^{\pi i/3} & \frac{1}{\sqrt{6}}e^{\pi i/3} \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \end{pmatrix}, \tag{3.14}$$

with

$$U_l^\dagger m_l^\dagger m_l U_l = \mathrm{diag}\left(R_{11}+2R_{12}, R_{11}-2R_{12}-2R_{13}, R_{11}-2R_{12}+4R_{13}\right). \tag{3.15}$$

Note that the unitary matrix U_l is determined up permutations of columns and phases of its column vectors, because the the charged lepton masses can not be predicted in the present approach. The same comment applies to the following cases with different remnant symmetry in the charged lepton sector. The mass matrix $m_l^\dagger m_l$ of Eq. (3.13) also respects the residual CP symmetry H^l_{CP}, which is determined by the so-called consistency equation:

$$X_{l\mathbf{r}}\rho^*_{\mathbf{r}}(abc)X^{-1}_{l\mathbf{r}} = \rho_{\mathbf{r}}(g'_1), \qquad X_{l\mathbf{r}}\rho^*_{\mathbf{r}}(c^2)X^{-1}_{l\mathbf{r}} = \rho_{\mathbf{r}}(g'_2), \qquad g'_1, g'_2 \in K_4^{(3)}. \tag{3.16}$$

By considering all possible values for g'_1 and g'_2, we find that only 16 of the 96 non-trivial CP transformations are acceptable,

$$\begin{aligned} H^l_{CP} = \Big\{&\rho_{\mathbf{r}}(a^2cd), \rho_{\mathbf{r}}(a^2b), \rho_{\mathbf{r}}(a^2c^3d^3), \rho_{\mathbf{r}}(a^2bc^2d^2), \rho_{\mathbf{r}}(a^2c^2), \rho_{\mathbf{r}}(a^2bcd^3), \rho_{\mathbf{r}}(a^2d^2), \rho_{\mathbf{r}}(a^2bc^3d), \\ &\rho_{\mathbf{r}}(a^2), \rho_{\mathbf{r}}(a^2bc^3d^3), \rho_{\mathbf{r}}(a^2c^2d^2), \rho_{\mathbf{r}}(a^2bcd), \rho_{\mathbf{r}}(a^2bd^2), \rho_{\mathbf{r}}(a^2cd^3), \rho_{\mathbf{r}}(a^2bc^2), \rho_{\mathbf{r}}(a^2c^3d)\Big\}. \end{aligned} \tag{3.17}$$

The invariance under the action of H^l_{CP} yields

$$X^\dagger_{l\mathbf{3}} m_l^\dagger m_l X_{l\mathbf{3}} = \left(m_l^\dagger m_l\right)^*, \tag{3.18}$$

which further constrains the charged lepton mass matrix $m_l^\dagger m_l$ of Eq. (3.13) for different preserved CP transformations. We find that the 16 elements of H^l_{CP} can be divided into two classes. For the case of $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(a^2cd)$, $\rho_{\mathbf{r}}(a^2b)$, $\rho_{\mathbf{r}}(a^2c^3d^3)$, $\rho_{\mathbf{r}}(a^2bc^2d^2)$, $\rho_{\mathbf{r}}(a^2c^2)$, $\rho_{\mathbf{r}}(a^2bcd^3)$, $\rho_{\mathbf{r}}(a^2d^2)$, $\rho_{\mathbf{r}}(a^2bc^3d)$, the remnant CP invariance condition of Eq. (3.18) is satisfied, and no new constraint is generated. For the remaining case of $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(a^2)$, $\rho_{\mathbf{r}}(a^2bc^3d^3)$, $\rho_{\mathbf{r}}(a^2c^2d^2)$, $\rho_{\mathbf{r}}(a^2bcd)$, $\rho_{\mathbf{r}}(a^2bd^2)$, $\rho_{\mathbf{r}}(a^2cd^3)$, $\rho_{\mathbf{r}}(a^2bc^2)$, $\rho_{\mathbf{r}}(a^2c^3d)$, the residual CP invariant condition of Eq. (3.18) leads to the constraint $R_{12} = R_{13}$. As a result, the charged lepton masses are predicted to be partially degenerate with $m_e = m_\tau$. Hence this case is not viable phenomenologically.

3.3.3 $G_l = Z_4^{(7)} = \{1, abd^2, c^2, abc^2d^2\}$

The underlying symmetry $\Delta(96) \rtimes G_{CP}$ is broken down to $G^l_{CP} \cong Z_4^{(7)} \rtimes H^l_{CP}$ in this case, and the element $X_{l\mathbf{r}}$ of H^l_{CP} fulfill

$$X_{l\mathbf{r}}\rho^*_{\mathbf{r}}(abd^2)X^{-1}_{l\mathbf{r}} = \rho_{\mathbf{r}}(g'), \qquad \text{with} \;\; g' = abd^2, abc^2d^2. \tag{3.19}$$

It is easy to check that the remnant CP symmetry H^l_{CP} can take the value

$$H^l_{CP} = \Big\{\rho_{\mathbf{r}}(a^2bd^2), \rho_{\mathbf{r}}(a^2), \rho_{\mathbf{r}}(a^2bc^2), \rho_{\mathbf{r}}(a^2c^2d^2), \rho_{\mathbf{r}}(a^2c^3d), \rho_{\mathbf{r}}(a^2bcd), \rho_{\mathbf{r}}(a^2cd^3), \rho_{\mathbf{r}}(a^2bc^3d^3), \rho_{\mathbf{r}}(a^2cd), \rho_{\mathbf{r}}(a^2bc^3d), \rho_{\mathbf{r}}(a^2c^3d^3), \rho_{\mathbf{r}}(a^2bcd^3), \rho_{\mathbf{r}}(a^2b), \rho_{\mathbf{r}}(a^2d^2), \rho_{\mathbf{r}}(a^2bc^2d^2), \rho_{\mathbf{r}}(a^2c^2)\Big\}\,. \quad (3.20)$$

The charged lepton mass matrix m_l respects both the residual family symmetry $Z_4^{(7)}$ and the generalised CP symmetry H^l_{CP}, i.e.

$$\rho^\dagger_{\mathbf{3}}(abd^2)m^\dagger_l m_l \rho_{\mathbf{3}}(abd^2) = m^\dagger_l m_l\,, \quad (3.21a)$$

$$X^\dagger_{l\mathbf{3}} m^\dagger_l m_l X_{l\mathbf{3}} = \left(m^\dagger_l m_l\right)^*\,, \quad (3.21b)$$

where Eq. (3.21a) is the invariance condition under $Z_4^{(7)}$, and it constrains the charged lepton mass matrix to take the following form

$$m^\dagger_l m_l = \begin{pmatrix} R_{11} & (1+i\sqrt{3})R_{12} & (1-i\sqrt{3})R_{13} \\ (1-i\sqrt{3})R_{12} & R_{11} & (1+i\sqrt{3})R_{13} \\ (1+i\sqrt{3})R_{13} & (1-i\sqrt{3})R_{13} & R_{11}-2R_{12}+2R_{13} \end{pmatrix}\,, \quad (3.22)$$

where R_{11}, R_{12} and R_{13} are real. The unitary matrix U_l which diagonalizes $m^\dagger_l m_l$ is then of the form

$$U_l = \begin{pmatrix} \frac{1}{\sqrt{2}}e^{\pi i/3} & \frac{1}{\sqrt{3}}e^{2\pi i/3} & -\frac{1}{\sqrt{6}}e^{2\pi i/3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}}e^{\pi i/3} & \frac{1}{\sqrt{6}}e^{\pi i/3} \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \end{pmatrix}\,, \quad (3.23)$$

with

$$U^\dagger_l m^\dagger_l m_l U_l = \mathrm{diag}\left(R_{11}+2R_{12}, R_{11}-2R_{12}-2R_{13}, R_{11}-2R_{12}+4R_{13}\right)\,. \quad (3.24)$$

The charged lepton mass matrix is further constrained by the residual CP symmetry as Eq. (3.21b). For the values of $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(a^2cd)$, $\rho_{\mathbf{r}}(a^2bc^3d)$, $\rho_{\mathbf{r}}(a^2c^3d^3)$, $\rho_{\mathbf{r}}(a^2bcd^3)$, $\rho_{\mathbf{r}}(a^2b)$, $\rho_{\mathbf{r}}(a^2d^2)$, $\rho_{\mathbf{r}}(a^2bc^2d^2)$, $\rho_{\mathbf{r}}(a^2c^2)$, it is easy to check that $m^\dagger_l m_l$ of Eq. (3.22) respects the CP invariant condition of Eq. (3.21b), and no new constraints are introduced. For the case of $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(a^2bd^2)$, $\rho_{\mathbf{r}}(a^2)$, $\rho_{\mathbf{r}}(a^2bc^2)$, $\rho_{\mathbf{r}}(a^2c^2d^2)$, $\rho_{\mathbf{r}}(a^2c^3d)$, $\rho_{\mathbf{r}}(a^2bcd)$, $\rho_{\mathbf{r}}(a^2cd^3)$, $\rho_{\mathbf{r}}(a^2bc^3d^3)$, the equality $R_{12} = R_{13}$ is required to be fulfilled. As a consequence, the degeneracy $m_e = m_\tau$ arises, and therefore this case is not viable. Comparing the charged lepton mass matrix $m^\dagger_l m_l$ predicted in Eq. (3.13) and Eq. (3.22), we see that the remnant family symmetries $G_l = K_4^{(3)}$ and $G_l = Z_4^{(7)}$ lead to the same constraints on the charged lepton mass, and hence the diagonalization matrix U_l and the charged lepton masses are predicted to be of the same forms in both cases.

3.3.4 $G_l = Z_8^{(1)} = \{1, abd, cd^2, abcd^3, c^2, abc^2d, c^3d^2, abc^3d^3\}$

The remnant family symmetry $G_l = Z_8^{(1)}$ imposes the following constraint on the charged lepton mass matrix:

$$\rho^\dagger_{\mathbf{3}}(abd)m^\dagger_l m_l \rho_{\mathbf{3}}(abd) = m^\dagger_l m_l\,. \quad (3.25)$$

The the mass matrix $m_l^\dagger m_l$ is determined to be of the form

$$m_l^\dagger m_l = \begin{pmatrix} R_{11} & (1+i\sqrt{3})R_{12} & (1-i\sqrt{3})R_{13} \\ (1-i\sqrt{3})R_{12} & R_{11}-2(1-\sqrt{3})(R_{12}-R_{13}) & -(1-\sqrt{3})(1+i\sqrt{3})R_{12}+(2-\sqrt{3})(1+i\sqrt{3})R_{13} \\ (1+i\sqrt{3})R_{13} & -(1-\sqrt{3})(1-i\sqrt{3})R_{12}+(2-\sqrt{3})(1-i\sqrt{3})R_{13} & R_{11}-2(2-\sqrt{3})(R_{12}-R_{13}) \end{pmatrix}, \tag{3.26}$$

where R_{11}, R_{12} and R_{13} are real parameters. It is diagonalized by the following unitary transformation U_l

$$\begin{aligned} U_l &= \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{4+\sqrt{2}-\sqrt{6}}\, e^{2\pi i/3} & 2e^{2\pi i/3} & -\sqrt{4-\sqrt{2}+\sqrt{6}}\, e^{2\pi i/3} \\ \sqrt{4+\sqrt{2}+\sqrt{6}}\, e^{\pi i/3} & -2e^{\pi i/3} & \sqrt{4-\sqrt{2}-\sqrt{6}}\, e^{\pi i/3} \\ \sqrt{4-2\sqrt{2}} & 2 & \sqrt{4+2\sqrt{2}} \end{pmatrix} \\ &= \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2}\, e^{\frac{2i\pi}{3}} \sin\frac{5\pi}{24} & e^{\frac{2i\pi}{3}} & -\sqrt{2}\, e^{\frac{2i\pi}{3}} \cos\frac{5\pi}{24} \\ \sqrt{2}\, e^{\frac{i\pi}{3}} \cos\frac{\pi}{24} & -\, e^{\frac{i\pi}{3}} & \sqrt{2}\, e^{\frac{i\pi}{3}} \sin\frac{\pi}{24} \\ \sqrt{2}\, \sin\frac{\pi}{8} & 1 & \sqrt{2}\, \cos\frac{\pi}{8} \end{pmatrix}, \end{aligned} \tag{3.27}$$

with

$$U_l^\dagger m_l^\dagger m_l U_l = \text{daig}\left(m_e^2, m_\mu^2, m_\tau^2\right), \tag{3.28}$$

where

$$\begin{aligned} m_e^2 &= R_{11} + (-2+3\sqrt{2}+2\sqrt{3}-\sqrt{6})R_{12} + (4-3\sqrt{2}-2\sqrt{3}+\sqrt{6})R_{13} \\ m_\mu^2 &= R_{11} - 2R_{12} - 2R_{13} \\ m_\tau^2 &= R_{11} + (-2-3\sqrt{2}+2\sqrt{3}+\sqrt{6})R_{12} + (4+3\sqrt{2}-2\sqrt{3}-\sqrt{6})R_{13}\,. \end{aligned} \tag{3.29}$$

The mass matrix $m_l^\dagger m_l$ also respects the CP symmetry H_{CP}^l which should be compatible with the remnant family symmetry $G_l = Z_8^{(1)}$, i.e.,

$$X_{l\mathbf{r}} \rho_{\mathbf{r}}^*(abd) X_{l\mathbf{r}}^{-1} = \rho_{\mathbf{r}}(g'), \qquad g' \in Z_8^{(1)}\,. \tag{3.30}$$

One can straightforwardly obtain that there are 16 possible choices for $X_{l\mathbf{r}}$,

$$\begin{aligned} H_{CP}^l = \Big\{ &\rho_{\mathbf{r}}(a^2bc^2d), \rho_{\mathbf{r}}(a^2c^2), \rho_{\mathbf{r}}(a^2bc^3), \rho_{\mathbf{r}}(a^2c^3d^3), \rho_{\mathbf{r}}(a^2bd^3), \rho_{\mathbf{r}}(a^2d^2), \rho_{\mathbf{r}}(a^2bcd^2), \rho_{\mathbf{r}}(a^2cd), \\ &\rho_{\mathbf{r}}(a^2bc^2d^3), \rho_{\mathbf{r}}(a^2c^2d^2), \rho_{\mathbf{r}}(a^2bc^3d^2), \rho_{\mathbf{r}}(a^2c^3d), \rho_{\mathbf{r}}(a^2bd), \rho_{\mathbf{r}}(a^2), \rho_{\mathbf{r}}(a^2bc), \rho_{\mathbf{r}}(a^2cd^3) \Big\}\,. \end{aligned} \tag{3.31}$$

The invariance under the action of the remnant CP symmetry H_{CP}^l implies that

$$X_{l\mathbf{3}}^\dagger m_l^\dagger m_l X_{l\mathbf{3}} = \left(m_l^\dagger m_l\right)^*\,. \tag{3.32}$$

We find no additional constraint for $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(a^2bc^2d)$, $\rho_{\mathbf{r}}(a^2c^2)$, $\rho_{\mathbf{r}}(a^2bc^3)$, $\rho_{\mathbf{r}}(a^2c^3d^3)$, $\rho_{\mathbf{r}}(a^2bd^3)$, $\rho_{\mathbf{r}}(a^2d^2)$, $\rho_{\mathbf{r}}(a^2bcd^2)$, $\rho_{\mathbf{r}}(a^2cd)$, while $X_{l\mathbf{r}} = \rho_{\mathbf{r}}(a^2bc^2d^3)$, $\rho_{\mathbf{r}}(a^2c^2d^2)$, $\rho_{\mathbf{r}}(a^2bc^3d^2)$, $\rho_{\mathbf{r}}(a^2c^3d)$, $\rho_{\mathbf{r}}(a^2bd)$, $\rho_{\mathbf{r}}(a^2)$, $\rho_{\mathbf{r}}(a^2bc)$, $\rho_{\mathbf{r}}(a^2cd^3)$ leads to $R_{12} = R_{13}$ such that the mass degeneracy $m_e = m_\tau$ follows.

3.4 Neutrino sector

3.4.1 $G_\nu = Z_2^{(2)} = \{1, d^2\}$

The symmetry $\Delta(96) \rtimes H_{CP}$ is spontaneously broken to $G_{CP}^\nu = Z_2^{(2)} \times H_{CP}^\nu$ in the neutrino sector. The residual CP symmetry H_{CP}^ν should be consistent with the residual family symmetry $G_\nu = Z_2^{(2)}$, and therefore its element $X_{\nu\mathbf{r}}$ has to fulfill the consistency equation

$$X_{\nu\mathbf{r}}\rho_{\mathbf{r}}^*(d^2)X_{\nu\mathbf{r}}^{-1} = \rho_{\mathbf{r}}(d^2)\,. \tag{3.33}$$

We find that 32 generalised CP transformations are acceptable,

$$H_{CP}^\nu = \left\{\rho_{\mathbf{r}}(c^m d^n), \rho_{\mathbf{r}}(a^2bc^m d^n)|m, n = 0, 1, 2, 3\right\}\,. \tag{3.34}$$

We can construct the light neutrino mass matrix m_ν from its invariance under both the remnant family symmetry $Z_2^{(2)}$ and the remnant CP symmetry H_{CP}^ν as follows:

$$\rho_{\mathbf{3}}^T(d^2)m_\nu\rho_{\mathbf{3}}(d^2) = m_\nu\,, \tag{3.35a}$$

$$X_{\nu\mathbf{3}}^T m_\nu X_{\nu\mathbf{3}} = m_\nu^*\,. \tag{3.35b}$$

The most general neutrino mass matrix satisfying Eq. (3.35a) is of the form

$$m_\nu = \alpha\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \beta\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 1 \end{pmatrix} + \gamma\begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \\ 2 & 0 & 1 \end{pmatrix} + \delta\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\,, \tag{3.36}$$

where α, β, γ and δ are complex parameters, and they are further constrained by the remnant CP symmetry as shown in Eq. (3.35b). In order to diagonalize this neutrino mass matrix, it is useful to first perform a THF transformation U_{THF} and yield

$$m_\nu' = U_{TFH}^T m_\nu U_{TFH} = \begin{pmatrix} 3\alpha + \sqrt{3}\,(\beta-\gamma) & 0 & \delta \\ 0 & 3\beta + 3\gamma + \delta & 0 \\ \delta & 0 & 3\alpha - \sqrt{3}\,(\beta-\gamma) \end{pmatrix}\,, \tag{3.37}$$

where

$$U_{TFH} = \frac{1}{6}\begin{pmatrix} -3-\sqrt{3} & 2\sqrt{3} & 3-\sqrt{3} \\ 3-\sqrt{3} & 2\sqrt{3} & -3-\sqrt{3} \\ 2\sqrt{3} & 2\sqrt{3} & 2\sqrt{3} \end{pmatrix}\,. \tag{3.38}$$

m_ν' can be further diagonalized by a unitary matrix U_ν' as

$$U_\nu'^T m_\nu' U_\nu' = \mathrm{diag}(m_1, m_2, m_3)\,. \tag{3.39}$$

As we shall show in the following, U_ν' can be written into the form

$$U_\nu' = \mathbb{U}R(\theta)P\,, \tag{3.40}$$

where $\mathbb{U}$ is a constant unitary matrix such that $\mathbb{U}^T m_\nu' \mathbb{U}$ becomes a real matrix, and $R(\theta)$ is a rotation matrix in the (1,3) sector with

$$R(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}\,. \tag{3.41}$$

Finally, the unitary matrix P is diagonal with entries ± 1 and $\pm i$ which encode the CP parity of the neutrino states and renders the light neutrino masses $m_{1,2,3}$ positive. Hence the neutrino mass matrix m_ν in Eq. (3.36) is diagonalized by the the unitary matrix U_ν as

$$U_\nu^T m_\nu U_\nu = \text{diag}(m_1, m_2, m_3)\,, \tag{3.42}$$

with

$$U_\nu = U_{TFH}\mathbb{U}R(\theta)P\,. \tag{3.43}$$

Notice that the neutrino diagonalization matrix U_ν is fixed up to permutations of the columns, since neutrino masses are unconstrained in the present framework. Now we turn to investigate the implication of the remnant CP invariant condition of Eq. (3.35b). The predictions for the neutrino diagonalization matrix U_ν and the light neutrino masses would be presented for different residual CP transformations in Eq. (3.34).

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(1), \rho_{\mathbf{r}}(d^2)$

  In this case, the residual CP invariant requirement of Eq. (3.35b) leads to the constraint

$$\text{Im}\alpha = \text{Im}\beta = \text{Im}\gamma = \text{Im}\delta = 0\,, \tag{3.44}$$

  where "Im" denotes the imaginary part, and hence all the four parameters α, β, γ and δ are real. The unitary transformation $\mathbb{U}$ is a unit matrix, i.e.

$$\mathbb{U} = \mathbb{1}_{3\times 3}\,. \tag{3.45}$$

  Therefore the neutrino diagonalization matrix U_ν is

$$U_\nu = \frac{1}{\sqrt{3}}\begin{pmatrix} -\sqrt{2}\cos\left(\frac{\pi}{12}-\theta\right) & 1 & \sqrt{2}\sin\left(\frac{\pi}{12}-\theta\right) \\ \sqrt{2}\sin\left(\frac{\pi}{12}+\theta\right) & 1 & -\sqrt{2}\cos\left(\frac{\pi}{12}+\theta\right) \\ \sqrt{2}\cos\left(\frac{\pi}{4}+\theta\right) & 1 & \sqrt{2}\sin\left(\frac{\pi}{4}+\theta\right) \end{pmatrix} P\,, \tag{3.46}$$

  where the rotation angle θ is determined to be

$$\tan 2\theta = \frac{\text{Re}\delta}{\sqrt{3}\left(\text{Re}\gamma - \text{Re}\beta\right)}\,. \tag{3.47}$$

  Finally the light neutrino masses are

$$\begin{aligned} m_1 &= \left|3\text{Re}\alpha + \text{sign}\left((\text{Re}\beta - \text{Re}\gamma)\cos 2\theta\right)\sqrt{3\left(\text{Re}\beta - \text{Re}\gamma\right)^2 + \left(\text{Re}\delta\right)^2}\right|\,, \\ m_2 &= \left|3\text{Re}\beta + 3\text{Re}\gamma + \text{Re}\delta\right|\,, \\ m_3 &= \left|3\text{Re}\alpha - \text{sign}\left((\text{Re}\beta - \text{Re}\gamma)\cos 2\theta\right)\sqrt{3\left(\text{Re}\beta - \text{Re}\gamma\right)^2 + \left(\text{Re}\delta\right)^2}\right|\,. \end{aligned} \tag{3.48}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^2d), \rho_{\mathbf{r}}(c^2d^3)$

  In this case, the parameters α, β, γ and δ are constrained to satisfy

$$\text{Re}\alpha = \text{Re}\delta = 0, \quad \text{Re}\gamma = \text{Re}\beta, \quad \text{Im}\delta = -3\left(\text{Im}\beta + \text{Im}\gamma\right)\,, \tag{3.49}$$

where "Re" denotes the real part. We find that the unitary transformation is given by

$$\mathbb{U} = \mathrm{diag}(e^{i\pi/4}, 1, e^{i\pi/4})\,. \tag{3.50}$$

The resulting neutrino diagonalization U_ν matrix reads

$$U_\nu = \frac{1}{\sqrt{3}}\begin{pmatrix} -\sqrt{2}\,e^{i\pi/4}\cos\left(\frac{\pi}{12}-\theta\right) & 1 & \sqrt{2}\,e^{i\pi/4}\sin\left(\frac{\pi}{12}-\theta\right) \\ \sqrt{2}\,e^{i\pi/4}\sin\left(\frac{\pi}{12}+\theta\right) & 1 & -\sqrt{2}\,e^{i\pi/4}\cos\left(\frac{\pi}{12}+\theta\right) \\ \sqrt{2}\,e^{i\pi/4}\cos\left(\frac{\pi}{4}+\theta\right) & 1 & \sqrt{2}\,e^{i\pi/4}\sin\left(\frac{\pi}{4}+\theta\right) \end{pmatrix}, \tag{3.51}$$

where the trivial phase matrix P has been omitted here, and it would be neglected as well in the following cases. The angle θ is

$$\tan 2\theta = \frac{\sqrt{3}\,(\mathrm{Im}\beta + \mathrm{Im}\gamma)}{\mathrm{Im}\beta - \mathrm{Im}\gamma}\,. \tag{3.52}$$

The light neutrino mass are predicted to be

$$\begin{aligned} m_1 &= \left|3\mathrm{Im}\alpha + \mathrm{sign}\left((\mathrm{Im}\beta - \mathrm{Im}\gamma)\cos 2\theta\right) 2\sqrt{3\,(\mathrm{Im}\alpha)^2 + 3\mathrm{Im}\beta\mathrm{Im}\gamma + 3\,(\mathrm{Im}\gamma)^2}\right|, \\ m_2 &= 6\,|\mathrm{Re}\beta|\,, \\ m_3 &= \left|3\mathrm{Im}\alpha - \mathrm{sign}\left((\mathrm{Im}\beta - \mathrm{Im}\gamma)\cos 2\theta\right) 2\sqrt{3\,(\mathrm{Im}\alpha)^2 + 3\mathrm{Im}\beta\mathrm{Im}\gamma + 3\,(\mathrm{Im}\gamma)^2}\right|. \end{aligned} \tag{3.53}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(a^2b), \rho_{\mathbf{r}}(a^2bd^2)$

  This residual CP symmetry implies that

$$\mathrm{Re}\gamma = \mathrm{Re}\beta, \quad \mathrm{Im}\alpha = \mathrm{Im}\delta = 0, \quad \mathrm{Im}\gamma = -\mathrm{Im}\beta\,. \tag{3.54}$$

  The unitary transformation $\mathbb{U}$ takes the form

$$\mathbb{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & i \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & i \end{pmatrix}. \tag{3.55}$$

  The light neutrino mass matrix is diagonalized by

$$U_\nu = \frac{1}{\sqrt{6}}\begin{pmatrix} -\sqrt{3}\cos\theta + i\sin\theta & \sqrt{2} & -\sqrt{3}\sin\theta - i\cos\theta \\ \sqrt{3}\cos\theta + i\sin\theta & \sqrt{2} & \sqrt{3}\sin\theta - i\cos\theta \\ -2i\sin\theta & \sqrt{2} & 2i\cos\theta \end{pmatrix}, \tag{3.56}$$

  with

$$\tan 2\theta = \frac{2\mathrm{Im}\beta}{\sqrt{3}\,\mathrm{Re}\alpha}\,. \tag{3.57}$$

  The light neutrino masses are given by

$$\begin{aligned} m_1 &= \left|\mathrm{Re}\delta - \mathrm{sign}\left(\mathrm{Re}\alpha\cos 2\theta\right)\sqrt{9\,(\mathrm{Re}\alpha)^2 + 12\,(\mathrm{Im}\beta)^2}\right|, \\ m_1 &= |6\mathrm{Re}\beta + \mathrm{Re}\delta|\,, \\ m_3 &= \left|\mathrm{Re}\delta + \mathrm{sign}\left(\mathrm{Re}\alpha\cos 2\theta\right)\sqrt{9\,(\mathrm{Re}\alpha)^2 + 12\,(\mathrm{Im}\beta)^2}\right|. \end{aligned} \tag{3.58}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(a^2bd), \rho_{\mathbf{r}}(a^2bd^3)$

  The invariance of the neutrino mass matrix under the residual CP transformation leads to

$$\mathrm{Re}\delta = \mathrm{Im}\alpha = 0, \quad \mathrm{Im}\gamma = \mathrm{Im}\beta, \quad \mathrm{Im}\delta = -6\mathrm{Im}\beta\,. \tag{3.59}$$

  The unitary transformation $\mathbb{U}$ is of the form

$$\mathbb{U} = \mathrm{diag}\left(1, 1, i\right)\,. \tag{3.60}$$

  The corresponding neutrino diagonalization matrix is given by

$$U_\nu = \frac{1}{2\sqrt{3}} \begin{pmatrix} -\sqrt{3}\,e^{i\theta} - e^{-i\theta} & 2 & i\left(\sqrt{3}\,e^{i\theta} - e^{-i\theta}\right) \\ \sqrt{3}\,e^{i\theta} - e^{-i\theta} & 2 & -i\left(\sqrt{3}\,e^{i\theta} + e^{-i\theta}\right) \\ 2e^{-i\theta} & 2 & 2ie^{-i\theta} \end{pmatrix}\,, \tag{3.61}$$

  where the rotation angle θ is

$$\tan 2\theta = -\frac{2\mathrm{Im}\beta}{\mathrm{Re}\alpha}\,. \tag{3.62}$$

  The light neutrino masses in this cases are

$$\begin{aligned} m_1 &= \left|\sqrt{3}\left(\mathrm{Re}\beta - \mathrm{Re}\gamma\right) + 3\mathrm{sign}\left(\mathrm{Re}\alpha\cos 2\theta\right)\sqrt{\left(\mathrm{Re}\alpha\right)^2 + 4\left(\mathrm{Im}\beta\right)^2}\right|\,, \\ m_2 &= 3\left|\mathrm{Re}\beta + \mathrm{Re}\gamma\right|\,, \\ m_3 &= \left|\sqrt{3}\left(\mathrm{Re}\beta - \mathrm{Re}\gamma\right) - 3\mathrm{sign}\left(\mathrm{Re}\alpha\cos 2\theta\right)\sqrt{\left(\mathrm{Re}\alpha\right)^2 + 4\left(\mathrm{Im}\beta\right)^2}\right|\,. \end{aligned} \tag{3.63}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(a^2bc), \rho_{\mathbf{r}}(a^2bcd^2)$

  The parameter α, β, γ and δ are constrained to satisfy

$$\begin{aligned} &\mathrm{Re}\delta = -3\left(\mathrm{Re}\beta + \mathrm{Re}\gamma\right), \quad \mathrm{Im}\alpha = \mathrm{Re}\alpha\,, \\ &\mathrm{Im}\gamma = \mathrm{Im}\beta - \sqrt{3}\,\left(\mathrm{Re}\beta + \mathrm{Re}\gamma\right), \quad \mathrm{Im}\delta = -\sqrt{3}\,\left(\mathrm{Re}\beta - \mathrm{Re}\gamma\right)\,. \end{aligned} \tag{3.64}$$

  The unitary transformation $\mathbb{U}$ takes the form

$$\mathbb{U} = \begin{pmatrix} e^{\frac{7i\pi}{8}}\cos\frac{\pi}{8} & 0 & e^{\frac{3i\pi}{8}}\sin\frac{\pi}{8} \\ 0 & e^{\frac{i\pi}{4}} & 0 \\ -e^{\frac{7i\pi}{8}}\sin\frac{\pi}{8} & 0 & e^{\frac{3i\pi}{8}}\cos\frac{\pi}{8} \end{pmatrix}\,. \tag{3.65}$$

  Therefore the neutrino diagonalization matrix U_ν is

$$U_\nu = \frac{1}{\sqrt{3}} \begin{pmatrix} e^{\frac{i\pi}{8}}\cos\left(\frac{\pi}{24} - \theta\right) - e^{\frac{5i\pi}{8}}\cos\left(\frac{\pi}{24} + \theta\right) & e^{\frac{i\pi}{4}} & -e^{\frac{i\pi}{8}}\sin\left(\frac{\pi}{24} - \theta\right) - e^{\frac{5i\pi}{8}}\sin\left(\frac{\pi}{24} + \theta\right) \\ -e^{\frac{i\pi}{8}}\sin\left(\frac{5\pi}{24} - \theta\right) + e^{\frac{5i\pi}{8}}\sin\left(\frac{5\pi}{24} + \theta\right) & e^{\frac{i\pi}{4}} & -e^{\frac{i\pi}{8}}\cos\left(\frac{5\pi}{24} - \theta\right) - e^{\frac{5i\pi}{8}}\cos\left(\frac{5\pi}{24} + \theta\right) \\ -e^{\frac{i\pi}{8}}\sin\left(\frac{\pi}{8} + \theta\right) + e^{\frac{5i\pi}{8}}\sin\left(\frac{\pi}{8} - \theta\right) & e^{\frac{i\pi}{4}} & e^{\frac{i\pi}{8}}\cos\left(\frac{\pi}{8} + \theta\right) + e^{\frac{5i\pi}{8}}\cos\left(\frac{\pi}{8} - \theta\right) \end{pmatrix}\,, \tag{3.66}$$

with

$$\tan 2\theta = \frac{\left(\sqrt{3}-1\right)\text{Re}\beta + \left(\sqrt{3}+1\right)\text{Re}\gamma}{-\sqrt{6}\ \text{Re}\alpha}\,. \tag{3.67}$$

The light neutrino masses are determined to be

$$\begin{aligned}
m_1 &= \left|(3+\sqrt{3})\text{Re}\beta + (3-\sqrt{3})\text{Re}\gamma + \text{sign}\left(\text{Re}\alpha\cos 2\theta\right)\sqrt{18(\text{Re}\alpha)^2 + \left((3-\sqrt{3})\text{Re}\beta + (3+\sqrt{3})\text{Re}\gamma\right)^2}\right|,\\
m_2 &= \left|2\sqrt{3}\left(2\text{Re}\beta + \text{Re}\gamma\right) - 6\text{Im}\beta\right|,\\
m_3 &= \left|(3+\sqrt{3})\text{Re}\beta + (3-\sqrt{3})\text{Re}\gamma - \text{sign}\left(\text{Re}\alpha\cos 2\theta\right)\sqrt{18(\text{Re}\alpha)^2 + \left((3-\sqrt{3})\text{Re}\beta + (3+\sqrt{3})\text{Re}\gamma\right)^2}\right|.
\end{aligned}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(a^2bcd), \rho_{\mathbf{r}}(a^2bcd^3)$

  The invariance under the residual CP symmetry implies that

$$\begin{aligned}
\text{Re}\delta = -3\left(\text{Re}\beta + \text{Re}\gamma\right), \quad \text{Im}\alpha = \text{Re}\alpha\,,\\
\text{Im}\gamma = \text{Im}\beta + \sqrt{3}\left(\text{Re}\beta + \text{Re}\gamma\right), \quad \text{Im}\delta = \sqrt{3}\left(\text{Re}\beta - \text{Re}\gamma\right)\,.
\end{aligned} \tag{3.68}$$

