UNIVERSITY OF SOUTHAMPTON

FACULTY OF SOCIAL, HUMAN & MATHEMATICAL SCIENCES

School of Education

CHILDREN'S LEARNING OF THE PARTITIVE QUOTIENT FRACTION SUB-CONSTRUCT AND THE ELABORATION OF THE DON'T NEED BOUNDARY FEATURE OF THE PIRIE-KIEREN THEORY

by

Lois Grace George

Thesis for the degree of Doctor of Philosophy

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ABSTRACT

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Using a qualitative, exploratory, microgenetic research design, this research study examines the strategies that children who had only been taught the part-whole fraction sub-construct used for finding the fraction associated with solving partitive quotient problems over a sequence of tasks. Additionally, a key feature of the Pirie-Kieren theory for growth of mathematical understanding, which has not been previously extended empirically, is elaborated.

Nine Year 5 children, who engaged in eight individual task-based interviews, over a six-week period, provide the data for this study. The research found that the research participants used four strategies for finding the fraction related to solving the partitive quotient problems. Further to this, one strategy appeared to resemble the conceptualisation for the part-whole relation. This finding suggests that previous part-whole learning impacts partitive quotient development. Another key contribution of this research is that it shows several ways in which part-whole knowledge impacted children's partitive quotient development. Regarding the theoretical contribution made by the current study, one of the significant findings challenges one aspect of the current Pirie-Kieren model, and therefore, an amendment to the existing model is proposed.

An implication of this study for teachers of mathematics is that there should be a greater focus on the concept of a unit when teaching the different fraction sub-construct contexts. In addition, to help learners grow in their understanding of mathematical concepts, it is suggested that there is an increased emphasis on assisting them to develop robust and thorough ideas of concepts when working in local contexts.

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List of Abbreviations

CoD	Commonwealth of Dominica
DNB	Don’t Need boundary
IM	Image Making
IH	Image Having
PN	Property Noticing
F	Formalising
Sol	Solution
PoS	Programme of study

DECLARATION OF AUTHORSHIP

I, Lois Grace George, declare that this thesis and the work presented in it are my own and have been generated by me as the result of my own original research.

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I confirm that:

1. This work was done wholly or mainly while in candidature for a research degree at this University;
2. Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated;
3. Where I have consulted the published work of others, this is always clearly attributed;
4. Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work;
5. I have acknowledged all main sources of help;
6. Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself;
7. None of this work has been published before submission.

Signed: ....

Date: ....

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This thesis represents not only a contribution to academic knowledge in the domain of mathematics education, and a significant milestone in my educational journey, but it is my hope that this serves as a springboard for further research in mathematics education in my beloved island Dominica and the wider Caribbean region, as well.

Chapter 1: Introduction

1.1 Overview

Fractions serve a critical purpose, from a theoretical, educational and practical perspective (National Mathematics Advisory Panel, 2008; Siegler *et al.*, 2013). According to Siegler *et al.* (2011), from a theoretical viewpoint, fractions are important because they provide students' first opportunity to learn that many properties of whole numbers do not extend to all types of numbers. In addition, fractions represent values that cannot be expressed with whole numbers. From an educational standpoint, fractions are important because of the strong predictive relationship between early fraction knowledge and later mathematics achievement (Bailey *et al.*, 2012; Siegler *et al.*, 2012). The Department for Education (2011) in England adds, 'proficiency of fractions is considered essential for accessing the secondary mathematics curriculum, in particular in the domains o*f measure*, *algebra* and *geometry* as well as *probability*' (p. 72). The pivotal role that fractions play in many subjects in a school's curriculum, such as Science, Home Economics and Geography, has also been documented (Gabriel *et al.*, 2013). Further to this, many occupations, ranging from doctor to clerical assistant, use a knowledge of fractions. From a practical perspective, fractions are an integral part of everyday life (Fuchs *et al.*, 2013) and are used in situations such as following a recipe, estimating a rebate or reading a map (Gabriel *et al.*, 2013).

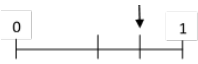
While the importance of fractions has been widely recognised by researchers and educators alike (Nunes, 2008; Siegler and Pyke, 2013), there is a general consensus, globally, that a large number of people at different ages and educational levels have difficulties in learning this concept (Behr *et al.*, 1992; Petit *et al.*, 2010; Siegler *et al.*, 2013). More specifically, research, focused on fraction learning in the United States (Siegler *et al.*, 2013), United Kingdom (Hallett *et al.*, 2010), Belgium (Gabriel *et al.*, 2013), Australia (Clarke *et al.*, 2006), Japan (Yoshida and Sawano, 2002), China and Taiwan (Chan *et al.*, 2007) have consistently documented children's difficulties in learning fractions. Additionally, the pattern of difficulty is longitudinally stable (Siegler and Pyke, 2013) and children who have early difficulties with fractions are likely to have problems in the future as well (Hecht and Vagi, 2012). This is despite the fact that fraction instruction typically begins by 1st (age 6) or 2nd grade (age 7) (Siegler *et al.*, 2010) and continues until the end of high-school and even into college. Taking into consideration the central role that fractions play, in and out of school, as well as the challenge students still face as they learn this topic, there have been very recent calls for more research to focus on the learning of particular aspects of fractions (Middleton *et al.*, 2015; Hackenberg *et al.*, 2016; Tsai and Li, 2016). The present research aims to answer these calls and focuses on primary school students' learning of fractions.

This introductory chapter seeks to provide a frame for this thesis and orient the reader to the specific aspects of fractions that constitute the focus of this empirical research. In this regard, it first delineates the research problem by discussing some of the key reasons why the learning of fractions is difficult for primary school students and, in so doing, exposes some of the gaps that still exist in the empirical literature on fraction learning in mathematics education. Following this, section 1.3 presents the empirical and theoretical contributions that the present research endeavours to make to the existing literature, in light of the gaps identified by the present researcher. In section 1.4, the researcher briefly presents her personal motivation for undertaking this research study. The chapter concludes with a complete statement of the research questions for this thesis and a summary of how the thesis is organised.

1.2 Research problem

Students' difficulty in learning fractions has several origins (Siegler *et al.*, 2013). A first source stems from the assumption adopted by many learners that the properties of whole numbers are properties of all numbers (Pitkethly and Hunting, 1996; Siegler *et al.*, 2011). Ni and Zhou (2005) label this as a 'whole number bias'. In the context of fractions, this bias leads learners to use whole number properties to make inferences on fractions. For example, many students apply the knowledge of whole number addition to fractions and find that $\frac{2}{3} + \frac{5}{6} = \frac{7}{9}$. In this regard, Kieren (1993) highlights that fractions are not merely an extension of whole numbers, but are a distinct entity. He also points out that there are unique properties in learning fractions, so individuals engaged in this learning will encounter new and distinct actions that correspond exclusively to fractions. For example, the property that multiplying makes numbers larger when working with whole numbers does not hold when students encounter problems involving multiplication by proper fractions (Kieren, 1993). For example, for the whole number multiplication problem 2 x 3, the answer, 6, is larger than either 2 or 3, in contrast to the fraction multiplication of $\frac{1}{2}$ x $\frac{1}{3}$ = $\frac{1}{6}$, where $\frac{1}{6}$ is less than either $\frac{1}{2}$ or $\frac{1}{3}$ and so does not become larger. Students, however, often do not recognise these key differences in whole number and fraction properties. This leads to great challenges in learning fractions.

Another difficulty that students experience as they engage in fraction learning, as identified by Lamon (2012), is that, as one moves from whole numbers into fractions, the variety and complexity of the situations that give meaning to the symbols increase dramatically. Extensive analysis of the domain of fractions in mathematics education has resulted in the widely-accepted notion that fractions represent a multi-faceted construct consisting of five sub-constructs (Behr *et al.*, 1983; Kieren, 1988; Ohlsson, 1988; Olive, 1999; Charalambous and Pitta-Pantazi, 2007; Middleton *et al.*, 2015). This means that one fraction can have five different meanings depending on the context. An example of this can be seen in some of the many interpretations of the fraction ¾ given in the list below:

- a shape divided into four equal pieces with three pieces shaded (*part-whole)*
- the amount that each person receives when three pizzas are shared among four friends (*quotient)*
- find ¾ of sixteen balls (*operator)*
- for every three boys in the class there are four girls (*ratio)*
- the number represented by the arrow on the number line 

(measure)

Section 2.2.2 of Chapter 2 elaborates the multi-faceted nature of the fraction construct, but the aforementioned examples of the various fraction sub-constructs provide an initial orientation of the conceptualisation of each sub-construct.

1.2.1 The challenge of learning the different sub-constructs

As researchers, curriculum developers and teachers have become increasingly aware that the fraction construct consists of multiple sub-constructs; the need to include all the fraction sub-constructs into fraction instruction has become a central focus of both research and curriculum development. In this regard, several key researchers (Behr and Post, 1992; Kieren, 1993; Brousseau *et al.*, 2004; Steffe and Olive, 2010; Lamon, 2012) in the domain of mathematics education have emphasised that children’s exposure to, and proficiency with, the different sub-constructs are needed for them to have a deeper and more complete understanding of the fraction domain. To this end, several primary school curricula in various localities (for example Romberg, 2001; Brousseau *et al.*, 2004; Cortina and Zúñiga, 2008; Clarke *et al.*, 2011b) have been reformulated to include the teaching and learning of all the fraction sub-constructs, but research has documented that children’s learning of the different fraction sub-constructs has been challenging. Clarke *et al.* (2011b), for example, state that ‘much of the confusion in teaching and learning fractions is evident when students cannot synthesise the many different interpretations (sub-constructs) of fractions’ (p. 24).

The recognition that many children find the learning of the different fraction sub-constructs challenging has uncovered an important research problem, as well as an associated gap in the empirical literature. This has resulted in the call for more research which investigates how one fraction sub-construct develops from existing knowledge of previous sub-construct(s) (Middleton *et al.*, 2015). This is a key focus of this thesis. Regarding the gap identified, Middleton *et al.* (2015) state that, although important findings as to how children operate with fractions have formed part of the research in the area of rational number knowing, a clear picture of how rational number knowledge, inclusive of how the different sub-constructs develop over time, has yet to be discovered. They further state that without a framework for interpreting students' understanding, the research base remains fragmented and primarily focused on further examination of understanding the origin and phenomenology of individual sub-constructs.

1.3 Contribution to existing literature

1.3.1 Empirical contribution

While the need for research focused on children's learning paths from one sub-construct to another has been recognised, since there are five fraction sub-constructs and there is no consensus on the order in which fractions should be learnt (Middleton *et al.*, 2015), there are a number of pathways that research can follow. In this regard, Kieren (1993) states that the fact that a complete understanding of fractions entails combining varied fraction knowledge 'provides the researcher… with a complex of considerations in research' (p. 66). In this regard, the present research aims to make two key contributions to the empirical research on fraction learning.

1.3.1.1 Fraction learning starting from the part-whole

The part-whole interpretation of a fraction $\frac{a}{b}$ represents a relationship or comparison of the number of parts, denoted by the numerator, a, to a fixed number of equal-sized parts that make up the unit or whole, b, represented by the denominator (Barnette-Clarke *et al.*, 2010). In many localities globally, the part-whole interpretation of fractions is the first meaning of fractions that children encounter in formal schooling (English and Halford, 1995; Sowder, 1995; Lamon, 2012; Torbeyns *et al.*, 2015). In this interpretation of fractions, the unit or whole can be either a continuous (measurable) quantity or a group of discrete (countable) things (Beckmann, 2011). When considering continuous quantities, a typically represents the number of shaded parts and b represents the total number of equal parts that an item is equally partitioned into.

Further to this, various empirical research studies have concluded that the part-whole is a good starting point for teaching children about fractions (Post *et al.*, 1982; English and Halford, 1995; Sowder, 1995; Charalambous and Pitta-Pantazi, 2007; Cramer and Whitney, 2010; Pantziara and Philippou, 2012). In this regard, Kieren (1988) states that the part-whole interpretation of fractions 'conveniently helps produce fractional language' (p. 177), while English and Halford (1995) suggest that it is the least complex fraction interpretation for children to learn. In addition, empirical research has shown that it is quite easy for young children to grasp (Charalambous and Pitta-Pantazi, 2007; Pantziara and Philippou, 2012).

Considering that, worldwide, the part-whole is still most often the first fraction sub-construct that children learn in the classroom, it is notable that very little research has been undertaken that focuses on how other meanings of fractions develop from initial part-whole knowledge (Amato, 2005). This is another gap that has been identified in the current empirical research on fraction learning.

1.3.1.2 The partitive quotient sub-construct

While much focus has been given in the academic literature to the part-whole sub-construct, Hackenberg (2010) notes that the partitive quotient meaning of fractions has received far less scholastic attention. The partitive quotient fraction sub-construct represents the single number that results from performing the operation of a ÷ b or the amount that each person receives when a continuous item, such as a pizza, is shared equally among a number of people. Section 2.2.2.2 presents a more detailed explanation of this fraction sub-construct.

The study of the partitive quotient sub-construct is important for several reasons. Kieren (1993) points out that a focus on learning the partitive quotient can facilitate a broader understanding of numerical order, the development of the concept of equivalence and comparison of quantities. In addition, McGee *et al.* (2006) argue that partitive quotient problems give children the opportunity to build and develop concepts such as algebraic fractions that are crucial to later high-school mathematics. van Galen *et al.* (2008) also highlight that as children engage with partitive quotients problems they 'develop a network of relationships around fractions that enables them to do arithmetic with fractions, but without requiring them to learn explicit arithmetic rules' (p. 73).

In addition to the important role that partitive quotients play in fraction learning, as well as the learning of other mathematical concepts, at least one empirically-grounded and well-established mathematics curriculum in the United States, '*Mathematics in Context*', suggests that partitive quotients build on students' knowledge of 'part-whole relationships and introduce operations with fractions' (Webb *et al.*, 2006, p. viii). This further concretises the present researcher's conclusion that the partitive quotient is an appropriate fraction sub-construct that children who have only been taught the part-whole meaning of fractions can learn next in their fraction learning and this forms part of the foci of the present research.

1.3.1.3 Mixed results on how the partitive quotient knowledge emerges alongside part-whole knowledge

To date, as far as the researcher is aware, only a small number of studies have explicitly focused on how another fraction sub-construct knowing emerges from part-whole learning and even fewer have investigated the impact of the part-whole meaning of fractions on the learning of the partitive quotient. An examination of these studies, which examined how the partitive quotient learning develops alongside existing part-whole knowledge, by the present researcher, reveals that there have been mixed findings. On one hand, researchers, for example, Kerslake (1986) and Charles and Nason (2000) found that the part-whole inhibits children's learning of other fraction sub-constructs, including the partitive quotient. Findings from Nunes (2008) and Naik and Subramaniam (2008), on the other hand, suggested that the part-whole and partitive quotient understandings emerge seamlessly alongside each other. Section 2.4.1 discusses the details of these studies. The contrasting findings suggest that there is need for further research in this area (Maxwell, 2013).

The present research aims to fill this existing gap identified in the research literature by focusing on investigating how the partitive quotient knowledge develops in children who have only been taught the part-whole meaning of fractions. The current study differs from existing empirical research in several ways. It differs from the work of Kerslake (1986) in that the research participants deemed suitable for the present study are primary school children, while Kerslake's were high-school children aged 12-14. The use of students of this age for this type of research is consistent with previous, more recent empirical work (Streefland, 1993; Nunes, 2008). The choice of age/grade level of the participants for the present research is discussed in section 4.5.1.

Another difference between the present research and that of Kerslake (1986) is that the present study focuses, in particular, on the development of the partitive quotient meaning of fractions, while Kerslake (1986) investigated many aspects of fraction knowledge, including the partitive quotient. This narrowing of focus in the current research is potentially advantageous because the results of this study aim to inform, specifically, if and how previous part-whole learning affects the development of the partitive quotient. The present researcher posits that the findings should therefore be relevant and useful to curriculum development and to the teaching of the different fraction interpretations.

The present research also differs from the research of Naik and Subramaniam (2008) and Nunes (2008) as it relates to methodology, the central focus of Chapter 4 of this thesis. Whereas both Nunes (2008) and Naik and Subramaniam (2008) used a teaching experiment approach, the present research engages children in one-to-one task-based interviews and no explicit teaching of the partitive quotient is undertaken as part of the data collection. The specific justification and discussion of this approach is presented in section 4.4.1. Nunes (2008) informs that 'working from the children's responses, the researcher guided them to the realisation this symbol (*referring to the fraction*) can be interpreted as 'one chocolate bar divided by two children' and that the line indicates a division. The children were then asked: 'If there is one chocolate bar to be shared among four children, what fraction will they receive?' (p. 34). It is not clear from this statement the extent of guidance given and if the children's responses following the instruction did not just merely copy the 'guidance' provided. The statement also puts into question the validity of Nunes' (2008) conclusion that the two fraction meanings coexisted and appeared in the children's arguments as they explained their answers. A similar comment applies to the work of Naik and Subramaniam (2008).

Thus far, this section discussed two gaps in the empirical literature and specific ways the present research aims to fill these gaps. The final sub-section 1.3.1.4, presented next, further refines a key focus of this research and thereby delineates a second empirical contribution this research hopes to make to existing mathematics education literature on fraction learning.

1.3.1.4 The focus on the fraction resulting from solving partitive quotient problems

Solving of partitive quotient problems, like many mathematical problems, involves a series of steps. When children engage in solving partitive quotient problems, they typically engage in various actions such as partitioning, distribution of the parts to the sharers and the description of the fraction amount that each person receives (Charles and Nason, 2000). Empson (1999) informs that children 'treat their partitioning actions and their descriptions of the results of partitioning as separate entities' (p. 3). Prior research has placed strong emphasis on children's strategies for partitioning (Pothier and Sawada, 1984, 1990; Lamon, 1996; Nunes *et al.*, 1996). Considering that, for children, various aspects of solving partitive quotient problems are distinct, the present researcher argues that focusing on finding the fraction amount aspect of the problem-solving process, which has not been the sole focus of prior research, is in order.

Further to this, two of the studies that investigated children's solving partitive quotient problems (Streefland, 1991; Charles and Nason, 2000) reported different explanations for one typical fraction answer that children provided. For example, in solving the problem of sharing three items among four people, prior research has reported that children often obtain the answer of 3/12. As stated earlier, Charles and Nason (2000) and Streefland (1991) have provided different explanations for this. These differences warrant a closer examination of how children find the fraction associated with solving partitive quotient problems. Sections 2.3.1 and 2.3.2 in Chapter 2 discuss in detail, these two empirical studies. By focusing explicitly on how children find the fraction related to solving partitive quotient problems, the present research aims to fill another gap identified by the present researcher in the empirical literature associated with the partitive quotient sub-construct. In so doing, the present research aims to add a second empirical contribution to the existing literature on fraction learning.

1.3.2 Summary

Building on the general research problem identified in section 1.2, there are several fundamental gaps in the empirical literature of the fraction sub-construct domain in mathematics education. The gaps identified show how broad and complex the domain of fractions is, but also its interconnectedness, which allows several gaps to be filled by this empirical study. More specifically, in view of:

(i) the importance of examining how one fraction sub-construct develops from existing fraction knowing;

(ii) the fact that globally, the part-whole fraction sub-construct is still predominantly the first sub-construct that children learn in school;

(iii) the scarcity of studies which focus on the development of knowledge starting from the part-whole notion;

(iv) the importance of developing children's partitive quotient knowledge and the mixed findings related to the impact of the part-whole on the development of the partitive quotient;

(v) the mixed findings related to how children find the fraction from solving partitive quotient problems and that very few studies have focused explicitly on this aspect of the problem-solving process;

the present research aims to study how children who have only been taught the part-whole fraction sub-construct find the fraction related to solving partitive quotient problems. Through this focus, this research hopes to make two significant contributions to the empirical literature on fraction knowledge, as previously explained in section 1.3.

The next section presents the second focus of this thesis and discusses the theoretical contribution that this study aims to make.

1.3.3 Theoretical contribution

The Pirie-Kieren theory (Pirie and Kieren, 1989b; Pirie and Kieren, 1994b; Towers *et al.*, 2000; Martin *et al.*, 2012) is currently deemed in the educational literature to be a well-established and widely adopted theoretical perspective on the nature of mathematical understanding (Martin, 2008). The proposed study seeks to add to the theoretical literature on mathematical understanding by elaborating on an essential feature of the Pirie-Kieren theory for growth of mathematical understanding. In this theory, growth of mathematical understanding is characterised as a whole, dynamic, levelled but non-linear, transcendentally recursive process (Pirie and Kieren, 1989a). The associated model shown in Figure 1-1 depicts eight nested circles and represents eight potential layers-of-action for describing the growth of understanding for a specific person, on any specific topic or concept (Martin, 2008) at any educational level.

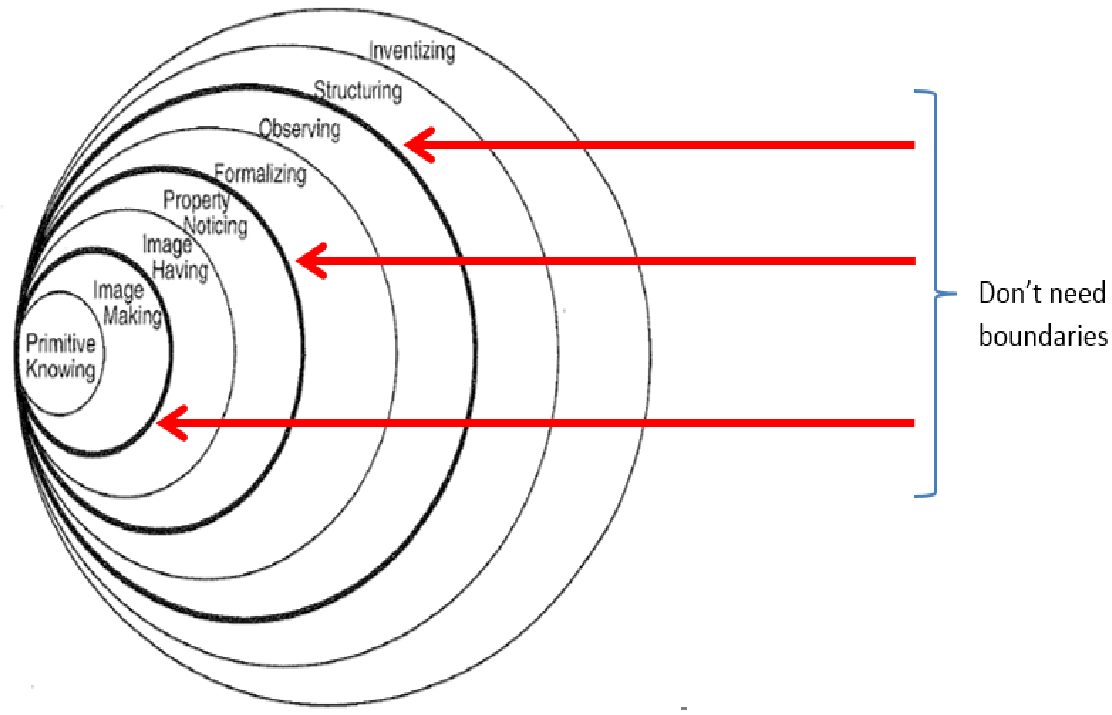


Figure 1-1 The Pirie-Kieren model showing the Don't Need boundaries (Pirie and Kieren, 1989a, p. 8)

Each layer within the model contains all previous layers nested within it. Growth is described as a dynamic organising process. Extending knowledge involves abstracting to a new outer layer, as well as returning to inner layers of understanding to recursively reconstruct knowledge.

The proposed research project endeavours to elaborate on the Don't Need boundaries (DNBs) shown in Figure 1-1, which are a key feature of the theory. The DNBs in the Pirie-Kieren model capture the idea that outside the boundary, one does not necessitate the specific inner understanding that gave rise to the outer knowing. Beyond these boundaries, the learner is able to work with conceptions that are no longer closely tied to previous forms of understanding.

Instead, these previous forms are embedded in the new outer layer of understanding and are readily accessible if needed (Pirie and Kieren, 1994b). Chapter 3 discusses, in detail, the Pirie-Kieren theory and the DNBs.

Recent research has questioned the validity of the DNBs. Wright (2014) adapted the Pirie-Kieren model to create a teaching model used in several numeracy development projects in New Zealand and states that the co-existence of recursion and DNBs appears contradictory. He emphasises that if a learner has crossed the first DNB, this suggests that they possess readily available images with which they can meet task demands without the need for physical action. It seems counter-positional to state that the learner may need to return to an inner layer from an outer layer in times of uncertainty. Further to this, apart from a description of the DNBs and the presentation of a few examples in empirical research, this particular feature of the theory has not been expanded in empirical research to date (L. C. Martin, 2015, personal communication, 13 May). The recent question concerning the DNB feature of the Pirie-Kieren model, as well as the dearth of research to elaborate this aspect of the theory, makes the contribution of this research project particularly timely and significant.

The present research aims to fill the gap identified by studying the growth of mathematical understanding of Year 5 children as they engage in solving partitive quotient fraction problems. In this regard, partitive quotients serve as a vehicle to study the DNBs of the Pirie-Kieren theory. An examination of the early work of the theory developers, Susan Pirie and Thomas Kieren, by the present researcher confirms that a large quantity of the validating evidence for the theory originated with children working on various types of fraction problems. These problems ranged from exploring the family of halves and thirds to equivalent fractions, basic operations and partitive quotient problems (Pirie, 1988; Pirie and Kieren, 1989a, 1992a; Kieren, 1993). Because of this, it is expected that this topic provides a viable medium through which to elaborate on this key feature of the Pirie-Kieren theory.

The two previous sections presented the empirical and theoretical contribution that this thesis aims to make. In the next section, the present researcher briefly outlines the personal motivation for conducting this research.

1.4 A personal motivation

While the theoretical and empirical contributions that this research aims to add to the mathematics education literature are important, by virtue of the arduous nature of doctoral research in general, this more than propels an individual to embark on this journey or sustain them once there. In this regard, the original motivation for the present researcher's interest in children's understanding was a practical one. Before embarking on the PhD journey, for fifteen years the researcher worked as a teacher of mathematics at the high-school level in the Commonwealth of Dominica (CoD), an island located in the Caribbean. The students who attended the school where she taught were generally low attaining and struggled enormously in learning mathematical concepts. The work with the present researcher's students ignited her desire to understand the development of mathematical understanding much more than she did at that time. While the present researcher had personal interests in investigating mathematical understanding, in deciding to undertake a doctoral degree she thought beyond herself to consider the mathematical needs of her small island. In this regard, she sourced and examined documents about student attainment from the Ministry of Education. She also spoke to mathematics education colleagues who worked there to ascertain some of the most pressing educational needs at that time. From the aforementioned engagement, the domain of number, especially at the primary school level of education, emerged as an area that could benefit from empirical research. This made definite not only the general topic area of this present thesis, but also the educational level that would be the focus.

1.5 Research questions

The specific questions that the research seeks to address are:

1. What strategies do Year 5 children who have only been taught the part-whole fraction sub-construct use:

    (i) to find the fraction associated with solving partitive quotient problems?

    (ii) to find the fraction associated with solving partitive quotient problems in a sequence of problem-solving sessions?

2. In what way(s) does evidence support or not support the Don't Need boundary (DNB) feature of the Pirie-Kieren theory for the growth of mathematical understanding?

1.6 Structure of the thesis

This thesis consists of nine chapters. This first chapter presented the research problem, the specific contributions that this research aims to make to the existing literature, and the associated research questions. Chapter 2 reviews in detail children's learning of the partitive quotient sub-construct, a key concept that this thesis focuses on, while Chapter 3 presents the theoretical framework adopted for this thesis. The details of the methodological approaches and the analytic framework adopted to guide this study are detailed in Chapters 4 and 5, respectively. Chapters 6 and 7 present the findings for Research Question 1 and 2, respectively. The research findings are discussed in Chapter 8. Finally, Chapter 9 presents the contributions to the academic literature, educational implications and recommendations for future research arising from the research documented in this thesis.

Chapter 2: Children's learning of the partitive quotient sub-construct

2.1 Overview

Chapter 2 builds on the previous chapter by providing an overview of fractions and a critical review of empirical research relevant to the present study. In so doing, it situates the current thesis within existing empirical research. More specifically, the overview of fractions discusses several descriptions of fractions at the primary school level, terminology-use related to fractions, and the five fraction sub-constructs introduced in Chapter 1. The critical analysis of literature includes a review of four key empirical research studies related to how children quantify a person's share in solving partitive quotient problems. It also examines previous literature that has focused on the development of the partitive quotient sub-construct from existing fraction knowledge and how the partitive quotient links to other fraction sub-constructs.

2.2 Overview of fractions

Streefland (1991) informs that the use of fractions traces back to 1700 BC and involved real-world sharing situations of food in trade and agriculture. Since these early days, the notion of fractions has expanded and developed to become an integral aspect of not just the everyday world, but also of the mathematics that children encounter in primary school. Carraher (1996) describes a fraction as 'a number of the form $\frac{a}{b}$ (where *a, b* are integers and *b* ≠ 0), with a set of well-defined and well-known operations and properties such as commutativity of addition and multiplication' (p. 241). Carraher (1996) argues that viewing a fraction solely as a number is narrow and misleading from the standpoint of pedagogy, history and psychology. He suggests that there is a need for a broader view regarding fractions in education. This is because learning fractions involves becoming aware of special relations between numbers and quantities, such as the numerator and denominator. It also entails learning to express these relations in various ways. Barnette-Clarke *et al.* (2010) offer a broader description of fractions as a non-negative '*symbolic expression* of the form $\frac{a}{b}$ representing the quotient of two quantities' (p. 15), provided that *b* ≠ 0. In the symbolic expression, $\frac{a}{b}$, *a* represents the dividend or numerator, while *b* represents the denominator or divisor. They justify the restriction of fractions to non-negative rational numbers by arguing that primary school children first study fractions well in advance of learning of negative numbers.

Lamon (2012) also provides a description for the term ‘fraction’. First, she explains that it refers to a notational system or a certain form for writing numbers. This form consists of bipartite symbols, $\frac{a}{b}$, where *a* and *b* are two integers, written with a bar between. A second description of fractions is that they are non-negative rational numbers. This second description largely mirrors that of Barnette-Clarke *et al.* (2010), but Lamon (2012) restricts *a* and *b* to integers, while Barnette-Clarke *et al.* (2010) do not. Lamon (2012) explicates the decision to limit *a* and *b* to integers by stating that, at the primary school level, *a* and *b* are whole numbers. In addition to this, Lamon (2012) does not indicate that $\frac{a}{b}$ represents a quotient, as do Barnette-Clarke *et al.* (2010). Although Kieren (1988) and others have suggested that fractions are produced by division, the inclusion of the phrase ‘represents a quotient’ by Barnette-Clarke *et al.* (2010), although mathematically correct, may be potentially confusing. This is because a quotient represents one of the five distinct fraction sub-constructs introduced in Chapter 1 and discussed in more detail below.