  The unitary transformation $\mathbb{U}$ is found to be

$$\mathbb{U} = \begin{pmatrix} e^{\frac{7i\pi}{8}}\sin\frac{\pi}{8} & 0 & e^{\frac{3i\pi}{8}}\cos\frac{\pi}{8} \\ 0 & e^{\frac{i\pi}{4}} & 0 \\ -e^{\frac{7i\pi}{8}}\cos\frac{\pi}{8} & 0 & e^{\frac{3i\pi}{8}}\sin\frac{\pi}{8} \end{pmatrix}\,. \tag{3.69}$$

  The resulting neutrino diagonalization matrix reads

$$U_\nu = \frac{1}{\sqrt{3}}\begin{pmatrix} e^{\frac{i\pi}{8}}\sin\left(\frac{5\pi}{24}+\theta\right) - e^{\frac{5i\pi}{8}}\sin\left(\frac{5\pi}{24}-\theta\right) & e^{\frac{i\pi}{4}} & -e^{\frac{i\pi}{8}}\cos\left(\frac{5\pi}{24}+\theta\right) - e^{\frac{5i\pi}{8}}\cos\left(\frac{5\pi}{24}-\theta\right) \\ -e^{\frac{i\pi}{8}}\cos\left(\frac{\pi}{24}+\theta\right) + e^{\frac{5i\pi}{8}}\cos\left(\frac{\pi}{24}-\theta\right) & e^{\frac{i\pi}{4}} & -e^{\frac{i\pi}{8}}\sin\left(\frac{\pi}{24}+\theta\right) - e^{\frac{5i\pi}{8}}\sin\left(\frac{\pi}{24}-\theta\right) \\ e^{\frac{i\pi}{8}}\sin\left(\frac{\pi}{8}-\theta\right) - e^{\frac{5i\pi}{8}}\sin\left(\frac{\pi}{8}+\theta\right) & e^{\frac{i\pi}{4}} & e^{\frac{i\pi}{8}}\cos\left(\frac{\pi}{8}-\theta\right) + e^{\frac{5i\pi}{8}}\cos\left(\frac{\pi}{8}+\theta\right) \end{pmatrix}, \tag{3.70}$$

  with

$$\tan 2\theta = \frac{(\sqrt{3}+1)\text{Re}\beta + (\sqrt{3}-1)\text{Re}\gamma}{\sqrt{6}\ \text{Re}\alpha}\,. \tag{3.71}$$

  The light neutrino masses are given by

$$\begin{aligned}
m_1 &= \left|(3-\sqrt{3})\text{Re}\beta + (3+\sqrt{3})\text{Re}\gamma + \text{sign}\left(\text{Re}\alpha\cos 2\theta\right)\sqrt{18(\text{Re}\alpha)^2 + \left((3+\sqrt{3})\text{Re}\beta + (3-\sqrt{3})\text{Re}\gamma\right)^2}\right|,\\
m_2 &= \left|2\sqrt{3}\left(2\text{Re}\beta + \text{Re}\gamma\right) + 6\text{Im}\beta\right|,\\
m_3 &= \left|(3-\sqrt{3})\text{Re}\beta + (3+\sqrt{3})\text{Re}\gamma - \text{sign}\left(\text{Re}\alpha\cos 2\theta\right)\sqrt{18(\text{Re}\alpha)^2 + \left((3+\sqrt{3})\text{Re}\beta + (3-\sqrt{3})\text{Re}\gamma\right)^2}\right|.
\end{aligned}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(a^2bc^2), \rho_{\mathbf{r}}(a^2bc^2d^2)$

  In this case, the parameters α, β, γ and δ are constrained to satisfy

$$\text{Re}\alpha = \text{Im}\delta = 0, \quad \text{Re}\gamma = \text{Re}\beta, \quad \text{Im}\gamma = -\text{Im}\beta \,. \tag{3.72}$$

  The unitary transformation $\mathbb{U}$ is given by

$$\mathbb{U} = \begin{pmatrix} e^{\frac{i\pi}{4}} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\frac{3i\pi}{4}} \end{pmatrix} . \tag{3.73}$$

  Hence the light neutrino mass matrix is diagonalized by

$$U_\nu = \frac{1}{2\sqrt{3}} \begin{pmatrix} -e^{\frac{i\pi}{4}} \left(\sqrt{3}\, e^{i\theta} + e^{-i\theta}\right) & 2 & e^{\frac{3i\pi}{4}} \left(\sqrt{3}\, e^{i\theta} - e^{-i\theta}\right) \\ e^{\frac{i\pi}{4}} \left(\sqrt{3}\, e^{i\theta} - e^{-i\theta}\right) & 2 & -e^{\frac{3i\pi}{4}} \left(\sqrt{3}\, e^{i\theta} + e^{-i\theta}\right) \\ 2e^{i\left(\frac{\pi}{4}-\theta\right)} & 2 & 2e^{i\left(\frac{3\pi}{4}-\theta\right)} \end{pmatrix} , \tag{3.74}$$

  where

$$\tan 2\theta = -\frac{\text{Re}\delta}{3\text{Im}\alpha} \,. \tag{3.75}$$

  The light neutrino masses are

$$\begin{aligned} m_1 &= \left| 2\sqrt{3}\, \text{Im}\beta + \text{sign}(\text{Im}\alpha \cos 2\theta) \sqrt{9\left(\text{Im}\alpha\right)^2 + \left(\text{Re}\delta\right)^2} \right| , \\ m_2 &= \left| 6\text{Re}\beta + \text{Re}\delta \right| , \\ m_3 &= \left| 2\sqrt{3}\, \text{Im}\beta - \text{sign}(\text{Im}\alpha \cos 2\theta) \sqrt{9\left(\text{Im}\alpha\right)^2 + \left(\text{Re}\delta\right)^2} \right| . \end{aligned} \tag{3.76}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(a^2bc^2d), \rho_{\mathbf{r}}(a^2bc^2d^3)$

  We find that the following relations should be satisfied in this case,

$$\text{Re}\alpha = \text{Re}\delta = 0, \quad \text{Im}\gamma = \text{Im}\beta, \quad \text{Im}\delta = -6\text{Im}\beta \,. \tag{3.77}$$

  The unitary transformation $\mathbb{U}$ is given by

$$\mathbb{U} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-\frac{i\pi}{4}} & 0 & e^{\frac{i\pi}{4}} \\ 0 & \sqrt{2} & 0 \\ -e^{-\frac{i\pi}{4}} & 0 & e^{\frac{i\pi}{4}} \end{pmatrix} . \tag{3.78}$$

  Hence the neutrino diagonalization matrix U_ν is

$$U_\nu = \frac{1}{\sqrt{6}} \begin{pmatrix} e^{\frac{i\pi}{4}} \left(\sin\theta + i\sqrt{3}\cos\theta\right) & \sqrt{2} & -e^{\frac{i\pi}{4}} \left(\cos\theta - i\sqrt{3}\sin\theta\right) \\ e^{\frac{i\pi}{4}} \left(\sin\theta - i\sqrt{3}\cos\theta\right) & \sqrt{2} & -e^{\frac{i\pi}{4}} \left(\cos\theta + i\sqrt{3}\sin\theta\right) \\ -2e^{\frac{i\pi}{4}} \sin\theta & \sqrt{2} & 2e^{\frac{i\pi}{4}} \cos\theta \end{pmatrix} , \tag{3.79}$$

with

$$\tan 2\theta = \frac{\mathrm{Re}\gamma - \mathrm{Re}\beta}{\sqrt{3}\,\mathrm{Im}\alpha}\,. \tag{3.80}$$

The light neutrino masses are

$$\begin{aligned}
m_1 &= \left|6\mathrm{Im}\beta + \mathrm{sign}\left(\mathrm{Im}\alpha\cos 2\theta\right)\sqrt{3\left(\mathrm{Re}\beta - \mathrm{Re}\gamma\right)^2 + 9\left(\mathrm{Im}\alpha\right)^2}\right|\,,\\
m_2 &= 3\left|\mathrm{Re}\beta + \mathrm{Re}\gamma\right|\,,\\
m_3 &= \left|6\mathrm{Im}\beta - \mathrm{sign}\left(\mathrm{Im}\alpha\cos 2\theta\right)\sqrt{3\left(\mathrm{Re}\beta - \mathrm{Re}\gamma\right)^2 + 9\left(\mathrm{Im}\alpha\right)^2}\right|\,.
\end{aligned} \tag{3.81}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(a^2bc^3), \rho_{\mathbf{r}}(a^2bc^3d^2)$

The remnant CP symmetry leads to

$$\begin{gathered}
\mathrm{Re}\delta = -3\left(\mathrm{Re}\beta + \mathrm{Re}\gamma\right), \quad \mathrm{Im}\alpha = -\mathrm{Re}\alpha\\
\mathrm{Im}\gamma = \mathrm{Im}\beta - \sqrt{3}\left(\mathrm{Re}\beta + \mathrm{Re}\gamma\right), \quad \mathrm{Im}\delta = -\sqrt{3}\left(\mathrm{Re}\beta - \mathrm{Re}\gamma\right)\,.
\end{gathered} \tag{3.82}$$

The unitary transformation $\mathbb{U}$ is

$$\mathbb{U} = \begin{pmatrix} e^{\frac{5i\pi}{8}}\sin\frac{\pi}{8} & 0 & e^{\frac{i\pi}{8}}\cos\frac{\pi}{8} \\ 0 & e^{\frac{i\pi}{4}} & 0 \\ -e^{\frac{5i\pi}{8}}\cos\frac{\pi}{8} & 0 & e^{\frac{i\pi}{8}}\sin\frac{\pi}{8} \end{pmatrix}\,. \tag{3.83}$$

Hence the light neutrino mass matrix is diagonalized by

$$U_\nu = \frac{1}{\sqrt{3}}\begin{pmatrix} e^{-\frac{i\pi}{8}}\sin\left(\frac{5\pi}{24}+\theta\right) - e^{\frac{3i\pi}{8}}\sin\left(\frac{5\pi}{24}-\theta\right) & e^{\frac{i\pi}{4}} & -e^{-\frac{i\pi}{8}}\cos\left(\frac{5\pi}{24}+\theta\right) - e^{\frac{3i\pi}{8}}\cos\left(\frac{5\pi}{24}-\theta\right) \\ -e^{-\frac{i\pi}{8}}\cos\left(\frac{\pi}{24}+\theta\right) + e^{\frac{3i\pi}{8}}\cos\left(\frac{\pi}{24}-\theta\right) & e^{\frac{i\pi}{4}} & -e^{-\frac{i\pi}{8}}\sin\left(\frac{\pi}{24}+\theta\right) - e^{\frac{3i\pi}{8}}\sin\left(\frac{\pi}{24}-\theta\right) \\ e^{-\frac{i\pi}{8}}\sin\left(\frac{\pi}{8}-\theta\right) - e^{\frac{3i\pi}{8}}\sin\left(\frac{\pi}{8}+\theta\right) & e^{\frac{i\pi}{4}} & e^{-\frac{i\pi}{8}}\cos\left(\frac{\pi}{8}-\theta\right) + e^{\frac{3i\pi}{8}}\cos\left(\frac{\pi}{8}+\theta\right) \end{pmatrix}, \tag{3.84}$$

where

$$\tan 2\theta = \frac{(1+\sqrt{3})\mathrm{Re}\beta + (\sqrt{3}-1)\gamma}{-\sqrt{6}\,\mathrm{Re}\alpha}\,. \tag{3.85}$$

The light neutrino masses are determined to be

$$\begin{aligned}
m_1 &= \left|(3-\sqrt{3})\mathrm{Re}\beta + (3+\sqrt{3})\mathrm{Re}\gamma + \mathrm{sign}\left(\mathrm{Re}\alpha\cos 2\theta\right)\sqrt{18(\mathrm{Re}\alpha)^2 + \left((3+\sqrt{3})\mathrm{Re}\beta + (3-\sqrt{3})\mathrm{Re}\gamma\right)^2}\right|\,,\\
m_2 &= \left|2\sqrt{3}\left(2\mathrm{Re}\beta + \mathrm{Re}\gamma\right) - 6\mathrm{Im}\beta\right|\,,\\
m_3 &= \left|(3-\sqrt{3})\mathrm{Re}\beta + (3+\sqrt{3})\mathrm{Re}\gamma - \mathrm{sign}\left(\mathrm{Re}\alpha\cos 2\theta\right)\sqrt{18(\mathrm{Re}\alpha)^2 + \left((3+\sqrt{3})\mathrm{Re}\beta + (3-\sqrt{3})\mathrm{Re}\gamma\right)^2}\right|\,.
\end{aligned}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(a^2bc^3d), \rho_{\mathbf{r}}(a^2bc^3d^3)$

The invariance of the neutrino mass matrix under the remnant CP transformations leads to

$$\begin{aligned}&\mathrm{Re}\delta=-3\left(\mathrm{Re}\beta+\mathrm{Re}\gamma\right),\quad \mathrm{Im}\alpha=-\mathrm{Re}\alpha\,,\\ &\mathrm{Im}\gamma=\mathrm{Im}\beta+\sqrt{3}\left(\mathrm{Re}\beta+\mathrm{Re}\gamma\right),\quad \mathrm{Im}\delta=\sqrt{3}\left(\mathrm{Re}\beta-\mathrm{Re}\gamma\right)\,.\end{aligned}\tag{3.86}$$

The unitary transformation $\mathbb{U}$ is

$$\mathbb{U}=\begin{pmatrix} e^{\frac{5i\pi}{8}}\cos\frac{\pi}{8} & 0 & e^{\frac{i\pi}{8}}\sin\frac{\pi}{8}\\ 0 & e^{\frac{i\pi}{4}} & 0\\ -e^{\frac{5i\pi}{8}}\sin\frac{\pi}{8} & 0 & e^{\frac{i\pi}{8}}\cos\frac{\pi}{8}\end{pmatrix}\,.\tag{3.87}$$

The neutrino diagonalization matrix is of the form

$$U_\nu=\frac{1}{\sqrt{3}}\begin{pmatrix} e^{-\frac{i\pi}{8}}\cos\left(\frac{\pi}{24}-\theta\right)-e^{\frac{3i\pi}{8}}\cos\left(\frac{\pi}{24}+\theta\right) & e^{\frac{i\pi}{4}} & -e^{-\frac{i\pi}{8}}\sin\left(\frac{\pi}{24}-\theta\right)-e^{\frac{3i\pi}{8}}\sin\left(\frac{\pi}{24}+\theta\right)\\ -e^{-\frac{i\pi}{8}}\sin\left(\frac{5\pi}{24}-\theta\right)+e^{\frac{3i\pi}{8}}\sin\left(\frac{5\pi}{24}+\theta\right) & e^{\frac{i\pi}{4}} & -e^{-\frac{i\pi}{8}}\cos\left(\frac{5\pi}{24}-\theta\right)-e^{\frac{3i\pi}{8}}\cos\left(\frac{5\pi}{24}+\theta\right)\\ -e^{-\frac{i\pi}{8}}\sin\left(\frac{\pi}{8}+\theta\right)+e^{\frac{3i\pi}{8}}\sin\left(\frac{\pi}{8}-\theta\right) & e^{\frac{i\pi}{4}} & e^{-\frac{i\pi}{8}}\cos\left(\frac{\pi}{8}+\theta\right)+e^{\frac{3i\pi}{8}}\cos\left(\frac{\pi}{8}-\theta\right)\end{pmatrix}\,,\tag{3.88}$$

with

$$\tan 2\theta=\frac{\left(\sqrt{3}-1\right)\mathrm{Re}\beta+\left(1+\sqrt{3}\right)\mathrm{Re}\gamma}{\sqrt{6}\;\mathrm{Re}\alpha}\,.\tag{3.89}$$

In the end, the light neutrino masses are

$$\begin{aligned}
m_1&=\left|(3+\sqrt{3})\mathrm{Re}\beta+(3-\sqrt{3})\mathrm{Re}\gamma+\mathrm{sign}\left(\mathrm{Re}\alpha\cos2\theta\right)\sqrt{18(\mathrm{Re}\alpha)^2+\left((3-\sqrt{3})\mathrm{Re}\beta+(3+\sqrt{3})\mathrm{Re}\gamma\right)^2}\right|,\\
m_2&=\left|2\sqrt{3}\left(2\mathrm{Re}\beta+\mathrm{Re}\gamma\right)+6\mathrm{Im}\beta\right|,\\
m_3&=\left|(3+\sqrt{3})\mathrm{Re}\beta+(3-\sqrt{3})\mathrm{Re}\gamma-\mathrm{sign}\left(\mathrm{Re}\alpha\cos2\theta\right)\sqrt{18(\mathrm{Re}\alpha)^2+\left((3-\sqrt{3})\mathrm{Re}\beta+(3+\sqrt{3})\mathrm{Re}\gamma\right)^2}\right|.
\end{aligned}$$

For the remaining twelve remnant CP transformations $X_{\nu\mathbf{r}}=\rho_{\mathbf{r}}(d)$, $\rho_{\mathbf{r}}(d^3)$, $\rho_{\mathbf{r}}(c)$, $\rho_{\mathbf{r}}(cd^2)$, $\rho_{\mathbf{r}}(cd)$, $\rho_{\mathbf{r}}(cd^3)$, $\rho_{\mathbf{r}}(c^2)$, $\rho_{\mathbf{r}}(c^2d^2)$, $\rho_{\mathbf{r}}(c^3)$, $\rho_{\mathbf{r}}(c^3d^2)$, $\rho_{\mathbf{r}}(c^3d)$ and $\rho_{\mathbf{r}}(c^3d^3)$, the light neutrino masses are predicted to be partially degenerate, i.e., two of the light neutrinos are of the same masses, therefore these cases are not phenomenologically viable.

3.4.2 $G_\nu=Z_2^{(9)}=\{1,a^2bd\}$

In this scenario, the residual CP symmetry H^ν_{CP}, which should be consistent with the residual family symmetry $G_\nu=Z_2^{(9)}$, and is determined by the consistency equation

$$X_{\nu\mathbf{r}}\rho^*_{\mathbf{r}}(a^2bd)X^{-1}_{\nu\mathbf{r}}=\rho_{\mathbf{r}}(a^2bd)\,.\tag{3.90}$$

One can easily check that there are 8 possible choices for $X_{\nu\mathbf{r}}$, i.e.,

$$H^\nu_{CP}=\left\{\rho_{\mathbf{r}}(1),\rho_{\mathbf{r}}(d^2),\rho_{\mathbf{r}}(c^2d),\rho_{\mathbf{r}}(c^2d^3),\rho_{\mathbf{r}}(a^2bd),\rho_{\mathbf{r}}(a^2bc^2),\rho_{\mathbf{r}}(a^2bd^3),\rho_{\mathbf{r}}(a^2bc^2d^2)\right\}\,.\tag{3.91}$$

The light neutrino mass matrix m_ν is constrained by the remnant family and remnant CP symmetries as

$$\rho^T_{\mathbf{3}}(a^2bd)m_\nu\rho_{\mathbf{3}}(a^2bd) = m_\nu\,, \tag{3.92a}$$
$$X^T_{\nu\mathbf{3}}m_\nu X_{\nu\mathbf{3}} = m^*_\nu\,. \tag{3.92b}$$

The most general neutrino mass matrix, which is invariant under the residual family symmetry $G_\nu = Z_2^{(9)}$ and satisfies Eq. (3.35a), takes the following form

$$m_\nu = \alpha\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \beta\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \gamma\begin{pmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ -1 & 1 & 0 \end{pmatrix} + \delta\begin{pmatrix} 2\left(2+\sqrt{3}\right) & 0 & -1 \\ 0 & 2 & -2-\sqrt{3} \\ -1 & -2-\sqrt{3} & 0 \end{pmatrix}, \tag{3.93}$$

where α, β, γ and δ are complex parameters, and they are further constrained by the remnant CP symmetry as shown in Eq. (3.92b). After perform the THF transformation U_{THF}, we have

$$\begin{aligned} m'_\nu &= U^T_{TFH}m_\nu U_{TFH} \\ &= \begin{pmatrix} 3\alpha+\sqrt{3}\,\gamma+\left(3+\sqrt{3}\right)\delta & -\left(3+\sqrt{3}\right)\delta & 0 \\ -\left(3+\sqrt{3}\right)\delta & 3\beta & 0 \\ 0 & 0 & 3\alpha-\sqrt{3}\,\gamma+\left(3+\sqrt{3}\right)\delta \end{pmatrix}. \end{aligned} \tag{3.94}$$

m'_ν can be further diagonalized by a (1,2) rotation U'_ν,

$$U'^T_\nu m'_\nu U'_\nu = \mathrm{diag}(m_1, m_2, m_3)\,. \tag{3.95}$$

Hence the light neutrino mass matrix is diagonalized by the unitary matrix U_ν with

$$U_\nu = U_{TFH}U'_\nu\,. \tag{3.96}$$

Analogous to the previous case, the neutrino diagonalization matrix U_ν is fixed up to permutations of the columns. In the following, we shall discuss the constraints of the different remnant CP transformation shown in Eq. (3.91) one by one.

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(1), \rho_{\mathbf{r}}(a^2bd)$

  In this case, the neutrino mass matrix is constrained to be real such that we have

$$\mathrm{Im}\alpha = \mathrm{Im}\beta = \mathrm{Im}\gamma = \mathrm{Im}\delta = 0\,. \tag{3.97}$$

  Then the unitary matrix U'_ν becomes a real rotation matrix with

$$U'_\nu = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{3.98}$$

  where the diagonal phase matrix P, which renders the light neutrino mass positive, has been omitted here and hereinafter. The rotation angle θ is given by

$$\tan 2\theta = \frac{2(1+\sqrt{3})\mathrm{Re}\delta}{\sqrt{3}\,(\mathrm{Re}\alpha-\mathrm{Re}\beta)+\mathrm{Re}\gamma+(1+\sqrt{3})\mathrm{Re}\delta}\,. \tag{3.99}$$

As a result, the neutrino mass matrix is diagonalized by

$$U_\nu = \frac{1}{2\sqrt{3}} \begin{pmatrix} -\left(1+\sqrt{3}\right)\cos\theta - 2\sin\theta & -\left(1+\sqrt{3}\right)\sin\theta + 2\cos\theta & \sqrt{3}-1 \\ \left(\sqrt{3}-1\right)\cos\theta - 2\sin\theta & \left(\sqrt{3}-1\right)\sin\theta + 2\cos\theta & -1-\sqrt{3} \\ 2\sqrt{2}\cos\left(\frac{\pi}{4}+\theta\right) & 2\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right) & 2 \end{pmatrix} . \tag{3.100}$$

The light neutrino masses are determined to be

$$\begin{aligned} m_1 &= \frac{1}{2}\left| a_{11} + a_{22} - \text{sign}\left((a_{22}-a_{11})\cos 2\theta\right)\sqrt{(a_{22}-a_{11})^2 + 4a_{12}^2} \right| , \\ m_2 &= \frac{1}{2}\left| a_{11} + a_{22} + \text{sign}\left((a_{22}-a_{11})\cos 2\theta\right)\sqrt{(a_{22}-a_{11})^2 + 4a_{12}^2} \right| , \\ m_3 &= \left| 3\text{Re}\alpha - \sqrt{3}\,\text{Re}\gamma + (3+\sqrt{3})\text{Re}\delta \right| , \end{aligned} \tag{3.101}$$

where

$$a_{11} = 3\text{Re}\alpha + \sqrt{3}\,\text{Re}\gamma + (3+\sqrt{3})\text{Re}\delta, \quad a_{22} = 3\text{Re}\beta, \quad a_{12} = -(3+\sqrt{3})\text{Re}\delta\,. \tag{3.102}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(d^2), \rho_{\mathbf{r}}(a^2bd^3)$

The invariance of the neutrino mass matrix under residual CP symmetry leads to

$$\text{Re}\delta = \text{Im}\beta = \text{Im}\gamma = 0, \quad \text{Im}\delta = \frac{1}{2}\left(-3+\sqrt{3}\right)\text{Im}\alpha\,. \tag{3.103}$$

The unitary transformation U'_ν diagonalizing the neutrino mass matrix m'_ν is of the form

$$U'_\nu = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -i\sin\theta & i\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.104}$$

Therefore the neutrino diagonalization matrix U_ν is

$$U_\nu = \frac{1}{2\sqrt{3}} \begin{pmatrix} -\left(1+\sqrt{3}\right)\cos\theta - 2i\sin\theta & -\left(1+\sqrt{3}\right)\sin\theta + 2i\cos\theta & \sqrt{3}-1 \\ \left(\sqrt{3}-1\right)\cos\theta - 2i\sin\theta & \left(\sqrt{3}-1\right)\sin\theta + 2i\cos\theta & -1-\sqrt{3} \\ 2e^{-i\theta} & 2ie^{-i\theta} & 2 \end{pmatrix} . \tag{3.105}$$

The light neutrino masses are

$$\begin{aligned} m_1 &= \frac{1}{2}\left| a_{11} + a_{22} - \text{sign}\left((a_{22}-a_{11})\cos 2\theta\right)\sqrt{(a_{22}-a_{11})^2 + 4a_{12}^2} \right| , \\ m_2 &= \frac{1}{2}\left| a_{11} + a_{22} + \text{sign}\left((a_{22}-a_{11})\cos 2\theta\right)\sqrt{(a_{22}-a_{11})^2 + 4a_{12}^2} \right| , \\ m_3 &= \left| 3\text{Re}\alpha - \sqrt{3}\,\text{Re}\gamma \right| , \end{aligned} \tag{3.106}$$

where

$$a_{11} = 3\text{Re}\alpha + \sqrt{3}\,\text{Re}\gamma, \quad a_{22} = -3\text{Re}\beta, \quad a_{12} = -3\text{Im}\alpha\,. \tag{3.107}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^2 d), \rho_{\mathbf{r}}(a^2 b c^2 d^2)$

  In this case, the parameters α, β, γ and δ are constrained by the remnant CP symmetry to satisfy

$$\mathrm{Re}\gamma = \mathrm{Im}\beta = 0, \quad \mathrm{Re}\delta = \frac{1}{2}\left(-3+\sqrt{3}\right)\mathrm{Re}\alpha, \quad \mathrm{Im}\delta = \frac{3\mathrm{Re}\alpha}{3+\sqrt{3}}\,. \tag{3.108}$$

  The unitary matrix U'_ν takes the form

$$U'_\nu = \begin{pmatrix} e^{\frac{i\pi}{4}}\cos\theta & e^{\frac{i\pi}{4}}\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & e^{\frac{i\pi}{4}} \end{pmatrix}. \tag{3.109}$$

  Then the neutrino mass matrix is diagonalized by

$$U_\nu = \frac{1}{2\sqrt{3}} \begin{pmatrix} -\left(1+\sqrt{3}\right)e^{\frac{i\pi}{4}}\cos\theta - 2\sin\theta & -\left(1+\sqrt{3}\right)e^{\frac{i\pi}{4}}\sin\theta + 2\cos\theta & \left(\sqrt{3}-1\right)e^{\frac{i\pi}{4}} \\ \left(\sqrt{3}-1\right)e^{\frac{i\pi}{4}}\cos\theta - 2\sin\theta & \left(\sqrt{3}-1\right)e^{\frac{i\pi}{4}}\sin\theta + 2\cos\theta & -\left(1+\sqrt{3}\right)e^{\frac{i\pi}{4}} \\ 2e^{\frac{i\pi}{4}}\cos\theta - 2\sin\theta & 2e^{\frac{i\pi}{4}}\sin\theta + 2\cos\theta & 2e^{\frac{i\pi}{4}} \end{pmatrix}, \tag{3.110}$$

  where

$$\tan 2\theta = \frac{2\sqrt{6}\,\mathrm{Re}\alpha}{\sqrt{3}\,\left(\mathrm{Re}\alpha + \mathrm{Re}\beta + \mathrm{Im}\alpha\right) + \mathrm{Im}\gamma}\,. \tag{3.111}$$

  The light neutrino masses are determined to be

$$\begin{aligned} m_1 &= \frac{1}{2}\left|a_{11} + a_{22} - \mathrm{sign}\left((a_{22}-a_{11})\cos 2\theta\right)\sqrt{\left(a_{22}-a_{11}\right)^2 + 4a_{12}^2}\right|, \\ m_2 &= \frac{1}{2}\left|a_{11} + a_{22} + \mathrm{sign}\left((a_{22}-a_{11})\cos 2\theta\right)\sqrt{\left(a_{22}-a_{11}\right)^2 + 4a_{12}^2}\right|, \\ m_3 &= \left|3\mathrm{Re}\alpha + 3\mathrm{Im}\alpha - \sqrt{3}\,\mathrm{Im}\gamma\right|, \end{aligned} \tag{3.112}$$

  where

$$a_{11} = -3\mathrm{Re}\alpha - 3\mathrm{Im}\alpha - \sqrt{3}\,\mathrm{Im}\gamma, \quad a_{22} = 3\mathrm{Re}\beta, \quad a_{12} = 3\sqrt{2}\,\mathrm{Re}\alpha\,. \tag{3.113}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^2 d^3), \rho_{\mathbf{r}}(a^2 b c^2)$

  This remnant CP symmetry implies that

$$\mathrm{Re}\gamma = \mathrm{Im}\beta = 0, \quad \mathrm{Re}\delta = \mathrm{Im}\delta = \frac{1}{2}\left(-3+\sqrt{3}\right)\mathrm{Re}\alpha\,. \tag{3.114}$$

  The unitary transformation U'_ν is given by

$$U'_\nu = \begin{pmatrix} e^{-\frac{i\pi}{4}}\cos\theta & e^{-\frac{i\pi}{4}}\sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & e^{\frac{i\pi}{4}} \end{pmatrix}. \tag{3.115}$$

Hence the neutrino diagonalization matrix U_ν is of the form

$$U_\nu = \frac{1}{2\sqrt{3}} \begin{pmatrix} -\left(1+\sqrt{3}\right) e^{-\frac{i\pi}{4}} \cos\theta - 2\sin\theta & -\left(1+\sqrt{3}\right) e^{-\frac{i\pi}{4}} \sin\theta + 2\cos\theta & \left(\sqrt{3}-1\right) e^{\frac{i\pi}{4}} \\ \left(\sqrt{3}-1\right) e^{-\frac{i\pi}{4}} \cos\theta - 2\sin\theta & \left(\sqrt{3}-1\right) e^{-\frac{i\pi}{4}} \sin\theta + 2\cos\theta & -\left(1+\sqrt{3}\right) e^{\frac{i\pi}{4}} \\ 2e^{-\frac{i\pi}{4}} \cos\theta - 2\sin\theta & 2e^{-\frac{i\pi}{4}} \sin\theta + 2\cos\theta & 2e^{\frac{i\pi}{4}} \end{pmatrix}, \tag{3.116}$$

with

$$\tan 2\theta = \frac{2\sqrt{6}\,\mathrm{Re}\alpha}{\sqrt{3}\left(\mathrm{Re}\alpha + \mathrm{Re}\beta - \mathrm{Im}\alpha\right) - \mathrm{Im}\gamma}\,. \tag{3.117}$$

Finally, the light neutrino masses are given by

$$\begin{aligned} m_1 &= \frac{1}{2}\left| a_{11} + a_{22} - \mathrm{sign}\left((a_{22}-a_{11})\cos 2\theta\right) \sqrt{\left(a_{22}-a_{11}\right)^2 + 4a_{12}^2} \right|, \\ m_2 &= \frac{1}{2}\left| a_{11} + a_{22} + \mathrm{sign}\left((a_{22}-a_{11})\cos 2\theta\right) \sqrt{\left(a_{22}-a_{11}\right)^2 + 4a_{12}^2} \right|, \\ m_3 &= \left| 3\mathrm{Re}\alpha - 3\mathrm{Im}\alpha + \sqrt{3}\,\mathrm{Im}\gamma \right|, \end{aligned} \tag{3.118}$$

where

$$a_{11} = -3\mathrm{Re}\alpha + 3\mathrm{Im}\alpha + \sqrt{3}\,\mathrm{Im}\gamma, \quad a_{22} = 3\mathrm{Re}\beta, \quad a_{12} = 3\sqrt{2}\,\mathrm{Re}\alpha\,. \tag{3.119}$$

3.4.3 $G_\nu = Z_2^{(10)} = \{1, a^2bd^2\}$

The residual CP symmetry H^ν_{CP} consistent with the remnant family symmetry $Z_2^{(9)}$, should satisfy the consistency equation:

$$X_{\nu\mathbf{r}}\rho^*_{\mathbf{r}}(a^2bd^2)X^{-1}_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(a^2bd^2)\,. \tag{3.120}$$

We find that only 8 of the 96 generalized CP transformations are acceptable,

$$H^\nu_{CP} = \left\{\rho_{\mathbf{r}}(1), \rho_{\mathbf{r}}(d^2), \rho_{\mathbf{r}}(c^2d), \rho_{\mathbf{r}}(c^2d^3), \rho_{\mathbf{r}}(a^2b), \rho_{\mathbf{r}}(a^2bd^2), \rho_{\mathbf{r}}(a^2bc^2d), \rho_{\mathbf{r}}(a^2bc^2d^3)\right\}\,. \tag{3.121}$$

The light neutrino mass matrix m_ν is subject to the following constraints:

$$\rho^T_{\mathbf{3}}(a^2bd^2)m_\nu\rho_{\mathbf{3}}(a^2bd^2) = m_\nu\,, \tag{3.122a}$$

$$X^T_{\nu\mathbf{3}}m_\nu X_{\nu\mathbf{3}} = m^*_\nu\,, \tag{3.122b}$$

where Eq. (3.122a) is the invariance condition of the neutrino mass matrix under the residual family symmetry $G_\nu = Z_2^{(10)}$, and it implies that the neutrino mass matrix is of the form

$$m_\nu = \alpha\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} + \beta\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \gamma\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \delta\begin{pmatrix} -2 & 0 & -1 \\ 0 & 2 & 1 \\ -1 & 1 & 0 \end{pmatrix}, \tag{3.123}$$

where α, β, γ and δ are complex parameters, and they are further constrained by the remnant CP symmetry shown in Eq. (3.122b). In order to diagonalize the neutrino mass matrix of Eq. (3.123), we first perform the following unitary transformation

$$m'_\nu = U^T_{TBP} m_\nu U_{TBP} = \begin{pmatrix} 3\alpha+\gamma & 0 & 0 \\ 0 & 3\beta+\gamma & \sqrt{6}\,\delta \\ 0 & \sqrt{6}\,\delta & 3\alpha-\gamma \end{pmatrix} ,$$

where

$$U_{TBP} = \begin{pmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \end{pmatrix} , \qquad (3.124)$$

which can be obtained by the permutating the first and the third rows of the tri-bimaximal mixing matrix. Furthermore, m'_ν can be diagonalized by another unitary matrix U'_ν,

$$U'^T_\nu m'_\nu U'_\nu = \text{diag}(m_1, m_2, m_3) \,. \qquad (3.125)$$

Therefore the unitary transformation U_ν diagonalizing the neutrino mass matrix m_ν in Eq. (3.123) is of the form

$$U_\nu = U_{TBP} U'_\nu. \qquad (3.126)$$

Here we would like to emphasize again that the neutrino diagonalization matrix U_ν is fixed up to permutations of the columns. In the following, we shall investigate the implications of the remnant CP invariant condition of Eq. (3.122b). The eight possible $X_{\nu\mathbf{r}}$ lead to four different phenomenological predictions.

- $X_{\nu\mathbf{r}} = \rho_\mathbf{r}(1), \rho_\mathbf{r}(a^2bd^2)$

  In this case, the neutrino mass matrix is constrained to be real. As a result, we have

$$\text{Im}\alpha = \text{Im}\beta = \text{Im}\gamma = \text{Im}\delta = 0 \,. \qquad (3.127)$$

  Hence m'_ν becomes a real symmetry matrix and can be diagonalized by a rotation matrix in the (2,3) sector with

$$U'_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} , \qquad (3.128)$$

  where

$$\tan 2\theta = \frac{2\sqrt{6}\,\text{Re}\delta}{3\text{Re}\alpha - 3\text{Re}\beta - 2\text{Re}\gamma} \,. \qquad (3.129)$$

  As a consequence, the neutrino diagonalization matrix U_ν is

$$U_\nu = \frac{1}{\sqrt{6}} \begin{pmatrix} -1 & \sqrt{2}\,\cos\theta + \sqrt{3}\,\sin\theta & \sqrt{2}\,\sin\theta - \sqrt{3}\,\cos\theta \\ -1 & \sqrt{2}\,\cos\theta - \sqrt{3}\,\sin\theta & \sqrt{2}\,\sin\theta + \sqrt{3}\,\cos\theta \\ 2 & \sqrt{2}\,\cos\theta & \sqrt{2}\,\sin\theta \end{pmatrix} . \qquad (3.130)$$

The light neutrino masses are determined to be

$$
\begin{aligned}
m_1 &= |3\mathrm{Re}\alpha + \mathrm{Re}\gamma| \,, \\
m_2 &= \frac{1}{2}\left|3\mathrm{Re}\alpha + 3\mathrm{Re}\beta - \mathrm{sign}\left((3\mathrm{Re}\alpha - 3\mathrm{Re}\beta - 2\mathrm{Re}\gamma)\cos 2\theta\right)\sqrt{(3\mathrm{Re}\alpha - 3\mathrm{Re}\beta - 2\mathrm{Re}\gamma)^2 + 24\left(\mathrm{Re}\delta\right)^2}\right| \,, \\
m_3 &= \frac{1}{2}\left|3\mathrm{Re}\alpha + 3\mathrm{Re}\beta + \mathrm{sign}\left((3\mathrm{Re}\alpha - 3\mathrm{Re}\beta - 2\mathrm{Re}\gamma)\cos 2\theta\right)\sqrt{(3\mathrm{Re}\alpha - 3\mathrm{Re}\beta - 2\mathrm{Re}\gamma)^2 + 24\left(\mathrm{Re}\delta\right)^2}\right| \,.
\end{aligned}
$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(d^2), \rho_{\mathbf{r}}(a^2b)$

  The invariance of the neutrino mass matrix under the action of residual CP transformation leads to

$$
\mathrm{Re}\delta = \mathrm{Im}\alpha = \mathrm{Im}\beta = \mathrm{Im}\gamma = 0\,. \tag{3.131}
$$

  The unitary transformation U'_ν is

$$
U'_\nu = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\,i\sin\theta & i\cos\theta \end{pmatrix}\,, \tag{3.132}
$$

  where

$$
\tan 2\theta = \frac{2\sqrt{6}\;\mathrm{Im}\delta}{3\left(\mathrm{Re}\alpha + \mathrm{Re}\beta\right)}\,. \tag{3.133}
$$

  Therefore the neutrino mass matrix is diagonalized by the following unitary matrix

$$
U_\nu = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 & \sqrt{2}\;\cos\theta + i\sqrt{3}\;\sin\theta & \sqrt{2}\;\sin\theta - i\sqrt{3}\;\cos\theta \\ -1 & \sqrt{2}\;\cos\theta - i\sqrt{3}\;\sin\theta & \sqrt{2}\;\sin\theta + i\sqrt{3}\;\cos\theta \\ 2 & \sqrt{2}\;\cos\theta & \sqrt{2}\;\sin\theta \end{pmatrix}\,. \tag{3.134}
$$

  The light neutrino masses are

$$
\begin{aligned}
m_1 &= |3\mathrm{Re}\alpha + \mathrm{Re}\gamma| \,, \\
m_2 &= \frac{1}{2}\left|-3\mathrm{Re}\alpha + 3\mathrm{Re}\beta + 2\mathrm{Re}\gamma + \mathrm{sign}\left((\mathrm{Re}\alpha + \mathrm{Re}\beta)\cos 2\theta\right)\sqrt{9\left(\mathrm{Re}\alpha + \mathrm{Re}\beta\right)^2 + 24\left(\mathrm{Im}\delta\right)^2}\right| \,, \\
m_3 &= \frac{1}{2}\left|-3\mathrm{Re}\alpha + 3\mathrm{Re}\beta + 2\mathrm{Re}\gamma - \mathrm{sign}\left((\mathrm{Re}\alpha + \mathrm{Re}\beta)\cos 2\theta\right)\sqrt{9\left(\mathrm{Re}\alpha + \mathrm{Re}\beta\right)^2 + 24\left(\mathrm{Im}\delta\right)^2}\right| \,.
\end{aligned} \tag{3.135}
$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^2d), \rho_{\mathbf{r}}(a^2bc^2d^3)$

  In this case, the parameters α, β, γ and δ are constrained to satisfy

$$
\mathrm{Re}\alpha = \mathrm{Re}\gamma = 0, \quad \mathrm{Im}\gamma = -3\mathrm{Im}\beta, \quad \mathrm{Im}\delta = -\mathrm{Re}\delta\,. \tag{3.136}
$$

  The unitary matrix U'_ν takes the form

$$
U'_\nu = \begin{pmatrix} e^{\frac{i\pi}{4}} & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\,e^{\frac{i\pi}{4}}\sin\theta & e^{\frac{i\pi}{4}}\cos\theta \end{pmatrix}\,, \tag{3.137}
$$

with

$$\tan 2\theta = -\frac{4\text{Re}\delta}{\sqrt{3}\ (\text{Im}\alpha + \text{Re}\beta + \text{Im}\beta)}\,. \tag{3.138}$$

Therefore the resulting neutrino diagonalization matrix is

$$U_\nu = \frac{1}{\sqrt{6}}\begin{pmatrix} -e^{\frac{i\pi}{4}} & \sqrt{2}\ \cos\theta + \sqrt{3}\ e^{\frac{i\pi}{4}}\sin\theta & \sqrt{2}\ \sin\theta - \sqrt{3}\ e^{\frac{i\pi}{4}}\cos\theta \\ -e^{\frac{i\pi}{4}} & \sqrt{2}\ \cos\theta - \sqrt{3}\ e^{\frac{i\pi}{4}}\sin\theta & \sqrt{2}\ \sin\theta + \sqrt{3}\ e^{\frac{i\pi}{4}}\cos\theta \\ 2e^{\frac{i\pi}{4}} & \sqrt{2}\ \cos\theta & \sqrt{2}\ \sin\theta \end{pmatrix}. \tag{3.139}$$

The light neutrino masses are

$$\begin{aligned} m_1 &= 3\left|\text{Im}\alpha - \text{Im}\beta\right|\,, \\ m_2 &= \frac{1}{2}\left|3\left(\text{Im}\alpha - \text{Re}\beta + \text{Im}\beta\right) - \text{sign}\left(\left(\text{Im}\alpha + \text{Re}\beta + \text{Im}\beta\right)\cos 2\theta\right)\sqrt{9\left(\text{Im}\alpha + \text{Re}\beta + \text{Im}\beta\right)^2 + 48\left(\text{Re}\delta\right)^2}\right|\,, \\ m_3 &= \frac{1}{2}\left|3\left(\text{Im}\alpha - \text{Re}\beta + \text{Im}\beta\right) + \text{sign}\left(\left(\text{Im}\alpha + \text{Re}\beta + \text{Im}\beta\right)\cos 2\theta\right)\sqrt{9\left(\text{Im}\alpha + \text{Re}\beta + \text{Im}\beta\right)^2 + 48\left(\text{Re}\delta\right)^2}\right|\,. \end{aligned}$$

- $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^2d^3), \rho_{\mathbf{r}}(a^2bc^2d)$

  In this case, the residual CP symmetry constrains the parameters as

$$\text{Re}\alpha = \text{Re}\gamma = 0, \quad \text{Im}\gamma = -3\text{Im}\beta, \quad \text{Im}\delta = \text{Re}\delta\,. \tag{3.140}$$

The unitary transformation U'_ν is given by

$$U'_\nu = \begin{pmatrix} e^{\frac{i\pi}{4}} & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\,e^{-\frac{i\pi}{4}}\sin\theta & e^{-\frac{i\pi}{4}}\cos\theta \end{pmatrix}, \tag{3.141}$$

where

$$\tan 2\theta = \frac{4\text{Re}\delta}{\sqrt{3}\ (\text{Im}\alpha - \text{Re}\beta + \text{Im}\beta)}\,. \tag{3.142}$$

Hence the neutrino mass matrix is diagonalized by

$$U_\nu = \frac{1}{\sqrt{6}}\begin{pmatrix} -e^{\frac{i\pi}{4}} & \sqrt{2}\cos\theta + \sqrt{3}e^{-\frac{i\pi}{4}}\sin\theta & \sqrt{2}\sin\theta - \sqrt{3}e^{-\frac{i\pi}{4}}\cos\theta \\ -e^{\frac{i\pi}{4}} & \sqrt{2}\cos\theta - \sqrt{3}e^{-\frac{i\pi}{4}}\sin\theta & \sqrt{2}\sin\theta + \sqrt{3}e^{-\frac{i\pi}{4}}\cos\theta \\ 2e^{\frac{i\pi}{4}} & \sqrt{2}\cos\theta & \sqrt{2}\sin\theta \end{pmatrix}. \tag{3.143}$$

Finally, the light neutrino masses are

$$\begin{aligned} m_1 &= 3\left|\text{Im}\alpha - \text{Im}\beta\right|\,, \\ m_2 &= \frac{1}{2}\left|3\left(\text{Im}\alpha + \text{Re}\beta + \text{Im}\beta\right) - \text{sign}\left(\left(\text{Im}\alpha - \text{Re}\beta + \text{Im}\beta\right)\cos 2\theta\right)\sqrt{9\left(\text{Im}\alpha - \text{Re}\beta + \text{Im}\beta\right)^2 + 48\left(\text{Re}\delta\right)^2}\right|\,, \\ m_3 &= \frac{1}{2}\left|3\left(\text{Im}\alpha + \text{Re}\beta + \text{Im}\beta\right) + \text{sign}\left(\left(\text{Im}\alpha - \text{Re}\beta + \text{Im}\beta\right)\cos 2\theta\right)\sqrt{9\left(\text{Im}\alpha - \text{Re}\beta + \text{Im}\beta\right)^2 + 48\left(\text{Re}\delta\right)^2}\right|\,. \end{aligned}$$

4 Lepton mixing predictions

In this section, we perform a comprehensive analysis of all possible lepton mixing matrices obtainable from the implementation of a $\Delta(96)$ family symmetry and its corresponding generalised CP symmetry by considering all possible residual symmetries $G^{\nu}_{\rm CP}$ and $G^{l}_{\rm CP}$ discussed in previous sections. In all the cases, both leptonic mixing angles and CP phases (including both Dirac and Majorana CP phases) are found to depend on only one parameter θ. As a consequence, the lepton mixing parameters are strongly correlated with each other in this context, and obviously it is highly nontrivial to be able to fit all the observed lepton mixing angles with the sole parameter θ. As a measure of to which extent the resulting lepton mixing angles can be close to the accurately measured values of θ_{12}, θ_{23} and θ_{13}, we use the χ^2 function defined in the conventional way. Since the octant of the atmospheric mixing angle has not been determined so far and $\sin^2\theta_{23}$ has two best fit values $\sin^2\theta_{23}=0.413$ and $\sin^2\theta_{23}=0.594$ [5], we define two different χ^2 functions. The smaller the minimum of the χ^2 function is, the better the corresponding PMNS matrix can explain the data. Notice that, without a particular model, the lepton mixing matrix is only determined up to permutations of rows and columns, since neither charged lepton nor neutrino masses are constrained in the present framework. Therefore all the possible permutation of rows and columns are taken into account for each symmetry breaking pattern and the corresponding global minimum of the χ^2 functions is calculated, and subsequently we choose the best one.

The analytical formulas for the mixing parameters and the best fitting results are summarized in Tables 1-6. Because the sign of the Jarlskog invariant $J_{\rm CP}$ depends on the ordering of rows and columns, while the sign of $\sin\alpha$ ($\tan\alpha$) and $\sin\beta$ ($\tan\beta$) depends on the CP parity of the neutrino states which is encoded in the matrix P, please see Eq. (3.40), all these quantities are presented in terms of absolute values. In addition, if we assign the LH lepton doublets to be the triplet $\overline{\mathbf{3}}$ instead of $\mathbf{3}$, the sign of the CP phases δ_{CP}, α and β would be changed. In the following, we shall present the resulting PMNS matrix and its predictions for the lepton mixing parameters for each possible symmetry breaking chains. It is remarkable that the best arrangements of the PMNS matrix for the first octant of θ_{23} and the second octant of θ_{23} turn out to be related by the exchange of the second and the third rows. Hence only the form of the PMNS matrix for the first octant of θ_{23} would be shown in the following if not mentioned explicitly.

4.1 $G_l=Z_3^{(2)}$, $G_\nu=Z_2^{(2)}$

In this case, the charged lepton mass matrix is diagonal, therefore the lepton mixing is completely determined by the neutrino sector, and the lepton mixing matrix coincides the neutrino diagonalization matrix U_ν up to permutations of rows and columns.

(I) $X_{\nu\mathbf{r}}=\rho_{\mathbf{r}}(1),\rho_{\mathbf{r}}(d^2)$

In this case, the lepton mixing matrix is determined to be

$$U_{PMNS} = \frac{1}{\sqrt{3}} \begin{pmatrix} -\sqrt{2}\cos\left(\frac{\pi}{12}-\theta\right) & 1 & \sqrt{2}\sin\left(\frac{\pi}{12}-\theta\right) \\ \sqrt{2}\cos\left(\frac{\pi}{4}+\theta\right) & 1 & \sqrt{2}\sin\left(\frac{\pi}{4}+\theta\right) \\ \sqrt{2}\sin\left(\frac{\pi}{12}+\theta\right) & 1 & -\sqrt{2}\cos\left(\frac{\pi}{12}+\theta\right) \end{pmatrix} P\,, \tag{4.1}$$

where P is a diagonal matrix with entry ± 1 or $\pm i$, and it would be neglected hereafter. In the present work, we shall work in the PDG convention [33], where the PMNS matrix is cast into the form

$$U_{PMNS} = V\,\mathrm{diag}(1, e^{i\frac{\alpha_{21}}{2}}, e^{i\frac{\alpha_{31}}{2}}), \tag{4.2}$$

with

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{pmatrix}, \tag{4.3}$$

where $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$, δ_{CP} is the Dirac CP phase, α_{21} and α_{31} are the Majorana CP phases. In the following, we shall redefine the Majorana phase and introduce $\alpha'_{31} = \alpha_{31} - 2\delta_{\mathrm{CP}}$ for the sake of convenience. As a result, the lepton mixing parameters are predicted to be

$$\begin{aligned} &\sin^2\theta_{13} = \frac{1}{3}\left[1 - \cos\left(\frac{\pi}{6} - 2\theta\right)\right], \qquad \sin^2\theta_{12} = \frac{1}{2+\cos\left(\frac{\pi}{6}-2\theta\right)}\,, \\ &\sin^2\theta_{23} = \frac{1+\sin 2\theta}{2+\cos\left(\frac{\pi}{6}-2\theta\right)}, \qquad \tan\delta_{CP} = \tan\alpha_{21} = \tan\alpha_{31} = 0\,. \end{aligned} \tag{4.4}$$

Obviously CP is predicted to be conserved in this scenario. The mixing parameters are strongly correlated with each other:

$$3\sin^2\theta_{12}\cos^2\theta_{13} = 1, \qquad \sin^2\theta_{23} = \frac{1}{2} \pm \frac{\sin\theta_{13}}{2\cos^2\theta_{13}}\sqrt{2 - 3\sin^2\theta_{13}}\,. \tag{4.5}$$

The correlations among the mixing angles are shown in Fig. 1. Excellent agreement with the present data [5] can be achieved. The best fitting value of θ is $\theta_{\mathrm{bf}} \simeq 0.0798$, the minimal value of χ^2 is $\chi^2_{\mathrm{min}} \simeq 9.548$, and the corresponding values for the mixing angles are:

$$\sin^2\theta_{13}(\theta_{\mathrm{bf}}) \simeq 0.0218, \quad \sin^2\theta_{12}(\theta_{\mathrm{bf}}) \simeq 0.341, \quad \sin^2\theta_{23}(\theta_{\mathrm{bf}}) \simeq 0.395\,, \tag{4.6}$$

where the atmospheric mixing angle is smaller than $\pi/4$ and therefore lies in the first octant. If we exchange the second and the third rows of the PMNS matrix in Eq. (4.1), the atmospheric mixing angle becomes

$$\sin^2\theta_{23} = \frac{1+\cos\left(\frac{\pi}{6}+2\theta\right)}{2+\cos\left(\frac{\pi}{6}-2\theta\right)}\,, \tag{4.7}$$

while the predictions for the remaining mixing parameters keep intact, as shown in Eq. (4.4). It is notable that the correlations in Eq. (4.5) are also satisfied in this case.

This pattern can also accommodate the observed lepton mixing data very well and it prefer the second octant θ_{23}. The best fitting values are

$$
\begin{gathered}
\theta_{\rm b.f.} \simeq 0.0798, \quad \chi^2_{\rm min} \simeq 9.303, \\
\sin^2\theta_{13}(\theta_{\rm bf}) \simeq 0.0218, \quad \sin^2\theta_{12}(\theta_{\rm bf}) \simeq 0.341, \quad \sin^2\theta_{23}(\theta_{\rm bf}) \simeq 0.605\,.
\end{gathered} \tag{4.8}
$$

(II) $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^2d), \rho_{\mathbf{r}}(c^2d^3)$

The PMNS matrix is given by

$$
U_{PMNS} = \frac{1}{\sqrt{3}} \begin{pmatrix} -\sqrt{2}\,e^{i\pi/4}\cos\left(\frac{\pi}{12}-\theta\right) & 1 & \sqrt{2}\,e^{i\pi/4}\sin\left(\frac{\pi}{12}-\theta\right) \\ \sqrt{2}\,e^{i\pi/4}\cos\left(\frac{\pi}{4}+\theta\right) & 1 & \sqrt{2}\,e^{i\pi/4}\sin\left(\frac{\pi}{4}+\theta\right) \\ \sqrt{2}\,e^{i\pi/4}\sin\left(\frac{\pi}{12}+\theta\right) & 1 & -\sqrt{2}\,e^{i\pi/4}\cos\left(\frac{\pi}{12}+\theta\right) \end{pmatrix}, \tag{4.9}
$$

where the trivial matrix P has been omitted. The lepton mixing parameters are

$$
\begin{gathered}
\sin^2\theta_{13} = \frac{1}{3}\left[1-\cos\left(\frac{\pi}{6}-2\theta\right)\right], \qquad \sin^2\theta_{12} = \frac{1}{2+\cos\left(\frac{\pi}{6}-2\theta\right)}, \\
\sin^2\theta_{23} = \frac{1+\sin 2\theta}{2+\cos\left(\frac{\pi}{6}-2\theta\right)}, \qquad \tan\delta_{CP} = \cot\alpha_{21} = \tan\alpha_{31} = 0\,.
\end{gathered} \tag{4.10}
$$

Compared with Eq. (4.4), the mixing parameters are predicted to be the same as those of case I except that now the Majorana phase α_{21} is $\pm\pi/2$ rather than zero.

(III) $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(a^2b), \rho_{\mathbf{r}}(a^2bd^2)$

The lepton mixing matrix is given by

$$
U_{PMNS} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2i\cos\theta & \sqrt{2} & -2i\sin\theta \\ -\sqrt{3}\sin\theta - i\cos\theta & \sqrt{2} & -\sqrt{3}\cos\theta + i\sin\theta \\ \sqrt{3}\sin\theta - i\cos\theta & \sqrt{2} & \sqrt{3}\cos\theta + i\sin\theta \end{pmatrix}. \tag{4.11}
$$

The lepton mixing parameters are determined to be

$$
\begin{gathered}
\sin^2\theta_{13} = \tfrac{1}{3}\left(1-\cos 2\theta\right), \qquad \sin^2\theta_{12} = \tfrac{1}{2+\cos 2\theta}, \quad \sin 2\theta_{23} = \tfrac{1}{2}, \\
|J_{CP}| = \tfrac{1}{6\sqrt{3}}\left|\sin 2\theta\right|, \qquad \cot\delta_{CP} = \tan\alpha_{21} = \tan\alpha_{31} = 0\,.
\end{gathered} \tag{4.12}
$$

Good agreement with the experimental data can be achieved in this case, and the best fitting results are listed in Table 1. Note that the solar mixing angle θ_{12} is related to the reactor mixing angle θ_{13} by $3\sin^2\theta_{12}\cos^2\theta_{13} = 1$, the atmospheric mixing angle is predicted to maximal and the Dirac CP is maximally broken. The correlations of $\sin^2\theta_{12}$ and $|J_{CP}|$ with respect to $\sin\theta_{13}$ are displayed in Fig. 2.

(IV) $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(a^2bd), \rho_{\mathbf{r}}(a^2bd^3)$

The lepton mixing matrix is

$$
U_{PMNS} = \frac{1}{2\sqrt{3}} \begin{pmatrix} \sqrt{3}\,e^{i\theta} - e^{-i\theta} & 2 & -i\left(\sqrt{3}\,e^{i\theta} + e^{-i\theta}\right) \\ 2e^{-i\theta} & 2 & 2ie^{-i\theta} \\ -\sqrt{3}\,e^{i\theta} - e^{-i\theta} & 2 & i\left(\sqrt{3}\,e^{i\theta} - e^{-i\theta}\right) \end{pmatrix}. \tag{4.13}
$$

The mixing parameters take the form

$$\sin^2\theta_{13}=\frac{1}{3}+\frac{1}{2\sqrt{3}}\cos 2\theta,\quad \sin^2\theta_{12}=\sin^2\theta_{23}=\frac{2}{4-\sqrt{3}\,\cos 2\theta},$$
$$|J_{CP}|=\frac{1}{6\sqrt{3}}\left|\sin 2\theta\right|,\quad |\tan\delta_{CP}|=\left|\frac{4-\sqrt{3}\,\cos 2\theta}{1-\sqrt{3}\,\cos 2\theta}\tan 2\theta\right|,$$
$$|\tan\alpha_{21}|=\left|\frac{\sin 2\theta}{\sqrt{3}-2\cos 2\theta}\right|,\quad |\tan\alpha'_{31}|=\left|\frac{4\sqrt{3}\,\sin 2\theta}{1-3\cos 4\theta}\right|. \tag{4.14}$$

We see that all the mixing parameters nontrivially depend on the parameter θ, and these results for the mixing parameters are illustrated in Fig. 3. However, this mixing pattern doesn't describe the the experimental data very well although not so bad. The minimum values of the χ^2 functions are somewhat large: 110.741 and 111.559 for $\theta_{23}<\pi/4$ and $\theta_{23}>\pi/4$ respectively, as shown in Table 1. The best fitting values $\theta_{\rm bf}$ is $\pi/2$, the reason is that for this value $\sin^2\theta_{13}$ is minimized as $\sin^2\theta_{13}(\theta_{\rm bf})=\left(2-\sqrt{3}\right)/6\simeq 0.0447$. In addition, all the three CP phases δ_{CP}, α_{21} and α_{31} become trivial with $\sin\delta_{CP}(\theta_{\rm bf})=\sin\alpha_{21}(\theta_{\rm bf})=\sin\alpha_{31}(\theta_{\rm bf})=0$ for $\theta_{\rm bf}=\pi/2$.

(V) $X_{\nu\mathbf{r}}=\rho_{\mathbf{r}}(a^2bc),\rho_{\mathbf{r}}(a^2bcd^2)$

In this case, the lepton mixing matrix is determined to be

$$U_{PMNS}=\frac{1}{\sqrt{3}}\begin{pmatrix} e^{-\frac{i\pi}{8}}\cos\left(\frac{\pi}{24}-\theta\right)-e^{\frac{3i\pi}{8}}\cos\left(\frac{\pi}{24}+\theta\right) & 1 & -e^{-\frac{i\pi}{8}}\sin\left(\frac{\pi}{24}-\theta\right)-e^{\frac{3i\pi}{8}}\sin\left(\frac{\pi}{24}+\theta\right)\\ -e^{-\frac{i\pi}{8}}\sin\left(\frac{5\pi}{24}-\theta\right)+e^{\frac{3i\pi}{8}}\sin\left(\frac{5\pi}{24}+\theta\right) & 1 & -e^{-\frac{i\pi}{8}}\cos\left(\frac{5\pi}{24}-\theta\right)-e^{\frac{3i\pi}{8}}\cos\left(\frac{5\pi}{24}+\theta\right)\\ -e^{-\frac{i\pi}{8}}\sin\left(\frac{\pi}{8}+\theta\right)+e^{\frac{3i\pi}{8}}\sin\left(\frac{\pi}{8}-\theta\right) & 1 & e^{-\frac{i\pi}{8}}\cos\left(\frac{\pi}{8}+\theta\right)+e^{\frac{3i\pi}{8}}\cos\left(\frac{\pi}{8}-\theta\right)\end{pmatrix}. \tag{4.15}$$

We can straightforwardly read out the lepton mixing parameters

$$\sin^2\theta_{13}=\frac{1}{3}-\frac{\sqrt{6}+\sqrt{2}}{12}\cos 2\theta,\qquad \sin^2\theta_{12}=\frac{4}{8+\left(\sqrt{6}+\sqrt{2}\right)\cos 2\theta},$$
$$\sin^2\theta_{23}=\frac{4+\left(\sqrt{6}-\sqrt{2}\right)\cos 2\theta}{8+\left(\sqrt{6}+\sqrt{2}\right)\cos 2\theta},\qquad |J_{CP}|=\frac{1}{6\sqrt{3}}\left|\sin 2\theta\right|,$$
$$|\tan\delta_{CP}|=\left|\frac{4\sqrt{2}+\left(1+\sqrt{3}\right)\cos 2\theta}{1-\sqrt{3}-\sqrt{2}\cos 2\theta}\tan 2\theta\right|,$$
$$|\tan\alpha_{21}|=\left|\frac{\sqrt{6}+\sqrt{2}+4\cos 2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}{\sqrt{6}+\sqrt{2}+4\cos 2\theta-\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}\right|,$$
$$|\tan\alpha'_{31}|=\left|\frac{4\sin 2\theta}{2-3\sqrt{3}+\left(2+\sqrt{3}\right)\cos 4\theta}\right|. \tag{4.16}$$

Note that all the mixing parameters are nontrivial functions of θ. The different mixing angles are correlated with each other as

$$3\sin^2\theta_{12}\cos^2\theta_{13}=1,$$

$$
\begin{aligned}
&3\sin^2\theta_{23}\cos^2\theta_{13}=3-\sqrt{3}-3\left(2-\sqrt{3}\right)\sin^2\theta_{13}, \quad && \theta_{23}<\pi/4,\\
&3\sin^2\theta_{23}\cos^2\theta_{13}=\sqrt{3}-3\left(\sqrt{3}-1\right)\sin^2\theta_{13}, && \theta_{23}>\pi/4\,.
\end{aligned}
\tag{4.17}
$$

For the measured value of θ_{13} with $\sin^2\theta_{13}=0.0227$, we obtain $\sin^2\theta_{12}\simeq 0.341$ and $\sin^2\theta_{23}\simeq 0.426$ ($\sin^2\theta_{23}\simeq 0.574$), which are in accordance with experimental data. The best fitting values are presented in Table 2 and the mixing parameters are plotted in Fig. 4, we see that the CP phases δ_{CP} and α_{21} no longer take regular values such as 0 or $\pm\pi/2$ although $|\alpha'_{31}|\simeq\pi/2$ is approximately fulfilled.

(VI) $X_{\nu\mathbf{r}}=\rho_{\mathbf{r}}(a^2bcd),\rho_{\mathbf{r}}(a^2bcd^3)$
The PMNS matrix takes the form

$$
U_{PMNS}=\frac{1}{\sqrt{3}}\begin{pmatrix}
-e^{-\frac{i\pi}{8}}\cos\left(\frac{\pi}{24}+\theta\right)+e^{\frac{3i\pi}{8}}\cos\left(\frac{\pi}{24}-\theta\right) & 1 & -e^{-\frac{i\pi}{8}}\sin\left(\frac{\pi}{24}+\theta\right)-e^{\frac{3i\pi}{8}}\sin\left(\frac{\pi}{24}-\theta\right)\\
e^{-\frac{i\pi}{8}}\sin\left(\frac{5\pi}{24}+\theta\right)-e^{\frac{3i\pi}{8}}\sin\left(\frac{5\pi}{24}-\theta\right) & 1 & -e^{-\frac{i\pi}{8}}\cos\left(\frac{5\pi}{24}+\theta\right)-e^{\frac{3i\pi}{8}}\cos\left(\frac{5\pi}{24}-\theta\right)\\
e^{-\frac{i\pi}{8}}\sin\left(\frac{\pi}{8}-\theta\right)-e^{\frac{3i\pi}{8}}\sin\left(\frac{\pi}{8}+\theta\right) & 1 & e^{-\frac{i\pi}{8}}\cos\left(\frac{\pi}{8}-\theta\right)+e^{\frac{3i\pi}{8}}\cos\left(\frac{\pi}{8}+\theta\right)
\end{pmatrix}.
\tag{4.18}
$$

The lepton mixing parameters read as

$$
\begin{aligned}
&\sin^2\theta_{13}=\frac{1}{3}-\frac{\sqrt{6}+\sqrt{2}}{12}\cos 2\theta, \qquad \sin^2\theta_{12}=\frac{4}{8+\left(\sqrt{6}+\sqrt{2}\right)\cos 2\theta},\\
&\sin^2\theta_{23}=\frac{4+\left(\sqrt{6}-\sqrt{2}\right)\cos 2\theta}{8+\left(\sqrt{6}+\sqrt{2}\right)\cos 2\theta}, \qquad |J_{CP}|=\frac{1}{6\sqrt{3}}\left|\sin 2\theta\right|,\\
&|\tan\delta_{CP}|=\left|\frac{4\sqrt{2}+\left(1+\sqrt{3}\right)\cos 2\theta}{1-\sqrt{3}-\sqrt{2}\cos 2\theta}\tan 2\theta\right|,\\
&|\tan\alpha_{21}|=\left|\frac{\sqrt{6}+\sqrt{2}+4\cos 2\theta-\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}{\sqrt{6}+\sqrt{2}+4\cos 2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}\right|,\\
&|\tan\alpha'_{31}|=\left|\frac{4\sin 2\theta}{2-3\sqrt{3}+\left(2+\sqrt{3}\right)\cos 4\theta}\right|.
\end{aligned}
\tag{4.19}
$$

Compared with Eq. (4.16), we see that $|\tan\alpha_{21}|$ is predicted to be the inverse of the corresponding one of case V, and all the remaining mixing parameters coincide exactly in both cases.

(VII) $X_{\nu\mathbf{r}}=\rho_{\mathbf{r}}(a^2bc^2),\rho_{\mathbf{r}}(a^2bc^2d^2)$
The PMNS matrix is of the form

$$
U_{PMNS}=\frac{1}{2\sqrt{3}}\begin{pmatrix}
ccce^{\frac{i\pi}{4}}\left(\sqrt{3}\,e^{i\theta}-e^{-i\theta}\right) & 2 & -e^{\frac{3i\pi}{4}}\left(\sqrt{3}\,e^{i\theta}+e^{-i\theta}\right)\\
2e^{i\left(\frac{\pi}{4}-\theta\right)} & 2 & 2e^{i\left(\frac{3\pi}{4}-\theta\right)}\\
-e^{\frac{i\pi}{4}}\left(\sqrt{3}\,e^{i\theta}+e^{-i\theta}\right) & 2 & e^{\frac{3i\pi}{4}}\left(\sqrt{3}\,e^{i\theta}-e^{-i\theta}\right)
\end{pmatrix}.
\tag{4.20}
$$

The lepton mixing parameters are fixed to be

$$
\sin^2\theta_{13}=\frac{1}{3}+\frac{1}{2\sqrt{3}}\cos 2\theta, \qquad \sin^2\theta_{12}=\sin^2\theta_{23}=\frac{2}{4-\sqrt{3}\cos 2\theta},
$$

$$|J_{CP}|=\frac{1}{6\sqrt{3}}\left|\sin 2\theta\right|\,,\qquad |\tan\delta_{CP}|=\left|\frac{4-\sqrt{3}\cos 2\theta}{1-\sqrt{3}\cos 2\theta}\tan 2\theta\right|\,,$$
$$|\tan\alpha_{21}|=\left|\frac{\sqrt{3}-2\cos 2\theta}{\sin 2\theta}\right|\,,\qquad |\tan\alpha'_{31}|=\left|\frac{4\sqrt{3}\sin 2\theta}{1-3\cos 4\theta}\right|\,. \tag{4.21}$$

Obviously the lepton mixing parameters are of the same forms in case VII and case IV except the Majorana phase α which fulfills $\alpha_{21}^{\text{VII}}=\alpha_{21}^{\text{IV}}-\pi/2$, where the superscripts "VII" and "IV" denote the different remnant symmetries.