2.2.1 The use of terminology associated with fractions

Terminology and its use have marked significance in mathematics education (Barnette-Clarke *et al.*, 2010). Relating to the use of the terminology associated with fractions, Payne (1976) notes that in mathematics education literature, there is great variety in the terms used that refer to fractions. He adds that terms such as fraction, fractional number and fraction symbol have sometimes been used interchangeably, based largely on the personal preference of the writers. As it pertains to primary school mathematics, in particular, another term that has been used interchangeably with ‘fraction’ is ‘rational number’ (Olive, 1999). Kieren (1995), a prominent researcher in the domain of mathematics education and fraction research who often uses the terms rational and fractional numbers interchangeably, provides his perspective on terminology use. He states:

> I am taking the liberty of using the terms fractional numbers, rational numbers and rational numbers of arithmetic, loosely and interchangeably; I am thinking about children perhaps aged 7 to 12, as they come to learn to deal with the non-negative rational numbers and their operations through using standard and nonstandard fractional language. (p. 35)

Several researchers, such as Lamon (2012) and Thompson and Saldanha (2003), have asserted that rational numbers and fractions, although closely linked, are not the same conceptually. Both

Lamon (2012) and Thompson and Saldanha (2003) therefore advocate for greater rigidity in how the terms relating to fractions are used. In this regard, Lamon (2012), while noting that there are some who may disagree with her stance, contends that using words carelessly can cause 'additional difficulties in communicating about an already-complicated topic' (p. 20).

Taking into consideration the aforementioned perspectives on fraction terminology use, the present research predominantly uses the term fraction and associated terms throughout this thesis. This is because, similar to Kieren (1993), this research study focuses on children (aged 8-10) in Year 5 of primary school as they develop an understanding of non-negative rational numbers of the form $\frac{a}{b}$. This thesis uses the term rational number when referring to the overarching construct under which fractions are subsumed or when using quotes by writers who use the terms interchangeably.

2.2.2 The multi-faceted nature of fractions

Chapter 1 introduced the notion that the fraction construct is multi-faceted. Over the past two decades, the measure, operator, part-whole, quotient and ratio sub-constructs have been widely accepted and adopted by researchers in the field, as the five sub-constructs comprising the fraction construct (Behr and Post, 1992; Olive, 1999; Tsai and Li, 2016). The next five sub-sections briefly describe each of these sub-constructs.

2.2.2.1 The part-whole sub-construct

The part-whole interpretation of a fraction $\frac{a}{b}$ indicates a comparison of the number of parts, denoted by the numerator, *a*, to a fixed number of equal-sized parts that make up the unit or whole, *b*, represented by the denominator (Barnett-Clarke *et al.*, 2010). In this interpretation of fractions, the unit or whole can be either a continuous (measurable) quantity, such as length of a pen, area of a floor, volume of water, pizza(s), pie(s) or a group of discrete (countable) items such as a box of pencils, a set of pages in a book or a carton of eggs (Beckmann, 2011). Figure 2-1 depicts various part-whole representations of the fraction ¾.

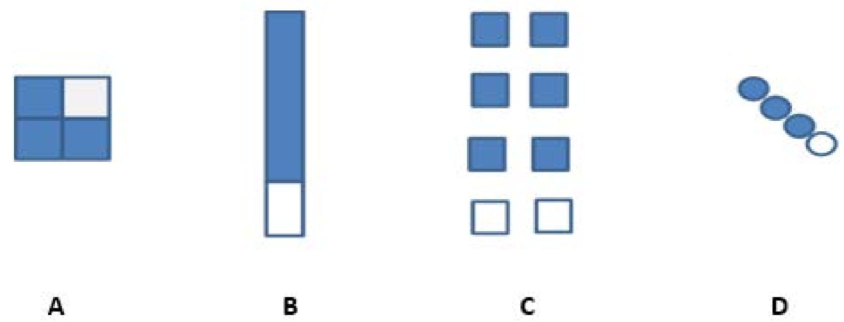


Figure 2-1 Continuous (A and B) and discrete (C and D) representations of ¾

English and Halford (1995), using diagrams and a series of steps, describe how a learner conceptualises the part-whole sub-construct. Figure 2-2 illustrates this, using the fraction 3/8.

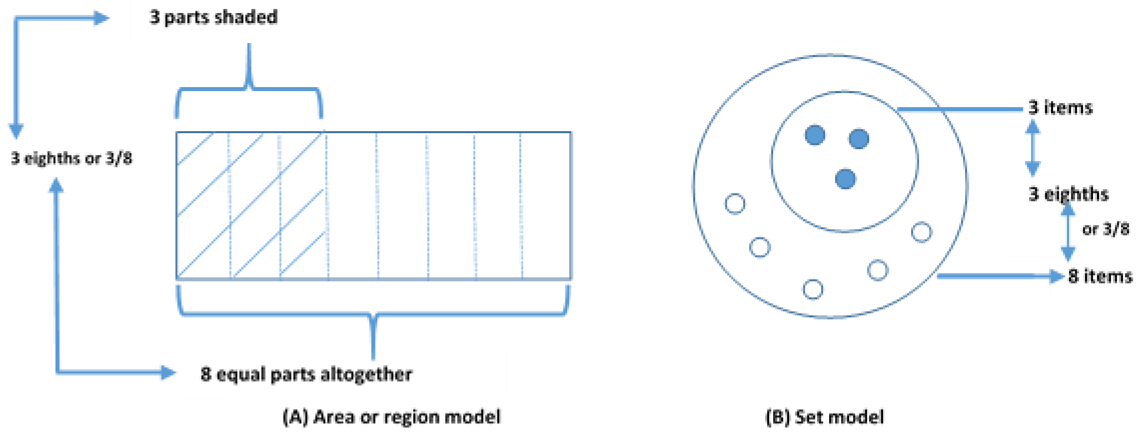


Figure 2-2 Steps involved in interpreting the part-whole sub-construct (English and Halford, 1995, p. 129)

Considering the area model (A) in Figure 2-2, in order for a learner to interpret the fraction represented by the shaded portion or part-whole, English and Halford (1995) suggest that they must:

1. Recognise that the parts in the whole are equal
2. Identify the total number of parts (eight)
3. Identify the number of shaded parts (three)
4. Coordinate the number of shaded parts and the total number of parts to obtain the fraction, $\frac{3}{8}$.

Nunes *et al.* (2004) add that, in part-whole situations, the numerator and denominator in the fractions denote quantities of the same type. Therefore, 3/8 in a part-whole situation means that a whole, such as a chocolate bar or a cake, is divided into eight equal parts, and three are taken. Both three and eight refer to the chocolate bar or the cake.

2.2.2.2 The quotient sub-construct

The quotient interpretation of fractions is considered to be the phenomenological source of fractions (Kieren, 1993; Streefland, 1993) since it is generally how children first engage with fractions, usually in informal sharing situations, before formal instruction. Further to this, Behr and Post (1993) describe the quotient sub-construct in procedural terms, taking into account the different meanings that the numerator and denominator could have. They state that to define a fraction as a quotient, one: (a) starts with two quantities; (b) treats one of them as a divisor and the other as the dividend; and (c) by the process of division obtains a single quantity result.

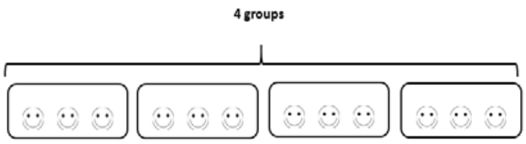
In this regard, a quotient is interpreted in two related ways (Mamede *et al.*, 2005). First, it can indicate a division operation where a fraction, $\frac{a}{b}$, can be seen to denote the single number that results from performing the operation of $a \div b$ (Lamon, 2012). An example of this is 3 ÷ 4 = ¾. Alternatively, $\frac{a}{b}$ represents a sharing situation. Within the context of sharing, there are two interpretations for division: partitive division and quotitive division. The literature associated with mathematics education in general, and empirical research, in particular, often overlook this distinction. The partitive quotient is therefore often equated with the quotient meaning of fractions and is the focus of most of the empirical research on the quotient fraction sub-construct.

Partitive division stems from sharing situations or distributing equally into a specified number of parts (Hiebert and Behr, 1988). More specifically, for continuous items, it represents the amount that each person receives when one shares items equally, for example cakes, pizzas or chocolates. Within the context of the partitive quotient sub-construct, the fraction ¾ may represent the amount of pizza that each person gets when four people share three pizzas fairly. In these situations, the denominator designates the number of recipients and the numerator designates the number of items shared. The values in the fraction, therefore, signify two variables of a different nature (Nunes *et al.*, 2004).

Nunes *et al.* (1996) inform that this action of dividing continuous items allows the question of 'how much' to be answered. They also add that two important relations emerge when a number results from this type of division. First, there is a reference to a unit, which is taken as the whole and second, for the fraction $\frac{a}{b}$, the size of b in the b-split of the continuous items.

Quotitive division, in contrast to partitive division, represents the number of groups of the same size that result from a given number of items. Table 2-1 presents a comparison between quotitive and partitive division from the context of sharing situations.

Table 2-1 Comparison between quotitive and partitive division

Quotitive Division	**Partitive Division**
Problem exemplar: In a class, students have been placed into groups of three. A parent dropped off 12 cupcakes to be shared. How many groups of three children would be able to get a cupcake? 	*Problem exemplar:* __*Discrete*__ Twelve cupcakes are to be shared equally among three children. How many cupcakes will each child receive?  __*Continuous*__ How much cake does each person get if three cakes are shared fairly among four people, and no cake is left over?
The question of 'how many groups can be formed' is answered.	The question of 'how much does each person get' is answered.
Discrete items are used in the sharing situations.	The items shared can be discrete or continuous.

Hiebert and Behr (1988) provide another example that illustrates the difference between partitive and quotitive division from outside the domain of sharing. They state that, on one hand, a partitive model for 12 ÷ 3 may be '12 miles divided by 3 hours'. The partitive quotient resulting from this is 4 miles per hour. A quotitive interpretation, on the other hand, would be '12 miles divided by 3 miles per hour', which equals 4 hours. An inspection of the unit for the partitive quotient shows that it consists of a composite quantity: miles per hour. This is an example of an intensive quantity. Another example of an intensive quantity would be pizza per person. The quotitive quotient consists of a single unit, which is hours. This is an example of an extensive quantity. In a similar way, the 12 miles and the 3 hours, in the partitive quotient problem, represent extensive quantities.

2.2.2.3 The ratio sub-construct

A ratio is a comparison of any two quantities (Lamon, 2005) and represents a notion which is unable to be written as a single number. In this regard, Behr *et al.* (1983) state, 'it is more correctly considered as a comparative index rather than as a number' (p. 94). For illustration, for this sub-construct, three pizzas for every four people represents $\frac{3}{4}$. This is distinct from the quotient interpretation, which interprets $\frac{3}{4}$ pizza per person as the amount of pizza one person receives when four people share three pizzas. A ratio could also represent a part-part relationship, which compares the same units. In this instance, $\frac{3}{4}$ may be interpreted as three green counters for every four red counters, when part of the counters is green and part is red.

2.2.2.4 The measure sub-construct

Lamon (2005) explicates that the measure sub-construct refers to some measure assigned to some interval or region, depending on whether one is using a one-or two-dimensional model. In a one-dimensional space, such as a number line, thermometer or ruler, this sub-construct measures the distance of a certain point from zero, and the unit is always an interval of length. The unit of measure can be divided into smaller sub-units, depending on the accuracy required for the problem. In a two-dimensional space, this sub-construct measures area. As a measure for a one-dimensional space, $\frac{3}{4}$ may be obtained and interpreted as follows:

1. An interval of length one is partitioned into four equal sub-intervals and each of the sub-intervals represents ¼.
2. The fraction $\frac{3}{4}$ is three intervals of length ¼.

On the number line below in Figure 2-3, every vertical line mark represents an interval of ¼. The blue bracket represents the interval 3/4, starting from zero, with the black arrow signifying the endpoint.

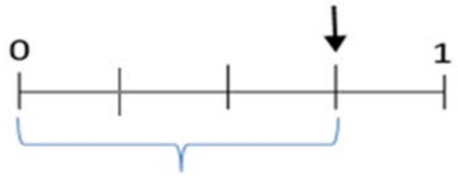


Figure 2-3 Number line depicting fourths and ¾

Lamon (2005) points out that although the measure sub-construct is distinct from the part-whole, depending on the task given, part-whole conceptualisation can be used. For example for a task: 'Locate the number 3/5 on a line', although nominally a measure problem, is essentially like a problem that asks a child to shade 3/5 of a pizza. In effect, the child makes five equal sub-intervals and marks the end of the third interval as 3/5. This observation by Lamon (2005) was later supported by Mitchell and Horne (2009), who assert that the nominal nature of a task may be different to the knowledge that a child uses to engage with the task.

2.2.2.5 The operator sub-construct

The operator fraction interpretation is about reducing and enlarging in size or amount, contracting and expanding, lengthening and shortening or multiplying and dividing. Lamon (2005) informs that the operator is a set of instructions for carrying out a process. For example, '$\frac{3}{4}$ *of*' is an operator that directs you to multiply by three and divide the result by four. To apply the process '$\frac{3}{4}$ *of*', one performs the operations of multiplication and division successively. For example:

$\frac{3}{4}$ *of* 16 = (3 x 16) ÷ 4 = 3 x (16 ÷ 4) = 12. Alternatively, it could be a single operation $\frac{3}{4}$(16).

Sometimes the use of the word '*of*' in the context of the quotient sub-construct may cause confusion with the operator sub-construct. For example, in a sharing situation, $\frac{3}{5}$ of a pizza often denotes a person's share. Since there is no instruction to transform the pizza in terms of size or amount, the application of the operator sub-construct is not appropriate in that situation. This underscores the importance of terminology use in mathematics that section 2.2.1 highlighted.

2.2.2.6 Summary

Section 2.2 briefly provided an overview of fractions by presenting several definitions of the term, delineating terminology use related to fractions and then briefly describing each of the five fraction sub-constructs. The next section reviews four empirical research studies, which reported findings related to how children found the fraction as they engaged in solving partitive quotient problems.

2.3 Empirical research on children's solving of partitive quotient problems

Partitive quotient problems, also known as equal-sharing or fair-sharing problems, afford rich environments at a concrete level for children to develop several aspects of fraction knowledge (Streefland, 1991; Empson, 1999; Empson *et al.*, 2006). Consequently, this sub-construct has been the focus of a large number of empirical research. As mentioned in Chapter 1, some of these studies (for example Pothier and Sawada, 1983; Pothier and Sawada, 1990; Charles and Nason, 2000) have investigated children's strategies for partitioning as they engaged in sharing different numbers of items among different numbers of people. Other research (for example Behr *et al.*, 1984; Kerslake, 1986; Behr and Post, 1993; Streefland, 1993; Mamede *et al.*, 2005; Nunes, 2008) involving partitive quotients has centred on the notions of fraction equivalence and ordering as children engaged in comparing the fraction amounts resulting from varied sharing situations.

While there are many research ventures associated with the partitive quotient, only a few of these empirical studies detail children's entire process of solving these problems, including the approaches used for finding the fraction amount that each sharer receives. This aspect of the solving of partitive quotient problems is the focus of the first research question of the present study. Four previously conducted empirical studies – Streefland (1991), Charles and Nason (2000), Yazgan (2010), and Empson *et al.* (2006) – included this specific detail. The remainder of this section discusses these four studies in detail.

2.3.1 Streefland's (1991) research work

Streefland (1991) conducted seminal empirical research upon piloting a primary school curriculum on fractions in the Netherlands. He states, 'the purpose of this research was the production and tracing of long-term processes' and involved 'experiments, interviews and comparative research' (p. 37) as part of its research design. Sixteen fourth grade students (ages 9–10) of a wide range of attainment made up the sample for this research. Exploration of fractions began with children exploring different partitive quotient problems in the context of sharing situations with food items, such as pancakes and pizzas, among varying numbers of people. In this study, the children's experience with fraction learning began with partitive quotients. This is in contrast to most localities, where children's learning of fractions begins with the part-whole sub-construct.

Several of the partitive quotient problems reported by Streefland (1991) involved a number of items less than the number of sharers. One such problem is that of sharing six pancakes among eight people. The children in Streefland's (1991) study used three different ways of quantifying each person's share for this problem. The resulting fraction(s) included: ¼ + ¼ + ¼; ¾ and ½ + ¼. Based on the fraction(s) that the children worked out, it appears that the children used the strategy of halving the number of items and people, and worked with two sets of the problem of sharing three pancakes among four children. Streefland (1991) corroborates this observation by the present researcher and states, 'Frans… wrote that this was the same as three pancakes for four people, only now it is doubled' (p. 72). In addition to this finding, Streefland (1991) reports that no child divided the pancakes into eighths. The present researcher therefore surmises that, for these students, there was no link, perhaps at least at that point, between number of sharers and number of partitions created, as was reported for other studies such as Empson *et al.* (2006). One possible explanation for this way of operating for Streefland's (1991) sample is that children had worked extensively with halves and quarters previously and with the problem of sharing three items among four people, so the use of this knowledge as a reference point is understandable. Streefland (1991) confirms this and states that, largely, children called upon their previous experiences to 'divide three pancakes among four children'.

On another occasion, Streefland's (1991) research participants engaged in solving the same problem of sharing six pancakes among eight people. In this scenario, waiters brought the pancakes to the table one by one, instead of all at once. The following solutions constituted some of the children's responses. Streefland (1991) classified the responses as follows:

- Repeated addition: 1/8 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8
- Repeated addition and compiling: 1/8 + 1/8 = 2/8; 2/8 + 1/8 = 3/8 etc.
- Adding and compiling while applying equivalences: 1/8 + 1/8 + 1/8 + 1/8 = 2/4.

The first approach is referred to by Streefland (1991) as 'French division', while the other two approaches are variations of this approach. Streefland (1991), in explaining children's responses, states that much work involving generative material and constructing and decomposing fractions has gone into the production of the above responses. This observation appears to support the view held by Lamon (2001), based on her own empirical research and that of other researchers such as Behr *et al.* (1984), that children need an extended time to engage with each fraction sub-construct for a deep understanding to develop. Streefland (1991) further reports that no one spontaneously converted the additions to multiplications such as: 1/8 + 1/8 + 1/8 = 3 x 1/8 = 3/8, nor would it have been an obvious step in the story of the 'French division'.

Another type of equal-sharing problem that Streefland (1991) used in his research involved a number of items that was greater than the number of people involved in the sharing situation. In this regard, the research sample engaged with the problem of sharing five eggs among four people. The children produced two correct responses and one that was incorrect, as follows:

- 1 + ¼ = 1 ¼ where each person got a whole and then ¼ from the last egg
- 1 + ¼ = 1 ¼ where each person got ¼ from each egg
- 1 + ¼ = 2/4 since 1 + 1 = 2 and 4 remains as is.

For the incorrect solution '1 + ¼ = 2/4', some of the children made the error of operating with the fraction symbols in the same way as natural numbers and performed the operation of addition on the numerators and denominators individually. More specifically, they added the two numerators of one to obtain the sum of two. The whole number one does not have an observable denominator and so the denominator of four remained unchanged, since 4 + 0 = 4. Streefland (1991) refers to this as IN-distractors and reports that most of the children in his sample operated in this way at several points during their engagement with the partitive quotient problems. Streefland (1991) also reports that this phenomenon was particularly stubborn, even when the teacher/researcher made specific efforts, from the commencement of the teaching experiment, to counteract it. This finding is in keeping with later research by Steffe (1994), who states that, in solving problems, children use their own methods based on their current knowledge. Steffe (1994) adds that children typically continue to use these approaches even though their teachers introduced explanations of accepted methods.

Streefland (1991) found five levels of resistance to IN-distractors based on the long-term individual learning processes of the students in his sample. These include:

Level 1: Absence of cognitive conflict: The student does not come into a cognitive conflict in situations where a child obtains different results to the same problem. This is because the child regards the solutions found on different levels as method-dependent. For example, a student finds that ½ + ½ = 2/4 and afterwards, by means of distribution states that ½ + ½ = 1. A child accepts that both answers are correct, because different methods were involved.
Level 2: Cognitive conflict takes place: The student encounters a cognitive conflict when faced with the dilemma in level 1. The learner rejects the IN-distractor solution as incorrect. S/he considers the solutions method-independent.
Level 3: Spontaneous refutation of IN-distractor errors: The student is inclined to make an IN-distractor error, but comes into conflict with him/herself and spontaneously refutes this error.
Level 4: Free of IN-distractors: The student's (written) work is free of IN-distractors.

Level 5: Resistant to IN-distractors: The student is under all circumstances resistant to IN-distractors.

Ni and Zhou (2005), in their research, also reported and expanded this way of operating by children, but labelled it differently to that of Streefland (1991). They use the term 'whole number bias' to denote 'a robust tendency to use the single-unit counting scheme to interpret instructional data on fractions' (p. 28). Two common examples of this include Streefland's (1991) illustration above, and confusing the number of pieces in a partitioned diagram representing the items being shared, with the size of each piece. An example of this is when a child states that 1/5 is bigger than ¼ since the denominator of five is greater than that of four. Another explanation for this response is that an item partitioned into five has more pieces than one partitioned into four. In Ni and Zhou's (2005) definition of whole number bias, they follow Caverni *et al.* (1990), who define the term 'bias' as a systematic and frequent deviation from a norm. They explicate that children's whole number bias reflects incomplete understanding of the concept or the principle under consideration. As it relates to fractions in particular, Ni and Zhou (2005) add that, while children are learning about the notions of fractions, they are 'often overtaken by the aspects of fraction situations that appear to be consistent with what they have known about whole numbers' (p. 29). Ni and Zhou's (2005) characterisation of the issue appears to be in line with that of Streefland (1991). Streefland (1991) asserts that the problem is grounded in 'children's propensity to associate what they are learning with the natural numbers' and 'the powerful "suction" exerted by this form, together with a concept of fractions that is not yet immune against it' (p. 219).

While the view that children's whole number knowledge interferes with their learning of varied fraction knowledge is widespread in the mathematics education domain, empirical research puts forward at least one rival explanation to this. Leslie Steffe and John Olive labelled this as the reorganisation hypothesis. This perspective suggests that children's whole number knowledge serves as the foundation and springboard for their fraction knowledge and can be reorganised to develop fraction concepts (Olive, 1999; Steffe, 2001; Biddlecomb, 2002; Norton and Hackenberg, 2010; Steffe and Olive, 2010).

Regarding children's engagement with partitive quotient problems in general, Streefland (1991) explains that, 'wrestling and struggling with this stubborn material, the children deal with one conflict after another and while not always to complete satisfaction, managed to reach decisions' (p. 65). He further adds that the initial exploration of partitive quotient problems is, for many children, filled with pitfalls, traps and considerations. For example, he reports that children confronted:

- What is a 'whole' and which number (number of items or number of sharers) do you have to use for it?
- Which relationship has to be described and what exactly are the parts produced to be compared with?
- How can 'a whole and a quarter', 'a quarter and a quarter and a quarter', 'three-quarters', a half and a quarter be abbreviated using symbols to produce:
  $1 + ¼ = 1 ¼$; $¼ + ¼ + ¼ = 3 \times ¼$; $½ + ¼ = ¾$, without giving in to tempting alternatives, such as $1 + ¼ = 2/4$ or $¼ + ¼ + ¼ = 3/12$.

The work of Streefland (1991) is important for the present study, because it provides many exemplars of partitive quotient task situations. In addition, the actual approaches used by the Streefland's (1991) sample, as they found the fractions related to solving the equal-sharing problems, provided key insights into how children actually engaged in quantifying each person's share. This is the specific aspect of the partitive quotient problem-solving pertinent to the first research question of this thesis.

2.3.2 Charles and Nason's (2000) research work

Similar to Streefland (1991), Charles and Nason (2000) also investigated how children solved partitive quotient problems. More specifically, using individual interviews, they examined how 12 Year 3 children (aged 7.9–8.3) from Eastern Australia solved various partitive quotient problems. The problems for this research consisted of a bank of 30 partitive quotient tasks, based on the work of Streefland (1991). The number of items shared ranged from one to six and the number of people from two to six. In each interview, a child received a unique set of problems. After a child solved the first problem, the researchers chose subsequent problems based on how the child engaged with the previous problem(s).

The work of Charles and Nason (2000) extended previous empirical research (for example Pothier and Sawada, 1983; Lamon, 1996) on the partitioning aspect of solving partitive quotient problems. In this regard, they state that the aims of their study were two-fold. First, they hoped to identify new partitioning strategies not previously reported in empirical literature for solving partitive quotient problems. Second, they sought to develop a taxonomy for classifying children's partitioning strategies identified in their study, as per the children's ability to facilitate the abstraction of the partitive quotient.

The concept of abstraction is important to investigate in the mathematics education domain (Mitchelmore and White, 2007). Charles and Nason (2000) define abstraction as 'a lasting change that enables the identification of the same concepts, structures and relationships in many different, but structurally-similar tasks' (pp. 192-193). The present researcher, however, based on an examination of a range of empirical research on the concept of abstraction (Mason, 1989; Dreyfus *et al.*, 2015) and her experience as a mathematics teacher for over fifteen years, surmises that it is very unlikely that abstraction was achieved in one individual task-based interview, even though the children solved multiple problems.

Dreyfus (2014) describes abstraction as a process by which learners try, succeed or fail to reach a conceptualisation of 'the structure of a concept or a strategy or a procedure, where structure refers to the elements and the relationships and connections between them' (p. 5). He adds that the study of the processes of abstraction requires heavy inputs of time and effort. Previous empirical studies on the learning of the partitive quotient (Clarke *et al.*, 2011b; Lamon, 2012) also suggest that multiple engagements with this sub-construct are needed for children to develop complete and sound understanding, which the conceptualisation of abstraction embodies. These considerations add credence to the point raised by the present researcher that one task-based interview was very likely insufficient for the abstraction of the partitive quotient to occur.

Despite the limitation concerning abstraction put forward by the present researcher, Charles and Nason's (2000) study is relevant to the present thesis for two main reasons. First, although their research focused largely on partitioning, as part of their research findings, they generally reported how their research participants found the fraction associated with quantifying each person's share in partitive quotient problems. These approaches provide the current researcher with some insight into various ways in which a different sample of children find the fraction in partitive quotient situations. The main difference between Streefland's (1991) sample and that of Charles and Nason (2000) is that the former group had not been previously exposed to fraction learning, while the latter, although not explicitly stated by Charles and Nason (2000), prior to their sample's engagement with partitive quotients would very likely have been exposed to some part-whole learning. The present researcher surmises this regarding Charles and Nason's (2000) participants, based on the reference made to the part-whole by Charles and Nason (2000) in relation to one of the approaches used by their research participants, which is discussed in the next sub-sections. In addition, researchers such as Clarke (2006), writing post-2000 about fraction learning in Australia's primary schools, state that 'in introducing students to fractions,… the part-whole comparison dominates the way in which fractions are presently taught and therefore learned' (p. 7).

A second reason why the work of Charles and Nason (2000) is significant to the present research is that the approaches for finding the fraction(s) from solving partitive quotient problems reported contribute to the building of the analytic framework for the data analysis pertaining to Research Question 1. This framework is presented in Chapter 5.

As previously stated, Charles and Nason (2000) included details regarding how the children in their study quantified each share. A close examination of their journal article by the present researcher shows that, while they report that the children in their study used 12 partitioning strategies, the children appeared to find the fraction associated with each person's share in five ways. These approaches to quantifying each person's share were not unique to the 12 partitioning strategies that Charles and Nason (2000) identified in their research. Consequently, the present researcher has decided not to present the specific names of the strategies identified by Charles and Nason (2000), mainly because the strategy names focus largely on partitioning, which is not the focus of the present research. In addition, including the names of the strategies identified by Charles and Nason (2000) may potentially serve as a distractor to readers of this thesis.

This section, thus far, has provided an overview of the empirical research conducted by Charles and Nason (2000). The remainder of this section describes the five approaches that the children in their study used to find the fraction related to solving partitive quotient problems. The present researcher labelled these approaches as 'Quantifying each person's share I, II, III,…'. At least one illustrative example from Charles and Nason's (2000) study accompanies the description of each approach.

2.3.2.1 Quantifying each person's share I

One way in which some children in Charles and Nason's (2000) study quantified each person's share is by 'the addition of 1/y pieces' (p. 199), when each item is partitioned into the number of sharers, which Charles and Nason (2000) labelled as '*y'.* They note that this strategy was previously reported by Streefland (1991). He described this way of working as 'French division'. One of the children in Charles and Nason's (2000) study, named Sally, used this approach when sharing two ice-cream barcakes among three people, where $y = 3$. She shared both items into three and gave a third of each barcake to each of the three people: 1/3 + 1/3. She then stated that each person received two-thirds.

Charles and Nason (2000) also report that 11 out of the 12 children in their study used this approach for sharing one item among a number of people. This approach, however, was not as common for sharing more than one item.

Two children from Charles and Nason's (2000) research, Joshua and Sally, used a variation of this first approach to finding the fraction in solving the partitive quotient problems. When this approach was used, children partitioned the items being shared in two different ways and then summed the fraction amounts from the different partitioning. In keeping with Streefland's (1991) 'French division', each fraction amount is a unit fraction. While the denominator of each fraction represents the number of pieces each item is shared into, in contrast to 'French division', the number of parts in each item is not necessarily the number of sharers. For the problem of 'share three pizzas among four people', the two children partitioned two items into two pieces each and the third into four pieces. Figure 2-4 illustrates this sharing.

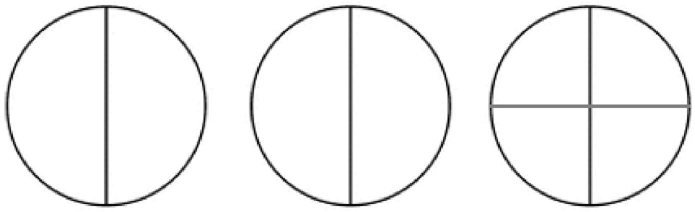


Figure 2-4 Illustration of partitioning for sharing three items among four people

The children then found the fraction that each person received by adding the two fractions, ½ + ¼, to obtain an answer of ¾.

2.3.2.2 Quantifying each person's share II

Another way in which some children in Charles and Nason's (2000) study quantified each person's share is by an immediate verbalisation of the answer as $\frac{\text{number of items}}{\text{number of people}}$ 'without partitioning and sharing of the objects' (p. 200). Caitlin, one of the research participants in Charles and Nason's (2000) study, when asked what each person would receive when four people shared ten liquorice straps, stated 'ten quarters' (p. 200). Charles and Nason (2000) report that only four children used this approach, but two of the four children had to confirm their answers by engaging with the concrete diagrams, which represented the items being shared.

Several empirical studies have confirmed the finding of Charles and Nason (2000) that children rarely make the generalisation of quantifying of each person's share as $\frac{\text{number of items}}{\text{number of people}}$. One example of this is Subramanian and Verma (2009), who also conducted research centred on children's solving of partitive quotient problems. They utilised teaching experiments spanning a period of three months with several gaps in between. The sample consisted of two groups of (10- to 12-year-old) children from India, who had almost no prior introduction to fractions. These groups consisted of 15 and 17 children each. Subramanian and Verma (2009) report that 'our attempts at getting children to say that 5/6 is the share of each child, when five rotis are equally shared among six children, were not always successful' (p. 142). Further to this, children rarely recognised the answer to the given sharing problem as number of items divided by number of people. To answer how much each received, children engaged in drawing different pictures to represent the sharing situations.

Another study which reported similar findings to that of Subramanian and Verma (2009) and Charles and Nason (2000), is that of Clarke *et al.* (2011b). Using a large research team, they conducted individual task-based interviews with 323 Grade 6 (11–12 years) students from Australia at the end of the school year. These task-based interviews focused on, among other things, the partitive quotient fraction sub-construct. They found, similar to Subramanian and Verma (2009), and later corroborated by Arieli-Attali and Cayton-Hodges (2014), that a very small percentage of the children interviewed knew the relationship of number of items divided by number of people, as one person's share, automatically.

Arieli-Attali and Cayton-Hodges (2014) conducted their research in the United States with a sample consisting of 14 students from Grades 3–5 (aged 8–10). In this study, among other things, they specifically investigated whether students recognised that they could quantify each person's share by making the numerator the number of items being shared and the denominator the number of people sharing. They found that none of the children interviewed recognised this.

The aforementioned findings are explained by Peck and Matassa (2016), who state that while the idea of sharing is natural for students, the process of finding the amount that each sharer receives is not. The process of partitive division has proven to be difficult for children as it involves problems in which the dividend and the divisor do not have the same units.

2.3.2.3 Quantifying each person's share III

A third approach used by some of the children to obtain the fraction amount that each person received involved finding the denominator of the fraction by adding the total number of pieces from their partitioning. Children obtained the numerator of the fraction by adding the number of pieces that they distributed to each of the sharers. Claudia, another of Charles and Nason's (2000) research participants, used this approach as she engaged in sharing two items among four people. After partitioning each of the two objects into half and distributing the pieces, she quantified each person's share as 'one part out of four equal parts' (pp. 204-205). The denominator of four appeared to be obtained by counting all the pieces from the partitioning of the items. Additionally, the numerator of one appeared to be obtained by counting how much each person received from each drawing.

This way of working by at least five (Claudia, Candice, Emma, Da and Sophie) of the 12 children in Charles and Nason's (2000) study is in contrast to Streefland (1991), whose sample appeared to act exclusively on the fraction symbols by adding all the numerators and denominators separately. For example, for the problem of sharing three items among four people, one child in Streefland's (1991) sample operated as follows: ¼ + ¼ + ¼ $= \frac{1+1+1}{4+4+4} = \frac{3}{12}$. Streefland (1991) confirms that his sample operated in this way when he informs that the attraction of operating with natural numbers proved too great for three children in the group.

This way of operating among the children in Streefland's (1991) sample resembles what Kieren (1993) referred to as technical-symbolic (T-S) knowing. This knowing is the result of working with symbolic expressions involving fractions and the observations made from this working. The present researcher opines that it was not necessarily surprising that Streefland's (1991) participants did not, at any time, utilise the quantification by part-whole approach, identified by Charles and Nason (2000). This may be because they had only previously been taught whole number knowing. In addition to this, the partitive quotient sub-construct was the first fraction sub-construct that they encountered in formal schooling, so part-whole interference in their finding of the fraction was not likely to be observed.

Interestingly, Charles and Nason (2000) did not characterise the approach used by Claudia and others, described above, as a demonstration of a whole number bias, although they referenced Streefland's (1991) work. They stated that this strategy had not been reported by previous empirical research studies and described it as 'quantification by a part-whole notion' (p. 200). In discussing this approach, Charles and Nason (2000) state that 'part-whole notions were sometimes misconstrued and the children therefore experienced difficulties quantifying each piece and each person's share' (p. 213). Further to this, they explicate that children who used this approach were unable to make the link between number of people and fraction name. In other words, the children did not know the name of each piece (for example, if there are three people involved in the sharing situation, the fraction name would be thirds), thus were unable to quantify each person's share correctly. They provide another explanation for children's way of working. They add that children:

> regularly suffered 'loss of whole'. That is, when attempting to quantify each person's share, they were not thinking in terms of the number of pieces in each one whole unit, but rather in terms of: (i) the number of pieces in all the units… For instance, if two objects were each partitioned into four pieces, the total number of pieces would be eight; therefore the fraction name would be eighths. Similarly, if two objects were each partitioned in half, there would be four pieces altogether, therefore each half would be seen as ¼. (pp. 214-215)

The issue of what a resulting answer to a fraction problem means is a significant one that empirical research often overlooks. Mitchell and Horne (2009) point out this limitation in some existing research and state that in many research projects, especially those that use pen and paper tasks to collect data, tasks are assigned as being part-whole, operator or measure. Students' performance, however, can only be analysed in terms of the strategy they offer, which may or may not be the same as the nominal context of the task. In this regard, the research of Charles and Nason (2000) provide credence to this assertion by Mitchell and Horne (2009) since some of the participants appeared to use a part-whole approach to finding the fraction associated with solving a partitive quotient problem.

2.3.2.4 Quantifying each person's share IV

A fourth approach used to find the fraction amount each person received from the sharing situations involved several steps. First, the child multiplied the number of items by the number of people sharing to find the total number of pieces to be shared. The second step was to divide this total number of pieces by the number of people sharing to find the numerator of the fraction. The number of people involved in the sharing was put as the denominator of the fraction. An example of this way of working involved Sophie sharing five pizzas among four people. Charles and Nason (2000) state that, when asked to predict each person's share, 'she reasoned that there were 20/4 in total, which when shared between the four people would give each person 5/4' (p. 201). They further state that only one child used this approach after successfully completing six partitive quotient problems.

2.3.2.5 Quantifying each person's share V

One child, Ben, in Charles and Nason's (2000) research, used a fifth approach to find the fraction amount that each person received. For this child, each person's share was a unit fraction in which the denominator represented the number of pieces each person received. An illustration of this involved sharing one pizza between two people. After sharing the pizza into eight, Ben 'incorrectly quantified each person's share as ¼ instead of 4/8' (p. 204). Regarding this, Charles and Nason (2000) state that he appeared 'to confuse the number of pieces in each person's share (4) with the number of pieces in the whole (8) when he was generating the fraction denominator' (p. 204).

2.3.2.6 Summary

The findings reported by Charles and Nason (2000) show that the children in this study used varied approaches to quantify a person's share when solving partitive quotient problems. In addition, several children offered multiple approaches to finding the fraction amount that each sharer received and not all children found a correct fraction to the partitive quotient problems that they solved. This coincides with Streefland's (1991) assertion that the initial exploration of partitive quotient problems is, for many children, filled with pitfalls, traps and considerations. A deeper look at the approaches that children use to quantify each person's share in the partitive quotient context is warranted, in order to investigate children's conceptualisations underpinning their responses. Empirical research, focusing solely on this aspect of partitive quotient engagement, has not yet been undertaken, as far as the present researcher is aware. As a result, in this regard, the present research hopes to contribute to the existing empirical research and, in so doing, to add to the work of researchers in the domain such as Streefland (1991), and Charles and Nason (2000).

2.3.3 Yazgan's (2010) research work

Yazgan (2010) also investigated how children solve partitive quotient problems. The aim of the study was to analyse to what extent fourth- and fifth-grade students can coordinate number of people sharing and number of things shared to solve partitive quotient problems. It involved 30 fourth graders and 28 fifth graders (aged 9–11) from Turkey. She also investigated which strategies in the taxonomy established by Charles and Nason (2000) provided this coordination. Children solved partitive quotient problems in pairs, which is one way in which the study of Yazgan (2010) differed from that of Charles and Nason (2000). Another difference between the two studies is that all the children in Yazgan's (2010) study solved the same three problems, whereas the children in Charles and Nason's (2000) study solved different combinations of partitive quotient problems. The three problems in Yazgan's (2010) study were sharing: (i) three items among four people; (ii) six items among nine people; and (iii) five items among three people. A third difference in the methodology of the two studies is that Charles and Nason (2000) used limited teaching episodes for the purpose of 'exploring and extending children's knowledge construction or helping the children to overcome cognitive impasses' (p. 196). They assert that during these episodes there was no direct teaching of partitioning strategies. Yazgan (2010), by contrast, states that 'she did not make any intervention or correction' (p. 1631) as the children in her study solved the problems. Instead, she asked the children questions about their approaches in order to understand the reasoning that they used in solving the problems.

The reporting of the approaches that children used to obtain the fraction for each person's share in Yazgan's (2010) study was not, generally, as explicit as in Charles and Nason's (2000). Notwithstanding this, a close examination of the findings of Yazgan (2010) showed four approaches to finding the fraction. These four approaches generally paralleled the first four approaches (see sections 2.3.2.1–2.3.2.4) identified by the present researcher in Charles and Nason's (2000) study. Table 2-1 presents the approaches for finding the fraction from solving partitive quotient problems reported by Charles and Nason (2000), and Yazgan (2010). The column labelled 'Comments' briefly summarises any differences noted by the researcher between the findings reported by the two studies.

Table 2-2 Summary of approaches reported by Charles and Nason (2000) and Yazgan (2010) as to how children find the fraction for quantifying each person's share in a partitive quotient problem

Approach label	**Approaches to quantifying each person's share as a fraction**	**Charles and Nason (2000)**	**Yazgan (2010)**	**Comments**
I	Addition of unit fractions or 1/y pieces, where *y* = number of pieces each items is shared into	✓	✓	When the items were partitioned differently, Yazgan's (2010) sample did not appear to add the two fractions, while those of Charles and Nason (2000) did.
II	Number of items/Number of people	✓	✓	
III	Number of items/(Number of items x Number of people)	✓	✓	Yazgan (2010) explained that some of the children 'made mistakes or were confused or were distracted by the number of total pieces while they were deciding the fraction' (p. 1639).
IV	(1) Number of items x Number of people = A (2) A/Number of people = B (3) B/Number of people	✓	✓	
V	1/Number of pieces given to each person	✓	×	

✓: Approach is reported ×: Approach is not reported

Table 2-1 shows that while Yazgan's (2010) study did not contribute any new approaches to finding the fraction, consistent with Charles and Nason (2000) it reported approach III, which appeared to be associated with the part-whole fraction sub-construct.

2.3.4 Empson *et al.*'s (2006) research work

Empson *et al.* (2006) also investigated how children solved partitive quotient problems, but concentrated on whether the coordination of the number of people sharing and the number of things being shared was multiplicative. This study involved a relatively large sample of 112 children, compared to most of the research studies reviewed in section 2.3 thus far. The children were in first (6–7 years), third (7-8 years), fourth (9–10 years) and fifth (10–11 years) grades in the United States and had been previously exposed to varying amounts of formal instruction associated with solving partitive quotient problems. This study builds upon previous empirical research conducted by Empson (1999), who investigated first-grade children's solving of equal-sharing tasks, based on children's informal knowledge of fractions in a classroom setting. The data for the 2006 study was collected, in a similar manner to that of Charles and Nason (2000), via standardised clinical one-to-one interviews.

Empson and her colleagues used tasks that included a combination of problems where the number of items shared were both greater and smaller than the number of people sharing. They chose this in order for the children to produce fractional amounts of varying sizes (greater and smaller than one). Regarding the selection of the number of people sharing in each of these problems, the researchers used the following criteria:

- To align with easy partitions into unit fractions that could be compounded into non-unit fractions
- To be a multiple of the number of items to be shared
- To have a common factor with the number of items to be shared other than itself.

Empson and her associates, like Charles and Nason (2000), reported how the children in their sample quantified each person's share as they solved the partitive quotient problems. In discussing the approaches used by the children in Empson *et al.*'s (2006) study, the present researcher proceeds in a similar manner to that used for the study of Charles and Nason (2000) and, as such, this discussion does not include the strategies for coordinating the number of items shared and the number of people involved in the sharing situations. Only the approaches for finding the fraction associated with each person's share are presented. In addition to anecdotal accounts of children's quantification of each person's share, Empson *et al.* (2006) presented algebraic descriptions of the share. These algebraic descriptions did not portray representations that the children themselves had used. Instead, they were summaries of children's actions and verbalisations created by the researchers. Empson *et al.* (2006) inform that this representation highlighted the mathematical features of the children's strategies and could be potentially useful to teachers and/or researchers when considering mathematical commonalities across the approaches used by the children.

The approaches used by Empson *et al.*'s (2006) sample to quantify each person's share were generally variations of 'French division' as explicated by Streefland (1991). In 'French division', each of the items are shared into the number of sharers then, each person receives one piece from each object. The fraction representing each person's share from each object, therefore, would be a unit fraction, $\frac{1}{b}$. These unit fractions are then summed, to obtain each person's share. Alternatively, depending on how a child distributes the parts to the sharers, for a given partitive quotient problem, it is possible that each person receives more than one piece from each object. Figure 2-5, which depicts one possible distribution of pieces when three items are shared among four people illustrates this.

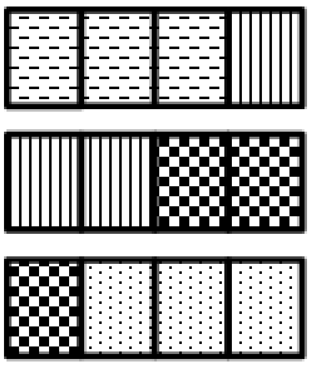


Figure 2-5 Distribution of shares (each shading represents a different person) for sharing three items among four people

The present researcher notes that it is not particularly surprising that the children in Empson *et al.*'s (2006) study used approaches that were variations of 'French division'. This is very likely because the researchers explicitly chose the number of people sharing in each of these problems so that the easy partitions, corresponding to unit fractions that could be subsequently compounded into non-unit fractions, could be obtained.

The next two sub-sections present in greater detail these two approaches used by the children in Empson *et al.*'s (2006) study to quantify each person's share. These two approaches are labelled 'Quantifying each person's share I' and 'Quantifying each person's share II', respectively.

2.3.4.1 Quantifying each person's share I

The most common approach used by the children in the study for finding the fraction was summarised by Empson *et al.* (2006) as: $1/n + 1/n + \ldots + 1/n$ [T times], where T is the number of items being shared and n is the number of pieces each person receives in each item. For this approach, each of the items was shared into the same number of pieces. An illustration of this approach from Empson *et al.*'s (2006) study involved a fifth grader sharing eight pies among 12 people. She drew eight circles, partitioned one into 12 and wrote the number 12 inside the remaining seven. She then wrote that each person would get 8/12.

This way of working resembles the approach labelled 'Quantifying each person's share I' from Charles and Nason's (2000) study (see section 2.3.2.1). For this approach, each person's share was found by the addition of unit fractions from each of the items. Empson *et al.* (2006) presented several variations of this approach in the reporting of their findings. One variation involved the summing of multiple fraction amounts when the items being shared were partitioned or grouped in different ways. One grouping used by some of the children in Empson *et al.*'s (2006) study involved sharing subsets of items among all the sharers. An illustration of this involved the sharing of eight pies among 12 people by a fifth grader. This student explained that two of the eight pies would be shared into six parts each, resulting in 12 parts. In other words, each person would receive one part or 1/6. Since there were four of these pairs (2 x 4 = 8 pies), each person would get 1/6 + 1/6 + 1/6 + 1/6 or four-sixths.

Another variation entailed a child associating a group of sharers with a group of things, for example three people for two things. An illustration of this involved a fifth grader sharing eight pies among 12 people. For this problem, the child determined that the situation was the same as three people sharing two pies and each person's share was therefore two-thirds. Figure 2-6 illustrates the distribution of pieces associated with this fraction amount.

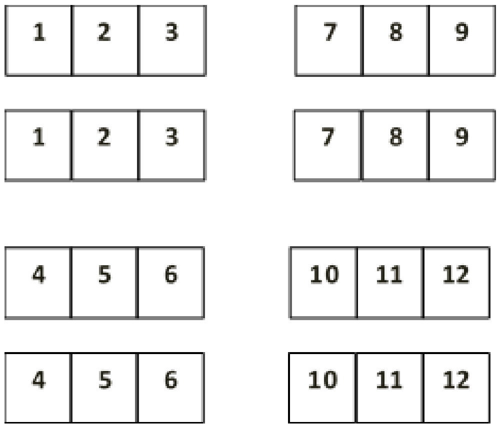


Figure 2-6 Illustration of the sharing for the problem 'share eight pies among twelve people'

This approach was previously reported in the research of Streefland (1991). Section 2.3.1 of this chapter presents an example of Frans sharing six pancakes among eight people by treating this problem as two groups sharing three pancakes among four people.

A third variation of combining of unit fractions is illustrated by the sharing of four candy bars among six children by a fourth grade child. The child first partitioned each candy bar into half, as illustrated in Figure 2-7, and then distributed one-half to each of the six people. This resulted in the distribution of three candy bars. He then repartitioned the remaining candy bar into eighths and the two-eighths left over after distribution were partitioned into six parts.

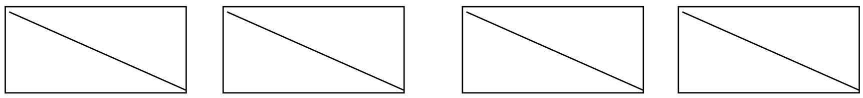


Figure 2-7 Partial illustration of the sharing for the problem 'share four candy bars among six people'

The child then stated that each person's share consisted of the three unit fractions: one-half, one-eighth and one-sixth. The one-sixth is incorrect, since it is supposed to be one-sixth of two-eighths. Other studies confirm this way of working by some children. Kieren (1988), for example, described children's explanations arising from this way of working as an 'engineering report', while Lamon (2012) explains that, in stating the fraction amount, children had difficulties keeping track of the unit when they partitioned the items into different numbers of pieces.

2.3.4.2 Quantifying each person's share II

Another approach used by the sample of Empson *et al.* (2006) to quantify each person's share was denoted as T(1/*n) = T/P,* where *P* is the number of people sharing. For this approach, a child was able to state the fraction amount that each person received straightaway. An example of this was presented for the problem of sharing nine cookies among 12 children. A child stated that they found the answer of nine-twelfths because they cut the nine cookies into twelfths.

2.3.5 Differences between the research of Charles and Nason (2000) and Empson *et al.* (2006)

While there are several similarities between the work of Empson *et al.* (2006) and Charles and Nason (2000), which were briefly discussed near the start of section 2.3.4, these two research studies differ in several respects. First, in the research of Empson *et al.* (2006), virtually all the children created a number of parts equal to the number of sharers, in contrast to Charles and Nason's (2000) study, where several children also created a number of parts that was a multiple of the number of sharers. The partitioning of items into a number of parts equal to the number of sharers by Empson *et al.* (2006) may account for the quantification of each person's share as unit fractions or combinations of the same. Another explanation for the use of this approach by Empson *et al.*'s (2006) sample may relate to the explicit teaching that the children had received regarding solving equal-sharing problems prior to engaging in the clinical interview.

Another difference in the research of the two groups of researchers is that, in contrast to Charles and Nason (2000), Empson and her research team did not mention as important any coordination involving the fraction name in children's solving of the partitive quotient problem or in finding the fraction associated with each person's share. In this regard, Empson *et al.* (2006) explicate that their focus was on reasoning about quantities in order to 'consider the conceptual sophistication of children's strategies independently of the use of conventional terms and notation' (p. 6). They suggest that this focus is advantageous in that it allows children to express their shares informally in terms of number of parts. In contrast to Empson *et al.* (2006), Charles and Nason (2000) analysed how children used fraction terminology.

The present researcher notes the importance of terminology in mathematical learning, including fractions. However, for children aged 7 to 9, who are still developing their knowledge of partitive quotients and who have little experience with the nuances of different terminology (for example fours, fourth, split in four) associated with the various fraction sub-constructs, the importance that researchers such as Charles and Nason (2000) ascribe to explicit terminology use may be misplaced or misguided.

2.3.6 Summary

This section reviewed in detail the key empirical research literature that is in keeping with the present research. This review showed that children use a variety of correct and incorrect approaches for finding the fraction associated with quantifying each person's share when solving partitive quotient problems. Three of the four studies reviewed in this section identified an incorrect approach to finding the fraction that generalises each person's share numerically as: $\frac{\text{number of items}}{\text{number of items x number of people}}$. In this regard, for the problem of 'share two items among three people', one common incorrect solution is 2/6. As the literature review revealed, this answer appears to be typical for some children, regardless of the age/grade, problem, locality or even teacher input. For this particular solution, however, Streefland (1991) and Charles and Nason (2000)/Yazgan (2010) provide different explanations of the basis of the children's actions in producing this result. Taking into account these contrasting findings, the present research study aims to investigate specifically what strategies children who have only been taught the part-whole fraction sub-construct use to quantify each person's share in solving partitive quotient problems.

2.4 The development of the partitive quotient fraction sub-construct

While the partitive quotient, as a separate fraction sub-construct, has been the focus of empirical research studies, few of these have investigated how this knowledge emerges from pre-existing knowledge, such as part-whole learning (Mamede *et al.*, 2005). In addition to this, few studies highlight or explicitly investigate what links the partitive quotient sub-construct may have with other fraction sub-constructs. The remainder of this section provides a review of the literature associated with how the partitive quotient develops from part-whole learning and the links made between the partitive quotient and other fraction sub-constructs.

2.4.1 Link to the part-whole sub-construct

In the mathematics education literature, the partitive quotient has been linked to the part-whole sub-construct empirically and theoretically. Each of these is discussed in turn in sections 2.4.1.1 and 2.4.1.2.

2.4.1.1 Empirical link to the part-whole sub-construct

The research of Naik and Subramaniam (2008) is one of the few studies that has explicitly focused on children who had previously only been exposed to the part-whole fraction interpretation in school. They investigated the impact of instruction aimed at developing the measure and partitive quotient interpretation of fractions. Two groups of Grade 5 children (aged 10–11 years) received

16 and 14 days of instruction, respectively, lasting 1.5 hours per day. In teaching the partitive quotient to the children, the researchers used the approach of summing unit fractions, consistent with Empson *et al.* (2006), to find each person's share. So for example, for the problem of share three items among eight people, each person's share was found to be: 1/8 + 1/8 + 1/8 = 3/8. Following the teaching sessions, the researchers interviewed 11 children who the teachers judged to have a weak understanding of fractions, in order to probe the nature of students' difficulties. Naik and Subramaniam (2008) concluded from this study that students exposed to fraction instruction who had previously been taught the part-whole fraction sub-construct can integrate the quotient perspective meaningfully with their existing knowledge.

In another study, Nunes (2008) conducted a teaching experiment that did not explicitly investigate how the partitive quotient meaning of fractions developed from part-whole knowledge, but reported one finding in that regard. She investigated two approaches used by children in solving different partitive quotient problems. The two approaches investigated included partitioning and correspondences. Partitioning is the process of dividing discrete or continuous objects into a number of disjoint and exhaustive parts, physically or mentally, whereas correspondences refer to children distributing or sharing out items such as chocolate bars to a number of recipients. In these situations, the dividend is in one domain of measures – the number of chocolate bars – and the divisor is in another domain – the number of children. The sample consisted of 62 children (aged 7.5–10 years) who had only been taught about half and quarters and the equivalence between half and two-quarters. They engaged in solving partitive quotient problems of the form 'what fraction of a continuous quantity, such as a chocolate bar, would each child receive if they had *a* chocolate bars shared by *b* children'. Nunes (2008) reported that, as children learnt about the partitive quotient, it did not replace their original part-whole interpretation of fractions or was adversely affected by it; the two meanings coexisted and appeared in the children's arguments as they explained their answers. This finding is in keeping with that of Naik and Subramaniam (2008).

In addition to Naik and Subramaniam (2008), another study, which explicitly examined how children who had only been taught the part-whole meaning of fractions develop other fraction knowledge and skills, is by Kerslake (1986). In her seminal research work, she focused on students' understanding of models of fractions, division of fractions, fractions as numbers and equivalent fractions. Based on this research, she suggests that the emphasis on the part-whole meaning of fractions may serve as a possible inhibitor of the development of other interpretations for fractions, since students see fractions as only parts of a shape or quantity and not as numbers. Charles and Nason (2000) also made similar assertions based on their empirical research. These findings are in contrast to Nunes (2008), and Naik and Subramaniam (2008).

Charalambous and Pitta-Pantazi (2007) investigated students' constructions of the different sub-constructs of fractions. More specifically, they examined the associations among the different sub-constructs of fractions. They found that the partitive quotient task, 'Three pizzas are evenly divided among four children. How much pizza will each child get?' was associated and related directly with the part-whole fraction sub-construct. They considered this reasonable because, 'due to the numbers used in the problem, students could also use a part-whole representation to solve this problem' (p. 307). This finding and conclusion by Charalambous and Pitta-Pantazi (2007) provide further support to the assertion by Mitchell and Horne (2009) that children may use different fraction sub-construct knowledge in responding to a given task than that which the task developers expected them to use. It also provides some confirmation of Lamon's (2005) observation regarding the measure sub-construct that, depending on the task given, part-whole conceptualisation can be used. This suggests that a correct response to a given fraction task does not necessarily infer an associated correct reasoning grounded in the sub-construct under consideration.

As it relates to children conceptualising partitive quotient problems through a part-whole lens, in addition to the findings by Charles and Nason (2000) and Yazgan (2010) presented in sections 2.3.2 and 2.3.3, respectively, Nunes (2008) reports a relevant and significant result. She states that one out of the 12 groups (to be more specific: one child in that one group) indicated that the fractions show the same part-whole relation, and this finding was not expected.

2.4.1.2 Theoretical link to the part-whole sub-construct

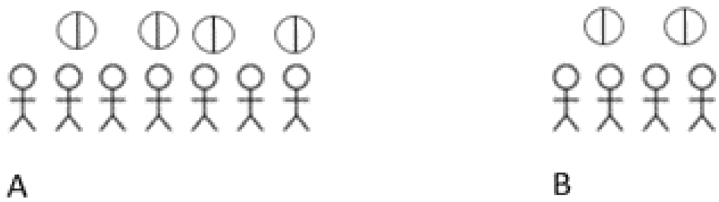
The part-whole and quotient sub-constructs also appear to be linked from a theoretical basis. Kieren (1993), a prominent researcher in the domain of fractions and the first researcher to suggest that the fraction construct consists of several sub-constructs, revised his initial list of sub-constructs of fractions to include later only the ratio, quotient, measure and operator sub-constructs. He removed the part-whole sub-construct as a separate fraction sub-construct. Rather, he opted to 'subsume it under the measure and quotient sub-constructs, as the dynamic comparison of a quantity to a dividable unit that allows for the generation of rational numbers as extensive quantities' (p. 57). Although this decision by Kieren (1993) is not analogous to providing evidence of the link between the quotient and part-whole constructs, it did initially provide a foundation on which to base subsequent research, such as that of Charalambous and Pitta-Pantazi (2007).

2.4.1.3 Summary

Children learn mathematical topics in a progressive manner, with one topic learnt after another. Middleton *et al.* (2015) suggest that, for children, moving from one fraction sub-construct to another is not straightforward. As it relates to the learning of the partitive quotient, in particular, the studies in this section thus far have shown mixed findings from empirical research on how it develops from existing part-whole knowledge. The present study aims to fill the existing gap in empirical literature concerning how one aspect of the partitive quotient sub-construct develops from existing part-whole knowledge.

2.4.2 Link to the ratio sub-construct

Children's solving of partitive quotient tasks has also been linked to the ratio interpretation of fractions. In this regard, Kieren (1993) states that the idea that fractions are simultaneously quotient and ratio is interesting. He investigated this possible complementarity of the ratio and quotient sub-constructs with children aged seven-, eight- and nine-years-old who were involved in doing fraction comparison tasks. An example of the tasks children engaged with is as follows:



a) How much does person A get?
b) How much does person B get?
c) Who gets more A or B or is the amount the same?
d) How do you know that?

Key: : Pizza cut in half : Person involved in the sharing situation

Kieren (1993) reports that one group of children said that B gets half. This suggests that they see the result as dividing an amount of quantity, which is in effect a quotient. Another group paired one pizza with every two persons by drawing two lines from each half-section of a pizza to two people. He states, 'this response shows 'ratio-like' thinking' (Kieren, 1993, p. 54). Empson *et al.* (2006) also found that some children used ratio-like approaches when solving partitive quotient problems, although the children did not 'use ratio terminology to communicate about these quantities' (p. 20). For this approach, children created ratio quantities by grouping a number of items with the number of people sharing. Figure 2-6 in section 2.3.4.1 illustrates this.

2.4.3 Link to the measure sub-construct

The partitive quotient sub-construct has also been linked to the measure sub-construct by van Galen *et al.* (2008), but in a less explicit and more limited way than the part-whole and ratio fraction sub-constructs. van Galen *et al.* (2008) state that there is no absolute difference between measuring and sharing situations, because the result of sharing can also be regarded as a measurement. They explicate that when eight people share six pizzas, each person gets ¾ pizza. In this scenario, there is little concern with how one cuts the pizzas, and 'pizza' is used as a unit of measurement. Since the measure sub-construct is more than just units, but is an actual conceptualisation embodying what a fraction means in a particular context, the present researcher concludes that this link between the measure and partitive quotient is tenuous at best. Further to this, as explicated in section 2.2.2.2, the partitive quotient represents an intensive quantity. This means that the unit for each person's share of ¾ is composite. In the aforementioned scenario, therefore, it is appropriate to use pizza per person, and not just pizza, as the units of the resulting fraction amount that each person receives from the sharing situation(s).

2.5 Summary

Based on the foundation laid in Chapter 1, Chapter 2 situates the present research squarely within the fraction domain of mathematics education literature. Specifically, it provides an overview of fractions in section 2.2 in which several descriptions of fraction as per primary school learning were given and terminology use associated with fractions was delineated. The discussion of the multi-faceted nature of fractions highlighted the complexity of the topic of fractions and the differences among the five fraction sub-constructs.

This chapter also reviewed empirical literature relating to children's solving of partitive quotient problems in sections 2.3 and 2.4. In so doing, it reinforced the gaps in the empirical research literature that the present study hopes to fill. These gaps identified relate to:

(i) the scarcity of empirical research and mixed findings associated with finding the fraction aspect of partitive quotient problem-solving
(ii) how partitive quotient knowledge develops from existing part-whole knowledge.

In this regard, the four empirical studies reviewed in section 2.3 reported six main approaches for finding the fraction by children engaged in solving partitive quotient problems. Two studies reported that one of the approaches used by their research participants was associated with the part-whole meaning of fractions, while two studies did not report any such associations. Section 2.4 also found that there were mixed findings on the effect of the part-whole on the learning of other fraction sub-constructs. In addition, it showed that there were links between the partitive quotient knowledge and other fraction sub-constructs, such as the part-whole, ratio and measure. The present research hopes to add to previous cross-sectional studies by investigating not only children's strategies for finding the fractions related to solving a partitive quotient problem, but also how these strategies change over time. By using this approach, the present research hopes to explore if the part-whole sub-construct impacts partitive quotient learning before formal instruction occurs and, if so, in what way(s).

The next chapter focuses broadly on mathematical understanding and specifically on the Pirie-Kieren theory for growth of mathematical understanding. It explains why this theory has been adopted as the theoretical framework for the second research question, which aims to elaborate the DNB feature of the Pirie-Kieren theory. The Pirie-Kieren theory uses mathematical topic/concepts as a lens through which mathematical understanding can be explored. The partitive quotient is, therefore, used in this research as the vehicle by which the exploration of the Pirie-Kieren theory takes place. In this way, the next chapter is inextricably linked to the current chapter.