(VIII) $X_{\nu\mathbf{r}}=\rho_{\mathbf{r}}(a^2bc^2d),\rho_{\mathbf{r}}(a^2bc^2d^3)$

In this case, the lepton flavor mixing matrix is determined to be

$$U_{PMNS}=\frac{1}{\sqrt{6}}\begin{pmatrix} ccc2e^{\frac{i\pi}{4}}\cos\theta & \sqrt{2} & -2e^{\frac{i\pi}{4}}\sin\theta \\ -e^{\frac{i\pi}{4}}\left(\cos\theta+i\sqrt{3}\sin\theta\right) & \sqrt{2} & e^{\frac{i\pi}{4}}\left(\sin\theta-i\sqrt{3}\cos\theta\right) \\ -e^{\frac{i\pi}{4}}\left(\cos\theta-i\sqrt{3}\sin\theta\right) & \sqrt{2} & e^{\frac{i\pi}{4}}\left(\sin\theta+i\sqrt{3}\cos\theta\right)\end{pmatrix}\,. \tag{4.22}$$

The lepton mixing parameters are given by

$$\sin^2\theta_{13}=\frac{1}{3}\left(1-\cos 2\theta\right)\,,\qquad \sin^2\theta_{12}=\frac{1}{2+\cos 2\theta},\qquad \sin^2\theta_{23}=\frac{1}{2}\,,$$
$$|J_{CP}|=\frac{1}{6\sqrt{3}}\left|\sin 2\theta\right|\,,\qquad \cot\delta_{CP}=\cot\alpha_{21}=\tan\alpha_{31}=0\,. \tag{4.23}$$

Obviously the phenomenological predictions of case VIII and case III only differ in the Majorana phase α_{21}. It is maximally broken in case VIII while it is completely conserved in case III. Analogous to case III, the experimentally preferred values of the mixing angles can be obtained.

(IX) $X_{\nu\mathbf{r}}=\rho_{\mathbf{r}}(a^2bc^3),\rho_{\mathbf{r}}(a^2bc^3d^2)$

The lepton mixing matrix takes the form

$$U_{PMNS}=\frac{1}{\sqrt{3}}\begin{pmatrix} -e^{-\frac{3i\pi}{8}}\cos\left(\frac{\pi}{24}+\theta\right)+e^{\frac{i\pi}{8}}\cos\left(\frac{\pi}{24}-\theta\right) & 1 & -e^{-\frac{3i\pi}{8}}\sin\left(\frac{\pi}{24}+\theta\right)-e^{\frac{i\pi}{8}}\sin\left(\frac{\pi}{24}-\theta\right) \\ e^{-\frac{3i\pi}{8}}\sin\left(\frac{5\pi}{24}+\theta\right)-e^{\frac{i\pi}{8}}\sin\left(\frac{5\pi}{24}-\theta\right) & 1 & -e^{-\frac{3i\pi}{8}}\cos\left(\frac{5\pi}{24}+\theta\right)-e^{\frac{i\pi}{8}}\cos\left(\frac{5\pi}{24}-\theta\right) \\ e^{-\frac{3i\pi}{8}}\sin\left(\frac{\pi}{8}-\theta\right)-e^{\frac{i\pi}{8}}\sin\left(\frac{\pi}{8}+\theta\right) & 1 & e^{-\frac{3i\pi}{8}}\cos\left(\frac{\pi}{8}-\theta\right)+e^{\frac{i\pi}{8}}\cos\left(\frac{\pi}{8}+\theta\right)\end{pmatrix}\,, \tag{4.24}$$

which is the complex conjugate of the PMNS matrix of case V shown in Eq. (4.15). The lepton mixing parameters are determined to be

$$\sin^2\theta_{13}=\frac{1}{3}-\frac{\sqrt{6}+\sqrt{2}}{12}\cos 2\theta,\qquad \sin^2\theta_{12}=\frac{4}{8+\left(\sqrt{6}+\sqrt{2}\right)\cos 2\theta},$$
$$\sin^2\theta_{23}=\frac{4+\left(\sqrt{6}-\sqrt{2}\right)\cos 2\theta}{8+\left(\sqrt{6}+\sqrt{2}\right)\cos 2\theta},\qquad |J_{CP}|=\frac{1}{6\sqrt{3}}\left|\sin 2\theta\right|\,,$$
$$|\tan\delta_{CP}|=\left|\frac{4\sqrt{2}+\left(1+\sqrt{3}\right)\cos 2\theta}{1-\sqrt{3}-\sqrt{2}\cos 2\theta}\tan 2\theta\right|\,,$$

$$|\tan\alpha_{21}| = \left|\frac{\sqrt{6}+\sqrt{2}+4\cos 2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}{\sqrt{6}+\sqrt{2}+4\cos 2\theta-\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}\right|,$$
$$|\tan\alpha'_{31}| = \left|\frac{4\sin 2\theta}{2-3\sqrt{3}+\left(2+\sqrt{3}\right)\cos 4\theta}\right|. \tag{4.25}$$

The above mixing parameters are exactly the same as the ones of case V. Hence the relations in Eq. (4.17) arise as well.

(X) $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(a^2bc^3d), \rho_{\mathbf{r}}(a^2bc^3d^3)$
The PMNS mixing matrix is given by

$$U_{PMNS} = \frac{1}{\sqrt{3}}\begin{pmatrix} e^{-\frac{3i\pi}{8}}\cos\left(\frac{\pi}{24}-\theta\right)-e^{\frac{i\pi}{8}}\cos\left(\frac{\pi}{24}+\theta\right) & 1 & -e^{-\frac{3i\pi}{8}}\sin\left(\frac{\pi}{24}-\theta\right)-e^{\frac{i\pi}{8}}\sin\left(\frac{\pi}{24}+\theta\right) \\ -e^{-\frac{3i\pi}{8}}\sin\left(\frac{5\pi}{24}-\theta\right)+e^{\frac{i\pi}{8}}\sin\left(\frac{5\pi}{24}+\theta\right) & 1 & -e^{-\frac{3i\pi}{8}}\cos\left(\frac{5\pi}{24}-\theta\right)-e^{\frac{i\pi}{8}}\cos\left(\frac{5\pi}{24}+\theta\right) \\ -e^{-\frac{3i\pi}{8}}\sin\left(\frac{\pi}{8}+\theta\right)+e^{\frac{i\pi}{8}}\sin\left(\frac{\pi}{8}-\theta\right) & 1 & e^{-\frac{3i\pi}{8}}\cos\left(\frac{\pi}{8}+\theta\right)+e^{\frac{i\pi}{8}}\cos\left(\frac{\pi}{8}-\theta\right) \end{pmatrix}, \tag{4.26}$$

which is the complex conjugate of the predicted PMNS matrix of case VI. The lepton mixing parameters are

$$\sin^2\theta_{13} = \frac{1}{3}-\frac{\sqrt{6}+\sqrt{2}}{12}\cos 2\theta, \qquad \sin^2\theta_{12} = \frac{4}{8+\left(\sqrt{6}+\sqrt{2}\right)\cos 2\theta},$$
$$\sin^2\theta_{23} = \frac{4+\left(\sqrt{6}-\sqrt{2}\right)\cos 2\theta}{8+\left(\sqrt{6}+\sqrt{2}\right)\cos 2\theta}, \qquad |J_{CP}| = \frac{1}{6\sqrt{3}}\left|\sin 2\theta\right|,$$
$$|\tan\delta_{CP}| = \left|\frac{4\sqrt{2}+\left(1+\sqrt{3}\right)\cos 2\theta}{1-\sqrt{3}-\sqrt{2}\cos 2\theta}\tan 2\theta\right|,$$
$$|\tan\alpha_{21}| = \left|\frac{\sqrt{6}+\sqrt{2}+4\cos 2\theta-\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}{\sqrt{6}+\sqrt{2}+4\cos 2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}\right|,$$
$$|\tan\alpha'_{31}| = \left|\frac{4\sin 2\theta}{2-3\sqrt{3}+\left(2+\sqrt{3}\right)\cos 4\theta}\right|. \tag{4.27}$$

They are exactly the same as the phenomenological predictions of case VI, as shown in Eq. (4.19).

For all the cases (case I — case X) discussed above, the second column of the mixing matrix is constrained to be $(1,1,1)^T/\sqrt{3}$ due to the protection of the remnant family symmetry $G_\nu = Z_2^{(2)}$. As a result, the relation $3\sin^2\theta_{12}\cos^2\theta_{13}=1$ is satisfied such that the solar mixing angle has a lower limit given by $\sin^2\theta_{12}\geq 1/3$.

4.2 $G_l = Z_3^{(2)}$, $G_\nu = Z_2^{(9)}$

In this scenario, only the residual CP symmetry $H^l_{CP} = \{\rho_{\mathbf{r}}(1), \rho_{\mathbf{r}}(ac), \rho_{\mathbf{r}}(a^2cd)\}$ is viable, and the hermitian combination $m_l^\dagger m_l$ is diagonal. Hence the lepton mixing is completely determined by the neutrino sector up to permutations of rows and columns.

(XI) $X_{\nu \mathbf{r}} = \rho_{\mathbf{r}}(1), \rho_{\mathbf{r}}(a^2bd)$

In this case, the lepton mixing matrix is determined to be

$$U_{PMNS} = \frac{1}{2\sqrt{3}} \begin{pmatrix} -1-\sqrt{3} & \left(\sqrt{3}-1\right)\sin\theta + 2\cos\theta & \left(\sqrt{3}-1\right)\cos\theta - 2\sin\theta \\ 2 & 2\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right) & 2\sqrt{2}\cos\left(\frac{\pi}{4}+\theta\right) \\ \sqrt{3}-1 & -\left(1+\sqrt{3}\right)\sin\theta + 2\cos\theta & -\left(1+\sqrt{3}\right)\cos\theta - 2\sin\theta \end{pmatrix}. \tag{4.28}$$

The lepton mixing parameters are given by

$$\begin{aligned} \sin^2\theta_{13} &= \frac{1}{12}\left[4-\sqrt{3}-\sqrt{3}\cos 2\theta - 2\left(\sqrt{3}-1\right)\sin 2\theta\right], \\ \sin^2\theta_{12} &= \frac{4-\sqrt{3}+\sqrt{3}\cos 2\theta + 2\left(\sqrt{3}-1\right)\sin 2\theta}{8+\sqrt{3}+\sqrt{3}\cos 2\theta + 2\left(\sqrt{3}-1\right)\sin 2\theta}, \\ \sin^2\theta_{23} &= \frac{4\left(1-\sin 2\theta\right)}{8+\sqrt{3}+\sqrt{3}\cos 2\theta + 2\left(\sqrt{3}-1\right)\sin 2\theta}, \\ \sin\delta_{CP} &= \sin\alpha_{21} = \sin\alpha_{31} = 0\,. \end{aligned} \tag{4.29}$$

Note that CP is fully conserved in this scenario. The results for the above predicted mixing angles are shown in Fig. 5, and the best fitting values for the mixing angles are collected in Table 3. We see that both $\sin^2\theta_{12}(\theta_{\rm bf})$ and $\sin^2\theta_{23}(\theta_{\rm bf})$ are slightly beyond the experimentally preferred 3σ range [5]. Therefore this mixing pattern can marginally accommodate the experimental data.

(XII) $X_{\nu \mathbf{r}} = \rho_{\mathbf{r}}(d^2), \rho_{\mathbf{r}}(a^2bd^3)$

The PMNS matrix is of the form

$$U_{PMNS} = \frac{1}{2\sqrt{3}} \begin{pmatrix} -\left(1+\sqrt{3}\right)\sin\theta + 2i\cos\theta & -\left(1+\sqrt{3}\right)\cos\theta - 2i\sin\theta & \sqrt{3}-1 \\ 2ie^{-i\theta} & 2e^{-i\theta} & 2 \\ \left(\sqrt{3}-1\right)\sin\theta + 2i\cos\theta & \left(\sqrt{3}-1\right)\cos\theta - 2i\sin\theta & -1-\sqrt{3} \end{pmatrix}. \tag{4.30}$$

The lepton mixing parameters are determined to be

$$\begin{aligned} &\sin^2\theta_{13} = \frac{1}{6}\left(2-\sqrt{3}\right), \qquad \sin^2\theta_{12} = \frac{1}{2} + \frac{\sqrt{3}\cos 2\theta}{2\left(4+\sqrt{3}\right)}, \qquad \sin^2\theta_{23} = \frac{2}{4+\sqrt{3}}, \\ &|J_{CP}| = \frac{1}{12\sqrt{3}}\left|\sin 2\theta\right|, \qquad \left|\tan\delta_{CP}\right| = \left|\frac{\left(4+\sqrt{3}\right)\tan 2\theta}{2\left(1+\sqrt{3}\right)}\right|, \\ &\left|\tan\alpha_{21}\right| = \left|\frac{8\left(3+\sqrt{3}\right)\sin 2\theta}{29+16\sqrt{3}+3\cos 4\theta}\right|, \quad \left|\tan\alpha'_{31}\right| = \left|\frac{2\left(1-\sqrt{3}\right)\sin 2\theta}{3-2\sqrt{3}+\left(5-2\sqrt{3}\right)\cos 2\theta}\right|. \end{aligned} \tag{4.31}$$

Obviously both the reactor mixing angle θ_{13} and the atmospheric mixing angle θ_{23} are independent of the parameter θ here. The best fit value of θ is $\theta_{\rm bf} = \pi/2$, since the minimal value of $\sin^2\theta_{12}$ is $\sin^2\theta_{12}\left(\theta_{\rm bf}\right) = 2/\left(4+\sqrt{3}\right)$. For $\theta_{\rm bf} = \pi/2$, all the three CP phases are trivial with $\sin\alpha_{21}\left(\theta_{\rm bf}\right) = \sin\alpha_{31}\left(\theta_{\rm bf}\right) = \sin\delta_{CP}\left(\theta_{\rm bf}\right) = 0$ whereas the resulting

$\sin^2\theta_{12}(\theta_{\rm bf})$ and $\sin^2\theta_{13}(\theta_{\rm bf})$ are slightly larger than their 3σ upper bounds [5], as shown in Table 3. The correlations of $|J_{CP}|$, $|\sin\delta_{CP}|$, $|\sin\alpha_{21}|$ and $|\sin\alpha'_{31}|$ with respect to $\sin\theta_{13}$ are displayed in Fig. 5.

(XIII) $X_{\nu\mathbf{r}}=\rho_{\mathbf{r}}(c^2d),\rho_{\mathbf{r}}(a^2bc^2d^2)$

The PMNS matrix is of the form

$$U_{PMNS}=\frac{1}{2\sqrt{3}}\begin{pmatrix}-1-\sqrt{3} & \left(\sqrt{3}-1\right)\sin\theta+2e^{-\frac{i\pi}{4}}\cos\theta & \left(\sqrt{3}-1\right)\cos\theta-2e^{-\frac{i\pi}{4}}\sin\theta \\ 2 & 2\sin\theta+2e^{-\frac{i\pi}{4}}\cos\theta & 2\cos\theta-2e^{-\frac{i\pi}{4}}\sin\theta \\ \sqrt{3}-1 & -\left(1+\sqrt{3}\right)\sin\theta+2e^{-\frac{i\pi}{4}}\cos\theta & -\left(1+\sqrt{3}\right)\cos\theta-2e^{-\frac{i\pi}{4}}\sin\theta\end{pmatrix}. \tag{4.32}$$

The lepton mixing parameters nontrivially depend on the parameter θ as follows:

$$\begin{aligned}
&\sin^2\theta_{13}=\frac{1}{12}\left[4-\sqrt{3}-\sqrt{3}\cos2\theta-\left(\sqrt{6}-\sqrt{2}\right)\sin2\theta\right],\\
&\sin^2\theta_{12}=\frac{4-\sqrt{3}+\sqrt{3}\cos2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin2\theta}{8+\sqrt{3}+\sqrt{3}\cos2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin2\theta},\\
&\sin^2\theta_{23}=\frac{4-2\sqrt{2}\sin2\theta}{8+\sqrt{3}+\sqrt{3}\cos2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin2\theta},\qquad |J_{CP}|=\frac{1}{12\sqrt{6}}\left|\sin2\theta\right|,\\
&|\tan\delta_{CP}|=\left|\frac{\left(6-2\sqrt{3}\right)\left(1-\cos4\theta\right)+\left(6\sqrt{2}+16\sqrt{6}\right)\sin2\theta+3\sqrt{2}\sin4\theta}{24+18\sqrt{3}+\left(24-8\sqrt{3}\right)\cos2\theta-6\sqrt{2}\sin2\theta+6\sqrt{3}\cos4\theta-\left(15\sqrt{2}+4\sqrt{6}\right)\sin4\theta}\right|,\\
&|\tan\alpha_{21}|=\left|\frac{2\left(2+\sqrt{3}\right)\left(1+\cos2\theta\right)+\left(\sqrt{6}+\sqrt{2}\right)\sin2\theta}{1-\cos2\theta+\left(\sqrt{6}+\sqrt{2}\right)\sin2\theta}\right|,\\
&|\tan\alpha'_{31}|=\left|\frac{2\left(2+\sqrt{3}\right)\left(1-\cos2\theta\right)-\left(\sqrt{6}+\sqrt{2}\right)\sin2\theta}{1+\cos2\theta-\left(\sqrt{6}+\sqrt{2}\right)\sin2\theta}\right|.
\end{aligned}\tag{4.33}$$

The relation among the different mixing parameters are shown in Fig. 6. The best fitting results are listed in Table 4. The experimental data can be marginally accommodated.

(XIV) $X_{\nu\mathbf{r}}=\rho_{\mathbf{r}}(c^2d^3),\rho_{\mathbf{r}}(a^2bc^2)$

The PMNS matrix is given by

$$U_{PMNS}=\frac{1}{2\sqrt{3}}\begin{pmatrix}-1-\sqrt{3} & i\left(1-\sqrt{3}\right)\sin\theta+2e^{-\frac{i\pi}{4}}\cos\theta & i\left(1-\sqrt{3}\right)\cos\theta-2e^{-\frac{i\pi}{4}}\sin\theta \\ 2 & -2i\sin\theta+2e^{-\frac{i\pi}{4}}\cos\theta & -2i\cos\theta-2e^{-\frac{i\pi}{4}}\sin\theta \\ \sqrt{3}-1 & i\left(1+\sqrt{3}\right)\sin\theta+2e^{-\frac{i\pi}{4}}\cos\theta & i\left(1+\sqrt{3}\right)\cos\theta-2e^{-\frac{i\pi}{4}}\sin\theta\end{pmatrix}. \tag{4.34}$$

It is related to the resulting PMNS matrix of case XIII via $U^{\rm XIV}_{PMNS}=U^{\rm XIII^*}_{PMNS}\,{\rm diag}\left(1,-i,-i\right)$. We can straightforwardly calculate the lepton mixing parameters

$$\begin{aligned}
&\sin^2\theta_{13}=\frac{1}{12}\left[4-\sqrt{3}-\sqrt{3}\cos2\theta-\left(\sqrt{6}-\sqrt{2}\right)\sin2\theta\right],\\
&\sin^2\theta_{12}=\frac{4-\sqrt{3}+\sqrt{3}\cos2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin2\theta}{8+\sqrt{3}+\sqrt{3}\cos2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin2\theta},
\end{aligned}$$

$$
\begin{aligned}
\sin^2\theta_{23} &= \frac{4-2\sqrt{2}\sin 2\theta}{8+\sqrt{3}+\sqrt{3}\cos 2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}, \qquad |J_{CP}| = \frac{1}{12\sqrt{6}}\left|\sin 2\theta\right|\,,\\
|\tan\delta_{CP}| &= \left|\frac{\left(6-2\sqrt{3}\right)(1-\cos 4\theta)+\left(6\sqrt{2}+16\sqrt{6}\right)\sin 2\theta+3\sqrt{2}\sin 4\theta}{24+18\sqrt{3}+\left(24-8\sqrt{3}\right)\cos 2\theta-6\sqrt{2}\sin 2\theta+6\sqrt{3}\cos 4\theta-\left(15\sqrt{2}+4\sqrt{6}\right)\sin 4\theta}\right|,\\
|\tan\alpha_{21}| &= \left|\frac{2\left(2+\sqrt{3}\right)(1+\cos 2\theta)+\left(\sqrt{6}+\sqrt{2}\right)\sin 2\theta}{1-\cos 2\theta+\left(\sqrt{6}+\sqrt{2}\right)\sin 2\theta}\right|,\\
|\tan\alpha'_{31}| &= \left|\frac{2\left(2+\sqrt{3}\right)(1-\cos 2\theta)-\left(\sqrt{6}+\sqrt{2}\right)\sin 2\theta}{1+\cos 2\theta-\left(\sqrt{6}+\sqrt{2}\right)\sin 2\theta}\right|. \qquad (4.35)
\end{aligned}
$$

Compared with Eq. (4.33), we see that the above mixing parameters are of the same form as those of case XIII.

4.3 $G_l = Z_3^{(2)}$, $G_\nu = Z_2^{(10)}$

Similar to previous cases, the lepton flavor mixing completely comes from the neutrino sector. The PMNS matrix is fixed up to permutations of rows and columns.

(XV) $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(1), \rho_{\mathbf{r}}(a^2bd^2)$

In this case, the lepton mixing matrix is of the following form

$$
U_{PMNS} = \frac{1}{\sqrt{6}}\begin{pmatrix} 2 & \sqrt{2}\,\cos\theta & \sqrt{2}\,\sin\theta \\ -1 & \sqrt{2}\,\cos\theta+\sqrt{3}\,\sin\theta & \sqrt{2}\,\sin\theta-\sqrt{3}\,\cos\theta \\ -1 & \sqrt{2}\,\cos\theta-\sqrt{3}\,\sin\theta & \sqrt{2}\,\sin\theta+\sqrt{3}\,\cos\theta \end{pmatrix}. \qquad (4.36)
$$

We see that the first column of the PMNS matrix is $(2,-1,-1)^T/\sqrt{6}$, this mixing pattern is the so-called TM_1 mixing. Since the PMNS matrix is real, there is no CP violation in this scenario. The lepton mixing parameters read

$$
\begin{aligned}
&\sin^2\theta_{13} = \frac{1}{6}\left(1-\cos 2\theta\right), \qquad \sin^2\theta_{12} = \frac{1+\cos 2\theta}{5+\cos 2\theta},\\
&\sin^2\theta_{23} = \frac{1}{2}-\frac{\sqrt{6}\sin 2\theta}{5+\cos 2\theta}, \qquad \tan\delta_{CP} = \tan\alpha_{21} = \tan\alpha_{31} = 0\,. \qquad (4.37)
\end{aligned}
$$

As a consequence, the mixing angles are related with each other as

$$
3\cos^2\theta_{12}\cos^2\theta_{13} = 2, \qquad \sin^2\theta_{23} = \frac{1}{2}\pm\frac{\sqrt{2-6\sin^2\theta_{13}}}{\cos^2\theta_{13}}\sin\theta_{13}\,. \qquad (4.38)
$$

The correlations among the mixing angles are presented in Fig. 7. As is shown in Table 5, we can find a value of θ for which the resulting mixing angles agree rather well with the experimental observations [5].

(XVI) $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(d^2), \rho_{\mathbf{r}}(a^2b)$

In this case, the PMNS matrix is determined to be

$$U_{PMNS} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2}\ \cos\theta & \sqrt{2}\ \sin\theta \\ -1 & \sqrt{2}\ \cos\theta + i\sqrt{3}\ \sin\theta & \sqrt{2}\ \sin\theta - i\sqrt{3}\ \cos\theta \\ -1 & \sqrt{2}\ \cos\theta - i\sqrt{3}\ \sin\theta & \sqrt{2}\ \sin\theta + i\sqrt{3}\ \cos\theta \end{pmatrix} . \tag{4.39}$$

The lepton mixing parameters are

$$\begin{aligned} &\sin^2\theta_{13} = \frac{1}{6}\left(1-\cos 2\theta\right), \qquad \sin^2\theta_{12} = \frac{1+\cos 2\theta}{5+\cos 2\theta}, \qquad \sin^2\theta_{23} = \frac{1}{2}, \\ &|J_{CP}| = \frac{1}{6\sqrt{6}}\left|\sin 2\theta\right|, \qquad \cot\delta_{CP} = \tan\alpha_{21} = \tan\alpha_{31} = 0 . \end{aligned} \tag{4.40}$$

Note that the atmospheric mixing angle and the Dirac CP phase are predicted to be maximal while both Majorana phases are trivial. The best fitting values are presented in Table 5, and excellent agreement with the experimental observations could be achieved.

(XVII) $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^2d), \rho_{\mathbf{r}}(a^2bc^2d^3)$

The PMNS matrix is given by

$$U_{PMNS} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2}\,e^{-\frac{i\pi}{4}}\cos\theta & \sqrt{2}\,e^{-\frac{i\pi}{4}}\sin\theta \\ -1 & \sqrt{3}\ \sin\theta + \sqrt{2}\,e^{-\frac{i\pi}{4}}\cos\theta & -\sqrt{3}\ \cos\theta + \sqrt{2}\,e^{-\frac{i\pi}{4}}\sin\theta \\ -1 & -\sqrt{3}\ \sin\theta + \sqrt{2}\,e^{-\frac{i\pi}{4}}\cos\theta & \sqrt{3}\ \cos\theta + \sqrt{2}\,e^{-\frac{i\pi}{4}}\sin\theta \end{pmatrix} . \tag{4.41}$$

The lepton mixing parameters are

$$\begin{aligned} &\sin^2\theta_{13} = \frac{1}{6}\left(1-\cos 2\theta\right), \quad \sin^2\theta_{12} = \frac{1+\cos 2\theta}{5+\cos 2\theta}, \quad \sin^2\theta_{23} = \frac{1}{2} - \frac{\sqrt{3}\sin 2\theta}{5+\cos 2\theta}, \\ &|J_{CP}| = \frac{1}{12\sqrt{3}}\left|\sin 2\theta\right|, \quad |\tan\delta_{CP}| = \left|\frac{5+\cos 2\theta}{1+5\cos 2\theta}\right|, \quad \cot\alpha_{21} = \cot\alpha'_{31} = 0 . \end{aligned} \tag{4.42}$$

We have maximal Majorana CP violation with $|\sin\alpha_{21}| = |\sin\alpha'_{31}| = 1$ in this case. There is a deviation of the atmospheric angle θ_{23} from maximal mixing. The three mixing angles are correlated with each other as

$$3\cos^2\theta_{12}\cos^2\theta_{13} = 2, \qquad \sin^2\theta_{23} = \frac{1}{2} \pm \frac{\sqrt{1-3\sin^2\theta_{13}}}{\cos^2\theta_{13}}\sin\theta_{13} . \tag{4.43}$$

With the measured reactor mixing angle $\sin^2\theta_{13} = 0.0227$, the other two mixing angles are determined to be $\sin^2\theta_{12} \simeq 0.318$, $\sin^2\theta_{23} \simeq 0.351$ or $\sin^2\theta_{23} \simeq 0.649$, which are in the experimentally preferred ranges. The above results for the mixing parameters are displayed in Fig. 8. It is remarkable that the Dirac CP phase is always nontrivial in this case, and its best fitting value fulfills $|\sin\delta_{\rm CP}(\theta_{\rm bf})| \simeq 0.738$.

(XVIII) $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^2d^3), \rho_{\mathbf{r}}(a^2bc^2d)$

The lepton mixing matrix takes the form

$$U_{PMNS} = \frac{1}{\sqrt{6}} \begin{pmatrix} 2 & \sqrt{2}\, e^{-\frac{i\pi}{4}} \cos\theta & \sqrt{2}\, e^{-\frac{i\pi}{4}} \sin\theta \\ -1 & \sqrt{2}\, e^{-\frac{i\pi}{4}} \cos\theta - \sqrt{3}\, i \sin\theta & \sqrt{2}\, e^{-\frac{i\pi}{4}} \sin\theta + \sqrt{3}\, i \cos\theta \\ -1 & \sqrt{2}\, e^{-\frac{i\pi}{4}} \cos\theta + \sqrt{3}\, i \sin\theta & \sqrt{2}\, e^{-\frac{i\pi}{4}} \sin\theta - \sqrt{3}\, i \cos\theta \end{pmatrix}, \quad (4.44)$$

which is related to the PMNS matrix of case XVII via $U^{\mathrm{XVIII}}_{PMNS} = U^{\mathrm{XVII}*}_{PMNS}\, \mathrm{diag}\,(1, -i, -i)$. The lepton mixing parameters read

$$\begin{aligned} &\sin^2\theta_{13} = \frac{1}{6}\left(1 - \cos 2\theta\right), \quad \sin^2\theta = \frac{1 + \cos 2\theta}{5 + \cos 2\theta}, \quad \sin^2\theta_{23} = \frac{1}{2} - \frac{\sqrt{3}\sin 2\theta}{5 + \cos 2\theta}, \\ &|J_{CP}| = \frac{1}{12\sqrt{3}}\,|\sin 2\theta|\,, \quad |\tan\delta_{CP}| = \left|\frac{5 + \cos 2\theta}{1 + 5\cos 2\theta}\right|, \quad \cot\alpha_{21} = \cot\alpha'_{31} = 0\,. \quad (4.45) \end{aligned}$$

They coincide with already predicted mixing parameters of case XVII, as shown in Eq. (4.42). As a result, the correlations in Eq. (4.43) are satisfied, and the experimental data can be accommodated very well.

For the above discussed case XV, case XVI, case XVII and case XVIII, the first column of the PMNS martrix is $(2, -1, -1)^T/\sqrt{6}$. As a consequence, the relation $3\cos^2\theta_{12}\cos^2\theta_{13} = 2$ is always fulfilled such that θ_{12} has a upper bound $\sin^2\theta_{12} \leq 1/3$.

4.4 $G_l = K_4^{(3)}$, $G_\nu = Z_2^{(9)}$

Now the contribution from the charged lepton sector to the lepton mixing is nontrivial, and it takes the form of Eq. (3.14). Combining the unitary transformation U_ν from the neutrino sector, which is studied in section 3.4.2, we can straightforwardly obtain the predictions for the lepton flavor mixing matrix.

(XIX) $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(1), \rho_{\mathbf{r}}(a^2bd)$

In this case, the PMNS matrix is

$$U_{PMNS} = \frac{1}{2} \begin{pmatrix} \sqrt{2} & i\sqrt{2}\cos\theta & i\sqrt{2}\sin\theta \\ 1 & -i\cos\theta + \sqrt{2}\, e^{-\frac{i\pi}{4}} \sin\theta & -i\sin\theta - \sqrt{2}\, e^{-\frac{i\pi}{4}} \cos\theta \\ 1 & -i\cos\theta - \sqrt{2}\, e^{-\frac{i\pi}{4}} \sin\theta & -i\sin\theta + \sqrt{2}\, e^{-\frac{i\pi}{4}} \cos\theta \end{pmatrix}. \quad (4.46)$$

The lepton mixing parameters are found to be

$$\begin{aligned} &\sin^2\theta_{13} = \frac{1}{4}\left(1 - \cos 2\theta\right), \quad \sin^2\theta_{12} = \frac{1 + \cos 2\theta}{3 + \cos 2\theta}, \quad \sin^2\theta_{23} = \frac{1}{2} - \frac{\sin 2\theta}{3 + \cos 2\theta}, \\ &|J_{CP}| = \frac{1}{16}\,|\sin 2\theta|\,, \quad |\tan\delta_{CP}| = \left|\frac{3 + \cos 2\theta}{1 + 3\cos 2\theta}\right|, \quad \tan\alpha_{21} = \tan\alpha'_{31} = 0\,. \quad (4.47) \end{aligned}$$

Notice that the Majorana CP is conserved, and the remaining other mixing parameters nontrivially depend on the parameter θ. The mixing angles are related by

$$2\cos^2\theta_{12}\cos^2\theta_{13}=1, \qquad \sin^2\theta_{23}=\frac{1}{2}\pm\frac{\sqrt{2-4\sin^2\theta_{13}}}{2\cos^2\theta_{13}}\sin\theta_{13}\,. \tag{4.48}$$

The correlations between different mixing parameters are illustrated in Fig. 9, we see that the correct values of θ_{13}, θ_{12} and θ_{23} can not be reproduced in this case. For the 3σ range of the reactor mixing angle $0.0156\leq\sin^2\theta_{13}\leq 0.0299$ [5], the atmospheric mixing angle is calculated to be in the range of $[0.377, 0.412]\cup[0.588, 0.622]$ which is compatible with the experimental data. However, the solar angle is constrained to vary in the range of $[0.484, 0.492]$ which has no overlap with the experimentally preferred 3σ region. Hence the minimal value of the χ^2 function is rather large: 204.875 and 204.610 for the first octant and the second octant θ_{23} respectively.

(XX) $X_{\nu\mathbf{r}}=\rho_{\mathbf{r}}(d^2), \rho_{\mathbf{r}}(a^2bd^3)$

The PMNS matrix is

$$U_{PMNS}=\frac{1}{2}\begin{pmatrix}\sqrt{2} & \sqrt{2}\,i\cos\theta & \sqrt{2}\,i\sin\theta \\ 1 & -i\cos\theta-\sqrt{2}\,e^{\frac{i\pi}{4}}\sin\theta & -i\sin\theta+\sqrt{2}\,e^{\frac{i\pi}{4}}\cos\theta \\ 1 & -i\cos\theta+\sqrt{2}\,e^{\frac{i\pi}{4}}\sin\theta & -i\sin\theta-\sqrt{2}\,e^{\frac{i\pi}{4}}\cos\theta\end{pmatrix}\,. \tag{4.49}$$

From Eq. (4.46), we see that this PMNS matrix is closely related to case XIX's prediction as $U^{\mathrm{XX}}_{PMNS}=U^{\mathrm{XIX}*}_{PMNS}\,\mathrm{diag}\,(1,-1,-1)$. The lepton mixing parameters are

$$\begin{gathered}\sin^2\theta_{13}=\frac{1}{4}\left(1-\cos 2\theta\right), \quad \sin^2\theta_{12}=\frac{1+\cos 2\theta}{3+\cos 2\theta}, \quad \sin^2\theta_{23}=\frac{1}{2}-\frac{\sin 2\theta}{3+\cos 2\theta}\,, \\ |J_{CP}|=\frac{1}{16}\left|\sin 2\theta\right|, \quad |\tan\delta_{CP}|=\left|\frac{3+\cos 2\theta}{1+3\cos 2\theta}\right|, \quad \tan\alpha_{21}=\tan\alpha'_{31}=0\,,\end{gathered} \tag{4.50}$$

which are the same as the phenomenological predictions of case XIX. Hence the observed values of θ_{12}, θ_{13} and θ_{23} can not be achieved simultaneously in this scenario.