Chapter 3: The theoretical framework: The Pirie-Kieren theory for growth of mathematical understanding

3.1 Overview

Theory plays an indispensable role in education, because it helps the education community to develop deep understanding, especially of the important, big questions; for example, what it means to understand something (Lester, 2005). The Pirie-Kieren theory for growth of mathematical understanding represents an established theory in mathematics education (Towers and Martin, 2014) that exemplifies this role of theory as it aims to provide deeper insights into mathematical understanding. This chapter explains why the present research has adopted this theory as its theoretical framework. The chapter begins by presenting some of the different conceptualisations of mathematical understanding in the mathematics education literature. Then a detailed outline of the Pirie-Kieren theory is presented. This includes the conceptualisation of mathematical understanding as per the theory, the model associated with the theory and some of the principal elements of the theory that are pertinent to this research. Finally, section 3.6 explains the decision to adopt the Pirie-Kieren theory as a theoretical framework for the present research.

3.2 The notion of mathematical understanding

The search for a meaningful cognitive description of understanding of mathematics has been ongoing for at least the past half-century (Meel, 2003; Simon, 2006). In fact, Davis (1996) wrote twenty years ago that 'the volume of literature that has been generated on the topic of understanding within the field of mathematics education is quite remarkable' (p. 196). Diverse viewpoints and frameworks within mathematics education literature suggest that there is no unilateral agreement on the meaning of understanding and how it can be characterised (Schroeder, 1987; Halford, 1993; Sierpinska, 1994; Simon, 2006). This section presents a sub-set of the explanations and frameworks of mathematical understanding that have been proposed within the domain of mathematics education. This provides the reader with a glimpse into not only the diversity in conceptualisations of understanding in mathematics education, but also how the discourse of understanding has changed over the past thirty years.

3.2.1 Understanding as types/categories

Researchers in the field of mathematics education, such as Herscovics and Bergeron (1989), Tall and Thomas (2002), and Kent and Foster (2015) describe Skemp's (1976) publication on mathematical understanding as seminal, and point to it as the springboard for renewed interest by researchers and others in mathematics education in examining and characterising the nature of mathematical understanding. In his article, Skemp (1976) proposed a dichotomous conceptualisation of understanding as either instrumental or relational. On one hand, a learner who has an instrumental understanding of a concept knows how to carry out a particular procedure for solving a problem, which Skemp (1976) described as 'rules without reasons' (p. 2). Relational understanding, on the other hand, is 'knowing both what to do and why' (p. 2). This type of understanding involves the process of 'building up a conceptual structure' (p. 14), called a schema, which an individual uses to carry out a limitless number of ways of doing something, such as solving a problem. This conceptualisation of understanding is significant because it highlights that learners are individuals with the capacity to develop their schema and understanding (Martin, 1999). This is in keeping with the constructivist view of learning that was just gaining traction at that time, and contrasted directly with the notion that learners are passive recipients of knowledge and understanding provided by others. Skemp (1979, 1982) later added two additional forms of understandings – logical (or formal) and symbolic understanding – to the existing two categories.

From the work of Skemp (1976), different researchers within the domain of mathematics education proposed other categorical conceptualisations and associated models of mathematical understanding. These include: (a) intuitive and formal; (b) procedural and conceptual; (c) concrete and symbolic; (d) intuitive, procedural and logico-physical abstraction and formalisation (for example Byers and Herscovics, 1977; Backhouse, 1978; Buxton, 1978; Herscovics and Bergeron, 1982; Hiebert and Lefevre, 1986; Herscovics and Bergeron, 1988). While this categorical view of mathematical understanding flourished in the late 1970s to early 1980s, and served as a starting point for the widespread discussion of mathematical understanding, currently this view of understanding is generally not held.

3.2.2 Understanding as a network of connections

Other researchers have characterised mathematical understanding from a representational perspective as a network of mental connections (Haylock, 1982; Davis, 1984; Nickerson, 1985; Hiebert, 1986; Nesher, 1986; Hiebert and Carpenter, 1992; Halford, 1993; Haylock, 2007). Currently, this conceptualisation of understanding is still maintained by many. Nesher (1986), for example, in investigating whether mathematical understanding and algorithmic performance were related, suggests that understanding involves making connections with prior knowledge, which she describes as an already existing system (consisting of several interrelated elements) of concepts. She adds that understanding consists of several different levels, which are not always ordered, and that the states that lead to undertanding are not known. In this explanation, Nesher (1986) suggests that understanding is multi-faceted, non-linear and complex. Additionally, although she speaks of ‘states leading to understanding’, which suggests that there is an endpoint, she highlights that the phenomenon of understanding is open ended and never going to be complete. This apparent contradiction underscores the ongoing difficulty in capturing the essence of understanding.

In later writing, in keeping with the characterisation of mathematical understanding as a network of connections, Hiebert and Carpenter (1992) state:

> A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections. (p. 67)

The connected nature of Hiebert and Carpenter’s (1992) view of understanding is in stark contrast to early static views proposed by Richard Skemp and some of his contemporaries. The interweaving of the numerous mental connections that constitute understanding, as conceptualised by Hiebert and Carpenter (1992), suggests that understanding is a complex phenomenon. The suggestion that a mathematical idea ‘can be understood’, similar to Nesher (1996), is suggestive of an endpoint and that understanding is a goal that one ultimately achieves.

Carpenter and Lehrer (1999) elaborated the earlier work of Hiebert (1986) on the conceptualisation of understanding. Like Haylock (2007), instead of formulating a framework for understanding and its development, they present activities that support and foster its development. In this regard, they focus their work towards the development of understanding within a classroom context and state that understanding is not an all-or-none phenomenon, hence it is more appropriate to think of understanding as emerging or developing. Consequently, they add, ‘we characterise understanding in terms of mental activity that contributes to the development of understanding rather than as a static attribute of an individual’s knowledge’ (p.

20). They propose five interrelated forms of mental activity from which mathematical understanding emerges: (a) constructing relationships; (b) extending and applying mathematical knowledge; (c) reflecting about experiences; (d) articulating what one knows; and (e) making mathematical knowledge one's own.

3.2.3 Understanding as overcoming epistemological obstacles

Sierpinska (1990) presents a different conceptualisation for mathematical understanding from the aforementioned viewpoints. In this regard, her characterisation does not resemble the static, categorical view of understanding proposed by Skemp and others, neither does it mirror the view of understanding as a network of growing connections proposed by James Hiebert, Thomas Carpenter and others. She informs that she conceptualises understanding, in a general sense, 'as an act (of grasping the meaning) and not as a process or way of knowing' (p. 25). From the perspective of education, however, she clarifies that understanding can be thought of as an act, as well as a process. Sierpinska (1994) further clarifies this in later writing when she states:

> An act of understanding is an experience that occurs at some point in time and is quickly over. But especially in education we also speak of 'understanding' as a cognitive activity that takes place over longer periods of time – then we use the term of 'process of understanding', in which 'acts of understanding' mark the significant steps while the acquired 'understandings' constitute props for further development. (p. 2)

In elaborating her conceptualisation of understanding, Sierpinska (1990) generally equates epistemological obstacles and understanding, informing that often these are two ways of speaking of the same thing. The former view is negative and looks backward, focusing on what is wrong or insufficient in our ways of knowing, while the second is positive and looks forward to new ways of knowing. She opines that, at times, some acts of understanding are acts of overcoming epistemological obstacles, while others are acts of acquiring new epistemological obstacles. In this regard, she states that depth of understanding, therefore, 'may be measured by the number and quality of acts of understanding one has experienced or by the number of epistemological obstacles one has overcome' (p. 35).

3.2.4 Understanding as a theory of reification

Sfard (1991), like Sierpinska (1990), puts forward a different viewpoint as to the characterisation of mathematical thinking and understanding. She creates a theoretical framework labelled the theory of reification for the development of mathematical thinking and more specifically, concept development of advanced mathematics. Herscovics and Bergeron (1989) point out that, regarding the use of the terms 'thinking' or 'understanding', the distinction is not important since understanding is the result of some form of thinking.

Sfard (1991), in discussing her theory, concludes that two distinct, but inseparable, complementary types of thinking are involved in the process of developing mathematical thinking. These are labelled 'structural', which refers to mathematical constructs which are invisible and abstract objects and 'operational', which describes 'processes, algorithms and actions' (p. 4). She stresses that this description conveys a duality rather than a dichotomy, as was espoused by Skemp (1976) and others. Sfard and Linchevski (1994) inform that a basic tenet of the theory of reification is that, in the process of concept development, both structural and operational conceptions are necessary, but that operational conceptions typically precede structural ones.

The process of concept development, which culminates in the formation of abstract objects is achieved in three steps that form a hierarchy. These include interiorisation, condensation and reification. Sfard (1991) informs that the hierarchical nature of the theory suggests that no stage can be skipped and that the order is important.

At the stage of interiorisation, a learner becomes familiar with the processes from which a new concept will emerge (for example counting, which leads to natural numbers). Sfard (1991) states, 'a process has been interiorised if it can be carried out through [mental] representations, and in order to be considered, analysed and compared, it needs no longer to be actually performed. In the case of negative number, it is the stage when a person becomes skilful in performing subtractions' (p. 18).

The phase of condensation involves 'squeezing lengthy sequences of operations into more manageable units' (p. 19). During this time, an individual becomes increasingly capable of 'thinking about a given process as a whole without feeling an urge to go into detail' (p. 19), and the process reduces to that of input-output, instead of a series of intermediary operations. She further informs that 'this is the point at which a new concept is officially born' (p. 19). Sfard (1991) notes that, even at that point, difficulties with the generation of the output may occur. This functions as a prompt for the idea of a new mathematical entity. This phase also allows processes to be combined, which facilitates comparisons and generalisations, as well as alternating between varying representations of the concept. For the concept of functions, a student interacts with a mapping as a whole, without considering its specific values. Ultimately, a learner is able to investigate, graph and combine different functions.

Reification, which is the final stage of concept development, is defined as 'an ontological shift – a sudden ability to see something familiar in a new light' (p. 19) and a process solidifies into a static object. Sfard (1991) adds that the new entity is soon detached from the process that produced it. It then begins to draw its meaning from the fact of its being a member of a certain category from which general properties can be extrapolated.

While the duality aspect of Sfard's' (1991) view of understanding is in line with the structural and operational components of Sierpinska's (1990, 1994) theory, the hierarchical nature of understanding sets it apart from the different conceptualisations of understanding delineated in this section thus far. This reinforces the nature of understanding as an ongoing process, but presents a starting point which involves the learner interacting with entities that are somewhat familiar, from which this process grows.

3.2.5 Understanding as a process of constructing Action-Process-Object-Schema

Dubinsky and his colleagues from the researchers in undergraduate mathematics education community (RUMEC) also created a theory for the development of mathematical understanding. This theory, labelled Action-Process-Object-Schema theory (APOS) focuses on advanced mathematics in undergraduate mathematics education.

In this theory, the development of understanding:

> begins with manipulating previously constructed mental or physical objects to form actions; actions are then interiorised to form processes which are then encapsulated to form objects. Objects can be de-encapsulated back to the processes from which they were formed. Finally, actions, processes and objects can be organised into schemas. (Asiala *et al.*, 1996, p. 8)

Meel (2003) points out that an action is necessary, since learners perform actions within their realm of experience. Repeated actions and reflection upon these actions result in interiorisation, 'where the action no longer remains driven by external influences since it becomes an internal construct called a process' (pp. 151–2). When a learner can reflect on a process and transform it by some actions, then the process becomes an object via encapsulation.

These ideas stem from the work of Jean Piaget on reflective abstraction in children's learning. Dubinsky (1991) defines reflective abstraction as 'a concept introduced by Piaget to describe the construction of logico-mathematical structures by an individual during the course of cognitive development' (p. 95). He adds that, stemming from Piaget, two features of reflective abstraction relevant to the APOS theory are that it has no absolute beginning but is present at the very earliest ages in the coordination of sensorimotor structures and continues on up through higher mathematics. Dubinsky and McDonald (2001) further point out, as it relates to the development of understanding, in reality the construction of the four components of the APOS theory is not a linear, ordered process.

The development of understanding in the APOS theory shares some distinct similarities with Sfard's (1991) theoretical framework. These include that understanding is a dynamic, ongoing and very complex process. Also, in both theories the development of understanding consists of varying stages, steps or levels which appear hierarchical but interconnected. Sfard (1990) maintains the strict hierarchical nature of the stages in her theory, while Dubinsky and his colleagues suggest that the development of understanding is non-linear. The present researcher surmises that this difference may be because of the different origins of the two theories. The APOS developed from working with actual learners engaged in doing mathematics, whereas Sfard (1991) described her theoretical framework at that time as 'hypothetical and simplified', but potentially useful as 'a tool for planning, integrating and interpreting empirical research' (p. 21).

3.2.6 Understanding as dynamic reorganisation

Von Glasersfeld (1987), a noted radical constructivist, presents another conceptualisation of understanding. He states:

> what determines the value of the conceptual structures is their experiential adequacy, their goodness of *fit* with experience, their *viability* as means for the solving of problems, among which is, of course, the never-ending problem of consistent organisation that we call *understanding.* (p. 5, italics in original)

This view of understanding suggests that understanding is a dynamic, ongoing process in which the individual learner organises his/her experiences which is likely to result in multiple meanings. Some parts of von Glasersfeld's (1987) explanations are similar to the conceptualisation of understanding put forward by Tall (1978), who conceptualised understanding as a single dynamic entity instead of separate, static categories. It also appears consistent with Nesher (1986) who sees understanding as never being complete. Further to this, von Glasersfeld (1987) states:

> the process of understanding is… analogous to the process of coming to know in the context of experience… it is a matter of building up out of available elements, conceptual structures that fit into such space as is left unencumbered by constraints. (p. 42)

In this respect, he clearly highlights that he conceptualises knowledge and understanding as having a constructed and connected nature. This is in keeping with previous conceptualisations of understanding put forward by Carpenter and Lehrer (1999) and others.

While this view of understanding is distinctly different to the fixed, categorical notions of understanding proposed by Skemp (1976), Byers and Herscovics (1977) and others, Davis (1996) opines that von Glasersfeld's (1987) conception of understanding offers only a first step in rethinking the notion and is inadequate, since two elements of traditional interpretations remain unchallenged. These are that the subjective constraints on understanding persist and a focus on formulated (stated, established, validated truths) aspects of one's knowing is retained. He adds that both of these shortfalls are overcome by enactive theorists who conceptualise understanding 'in terms of effective action, rather than conceptual structures whereby words and concepts are interpreted as patterns of acting' (p. 200). In this regard, understandings are not solely dynamic: they are also 'relationally, contextually and temporally specific' (p. 200). This means that, in different situations, an individual's understanding as evidenced in one's actions may differ considerably. Davis (1996) also stresses that understanding should move away from a focus on formulated (stated, established, abstracted, validated truths) knowledge to the experiences, quite often invisible and unverbalised, that might support them. This conceptualisation of understanding, as explicated by Davis (1996), deviates from the views of understanding previously discussed and forms part of the foundation of the Pirie-Kieren theory for growth of mathematical understanding.

3.2.7 Summary

Section 3.2 presented an overview of some of the models, theories and views of mathematical understanding in mathematics education literature. The models presented from the 1970s to the early 1980s generally seem to conceptualise mathematical understanding as static categories.

From the models presented, the 1980s saw mathematical understanding conceptualised as a network of mental connections. The late 1980s and 1990s saw another shift in the characterisation of understanding with the emergence of the theories of Anna Sierpinska, Anna Sfard, Ed Dubinsky and others. The next section focuses on another characterisation of mathematical understanding from the perspectives of Susan Pirie and Thomas Kieren, the developers of the Pirie-Kieren theory for growth of mathematical understanding.

3.3 Mathematical understanding as per the Pirie-Kieren theory

In discussing some of the prevalent conceptualisations of understanding prior to 1988, Pirie (1988) argued that linear, categorical conceptions of understanding are inadequate to describe and encapsulate the complex phenomenon of understanding. She considered these characterisations deficient, because her extensive observation of learners engaged in doing mathematics portrayed understanding differently – as a whole, dynamic process. Her dissatisfaction with the views of understanding up to that point, as well as key epistemological considerations as to how children learn, led her, together with Thomas Kieren, to develop the Pirie-Kieren theory for growth of mathematical understanding. This theory was originally founded on the constructivist perspective of understanding as an ongoing process in which reflection and reorganisation of an individual's conceptual structures occur (von Glasersfeld, 1987).

While this view of understanding served as an initial basis for conceptualising mathematical understanding, Pirie and Kieren (1989a) felt that other components needed to be included to capture a more complete picture of how mathematical understanding grows as learners engage in doing mathematics. Consequently, in addition to being 'compatible with the constructivist view outlined by von Glasersfeld (1987),… the Pirie-Kieren theory also shares the enactivist view of learning and understanding' (Pirie and Martin, 2000, pp. 127-128), as outlined by Maturana and Varela (1987). This view conceptualises understanding and learning as an interactive process and not as an achievable state or a static categorical depiction (Davis, 1996). From the combined perspectives of constructivism and enactivism, Pirie and Kieren (1994b) described growth of mathematical understanding as a 'whole, dynamic, levelled but not linear, transcendently recursive process' (p. 166).

This description of mathematical understanding included distinct new features such as levelled but non-linear, transcendently recursive and dynamic whole. The next three sub-sections discuss these new features of mathematical understanding in the context of the Pirie-Kieren theory.

3.3.1 Mathematical understanding as levelled but non-linear

Following Vitale (1989), Pirie and Kieren (1989a) characterised mathematical understanding as 'levelled' to encapsulate one aspect of its recursive nature, that each level of understanding was not identical to the previous. The word 'level' has been previously applied to models of understanding (for example Buxton, 1978) to suggest a hierarchical structure, but Pirie and Kieren (1994b) do not see growth of understanding as a monodirectional or hierarchical process, like Sfard (1991). Instead, they see growth as represented by a back and forth movement between layers of understanding. This encapsulates the non-linear nature of mathematical understanding. As a result of the confusion and misunderstanding caused by the use of the word 'level' in describing various modes of understanding, Pirie and Kieren have used the more neutral and descriptive term 'layer' or 'mode', quite often alongside the term 'level'.

The Pirie-Kieren model has eight potential layers-of-action for describing the growth of understanding for a specific person, on any specific topic or concept (Martin, 2008). The labels assigned to each of these layers, starting with the innermost layer, are: Primitive Knowing; Image Making; Image Having; Property Noticing; Formalising; Observing; Structuring; and Inventising. Sections 3.4.1-3.4.5 discuss the first five layers of the Pirie-Kieren model in detail. Figure 3-1 is a diagrammatic representation of the Pirie-Kieren model, depicting the layers as eight nested circles.

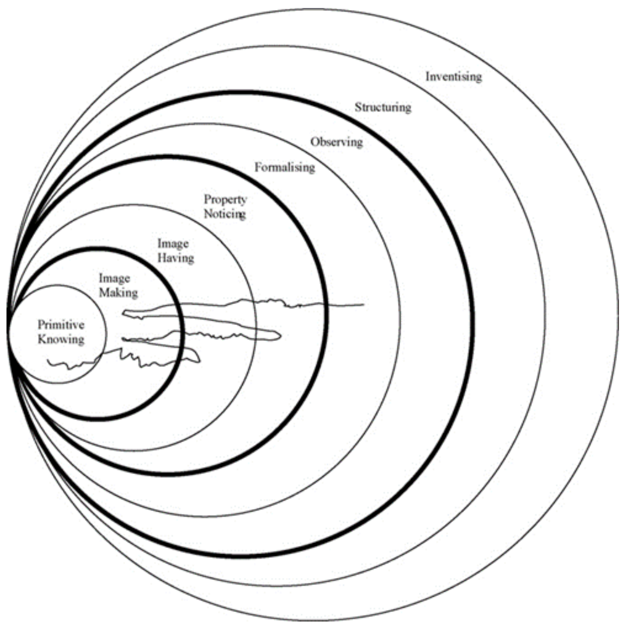


Figure 3-1 The Pirie-Kieren model showing an individual's hypothetical growth of understanding (Martin, 2008, p. 66)

An individual's growth of understanding, which consists of the various back and forth movements between the layers of understanding, is depicted on the two-dimensional model via a mapping process. A mapping process, also known as a mapping diagram or mapping, is 'the name given to the trace of a student's growth of understanding on a diagram of the embedded modes of understanding presented in the Dynamical theory for the Growth of Mathematical Understanding' (Towers, 1998, p. 106).

3.3.2 Mathematical understanding as transcendently recursive

Applying the concept of recursion to the complex phenomenon of mathematical understanding, Pirie and Kieren (1989a) adopted the definition of recursion as put forward by Vitale (1989). Vitale posits that an entity (e.g. mathematical understanding) is recursive if each level is in some ways defined in terms of itself (self-referencing, self-similar), yet each level is not the same as the previous level (level-stepping). Kieren and Pirie (1991) consider this definition of recursion put forward by Vitale (1989) as useful for three reasons. First, it conceptualises a whole entity as opposed to only particular steps being recursive. Since Pirie and Kieren (1989a) consider mathematical understanding to be a dynamical whole, the applicability of this description of recursion is understandable. Second, it highlights two features: non-reducible levels and self-similarity in structure, which are applicable to knowledge and understanding. Kieren and Pirie (1991) argue that, because one has a history of experience in mathematics, one's knowledge and understanding at any point in time should entail previous knowledge in some temporally 'levelled' way. Additionally, since mathematical understanding should be a growing process, any state of such understanding should have structural similarity to previous states.

Kieren and Pirie (1991) point out that when applying this definition of recursion to examining children's mathematical experience, without nominally violating the constraints of Vitale's definition, one could view a new level of mathematical knowledge as simply an 'embroidery on the old' (p. 80). This suggests that 'one's current mathematical knowledge could be "reduced" to one's previous mathematical knowledge' (p. 80). For this reason, Pirie and Kieren (1989a) added an idea taken from Margenau (1987) called 'transcendence with compatibility'. Kieren and Pirie (1991) explicate that, in terms of recursion, we take this to mean that a new level of knowledge transcends, but is compatible with, the knowledge previously held (they are not merely extensions or new pieces tacked onto existing pieces). Additionally, more importantly, the 'new' transcendent knowing frees one from the actions of the previous layer (Pirie and Kieren, 1989a). In this regard, this particular conceptualisation of understanding bears a striking resemblance to the process of interiorisation in Sfard's (1991) theory of reification and Dubinsky's APOS theory.

Interiorisation occurs as the repeated physical actions of an individual and reflection upon these actions cause him/her to get to a point whereby s/he no longer needs to perform these actions, because they are now an internal construct.

3.3.3 Mathematical understanding as dynamic and whole

The characterisation of understanding as dynamic initially encapsulated the continuous process of re-organising an individual's knowledge structures within a particular environment. This characterisation is in keeping with previous models of understanding (such as Davis, 1984; Nesher, 1986). As it relates to the dynamic characterisation of understanding, Manu (2005) informs, based on personal correspondence with Susan Pirie, one of the theory's developers, that 'the word "dynamical" is now preferred because it encompasses and reflects the interactive nature of learning and understanding, which is continuously affected by the environment' (p. 73).

Further to this, the nature of mathematical understanding as a dynamic whole refers to both a condition and consequence of recursion called 'level-connectedness' (Manu, 2005). This suggests that a learner operating at an outer layer can return to a previous layer of understanding to reconstruct the basis for outer layer experience. In this way, the whole or totality in understanding, at any given time, structurally resembles previous states, but cannot be reduced to them (Pirie and Kieren, 1991a). Pirie and Kieren (1992b) add that, from the perspective of dynamic whole, 'one can understand a piece of mathematics many ways at once' (p. 244).

3.3.4 Summary

Section 3.3 discussed the notion of understanding as conceptualised by the Pirie-Kieren theory. This discussion showed that several different philosophies, concepts and ideas informed this conceptualisation and are woven together to capture the complexity and diversity of the phenomenon of understanding. It also highlighted how difficult it is to capture a comprehensive picture of this notion, since so many different elements have to be incorporated into this portrayal. While the development of the Pirie-Kieren theory began with a discourse of the theorists' conceptualisations of understanding, it quickly expanded beyond this to include an associated model and specific features of the theory. The Pirie-Kieren model for growth of mathematical understanding was developed to capture and describe the growth of understanding relating to an individual's engagement with a topic or concept, and is discussed in the next section.

3.4 The Pirie-Kieren model for growth of mathematical understanding

The Pirie-Kieren model (see Figure 3-1 for a diagrammatic representation) has eight potential layers-of-action, depicted by eight nested circles. This section explains the first five layers of this model, since these are the layers applicable to the present research. It also presents examples from the existing literature on the Pirie-Kieren theory to illustrate a child's working within each layer of understanding.

3.4.1 Layer 1: Primitive Knowing

Primitive knowing is the first stage or starting point in the process of coming to understand any topic. It does not suggest, in any way, low-level mathematics and includes the history and/or the experiences that a person brings to learning a given topic. Undoubtedly, this knowledge for any learner would be extended and diverse. As a result, an observer such as a teacher or researcher can never know exactly what primitive knowing each learner might have. They can, however, make reasonable assumptions about what the learner already knows, based on the learner's prior schooling experiences in former grades or his/her own previous experience of teaching that particular topic at that grade level. Primitive knowing resembles informal knowledge (Mack, 1990, 1993) and prior knowledge (Krainer, 2004; Hailikari *et al.*, 2007).

For Grade 3 (8-year-old) children initially engaging with the topic of 'fractions with denominators of 2^n, where n = 1, 2, …, the teacher might assume that the child is able to divide objects into two equal parts and has knowledge of "half" and "quarter" and associated words in a range of contexts' (Pirie and Kieren, 1992b).

3.4.2 Layer 2: Image Making

At the second layer, labelled Image Making, a learner engages in activities aimed at helping to develop particular images for the current topic and making changes to previously held images. Martin and Towers (2009) explicate that the term 'image,' as per the Pirie-Kieren theory, means any physical or mental ideas or representations that a learner has about a topic, and is not restricted to visual or pictorial ones. Additionally, for any topic, there are always multiple images (Pirie and Kieren, 1994b). Martin and Towers (2009) emphasise further that although physical representations of ideas can aid the process of making images, within the Pirie-Kieren theory, external representations (e.g. diagrams) are not labelled as images. Rather, it is the sense that is made of these representations, the thinking and/or acting around the concept, as well as the personal meaning created by the learner for a concept, which is called an image.

Stemming from the example of primitive knowing presented in section 3.4.1, a child operating within the Image Making layer may fold rectangular sheets of paper, representing a 'unit', into two, then four… to form various 'half' fractions (Pirie and Kieren, 1992b). The activities of folding help children to form the image of 'half' and associated fractions, such as quarters and eighths.

Pirie and Kieren (1994b) provide another illustration of a child operating within the Image Making layer from their research. Teresa, a 12-year-old student, used a kit containing halves, thirds, fourths, sixths, eighths, twelfths and twenty-fourths, based on a common standard rectangular sheet, as a unit to make different combinations of quantities for the fraction amount of ¾. Her Image Making activities included placing different combinations of pieces from the kit on a ¾ section of the rectangle, in a mix-and-match way, to see which would combine to make ¾. Davis (1996) describes an individual's actions within this layer as 'a sort of random "mucking about" or playing' (p. 203).

3.4.3 Layer 3: Image Having

After engaging in the actions of Image Making and reviewing, reflecting on and noting (at least to oneself), the character of these actions, a learner can substitute these actions with a mental plan of them or their effects (Kieren *et al.*, 1999). This forms the next layer of the Pirie-Kieren model, Image Having. Within this layer, the child no longer has the need for particular physical actions or examples. It is important to note that Image Having does not imply necessarily having the 'right' or complete picture. Meel (2003) points out that the freedom to imagine a concept unconstrained by the physical processes that gave rise to the associated image(s) is valuable to the growth of understanding. This is because the learner, at that time, begins to recognise global properties of the inspected mathematical images.

Kieren (2001) illustrates operating within the Image Having layer by discussing the work of a pair of eight-year-old children as they engaged in a task requiring them to find fractions that sum to $\frac{5}{4}$. Prior to moving to working within the Image Having layer, the two children, similar to Teresa in section 3.4.2, worked with their fraction kits to explore different combinations of fractions for various fraction amounts. Kieren (2001), in discussing the children's working within the Image Having layer, states that 'the girls generate and collect together instances and implicitly and explicitly have "a rule" to generate and test for membership (in the set of $\frac{5}{4}$)' (p. 230). At that juncture, the girls work in a more systematic manner, which suggests that they have formed some way of knowing or determining which two fractions would sum to $\frac{5}{4}$.

3.4.4 Layer 4: Property Noticing

Evidence of operating at the fourth layer of understanding, Property Noticing, includes examining, manipulating or combining aspects of images to construct context specific, relevant properties (Pirie and Kieren, 1994b). This involves noting distinctions, combinations or connections between images, predicting how images might be achieved, and expressing and recording such relationships (Pirie and Kieren, 1992). It is important to note that these actions are still tied to local, informal examples (Kieren *et al.*, 1999).

Pirie and Kieren (1994b) provide an example of Teresa operating at the Property Noticing layer for the same task presented in section 3.4.2: Use your kit to find as many quantities or combinations of quantities that make exactly three-fourths. After engaging in these tasks for some time, Teresa stated, 'you can do $\frac{2}{3} + \frac{5}{6}$ because twelfths fit on both' (p. 168). In making this statement, she demonstrates that she has found a key property of adding fractions with different denominators. This involves finding the common denominator of the fractions as a first step to finding the corresponding equivalent fractions. In further work, when presented with the question:

> If you have an imaginary fraction kit – it has halves, fourths, fifths, tenths and twentieths – what is $\frac{1}{2} + \frac{3}{4} + \frac{2}{5} + \frac{7}{10}$? Teresa says: Twentieths will fit on all of them. Two times ten makes twenty, so one times ten or ten-twentieths. Four times five makes twenty so three times five is fifteen-twentieths. (p. 168)

In this statement, Teresa verbalises another key property of adding fractions with different denominators. This is finding equivalent fractions with the same denominator of the addends. Pirie and Kieren (1994b) point out that operating within this kind of local, context-based know how, Teresa is considered to be operating at the Property Noticing layer and was able to add sets of many fractions.

3.4.5 Layer 5: Formalising

Formalising is the label assigned to the fifth layer of understanding. Working within this layer, a learner, after consciously thinking about the noted properties and abstracting commonalities, is able to express generalities associated with a mathematical concept without specific reference to a particular example, action or image. These general observations could take the form of generalisations, observed patterns, formal mathematical methods, definitions and/or algorithms, which may or may not use algebraic symbolising.