(XXI) $X_{\nu\mathbf{r}}=\rho_{\mathbf{r}}(c^2d), \rho_{\mathbf{r}}(a^2bc^2d^2)$

In this case, the lepton mixing matrix takes the form

$$U_{PMNS}=\frac{1}{2}\begin{pmatrix}-i\left(\sqrt{2}\cos\theta+\sin\theta\right) & 1 & -i\left(\cos\theta-\sqrt{2}\sin\theta\right) \\ i\sqrt{2}\sin\theta & \sqrt{2} & i\sqrt{2}\cos\theta \\ i\left(\sqrt{2}\cos\theta-\sin\theta\right) & 1 & -i\left(\cos\theta+\sqrt{2}\sin\theta\right)\end{pmatrix}\,. \tag{4.51}$$

The lepton mixing parameters are

$$\begin{gathered}\sin^2\theta_{13}=\frac{1}{8}\left(3-\cos 2\theta-2\sqrt{2}\sin 2\theta\right), \qquad \sin^2\theta_{12}=\frac{2}{5+\cos 2\theta+2\sqrt{2}\sin 2\theta}, \\ \sin^2\theta_{23}=\frac{2+2\cos 2\theta}{5+\cos 2\theta+2\sqrt{2}\sin 2\theta}, \qquad \tan\delta_{CP}=\tan\alpha_{21}=\tan\alpha_{31}=0\,.\end{gathered} \tag{4.52}$$

We have CP conservation in this scenario. The three mixing angles are related with each as

$$
\begin{aligned}
& 4\sin^2\theta_{12}\cos^2\theta_{13}=1, \\
& 9\sin^2\theta_{23}\cos^2\theta_{13}=3-2\sin^2\theta_{13}\pm 2\sin\theta_{13}\sqrt{6-8\sin^2\theta_{13}}\;, \qquad \theta_{23}<\pi/4, \\
& 9\sin^2\theta_{23}\cos^2\theta_{13}=6-7\sin^2\theta_{13}\mp 2\sin\theta_{13}\sqrt{6-8\sin^2\theta_{13}}\;, \qquad \theta_{23}<\pi/4\,.
\end{aligned}
\tag{4.53}
$$

As a consequence, we have $\sin^2\theta_{12}=1/\left(4\cos^2\theta_{13}\right)$ and the solar mixing angle is predicted to be close to its 3σ lower bound. The predicted mixing angles in Eq. (4.52) are shown in Fig. 10. The best fitting results are presented in Table 6. The minimum value of the χ^2 function is rather small: 14.811 for $\theta_{23}<\pi/4$ and 15.138 for $\theta_{23}>\pi/4$. Hence excellent agreement with the experimental data can be achieved.

(XXII) $X_{\nu\mathbf{r}}=\rho_{\mathbf{r}}(c^2d^3), \rho_{\mathbf{r}}(a^2bc^2)$

The PMNS matrix is given

$$
U_{PMNS}=\frac{1}{2}\begin{pmatrix}
\sqrt{2} & \sqrt{2}\cos\theta & \sqrt{2}\sin\theta \\
1 & -\cos\theta-i\sqrt{2}\sin\theta & -\sin\theta+i\sqrt{2}\cos\theta \\
1 & -\cos\theta+i\sqrt{2}\sin\theta & -\sin\theta-i\sqrt{2}\cos\theta
\end{pmatrix}\,. \tag{4.54}
$$

The lepton mixing parameters are

$$
\begin{aligned}
& \sin^2\theta_{13}=\frac{1}{4}\left(1-\cos 2\theta\right), \qquad \sin^2\theta_{12}=\frac{1+\cos 2\theta}{3+\cos 2\theta}, \qquad \sin^2\theta_{23}=\frac{1}{2}\,, \\
& |J_{CP}|=\frac{1}{8\sqrt{2}}\left|\sin 2\theta\right|, \qquad \cot\delta_{CP}=\tan\alpha_{21}=\tan\alpha_{31}=0\,.
\end{aligned}
\tag{4.55}
$$

We see that the atmospheric neutrino mixing is maximal, and the Dirac CP is maximally violated. In addition, the solar mixing angle and the reactor mixing angle are predicted to be of the same form as the corresponding ones of case XIX. Therefore we have the equality $2\cos^2\theta_{12}\cos^2\theta_{13}=1$ and thus the correct values of θ_{12} and θ_{13} can not be reproduced simultaneously, as shown in Fig. 9. Moderate corrections to θ_{12} and θ_{13} are necessary in order to match the experimental best fit value.

In short summary, for the above four cases XIX, XX, XXI and XXII, the PMNS matrix is predicted to have one column of the form $\left(\sqrt{2},1,1\right)^T/2$ or $\left(1,\sqrt{2},1\right)^T/2$ which is in common (up to permutation) with the bimaximal mixing pattern. Only the case XXI can accommodate the observed the three lepton mixing angles, and the remaining three cases can not produce the correct values of the θ_{12} and θ_{13} simultaneously.

4.5 $G_l=Z_8^{(1)}$, $G_\nu=Z_2^{(9)}$

In this scenario, the charged lepton matrix $m_l^\dagger m_l$ is diagonalized by the unitary transformation U_l shown in Eq. (3.27). The constraints on the light neutrino mass matrix and its diagonalization for $G_\nu=Z_2^{(9)}$ has been discussed in section 3.4.2. The resulting PMNS matrix for different remnant CP symmetry in the neutrino sector can be easily obtained as follows.

(XXIII) $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(1), \rho_{\mathbf{r}}(a^2bd)$

In this case, the lepton mixing matrix is

$$U_{PMNS} = \frac{1}{2}\begin{pmatrix} \sqrt{2} & i\sqrt{2}\cos\theta & i\sqrt{2}\sin\theta \\ 1 & \sqrt{2}\sin\theta - i\cos\theta & -\sqrt{2}\cos\theta - i\sin\theta \\ 1 & -\sqrt{2}\sin\theta - i\cos\theta & \sqrt{2}\cos\theta - i\sin\theta \end{pmatrix}, \tag{4.56}$$

which is related to the PMNS matrix of the above case XXII by $U^{\rm XXIII}_{PMNS} = U^{\rm XXII}_{PMNS}\,\mathrm{diag}\,(1, i, i)$. The lepton mixing parameters are

$$\begin{aligned} &\sin^2\theta_{13} = \frac{1}{4}\left(1 - \cos 2\theta\right), \qquad \sin^2\theta_{12} = \frac{1+\cos 2\theta}{3+\cos 2\theta}, \qquad \sin^2\theta_{23} = \frac{1}{2} \\ &|J_{CP}| = \frac{1}{8\sqrt{2}}\left|\sin 2\theta\right|, \qquad \cot\delta_{CP} = \tan\alpha_{21} = \tan\alpha_{31} = 0\,, \end{aligned} \tag{4.57}$$

which coincide exactly with the phenomenological predictions of case XXII.

(XXIV) $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(d^2), \rho_{\mathbf{r}}(a^2bd^3)$

In this case, the lepton mixing matrix is

$$U_{PMNS} = \frac{1}{2}\begin{pmatrix} -i\left(\sqrt{2}\cos\theta + \sin\theta\right) & 1 & -i\left(\cos\theta - \sqrt{2}\sin\theta\right) \\ i\sqrt{2}\sin\theta & \sqrt{2} & i\sqrt{2}\cos\theta \\ i\left(\sqrt{2}\cos\theta - \sin\theta\right) & 1 & -i\left(\cos\theta + \sqrt{2}\sin\theta\right) \end{pmatrix}. \tag{4.58}$$

It is exactly the same as the PMNS matrix of case XXI. Hence the same lepton mixing parameters are predicted as

$$\begin{aligned} &\sin^2\theta_{13} = \frac{1}{8}\left(3 - \cos 2\theta - 2\sqrt{2}\sin 2\theta\right), \qquad \sin^2\theta_{12} = \frac{2}{5 + \cos 2\theta + 2\sqrt{2}\sin 2\theta}, \\ &\sin^2\theta_{23} = \frac{2 + 2\cos 2\theta}{5 + \cos 2\theta + 2\sqrt{2}\sin 2\theta}, \qquad \tan\delta_{CP} = \tan\alpha_{21} = \tan\alpha_{31} = 0\,. \end{aligned} \tag{4.59}$$

As has been already shown, the experimental data can be accommodated very well.

(XXV) $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^2d), \rho_{\mathbf{r}}(a^2bc^2d^2)$

In this case, the lepton mixing matrix is determined to be

$$U_{PMNS} = \frac{1}{2}\begin{pmatrix} \sqrt{2} & i\sqrt{2}\cos\theta & i\sqrt{2}\sin\theta \\ 1 & -i\cos\theta + \sqrt{2}\,e^{-\frac{i\pi}{4}}\sin\theta & -i\sin\theta - \sqrt{2}\,e^{-\frac{i\pi}{4}}\cos\theta \\ 1 & -i\cos\theta - \sqrt{2}\,e^{-\frac{i\pi}{4}}\sin\theta & -i\sin\theta + \sqrt{2}\,e^{-\frac{i\pi}{4}}\cos\theta \end{pmatrix}. \tag{4.60}$$

It coincides with the PMNS matrix of case XIX. As a consequence, the same lepton mixing parameters as shown in Eq. (4.47) arise:

$$\begin{aligned} &\sin^2\theta_{13} = \frac{1}{4}\left(1 - \cos 2\theta\right), \quad \sin^2\theta_{12} = \frac{1+\cos 2\theta}{3+\cos 2\theta}, \quad \sin^2\theta_{23} = \frac{1}{2} - \frac{\sin 2\theta}{3+\cos 2\theta}, \\ &|J_{CP}| = \frac{1}{16}\left|\sin 2\theta\right|, \quad |\tan\delta_{CP}| = \left|\frac{3+\cos 2\theta}{1+3\cos 2\theta}\right|, \quad \tan\alpha_{21} = \tan\alpha'_{31} = 0\,. \end{aligned} \tag{4.61}$$

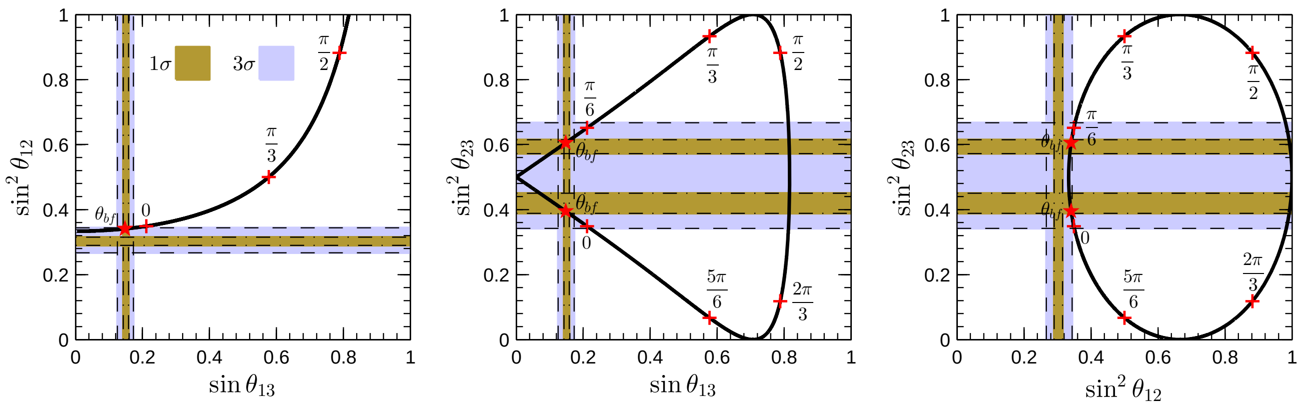

Figure 1: The relation among the lepton mixing angles in the case I and case II. The best fit value θ_{bf} for θ is labelled as a star. We also indicate the points for $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$ by the sign "+" on the curves, and only one θ value is displayed if there are points matching together. The shown 3σ ranges for the mixing angles and their best fit values are taken from Ref. [5].

(XXVI) $X_{\nu\mathbf{r}} = \rho_{\mathbf{r}}(c^2d^3), \rho_{\mathbf{r}}(a^2bc^2)$

In this case, the PMNS matrix is given by

$$U_{PMNS} = \frac{1}{2}\begin{pmatrix} \sqrt{2} & \sqrt{2}\cos\theta & \sqrt{2}\sin\theta \\ 1 & -\cos\theta - \sqrt{2}\,e^{-\frac{i\pi}{4}}\sin\theta & -\sin\theta + \sqrt{2}\,e^{-\frac{i\pi}{4}}\cos\theta \\ 1 & -\cos\theta + \sqrt{2}\,e^{-\frac{i\pi}{4}}\sin\theta & -\sin\theta - \sqrt{2}\,e^{-\frac{i\pi}{4}}\cos\theta \end{pmatrix}. \tag{4.62}$$

Compared with the case XX's PMNS matrix in Eq. (4.49), we have $U^{\mathrm{XXVI}}_{PMNS} = U^{\mathrm{XX}}_{PMNS}\,\mathrm{diag}\,(1, -i, -i)$. The lepton mixing parameters read as

$$\sin^2\theta_{13} = \frac{1}{4}\left(1-\cos 2\theta\right), \qquad \sin^2\theta_{12} = \frac{1+\cos 2\theta}{3+\cos 2\theta}, \qquad \sin^2\theta_{23} = \frac{1}{2} - \frac{\sin 2\theta}{3+\cos 2\theta},$$
$$|J_{CP}| = \frac{1}{16}\left|\sin 2\theta\right|, \qquad |\tan\delta_{CP}| = \left|\frac{3+\cos 2\theta}{1+3\cos 2\theta}\right|, \qquad \tan\alpha_{21} = \tan\alpha'_{31} = 0\,. \tag{4.63}$$

They are of the same form as the corresponding ones of case XX, as shown in Eq. (4.50). Hence the relation $2\cos^2\theta_{12}\cos^2\theta_{13} = 1$ is fulfilled such that the observed values of θ_{12} and θ_{13} can not be accommodated simultaneously. In summary, compared with scenario of $G_l = K_4^{(3)}$, $G_\nu = Z_2^{(9)}$, no new predictions for the lepton mixing parameters are obtained. and only the case XXIV is in accordance with the present data.

5 Conclusions

We have performed a comprehensive study of the $\Delta(96)$ family symmetry combined with the generalised CP symmetry $H_{\rm CP}$. We have investigated the lepton mixing parameters which can be obtained from the original symmetry $\Delta(96) \rtimes H_{\rm CP}$ breaking to different remnant symmetries in the neutrino and charged lepton sectors, namely G_ν and

	I, II	III, VIII	IV, VII
$\sin^2\theta_{13}$	$\frac{1}{3}\left[1-\cos\left(\frac{\pi}{6}-2\theta\right)\right]$	$\frac{1}{3}\left(1-\cos 2\theta\right)$	$\frac{1}{3}+\frac{1}{2\sqrt{3}}\cos 2\theta$
$\sin^2\theta_{12}$	$\frac{1}{2+\cos\left(\frac{\pi}{6}-2\theta\right)}$	$\frac{1}{2+\cos 2\theta}$	$\frac{2}{4-\sqrt{3}\cos 2\theta}$
$\sin^2\theta_{23}(\theta_{23}<\pi/4)$	$\frac{1+\sin 2\theta}{2+\cos\left(\frac{\pi}{6}-2\theta\right)}$	$\frac{1}{2}$	$\frac{2}{4-\sqrt{3}\cos 2\theta}$
$\sin^2\theta_{23}(\theta_{23}>\pi/4)$	$\frac{1+\cos\left(\frac{\pi}{6}+2\theta\right)}{2+\cos\left(\frac{\pi}{6}-2\theta\right)}$	$\frac{1}{2}$	$\frac{2-\sqrt{3}\cos 2\theta}{4-\sqrt{3}\cos 2\theta}$
$\lvert J_{\rm CP}\rvert$	0	$\frac{1}{6\sqrt{3}}\lvert\sin 2\theta\rvert$	$\frac{1}{6\sqrt{3}}\lvert\sin 2\theta\rvert$
$\lvert\tan\delta_{\rm CP}\rvert$	0	$+\infty$	$\left\lvert\frac{4-\sqrt{3}\cos 2\theta}{1-\sqrt{3}\cos 2\theta}\tan 2\theta\right\rvert$
$\lvert\tan\alpha_{21}\rvert$	0, case I	0, case III	$\left\lvert\frac{\sin 2\theta}{\sqrt{3}-2\cos 2\theta}\right\rvert$, case IV
	$+\infty$, case II	$+\infty$, case VIII	$\left\lvert\frac{\sqrt{3}-2\cos 2\theta}{\sin 2\theta}\right\rvert$, case VII
$\lvert\tan\alpha'_{31}\rvert$	0	0	$\left\lvert\frac{4\sqrt{3}\sin 2\theta}{1-3\cos 4\theta}\right\rvert$

Best Fitting

	$\theta_{23}<\pi/4$	$\theta_{23}>\pi/4$	$\theta_{23}<\pi/4$	$\theta_{23}>\pi/4$	$\theta_{23}<\pi/4$	$\theta_{23}>\pi/4$
$\chi^2_{\rm min}$	9.548	9.303	14.527	27.254	110.741	111.559
$\theta_{\rm bf}$	0.0798		± 0.184		$\pm\pi/2$	
$\sin^2\theta_{13}(\theta_{\rm bf})$	0.0218		0.0222		0.0447	
$\theta_{13}(\theta_{\rm bf})/^\circ$	8.498		8.576		12.200	
$\sin^2\theta_{12}(\theta_{\rm bf})$	0.341		0.341		0.349	
$\theta_{12}(\theta_{\rm bf})/^\circ$	35.715		35.724		36.206	
$\sin^2\theta_{23}(\theta_{\rm bf})$	0.395	0.605	0.5		0.349	0.651
$\sin^2\theta_{23}(\theta_{\rm bf})/^\circ$	38.936	51.072	45		36.206	53.794
$\lvert\sin\delta_{\rm CP}(\theta_{\rm bf})\rvert$	0		1		0	
$\delta_{\rm CP}(\theta_{\rm bf})/^\circ$	0		90		0	
$\lvert\sin\alpha_{21}(\theta_{\rm bf})\rvert$	0, case I		0, case III		0, case IV	
	1, case II		1, case VIII		1, case VII	
$\alpha_{21}(\theta_{\rm bf})/^\circ$	0, case I		0, case III		0, case IV	
	90, case II		90, case VIII		90, case VII	
$\lvert\sin\alpha'_{31}(\theta_{\rm bf})\rvert$	0		0		0	
$\alpha'_{31}(\theta_{\rm bf})/^\circ$	0		0		0	

Table 1: The results of the mixing parameters for the cases I, II, III, IV, VII and VIII, where "$+\infty$" for $\lvert\tan\delta_{\rm CP}\rvert$, $\lvert\tan\alpha\rvert$ and $\lvert\tan\beta\rvert$ implies that the absolute value of the corresponding CP phase is $\pi/2$. Notice that the Dirac CP phase $\delta_{\rm CP}$ is determined up to $\delta_{\rm CP}$, $\pi+\delta_{\rm CP}$, $\pi-\delta_{\rm CP}$ and $2\pi-\delta_{\rm CP}$ in the present context, and only one representative value is displayed in this table. The same convention is taken for the Majorana CP phases α_{21} and α'_{31}.

	V, VI, IX, X	
$\sin^2\theta_{13}$	$\frac{1}{3}-\frac{\sqrt{6}+\sqrt{2}}{12}\cos 2\theta$	
$\sin^2\theta_{12}$	$\frac{4}{8+\left(\sqrt{6}+\sqrt{2}\right)\cos 2\theta}$	
$\sin^2\theta_{23}(\theta_{23}<\pi/4)$	$\frac{4+\left(\sqrt{6}-\sqrt{2}\right)\cos 2\theta}{8+\left(\sqrt{6}+\sqrt{2}\right)\cos 2\theta}$	
$\sin^2\theta_{23}(\theta_{23}>\pi/4)$	$\frac{4+2\sqrt{2}\cos 2\theta}{8+\left(\sqrt{6}+\sqrt{2}\right)\cos 2\theta}$	
$\lvert J_{\rm CP}\rvert$	$\frac{1}{6\sqrt{3}}\lvert\sin 2\theta\rvert$	
$\lvert\tan\delta_{\rm CP}\rvert$	$\left\lvert\frac{4\sqrt{2}+\left(1+\sqrt{3}\right)\cos 2\theta}{1-\sqrt{3}-\sqrt{2}\cos 2\theta}\tan 2\theta\right\rvert$	
$\lvert\tan\alpha_{21}\rvert$	$\left\lvert\frac{\sqrt{6}+\sqrt{2}+4\cos 2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}{\sqrt{6}+\sqrt{2}+4\cos 2\theta-\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}\right\rvert$, cases V, IX	
	$\left\lvert\frac{\sqrt{6}+\sqrt{2}+4\cos 2\theta-\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}{\sqrt{6}+\sqrt{2}+4\cos 2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}\right\rvert$, cases VI, X	
$\lvert\tan\alpha'_{31}\rvert$	$\left\lvert\frac{4\sin 2\theta}{2-3\sqrt{3}+\left(2+\sqrt{3}\right)\cos 4\theta}\right\rvert$	
Best Fitting		
	$\theta_{23}<\pi/4$	$\theta_{23}>\pi/4$
$\chi^2_{\rm min}$	9.124	9.838
$\theta_{\rm bf}$	± 0.130	
$\sin^2\theta_{13}(\theta_{\rm bf})$	0.0222	
$\theta_{13}(\theta_{\rm bf})/^\circ$	8.574	
$\sin^2\theta_{12}(\theta_{\rm bf})$	0.341	
$\theta_{12}(\theta_{\rm bf})/^\circ$	35.724	
$\sin^2\theta_{23}(\theta_{\rm bf})$	0.426	0.574
$\theta_{23}(\theta_{\rm bf})/^\circ$	40.754	49.246
$\lvert\sin\delta_{\rm CP}(\theta_{\rm bf})\rvert$	0.725	
$\delta_{\rm CP}(\theta_{\rm bf})/^\circ$	46.512	
$\lvert\sin\alpha_{21}(\theta_{\rm bf})\rvert$	0.682 or 0.731	
$\alpha_{21}(\theta_{\rm bf})/^\circ$	43.023 or 46.977	
$\lvert\sin\alpha'_{31}(\theta_{\rm bf})\rvert$	0.999	
$\alpha'_{31}(\theta_{\rm bf})/^\circ$	87.755	

Table 2: The results of the mixing parameters for the cases V, VI, IX and X, where "$+\infty$" for $\lvert\tan\delta_{\rm CP}\rvert$, $\lvert\tan\alpha\rvert$ and $\lvert\tan\beta\rvert$ implies that the absolute value of the corresponding CP phase is $\pi/2$. Notice that the Dirac CP phase $\delta_{\rm CP}$ is determined up to $\delta_{\rm CP}$, $\pi+\delta_{\rm CP}$, $\pi-\delta_{\rm CP}$ and $2\pi-\delta_{\rm CP}$ in the present context, and only one representative value is displayed in this table. The same convention is taken for the Majorana CP phases α_{21} and α'_{31}.

	XI		XII	
$\sin^2\theta_{13}$	$\frac{1}{12}\left[4-\sqrt{3}-\sqrt{3}\cos 2\theta-2\left(\sqrt{3}-1\right)\sin 2\theta\right]$		$\frac{1}{6}\left(2-\sqrt{3}\right)$	
$\sin^2\theta_{12}$	$\frac{4-\sqrt{3}+\sqrt{3}\cos 2\theta+2\left(\sqrt{3}-1\right)\sin 2\theta}{8+\sqrt{3}+\sqrt{3}\cos 2\theta+2\left(\sqrt{3}-1\right)\sin 2\theta}$		$\frac{1}{2}+\frac{\sqrt{3}\cos 2\theta}{2\left(4+\sqrt{3}\right)}$	
$\sin^2\theta_{23}(\theta_{23}<\pi/4)$	$\frac{4(1-\sin 2\theta)}{8+\sqrt{3}+\sqrt{3}\cos 2\theta+2\left(\sqrt{3}-1\right)\sin 2\theta}$		$\frac{2}{4+\sqrt{3}}$	
$\sin^2\theta_{23}(\theta_{23}>\pi/4)$	$\frac{4+\sqrt{3}+\sqrt{3}\cos 2\theta+2\left(1+\sqrt{3}\right)\sin 2\theta}{8+\sqrt{3}+\sqrt{3}\cos 2\theta+2\left(\sqrt{3}-1\right)\sin 2\theta}$		$\frac{2+\sqrt{3}}{4+\sqrt{3}}$	
$\lvert J_{\rm CP}\rvert$	0		$\frac{1}{12\sqrt{3}}\lvert\sin 2\theta\rvert$	
$\lvert\tan\delta_{\rm CP}\rvert$	0		$\left\lvert\frac{\left(4+\sqrt{3}\right)\tan 2\theta}{2\left(1+\sqrt{3}\right)}\right\rvert$	
$\lvert\tan\alpha_{21}\rvert$	0		$\left\lvert\frac{8\left(3+\sqrt{3}\right)\sin 2\theta}{29+16\sqrt{3}+3\cos 4\theta}\right\rvert$	
$\lvert\tan\alpha'_{31}\rvert$	0		$\left\lvert\frac{2\left(1-\sqrt{3}\right)\sin 2\theta}{3-2\sqrt{3}+\left(5-2\sqrt{3}\right)\cos 2\theta}\right\rvert$	
Best Fitting				
	$\theta_{23}<\pi/4$	$\theta_{23}>\pi/4$	$\theta_{23}<\pi/4$	$\theta_{23}>\pi/4$
$\chi^2_{\rm min}$	48.862	54.822	110.741	111.559
$\theta_{\rm bf}$	0.0764	0.0711	$\pm\pi/2$	
$\sin^2\theta_{13}(\theta_{\rm bf})$	0.0278	0.0288	0.0447	
$\theta_{13}(\theta_{\rm bf})/^\circ$	9.592	9.774	12.200	
$\sin^2\theta_{12}(\theta_{\rm bf})$	0.360		0.349	
$\theta_{12}(\theta_{\rm bf})/^\circ$	36.883		36.206	
$\sin^2\theta_{23}(\theta_{\rm bf})$	0.291	0.705	0.349	0.651
$\theta_{23}(\theta_{\rm bf})/^\circ$	32.624	57.129	36.206	53.794
$\lvert\sin\delta_{\rm CP}(\theta_{\rm bf})\rvert$	0		0	
$\delta_{\rm CP}(\theta_{\rm bf})/^\circ$	0		0	
$\lvert\sin\alpha_{21}(\theta_{\rm bf})\rvert$	0		0	
$\alpha_{21}(\theta_{\rm bf})/^\circ$	0		0	
$\lvert\sin\alpha'_{31}(\theta_{\rm bf})\rvert$	0		0	
$\alpha'_{31}(\theta_{\rm bf})/^\circ$	0		0	

Table 3: The results of the mixing parameters for the case XI and case XII, where "$+\infty$" for $|\tan\delta_{\rm CP}|$, $|\tan\alpha|$ and $|\tan\beta|$ implies that the absolute value of the corresponding CP phase is $\pi/2$. Notice that the Dirac CP phase $\delta_{\rm CP}$ is determined up to $\delta_{\rm CP}$, $\pi+\delta_{\rm CP}$, $\pi-\delta_{\rm CP}$ and $2\pi-\delta_{\rm CP}$ in the present context, and only one representative value is displayed in this table. The same convention is taken for the Majorana CP phases α_{21} and α'_{31}.

	XIII, XIV	
$\sin^2\theta_{13}$	$\frac{1}{12}\left[4-\sqrt{3}-\sqrt{3}\cos 2\theta-\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta\right]$	
$\sin^2\theta_{12}$	$\frac{4-\sqrt{3}+\sqrt{3}\cos 2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}{8+\sqrt{3}+\sqrt{3}\cos 2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}$	
$\sin^2\theta_{23}(\theta_{23}<\pi/4)$	$\frac{4-2\sqrt{2}\sin 2\theta}{8+\sqrt{3}+\sqrt{3}\cos 2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}$	
$\sin^2\theta_{23}(\theta_{23}>\pi/4)$	$\frac{4+\sqrt{3}+\sqrt{3}\cos 2\theta+\left(\sqrt{6}+\sqrt{2}\right)\sin 2\theta}{8+\sqrt{3}+\sqrt{3}\cos 2\theta+\left(\sqrt{6}-\sqrt{2}\right)\sin 2\theta}$	
$\lvert J_{\rm CP}\rvert$	$\frac{1}{12\sqrt{6}}\lvert\sin 2\theta\rvert$	
$\lvert\tan\delta_{\rm CP}\rvert$	$\left\lvert\frac{\left(6-2\sqrt{3}\right)(1-\cos 4\theta)+\left(6\sqrt{2}+16\sqrt{6}\right)\sin 2\theta+3\sqrt{2}\sin 4\theta}{24+18\sqrt{3}+\left(24-8\sqrt{3}\right)\cos 2\theta-6\sqrt{2}\sin 2\theta+6\sqrt{3}\cos 4\theta-\left(15\sqrt{2}+4\sqrt{6}\right)\sin 4\theta}\right\rvert$	
$\lvert\tan\alpha_{21}\rvert$	$\left\lvert\frac{2\left(2+\sqrt{3}\right)(1+\cos 2\theta)+\left(\sqrt{6}+\sqrt{2}\right)\sin 2\theta}{1-\cos 2\theta+\left(\sqrt{6}+\sqrt{2}\right)\sin 2\theta}\right\rvert$	
$\lvert\tan\alpha'_{31}\rvert$	$\left\lvert\frac{2\left(2+\sqrt{3}\right)(1-\cos 2\theta)-\left(\sqrt{6}+\sqrt{2}\right)\sin 2\theta}{1+\cos 2\theta-\left(\sqrt{6}+\sqrt{2}\right)\sin 2\theta}\right\rvert$	
Best Fitting		
	$\theta_{23}<\pi/4$	$\theta_{23}>\pi/4$
$\chi^2_{\rm min}$	51.645	57.745
$\theta_{\rm bf}$	0.113	0.103
$\sin^2\theta_{13}(\theta_{\rm bf})$	0.0290	0.0301
$\theta_{13}(\theta_{\rm bf})/^\circ$	9.805	9.989
$\sin^2\theta_{12}(\theta_{\rm bf})$	0.359	
$\theta_{12}(\theta_{\rm bf})/^\circ$	36.835	
$\sin^2\theta_{23}(\theta_{\rm bf})$	0.289	0.706
$\theta_{23}(\theta_{\rm bf})/^\circ$	32.514	57.161
$\lvert\sin\delta_{\rm CP}(\theta_{\rm bf})\rvert$	0.212	0.189
$\delta_{\rm CP}(\theta_{\rm bf})/^\circ$	12.235	10.886
$\lvert\sin\alpha_{21}(\theta_{\rm bf})\rvert$	0.998	
$\alpha_{21}(\theta_{\rm bf})/^\circ$	86.732	
$\lvert\sin\alpha'_{31}(\theta_{\rm bf})\rvert$	0.520	0.469
$\alpha'_{31}(\theta_{\rm bf})/^\circ$	31.358	27.944

Table 4: The results of the mixing parameters for the case XIII and case XIV, where "$+\infty$" for $|\tan\delta_{\rm CP}|$, $|\tan\alpha|$ and $|\tan\beta|$ implies that the absolute value of the corresponding CP phase is $\pi/2$. Notice that the Dirac CP phase $\delta_{\rm CP}$ is determined up to $\delta_{\rm CP}$, $\pi+\delta_{\rm CP}$, $\pi-\delta_{\rm CP}$ and $2\pi-\delta_{\rm CP}$ in the present context, and only one representative value is displayed in this table. The same convention is taken for the Majorana CP phases α_{21} and α'_{31}.

	XV		XVI		XVII, XVIII	
$\sin^2\theta_{13}$	$\frac{1}{6}(1-\cos 2\theta)$					
$\sin^2\theta_{12}$	$\frac{1+\cos 2\theta}{5+\cos 2\theta}$					
$\sin^2\theta_{23}(\theta_{23}<\pi/4)$	$\frac{1}{2}-\frac{\sqrt{6}\sin 2\theta}{5+\cos 2\theta}$		$\frac{1}{2}$		$\frac{1}{2}-\frac{\sqrt{3}\sin 2\theta}{5+\cos 2\theta}$	
$\sin^2\theta_{23}(\theta_{23}>\pi/4)$	$\frac{1}{2}+\frac{\sqrt{6}\sin 2\theta}{5+\cos 2\theta}$		$\frac{1}{2}$		$\frac{1}{2}+\frac{\sqrt{3}\sin 2\theta}{5+\cos 2\theta}$	
$\lvert J_{\rm CP}\rvert$	0		$\frac{1}{6\sqrt{6}}\lvert\sin 2\theta\rvert$		$\frac{1}{12\sqrt{3}}\lvert\sin 2\theta\rvert$	
$\lvert\tan\delta_{\rm CP}\rvert$	0		$+\infty$		$\left\lvert\frac{5+\cos 2\theta}{1+5\cos 2\theta}\right\rvert$	
$\lvert\tan\alpha_{21}\rvert$	0				$+\infty$	
$\lvert\tan\alpha'_{31}\rvert$	0				$+\infty$	
Best Fitting						
	$\theta_{23}<\pi/4$	$\theta_{23}>\pi/4$	$\theta_{23}<\pi/4$	$\theta_{23}>\pi/4$	$\theta_{23}<\pi/4$	$\theta_{23}>\pi/4$
$\chi^2_{\rm min}$	22.270	25.815	6.993	19.720	7.264	7.726
$\theta_{\rm bf}$	0.237	0.228	± 0.266		0.256	0.253
$\sin^2\theta_{13}(\theta_{\rm bf})$	0.0184	0.0170	0.0230		0.0214	0.0210
$\theta_{13}(\theta_{\rm bf})/^\circ$	7.786	7.502	8.731		8.402	8.325
$\sin^2\theta_{12}(\theta_{\rm bf})$	0.321		0.318		0.319	
$\theta_{12}(\theta_{\rm bf})/^\circ$	34.503		34.303		34.376	
$\sin^2\theta_{23}(\theta_{\rm bf})$	0.310	0.683	0.5		0.356	0.643
$\theta_{23}(\theta_{\rm bf})/^\circ$	33.850	55.734	45		36.604	53.319
$\lvert\sin\delta_{\rm CP}(\theta_{\rm bf})\rvert$	0		1		0.738	
$\delta_{\rm CP}(\theta_{\rm bf})/^\circ$	0		90		47.612	
$\lvert\sin\alpha_{21}(\theta_{\rm bf})\rvert$	0				1	
$\alpha_{21}(\theta_{\rm bf})_{21}/^\circ$	0				90	
$\lvert\sin\alpha'_{31}(\theta_{\rm bf})\rvert$	0				1	
$\alpha'_{31}(\theta_{\rm bf})/^\circ$	0				90	

Table 5: The results of the mixing parameters for the cases XV, XVI, XVII and XVIII, where "$+\infty$" for $|\tan\delta_{\rm CP}|$, $|\tan\alpha|$ and $|\tan\beta|$ implies that the absolute value of the corresponding CP phase is $\pi/2$. Notice that the Dirac CP phase $\delta_{\rm CP}$ is determined up to $\delta_{\rm CP}$, $\pi+\delta_{\rm CP}$, $\pi-\delta_{\rm CP}$ and $2\pi-\delta_{\rm CP}$ in the present context, and only one representative value is displayed in this table. The same convention is taken for the Majorana CP phases α_{21} and α'_{31}.

	XIX, XX, XXV, XXVI	XXII, XXIII	XXI, XXIV
$\sin^2\theta_{13}$	$\frac{1}{4}(1-\cos 2\theta)$		$\frac{1}{8}\left(3-\cos 2\theta-2\sqrt{2}\sin 2\theta\right)$
$\sin^2\theta_{12}$	$\frac{1+\cos 2\theta}{3+\cos 2\theta}$		$\frac{2}{5+\cos 2\theta+2\sqrt{2}\sin 2\theta}$
$\sin^2\theta_{23}(\theta_{23}<\pi/4)$	$\frac{1}{2}-\frac{\sin 2\theta}{3+\cos 2\theta}$	$\frac{1}{2}$	$\frac{2+2\cos 2\theta}{5+\cos 2\theta+2\sqrt{2}\sin 2\theta}$
$\sin^2\theta_{23}(\theta_{23}>\pi/4)$	$\frac{1}{2}+\frac{\sin 2\theta}{3+\cos 2\theta}$	$\frac{1}{2}$	$\frac{3-\cos 2\theta+2\sqrt{2}\sin 2\theta}{5+\cos 2\theta+2\sqrt{2}\sin 2\theta}$
$\lvert J_{\rm CP}\rvert$	$\frac{1}{16}\lvert\sin 2\theta\rvert$	$\frac{1}{8\sqrt{2}}\lvert\sin 2\theta\rvert$	0
$\lvert\tan\delta_{\rm CP}\rvert$	$\left\lvert\frac{3+\cos 2\theta}{1+3\cos 2\theta}\right\rvert$	$+\infty$	0
$\lvert\tan\alpha_{21}\rvert$	0		
$\lvert\tan\alpha'_{31}\rvert$	0		

Best Fitting

	$\theta_{23}<\pi/4$	$\theta_{23}>\pi/4$	$\theta_{23}<\pi/4$	$\theta_{23}>\pi/4$	$\theta_{23}<\pi/4$	$\theta_{23}>\pi/4$
$\chi^2_{\rm min}$	204.875	204.610	209.331	222.058	14.811	15.138
$\theta_{\rm bf}$	0.227		± 0.229		0.439	
$\sin^2\theta_{13}(\theta_{\rm bf})$	0.0253		0.0257		0.0230	
$\theta_{13}(\theta_{\rm bf})/^\circ$	9.147		9.233		8.741	
$\sin^2\theta_{12}(\theta_{\rm bf})$	0.487		0.487		0.256	
$\theta_{12}(\theta_{\rm bf})/^\circ$	44.257		44.243		30.390	
$\sin^2\theta_{23}(\theta_{\rm bf})$	0.388	0.612	0.5		0.419	0.581
$\theta_{23}(\theta_{\rm bf})/^\circ$	38.506	51.492	45		40.358	49.668
$\lvert\sin\delta_{\rm CP}(\theta_{\rm bf})\rvert$	0.726		1		0	
$\delta_{\rm CP}(\theta_{\rm bf})/^\circ$	46.525		90		0	
$\lvert\sin\alpha_{21}(\theta_{\rm bf})\rvert$	0					
$\alpha_{21}(\theta_{\rm bf})/^\circ$	0					
$\lvert\sin\alpha'_{31}(\theta_{\rm bf})\rvert$	0					
$\alpha'_{31}(\theta_{\rm bf})/^\circ$	0					

Table 6: The results of the mixing parameters for the cases XIX, XX, XXI, XXII, XXIII, XXIV, XXV and XXVI, where "$+\infty$" for $|\tan\delta_{\rm CP}|$, $|\tan\alpha|$ and $|\tan\beta|$ implies that the absolute value of the corresponding CP phase is $\pi/2$. Notice that the Dirac CP phase $\delta_{\rm CP}$ is determined up to $\delta_{\rm CP}$, $\pi+\delta_{\rm CP}$, $\pi-\delta_{\rm CP}$ and $2\pi-\delta_{\rm CP}$ in the present context, and only one representative value is displayed in this table. The same convention is taken for the Majorana CP phases α_{21} and α'_{31}.

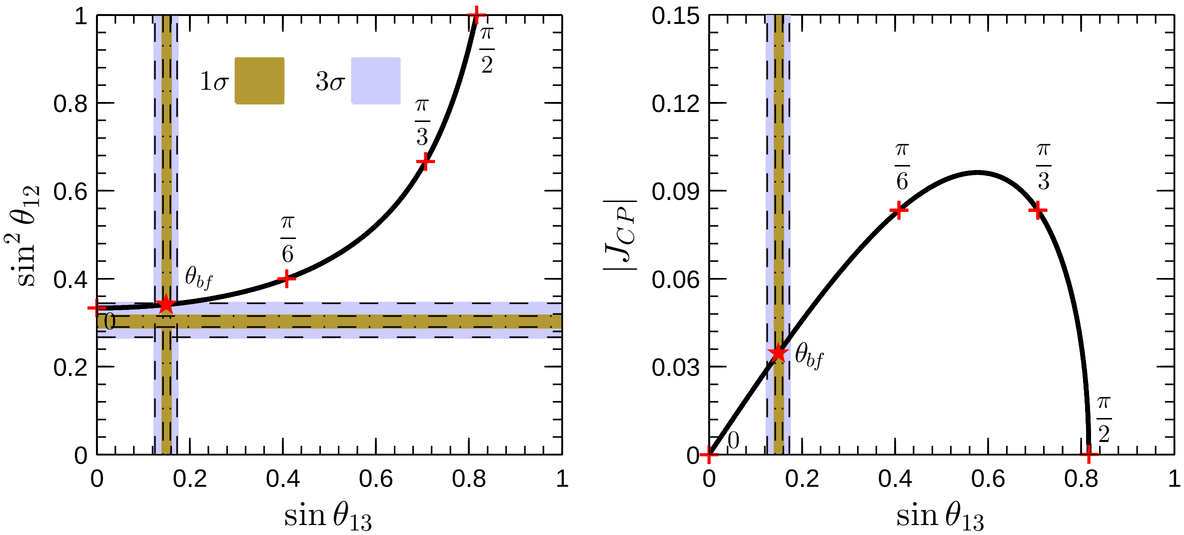

Figure 2: The correlations of $\sin\theta_{13}$, $\sin^2\theta_{12}$ and $|J_{CP}|$ in case III and case VIII. The best fit value θ_{bf} for θ is labelled as a star. We also indicate the points for $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$ by the sign "+" on the curves, and only one θ value is displayed if there are points matching together. The shown 3σ ranges for the mixing angles and their best fit values are taken from Ref. [5].

G_l subgroups in the neutrino and the charged lepton sector respectively, and the remnant CP symmetries from the breaking of $H_{\rm CP}$ are $H^{\nu}_{\rm CP}$ and $H^{l}_{\rm CP}$, respectively, where all cases correspond to a preserved symmetry smaller than the full Klein symmetry, as in the semi-direct approach, leading to predictions which depend on a single undetermined parameter, which may be fitted to the reactor angle for example.

The semi-direct approach to $\Delta(96) \rtimes H_{\rm CP}$, in which a smaller symmetry than the full Klein symmetry is preserved, clearly leads to a very rich set of possible cases which we have systematically studied here. We have discussed 26 possible cases, including a global χ^2 determination of the best fit parameters and the correlations between mixing parameters, in each case. Excellent agreement with the presently observed lepton mixing angles can be achieved in some cases. It is remarkable that the CP phases are predicted to take irregular values rather than 0, π or $\pm\pi/2$ in cases V, VI, IX and X, as shown in Table 2. It remains to be seen if any of these possibilities will closely correspond to the observed future precise determination of leptonic mixing angles and CP violating parameters in the future.

Acknowledgements

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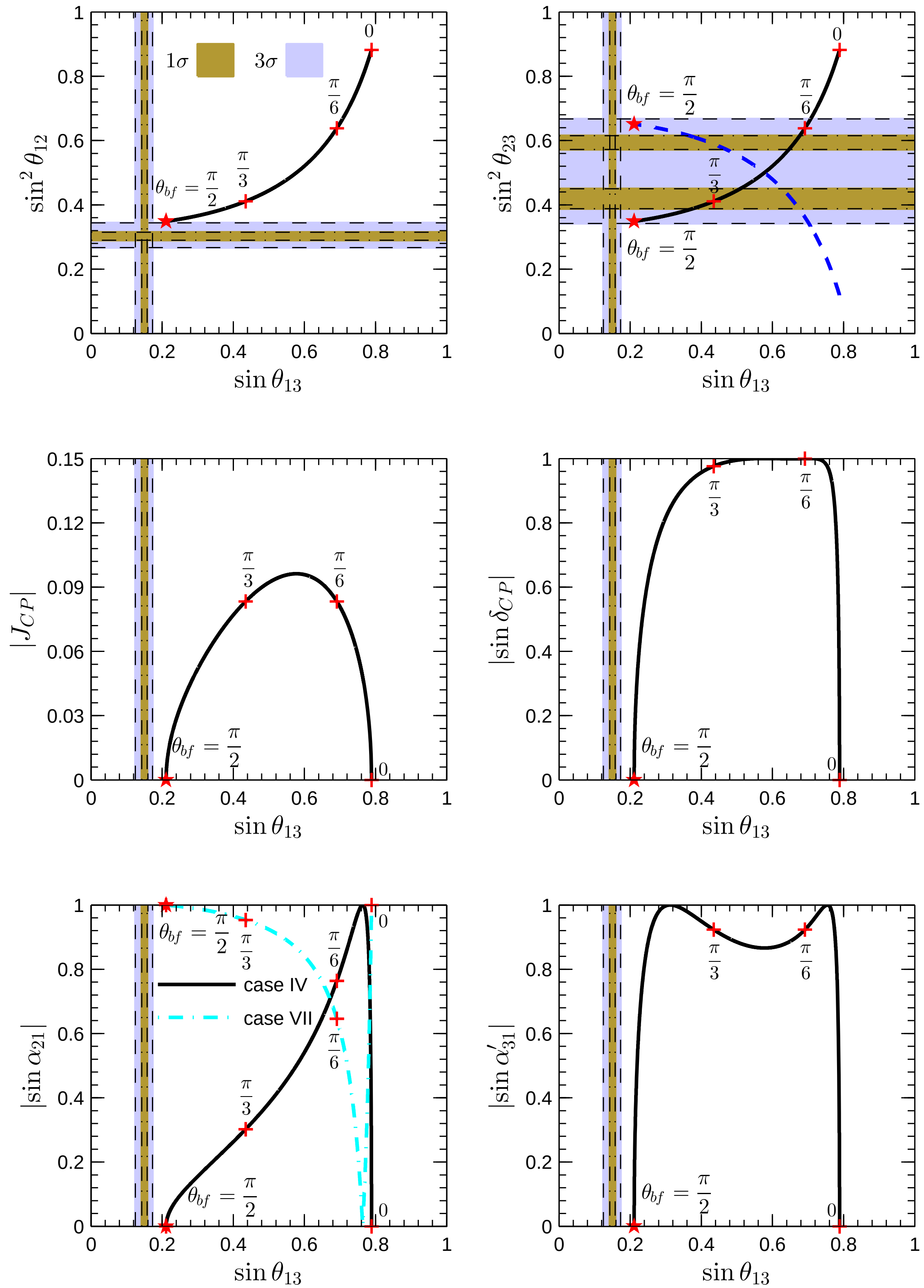

Figure 3: The relation among the lepton mixing parameters in case IV and case VII. The best fit value θ_{bf} for θ is labelled as a star. We also indicate the points for $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$ by the sign "+" on the curves, and only one θ value is displayed if there are points matching together. The shown 3σ ranges for the mixing angles and their best fit values are taken from Ref. [5].

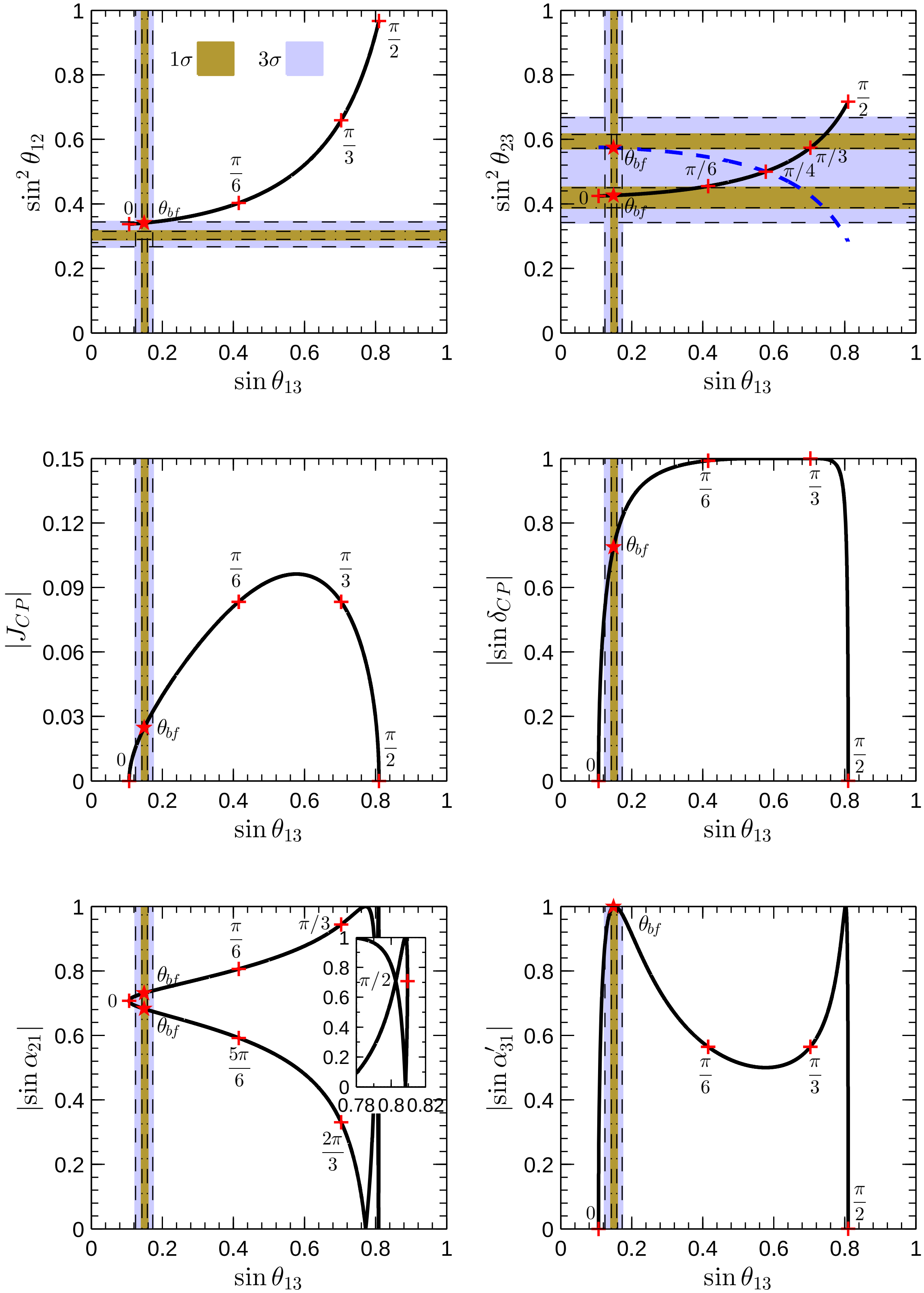

Figure 4: The correlation between the different lepton mixing parameters in case V, case VI, case IX and case X. The best fit value θ_{bf} for θ is labelled as a star. We also indicate the points for $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$ by the sign "+" on the curves, and only one θ value is displayed if there are points matching together. The shown 3σ ranges for the mixing angles and their best fit values are taken from Ref. [5]. Note that the plots for $|\sin\alpha_{21}|$ with respect to $\sin\theta_{13}$ in cases V, IX and case VI, X coincide with each other since they are related by the transformation $\theta \to -\theta$, as shown in Table 2.

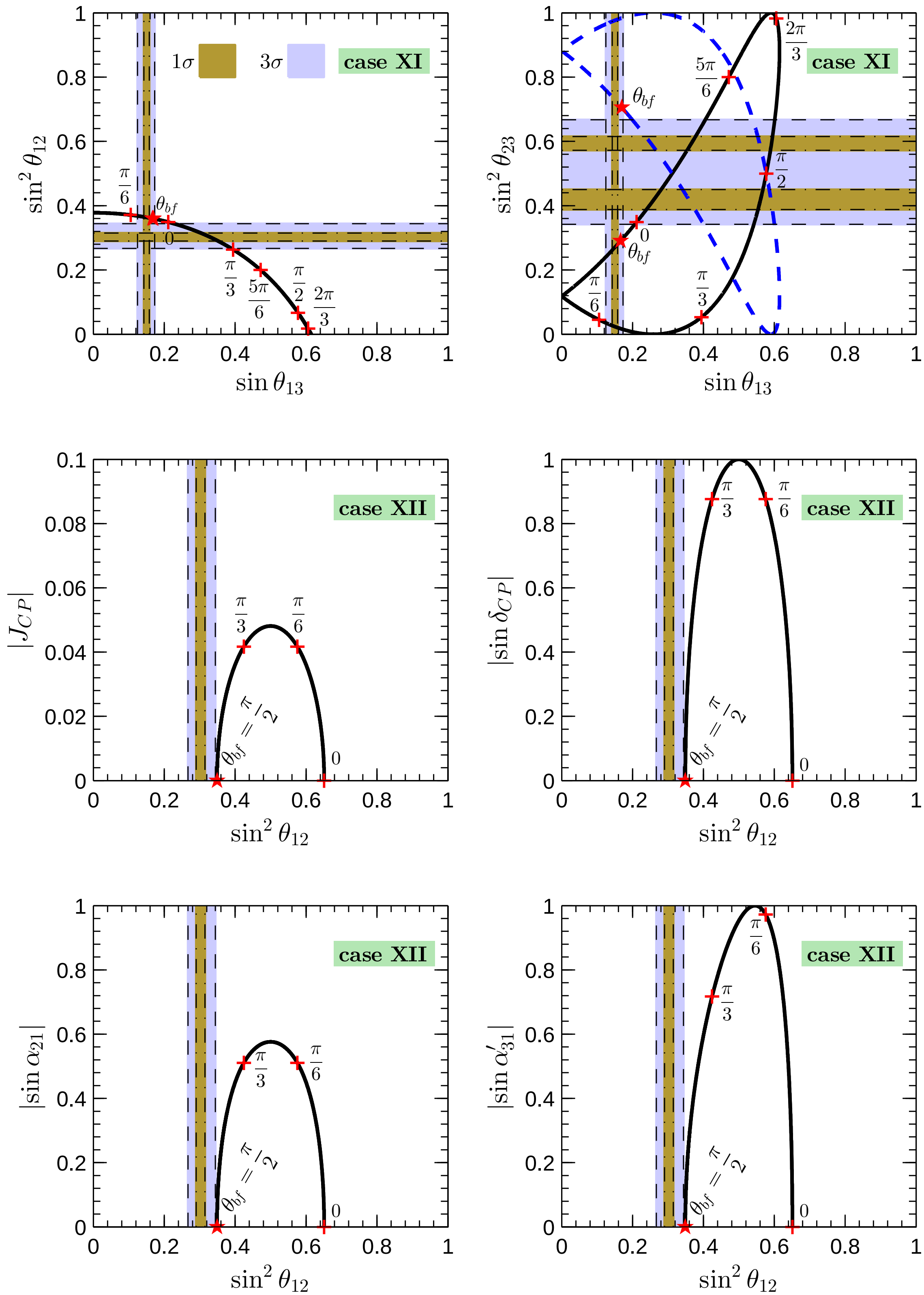

Figure 5: The relation among the lepton mixing parameters in case XI and case XII. The best fit value θ_{bf} for θ is labelled as a star. We also indicate the points for $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$ by the sign "+" on the curves, and only one θ value is displayed if there are points matching together. The shown 3σ ranges for the mixing angles and their best fit values are taken from Ref. [5].

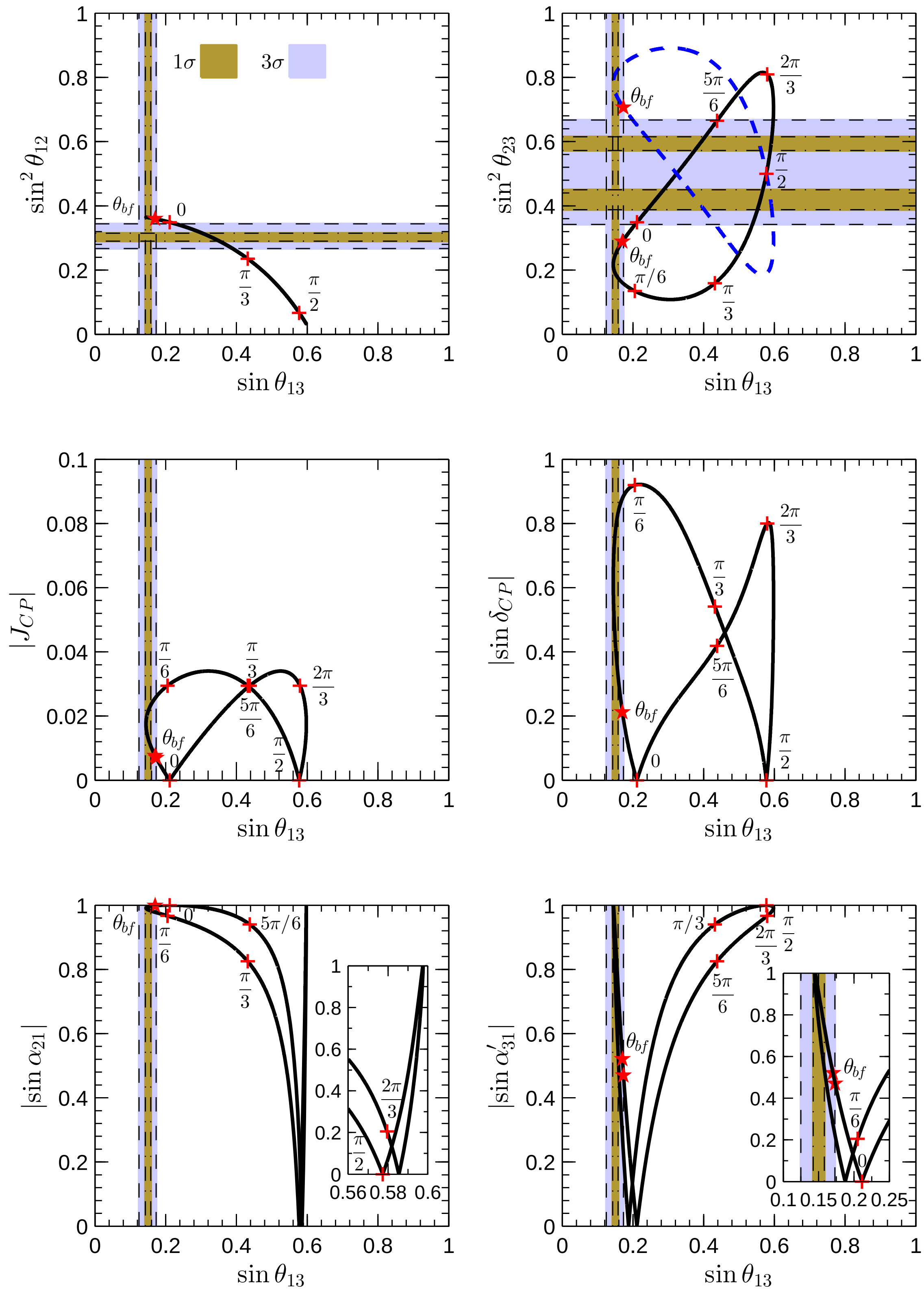

Figure 6: The relation among the lepton mixing parameters in case XIII and case XIV. The best fit value θ_{bf} for θ is labelled as a star. We also indicate the points for $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$ by the sign "+" on the curves, and only one θ value is displayed if there are points matching together. The shown 3σ ranges for the mixing angles and their best fit values are taken from Ref. [5].

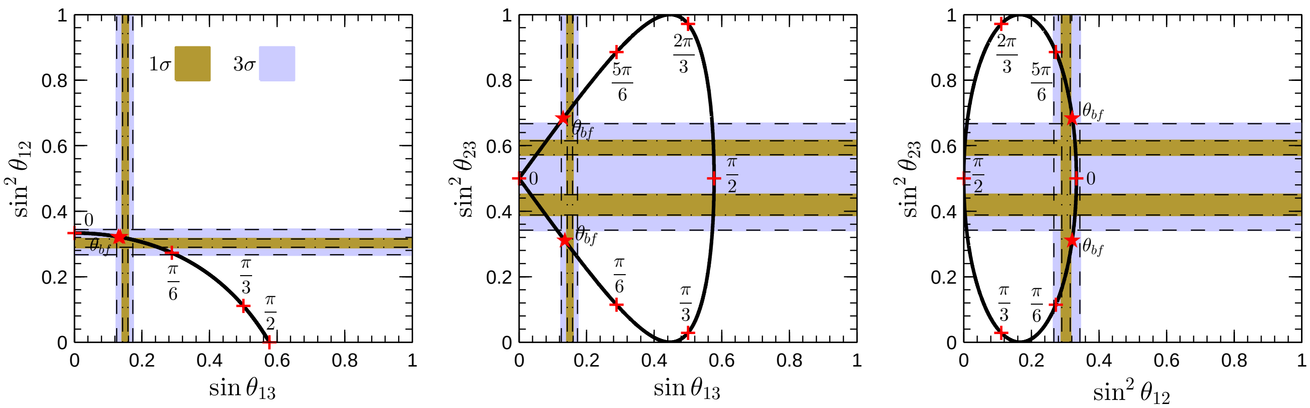

Figure 7: The relation among the lepton mixing angles in case XV. Note that the curves for $\sin^2\theta_{12}$ with respect to $\sin\theta_{13}$ coincidence in case XV, case XVI, case XVII and case XVIII because they are predicted to be of the same form, as shown in Table 5. The best fit value θ_{bf} for θ is labelled as a star. We also indicate the points for $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$ by the sign "+" on the curves, and only one θ value is displayed if there are points matching together. The shown 3σ ranges for the mixing angles and their best fit values are taken from Ref. [5].

A Group theory of $\Delta(96)$

$\Delta(96)$ belongs to the group series $\Delta(6n^2)$ with $n = 4$, and it is a non-abelian finite subgroup of SU(3) of order 96. $\Delta(96)$ is isomorphic to $(Z_4 \times Z_4) \rtimes S_3$, where S_3 is isomorphic to $Z_3 \rtimes Z_2$, and it can be conveniently defined by four generators a, b, c and d obeying the relations [34]:

$$\begin{aligned} a^3 \;=\; b^2 \;=\; (ab)^2 \;=\; c^4 \;=\; d^4 \;=\; 1 \\ cd \;=\; dc \\ aca^{-1} = c^{-1}d^{-1}, \quad ada^{-1} = c \\ bcb^{-1} = d^{-1}, \quad bdb^{-1} = c^{-1}\,, \end{aligned} \tag{A.1}$$

The elements a and b are the generators of S_3 while c and d generate $Z_4 \times Z_4$, and the last two lines define the semidirect product " $\rtimes$ ". Note that the generator $d = bc^3b$ is not independent. In order to see clearly the connection between the lepton flavor mixing and $\Delta(96)$ family symmetry, it is useful to express $\Delta(96)$ in terms of the "canonical" S, T and U generators [35], where S and U usually generate the remnant Klein group $Z_2^S \times Z_2^U$ in the neutrino sector while T is the generator of the residual symmetry group Z_3^T in the charged lepton sector. They satisfy the multiplication rules

$$\begin{aligned} &S^2 = T^3 = U^2 = (ST)^3 = 1, \qquad SU = US, \\ &(TU)^8 = 1, \qquad (TUT^2U)^3 = 1, \qquad (UTSUT^2UT)^2 = 1\,. \end{aligned} \tag{A.2}$$

Note that the generators S and T alone generate the well-known group A_4. The identities relating the two sets of generators are as follows,

$$S = d^2, \qquad T = ac, \qquad U = a^2bd,$$

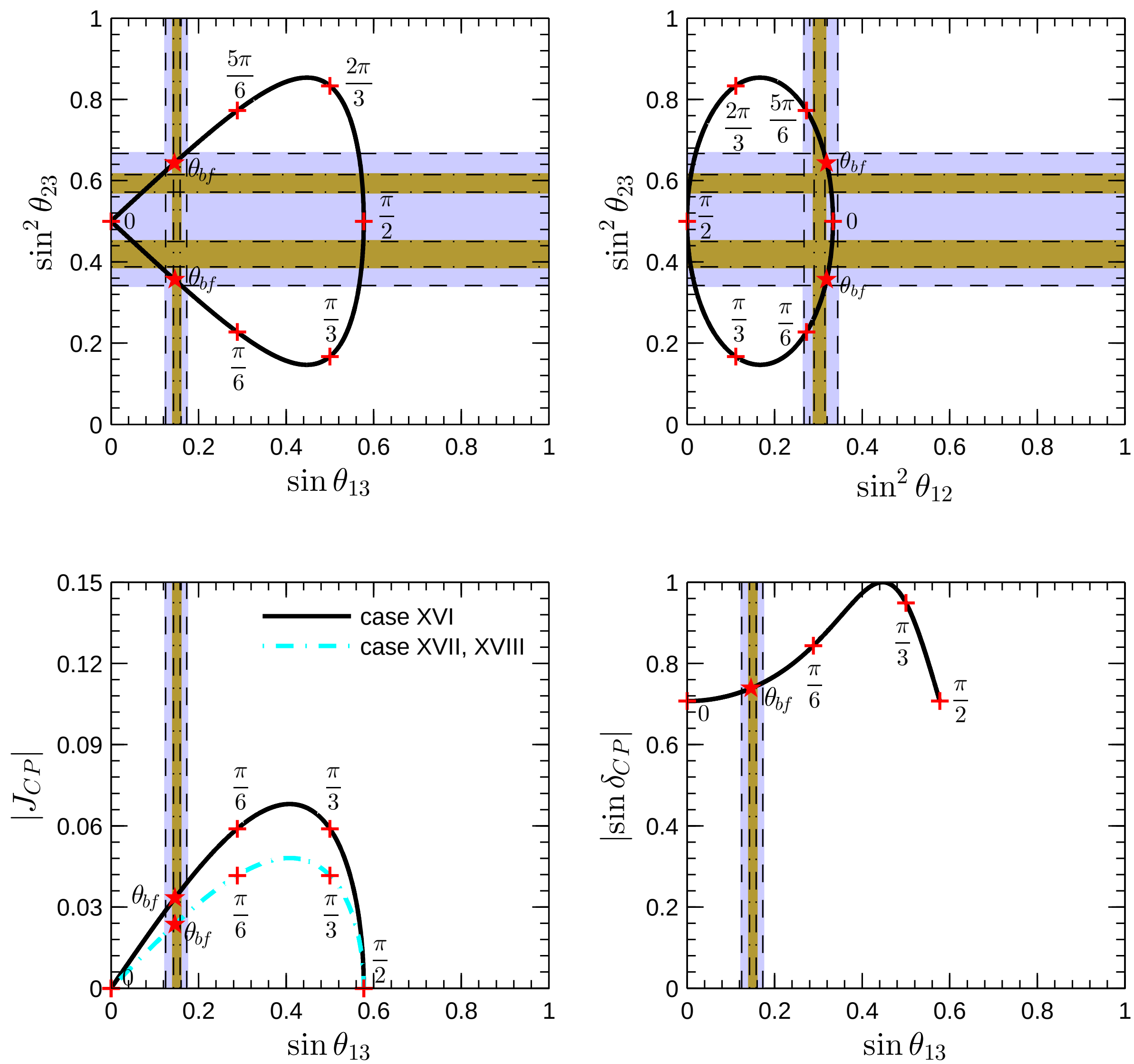

Figure 8: The relation among the lepton mixing parameters in case XVII and case XVIII. Notice that the curves of $\sin^2\theta_{23}$ versus $\sin\theta_{13}$ (or $\sin^2\theta_{23}$ versus $\sin\theta_{12}$) for $\theta_{23}(\theta_{\rm bf}) < \pi/4$ and $\theta_{23}(\theta_{\rm bf}) > \pi/4$ coincide with each other. The best fit value θ_{bf} for θ is labelled as a star. We also indicate the points for $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$ by the sign "+" on the curves, and only one θ value is displayed if there are points matching together. The shown 3σ ranges for the mixing angles and their best fit values are taken from Ref. [5]. Note that the plot for $\sin^2\theta_{12}$ with respect to $\sin\theta_{13}$ is the same as that for case XV and can be found in Fig. 7.