In several ways, Pirie and Kieren's notion of formalising bears some similarity to Sfard's (1991) notion of reification. First, reification results from the condensation phase, where generalisations and comparisons, which resemble Property Noticing working, allow a new entity to be formed. While Sfard (1991) suggests that the working at the previous stage of condensation is largely responsible for the reification that occurs, Pirie and Kieren suggest that it is the sum total of all that has transpired in all the previous layers that ultimately faciliates formalising. A second similiarity is that, at the stage of reification, the newly formed entity is detached from the process which produced it and begins to draw its meaning from the fact that it is a member of a certain category from which general properties can be extrapolated. This is generally consistent with Pirie-Kieren's Formalising layer of understanding. One difference, however, that is linked to this similarity is that for the Pirie-Kieren theory, a learner can return to previous, inner layers, while for Sfard (1991), after reification, a learner does not return to previous stages. A third likeness between the theories of Sfard (1991) and Pirire and Kieren (1989a) is that moving to the stage of reification and the shift to working within the Formalising layer requires a metamorphic shift in the attention of the learner. However, Sfard (1991) does not detail the actual actions or mechanisms in this shift (Pirie and Kieren, 1992b). A point of difference in the nature of the metamorphic change is that, for Sfard (1991), this is sudden, while the Pirie-Kieren theory appears to suggest, based on the transcendent, recursive feature of the theory, that formalising is a result of the different workings in previous layers and so is not necessarily a sudden occurrence.

Pirie and Kieren (1994b) inform that Teresa (from section 3.4.4), when operating in the Formalising layer of understanding for adding fractions does not base her method on a particular example or actual pieces. Instead, it is a method, which applies independently of her previous actions and is applicable for 'all' fractional numbers.

3.4.6 Summary

Section 3.4 focused on describing the first five layers of the Pirie-Kieren model for growth of mathematical understanding. The next section discusses two of the four features of the Pirie-Kieren theory that are relevant to this thesis.

3.5 Key features of the Pirie-Kieren theory

There are four key features of the Pirie-Kieren theory for growth in mathematical understanding. These include 'folding back', 'teacher interventions', 'DNBs' and the 'complementarities of acting and expressing'. This section explicates the features of folding back and the DNBs, since they are most pertinent to this thesis.

3.5.1 Folding back

Pirie and Kieren (1991b) assert that the most critical feature of the Pirie-Kieren theory is that of folding back, and describe this as:

> A person functioning at an outer level of understanding when challenged may invoke or fold back to inner, perhaps more specific, local or intuitive understandings. This return to inner level activity is not the same as the original activity at that level. It is now stimulated and guided by outer level knowing. The metaphor of folding back is intended to carry with it notions of superimposing one's current understanding on an earlier understanding, and the idea that understanding is somehow 'thicker' when inner levels are revisited. This folding back allows for the reconstruction and elaboration of inner level understanding to support and lead to new outer level understanding. (p. 172)

Figure 3-2 shows the Pirie-Kieren model for growth of mathematical understanding illustrating two instances of folding back: (1) from Image Having to Image Making; and (2) from Property Noticing to Image Making.

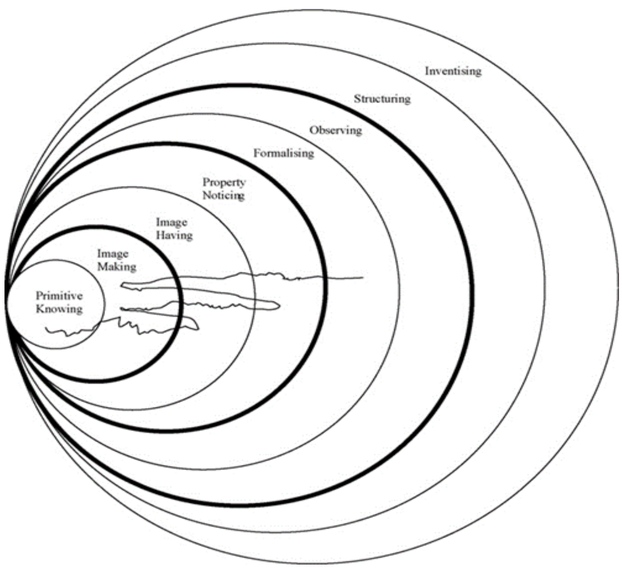


Figure 3-2 The Pirie-Kieren model showing folding back (Martin, 2008, p. 66)

Kieren and Pirie (1992) inform that:

> an illustration of folding back would be the student who, having spent some time working with physical shapes or accurate drawings on squared paper (Image Making, Image Having and Property Noticing) derives the generalisation 'length times breadth' for the area of a rectangle (Formalising) but, on being asked what the area of a triangle might be, replies 'I don't know – I'll have to go back to drawing some to see if there is a formal way of doing that too.' The student is folding back to an inner understanding (Image Making) in order to extend her outer, formalised understanding. (p. 3)

Folding back, as exemplified in the previous illustration, occurs when one faces a problem, question or circumstance at any level of understanding activity that is not immediately resolvable. The result of folding back is, ideally, that the individual is able to extend their current, inadequate and incomplete understanding. According to Martin (2008), this occurs when a learner reflects on and then reorganises their earlier constructs for the concept, or even generates and creates new images, should their existing constructs be insufficient to solve the problem. It is noteworthy that the return to inner layer activity is not synonymous, in either quality or intent, as the earlier original inner layer understanding of that topic. He further adds that when a learner returns to an inner layer, the person is effectively building a 'thicker' understanding at that inner layer to support and extend their understanding at the outer layer that they subsequently return to.

Martin (1999) explored and expanded the notion of folding back and produced a framework, containing three main categories for describing in detail this element of the theory. The three categories are sources of folding back, forms of folding back and outcomes of folding back. From his empirical research, Martin (1999) identified four sources of folding back. These include teacher, peer, curriculum materials and the learner him/herself. In addition to the four sources of folding back, Martin (1999) discovered four forms of folding back. These include 'working at an inner layer', 'collecting at an inner layer', 'moving out of the topic and working there' and 'causing a discontinuity'.

3.5.2 Don't Need boundaries

One of the strong points of mathematics that any comprehensive model of mathematical understanding needs to reflect is the capacity to operate mentally and symbolically without reference to the meanings of basic concepts or images (Pirie and Kieren, 1992b). In the Pirie-

Kieren model, three bold rings labelled DNBs depict this notion. Figure 3-3 shows the position of the DNBs in the Pirie-Kieren model, between the layers of Image Making and Image Having; Property Noticing and Formalising; Observing and Structuring.

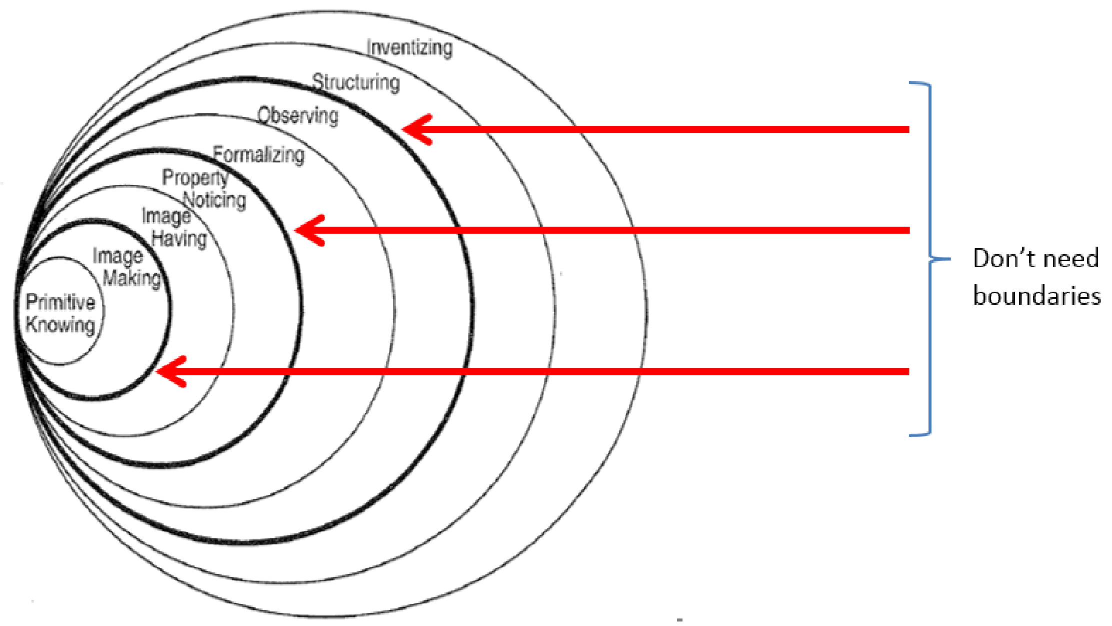


Figure 3-3 The Pirie-Kieren model showing the DNBs (Pirie and Kieren, 1989a, p. 8)

Pirie and Kieren (1994b) explain that the 'Don't Need' label suggests that a learner working outside a DNB does not necessitate the specific inner layer understanding that gave rise to the outer knowing. They further state that this means that a learner can operate at a level of abstraction without the need mentally or physically to reference specific images, but clarify that this does not suggest that s/he cannot return to inner layers of understanding, if needed. Indeed, the feature of folding back suggests just that. According to Pirie and Kieren (1994b):

> Beyond these boundaries, the learner is able to work with notions that are no longer obviously tied to previous forms of understanding, but these previous forms are embedded in the new layer of understanding and are readily accessible if needed. (p. 172)

A close reading of the aforementioned description of the DNB feature of the Pirie-Kieren theory and its compatibility with folding back by the present researcher suggests a contradiction. It also raises the question: if after a learner crosses a DNB s/he can operate at a level of abstraction that does not require previous inner layer activities, why then is there the need to fold back after having crossed a DNB? This seeming inconsistency noted by the present researcher has also been pointed out by Wright (2014) and forms the basis of his contention regarding the DNB feature of the Pirie-Kieren theory.

Stemming from this first contention, the present researcher posits that the solid nature of the rings currently representing the DNBs of the Pirie-Kieren model may be questionable. According to the Pirie-Kieren theory, as a learner's mathematical understanding grows, crossing from inner to outer layers of understanding and vice versa is wholly in order. The present researcher suggests, therefore, that rings of a porous nature, that capture the bi-directional movements across the layers of understanding may be more appropriate to capture the true essence of the DNB than the present solid rings. Figure 3-4 shows an amended Pirie-Kieren model with the DNBs represented as dotted/broken rings instead of the solid rings to depict, diagrammatically, the aforementioned suggestion.

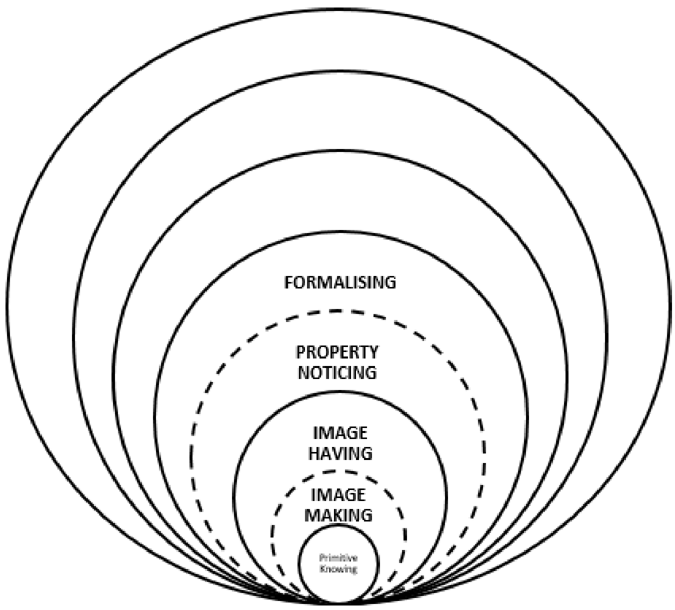


Figure 3-4 The Pirie-Kieren model showing an amendment to the DNBs reflective of the academic literature

Pirie and Kieren (1994a) illustrate the crossing of a DNB from the Property Noticing to Formalising layer as Tanya, an eight-year-old works on adding fractions. Tanya, when faced with the problem of adding $\frac{1}{2} + \frac{2}{3}$, states, ‘Well twenty-fourths could cover both so:

x 12

x 8

$$\frac{1}{2} \quad + \quad \frac{2}{3} \quad \frac{16}{24} \quad + \quad \frac{12}{24} = \frac{28}{24}$$

x 8

x 12

’ (p. 41).

In the aforementioned excerpt, Tanya, while solving a particular problem, noticed a key property when adding fractions with different denominators. This was to find the lowest common multiple of the two denominator. The stated property stems from the problem that she is currently working on. Tanya is working within the Property Noticing layer of the Pirie-Kieren model.

A short while later, the following exchange occurred between Tanya and her teacher (T):

T: What do you think about addition (fractions) now?

Tanya: It’s easy! You just make fractions that work for them all. Say you had $\frac{2}{3}, \frac{4}{7}$ and $\frac{5}{6}$. Well, sixths would work for 2/3 and 5/6, so you’d have to make forty-seconds.

T: Why is that?

Tanya: Well I just know that forty-seconds will fit because sixths times sevenths makes forty-seconds.

T: So?

Tanya:

$$\frac{2}{3} + \frac{4}{7} + \frac{5}{6}$$

$$= \frac{4}{6} + \frac{4}{7} + \frac{5}{6}$$

$$= \frac{28}{42} + \frac{24}{42} + \frac{35}{42}$$

And like that! (p. 41)

From the second excerpt, Pirie and Kieren (1994a) inform that Tanya now understands addition as a method, based on the forms of the number, and uses numbers in their own right. In addition to this, they state that Tanya now has a method for adding that is not confined to fractions that she knows physically, but which works for any fractions. For Tanya, a seventh is an abstract quantity, which she is not familiar with physically or image-wise, even though it is numerically expressed. She is therefore applying a method that she thinks works for all fractions. From the first to the second excerpt, Tanya crosses the DNB from Property Noticing to Formalising.

Davis (1996) also provides a general example of crossing a DNB, based on the topic of fraction addition. He states that a learner whose understanding of fraction addition is tied to his/her actions in manipulating materials, such as cut-outs from a fraction kit, may begin to talk about the additive process in terms of those materials without actually manipulating them. This suggests that the learner crossed the DNB between Image Making and Image Having. Following this, perhaps on some other occasion s/he might begin formally operating on mathematical symbols without having to refer to the previous manipulatives activity. This suggests that the child is now working within the Formalising layer, having crossed a DNB. He points out in each illustration that the learner has moved to a new, outer level of understanding in which previous actions and processes, although still available should s/he need to fold back, are not explicit parts of the current outer understandings.

Freudenthal (1978), in early writing, states ‘If the learning process is to be observed, the moments that count are its discontinuities, the jumps in the learning process’ (p. 78). Sfard and Linchevski (1994) add that ‘those crucial junctions in the development of mathematics where a transition from one level to another takes place are the most problematic, and clearly the most interesting’ (p. 195). The present researcher’s interest in studying learners’ understanding and the DNB feature of the Pirie-Kieren model in particular emerged out of her own lack of understanding of those jumps, which Freudenthal (1978) refers to, in her own students as she taught mathematics at high-school level.

3.6 The appropriateness of the Pirie-Kieren theory for the present research

The present researcher made several considerations regarding the adoption of the Pirie-Kieren theory for the present research. This section discusses these considerations.

3.6.1 Contributing to existing theoretical literature

From the first introduction of the Pirie-Kieren theory to mathematics education literature in 1989, the theory has evolved and grown dynamically (Martin, 2008) and is presently considered to be an established and recognised theoretical perspective on the nature of mathematical understanding in the educational literature (Meel, 2003; Towers and Martin, 2006; Martin, 2008). To date, numerous empirical studies, including very current work, have utilised the Pirie-Kieren theory as a theoretical and analytic tool for studying the growth of understanding (for example Kieren *et al.*, 1995; Martin *et al.*, 2005a; Codes *et al.*, 2013). Some of the topics investigated in the field of mathematics education include fractions, algebra, the Fibonacci sequence and infinite numerical series. Some new fields of inquiry also used the Pirie-Kieren theory as a theoretical and analytic tool. Some of these include: teacher preparation (Borgen, 2006; Nillas, 2010); the development of teaching models (Higgins and Parsons, 2009; Wright, 2014); teacher actions (Warner, 2008); the context of bilingualism (Manu, 2005); the nature of collective mathematical understanding (Martin and Towers, 2014); and workplace training (Martin *et al.*, 2005b).

Despite the extensive development and empirical use of the Pirie-Kieren theory over the past two decades or so, the work of gaining deeper insights into the complex phenomenon of understanding and the development of the Pirie-Kieren theory is still ongoing. In this regard, Pirie (1988), in early writing that is still very relevant presently, remarks:

> In all actuality, we can never fully comprehend 'understanding' itself... with each step that we take forward in order to bring us nearer to our goal, the goal itself recedes and the successive models that we create can be no more approximations, that can never reach the goal, which will always continue to possess undiscovered properties. What we can, however, do is attempt to categorize, partition and elaborate component facets of understanding in such a way as to give ourselves deeper insights into the thinking of children. (p. 2)

In this regard, since the phenomenon of understanding has not been fully explored, both in general and within the context of the Pirie-Kieren theory in particular, additional empirical research focused on further refinement and elaboration of the theory is appropriate.

Further to the aforementioned, as it relates to theory development and elaboration within a particular domain, Lesh and Sriraman (2005) comment that an important challenge for newly evolving fields, such as mathematics education, 'is to develop a sense of its own identity' (p. 490). They add that the domain of mathematics education lacks a distinct identity in terms of 'theory, methodologies, tools, or coherent and well-defined collections of priority problems to be addressed' (p. 490), since it is standard practice for mathematics researchers in their empirical research to adopt theories from other domains such as psychology or information processing. While the present researcher does not reject in any way interdisciplinary approaches to conducting research, she considers it noteworthy that the Pirie-Kieren theory originated and subsequently developed from the theorists' engagement with children involved in solving mathematics problems. Further to this, she posits that continual refinement of a discipline's theories is important. Lesh and Sriraman (2005) confirm this viewpoint and state, regarding mathematics education, that 'the development of theory is absolutely essential in order for significant advances to be made in the thinking of communities' (p. 501). In addition, Sriraman and English (2010) suggest that 'it might be better… to develop models of thinking, teaching and learning, which are testable and refine-able over time' (p. 18). There is no current evidence to dispute the continued viability of the Pirie-Kieren theory as a central theory for growth of mathematical understanding in the domain of mathematics education. Consequently, based on the aforementioned discussion, the present researcher argues that the decision to elaborate one of the key features of the Pirie-Kieren theory is wholly in order, and represents an important undertaking.

3.6.2 The present researcher's philosophical perspective

At the core of any research study are beliefs and assumptions about how a researcher looks at, interprets and acts in the world (Denzin and Lincoln, 2011). These assumptions and views are characterised by epistemological (knowledge), ontological (reality) and methodological considerations, which together comprise a researcher's paradigm – a basic set of beliefs that guide action (Guba, 1990). These assumptions and views are the starting points of a research inquiry and significantly impact not only the questions a researcher chooses to ask, but also how s/he undertakes a given research project. This section explicates the present researcher's decision to focus on the Pirie-Kieren theory for the present research, based on ontological and epistemological considerations. It also discusses some of the strengths of the theory.

Ontology refers to what one thinks reality looks like and how one views the world (Denzin and Lincoln, 2011). From the present researcher's experiences as a learner and teacher of mathematics, one assumption that she holds regarding learners and their associated understandings is that individuals do not encounter a particular learning situation or topic as a blank slate. Their previous experiences in and out of school, the tasks given, their peers, among other factors, influence and shape how they grasp the topic. Further to this, what an individual learns is not necessarily the same as what the teacher conceptualises.

As it relates to how children grow in their understanding of any topic, it is the present researcher's belief that the learning pathway of each child is different. This means that understanding develops via multiple routes. Further to this, as it relates to solving mathematical problems, the approaches that children use are not uniform. Some of these approaches may differ considerably from well-established methods, but quite often children have a distinct reason for proceeding in a particular way.

Another assumption that the present researcher holds regarding understanding is that it cannot be transferred to children. Instead, children, like researchers, construct their understanding as a result of an interplay of different factors, such as previous knowledge, past experiences and social interactions. Reality therefore is multi-faceted, complex and unpredictable.

In addition to this, the present researcher notes that an individual's understanding is largely and often implicit and unseen. In this regard, Pirie (1988) states that, realistically, it is not possible to comprehend, fully, 'understanding' itself. Researchers however, can examine its different features in order to gain deeper insights into children's conceptualisations. In this regard, the present researcher recognises that what a child says and does is just a sub-set of his/her thinking on a topic.

In choosing the Pirie-Kieren theory, the present researcher also considered epistemological issues, which Davis (1996) states is of central concern to mathematics educators. Epistemological considerations bring to the fore questions about how the social world should be studied. It also focuses on what assumptions the researcher has about the relationship between the inquirer and the known (Denzin and Lincoln, 2011) or, within the context of this research, the researcher and the child. In addition to this, Mason (2002) states that epistemological questions explore what the researcher considers to be 'knowledge or evidence of the entities or social "reality"' (p. 16).

Concerning what knowledge looks like when studying growth of mathematical understanding, which is mostly implicit and unseen, an observer needs a learner's actions, as well as his/her verbalisations, in order to make inferences about his/her understanding. Even with a learner's actions and verbalisations, an observer can only make inferences about what a child's verbalisations and actions mean. Growth of mathematical understanding, therefore, cannot be wholly or objectively known.

While this is the case, previous empirical research has shown how to access these entities. In this regard, Davis (1996) states that the Pirie-Kieren theory can be interpreted as a framework for listening, since it places the teacher/researcher in a necessarily attentive relationship to the learner. In addition, it provides a language for explaining, and the means to observe and account for, the growth of mathematical understanding (Manu, 2005) and thus describes it as a theory for, and not of, mathematical understanding. As a 'theory for' it provides an explanation of mathematical understanding, but does not suggest that this is the only plausible explanation. The extensive use of the Pirie-Kieren theory as a theoretical framework in previous empirical research shows that it is possible to investigate learners' growth of understanding.

In regards to the relationship between the inquirer and the known, the present researcher in no way assumes that a researcher sits objectively at a distance from the research entities whom s/he engages with. The present researcher believes that she, together with the learner(s), are co-creators of knowledge. This assumption links not only to epistemology, but extends to the view of human nature and the connection between humans and their environments.

As the present researcher reviewed and reflected on the different views and models of mathematical understanding as part of preliminary work done for this research project, she found that there appeared to be a distinct match between the present researcher's ontological and epistemological persuasions and that of the Pirie-Kieren theory. The present researcher concedes that although, at the start of this research project, she did not conceptualise mathematical understanding in explicit terms such as transcendently recursive, the initial compatibility from a philosophical perspective allowed her to gain access to the other aspects of the theory. Further to this, considering her interest in studying the phenomenon of understanding, as was briefly discussed in section 1.4 of Chapter 1, the prospect of elaborating one of the central features of a well-established theory for the growth of mathematical understanding seemed entirely worthwhile.

3.6.3 The other research interests of the thesis

In addition to choosing the Pirie-Kieren theory as a framework for the present research based on the opportunity that it presented to contribute to theory elaboration within the domain of mathematics education and on ontological and epistemological grounds, the present researcher considered its complementarity to the other research interest of this thesis. Towers (1998), in comparing the Pirie-Kieren theory and the associated model to the other existing models and theories for mathematical understanding, noted almost fifteen years ago that the Pirie-Kieren theory and associated model is the most explicitly formulated and detailed. Currently, the present researcher posits that the APOS theory is the only other existing theory for the development of mathematical understanding that compares to the Pirie-Kieren theory in this respect, but it is only applicable for the growth of understanding of advanced mathematics concepts. The present research, in addition to elaborating the DNB feature of the Pirie-Kieren theory, which (L. C. Martin, 2015, personal communication, 13 May) has confirmed has not been previously elaborated in empirical research, focuses on primary school children's solving of partitive quotient problems. The theoretical framework chosen for this thesis, therefore, had to complement this other aspect of the present research. The choice of the Pirie-Kieren theory was obvious, therefore, as the framework to investigate the growth of mathematical understanding. In this regard, the present research uses the growth of understanding of partitive quotients as a vehicle to explore the DNB feature of the Pirie-Kieren theory and model.

3.7 Summary

This chapter discussed aspects relevant to the theoretical framework adopted for this research. In particular, it provided an overview of some of the views and frameworks of mathematical understanding from mathematical education literature. It also presented in detail a description of the relevant features of the Pirie-Kieren theory as per the present research and the associated Pirie-Kieren model. The chapter concluded by discussing the decision to adopt the theory for the present research and to elaborate the DNB feature of the Pirie-Kieren theory. It also briefly delineated how the use of the Pirie-Kieren model complements the other research interest of this thesis.

The next chapter adds another dimension to what has already been discussed in this thesis by focusing on the specific details regarding how the present research seeks to fill the gaps identified in the literature empirically. In this regard, the methodological considerations of the present research are discussed in detail.

Chapter 4: Methodology

4.1 Overview

Consistent with Denzin and Lincoln (2011), Bryman (2016) asserts that a researcher's paradigm, which is a basic set of beliefs characterised by epistemological, ontological and methodological considerations that guide action, is an important element to reflect upon at the start of a research project. This is because of its extensive influence on the research process. Section 3.6.2 in the previous chapter discussed the present researcher's paradigm and its compatibility with the Pirie-Kieren theory. An examination of the elements of this stance shows that, in addition to being compatible with the Pirie-Kieren theory, it is wholly in line with the interpretivist paradigm. This chapter serves a key unifying purpose in this thesis by bringing together all the elements of the research process (for example, research strategy, design, methods, research phenomena/questions, theory) into a cohesive whole. Using the interpretivist paradigm as the foundation for the present research, this chapter discusses the methodological approaches and research design that guide this study.

4.2 Research strategy

A research strategy, according to Bryman (2016), is 'a general orientation to the conduct of social research' (p. 35). Several considerations can be taken into account in choosing a particular research approach. These include the nature of the research problem, data generation, research methods, research outputs, data analysis and the perspective of the researcher and the researched (Snape and Spencer, 2013). While several considerations can be made in choosing a research approach, many researchers, such as Shavelson and Towne (2002) and Yin (2014), consider that the most important concerns are the nature of the research problem and the associated research questions being investigated.

In keeping with the research problem, Snape and Spencer (2013) suggest that qualitative methods are 'particularly well suited to exploring issues that hold some complexity and to studying processes that occur over time' (p. 5). Both research questions for this thesis are associated with the phenomena that Snape and Spencer (2013) describe. These include the growth of mathematical understanding and children's strategies for finding the fraction related to solving partitive quotient problems. Mathematical understanding is a phenomenon that has long been categorised by researchers in mathematics education as complex and involves inner experiences of people that cannot be studied using quantitative methods (Pirie, 1988; Pirie and Kieren, 1989b; Pirie and Kieren, 1991a; Clarke, 2015). In addition, growth of an individual's understanding and the development of strategies for solving problems involve processes that occur over time. Both of these characteristics of the research foci of this thesis are in line with Snape and Spencer's (2013) reason for using a qualitative approach. Consequently, the present researcher deems that this approach is appropriate for the research phenomena under investigation in this thesis.

Further to the general research foci for this thesis, the specific research questions are framed in terms of 'what are' and 'in what ways'. This does not involve the formulation of a hypothesis, followed by the testing of this hypothesis using experimental/manipulated settings or the use of precise measurement procedures, which are characteristic of a quantitative research approach. For the present research, the researcher aims to gain insight into children's development of understanding of the partitive quotient fraction sub-construct, hence the particular framing of the research questions. The approach to achieving this involves the researcher situating herself within the learners' world in order to obtain deep knowledge and understanding of their particular ways and processes of working. Snape and Spencer (2013) inform that research, which aims to focus on the perspective of the research participants by penetrating their frames of meaning, is congruent with a qualitative research approach. They further add that 'answering "what is", "how" and "why" questions' (p. 4), with the aim to provide an in-depth and interpreted understanding of research participants' experiences and perspectives, is wholly within the purview of qualitative research. On these bases, the present researcher surmises that the qualitative research approach is more suited to the current research than a quantitative one.

Another consideration in selecting a research strategy relates to the type of data that is required for responding to the research questions. In order to explore growth of mathematical understanding and the development of the partitive quotient, there is a clear need to collect comprehensive data in the form of students' actions and verbalisations as they engage in solving problems. Section 4.6 of this chapter discusses the specific details of the data collection for this thesis. This thick, rich data is consistent with qualitative research, as opposed to quantitative research, which collects large amounts of numerical data.

Because of the type of data that is required to investigate the phenomena in the present study, there is a pragmatic limit to the volume of rich data that the present researcher, working alone, can collect and analyse. It appears, therefore, that the aims of this study could only be achieved optimally by closely and intensively studying a small number of children as they solved partitive quotient problems. The study of a small number of purposively-selected research participants is a distinctive feature of qualitative, not quantitative research, which collects data from a large sample of representative cases (Hennink *et al.*, 2010). The specific details as to the choice and characteristics of the research participants are discussed in section 4.5 of this chapter.

Following from the type of data, described above, that is needed to explore the specific phenomena related to this thesis, the analysis of the data can only be achieved by the application of qualitative methods. Understanding is not readily observable and must therefore be inferred from what an individual says and/or does (Pirie and Kieren, 1994b; Steffe and Olive, 2010). This makes the phenomenon of mathematical understanding inherently difficult to study using quantitative methods or large-scale studies (Clarke, 2015). Fine-grained analysis of children's engagement with partitive quotient problems is needed instead, which is associated with a qualitative research approach.

A final consideration in choosing the research strategy for the present research is previous empirical literature, which has explored similar phenomena to the present research. An examination of this research domain (for example Schoenfeld *et al.*, 1993; Charles and Nason, 2000; Martin, 2008) shows that a qualitative approach has been used for these studies. Based on the aforementioned considerations, the present researcher concludes that a qualitative research approach is appropriate for the present empirical study.

4.2.1 Critiques of a qualitative research strategy

While the adoption of a qualitative approach in mathematics education research is presently commonplace (Silver, 2004; Sharma, 2013), several criticisms have been levelled at this approach. Many of these criticisms originate from the domain of quantitative research and, in this regard, Groth (2010) states that how one responds to questions about the utility of qualitative research stems largely from one's philosophical orientation towards research. This section discusses three main critiques of qualitative research.

Bryman (2016) states that qualitative research is often criticised as being too subjective in that the findings are overly dependent on the 'researcher's often unsystematic views about what is significant and important' (p. 405). In line with this first critique, qualitative research is regarded as difficult to replicate, because it is 'unstructured and often reliant upon the researcher's ingenuity' (p. 405). Since the researcher is the main data-collecting instrument, what is observed, heard and focused on in the field are the choosing of the researcher, based on what s/he empathises with and/or considers significant. Sharma (2013) agrees that the values, assumptions, beliefs and knowledge of a researcher influence the data collected but, in keeping with Maxwell (2013), points out that instead of considering this to be a limitation this should be acknowledged and discussed explicitly. Cohen *et al.* (2011) add another aspect to the influence of the researcher by highlighting that, in some research situations, there is an imbalance of power between the research and the researched. This has the potential to influence research participants' behaviour and events. Section 4.2.1.1 addresses the issues related to the first criticism of qualitative research in the context of the present research.

Another main critique of qualitative research is that the findings are restricted, because they are not generalisable. There is no consensus on the issue of generalisation relating to qualitative research, however. According to Bryman (2016), quantitative researchers argue that when data collection only involves a small number of individuals within a particular locality or organisation, it is impossible to generalise to other settings. This is because one or two cases are not representative of all cases. Cohen *et al.* (2011) argue in defence of qualitative research that this type of research 'does not aim to generalise, but only to represent the phenomenon being investigated fairly and fully' (p. 181). This, they suggest, renders the issues of replicability, reliability and predictability, as conceptualised in quantitative research, of limited relevance to qualitative research.

Lewis *et al.* (2014) propose a different position to that of Cohen *et al.* (2011) and posit that although a main aim of qualitative research is not to generalise, stemming from a reconceptualisation of the notion of generalisation by an increasing number of qualitative researchers, various types of generalisations can be made and justified within the domain of qualitative research. These include, but are not limited to 'generalising from the context of the research study itself to other settings or contexts' (p. 351), to the parent population of the study and/or to theory. They, caution, however, that although they wholly support the viewpoint that 'the findings of qualitative research can be generalised… the framework within which this can occur needs careful explication' (p. 348).

A third critique of qualitative research does not specifically originate from quantitative researchers and relates to the lack of transparency in how the research is conducted. In this regard, Bryman (2016) points out that key aspects of qualitative research, such as critical decisions regarding selecting research participants and/or data analysis procedures are sometimes opaque. He does concede that, increasingly, qualitative researchers are addressing these important aspects of transparency.

4.2.1.1 Establishing quality in qualitative research

The third critique raised by Bryman (2016) links to that of reliability, which qualitative researchers preferably term as trustworthiness, credibility, transferability, dependability and confirmability (Lincoln and Guba, 1985). Cohen *et al.* (2011) describe reliability from the perspective of qualitative research 'as a fit between what researchers record as data and what actually occurs in the natural setting that is being researched' (p. 202). They clarify that the aim of this is not to strive for uniformity, since two researchers studying the same setting may produce different findings, both of which are reliable. Alternatively, there may be varied interpretations of a given qualitative data set by different researchers.

Several suggestions, which the present researcher has considered and adopted in conducting this empirical study, have been proposed by qualitative researchers to address the issue of trustworthiness related to qualitative research. Guba and Lincoln (1994) discuss the issue of credibility, which according to Bryman (2016), brings to the fore that multiple accounts of a particular facet of social reality exist. In order to address this concern they recommend that researchers use good standards of research in conducting empirical research. Spencer *et al.* (2003) suggest 18 criteria that represent good standards for conducting qualitative research relating to the research design, data collecting, sampling, data analysis, reporting, ethics, auditability, reflexivity and neutrality. Bryman (2016) describes this list as ‘probably the most comprehensive list of criteria around’ (p. 395). Some of the elements of this framework include:

- How defensible is the research design/sample design/target selection of cases?
- Sample composition/case inclusion – how well is the eventual coverage described?
- How well was the data collection carried out?
- How well has the approach to, and the analysis been conveyed?
- Contexts of data sources – how well are they retained and conveyed?
- How clear and coherent is the reporting; the assumptions/theoretical perspectives/values that shaped the researcher and the links between data, interpretations and conclusions? (Bryman, 2016, p. 395)

To address the issue of trustworthiness, the present researcher has adopted several criteria of Spencer *et al.* (2003) that she considered applicable to the present study. An example of this adoption is the comprehensive discussion of the overall research strategy, design, method and participants, including convincing justifications for the selection of each, in the present chapter.

The second criterion proposed by Guba and his associate relates to the issue of transferability. This refers to whether research findings are consistent in another context, or in the same context at another time. They posit that thick descriptions, for example verbatim transcripts of interviews, not just researcher notes on significant aspects, provide others with the requisite information to make decisions regarding whether findings are transferable. The present research satisfies this second criterion by presenting a large number of verbatim transcripts of task-based interviews in Chapters 6 and 7, as well as in the appendices of this thesis, in order for others, such as examiners, to make decisions relating to the transferability of the current research findings.

Anfara *et al.* (2002), in discussing the issue of dependability as a component of trustworthiness, suggest that researchers should make the analytic processes that they utilise clear to readers. In this regard, researchers should explain explicitly how they derived their themes and show that these themes have 'some congruence or verisimilitude with the reality of the phenomenon studied' (p. 29). They also recommend the use of triangulation, which Cohen *et al.* (2011) describe as the use of multiple strategies of data gathering and/or sources of data to illustrate a coherent picture of a phenomenon under investigation. Both of these suggestions put forward by Anfara *et al.* (2002) and Cohen *et al.* (2011) have been adopted by the present research. More specifically, Chapter 5 of this thesis describes how the research data was analysed, including the derivation of the themes. Multiple tasks, task-based interviews, research participants and sites from which research participants were sourced were utilised to satisfy the criterion of triangulation. Each of these are discussed further in sections 4.4-4.6.

Guba and Lincoln (1994), writing earlier than Anfara *et al.* (2002), go beyond their recommendation and suggest the use of an auditing approach whereby all stages in the research process are recorded and accounted for. This approach facilitates the examination and evaluation by auditors, who are parties external to the researcher. Throughout the present research, the research supervisors of this thesis assumed the important role of auditors who continually ensured that proper procedures were being followed. While the auditing approach is laudable, Bryman (2016) points out that it is very demanding, especially for auditors, since qualitative research produces massive data sets.

The fourth criterion of confirmability concentrates on finding the appropriate balance between researcher objectivity and subjectivity. While it is understood that it is impossible to be wholly objective in social research, the researcher must demonstrate s/he has not 'overly allowed personal values or theoretical inclinations… to sway the conduct of the research and the findings deriving from it' (Bryman, 2016, p. 393). Section 8.4, which focuses on delineating the role of the researcher, as well as the trustworthiness of the research findings, addresses this fourth criterion of confirmability in relation to the present research study.

4.2.2 Summary

This section has shown that there is a distinct match between the research phenomena/questions under investigation in this thesis and the qualitative research approach. It has also brought to the fore some key criticisms of the adopted approach, as well as some broad suggestions for engaging with or addressing the issues identified. Finally, it discussed the issue of establishing quality as it relates to conducting qualitative research.

The next two sections build on the previous sections and focus on explicating two further key decisions, which are essential in conducting empirical research. These include the specific research design and research method adopted for the present research.

4.3 Research design

According to Bryman (2016), 'a research design provides a framework for the collection and analysis of data' (p. 46) and reflects decisions taken by a researcher regarding the core objective(s) of the research. Consequently, considering the range of research foci available in the domain of social science, various research designs are available for conducting research.

4.3.1 An exploratory design

Yin (2014) informs that each research design can be used for three purposes: exploratory, descriptive, and explanatory. He asserts that the most important condition for differentiating among the various research designs relates to the type of research question(s) being investigated. He suggests that questions that focus on exploring entities that have not been the focus of previous empirical work or have been studied in a limited way lend themselves to an exploratory design. Stebbins (2008), a key writer on the topic of exploratory research, agrees that the aforementioned is a key reason for conducting exploratory research. The research questions of the present thesis focus on elaborating a key feature of the Pirie-Kieren theory, which has not yet been previously elaborated in empirical literature. In addition, the current research centres on children's solving of partitive quotient problems from a viewpoint, which has not been extensively explored by empirical research. Previous empirical studies have primarily investigated how children, in general, solve these fraction problems. The present research, in contrast, focuses on how children who have only been taught the part-whole meaning of fractions engage with solving partitive quotient problems. This research also focuses explicitly on how the fraction is obtained while solving these types of problems. While this aspect has formed part of previous empirical inquiries, it has not been the exclusive focus. Consequently, the research questions of this thesis inherently lend themselves to an exploratory design as proposed by Yin (2014) and Stebbins (2008).

Yin (2014) further posits that a key consideration in characterising research as exploratory is that one of the goals of the research is to develop relevant propositions for further investigation. This view of exploratory research as basically a preliminary stage in a research project erroneously characterises exploratory research as simplistic and does not do justice to its actual utility in the domain of social science research (Stebbins, 2001; Jupp, 2006). In this regard, Stebbins (2001) argues that this view of exploratory research 'leads to an unnecessarily stiff and stifling view of social study' (p. v). He also provides several reasons pertinent to this thesis why exploratory research is important. First, he emphasises that people do not know all there is to know about the social world that they occupy. This claim on the surface seems simplistic, but the present researcher posits that it is powerful in its simplicity. Decades of research in education have shown that exploratory work, which involves a researcher following different, unchartered pathways of inquiry, has led to a substantial volume of current knowledge (Pothier and Sawada, 1983; Harel and Sowder, 1998; Flores and Kaylor, 2007) on topics such as fractions, for example. It has also provided significant starting points for future research work. Second, Stebbins (2001) further argues that it is through exploratory procedures that people discover, as a result of their own experiences and powers of reasoning, optimal and/or alternative ways of working, as well as the limitations of approaches that have been previously utilised. Third, he also claims that this type of research is useful for developing and elaborating theory.

In keeping with the notion that exploratory research produces propositions for further inquiry, Reiter (2013) points out that exploratory research admits up front that results are tentative and, in so doing, makes non-exclusive, plausible and fruitful claims about reality and explanations of reality. One example of this is Harel and Sowder (1998), who investigated students' proof schemes. They described their research as exploratory because the results needed to be validated by other empirical research. In this regard, in terms of the present research, not only will the research produce further potentially viable topics of inquiry but, in keeping with Reiter (2013), the findings will also need to be validated by further research. Since the present research, in line with Stebbins (2001), aims to provide further insight into Pirie-Kieren's theory for growth of mathematical understanding and children's solving of partitive quotient problems and generate results for further inquiry consistent with Yin's (2014) conceptualisation of exploratory research, an exploratory design is appropriate for this research.

4.3.1.1 Shortcomings of an exploratory research design

While there are distinct advantages to conducting exploratory research, there are also some shortcomings. Stebbins (2001) informs that exploratory research is 'labour intensive' and 'requires lengthy periods in the field' (p. vi). In addition to the fieldwork requirements, Reiter (2013) claims that exploratory research places great demands on the researcher with regards to research planning and the intellectual engagement with the entity being researched. While Reiter (2013) has singled this out as a specific shortcoming of exploratory research, the present researcher suggests that this is a common characteristic of conducting empirical research in general and is wholly surmountable.

Stemming from the previous point regarding fieldwork, Stebbins (2001) suggests that remaining motivated for the duration of the data collection may prove challenging for researchers. Other potential drawbacks to exploratory research put forward by Stebbins (2001) are that it is generally messy, open ended and fraught with possible disappointment. For researchers who adhere to a positivist paradigm for conducting research, which is rule-bound and ordered, this research design could be extremely challenging. For qualitative researchers, like the present researcher, however, these characteristics of an exploratory research design are in accordance with a qualitative approach that has been adopted as the research strategy for the current thesis (see section 4.2). Consequently, Stebbins's (2001) point is not considered as a shortcoming that could hamper the present research study.

4.3.1.2 Summary

While the shortcomings of exploratory research are important to note, they are not insurmountable. Several of the drawbacks of exploratory research are also characteristics of a qualitative research approach, thus were already taken into consideration in planning for the present research. Further to this, the benefits of using this approach appear to be greater than the disadvantages. As a result, since the present research focuses on aspects that have not been previously or extensively researched, and the findings aim potentially to provide further pathways of inquiry, the present researcher argues that an exploratory research design is appropriate for the present research.

Since exploratory research is such that all research designs might be relevant (Yin, 2014), the following section details the methodological deliberations that resulted in the selection of a microgenetic approach for the present study.

4.3.2 A microgenetic design

The microgenetic design is an approach to conducting research whereby 'a researcher gathers intensive data from learners over an extensive period of time to generate a very rich picture of moment-to-moment learning processes' (Chinn, 2006, p. 439). Each research participant typically encounters similar tasks and/or measures repeatedly, across a period of time, to permit systematic comparisons across and within individuals. Siegler (2006) states that the data collected may be quantitative, qualitative or a combination of both, depending on the nature of the research inquiry. Chinn (2006), by contrast, asserts that microgenetic designs are quantitative in nature, but 'gives careful attention to individual learning events that is otherwise found mainly in qualitative research' (p. 454). This divergence in conceptualisations of what constitutes a particular research design is characteristic of every research design and appears to be typical of new and emerging fields of research as well.

The microgenetic research design evolved from the recognition by researchers of the inadequacy of existing research designs, such as cross-sectional, longitudinal and teaching experiments, to answer important empirical questions concerning the learning processes of individuals (Chinn, 2006; Siegler, 2006). In a cross-sectional study, a learner is interviewed once at a particular point in time, while for a longitudinal study the same children are interviewed over time, for example at the ages of eight, nine and then ten years. In a teaching experiment, a researcher focuses on evaluating the effectiveness of a particular approach to teaching a concept or topic. While the cross-sectional and longitudinal research designs provide useful information about knowledge acquisition and change across varying points in time, Siegler (2006) asserts that the observations are too widely spaced to provide comprehensive information about the learning process. In a teaching experiment, while there is some focus on learning, the main emphasis is on evaluating the effectiveness of whatever aspect of the learning environment (for example, teaching strategy, teaching tools, etc.) is under investigation. The microgenetic design allows a researcher closely to study the processes of change by the application of fine-grained data analysis techniques and to answer questions pertaining to how the process of learning occurs (Siegler, 2006). In this regard, the microgenetic approach addresses a key limitation of the aforementioned research designs and permits a broader set of research questions on individuals' learning to be explored.

Other strengths of the microgenetic design have been offered. Kuhn (1995) states that this research design provides the researcher with the opportunity to observe both the developing knowledge base for a particular topic or concept and the strategies by which this knowledge is acquired. Further to this, 'it is well suited to capture the dynamic of competition among strategies' (p. 134). Fogel *et al.* (2006) add that a microgenetic research design 'enhances our understanding of how individuals and relationships change over time' (p. 69) and allows individual differences in change processes of varying kinds to be examined.

Siegler and Crowley (1991) state that three critical principles define the microgenetic approach. These include that:

1. Observations must span the period of change from the beginning of the change to the time at which it reaches a relatively stable state;
2. Density of observations must be high in comparison with the rate of the change of the phenomenon;
3. Observed behaviours are analysed intensively to infer the processes that gave rise to the varying aspects of change.

The use of the microgenetic approach is varied, in existing empirical literature. It has been applied as a single research design (Voutsina, 2012), or in conjunction with other designs such longitudinal designs (Schlagmüller and Schneider, 2002) or experiments (Luwel *et al.*, 2008). It has also been used to investigate research participants of varying ages, in different domains and settings and using a variety of theoretical orientations. In addition, several empirical studies (Schoenfeld *et al.*, 1993; Martin and Towers, 2014), while not adopting all the elements of the microgenetic design, have utilised certain aspects such as the data collection specifications or the fine-grained approach to analysis. Chinn (2006) describes these studies as both 'microgenetic' and 'having a very strong microgenetic flavour' (p. 443).

4.3.2.1 Disadvantages of a microgenetic design

As per other research designs, there are several disadvantages of utilising a microgenetic approach. Several researchers such as Shaffer and Kipp (2013), Chinn (2006) and Siegler (2006) identify as a key limitation the time-consuming nature of the approach due to the high-density observations. In addition, the demands of high-density observation lead towards either a small number of participants, a small number of sessions or both (Siegler, 2006). This could be particularly problematic for quantitative empirical studies, but the present research study has adopted a qualitative research strategy that typically involves a small number of research participants.

Shaffer and Kipp (2013) note two other potential disadvantages of a microgenetic design. They state that the intensive engagement of learners with situations to stimulate development may not be reflective of standard real-world encounters and may produce changes in their behaviour that may not persist in the long run. While this may be the case, many studies in education (for example Middleton *et al.*, 2008) are conducted to gain insight into a particular phenomenon and may not be specifically interested in mimicking real-world settings. The application of the findings to real-life situations, such as the classroom, may be the foci of follow-up empirical research or entirely separate research studies. Shaffer and Kipp (2013) also suggest that the frequency of observations required by the use of a microgenetic design may affect the developmental outcomes of the children involved. In relation to this point, the present researcher notes that the design may not only affect learners negatively, but positive or no effects may also be observed. In addressing this particular limitation, Shaffer and Kipp (2013) posit that practice effects in microgenetic research may be lessened if more naturalistic observational techniques are used.

Up to this point, this section has delineated the characteristics of the microgenetic design. In the remainder of this section, the present researcher argues for the adoption of a microgenetic research design for the present research. Siegler and Crowley (1991), similar to Bryman (2016) claim that two key considerations in choosing a research design stem from the nature of the research problem/questions and considerations associated with the data collection and analysis for the research. Each of the two concerns is dealt with in turn.

The present research focuses on children's development of the partitive quotient over a period of time. The particular duration of the data collection is explicated in section 4.6 of this chapter. It is therefore interested in the process of learning and in particular, changes in the strategies that children use to find the fraction associated with solving partitive quotient problems, and growth of understanding conceptualised as movements through the various layers of the Pirie-Kieren model. Both these foci reflect core rationales for the use of a microgenetic design as put forward by Siegler (2006). Another focus of the present research is how the partitive quotient develops from children's existing fraction knowledge. Chinn (2006) states that the microgenetic approach is 'designed specifically to address questions' such as 'how do various components of students' prior knowledge interact with new information to produce change… do learners build on prior knowledge… or do they partly sidestep old knowledge and try to set it aside as they construct new understandings' (p. 441)? This is a specific interest of the present thesis. Consequently, from the perspective of the research foci of the present research, the microgenetic approach appears to be a more appropriate research design than a cross-sectional or longitudinal one.

Section 4.2, which discussed the research strategy as per the present research, indicated that thick, rich data was required for investigating the present study's research interests. Following from the type of data that is needed, section 4.2 also discussed the data analysis approach, which suits both the data collected and the research questions. As it pertains to the type of analysis that is needed for studying children's learning, development of understanding and/or mathematical problem solving, Siegler (2006) maintains that, if children shifted directly from less advanced levels of understanding to more advanced ones, microgenetic analysis would be unnecessary. He adds:

> Cognitive change involves regression as well as progression, odd transitional states that are present only briefly but that are crucial for the changes to occur… and many other surprising features. Simply put, the only way to way to find out how children learn is to study them closely while they are learning. (p. 468)

Further to this, empirical work on individuals’ understanding and problem solving, for example, Sfard and Linchevski (1994) and Abrahamson and Trninic (2015), agrees with Siegler’s (2006) assertion. In this regard, Sfard and Linchevski (1994) state that a painstakingly detailed scrutiny of a student's behaviours and utterances called microgenetic analysis is necessary to have some insight into his or her thinking. Sfard and her colleague used this 'fine-grained' analysis within their theoretical framework to investigate the notion of understanding. Data analysis considerations of the present study suggest that microgenetic analysis is an appropriate approach that can be utilised.

In addition to the reasons discussed that support the adoption of the microgenetic design for the present research, this research design has also been applied in previous empirical work focused on children’s learning of mathematical concepts and engagement with mathematical problems. Voutsina (2012), for example, utilised a qualitative, microgenetic research design to investigate the process of change in 10 five- to six-year-old children. More specifically, in five sessions that occurred over five consecutive days, she explored children’s successful problem-solving approaches when engaging with multi-step tasks of the form ‘find all the possible number bonds that add up to a specific number’. Middleton *et al.* (2001) conducted a ‘microgenetic study of a group of four children’ (p. 265) in the context of a larger five-week teaching experiment involving a classroom of 20 fifth-grade students. The microgenetic study explored children’s individual understanding of the quotient. Schoenfeld *et al.* (1993) used microgenetic analysis focused on the qualitative changes in the mathematical knowledge of one student as she explored the topic of graphs and equations of simple algebraic functions over seven hours, spread over a seven-week period. The data analysis provided a fine-grained characterisation of this student’s knowledge structures related to the domain and a description of the nature of the change in those knowledge structures resulting from her interactions with the learning environment.

4.3.2.2 Summary

Based on the discussion in section 4.3.2, a microgenetic design appears to be suitable for the research questions of this thesis and for the data collection and analysis as well. Although there are some key limitations to the use of a microgenetic approach, not all are applicable to the present study and the benefits to be derived from the use of the approach appear to outweigh the limitations.

As a result of the all the considerations of section 4.3, the present researcher, using a qualitative research strategy, adopts an exploratory, microgenetic research design. Once a research design has been chosen for an empirical study, the next major consideration centres on decisions pertaining to the research methods of the research. The next section discusses this decision in detail.

4.4 Research method

A research method is a technique for collecting data (Bryman, 2016). Consistent with choosing a research strategy or design, which were discussed in sections 4.2 and 4.3 respectively, a key consideration in choosing an appropriate data collection tool relates to the research problem and questions being explored. Understanding, a key focus of this thesis, is an internal, unobserved process. Taking this into consideration, Brownell and Sims (1946) state that ‘understanding is inferred from what the pupil says and does… in situations confronting him’ (p. 41). This assertion by Brownell and Sims (1946) agrees with the approach used by the theory developers of the Pirie-Kieren theory, which has been adopted for the present research. In this regard, Pirie and Kieren (1994b) assert that they ‘make no claims as to what might have gone on “in the students' heads” and so analysis can only ever be based on what the teacher observes’ (p. 78). This approach to studying learners’ mathematical thinking, problem solving and understanding is also consistent with more contemporary empirical research. In this respect, Steffe and Olive (2010) state that to construct meaning for a mathematical term or concept, such as fractions, we look to what children say and do as the source of our construction of such meaning.

Davis (1996) adds another dimension to the discourse related to the choice of research method for the present research. Further to considering the research method through the lens of the data to be collected, he emphasises that studying growth of understanding ‘demands not just that one be more mindful of student articulations and actions, but… one must present occasions for learners to express themselves verbally and in action’ (p. 205). The task-based interview is an optimal research method, which provides this avenue for learners to express themselves in this way. The researcher is then able to examine and probe the actions and utterances of the learners as they solve mathematical problems, in order to gain deep insights into learners' reasoning and ways of working.

The task-based interview is a particular form of clinical interview in which a subject or group of subjects talk while working on a mathematical task or set of tasks (Maher and Sigley, 2014). The 'talk' that learners typically engage in may include thinking aloud as they engage in completing the tasks or responding to probes by the researcher. Maher and Sigley (2014) inform that task-based interviews are used to investigate learners' developing mathematical knowledge, ways of reasoning and how connections are built to other ideas as they extend their knowledge. The research questions of the current study focus on how the partitive quotient emerges from existing fraction knowledge, learners' growth of mathematical understanding and their solving of partitive quotient problems. These foci are wholly aligned to the uses of task-based interviews as put forward by Maher and Sigley (2014). This further confirms that the task-based interview appears to be an appropriate method for collecting the data for this thesis.

Task-based interviews have been utilised for some time now in mathematics education research to gain knowledge about the existing knowledge of an individual or group, their structures and ways of reasoning of particular mathematical ideas, growth in knowledge and strategies in solving problems (Empson *et al.*, 2006; Maher and Sigley, 2014). Consequently, previous empirical research in mathematics education (Charles and Nason, 2000; Empson *et al.*, 2006; Nillas, 2010; Clarke *et al.*, 2011b; Mitchell, 2012; Clarke, 2015) provides additional verification that the task-based interview is a suitable method through which to investigate children's understanding and problem-solving strategies.

Taking into consideration the research questions under investigation in this thesis, the data that is suitable for answering the research questions and previous empirical literature related to learners' growth in knowledge and strategies in solving mathematical problems, the task-based interview is adopted as the main research method for collecting the data for the present research.

4.4.1 One-to-one task-based interviews

The aforementioned discussion established that the task-based interview is appropriate for the present research. Maher and Sigley's (2014) description of the task-based interview suggests that these interviews can be conducted individually or in small groups. This sub-section delineates the choice of grouping for the task-based interview.

Middleton *et al.* (2008) argue that the epistemological focus of a research design governs its form and function. They inform that, in their research, their interest was 'to understand children's thinking at the individual level and then build their theory and instructional sequences on that basis' (p. 18). They consider that others may be concerned with the nature of classroom discourse and use it as the starting point for their theory development and curriculum delivery recommendations. The present researcher, similar to Middleton *et al.* (2008), is interested in gaining insights into individual learner's solving of partitive quotient problems. The findings from this could then be used to inform aspects concerning the learning and teaching of the partitive quotient fraction sub-construct.

In addition to this, previous empirical literature, which examined similar phenomenon as the present research, supports the use of one-to-one task-based interviews. In particular, Davis and Maher (1990) used a task-based interview with a teacher and one child in Year 5 involved in solving a problem associated with the division of fractions, while Charles and Nason (2000) used individual task-based interviews to explore children's solving of partitive quotient problems.

4.4.1.1 Limitations of the one-to-one task-based interview

While the reasons for adopting the one-to-one task-based interview as the research method for the present research are compelling, the limitations and benefits need to be weighed. Clarke *et al.* (2011a) inform that a main drawback to conducting individual task-based interviews is that it is time consuming. Despite this, they maintain that the potential benefits that may be derived from using this approach outweigh the limitations. These include building teacher expertise through enhancing teachers' knowledge of individual's understanding of mathematics, providing an understanding of typical learning paths in various mathematical domains, providing a model for teachers' interactions and discussions with children and building both teachers' pedagogical content knowledge and subject matter knowledge.

Cohen *et al.* (2011) also propose other limitations of interviews that are relevant to this thesis. Writing in the context of interviews in general and those involving children in particular, they state that interviews are open to researcher bias and may be inconvenient for research participants, and that interviewee fatigue may hinder the interview since children have short attention spans. All these limitations can be applied to one-to-one task-based interviews. The researcher may be able to minimise limitations regarding interviewee fatigue and inconvenience to participants by scheduling both the time and duration of the interview in conjunction with the research participants. S/he can also enquire about the participant's comfort at appropriate junctures within the interview. Although researcher bias cannot be completely eliminated, the researcher can organise tasks and questions to minimise her influence on the participants. Gaining prior experience in conducting interviews with children may also assist in addressing researcher bias.

The limitations of a particular research method do not invalidate its usefulness and appropriateness for a research undertaking. Instead, cognisance and minimisation of these limitations as far as possible, result ultimately in strengthening a research study. Taking into account the objectives of the present research, previous empirical literature, as well as the benefits and limitations of the method, the present researcher surmises that the one-to-one task-based interview is an appropriate data collection method of this research.

4.4.2 Alternative research methods

In choosing a particular method for empirical research, alternative approaches need also to be considered. Observation is another research method that is sometimes utilised in education research. Cohen *et al.* (2011) defines observation as systematically looking and noting people, events, behaviour, settings, routines and artefacts. While the roles that a researcher can assume in observation lie on a continuum from no interaction to full immersion with the research participants, in using observation as a research method for this thesis, the examination and probing of actions and utterances of children as they solved partitive quotient problems would be distinctly limited. Although in using observation a researcher may fully immerse him/herself in the setting of those being researched, they are not able to question the research participants directly in the same way that an interview allows. Clement (2000) notes this as a distinct disadvantage, since learners often have many interesting conceptualisations of concepts and reasoning processes that are not accessible by techniques that do not allow a researcher to examine and probe learners' responses and actions further. Based on this shortcoming, the present researcher considers observation to be an unsuitable research method for the present research. The shortcoming is also applicable to the administration of open-ended tests as a research method. In addition, since the researcher had the specific aim of gaining insights into children's solving of partitive quotient problems over time, observation of individual children within a classroom setting would not have been practical since limited time is allocated to each topic. Moreover, the disruption of a teaching cycle to facilitate the needs of the present research would be neither ethical nor pragmatic.

4.4.3 Summary

Thus far, section 4.4, from varying perspectives, has presented arguments to support the present researcher's assertion that the one-to one task-based interview is an ideal research method for the present research. Beyond the choice of the one-to-one task-based interview as the main data collecting tool for this study, several decisions need to be made regarding the specific elements of the interview. These include the structure of the interview, the types of questions posed and the techniques for obtaining the required depth of response (Brenner, 2006). The next two sub-sections further develop the discussion on the data collection tool and delineate, in turn, the choice of the format of the interview and the need for multiple tasks and task-based interviews as per the present research.

4.4.4 A semi-structured task-based interview

A task-based interview, like any other, can assume different formats (structured, semi-structured or open ended). Since children grow in their understanding along different paths, a structured interview would be too rigid to allow for the 'following' of a child's solution strategy in various problem-solving situations. Goldin (1993), in corroboration of the aforementioned point put forward by the present researcher, states that flexibility by a researcher in a task-based interview is 'essential to allow for the enormous differences that occur in individual problem-solving behaviours… and to avoid "leading the child in a predetermined direction"' (p. 305).

While Goldin (1993) emphasises that a task-based interview should be flexible, he does not recommend an open-ended format. Instead, he advocates the use of a semi-structured format in which an interview protocol is used. This protocol serves as a guide for conducting the task-based interview(s), yet allows for flexibility in questioning based on what the research participant says. As it relates to flexibility, it allows the researcher/interviewer to make modifications, depending on the judgment of the researcher (Maher and Sigley, 2014). These modifications or contingencies may include branching sequences of heuristic questions, related problems in a sequence or retrospective questions, among others (Goldin, 2000). Additionally, the researcher can react responsively to data as it is collected by asking new questions in order to clarify and extend the investigation (Clement, 2000).

In regards to the structure that an interview protocol provides, Goldin (1993) suggests that it is beneficial in several ways. He states that it allows for: (i) reproducibility, whereby to a certain imperfect degree, the 'same' interview can be administered to different research participants in different contexts; (ii) comparability of interview outcomes between different children; and (c) subsequent research to investigate the generalisability of observations made for individual cases.

The flexibility to follow a line of questioning that is not rigidly pre-planned, and the opportunity to observe a student in action that the semi-structured interview allows, makes it an appropriate format for the data collection method for this study and therefore is utilised in the construction of the task-based interview.

4.4.5 Multiple partitive quotient problems and task-based interviews

There are both empirical and theoretical reasons why multiple tasks are needed for the present research. First, Chinn (2006) explicitly states, 'microgenetic studies… need to use multiple measures' (p. 451). Each research participant engages with similar tasks in multiple sessions over periods of time of varying lengths. In addition, since this thesis uses a microgenetic research design to explore how children's problem-solving strategies and understanding of various aspects of the partitive quotient sub-construct change over time, the need to use multiple partitive quotient problems and task-based interviews is clear.

In addition to the research design adopted for this research, the research focus also dictates that multiple tasks and problem-solving sessions are needed. Previous empirical research for this topic area and age group, such as by Charles and Nason (2000) and Empson *et al.* (2006), have been classified as cross-sectional. As such, their findings speak to performance and cannot apply to learning. This thesis, however, although also focused on children solving partitive quotient problems, is interested in learning and the growth of understanding. Both of these phenomena occur over a period of time and so empirical research interested in investigating such entities should align with how they develop in actuality.

Further to this, Research Question 2 also requires multiple partitive quotient problems for optimal exploration. This question seeks to explore the DNBs of the Pirie-Kieren theory for growth of mathematical understanding. As explicated in Chapter 3, there are three DNBs, two of which are of interest to this thesis. These are located between the Image Making/Image Having and Property Noticing/Formalising layers of understanding in the Pirie-Kieren model. In order to facilitate the exploration of and elaboration on this feature of the Pirie-Kieren theory, several instances of children traversing these layers and hence boundaries are needed. The more problems that children solve, the more likely that they will encounter the Property Noticing and Formalising layers of understanding. Both Charles and Nason (2000) and Vergnaud (1987), in discussing how children develop understanding of a mathematical topic, imply that multiple engagements with a concept/ topic are needed before a child can shift to operating within the Formalising layer of understanding. Vergnaud (1987) states that students develop their knowledge in a fairly wide variety of situations whereby they catch the simplest properties and relationships first, then more difficult ones, until they master the whole system. Before they reach that stage, they do master and sometimes express local and non-coherent properties. It stands to reason, then, that in order for the consistencies and/or similarities to be made by children, multiple tasks over multiple sessions are required.