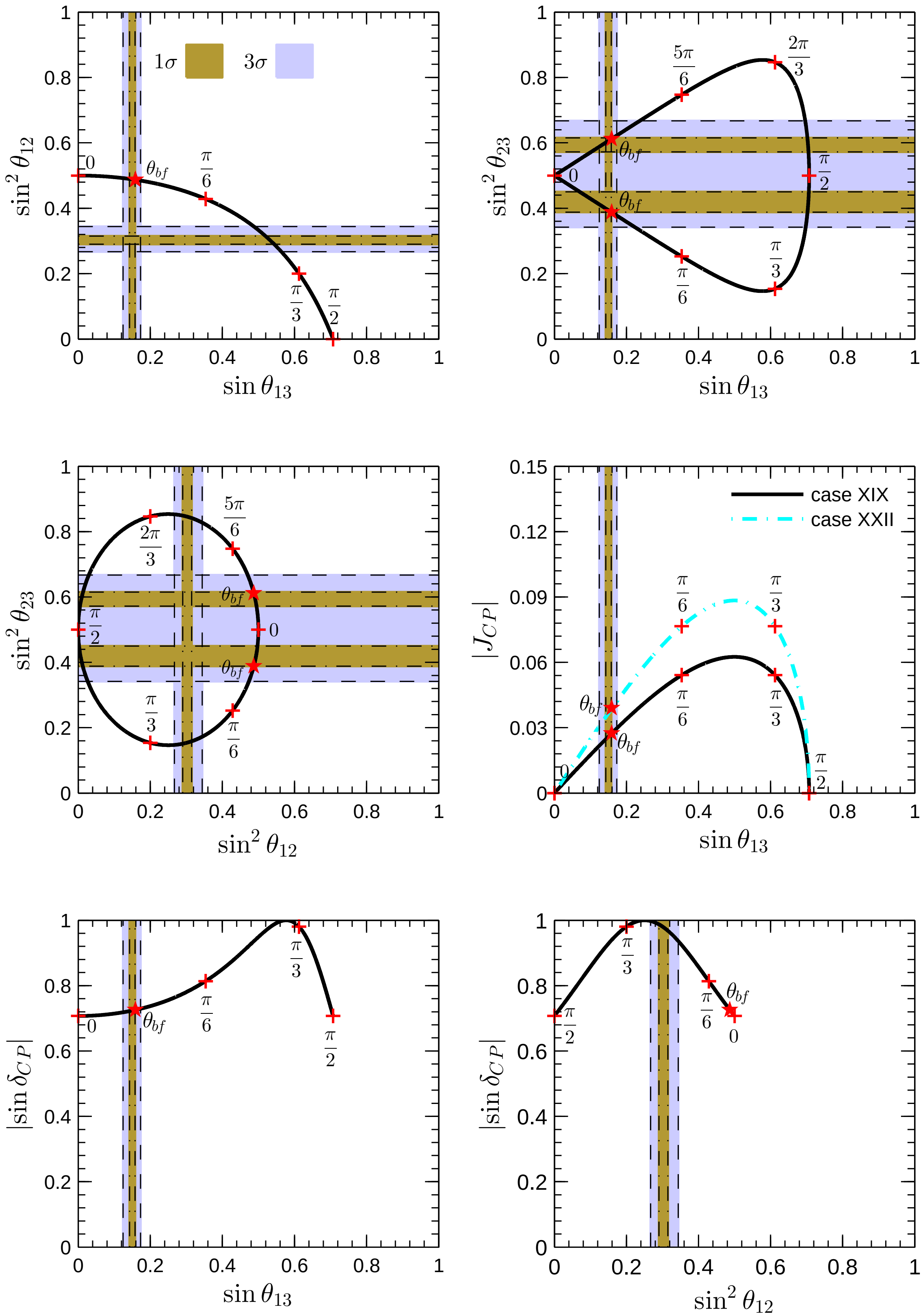

Figure 9: The relation among the lepton mixing parameters in cases XIX, XX, XXV, XXVI and cases XXII, XXIII. The best fit value θ_{bf} for θ is labelled as a star. We also indicate the points for $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$ by the sign "+" on the curves, and only one θ value is displayed if there are points matching together. The shown 3σ ranges for the mixing angles and their best fit values are taken from Ref. [5]. Note that the figure for $\sin^2\theta_{12}$ versus $\sin\theta_{13}$ is the same in all the cases considered here.

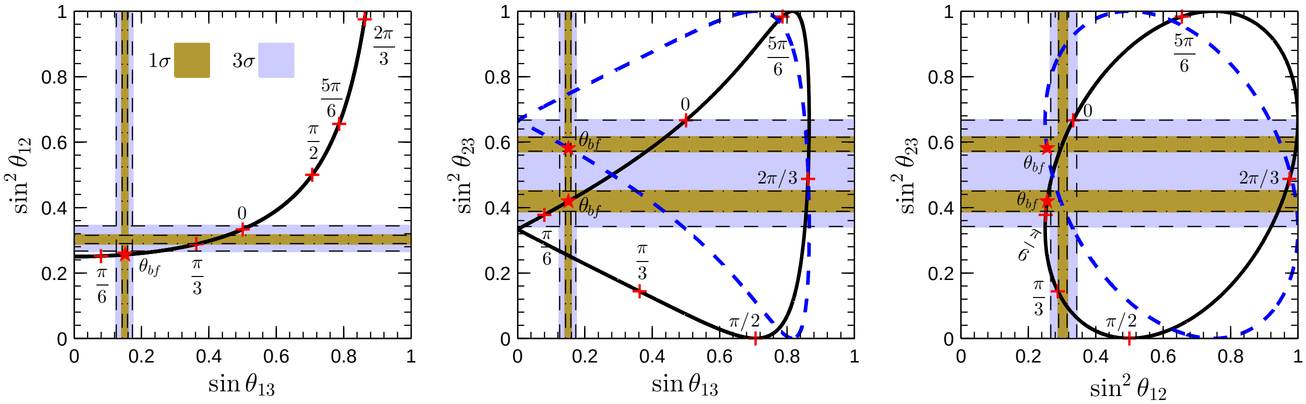

Figure 10: The relation among the lepton mixing angles in case XXI and case XXIV. The best fit value θ_{bf} for θ is labelled as a star. We also indicate the points for $\theta = 0, \pi/6, \pi/3, \pi/2, 2\pi/3$ and $5\pi/6$ by the sign "+" on the curves, and only one θ value is displayed if there are points matching together. The shown 3σ ranges for the mixing angles and their best fit values are taken from Ref. [5].

$$a = SUT^2U, \qquad b = T^2UT, \qquad c = UTSUT, \qquad d = TUT^2SUT\,. \tag{A.3}$$

The $\Delta(96)$ group has ten conjugacy classes:

$1C_1 : 1$

$3C_4 : cd^2 = (T^2U)^2, cd^3 = (UT^2)^2, c^2d^3 = TUT^2UT\,,$

$3C_2 : c^2 = TST^2, d^2 = S, c^2d^2 = T^2ST\,,$

$3C_4' : c^2d = T^2UTUT^2, c^3d = (TU)^2, c^3d^2 = (UT)^2\,,$

$6C_4 : c = UTSUT, d = TUT^2SUT, cd = S(UT^2)^2, c^3 = S(UT)^2, d^3 = TSUT^2UT,$
$c^3d^3 = UT^2SUT^2\,,$

$32C_3 : a = SUT^2U, ac = T, ac^2 = UT^2SU, ac^3 = T^2ST^2, ad = T(TU)^2, ad^2 = SUT^2US,$
$ad^3 = T^2UTSU, acd = UTSUT^2, acd^2 = TS, acd^3 = SUTUT^2, ac^2d = T^2SUTU,$
$ac^2d^2 = UT^2U, ac^2d^3 = T(UT^2)^2, ac^3d = (UT)^2T, ac^3d^2 = ST, ac^3d^3 = (T^2U)^2T,$
$a^2 = UTSU, a^2c = SUT^2UT, a^2c^2 = SUTUS, a^2c^3 = UT^2UT, a^2d = TSUT^2U,$
$a^2d^2 = UTU, a^2d^3 = T^2(UT)^2, a^2cd = T^2, a^2cd^2 = UT^2SUT, a^2cd^3 = T^2S,$
$a^2c^2d = TUT^2SU, a^2c^2d^2 = SUTU, a^2c^2d^3 = TUT^2U, a^2c^3d = ST^2,$
$a^2c^3d^2 = (TU)^2T^2, a^2c^3d^3 = TST\,,$

$12C_2 : ab = TSUT^2, abc = UT^2SUTU, abc^2 = TUT^2, abc^3 = UT^2UTU, a^2b = UTSUT^2UT,$
$a^2bd = U, a^2bd^2 = UTUT^2UST, a^2bd^3 = SU, b = T^2UT, bcd = UTSUT^2U,$
$bc^2d^2 = T^2SUT, bc^3d^3 = UTUT^2U\,,$

$12C_8 : abd = UTS, abcd = T^2U, abc^2d = SUT, abc^3d = ST^2US, a^2bc^3 = T^2SUT^2,$
$a^2bc^3d = STUT, a^2bc^3d^2 = ST^2UST^2, a^2bc^3d^3 = TUT, bc = TSU, bc^2d = UT^2,$
$bc^3d^2 = STU, bd^3 = UTST\,,$

$12C_4 : abd^2 = STUT^2, abcd^2 = UT(UT^2)^2, abc^2d^2 = TUT^2S, abc^3d^2 = UT^2UTUS,$

	S	T	U
$\mathbf{1}$, $\mathbf{1}'$	1	1	± 1
$\mathbf{2}$	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} \omega^2 & 0 \\ 0 & \omega \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
$\mathbf{3}$, $\mathbf{3}'$	$S_{\mathbf{3}}$	$T_{\mathbf{3}}$	$\pm U_{\mathbf{3}}$
$\overline{\mathbf{3}}$, $\overline{\mathbf{3}}'$	$S_{\mathbf{3}}$	$T^*_{\mathbf{3}}$	$\pm U_{\mathbf{3}}$
$\widetilde{\mathbf{3}}$, $\widetilde{\mathbf{3}}'$	$\mathbb{1}_{3\times 3}$	$T_{\mathbf{3}}$	$\mp P_{12} S_{\mathbf{3}}$
$\mathbf{6}$	$\begin{pmatrix} S_{\mathbf{3}} & 0 \\ 0 & S_{\mathbf{3}} \end{pmatrix}$	$\begin{pmatrix} T_{\mathbf{3}} & 0 \\ 0 & T_{\mathbf{3}} \end{pmatrix}$	$\begin{pmatrix} \widetilde{U} & \widetilde{U} - P_{12}S_{\mathbf{3}} \\ \widetilde{U} - P_{12}S_{\mathbf{3}} & -\widetilde{U} \end{pmatrix}$

Table 7: The representation matrices for the $\Delta(96)$ generators S, T and U in different irreducible representations, where $\omega = e^{2\pi i/3}$ is the cube root of unit, $\mathbb{1}_{3\times 3}$ denotes the 3×3 unity matrix, and the matrices $S_{\mathbf{3}}$, $T_{\mathbf{3}}$, $U_{\mathbf{3}}$, P_{12} and $\widetilde{U}$ are given in Eq. (A.5).

$$
\begin{aligned}
& a^2bc^2 = UTUT^2UT, a^2bc^2d = UTST^2, a^2bc^2d^2 = UT^2UTUT^2, a^2bc^2d^3 = UT^2ST, \\
& bc^2 = ST^2UT, bc^3d = UTUT^2US, bd^2 = T^2UTS, bcd^3 = UT^2(UT)^2\,, \\
12C_8' :\ & abd^3 = UT, abcd^3 = T^2SU, abc^2d^3 = UT^2ST^2, abc^3d^3 = ST^2U, a^2bc = T^2UT^2, \\
& a^2bcd = TSUT, a^2bcd^2 = ST^2UT^2, a^2bcd^3 = STUST, bc^3 = STUS, bd = SUT^2, \\
& bcd^2 = TU, bc^2d^3 = UT^2S\,.
\end{aligned} \tag{A.4}
$$

Note that the conjugacy class is denoted in the notation of kC_n, where k stands for the number of elements in the class and the subscript n indicates the order of the elements. $\Delta(96)$ has two singlet irreducible representations $\mathbf{1}$ and $\mathbf{1}'$, one doublet irreducible representation $\mathbf{2}$, six triplet irreducible representations $\mathbf{3}$, $\mathbf{3}'$, $\overline{\mathbf{3}}$, $\overline{\mathbf{3}}'$, $\widetilde{\mathbf{3}}$, $\widetilde{\mathbf{3}}'$, and one sextet $\mathbf{6}$. We note that $\overline{\mathbf{3}}$ and $\overline{\mathbf{3}}'$ are the complex conjugate representations of $\mathbf{3}$ and $\mathbf{3}'$ respectively, and the representations $\mathbf{3}$, $\mathbf{3}'$, $\overline{\mathbf{3}}$, $\overline{\mathbf{3}}'$ and $\mathbf{6}$ are the faithful representations of $\Delta(96)$, while $\widetilde{\mathbf{3}}$ and $\widetilde{\mathbf{3}}'$ are not. Our choice of the basis for the representation matrices of S, T and U is listed in Table 7, where we have defined

$$
\begin{aligned}
& P_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{U} = \tfrac{1}{3}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \quad S_{\mathbf{3}} = \tfrac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \\
& T_{\mathbf{3}} = \begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad U_{\mathbf{3}} = \tfrac{1}{3}\begin{pmatrix} -1-\sqrt{3} & -1 & -1+\sqrt{3} \\ -1 & -1+\sqrt{3} & -1-\sqrt{3} \\ -1+\sqrt{3} & -1-\sqrt{3} & -1 \end{pmatrix}.
\end{aligned} \tag{A.5}
$$

Then we can straightforwardly obtain the character table of $\Delta(96)$ as shown in Table 8. Furthermore, the Kronecker products between various irreducible representations are as follows:

$$
\begin{aligned}
& \mathbf{1}' \otimes \mathbf{2} = \mathbf{2},\ \mathbf{1}' \otimes \mathbf{r} = \mathbf{r}',\ \mathbf{1}' \otimes \mathbf{r}' = \mathbf{r},\ \mathbf{1}' \otimes \mathbf{6} = \mathbf{6},\ \mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{1}' \oplus \mathbf{2}, \\
& \mathbf{2} \otimes \mathbf{r} = \mathbf{2} \otimes \mathbf{r}' = \mathbf{r} \oplus \mathbf{r}',\ \mathbf{2} \otimes \mathbf{6} = \mathbf{6} \oplus \mathbf{6},\ \mathbf{3} \otimes \mathbf{3} = \mathbf{3}' \otimes \mathbf{3}' = \overline{\mathbf{3}} \oplus \overline{\mathbf{3}}' \oplus \widetilde{\mathbf{3}}',
\end{aligned}
$$

	Conjugate Classes									
	$1C_1$	$3C_4$	$3C_2$	$3C'_4$	$6C_4$	$32C_3$	$12C_2$	$12C_8$	$12C_4$	$12C'_8$
G	1	cd^2	c^2	c^2d	c	a	b	bc	bc^2	bd
$\mathbf{1}$	1	1	1	1	1	1	1	1	1	1
$\mathbf{1}'$	1	1	1	1	1	1	-1	-1	-1	-1
$\mathbf{2}$	2	2	2	2	2	-1	0	0	0	0
$\mathbf{3}$	3	$-1+2i$	-1	$-1-2i$	1	0	-1	i	1	$-i$
$\mathbf{3}'$	3	$-1+2i$	-1	$-1-2i$	1	0	1	$-i$	-1	i
$\overline{\mathbf{3}}$	3	$-1-2i$	-1	$-1+2i$	1	0	-1	$-i$	1	i
$\overline{\mathbf{3}}'$	3	$-1-2i$	-1	$-1+2i$	1	0	1	i	-1	$-i$
$\widetilde{\mathbf{3}}$	3	-1	3	-1	-1	0	-1	1	-1	1
$\widetilde{\mathbf{3}}'$	3	-1	3	-1	-1	0	1	-1	1	-1
$\mathbf{6}$	6	2	-2	2	-2	0	0	0	0	0

Table 8: Character table of the $\Delta(96)$ group, where kC_n denotes the classes with k elements which have order n, G is a representative of the class kC_n in terms of the generators a, b, c and d.

$$\begin{aligned}
&\mathbf{3}\otimes\mathbf{3}'=\overline{\mathbf{3}}\oplus\overline{\mathbf{3}}'\oplus\widetilde{\mathbf{3}},\ \mathbf{3}\otimes\overline{\mathbf{3}}=\mathbf{3}'\otimes\overline{\mathbf{3}}'=\mathbf{1}\oplus\mathbf{2}\oplus\mathbf{6},\ \mathbf{3}\otimes\overline{\mathbf{3}}'=\mathbf{3}'\otimes\overline{\mathbf{3}}=\mathbf{1}'\oplus\mathbf{2}\oplus\mathbf{6},\\
&\mathbf{3}\otimes\widetilde{\mathbf{3}}=\mathbf{3}'\otimes\widetilde{\mathbf{3}}'=\overline{\mathbf{3}}'\oplus\mathbf{6},\ \mathbf{3}\otimes\widetilde{\mathbf{3}}'=\mathbf{3}'\otimes\widetilde{\mathbf{3}}=\overline{\mathbf{3}}\oplus\mathbf{6},\\
&\mathbf{3}\otimes\mathbf{6}=\mathbf{3}'\otimes\mathbf{6}=\mathbf{3}\oplus\mathbf{3}'\oplus\widetilde{\mathbf{3}}\oplus\widetilde{\mathbf{3}}'\oplus\mathbf{6},\ \overline{\mathbf{3}}\otimes\overline{\mathbf{3}}=\overline{\mathbf{3}}'\otimes\overline{\mathbf{3}}'=\mathbf{3}\oplus\mathbf{3}'\oplus\widetilde{\mathbf{3}}',\\
&\overline{\mathbf{3}}\otimes\overline{\mathbf{3}}'=\mathbf{3}\oplus\mathbf{3}'\oplus\widetilde{\mathbf{3}},\ \overline{\mathbf{3}}\otimes\widetilde{\mathbf{3}}=\overline{\mathbf{3}}'\otimes\widetilde{\mathbf{3}}'=\mathbf{3}'\oplus\mathbf{6},\ \overline{\mathbf{3}}\otimes\widetilde{\mathbf{3}}'=\overline{\mathbf{3}}'\otimes\widetilde{\mathbf{3}}=\mathbf{3}\oplus\mathbf{6},\\
&\overline{\mathbf{3}}\otimes\mathbf{6}=\overline{\mathbf{3}}'\otimes\mathbf{6}=\overline{\mathbf{3}}\oplus\overline{\mathbf{3}}'\oplus\widetilde{\mathbf{3}}\oplus\widetilde{\mathbf{3}}'\oplus\mathbf{6},\ \widetilde{\mathbf{3}}\otimes\widetilde{\mathbf{3}}=\widetilde{\mathbf{3}}'\otimes\widetilde{\mathbf{3}}'=\mathbf{1}\oplus\mathbf{2}\oplus\widetilde{\mathbf{3}}\oplus\widetilde{\mathbf{3}}',\\
&\widetilde{\mathbf{3}}\otimes\widetilde{\mathbf{3}}'=\mathbf{1}'\oplus\mathbf{2}\oplus\widetilde{\mathbf{3}}\oplus\widetilde{\mathbf{3}}',\ \widetilde{\mathbf{3}}\otimes\mathbf{6}=\widetilde{\mathbf{3}}'\otimes\mathbf{6}=\mathbf{3}\otimes\mathbf{3}'\oplus\overline{\mathbf{3}}\oplus\overline{\mathbf{3}}'\oplus\mathbf{6},\\
&\mathbf{6}\otimes\mathbf{6}=\mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{2}_S\oplus\mathbf{2}_A\oplus\mathbf{3}\oplus\mathbf{3}'\oplus\overline{\mathbf{3}}\oplus\overline{\mathbf{3}}'\oplus\widetilde{\mathbf{3}}\oplus\widetilde{\mathbf{3}}'\oplus\mathbf{6}_S\oplus\mathbf{6}_A
\end{aligned} \tag{A.6}$$

where $\mathbf{r}$ denotes the triplet representations $\mathbf{3}$, $\overline{\mathbf{3}}$ and $\widetilde{\mathbf{3}}$, and the subscript $S(A)$ denotes symmetric (antisymmetric) combinations. Given the explicit forms of the generators in Table 7, one can calculate the corresponding Clebsch-Gordan (CG) coefficients. In the following, we report the CG coefficients of $\Delta(96)$ in the form $\alpha\otimes\beta$, where the α_i denote the elements of the representation on the left of the product, and β_i indicate those of the representation on the right of the product.

- $\mathbf{1}'\otimes\mathbf{2}=\mathbf{2}$

$$\mathbf{2}\sim\begin{pmatrix}\alpha_1\beta_1\\-\alpha_1\beta_2\end{pmatrix}.$$

- $\mathbf{1}'\otimes\mathbf{r}=\mathbf{r}'$ with $\mathbf{r}=\mathbf{3},\overline{\mathbf{3}},\widetilde{\mathbf{3}}$

$$\mathbf{r}'\sim\begin{pmatrix}\alpha_1\beta_1\\\alpha_1\beta_2\\\alpha_1\beta_3\end{pmatrix}.$$

- $\mathbf{1}'\otimes\mathbf{r}'=\mathbf{r}$ with $\mathbf{r}=\mathbf{3},\overline{\mathbf{3}},\widetilde{\mathbf{3}}$

$$\mathbf{r}\sim\begin{pmatrix}\alpha_1\beta_1\\ \alpha_1\beta_2\\ \alpha_1\beta_3\end{pmatrix}\,.$$

- $\mathbf{1}'\otimes\mathbf{6}=\mathbf{6}$

$$\mathbf{6}\sim\begin{pmatrix}\alpha_1\beta_4\\ \alpha_1\beta_5\\ \alpha_1\beta_6\\ -\alpha_1\beta_1\\ -\alpha_1\beta_2\\ -\alpha_1\beta_3\end{pmatrix}\,.$$

- $\mathbf{2}\otimes\mathbf{2}=\mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{2}$

$$\mathbf{1}\sim\alpha_1\beta_2+\alpha_2\beta_1,\quad \mathbf{1}'\sim\alpha_1\beta_2-\alpha_2\beta_1,\quad \mathbf{2}\sim\begin{pmatrix}\alpha_2\beta_2\\ \alpha_1\beta_1\end{pmatrix}\,.$$

- $\mathbf{2}\otimes\mathbf{r}=\mathbf{r}\oplus\mathbf{r}'$ with $\mathbf{r}=\mathbf{3},\widetilde{\mathbf{3}}$

$$\mathbf{r}\sim\begin{pmatrix}c\alpha_1\beta_2+\alpha_2\beta_3\\ \alpha_1\beta_3+\alpha_2\beta_1\\ \alpha_1\beta_1+\alpha_2\beta_2\end{pmatrix},\quad \mathbf{r}'\sim\begin{pmatrix}\alpha_1\beta_2-\alpha_2\beta_3\\ \alpha_1\beta_3-\alpha_2\beta_1\\ \alpha_1\beta_1-\alpha_2\beta_2\end{pmatrix}\,.$$

- $\mathbf{2}\otimes\mathbf{r}'=\mathbf{r}\oplus\mathbf{r}'$ with $\mathbf{r}=\mathbf{3},\widetilde{\mathbf{3}}$

$$\mathbf{r}\sim\begin{pmatrix}\alpha_1\beta_2-\alpha_2\beta_3\\ \alpha_1\beta_3-\alpha_2\beta_1\\ \alpha_1\beta_1-\alpha_2\beta_2\end{pmatrix},\quad \mathbf{r}'\sim\begin{pmatrix}\alpha_1\beta_2+\alpha_2\beta_3\\ \alpha_1\beta_3+\alpha_2\beta_1\\ \alpha_1\beta_1+\alpha_2\beta_2\end{pmatrix}\,.$$

- $\mathbf{2}\otimes\overline{\mathbf{3}}=\overline{\mathbf{3}}\oplus\overline{\mathbf{3}}'$

$$\overline{\mathbf{3}}\sim\begin{pmatrix}\alpha_1\beta_3+\alpha_2\beta_2\\ \alpha_1\beta_1+\alpha_2\beta_3\\ \alpha_1\beta_2+\alpha_2\beta_1\end{pmatrix},\quad \overline{\mathbf{3}}'\sim\begin{pmatrix}\alpha_1\beta_3-\alpha_2\beta_2\\ \alpha_1\beta_1-\alpha_2\beta_3\\ \alpha_1\beta_2-\alpha_2\beta_1\end{pmatrix}\,.$$

- $\mathbf{2}\otimes\overline{\mathbf{3}}'=\overline{\mathbf{3}}\oplus\overline{\mathbf{3}}'$

$$\overline{\mathbf{3}}\sim\begin{pmatrix}\alpha_1\beta_3-\alpha_2\beta_2\\ \alpha_1\beta_1-\alpha_2\beta_3\\ \alpha_1\beta_2-\alpha_2\beta_1\end{pmatrix},\quad \overline{\mathbf{3}}'\sim\begin{pmatrix}\alpha_1\beta_3+\alpha_2\beta_2\\ \alpha_1\beta_1+\alpha_2\beta_3\\ \alpha_1\beta_2+\alpha_2\beta_1\end{pmatrix}\,.$$

- $\mathbf{2}\otimes\mathbf{6}=\mathbf{6}\oplus\mathbf{6}$

$$\mathbf{6}\sim\begin{pmatrix}\alpha_1\beta_2+\alpha_2\beta_3\\ \alpha_1\beta_3+\alpha_2\beta_1\\ \alpha_1\beta_1+\alpha_2\beta_2\\ \alpha_1\beta_5+\alpha_2\beta_6\\ \alpha_1\beta_6+\alpha_2\beta_4\\ \alpha_1\beta_4+\alpha_2\beta_5\end{pmatrix},\qquad \mathbf{6}\sim\begin{pmatrix}\alpha_1\beta_5-\alpha_2\beta_6\\ \alpha_1\beta_6-\alpha_2\beta_4\\ \alpha_1\beta_4-\alpha_2\beta_5\\ \alpha_2\beta_3-\alpha_1\beta_2\\ \alpha_2\beta_1-\alpha_1\beta_3\\ \alpha_2\beta_2-\alpha_1\beta_1\end{pmatrix}.$$

- $\mathbf{3}\otimes\mathbf{3}=\mathbf{3}'\otimes\mathbf{3}'=\overline{\mathbf{3}}\oplus\overline{\mathbf{3}}'\oplus\widetilde{\mathbf{3}}'$

$$\overline{\mathbf{3}}\sim\begin{pmatrix}\alpha_2\beta_3-\alpha_3\beta_2\\ \alpha_3\beta_1-\alpha_1\beta_3\\ \alpha_1\beta_2-\alpha_2\beta_1\end{pmatrix},\;\; \overline{\mathbf{3}}'\sim\begin{pmatrix}2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2\\ 2\alpha_2\beta_2-\alpha_1\beta_3-\alpha_3\beta_1\\ 2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1\end{pmatrix},\;\; \widetilde{\mathbf{3}}'\sim\begin{pmatrix}\alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1\\ \alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2\\ \alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_3\end{pmatrix}.$$

- $\mathbf{3}\otimes\mathbf{3}'=\overline{\mathbf{3}}\oplus\overline{\mathbf{3}}'\oplus\widetilde{\mathbf{3}}$

$$\overline{\mathbf{3}}\sim\begin{pmatrix}2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2\\ 2\alpha_2\beta_2-\alpha_1\beta_3-\alpha_3\beta_1\\ 2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1\end{pmatrix},\;\; \overline{\mathbf{3}}'\sim\begin{pmatrix}\alpha_2\beta_3-\alpha_3\beta_2\\ \alpha_3\beta_1-\alpha_1\beta_3\\ \alpha_1\beta_2-\alpha_2\beta_1\end{pmatrix},\;\; \widetilde{\mathbf{3}}\sim\begin{pmatrix}\alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1\\ \alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2\\ \alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_3\end{pmatrix}.$$

- $\mathbf{3}\otimes\overline{\mathbf{3}}=\mathbf{3}'\otimes\overline{\mathbf{3}}'=\mathbf{1}\oplus\mathbf{2}\oplus\mathbf{6}$

$$\mathbf{1}\sim\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3,\;\; \mathbf{2}\sim\begin{pmatrix}\alpha_1\beta_2+\alpha_2\beta_3+\alpha_3\beta_1\\ \alpha_1\beta_3+\alpha_2\beta_1+\alpha_3\beta_2\end{pmatrix},\;\; \mathbf{6}\sim\begin{pmatrix}\sqrt{3}\,(\alpha_1\beta_3-\alpha_3\beta_2)\\ \sqrt{3}\,(\alpha_3\beta_1-\alpha_2\beta_3)\\ \sqrt{3}\,(\alpha_2\beta_2-\alpha_1\beta_1)\\ 2\alpha_2\beta_1-\alpha_1\beta_3-\alpha_3\beta_2\\ 2\alpha_1\beta_2-\alpha_2\beta_3-\alpha_3\beta_1\\ 2\alpha_3\beta_3-\alpha_1\beta_1-\alpha_2\beta_2\end{pmatrix}.$$

- $\mathbf{3}\otimes\overline{\mathbf{3}}'=\mathbf{3}'\otimes\overline{\mathbf{3}}=\mathbf{1}'\oplus\mathbf{2}\oplus\mathbf{6}$

$$\mathbf{1}'\sim\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3,\;\; \mathbf{2}\sim\begin{pmatrix}\alpha_1\beta_2+\alpha_2\beta_3+\alpha_3\beta_1\\ -\alpha_1\beta_3-\alpha_2\beta_1-\alpha_3\beta_2\end{pmatrix},\;\; \mathbf{6}\sim\begin{pmatrix}2\alpha_2\beta_1-\alpha_1\beta_3-\alpha_3\beta_2\\ 2\alpha_1\beta_2-\alpha_2\beta_3-\alpha_3\beta_1\\ 2\alpha_3\beta_3-\alpha_1\beta_1-\alpha_2\beta_2\\ \sqrt{3}\,(\alpha_3\beta_2-\alpha_1\beta_3)\\ \sqrt{3}\,(\alpha_2\beta_3-\alpha_3\beta_1)\\ \sqrt{3}\,(\alpha_1\beta_1-\alpha_2\beta_2)\end{pmatrix}.$$

- $\mathbf{3}\otimes\widetilde{\mathbf{3}}=\mathbf{3}'\otimes\widetilde{\mathbf{3}}'=\overline{\mathbf{3}}'\oplus\mathbf{6}$

$$\overline{\mathbf{3}}'\sim\begin{pmatrix}\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2\\ \alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1\\ \alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_3\end{pmatrix},\qquad \mathbf{6}\sim\begin{pmatrix}\sqrt{3}\,(\alpha_3\beta_1-\alpha_2\beta_2)\\ \sqrt{3}\,(\alpha_1\beta_1-\alpha_3\beta_2)\\ \sqrt{3}\,(\alpha_2\beta_1-\alpha_1\beta_2)\\ 2\alpha_1\beta_3-\alpha_2\beta_2-\alpha_3\beta_1\\ 2\alpha_2\beta_3-\alpha_1\beta_1-\alpha_3\beta_2\\ 2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1\end{pmatrix}.$$

- $\mathbf{3}\otimes\widetilde{\mathbf{3}}'=\mathbf{3}'\otimes\widetilde{\mathbf{3}}=\overline{\mathbf{3}}\oplus\mathbf{6}$

$$\overline{\mathbf{3}}\sim\begin{pmatrix}\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2\\ \alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1\\ \alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_3\end{pmatrix},\qquad \mathbf{6}\sim\begin{pmatrix}2\alpha_1\beta_3-\alpha_2\beta_2-\alpha_3\beta_1\\ 2\alpha_2\beta_3-\alpha_1\beta_1-\alpha_3\beta_2\\ 2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1\\ \sqrt{3}\,(\alpha_2\beta_2-\alpha_3\beta_1)\\ \sqrt{3}\,(\alpha_3\beta_2-\alpha_1\beta_1)\\ \sqrt{3}\,(\alpha_1\beta_2-\alpha_2\beta_1)\end{pmatrix}.$$

- $\mathbf{3}\otimes\mathbf{6}=\mathbf{3}\oplus\mathbf{3}'\oplus\widetilde{\mathbf{3}}\oplus\widetilde{\mathbf{3}}'\oplus\mathbf{6}$

$$\mathbf{3}\sim\begin{pmatrix}2\alpha_2\beta_5-\alpha_1\beta_6-\alpha_3\beta_4+\sqrt{3}\,(\alpha_3\beta_1-\alpha_1\beta_3)\\ 2\alpha_1\beta_4-\alpha_2\beta_6-\alpha_3\beta_5+\sqrt{3}\,(\alpha_2\beta_3-\alpha_3\beta_2)\\ 2\alpha_3\beta_6-\alpha_1\beta_5-\alpha_2\beta_4+\sqrt{3}\,(\alpha_1\beta_2-\alpha_2\beta_1)\end{pmatrix},$$

$$\mathbf{3}'\sim\begin{pmatrix}2\alpha_2\beta_2-\alpha_1\beta_3-\alpha_3\beta_1+\sqrt{3}\,(\alpha_1\beta_6-\alpha_3\beta_4)\\ 2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2+\sqrt{3}\,(\alpha_3\beta_5-\alpha_2\beta_6)\\ 2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1+\sqrt{3}\,(\alpha_2\beta_4-\alpha_1\beta_5)\end{pmatrix},$$

$$\widetilde{\mathbf{3}}\sim\begin{pmatrix}\alpha_1\beta_6+\alpha_2\beta_5+\alpha_3\beta_4+\sqrt{3}\,(\alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1)\\ \alpha_1\beta_4+\alpha_2\beta_6+\alpha_3\beta_5-\sqrt{3}\,(\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2)\\ -2\,(\alpha_1\beta_5+\alpha_2\beta_4+\alpha_3\beta_6)\end{pmatrix},$$

$$\widetilde{\mathbf{3}}'\sim\begin{pmatrix}\alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1-\sqrt{3}\,(\alpha_1\beta_6+\alpha_2\beta_5+\alpha_3\beta_4)\\ \alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2+\sqrt{3}\,(\alpha_1\beta_4+\alpha_2\beta_6+\alpha_3\beta_5)\\ -2\,(\alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_3)\end{pmatrix},$$

$$\mathbf{6}\sim\begin{pmatrix}2\alpha_2\beta_5-\alpha_1\beta_6-\alpha_3\beta_4+\sqrt{3}\,(\alpha_1\beta_3-\alpha_3\beta_1)\\ 2\alpha_1\beta_4-\alpha_2\beta_6-\alpha_3\beta_5+\sqrt{3}\,(\alpha_3\beta_2-\alpha_2\beta_3)\\ 2\alpha_3\beta_6-\alpha_1\beta_5-\alpha_2\beta_4+\sqrt{3}\,(\alpha_2\beta_1-\alpha_1\beta_2)\\ -2\alpha_2\beta_2+\alpha_1\beta_3+\alpha_3\beta_1+\sqrt{3}\,(\alpha_1\beta_6-\alpha_3\beta_4)\\ -2\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2+\sqrt{3}\,(\alpha_3\beta_5-\alpha_2\beta_6)\\ -2\alpha_3\beta_3+\alpha_1\beta_2+\alpha_2\beta_1+\sqrt{3}\,(\alpha_2\beta_4-\alpha_1\beta_5)\end{pmatrix}.$$

- $\mathbf{3}' \otimes \mathbf{6} = \mathbf{3} \oplus \mathbf{3}' \oplus \widetilde{\mathbf{3}} \oplus \widetilde{\mathbf{3}}' \oplus \mathbf{6}$

$$\mathbf{3} \sim \begin{pmatrix} 2\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1 + \sqrt{3}\,(\alpha_1\beta_6 - \alpha_3\beta_4) \\ 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 + \sqrt{3}\,(\alpha_3\beta_5 - \alpha_2\beta_6) \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 + \sqrt{3}\,(\alpha_2\beta_4 - \alpha_1\beta_5) \end{pmatrix}\,,$$

$$\mathbf{3}' \sim \begin{pmatrix} 2\alpha_2\beta_5 - \alpha_1\beta_6 - \alpha_3\beta_4 + \sqrt{3}\,(\alpha_3\beta_1 - \alpha_1\beta_3) \\ 2\alpha_1\beta_4 - \alpha_2\beta_6 - \alpha_3\beta_5 + \sqrt{3}\,(\alpha_2\beta_3 - \alpha_3\beta_2) \\ 2\alpha_3\beta_6 - \alpha_1\beta_5 - \alpha_2\beta_4 + \sqrt{3}\,(\alpha_1\beta_2 - \alpha_2\beta_1) \end{pmatrix}\,,$$

$$\widetilde{\mathbf{3}} \sim \begin{pmatrix} \alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1 - \sqrt{3}\,(\alpha_1\beta_6 + \alpha_2\beta_5 + \alpha_3\beta_4) \\ \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 + \sqrt{3}\,(\alpha_1\beta_4 + \alpha_2\beta_6 + \alpha_3\beta_5) \\ -2\,(\alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_3) \end{pmatrix}\,,$$

$$\widetilde{\mathbf{3}}' \sim \begin{pmatrix} \alpha_1\beta_6 + \alpha_2\beta_5 + \alpha_3\beta_4 + \sqrt{3}\,(\alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1) \\ \alpha_1\beta_4 + \alpha_2\beta_6 + \alpha_3\beta_5 - \sqrt{3}\,(\alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2) \\ -2\,(\alpha_1\beta_5 + \alpha_2\beta_4 + \alpha_3\beta_6) \end{pmatrix}\,,$$

$$\mathbf{6} \sim \begin{pmatrix} 2\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1 + \sqrt{3}\,(\alpha_3\beta_4 - \alpha_1\beta_6) \\ 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 + \sqrt{3}\,(\alpha_2\beta_6 - \alpha_3\beta_5) \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 + \sqrt{3}\,(\alpha_1\beta_5 - \alpha_2\beta_4) \\ 2\alpha_2\beta_5 - \alpha_1\beta_6 - \alpha_3\beta_4 + \sqrt{3}\,(\alpha_1\beta_3 - \alpha_3\beta_1) \\ 2\alpha_1\beta_4 - \alpha_2\beta_6 - \alpha_3\beta_5 + \sqrt{3}\,(\alpha_3\beta_2 - \alpha_2\beta_3) \\ 2\alpha_3\beta_6 - \alpha_1\beta_5 - \alpha_2\beta_4 + \sqrt{3}\,(\alpha_2\beta_1 - \alpha_1\beta_2) \end{pmatrix}\,.$$

- $\overline{\mathbf{3}} \otimes \overline{\mathbf{3}} = \overline{\mathbf{3}}' \otimes \overline{\mathbf{3}}' = \mathbf{3} \oplus \mathbf{3}' \oplus \widetilde{\mathbf{3}}'$

$$\mathbf{3} \sim \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix},\;\; \mathbf{3}' \sim \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix},\;\; \widetilde{\mathbf{3}}' \sim \begin{pmatrix} \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 \\ \alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1 \\ \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_3 \end{pmatrix}\,.$$

- $\overline{\mathbf{3}} \otimes \overline{\mathbf{3}}' = \mathbf{3} \oplus \mathbf{3}' \oplus \widetilde{\mathbf{3}}$

$$\mathbf{3} \sim \begin{pmatrix} 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix},\;\; \mathbf{3}' \sim \begin{pmatrix} \alpha_2\beta_3 - \alpha_3\beta_2 \\ \alpha_3\beta_1 - \alpha_1\beta_3 \\ \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix},\;\; \widetilde{\mathbf{3}} \sim \begin{pmatrix} \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 \\ \alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1 \\ \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_3 \end{pmatrix}\,.$$

- $\overline{\mathbf{3}} \otimes \widetilde{\mathbf{3}} = \overline{\mathbf{3}}' \otimes \widetilde{\mathbf{3}}' = \mathbf{3}' \oplus \mathbf{6}$

$$\mathbf{3}' \sim \begin{pmatrix} \alpha_1\beta_2 + \alpha_2\beta_3 + \alpha_3\beta_1 \\ \alpha_1\beta_3 + \alpha_2\beta_1 + \alpha_3\beta_2 \\ \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 \end{pmatrix}, \quad \mathbf{6} \sim \begin{pmatrix} \sqrt{3}\,(\alpha_1\beta_2 - \alpha_3\beta_1) \\ \sqrt{3}\,(\alpha_3\beta_2 - \alpha_2\beta_1) \\ \sqrt{3}\,(\alpha_2\beta_2 - \alpha_1\beta_1) \\ 2\alpha_2\beta_3 - \alpha_1\beta_2 - \alpha_3\beta_1 \\ 2\alpha_1\beta_3 - \alpha_2\beta_1 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_1 - \alpha_2\beta_2 \end{pmatrix}.$$

- $\overline{\mathbf{3}} \otimes \widetilde{\mathbf{3}}' = \overline{\mathbf{3}}' \otimes \widetilde{\mathbf{3}} = \mathbf{3} \oplus \mathbf{6}$

$$\mathbf{3} \sim \begin{pmatrix} \alpha_1\beta_2 + \alpha_2\beta_3 + \alpha_3\beta_1 \\ \alpha_1\beta_3 + \alpha_2\beta_1 + \alpha_3\beta_2 \\ \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 \end{pmatrix}, \quad \mathbf{6} \sim \begin{pmatrix} 2\alpha_2\beta_3 - \alpha_1\beta_2 - \alpha_3\beta_1 \\ 2\alpha_1\beta_3 - \alpha_2\beta_1 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_1 - \alpha_2\beta_2 \\ \sqrt{3}\,(\alpha_3\beta_1 - \alpha_1\beta_2) \\ \sqrt{3}\,(\alpha_2\beta_1 - \alpha_3\beta_2) \\ \sqrt{3}\,(\alpha_1\beta_1 - \alpha_2\beta_2) \end{pmatrix}.$$

- $\overline{\mathbf{3}} \otimes \mathbf{6} = \overline{\mathbf{3}} \oplus \overline{\mathbf{3}}' \oplus \widetilde{\mathbf{3}} \oplus \widetilde{\mathbf{3}}' \oplus \mathbf{6}$

$$\overline{\mathbf{3}} \sim \begin{pmatrix} 2\alpha_2\beta_4 - \alpha_1\beta_6 - \alpha_3\beta_5 + \sqrt{3}\,(\alpha_3\beta_2 - \alpha_1\beta_3) \\ 2\alpha_1\beta_5 - \alpha_2\beta_6 - \alpha_3\beta_4 + \sqrt{3}\,(\alpha_2\beta_3 - \alpha_3\beta_1) \\ 2\alpha_3\beta_6 - \alpha_1\beta_4 - \alpha_2\beta_5 + \sqrt{3}\,(\alpha_1\beta_1 - \alpha_2\beta_2) \end{pmatrix},$$

$$\overline{\mathbf{3}}' \sim \begin{pmatrix} 2\alpha_2\beta_1 - \alpha_1\beta_3 - \alpha_3\beta_2 + \sqrt{3}\,(\alpha_1\beta_6 - \alpha_3\beta_5) \\ 2\alpha_1\beta_2 - \alpha_2\beta_3 - \alpha_3\beta_1 + \sqrt{3}\,(\alpha_3\beta_4 - \alpha_2\beta_6) \\ 2\alpha_3\beta_3 - \alpha_1\beta_1 - \alpha_2\beta_2 + \sqrt{3}\,(\alpha_2\beta_5 - \alpha_1\beta_4) \end{pmatrix},$$

$$\widetilde{\mathbf{3}} \sim \begin{pmatrix} \alpha_1\beta_5 + \alpha_2\beta_6 + \alpha_3\beta_4 - \sqrt{3}\,(\alpha_1\beta_2 + \alpha_2\beta_3 + \alpha_3\beta_1) \\ \alpha_1\beta_6 + \alpha_2\beta_4 + \alpha_3\beta_5 + \sqrt{3}\,(\alpha_1\beta_3 + \alpha_2\beta_1 + \alpha_3\beta_2) \\ -2\,(\alpha_1\beta_4 + \alpha_2\beta_5 + \alpha_3\beta_6) \end{pmatrix},$$

$$\widetilde{\mathbf{3}}' \sim \begin{pmatrix} \alpha_1\beta_2 + \alpha_2\beta_3 + \alpha_3\beta_1 + \sqrt{3}\,(\alpha_1\beta_5 + \alpha_2\beta_6 + \alpha_3\beta_4) \\ \alpha_1\beta_3 + \alpha_2\beta_1 + \alpha_3\beta_2 - \sqrt{3}\,(\alpha_1\beta_6 + \alpha_2\beta_4 + \alpha_3\beta_5) \\ -2\,(\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3) \end{pmatrix},$$

$$\mathbf{6} \sim \begin{pmatrix} 2\alpha_1\beta_5 - \alpha_2\beta_6 - \alpha_3\beta_4 + \sqrt{3}\,(\alpha_3\beta_1 - \alpha_2\beta_3) \\ 2\alpha_2\beta_4 - \alpha_1\beta_6 - \alpha_3\beta_5 + \sqrt{3}\,(\alpha_1\beta_3 - \alpha_3\beta_2) \\ 2\alpha_3\beta_6 - \alpha_1\beta_4 - \alpha_2\beta_5 + \sqrt{3}\,(\alpha_2\beta_2 - \alpha_1\beta_1) \\ -2\alpha_1\beta_2 + \alpha_2\beta_3 + \alpha_3\beta_1 + \sqrt{3}\,(\alpha_3\beta_4 - \alpha_2\beta_6) \\ -2\alpha_2\beta_1 + \alpha_1\beta_3 + \alpha_3\beta_2 + \sqrt{3}\,(\alpha_1\beta_6 - \alpha_3\beta_5) \\ -2\alpha_3\beta_3 + \alpha_1\beta_1 + \alpha_2\beta_2 + \sqrt{3}\,(\alpha_2\beta_5 - \alpha_1\beta_4) \end{pmatrix}.$$

- $\overline{\mathbf{3}}' \otimes \mathbf{6} = \overline{\mathbf{3}} \oplus \overline{\mathbf{3}}' \oplus \widetilde{\mathbf{3}} \oplus \widetilde{\mathbf{3}}' \oplus \mathbf{6}$

$$\overline{\mathbf{3}} \sim \begin{pmatrix} 2\alpha_2\beta_1 - \alpha_1\beta_3 - \alpha_3\beta_2 + \sqrt{3}\left(\alpha_1\beta_6 - \alpha_3\beta_5\right) \\ 2\alpha_1\beta_2 - \alpha_2\beta_3 - \alpha_3\beta_1 + \sqrt{3}\left(\alpha_3\beta_4 - \alpha_2\beta_6\right) \\ 2\alpha_3\beta_3 - \alpha_1\beta_1 - \alpha_2\beta_2 + \sqrt{3}\left(\alpha_2\beta_5 - \alpha_1\beta_4\right) \end{pmatrix} ,$$

$$\overline{\mathbf{3}}' \sim \begin{pmatrix} 2\alpha_2\beta_4 - \alpha_1\beta_6 - \alpha_3\beta_5 + \sqrt{3}\left(\alpha_3\beta_2 - \alpha_1\beta_3\right) \\ 2\alpha_1\beta_5 - \alpha_2\beta_6 - \alpha_3\beta_4 + \sqrt{3}\left(\alpha_2\beta_3 - \alpha_3\beta_1\right) \\ 2\alpha_3\beta_6 - \alpha_1\beta_4 - \alpha_2\beta_5 + \sqrt{3}\left(\alpha_1\beta_1 - \alpha_2\beta_2\right) \end{pmatrix} ,$$

$$\widetilde{\mathbf{3}} \sim \begin{pmatrix} \alpha_1\beta_2 + \alpha_2\beta_3 + \alpha_3\beta_1 + \sqrt{3}\left(\alpha_1\beta_5 + \alpha_2\beta_6 + \alpha_3\beta_4\right) \\ \alpha_1\beta_3 + \alpha_2\beta_1 + \alpha_3\beta_2 - \sqrt{3}\left(\alpha_1\beta_6 + \alpha_2\beta_4 + \alpha_3\beta_5\right) \\ -2\left(\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3\right) \end{pmatrix} ,$$

$$\widetilde{\mathbf{3}}' \sim \begin{pmatrix} \alpha_1\beta_5 + \alpha_2\beta_6 + \alpha_3\beta_4 - \sqrt{3}\left(\alpha_1\beta_2 + \alpha_2\beta_3 + \alpha_3\beta_1\right) \\ \alpha_1\beta_6 + \alpha_2\beta_4 + \alpha_3\beta_5 + \sqrt{3}\left(\alpha_1\beta_3 + \alpha_2\beta_1 + \alpha_3\beta_2\right) \\ -2\left(\alpha_1\beta_4 + \alpha_2\beta_5 + \alpha_3\beta_6\right) \end{pmatrix} ,$$

$$\mathbf{6} \sim \begin{pmatrix} 2\alpha_1\beta_2 - \alpha_2\beta_3 - \alpha_3\beta_1 + \sqrt{3}\left(\alpha_2\beta_6 - \alpha_3\beta_4\right) \\ 2\alpha_2\beta_1 - \alpha_1\beta_3 - \alpha_3\beta_2 + \sqrt{3}\left(\alpha_3\beta_5 - \alpha_1\beta_6\right) \\ 2\alpha_3\beta_3 - \alpha_1\beta_1 - \alpha_2\beta_2 + \sqrt{3}\left(\alpha_1\beta_4 - \alpha_2\beta_5\right) \\ 2\alpha_1\beta_5 - \alpha_2\beta_6 - \alpha_3\beta_4 + \sqrt{3}\left(\alpha_3\beta_1 - \alpha_2\beta_3\right) \\ 2\alpha_2\beta_4 - \alpha_1\beta_6 - \alpha_3\beta_5 + \sqrt{3}\left(\alpha_1\beta_3 - \alpha_3\beta_2\right) \\ 2\alpha_3\beta_6 - \alpha_1\beta_4 - \alpha_2\beta_5 + \sqrt{3}\left(\alpha_2\beta_2 - \alpha_1\beta_1\right) \end{pmatrix} .$$

- $\widetilde{\mathbf{3}} \otimes \widetilde{\mathbf{3}} = \widetilde{\mathbf{3}}' \otimes \widetilde{\mathbf{3}}' = \mathbf{1} \oplus \mathbf{2} \oplus \widetilde{\mathbf{3}} \oplus \widetilde{\mathbf{3}}'$

$$\mathbf{1} \sim \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_3, \qquad \mathbf{2} \sim \begin{pmatrix} \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 \\ \alpha_1\beta_3 + \alpha_2\beta_2 + \alpha_3\beta_1 \end{pmatrix},$$

$$\widetilde{\mathbf{3}} \sim \begin{pmatrix} \alpha_1\beta_3 - \alpha_3\beta_1 \\ \alpha_3\beta_2 - \alpha_2\beta_3 \\ \alpha_2\beta_1 - \alpha_1\beta_2 \end{pmatrix}, \qquad \widetilde{\mathbf{3}}' \sim \begin{pmatrix} 2\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1 \\ 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix} .$$

- $\widetilde{\mathbf{3}} \otimes \widetilde{\mathbf{3}}' = \mathbf{1}' \oplus \mathbf{2} \oplus \widetilde{\mathbf{3}} \oplus \widetilde{\mathbf{3}}'$

$$\mathbf{1}' \sim \alpha_1\beta_2 + \alpha_2\beta_1 + \alpha_3\beta_3, \qquad \mathbf{2} \sim \begin{pmatrix} \alpha_1\beta_1 + \alpha_2\beta_3 + \alpha_3\beta_2 \\ -\alpha_1\beta_3 - \alpha_2\beta_2 - \alpha_3\beta_1 \end{pmatrix},$$

$$\widetilde{\mathbf{3}} \sim \begin{pmatrix} 2\alpha_2\beta_2 - \alpha_1\beta_3 - \alpha_3\beta_1 \\ 2\alpha_1\beta_1 - \alpha_2\beta_3 - \alpha_3\beta_2 \\ 2\alpha_3\beta_3 - \alpha_1\beta_2 - \alpha_2\beta_1 \end{pmatrix}, \qquad \widetilde{\mathbf{3}}' \sim \begin{pmatrix} \alpha_1\beta_3 - \alpha_3\beta_1 \\ \alpha_3\beta_2 - \alpha_2\beta_3 \\ \alpha_2\beta_1 - \alpha_1\beta_2 \end{pmatrix} .$$

- $\widetilde{\mathbf{3}}\otimes\mathbf{6}=\mathbf{3}\otimes\mathbf{3}'\oplus\overline{\mathbf{3}}\oplus\overline{\mathbf{3}}'\oplus\mathbf{6}$

$$\mathbf{3}\sim\begin{pmatrix}2\alpha_3\beta_4-\alpha_1\beta_6-\alpha_2\beta_5+\sqrt{3}\,(\alpha_2\beta_2-\alpha_1\beta_3)\\2\alpha_3\beta_5-\alpha_1\beta_4-\alpha_2\beta_6+\sqrt{3}\,(\alpha_2\beta_3-\alpha_1\beta_1)\\2\alpha_3\beta_6-\alpha_1\beta_5-\alpha_2\beta_4+\sqrt{3}\,(\alpha_2\beta_1-\alpha_1\beta_2)\end{pmatrix}\,,$$

$$\mathbf{3}'\sim\begin{pmatrix}2\alpha_3\beta_1-\alpha_1\beta_3-\alpha_2\beta_2+\sqrt{3}\,(\alpha_1\beta_6-\alpha_2\beta_5)\\2\alpha_3\beta_2-\alpha_1\beta_1-\alpha_2\beta_3+\sqrt{3}\,(\alpha_1\beta_4-\alpha_2\beta_6)\\2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1+\sqrt{3}\,(\alpha_1\beta_5-\alpha_2\beta_4)\end{pmatrix}\,,$$

$$\overline{\mathbf{3}}\sim\begin{pmatrix}2\alpha_3\beta_5-\alpha_1\beta_4-\alpha_2\beta_6+\sqrt{3}\,(\alpha_1\beta_1-\alpha_2\beta_3)\\2\alpha_3\beta_4-\alpha_1\beta_6-\alpha_2\beta_5+\sqrt{3}\,(\alpha_1\beta_3-\alpha_2\beta_2)\\2\alpha_3\beta_6-\alpha_1\beta_5-\alpha_2\beta_4+\sqrt{3}\,(\alpha_1\beta_2-\alpha_2\beta_1)\end{pmatrix}\,,$$

$$\overline{\mathbf{3}}'\sim\begin{pmatrix}2\alpha_3\beta_2-\alpha_1\beta_1-\alpha_2\beta_3+\sqrt{3}\,(\alpha_2\beta_6-\alpha_1\beta_4)\\2\alpha_3\beta_1-\alpha_1\beta_3-\alpha_2\beta_2+\sqrt{3}\,(\alpha_2\beta_5-\alpha_1\beta_6)\\2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1+\sqrt{3}\,(\alpha_2\beta_4-\alpha_1\beta_5)\end{pmatrix}\,,$$

$$\mathbf{6}\sim\begin{pmatrix}\alpha_1\beta_6+\alpha_2\beta_5+\alpha_3\beta_4\\\alpha_1\beta_4+\alpha_2\beta_6+\alpha_3\beta_5\\\alpha_1\beta_5+\alpha_2\beta_4+\alpha_3\beta_6\\\alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1\\\alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2\\\alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_3\end{pmatrix}\,.$$

- $\widetilde{\mathbf{3}}'\otimes\mathbf{6}=\mathbf{3}\oplus\mathbf{3}'\oplus\overline{\mathbf{3}}\oplus\overline{\mathbf{3}}'\oplus\mathbf{6}$

$$\mathbf{3}\sim\begin{pmatrix}2\alpha_3\beta_1-\alpha_1\beta_3-\alpha_2\beta_2+\sqrt{3}\,(\alpha_1\beta_6-\alpha_2\beta_5)\\2\alpha_3\beta_2-\alpha_1\beta_1-\alpha_2\beta_3+\sqrt{3}\,(\alpha_1\beta_4-\alpha_2\beta_6)\\2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1+\sqrt{3}\,(\alpha_1\beta_5-\alpha_2\beta_4)\end{pmatrix}\,,$$

$$\mathbf{3}'\sim\begin{pmatrix}2\alpha_3\beta_4-\alpha_1\beta_6-\alpha_2\beta_5+\sqrt{3}\,(\alpha_2\beta_2-\alpha_1\beta_3)\\2\alpha_3\beta_5-\alpha_1\beta_4-\alpha_2\beta_6+\sqrt{3}\,(\alpha_2\beta_3-\alpha_1\beta_1)\\2\alpha_3\beta_6-\alpha_1\beta_5-\alpha_2\beta_4+\sqrt{3}\,(\alpha_2\beta_1-\alpha_1\beta_2)\end{pmatrix}\,,$$

$$\overline{\mathbf{3}}\sim\begin{pmatrix}2\alpha_3\beta_2-\alpha_1\beta_1-\alpha_2\beta_3+\sqrt{3}\,(\alpha_2\beta_6-\alpha_1\beta_4)\\2\alpha_3\beta_1-\alpha_1\beta_3-\alpha_2\beta_2+\sqrt{3}\,(\alpha_2\beta_5-\alpha_1\beta_6)\\2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1+\sqrt{3}\,(\alpha_2\beta_4-\alpha_1\beta_5)\end{pmatrix}\,,$$

$$\overline{\mathbf{3}}'\sim\begin{pmatrix}2\alpha_3\beta_5-\alpha_1\beta_4-\alpha_2\beta_6+\sqrt{3}\,(\alpha_1\beta_1-\alpha_2\beta_3)\\2\alpha_3\beta_4-\alpha_1\beta_6-\alpha_2\beta_5+\sqrt{3}\,(\alpha_1\beta_3-\alpha_2\beta_2)\\2\alpha_3\beta_6-\alpha_1\beta_5-\alpha_2\beta_4+\sqrt{3}\,(\alpha_1\beta_2-\alpha_2\beta_1)\end{pmatrix}\,,$$

$$\mathbf{6} \sim \begin{pmatrix} \alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1 \\ \alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2 \\ \alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_3 \\ -\alpha_1\beta_6-\alpha_2\beta_5-\alpha_3\beta_4 \\ -\alpha_1\beta_4-\alpha_2\beta_6-\alpha_3\beta_5 \\ -\alpha_1\beta_5-\alpha_2\beta_4-\alpha_3\beta_6 \end{pmatrix} .$$

- $\mathbf{6}\otimes\mathbf{6}=\mathbf{1}\oplus\mathbf{1}'\oplus\mathbf{2}_S\oplus\mathbf{2}_A\oplus\mathbf{3}\oplus\mathbf{3}'\oplus\overline{\mathbf{3}}\oplus\overline{\mathbf{3}}'\oplus\widetilde{\mathbf{3}}\oplus\widetilde{\mathbf{3}}'\oplus\mathbf{6}_S\oplus\mathbf{6}_A$

$$\mathbf{1} \sim \alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_3+\alpha_4\beta_5+\alpha_5\beta_4+\alpha_6\beta_6\,,$$

$$\mathbf{1}' \sim \alpha_1\beta_5+\alpha_2\beta_4+\alpha_3\beta_6-\alpha_4\beta_2-\alpha_5\beta_1-\alpha_6\beta_3\,,$$

$$\mathbf{2}_S \sim \begin{pmatrix} \alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2+\alpha_4\beta_4+\alpha_5\beta_6+\alpha_6\beta_5 \\ \alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1+\alpha_4\beta_6+\alpha_5\beta_5+\alpha_6\beta_4 \end{pmatrix} ,$$

$$\mathbf{2}_A \sim \begin{pmatrix} \alpha_1\beta_4+\alpha_2\beta_6+\alpha_3\beta_5-\alpha_4\beta_1-\alpha_5\beta_3-\alpha_6\beta_2 \\ -\alpha_1\beta_6-\alpha_2\beta_5-\alpha_3\beta_4+\alpha_4\beta_3+\alpha_5\beta_2+\alpha_6\beta_1 \end{pmatrix} ,$$

$$\mathbf{3} \sim \begin{pmatrix} 2\alpha_2\beta_5-\alpha_1\beta_6-\alpha_3\beta_4-2\alpha_5\beta_2+\alpha_4\beta_3+\alpha_6\beta_1+\sqrt{3}\,(\alpha_1\beta_3-\alpha_3\beta_1+\alpha_4\beta_6-\alpha_6\beta_4) \\ 2\alpha_1\beta_4-\alpha_2\beta_6-\alpha_3\beta_5-2\alpha_4\beta_1+\alpha_5\beta_3+\alpha_6\beta_2+\sqrt{3}\,(\alpha_3\beta_2-\alpha_2\beta_3+\alpha_6\beta_5-\alpha_5\beta_6) \\ 2\alpha_3\beta_6-\alpha_1\beta_5-\alpha_2\beta_4-2\alpha_6\beta_3+\alpha_4\beta_2+\alpha_5\beta_1+\sqrt{3}\,(\alpha_2\beta_1-\alpha_1\beta_2+\alpha_5\beta_4-\alpha_4\beta_5) \end{pmatrix} ,$$

$$\mathbf{3}' \sim \begin{pmatrix} 2\alpha_2\beta_2-\alpha_1\beta_3-\alpha_3\beta_1+2\alpha_5\beta_5-\alpha_4\beta_6-\alpha_6\beta_4+\sqrt{3}\,(\alpha_3\beta_4-\alpha_1\beta_6+\alpha_4\beta_3-\alpha_6\beta_1) \\ 2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2+2\alpha_4\beta_4-\alpha_5\beta_6-\alpha_6\beta_5+\sqrt{3}\,(\alpha_2\beta_6-\alpha_3\beta_5+\alpha_6\beta_2-\alpha_5\beta_3) \\ 2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1+2\alpha_6\beta_6-\alpha_4\beta_5-\alpha_5\beta_4+\sqrt{3}\,(\alpha_1\beta_5-\alpha_2\beta_4+\alpha_5\beta_1-\alpha_4\beta_2) \end{pmatrix} ,$$

$$\overline{\mathbf{3}} \sim \begin{pmatrix} 2\alpha_1\beta_4-\alpha_2\beta_6-\alpha_3\beta_5-2\alpha_4\beta_1+\alpha_5\beta_3+\alpha_6\beta_2+\sqrt{3}\,(\alpha_2\beta_3-\alpha_3\beta_2+\alpha_5\beta_6-\alpha_6\beta_5) \\ 2\alpha_2\beta_5-\alpha_1\beta_6-\alpha_3\beta_4-2\alpha_5\beta_2+\alpha_4\beta_3+\alpha_6\beta_1+\sqrt{3}\,(\alpha_3\beta_1-\alpha_1\beta_3+\alpha_6\beta_4-\alpha_4\beta_6) \\ 2\alpha_3\beta_6-\alpha_1\beta_5-\alpha_2\beta_4-2\alpha_6\beta_3+\alpha_4\beta_2+\alpha_5\beta_1+\sqrt{3}\,(\alpha_1\beta_2-\alpha_2\beta_1+\alpha_4\beta_5-\alpha_5\beta_4) \end{pmatrix} ,$$

$$\overline{\mathbf{3}}' \sim \begin{pmatrix} 2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2+2\alpha_4\beta_4-\alpha_5\beta_6-\alpha_6\beta_5+\sqrt{3}\,(\alpha_3\beta_5-\alpha_2\beta_6+\alpha_5\beta_3-\alpha_6\beta_2) \\ 2\alpha_2\beta_2-\alpha_1\beta_3-\alpha_3\beta_1+2\alpha_5\beta_5-\alpha_4\beta_6-\alpha_6\beta_4+\sqrt{3}\,(\alpha_1\beta_6-\alpha_3\beta_4+\alpha_6\beta_1-\alpha_4\beta_3) \\ 2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1+2\alpha_6\beta_6-\alpha_4\beta_5-\alpha_5\beta_4+\sqrt{3}\,(\alpha_2\beta_4-\alpha_1\beta_5+\alpha_4\beta_2-\alpha_5\beta_1) \end{pmatrix} ,$$

$$\widetilde{\mathbf{3}} \sim \begin{pmatrix} \alpha_1\beta_6+\alpha_2\beta_5+\alpha_3\beta_4+\alpha_4\beta_3+\alpha_5\beta_2+\alpha_6\beta_1 \\ \alpha_1\beta_4+\alpha_2\beta_6+\alpha_3\beta_5+\alpha_4\beta_1+\alpha_5\beta_3+\alpha_6\beta_2 \\ \alpha_1\beta_5+\alpha_2\beta_4+\alpha_3\beta_6+\alpha_4\beta_2+\alpha_5\beta_1+\alpha_6\beta_3 \end{pmatrix} ,$$

$$\widetilde{\mathbf{3}}' \sim \begin{pmatrix} \alpha_1\beta_3+\alpha_2\beta_2+\alpha_3\beta_1-\alpha_4\beta_6-\alpha_5\beta_5-\alpha_6\beta_4 \\ \alpha_1\beta_1+\alpha_2\beta_3+\alpha_3\beta_2-\alpha_4\beta_4-\alpha_5\beta_6-\alpha_6\beta_5 \\ \alpha_1\beta_2+\alpha_2\beta_1+\alpha_3\beta_3-\alpha_4\beta_5-\alpha_5\beta_4-\alpha_6\beta_6 \end{pmatrix},$$

$$\mathbf{6}_S \sim \begin{pmatrix} 2\alpha_2\beta_5-\alpha_1\beta_6-\alpha_3\beta_4+2\alpha_5\beta_2-\alpha_4\beta_3-\alpha_6\beta_1 \\ 2\alpha_1\beta_4-\alpha_2\beta_6-\alpha_3\beta_5+2\alpha_4\beta_1-\alpha_5\beta_3-\alpha_6\beta_2 \\ 2\alpha_3\beta_6-\alpha_1\beta_5-\alpha_2\beta_4+2\alpha_6\beta_3-\alpha_4\beta_2-\alpha_5\beta_1 \\ 2\alpha_2\beta_2-\alpha_1\beta_3-\alpha_3\beta_1-2\alpha_5\beta_5+\alpha_4\beta_6+\alpha_6\beta_4 \\ 2\alpha_1\beta_1-\alpha_2\beta_3-\alpha_3\beta_2-2\alpha_4\beta_4+\alpha_5\beta_6+\alpha_6\beta_5 \\ 2\alpha_3\beta_3-\alpha_1\beta_2-\alpha_2\beta_1-2\alpha_6\beta_6+\alpha_4\beta_5+\alpha_5\beta_4 \end{pmatrix},$$

$$\mathbf{6}_A \sim \begin{pmatrix} \alpha_1\beta_3-\alpha_3\beta_1+\alpha_6\beta_4-\alpha_4\beta_6 \\ \alpha_3\beta_2-\alpha_2\beta_3+\alpha_5\beta_6-\alpha_6\beta_5 \\ \alpha_2\beta_1-\alpha_1\beta_2+\alpha_4\beta_5-\alpha_5\beta_4 \\ \alpha_3\beta_4-\alpha_4\beta_3+\alpha_6\beta_1-\alpha_1\beta_6 \\ \alpha_2\beta_6-\alpha_6\beta_2+\alpha_5\beta_3-\alpha_3\beta_5 \\ \alpha_1\beta_5-\alpha_2\beta_4+\alpha_4\beta_2-\alpha_5\beta_1 \end{pmatrix}.$$

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