Another reason why multiple problems and task-based interviews should be used is because previous empirical work documents that a range of solution strategies is possible for partitive quotient problems. Children would therefore need to solve several partitive quotient problems in which the number of items and people sharing varies to demonstrate these different strategies. Due to ethical considerations, as well as the young age and associated attention span of the children in this study, the number of problems that they could solve in one session is limited. Therefore, the optimal approach is to provide multiple task-based interviews in which they could solve a range of problems.

From a theoretical perspective, Vergnaud (1987) advises that ‘it is a good theoretical and methodological choice to study a set of situations, related to one another in the same conceptual field’, given that ‘a concept refers to more than one kind of situation and as the analysis of a situation requires more than one concept’ (p. 231). This is because no single, isolated task offers a perfect window into children’s minds (Sophian, 1997). Previous empirical work associated with the Pirie-Kieren theory, as well as the partitive quotient fraction sub-construct (for example Streefland, 1991; Charles and Nason, 2000; Kastberg, 2002; Taylor, 2008; Duzenli-Gokalp and Sharma, 2010; Mitchell, 2012) confirms Vergnaud’s (1987) position.

The use of multiple tasks and task-based interviews for the present research is also important in relation to establishing the trustworthiness of the present research. Maxwell (2013) suggests that repeated interviews enable a researcher to collect rich data that is detailed and sufficiently varied to provide a complete and revealing picture of the phenomenon being studied. He adds that repeated interviews not only ‘provides more, and more different kinds of data, but (it) also enables (a researcher) to check and confirm…observations and inferences.’ (p. 126).

The use of multiple tasks and task-based interviews for the present research also satisfies the criterion of triangulation, which is a key aspect to establishing quality in qualitative research. The collection of multiple sources of data such as children’s verbalisations, actions and their written work, as well as researcher field notes all address the need for triangulation for the present research.

4.4.6 Summary

Section 4.4 focused on comprehensively explicating the decisions regarding the choice of one-to-one task-based interview as the method of collecting the data for this research. The next section centres on delineating the decisions made and issues confronted regarding the research participants for the present study.

4.5 Research participants

Bryman (2016) informs that research questions typically indicate what categories of people a research study needs to focus on and sample. While the present research focuses on the study of primary school children, this section specifically discusses the considerations made in choosing the precise age/grade level of the children on which to focus and then how the specific children within that particular age/grade level were selected. This section also discusses issues relating to choosing an appropriate number of research participants and the ethics associated with conducting research that involves children.

4.5.1 Age/Grade level of research participants

For the present study, there were two main considerations in selecting the age/grade level of the research participants. These included the prior learning of the participants in school, as well as previous empirical research on children's solving of partitive quotient problems. Each of these is discussed in turn.

This study aimed to investigate children who had only been taught the part-whole fraction sub-construct. It was therefore imperative for the present researcher to take into account children's previous formal learning in choosing the research participants. Based on an inspection of the programme of study (PoS) for mathematics of the CoD (see Appendix A), it appears that students begin learning about fractions in Grade K (age 4–5). A further examination of the PoS by the present researcher reveals that, according to the stated objectives, by the end of Grade 4, students (age 8–9) should have a well-developed understanding of the part-whole meaning of fractions. This is because, by this time, they would have had five years of exposure to this fraction sub-construct in school. At this stage, no other fraction sub-construct would have had been taught to the students. The two teachers from whose classes the research participants were taken confirmed this conclusion by the present researcher. Further to this, previous research relating to solving of partitive quotient problems, for example Streefland (1991) and Empson *et al.* (2006), conducted in the United States, Australia, Netherlands, United Kingdom, among other localities, showed that this research was mainly conducted with children aged eight to ten years. Based on the aforementioned considerations, the present researcher argues that Year 5 children (aged 8–10) are an appropriate sample for the present research and so were chosen as the research participants for this study.

4.5.2 Selection of research participants

The present researcher used a non-probability, purposive sampling approach for selecting the specific research participants. Bryman (2016) informs that the objective of purposive sampling is to sample participants in a strategic way so that those sampled are relevant to the research questions under exploration.

Children were selected for the study based on the type of data which was to be collected (verbalisations and actions) and practical considerations. Selection of the research participants for this study was undertaken in consultation with the children's class teacher, based on the following criteria:

- middle to high cognitive levels based on previous mathematics performance in Grades 4 and 5
- children who are likely to verbalise their thoughts as they work on problems
- children likely to want to participate, hence no children with mathematics anxiety as reported by the teacher, parent and/or child
- regular attendance record
- ability to secure parental consent

Parental consent for participation in the present study was received for all the children. All the children themselves consented to take part in the study as well. Appendix B shows copies of consent forms completed by the parents and children who participated in this study. Appendix C shows the letter which was sent to parents informing them of the particulars of the research. Section 4.5.4, which focuses on ethical considerations, addresses the issue of consent more expansively.

4.5.3 Number of research participants

There is no fixed set of guidelines that a researcher can follow regarding the number of participants to include for qualitative research (Bryman, 2016). For the present research, several factors that have been previously discussed were considered in deciding on the final number of participants. These include the number of research participants used in previous empirical research, the research strategy adopted for this study and the fine-grained data analysis required for the data collected, as well as the funding and manpower limitations of the proposed research. Based on the previously mentioned considerations, the present researcher decided that eight children would be approached to participate in the pilot study and 12 for the main study, cognisant of the fact that sample attrition may occur. The specific details of the pilot and main study are discussed in detail in section 4.6.

4.5.4 Ethical considerations

Most of the ethical issues associated with this thesis pertain to the issue of video data involving young children. The three major ethical issues addressed in this regard are informed consent, anonymity and confidentiality.

Written informed consent was obtained from both the parents of the research participants and from the participants themselves by the completion of relevant consent forms (see Appendix B) before the commencement of data collection. The present researcher also sought and obtained written permission from each of the school principals in which the research took place and the Chief Education Officer of the Ministry of Education and Human Resource Development in the CoD. At the start of each task-based interview, each research participant was asked if they wanted to participate in the particular session to confirm further that the very important issue of informed consent was taken into serious consideration.

To maintain each participant's anonymity, all research reporting uses pseudonyms for each of the children who participated in this research study. Additionally, all identifying elements, such as children's names have been removed from any written work, such as writings/drawings/pictures presented in this thesis. In some cases, the written work has not been used directly, but rather has been typed up by the researcher for illustration.

To satisfy the confidentiality requirement, the research video data collected were shared only with the research supervisors until written up in fully analysed form. In giving consent to participate, parents and participants were made aware that findings from the research would be shared with the academic community. Also, the video recordings are stored securely to respect privacy and confidentiality.

In addition to the aforementioned, before each phase of data collection was undertaken, the British Education Research Association (BERA) ethical guidelines (British Educational Research Association, 2011), as well as those of the University of Southampton Research Governance Office (RGO), were thoroughly reviewed. Further to this, ethical approval was sought and obtained from the University of Southampton Research Governance Office (RGO) (see Appendix D).

Another ethical issue associated with research involving children that the present researcher considered, but was not linked to that of video data, is how to counter the unequal power between an adult researcher and child participants (Punch, 2008). Ginsburg (1997) explicates this issue and states that most children have had no experience with one-to-one interviews and so probably focus on the interviewer's status as an adult. In general, this means 'an adult who is powerful, knowledgeable and who must be obeyed' (p. 110). Further to this, a child may believe that classroom rules apply to the interview, which typically means that children expect to be 'tested' about their knowledge, and that right answers are positive, while wrong ones are negative. In order to address this issue, the present researcher followed Ginsburg's (1997) advice, based on his extensive experience in interviewing children. Some of the suggestions made by Ginsburg (1997) include:

- Establish trust
- Make informal contact
- Explain the purpose of the interview
- Use the child's language
- Be warm and supportive
- Encourage effort and verbalisation
- Monitor affect.

Specific examples of how the present researcher incorporated these suggestions in the research include that she attempted to begin to 'develop a relationship of trust and mutual respect' that allowed 'an intimacy centred on the child's thought' (p. 113), even before the interviews commenced. The researcher spent up to a week in the research participants' classroom for the children to become familiar with her presence. She also interacted with them during the break and lunch periods so that some level of comfort with the researcher would develop before the task-based interviews were conducted. At the start of each task-based interview, each child was asked if they were happy to engage in solving fraction problems (see Appendix F.1). They were also reminded that the researcher was interested in how they solve problems, and not whether the answers are right or wrong.

4.6 Research design: Data collection

Sections 4.2–4.5 discussed the decisions taken by the present researcher as they relate to the research design, methods and participants. Stemming from this discussion, section 4.6 presents the specific details of how the data was collected for the current research, while section 4.7 discusses some preliminary processes and procedures the present research used in the data analysis.

In order to check the efficacy of the research design, data collection for this research occurred in two phases, with a pilot study preceding the main study. Section 4.6.1 discusses the need to conduct a pilot study for the present research and the main components of the pilot study design, as well as how it informed the main study data collection. Following this, the details of the main study data collection are presented in section 4.6.2.

4.6.1 Pilot study

A pilot study, which Polit *et al.* (2001) describe as a small-scale version of a study done in preparation for the main study, is a vital component of good research study design (van Teijlingen and Hundley, 2001). van Teijlingen and Hundley (2001) suggest several reasons and associated advantages for conducting a pilot study, which are relevant to the present research. These include that the pilot study may provide valuable information:

- about where the main research project could fail by trying to identify potential practical and logistical problems in the research procedure
- as to whether the research design, methods, protocols, instruments, sampling frame, data analysis frame work are feasible, appropriate and effective
- concerning local politics or problems that may affect the research process.

Concerning qualitative research in particular, Maxwell (2013) states that pilot studies assist researchers to develop an understanding of the individuals, perspectives and the phenomena that are being studied. In addition, regarding qualitative research, Holloway (1997) states that conducting a pilot study is advantageous because it provides practice and experience to an individual with limited experience in using the interview technique.

While there are several advantages to conducting a pilot study, some limitations have also been documented. van Teijlingen and Hundley (2001) state that three potential problems include 'the possibility of making inaccurate predictions or assumptions on the basis of pilot data; problems arising from contamination and related to funding' (p. 2). They emphasise, regarding the first problem, that a pilot study does not ensure a successful main study. Due to its small scale, key aspects may be overlooked or do not become evident until the larger-scale research is conducted. For the present study, the size of the pilot and main study is comparable, so this shortcoming has limited applicability. Since for one of the schools in which the data is collected, the research participants for the pilot study and the main study are in the same class, contamination is a potential threat to the present study. To address this, children were asked not to discuss with their classmates the problems that they solved or their ways of solving.

The issue of funding was also relevant to the present research. Funding agencies are less inclined to fund a pilot study for a small-scale qualitative study. This can be addressed, however, with advanced planning and appropriate budgeting by the researcher.

An examination of previous empirical literature, for example, Pantziara and Philippou (2012), Van Steenbrugge (2012) and Voutsina (2012), in the domain of mathematics education, reveals that the use of pilot studies is a common practice. Taking into consideration the distinct benefits and limitations that conducting a pilot study affords, and the approach adopted by previous research studies in the domain of mathematics education, prior to finalising the research design for this empirical study the present researcher conducted a pilot study for the following reasons. These include to trial:

- the microgenetic approach for both data collection and analysis
- aspects of the task-based interview, such as number, sequence and details (number of items and people, support materials e.g. cut-outs of items, diagrams of items) of the tasks, the semi-structured protocol, circular vs rectangular diagrams and one-to-one vs pair groupings
- the previous knowledge relating to the part-whole and partitive quotient of the research participants
- the number of research participants in order to make decisions regarding the volume of data that could be managed by a single researcher
- student grouping (individuals and pairs)
- the video equipment for data collection.

4.6.1.1 Research participants (pilot study)

Eight children (aged 8–10; three girls and five boys) were invited to participate in the three-week pilot study for this research. Of these eight children, six (three girls and three boys) completed the pilot study. These children were selected from one Year 5 class at a government-run primary school located in the Western district of the CoD, according to the criteria presented in section 4.5.2.

4.6.1.2 Task-based interviews

The task-based interviews were conducted in the school's library, generally during the lunch period after the children had eaten their lunch. The school principal recommended both the timing and location of the task-based interviews. To ensure that the children were comfortable with the video recording aspect of the data collection, this was explicitly discussed with each child during their first session. This discussion included the purpose for using the cameras, as well as how they worked. At that point, participants were invited to handle the camera and do a short piece of recording. After this first session, each child was invited to look at some of the video that was recorded and ask questions. The present researcher found that by the second task-based interview of the pilot study, the children were not bothered or overly pre-occupied with the video camera. Each interview involved the completion of at least one partitive quotient problem. All interviews were video-recorded using a single camera facing each research participant to capture their verbalisations and actions, including their written work, for later analysis.

For the data collection associated with the pilot study, two children completed the partitive quotient tasks individually and the remaining four were divided into pairs. Children were scheduled to be interviewed twice per week, resulting in six interviews in total. As a result of conflicting school programmes, the unavailability of one of the pair of children and/or illness, however, this was not always possible. Each individual child was interviewed six times and each pair of children was interviewed four times. On average, the interviews were approximately 30 minutes in duration.

4.6.1.2.1 Semi-structured task-based interview protocol

A semi-structured protocol was developed for use in the task-based interviews. The format follows guidelines put forward by Ginsburg *et al.* (1998), who has conducted extensive research that focused on interviewing children. The main sections in each task-based interview (the introduction, follow-up questions and conclusion) show the structure of each interview. In the introduction, the interviewer presents a problem or task that provides the child with something specific to engage. As part of the introduction, the interviewer ensures that the child understands the task by restating the task, changing the wording or questioning the child about it. Ginsburg *et al.* (1998) further advise that, once the child gives a response to the task or problem provided, follow-up questions are asked in order for the interviewer to understand the child's thinking and to interpret the child's words and/or actions correctly. Such questions focus on how the child solved the problems and what s/he means by a particular answer. Specific questions may include, but are not limited to: 'How did you figure that out?', 'How do you know?', 'What do you mean?' and/or 'How did you solve that problem?'

Following Ginsburg *et al.* (1998), for the present study, to ensure that the research participants understood the task, the question 'how much cake does each person get' was often reworded to 'what fraction does each person get' so that the participants knew that a fraction was expected. Both these ways of phrasing the partitive quotient problem are consistent with previous empirical research (for example Lamon, 2005; Nunes, 2008). Section 4.6.1.2.2 discusses further the partitive quotient tasks used for the current research.

Another technique used by the researcher to assist the research participants in understanding the task(s), especially at the start of the data collection period, was to question them about the task and point out important features. An example of this is the present researcher emphasising to the children that separate items were being shared by specifically pointing to the separate items on the task sheet or to the individual cut-outs that were provided for the children's use in the problem-solving process, when introducing the task.

For the present research, the semi-structured task-based interview protocol served an important role in mitigating against the tensions confronted by the present researcher during the data collection, as it relates to the child's conceptualisation of the task/the task requirements and the teacher/researcher relationship, thereby increasing the trustworthiness of the findings of the present research. First, while the researcher aimed to ensure that the research participants understood the task, it is important to note that the guidance provided was not to steer the children towards providing a particular response to the problem. Rather, the researcher's intention was to allow the research participants to solve each problem, based on their conceptualisation of the task, after the task was presented using the same structure/protocol, in a typical wording/format as previous empirical literature (for example Lamon, 2005; Nunes, 2008). The variations in the children's responses and potential deviations from previous empirical literature could therefore not be attributed to the task administration during the task-based interviews. This approach used is consistent with Lamon (2012) who suggests that this way of operating helps teachers and researchers to gain insights into what children do when they are not steered along a particular path, and how instruction might be structured, in light of this knowledge to achieve the desirable outcomes.

Second, the present researcher, who also is a teacher of mathematics, also confronted the tension between allowing the child to take his or her path and the instinct to guide the child to the correct answer through questioning. The semi-structured task-based interview protocol served an important role in keeping the present researcher on track regarding the questioning to ensure that the aims of the present research study were achieved. The opportunity to trial its use during the pilot phase of the data collection allowed the present researcher to become comfortable conducting the interviews in the role of 'researcher' as opposed to 'teacher' and in so doing, the trustworthiness of the data collected and the research findings was increased.

Appendix F.1 details the protocol used for the task-based interviews and lists the follow-up questions asked by the researcher in order to 'follow' a child's solving of the problem. These questions capture where the variation within each interview occurred. The interview schedule also follows the approach that Ginsburg *et al.* (1998) advocate, which emphasises that the interviewer does not correct wrong answers or *teach* during the interview.

4.6.1.2.2 Partitive quotient tasks

The partitive quotient is exemplified by the solving of problems associated with sharing *x* number of continuous items among *y* people. Two groups of tasks were adapted from previous empirical work (Streefland, 1991; Charles and Nason, 2000; Lamon, 2005; Nunes, 2008) for the task-based interviews. Appendix F.2 shows an example of partitive quotient tasks from previous empirical work. The adaption largely related to the context of the sharing situation, and to the addition of the second part of the task, ‘How else can you…’. Appendix F.3 presents a description of the tasks used for the pilot study.

Each interview task consisted of two main questions. The first part involved sharing a number of items, such as cakes, equally and completely among a given number of people and finding how much each person received. Since this task resembled those of previous empirical literature (see Appendix F.2) and the validity of those tasks had been previously established, the present researcher surmised that if the research participants of the present study provided solutions/responses that differed from previous findings, it is highly likely that the difference(s) could not be attributed to the characteristics of the tasks. In this regard, the trustworthiness of the data collected and the findings of the present research study is increased.

The second part of the partitive quotient task entailed the research participants finding other ways of sharing the items among the people when asked, ‘How else can you share the items among the people and how much does each person receive’. The initial reason for posing this follow-up question was to elicit other ways of solving the given partitive quotient problem in order to answer a research question that was eventually removed from the thesis. She explicitly communicated this objective to the research participants at the start of the data collection period when she states, ‘Today, you are going to solve problems where you will find how much cake/pizza each person would get if the cake/pizza is shared equally among the people and no (cake/pizza) is left over’ (see Appendix F.1). Further to this, from the first task-based interview (see Appendix F.1), the researcher stressed to the research participants that the focus was not on right or wrong answers when she states ‘I will not tell you whether your answer is right or wrong. This is not important here. What I am interested in is how you solve each problem’. The research participants appeared to fully grasp the present researcher’s intention to view this follow up question of ‘How else…’ as part of the structure of the task, whereby further ways of solving the stated problem were elicited. By the second task-based interview, after providing the first solution, most of the children proceeded to offer additional solutions to the task without being prompted by the researcher.

4.6.1.2.3 Materials used within tasks

For each task-based interview, materials, in the form of A4 paper with diagrams of circles and rectangles printed on it, plain A4 paper, pencil, sharpener, cut-outs of circular and rectangular diagrams and crayons, were provided to the research participants. In addition, for the first task-based interview, actual circular and rectangular cakes were provided to the research participants in order for them to fully understand the task that they were to engage with and to conceptualise that individual cakes were being shared.

4.6.1.3 Assessment of fraction knowledge

The present research centres on children who had only been exposed to the part-whole fraction sub-construct in formal teaching, and therefore had not previously engaged in solving partitive quotient problems. Consequently, ensuring that this criterion was met was an important element of the present research. In previous empirical work (for example Charles and Nason, 2000; Nunes, 2008), the researchers did not provide explicit information as to the previous fraction knowledge of their participants.

For the present research, an inspection of the curriculum documents (see Appendix A) and primary school text books from Year K-5 showed that, based on these documents, the only fraction sub-construct Year 5 children from the CoD would have been formerly taught was the part-whole. A discussion with two Year 5 teachers confirmed the aforementioned conclusion. In addition to this, an individual assessment of fraction knowledge consisting of written and verbal tasks (see Appendix E) was administered to the research participants by the researcher before they engaged in solving the partitive quotient tasks. This was done to ascertain whether the partitive quotient was a novel task to the Year 5 children and the extent of the research participants' part-whole knowledge as high and middle attaining students. In addition, the aim was to establish, further, the trustworthiness of the research study.

4.6.2 Main study research design as informed by the pilot study

As anticipated, the pilot study informed several key aspects of the main study. The specific ways in which the pilot either confirmed aspects of the research design or informed changes are discussed in the sub-sections below.

4.6.2.1 The microgenetic approach for both data collection and analysis

The three criteria of the microgenetic research design, previously discussed in section 4.3.2, were confirmed during the pilot study. The three weeks that children engaged with the tasks allowed for the strategies in finding the fraction for the partitive quotient problem, a novel task for the children, to appear and to stabilise. Changes in children's growth of understanding of the partitive quotient fraction sub-construct were also observed, as per the layers of the Pirie-Kieren model. The pilot study also confirmed that the density of observations was high in comparison to the rate of change of children's growing understanding. In addition, it verified that fine-grained analysis of children's actions and verbalisations was needed to track the growth in understanding as per the Pirie-Kieren theory, as well as changes in the strategies children used to find the fraction resulting from solving partitive quotient problems.

4.6.2.2 Task -based interviews

4.6.2.2.1 Number of task-based interviews

In the three-week pilot study, the maximum number of sessions that took place per research participant was six sessions. One child completed six sessions. This child, during the sixth session, began to operate at the Formalising layer of the Pirie-Kieren model in solving the partitive quotient task. Another child expressed boredom by the sixth session. Taking into consideration the aforementioned, as well as the fact that term 2 in the CoD, during which the data collection was to occur, ended on 2 April 2015, the researcher decided to conduct eight individual task-based interviews in the main study over a six-week period, in which each research participant would engage in two task-based interviews per week. The pilot study showed that the semi-structured task-based interview protocol was effective for conducting the task-based interviews and was therefore utilised for the data collection for the main study.

4.6.2.2.2 Partitive quotient tasks

The tasks utilised in the pilot study facilitated the collection of the data required to answer the research questions associated with this thesis. Additionally, as it relates to the question 'How else can you share the items among the people and how much does each person receive' the pilot study showed that this question served to increase the validity of the findings for the two research questions of this thesis. First, this follow up question facilitated investigating children's strategy change in finding the fraction associated with solving partitive quotient problems within a task-based interview. This potentially provided a broader window through which to investigate how children engaged in solving partitive quotient problems than was previously presented in empirical literature. It also served a purpose of triangulation in order to establish the trustworthiness of the data collected and associated findings related to the strategies the research participants used. The answering of the second research question that involved the exploration of DNB feature of the Pirie-Kieren theory, using the partitive quotient as the vehicle for this investigation was also facilitated. Based on these considerations, the structure of the task, comprising two parts, remained unchanged for the main study.

Further to the aforementioned, although the structure of the task remained unchanged, since the duration of the main study was longer than the pilot study, more tasks were needed. On the other hand, it was expected that each research participant would spend more time on each task than was spent in the pilot study. This is to explore the solving process of various aspects of the partitive quotient tasks in greater depth for each task. Consequently, the researcher opted to use eight tasks in the main study.

For the pilot study, the number of items to be shared ranged from one to four and the number of children ranged from two to six. This preliminary research demonstrated that the Year 5 children included in the sample were able to work relatively comfortably with up to six people and four items. Although sharing four cakes among six children was challenging for the research participants, they were all ultimately able to provide a solution to the task and, in some cases, multiple solutions. The interaction with the students and a discussion with the Year 5 class teacher revealed that a large number of children had difficulties with the recall of their multiplication facts up to 12. Bezuk and Cramer (1989) suggest, based on their work on the Rational Number Project spanning several years, that in primary grades fraction concept activities can include fractions with denominators no larger than 12. As a result of the aforementioned considerations, the same number of items would be used, but the maximum number of people used in the partitive quotient tasks would be eight. Table 4-1 shows the number of items and people that were used for each the eight tasks, labelled T01-T08. Appendix F.4 shows an example of a task sheet used in the task-based interviews.

Table 4-1 The number of items and people that are used for the eight tasks

Tasks	**T01**	**T02**	**T03**	**T04**	**T05**	**T06**	**T07**	**T08**
Number of items	2	4	3	4	2	3	2	3
Number of people	3	3	5	6	7	6	5	8

An inspection of the number of people sharing shows that a combination of even and odd numbers are used. The smallest number of items that was included is two, since for one item the number of ways of solving the problem is limited. For tasks T04 and T06, the number of items and people involved in the fair-sharing task had a common factor, which could potentially elicit a wide variety of ways of engaging with the tasks. One task, T02, involved a number of items more than the number of people in order to elicit the generation of improper fractions or mixed numbers.

4.6.2.2.3 Materials used within tasks

Previous empirical research associated with the partitive quotient fraction sub-construct have consistently documented the use of diagrams or cut-outs to represent the items based on continuous area/region models (Pothier and Sawada, 1990; Streefland, 1991; Lamon, 1996; Charles and Nason, 2000; Tunç-Pekkan, 2015). A region model is a two-dimensional geometric form such as a square, rectangle or circle. As it relates to the specific area models that can be used in fraction teaching and learning, previous empirical research have found mixed results on the impact that each model had on children's learning of fractions. Although the circle has been used widely in fraction research and instruction (Cramer *et al.*, 2002), many primary school children found it challenging to partition circles (Ball, 1993), especially when odd numbers are involved (Myers *et al.*, 2009). By contrast, other research studies such as by Cramer *et al.* (2002) and Bray and Abreu-Sanchez (2010) found that the circle was a powerful model which could be used to facilitate the learning of various fraction knowledge. Regarding the bar model, which is represented by a rectangle, several research studies have found that children are able to work more easily with this model than with the circular model (Charles *et al.*, 1999; Keijzer and Terwel, 2001). For the tasks given, the children involved in the pilot study of the present research were able to partition rectangles more easily than circles. Consequently, based on the aforementioned considerations, the rectangular region model was selected to represent cakes and pizzas in the fair-sharing tasks in this research.

In addition to the diagrams, rectangular and circular cut-outs that represented the items to be shared, were provided for the children to use during their problem-solving process in the pilot study. Also, in the first-task-based interview, individual cakes were provided. This was done, in addition to restating the task, changing the wording or questioning the child about it as part of the introduction to each task-based interview, to ensure that the research participants understood the task presented, but at the same time were free to solve it as per their conceptualisation(s). Based on the pilot study, the present researcher found that children verbalised that they understood that individual cakes were being shared and primarily used the diagrams of the cakes/pizzas when solving the problems versus the blank sheets of paper or the cut-outs. Consequently, to simplify the task-based interview and to avoid potentially overwhelming the research participants with various support materials, cut-outs and real cakes were not provided to the research participants for the main study. Instead, for each interview in the main study, each child was provided with a single task sheet, on which was printed the task and diagrams of rectangles, along with a pencil for solving of the given problem (see Appendix F.4). The researcher further informed the children that blank paper was available if they felt that it was needed (see Appendix F.1).

4.6.2.3 Assessment of fraction knowledge

During the pilot study phase of the data collection, an assessment of each of the research participant's fraction knowledge was undertaken. This assessment revealed that the research participants selected by the Year 5 class teacher had fraction knowledge in line with the Commonwealth of Dominica's Grade K-4 programme of study (PoS) for mathematics and had only been familiar with the part-whole fraction sub-construct and not with the partitive quotient. When each child was asked if s/he had ever solved problems of the form: 'if three cakes are shared equally among four people and no cake is left over, how much cake (what fraction) would each person receive', in school, they all gave a negative response. The assessment also showed that the high and middle attaining children had a well-formed knowledge of the part-whole and were able to state the fraction associated with a part-whole representation and draw a part-whole representation of a fraction given in the form of $\frac{a}{b}$.

In addition to considering the outcomes of the assessment administered, ethical and time considerations were taken into account in making a decision as to whether this assessment would form part of the main study. While the assessment of fraction knowledge administered during the pilot study confirmed that the Year 5 children selected by the class teacher had only been taught the part-whole meaning of fractions in school, the individual assessments were very time-consuming. Furthermore, the six-week data collection period for the main study, inclusive of approximately one week for contingencies, could only accommodate the solving of the eight partitive quotient problems. The final consideration related to the concern that the assessment could potentially, inadvertently, cause some children anxiety and to be overly concerned about whether their responses were correct, although they were told explicitly that (see Appendix F.1) this was not the focus of the study. Although this anxiety did not appear to manifest during the pilot study, based on this ethical concern and the aforementioned considerations, the assessment was not included as part of the main data collection.

4.6.2.4 Number of research participants

Six children participated in the pilot study. A preliminary examination of the pilot study data showed that these children provided adequate data to explore the research questions under consideration. In order to improve the validity and potential generalisability of some of the findings of this research, however, as well as to provide sufficient viable directions for future research, springboarding from this exploratory research there is a need to increase the number of research participants in the sample. The fine-grained analysis that this research requires, on the other hand, limits the number of research participants that a research project of this size and resource capabilities can manage. Taking into consideration the aforementioned factors, the target number of research participants for the main study was increased to 12 children, with the expectation that there might be some attrition of research participants.

This increase in the number of research participants affects the number of schools included in the research. Research participants for the pilot study were sourced from one government-run primary school in the Western district of the CoD. This school would not have 12 children at the middle- to high-attaining levels needed for the main study and so there was a need to source participants from another school. In this regard, the present researcher approached the principal of a private primary school, also located in the Western district of the CoD, who agreed to participate in the main study data collection.

4.6.2.5 Video recording considerations

For the pilot study, the task-based interviews occurred during the lunch period. As a result, the video recordings contained much background noise. For the main study, desk microphones were to be used in addition to the video equipment used in the pilot study. The present researcher also aimed to remove the background noise on the video recordings to facilitate easier viewing of the data. The pilot study showed that the placement of the video camera was adequate and so this aspect remains unchanged in the main study.

4.6.2.6 Summary of main study data collection

Twelve children, aged nine to ten years, were invited to participate in the data collection for the main study for this research. Of these 12 children, nine (three girls and six boys) completed the main study data collection, during which they solved eight partitive quotient problems over a six-week period. All interviews were video-recorded for later analysis. By the end of the six-week period, there were 70 video recordings and 72 pieces of written work. There are two more written pieces of work than videos, because two video recordings had no audio due to technical difficulties. The duration of each interview depended on how long a child took to solve the given problem for the particular interview, but because of the age of the research participants, associated attention span and ethical considerations, the maximum duration of any task-based interview was 30 minutes.

4.7 Data analysis procedures

There are varied processes and procedures that can form part of the analysis of data collected. In deciding the steps to follow in carrying out the data analysis for this project, the researcher considered the nature of the research questions of this study and the theoretical framework, as well as the procedures adopted by previous empirical research relating to the domain of development of mathematical ideas. One guide that the present researcher examined in regards to this, which is based on over a decade of investigating the development of learners' mathematical ideas and reasoning, is by Powell *et al.* (2003). Their focus of interest, development of mathematical ideas and understanding, fitted that of the current project and so this guide seemed initially appropriate for the present research. In addition to having a similar focus of interest to the present research, the approach of Powell *et al.* (2003) has been used by other researchers in the domain of growth of mathematical ideas and understanding, such as Tzur (2007) and Martin and Towers (2016).

Further to this, an examination of the guide revealed that it can be used flexibly. The guide consists of a sequence of seven interacting phases, but Powell *et al.* (2003) assert that the arrangement of the phases does not prescribe a linear, step-by-step procedure to undertake video data analysis. Instead, it encourages researchers to decide, in the context of their research, the salient ideas to concentrate on, as well as the implementation sequence of the proposed phases. This flexibility appealed to the present researcher because of the exploratory nature of the present study. These phases comprise: viewing attentively the video data; describing the video data; identifying critical events; transcribing; coding; constructing a storyline; and composing a narrative.

Based on the aforementioned considerations, the qualitative data analysis process began as per the first four phases of the data analysis process put forward by Powell *et al.* (2003). The remainder of this section briefly describes the four phases implemented in the present research. This discussion is important to this thesis, because it addresses the issues relating to establishing quality in qualitative research, such as transparency, discussed in section 4.2.1 of this chapter.

During the first phase of viewing the video data attentively, the researcher familiarised herself with the data by viewing each video recording on at least two occasions, without pausing, prior to focusing on any aspects associated with the research questions. The next stage of the analysis of the data involved making a decision regarding transcription. Although Powell *et al.* (2003) include transcription as a phase in the data analysis process, they highlight that there are mixed opinions regarding this issue. On one hand, Maher and Alston (1991) posit that 'careful analysis of videotape transcripts of children doing mathematics enables a detailed study of how children deal with mathematical ideas that arise from the problem situation' (pp. 71–72). Derry *et al.* (2010) also state that the creation of intermediate representations of video records, such as transcripts, with verbalisations, as well as non-verbal information, is essential because such representations allow the researcher to identify which segments to analyse and to begin to see patterns within and across segments. On the other hand, researchers such as Pirie (1996) advise against data transcription of video recordings. She opts instead to work only with videos. She concedes, though, that 'working with the tapes as opposed to transcripts is neither intrinsically better nor worse' (p. 5), and there are distinct advantages and disadvantages to working with both.

For the present research, transcription of the video recordings was a critical step, prior to the application of the analytic framework for data analysis. The main reason for this is in line with Powell *et al.* (2003), who point out that since video data results in a sizeable amount of information, owing to its denseness, for analytical purposes there is a need to transcribe video data. Considering that the data collected for this research project consisted of 72 video recordings from eight task-based interviews with nine research participants, transcribing the videos was an obvious next step in the data analysis process for the present researcher. Seventy transcripts resulted from this process because for two video recordings, due to technical difficulties, there was no audio, so the present researcher was unable to transcribe them. Appendix G presents an excerpt from a transcript from the current research.

In the next phase of the analysis, the researcher briefly described the video data and identified the critical events, which Powell *et al.* (2003) describe as connected sequences of utterances and actions that are somehow significant to a study's research agenda. To describe the video data, at the end of each solution to a task the present researcher created a brief summary of the major aspects of the child's solution as per the research questions. These major aspects included partitioning, distributing pieces and quantifying each person's share.

4.8 Summary

Chapter 4 provided an in-depth description of the methodological approaches and research design that guided the present study. The next chapter is inextricably linked to the present chapter and elaborates on another aspect of the methodological considerations of this thesis. Specifically, it discusses the analytic framework that the present researcher adopted to address the research concerns in this thesis.

Chapter 5: Framework for data analysis

Extending knowledge in any field of endeavour often involves present empirical research, utilising and building on previous research in various ways. In this regard, the analytic frameworks as per Research Questions 1 and 2, discussed in sections 5.1 and 5.2 respectively, not only assist in explicitly facilitating the answering of the two research questions of this thesis, but they also firmly connect the previous empirical literature from Chapters 2 and 3, to the present research data. In so doing, they bring this thesis together into a cohesive whole.

5.1 Research Question 1: Description and application of the analytical framework

The first research question of this thesis consists of three components of inquiry:

- finding the strategies that children used to find the fraction for solving partitive quotient problems;
- finding the strategies that children used to find the fraction for solving partitive quotient problems over a sequence of problem-solving sessions;
- the impact of the part-whole meaning of fractions on the learning of the partitive quotient.

The framework for analysing the data for the first research question, therefore consists of a summary of strategies for finding the fraction from solving partitive quotient problems based on the empirical literature reviewed in Chapter 2. This is presented next in Table 5-1.

Table 5-1 Summary of approaches for finding the fraction from solving partitive quotient problems reported by previous empirical literature in section 2.3

Strategy label	Strategy description from previous literature	Example of strategy from previous literature	Streefland (1991)	Charles and Nason (2000)	Yazgan (2010)	Empson *et al.* (2006)
I	Addition of unit fractions, *1/y*, *x* times, for the problem of share *x* items amongst *y* people	**Problem:** Share six pancakes among eight people **Answer:** *Repeated addition:*	✓	✓	✓	✓

Strategy label	Strategy description from previous literature	Example of strategy from previous literature	Streefland (1991)	Charles and Nason (2000)	Yazgan (2010)	Empson *et al.* (2006)
		1/8 + 1/8 + 1/8 + 1/8 + 1/8 + 1/8 **Variations of strategy I** *Repeated addition and compiling:* 1/8 + 1/8 = 2/8; 2/8 + 1/8 = 3/8 etc. *Adding and compiling while applying equivalences:* 1/8 + 1/8 + 1/8 + 1/8 = 2/4 etc. **Problem:** Share three pancakes among four people **Answer:** ¼ + ¼ + ¼; ¾; ½ + ¼ *When number of items is more than number of people sharing* **Problem**: Share five eggs among four people **Answer:** *Whole + Unit fraction* 1 + ¼ = 1 ¼ *Sum of unit fractions* ¼ + ¼ + ¼ + ¼ + ¼ = 1 + ¼ = 1 ¼				

Strategy label	Strategy description from previous literature	Example of strategy from previous literature	Streefland (1991)	Charles and Nason (2000)	Yazgan (2010)	Empson *et al.* (2006)
II TI/TPe	Number of items/ Number of people	**Problem**: Share ten licorice straps among four people **Answer:** $\frac{10}{4}$	✓	✓	✓	✓
III TPPe/ TPAI	Quantification by a part-whole notion (1) Add the number of pieces given to each sharer (numerator) (2) Add all the pieces from the partitioning of the items (denominator) Total number of pieces given to each person/Total number of pieces in all the items	**Problem**: Share three items among four people **Answer**: $\frac{3}{12}$	×	✓	✓	×
IV	(1) Multiply the number of items by the number of people sharing to find the total number of pieces to be shared (2) Divide this total number of pieces by the number of people sharing (numerator) (3) Number of people involved in the sharing (denominator)	**Problem:** Share five pizzas among four people **Answer**: (20/4)/4 → 5/4	×	✓	✓	×

Strategy label	Strategy description from previous literature	Example of strategy from previous literature	Streefland (1991)	Charles and Nason (2000)	Yazgan (2010)	Empson *et al.* (2006)
V 1/TPPe	1/ number of pieces given to each person	**Problem:** Share one pizza between two people **Answer**: ¼, after sharing the pizza into eight pieces	×	✓	×	×
VI	IN-distractors or whole number bias (1) Add the numerators from unit fractions (2) Add the denominators from unit fractions Sum of numerators/ Sum of denominators	**Problem**: Share three items among four people **Answer:** $\frac{1+1+1}{4+4+4} = \frac{3}{12}$ **Problem**: Share five eggs among four people **Answer:** 1 + ¼ = 2/4 since 1 + 1 = 2 and 4 remains as is	✓	×	×	×

✓: Approach is reported ×: Approach is not reported

The information in Table 5-1 directly facilitates answering the first two components of Research Question 1, related to finding the strategies that the children used to find the fraction for solving partitive quotient problems, in general, and over a sequence of problem-solving sessions. More specifically, the second column of Table 5-1, 'Strategy description from previous literature', brings together under one umbrella the strategies reported in previous empirical research. Using these strategies, which form the set of possible strategies for finding the fraction, as a starting point, the present researcher undertook coding of the present research data. In this regard, she closely examined each strategy used by the present research participants and, if it resembled a strategy recorded in Table 5-1, it was coded thus.

An illustration of the coding of present research data in this manner is presented next. The coding is based on the excerpt of David engaged in solving the problem of sharing four cakes among six children (see Appendix G). In the excerpt, the section highlighted in yellow shows David's verbalisation and actions as he quantified each person's share. David counts the total number of pieces in the two rectangles which represent the cakes and finds this to be six. This six becomes the denominator of the fraction. The numerator of two represents the number of pieces that he gave to each person after partitioning as per the following verbalisation: 'This child gets one' [writes 1 in the first partition of the first rectangle] and 'I'm going to do the same thing I did for the first cake, for the second cake'... 'and then each kid gets another piece.' [Writes 1, 2, 3 in each of the three partitions while speaking.]

Short description of strategy (Fraction): Total number of pieces given to each person/Total number of pieces in all the items

Label: TPPe/TPAI

A close inspection of strategy III in Table 5-1 and David's problem solving shows that there is a match in the ways of working.

While the framework in Table 5-1 served as an initial starting point to code the present research data, in instances where the strategy used by a research participant did not exactly resemble any of the strategies listed in Table 5-1, a new short description and an associated code were created by the present researcher. This represents a key strength of this framework in that, while it provided a springboard from which to approach the present research data, it did not constrain the engagement with the data. An illustration of coding a different strategy than previously reported is as follows:

Short description of strategy (Fraction): Total number of pieces given to each person/Total number of items

Label: TPPe/TI

This coding is based on the excerpt in Appendix G.1, where Jack engages in solving the problem of sharing four cakes among six children.

The application of the aforementioned framework resulted in the identification of four main strategies (see Appendix H for strategy descriptions and associated labels) for finding the fraction associated with quantifying each person's share. This directly answers the first part of Research Question 1 of this thesis. Following from this, the present researcher closely examines the strategies identified, but this is done within each task and across the eight tasks. This provides the data for addressing the second part of the first research question. Section 6.2 of Chapter 6 discusses each of the strategies identified. It also presents associated illustrative extracts of data alongside each strategy described.

In addition to enabling the answering of the first two components of Research Question 1, the framework presented in Table 5-1 facilitates investigating the impact of the part-whole meaning of fractions on the learning of the partitive quotient in two ways. The first way involves a comparison between the approaches used by the research participants of one empirical study in Table 5-1 and the present research participants. In particular, Streefland's (1991) research sample was a group of children who had not been previously taught the part-whole fraction sub-construct, while the research participants of the present study are children who have only been taught the part-whole meaning of fractions. A comparison of the strategies used by the children in the present study and that of Streefland (1991) assisted the present researcher in identifying if and how the part-whole impacts on the learning of the partitive quotient fraction sub-construct by pinpointing differences in how these two samples found the fraction.

A second way in which the analytic framework facilitates addressing whether the part-whole meaning of fractions impacts on the learning of the partitive quotient is that two studies in the framework (Charles and Nason, 2000; Yazgan, 2010) made reference to the part-whole affecting the development of partitive quotient knowledge, although this was not their primary focus. These two studies are key aspects of the framework, because they describe children's way of working when viewing a partitive quotient problem through a part-whole conceptualisation. In so doing, they provide the current research with an initial lens through which to view the workings of its research participants, noting similarities and/or differences in ways of engagement with the problems. This potentially allows the present research to build on and expand the previous empirical research findings.

5.1.1 Summary

Section 5.1 detailed how the present research uses the previous empirical literature summarised in Table 5-1 as an effective and appropriate framework for answering all the components of Research Question 1 of this thesis. The specific findings related to this research question are presented in Chapter 6.

The next section explains how the Pirie-Kieren theory and associated model for growth of mathematical understanding were used as the analytic framework for the data analysis related to the second research question of this thesis.

5.2 Research Question 2: Using the Pirie-Kieren theory and model as the framework of analysis

Chapter 3, which presented the theoretical framework of this thesis, firmly established that a central focus of this research, particularly, Research Question 2, involves the Pirie-Kieren theory and its associated model for the dynamical growth of mathematical understanding.

A key aspect of the Pirie-Kieren model is on an individual's mathematical constructions and the mapping or tracing of his/her development of mathematical images for a particular concept/topic. Two of the four layers of understanding of the Pirie-Kieren model are associated with this notion of images. These two layers are Image Making and Image Having. Before the present researcher used the Pirie-Kieren model to map children’s growth of mathematical understanding, she considered and delineated the issue of images as per the present research. This is in keeping with Davis (1996), who states, ‘a researcher using the model must be aware of the sorts of activities and expressions that might be considered proper to each level’ (p. 203). To facilitate this, the present researcher closely examined several examples of learners working at the Image Making and Image Having layers from previous research associated with the Pirie-Kieren theory. The next section discusses a book chapter by Kieren *et al.* (1999) to show how the present researcher engaged with the issue of images to arrive at the specific images associated with solving partitive quotient problems, which form the unit of analysis through which the elaboration of the DNBs occurs. Appendix I lists all the instances of the term ‘image’ in Kieren *et al.* (1999).

5.2.1 Images as per the Pirie-Kieren theory

Kieren *et al.* (1999) describe three instances of learners engaged in solving mathematics problems. In the first instance, Jo and Kay, two students enrolled in a mathematics teaching methodology course, engaged in solving an arithmagon problem for the given prompt: ‘A secret number has been assigned to the corner of a triangle. Each side shows the sum of the secret numbers at the two vertices. Find the secret numbers’. In the second instance, Stacey and Kerry, fourth-year university students, engage in solving the same arithmagon problem as Jo and Kay. The third instance of problem solving focuses on Kara, an eight-year-old student who is involved in using a half-fraction kit to form various fractions, such as three-fourths.

The remainder of this section discusses the growth of understanding for the individuals engaged in the mathematical problem-solving situations, using the Pirie-Kieren model for growth of mathematical understanding. More specifically, the next sub-section discusses relevant aspects of Jo’s and Kay’s pathways of understanding as presented by Kieren *et al.* (1999) relating to the issue of image(s). Following this, section 5.2.1.2 briefly discusses images in the context of Kara's problem solving. Finally, this section presents the image(s) adopted for the present research.

5.2.1.1 Jo's and Kay's pathways of understanding

The mapping for both Jo and Kay commences in the Image Having layer. As discussed in section 3.4.3, when working within the Image Having layer, a learner possesses a general mental plan or image about a mathematical topic or problem and is able to use it accordingly. In the case of Jo and Kay, from the start of the problem-solving session both students verbalised that they had an image of arithmagons. This image referred to a plan/method/strategy that could be utilised for solving the arithmagon problem. This approach involved solving systems of linear equations. This conceptualisation of image is consistent with Martin and LaCroix (2008) who inform that 'by "images" the theory means any ideas the learner may have about the topic, any "mental" representations, not just visual or pictorial ones' (p. 123). The term 'idea' in this explanation is decidedly broad and so could be potentially confusing. However, an examination by the present researcher of literature associated with the Pirie-Kieren theory for growth of mathematical understanding associated with the terminology 'image' revealed that 'idea', in this context, could mean a visual representation, a definition/description/explanation of a mathematical concept or an approach for finding the answer to a mathematical problem.

It is important to note though, that images about a mathematical topic, concept or problem are not necessarily complete or correct. In the case of Jo and Kay, even though both individuals had an image of arithmagons or knew that a system of linear equations should be used to solve the arithmagon problem, Kay possessed a more complete image of how to apply the solving of linear equations to the specific problem. Jo, on the other hand, worked with pairs of equations in a local, non-systematic way. This suggests that even though the two students have the same overall image or are applying the same approach to the given problem, there may be variations in the use of the image or application of the approach.

The use of image to mean a mental plan or a particular strategy/ approach for solving a given problem is in line with other literature associated with the Pirie-Kieren theory for growth of mathematical understanding. In more recent research, Martin and Towers (2014) give several examples of Image Making and Image Having associated with three Grade 6 students working on a problem of calculating the area of a parallelogram, a figure with which they had not worked previously. They state:

> No single student seems to have a ready-formulated image for the area of a parallelogram (such an image might be demonstrated through saying something like 'turn it into a rectangle and multiply the length and height'). Instead, Natalie begins the process of *Image Making*—she is exploring, playing with the shape, to see what might be a useful way to proceed. (p. 7)

In this example, image can clearly be taken to mean an approach for finding the area of a parallelogram. Further to this, for the same students, Martin and Towers (2014) distinguish between Image Making and Image Having. They state:

> We see this transitioning occur in the extract above – they now have an idea that they can use to solve the problem: find the area of the inner 'square' and multiply it by two. They are no longer trying to find an approach that might work; instead, they think they have one that is now ready to be applied and they continue working with this image, using it to calculate the area of the parallelogram. (p. 9)

From the previous quote, the term 'idea' appears to be used in place of image, initially. Nevertheless, it is clear that 'image' here refers to an approach as to how to find the area of a parallelogram.

5.2.1.2 Kara's pathways of understanding

For the case of Kara, Kieren *et al.* (1999) state that 'Kara is observed to *have an image* of fractions in that she can read simple fraction language, create fractional symbols, and can solve problems using halves and fourths' (p. 229). Consequently, the mapping for Kara's growth of understanding begins in the Image Having layer. This explanation of 'image' seems to be different from that of Jo and Kay. 'Image' here means a definition/description/explanation/representation of a mathematical concept – in this case, a fraction. This conceptualisation of image, though different from that of Jo and Kay, is still consistent with the explanation of Martin and LaCroix (2008) that '"image" means any ideas the learner may have about the topic, any "mental" representations, not just visual or pictorial ones' (p. 123).

5.2.2 Images for the present research

For the present research, nine Year 5 children engaged in solving partitive quotient problems of the type: 'Share *two* cakes/pizzas fairly among *three* children. How much (*fraction*) cake does each child receive'? After the children offered one solution, the present researcher asked: 'How else can you share the cakes/pizzas among the three children? How much does each person receive from the sharing'? This section discusses the image(s) used for the present research.

An inspection of the video recordings and the written work of the research participants by the present researcher shows that the children typically engaged with each of the partitive quotient tasks as per the excerpt below. In this excerpt, one research participant, Kenny, shares two cakes fairly among three children.

Excerpt 5-1

> **Researcher:** Here is our task for today. [Researcher reads from the task sheet.] Share two cakes, two rectangular cakes among three children so that each child gets the same amount of cake and no cake is left over. How much cake would each child get if each person gets the same amount of cake and no cake is left over?
>
> **Kenny:** There are two cakes. You have to share them among three children, so you cut the cake into three [partitions the first rectangle into three]. So that means one child would get, one child would get this piece, another child would get this and the third child would get this one [points on each of the three partitions of the first rectangle in turn]. And then, you do the same [partitions the second rectangle into three], to the second cake so that they each get two pieces. So the first child would get this piece, the second would get this piece, the third child would get this piece. So that means there... each child takes two cakes, gets two cakes. Two cakes.
>
> **Researcher:** Two cakes. In terms/ [/: Both persons in the interview speak at the same time.]
>
> **Kenny:** /Two pieces of cakes.
>
> **Researcher:** In terms of fractions, not pieces of cakes, what would that be?
>
> **Kenny:** A child would get two-sixths. [Writes 2/6.] Two-sixths. Out of the six pieces, one child would get two.

The excerpt above shows three aspects of the problem-solving process: partitioning, which is the process of dividing (mentally or otherwise) continuous objects such as cakes or pizzas into a number of disjoint and exhaustive parts, distributing a particular number of partitions to each person, and then stating the fraction amount that each person receives.

As stated earlier, quite broadly, 'image' as per the Pirie-Kieren theory 'means any ideas the learner may have about the topic or problem, any "mental" representations, not just visual or pictorial ones' (Martin and LaCroix, p. 123). An examination of Kieren *et al.* (1999), as well as other literature associated with the Pirie-Kieren theory, suggests that the conceptualisation of image that seems most suitable to the present research study is in line with the interpretation put forward in explaining the growth of understanding of Jo and Kay as they engaged with the arithmagon problem. This is the approach for finding the answer to a mathematical problem. Consistent with the scenarios involving Jo and Kay, the ways in which children in the present research worked with the same images differed.

More specifically, while children's solving of the partitive quotient problems showed three distinct aspects, a close examination of the research data by the present researcher revealed that children in this study appeared to use different images to find the number of partitions to share the items, depicted as rectangles. Four such images are presented next, followed by associated illustrative excerpts:

- Number of partitions in each rectangular diagram = Number of people sharing
- Number of partitions in each rectangular diagram = Multiple of the number of people sharing
- Number of partitions in each rectangular diagram = Number associated with the half family
- Number of partitions in multiple rectangular diagrams = Number of people sharing.

Excerpt 5-1, in which Kenny shared two cakes fairly among three children, illustrates the first enactment of this image: 'Number of partitions in each rectangular diagram = Number of people sharing'. The next excerpt of Samuel solving the same problem illustrates 'Number of partitions in each rectangular diagram = Multiple of the number of people sharing'. In this very short excerpt, Samuel shares two cakes among seven people.

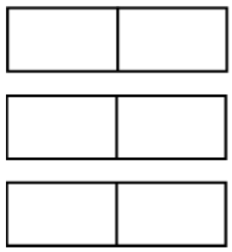
Excerpt 5-2

Samuel: If I divide each cake into twenty-one pieces, … in the first cake, child one or A will get three pieces because seven can go into twenty-one three times.

Excerpt 5-3 illustrates 'Number of partitions in each rectangular diagram = Number associated with the half family' as Harry shares three pizzas fairly among six people. It also exemplifies 'Number/Sum of partitions in multiple rectangular diagrams = Number of people sharing'.

Excerpt 5-3

Harry: Miss is three pizzas and six child – six people. Then you can cut, share them in halves. [Partitions each diagram in two.]



In addition to the different images that children held to find the number of partitions to share the items, these images appeared to develop as they solved one problem in several ways and eight similar problems in eight task-based interviews. The next two excerpts illustrate this as Mary shares three cakes among five children. Excerpt 5-4 presents Mary's first solution to the problem.

Excerpt 5-4

SOLUTION 1

Mary: I think I could try fifteen [referring to the total number of partitions for all three diagrams]? Because three fives a fifteen. So each cake would have five. [Mary partitions each of the three diagrams into five while counting aloud.]

For this first solution illustrated in the excerpt above, Mary multiplies the number of items, three, by the number of people sharing, five to obtain a total number of pieces of 15. The following excerpt presents one of Mary's subsequent solutions to the same problem of sharing three cakes among five children.

Excerpt 5-5

SUBSEQUENT SOLUTIONS

> **Mary:** Trying to find a different way of getting, five children a piece at least more than one slice of three cakes. [Long pause while staring ahead intently.]
>
> **Mary:** I think thirty.
>
> **Researcher:** Tell me why you chose thirty.
>
> **Mary:** Because since fifteen can work, fifteen times two is thirty.
>
> ………………………………. [There is an exchange, not relevant to the illustration, separating the two parts of the excerpt.]
>
> **Researcher:** Do you think there are other ways that you could share the cake?
>
> **Mary:** Well, if I had to share the cake again I would probably add fifteen again.
>
> **Researcher:** Okay. You would add fifteen to what?
>
> **Mary:** Thirty. Which would be forty-five.

Excerpt 5-5 shows that Mary uses two approaches or images to find the total number of partitions. She multiplies 15 by two to get 30 and adds 15 to 30 to obtain 45. Excerpts 5-4 and 5-5 clearly depict that Mary's understanding is developing and changing as per the number of partitions to share each diagram as she engages in solving the problem. It also shows that her image for finding the number of partitions for the first solution was distinctly different from that used to find the number of partitions associated with the items in subsequent solutions.

The variations in images used by the research participants and the varying developments of these images allow for the exploration of DNB crossings from multiple perspectives. This bodes well for this thesis because, although this research is exploratory in nature, it provides richer data with which to elaborate this feature of the Pirie-Kieren theory.

Research Question 1 focused on the fraction aspect of the solving of the partitive quotient problem and so the focus of the data analysis for Research Question 2 is the notion of image associated with partitioning, particularly, the approaches the children used to find the number of partitions they would share the items (cakes/pizzas).

5.2.3 Using the Pirie-Kieren model as an analytic tool

Once the decision was taken to focus on the images associated with the number of partitions each item is shared into, the research data was again reviewed for the purpose of explicitly studying the DNB phenomenon of the Pirie-Kieren theory. To faciliate this, using the transcripts, along with associated video recordings, for each of the eight task-based interviews, the present researcher coded the actions and verbalisations of the research participants according to the first four layers of the Pirie-Kieren model: Image Making; Image Having; Property Noticing; and Formalising. Table 5-2 shows the actions and verbalisations that the present researcher identified that were characteristic of operating within each of the first four layers of the Pirie-Kieren model, for the present research, for the given topic of partitive quotients.

Table 5-2 Examples of actions and verbalisations from the present research characteristic of the Image Making–Formalising layers of the Pirie-Kieren model

Layer	Actions and verbalisations associated with each layer
Image Making (IM)	**Partitive quotient problem specific** • Partitioning analogs with lines drawn on analogs • Partitioning motion with fingers/pencil/head • Counting, recounting partitions • Inspecting partitioned analogs • Distributing pieces of cake/pizza by finger motions from the analog to the picture of the people, drawing lines from partitions to people sharing, shading partitions, numbering/labelling partitions **General** • Spends an extended period of time staring ahead or looking at the task sheet • Speaks extremely slowly and haltingly • Lacks confidence in verbalisations and actions (e.g. counting, partitioning)

Layer	Actions and verbalisations associated with each layer
	• Unable to give an explanation or a coherent explanation for the choice of # of partitions or method of solving • An explicit statement from the learner that s/he is Image Making such as: 'I am exploring, I am just trying this to see, I am not sure if this will work, I do not have a way to share the cakes'
Image Having **(IH)**	**Child:** 'I know the answer already' or 'I know how to share the cake'. This suggests that they have an approach as to how to partition the items, which they go on to detail. They may also respond immediately to the question posed and state for example, for the problem of sharing three cakes among five children: 'I am sharing each of my cakes into five pieces'. ***Images*** • Number of partitions in each analog = Number of people sharing • Number of partitions in each analog = Multiple of the number of people sharing • Number of partitions in each analog is divisible by number of people sharing • Number of partitions in each analog = Number associated with the half family • Number of partitions in multiple analogs = Number of people sharing • Number of partitions in a subset of analogs = Number of people sharing • Number of partitions in each analog = Number of items multiplied by the number of people • Total number of partitions in all the analogs = Number of items multiplied by the number of people sharing • Partitions analogs into half and distributes the parts. The remaining partitions are partitioned into the number of people sharing
Property Noticing (PN)	• When there are more items than people, each person gets a whole item • When the number of items and people sharing are even, the number of partitions in each analog is halved

Layer	Actions and verbalisations associated with each layer
	• For subsequent solutions, the number of pieces gets smaller and increases in number • Even though the number and size of the pieces each person receives may differ for each solution to a given problem, each person gets the same/equivalent amounts from the different sharings • Linking *first solution* to *subsequent solutions*: Number of partitions = Multiples of number of people sharing • When the number of people sharing is the same in different problems, the same number of partitions can be used.
Formalising **(F)**	A child states his method in a general or broad way, *not linked to a specific example*. For example, he may state, 'For any problem I can keep sharing the cake into multiples of people' or 'we can go on and on if we keep adding our first number' or 'to get the number of pieces in all your items for your first solution multiply the two numbers in the problem'.

5.2.4 Creating mappings of children's understanding as per the Pirie-Kieren model

Each occurrence where the present researcher coded a learner's actions and/or verbalisations at a layer of understanding, as per the Pirie-Kieren model, is called a point or mapping point. It is important to note that a point does not connote a single word or action. It represents an episode of varying length, whether several minutes or mere seconds, which portrays the characteristics of a particular layer of understanding. Each learner's growth of understanding consisted of different numbers of mapping points, depending on their engagement with the partitive quotient problems.

The collection of mapping points that show the trace of a student's growth of understanding on a diagram of the Pirie-Kieren model is called a mapping diagram, or mapping (Towers, 1998). The present researcher created a mapping diagram for each learner's growth of understanding over the eight tasks. Table 5-3 is an annotated table with the mapping points for Mary's growth of understanding over the eight task-based interviews. Appendix J presents the associated transcript and researcher explanations for the coding for T02, T03 and T04, which refer to the second, third and fourth task-based interviews, respectively.

Table 5-3 Mapping diagram for Mary's growth of understanding

T01 2/3	T02 4/3	T03 3/5	T04 4/6	T05 2/7	T06 3/6	T07 2/5	T08 3/8
IM (Sol 1) IM (Sol 2) IM (Sol 3) IM (Sol 4)	IM (Sol 1) IM (Sol 2) IM (Sol 3)	IH (Sol 1) DNB First solution IM (Sol 2) IH (Sol 3) DNB subsequent solution IH (Sol 4) PN F (General) DNB First solution	F (Sol 1) F (Sol 2) DNB subsequent solution F (General)	F (Sol 1) F (Sol 2) F (Sol 3)	F (Sol 1) F (Sol 2) F (Sol 3)	F (Sol 1) F (Sol 2) F (Sol 3) F (Sol 4)	F (Sol 1) F (Sol 2) F (Sol 3)

Key :

IM: Image Making
IH: Image Having
PN: Property Noticing
F: Formalising
Sol: Solution
DNB: Don't Need boundary

An examination of the mapping diagram for Mary's growth of understanding in Table 5-3 shows that there are four DNB crossings. Two of these are associated with the first solution and two with subsequent solutions. This information is summarised in Table 5-4.

Table 5-4 Mary's DNB crossings details

Solution	Layer of Pirie-Kieren model	Task-based interview
First	Image Making to Image Having *(yellow highlight)*	T02→T03 (Sol 1)
	To Formalising *(aqua highlight)*	T03
Subsequent	Image Making to Image Having *(green highlight)* To Formalising (grey highlight)	T03 T04

The nine children in this research study displayed different pathways of growth of understanding and so the mapping diagram in Table 5-3 is not representative of all the research participants' growth of understanding. The presentation of the research data in Tables 5-3 and 5-4 and the associated transcript in Appendix J, however, serves several purposes. First, it presents one example of the present researcher's assigning of children's actions and verbalisations as per the Pirie-Kieren model and the associated reasoning. In this regard, this addresses some of the issues of transparency and validity for this research. This also allowed others, such as the present researcher's supervisors to review and comment on the validity of the researcher's mappings and explanations. Third, this example shows examples of actions and verbalisations that are characteristic of each of the four layers of understanding of the Pirie-Kieren model. It also presents several instances of different DNB crossings, for example, from Image Making in T02 (Sol 1) to Image Having in T03 (Sol 1) highlighted in yellow and within a task, for example, from Image Making (T03, Sol 2) to Image Having (T03, Sol 3) highlighted green.

A further examination of the mapping diagram in Table 5-3 shows that it does not exactly resemble Pirie-Kieren's model for growth of mathematical understanding. Similar to Towers (1998) who elaborated teacher intervention of the Pirie-Kieren theory, when the present researcher began mapping each learner's growing understanding on a standard Pirie-Kieren model she quickly found that the layout was unsuitable for the number of points that there were and the details/annotations that she was including. Further to this, because of the structure of the tasks it was necessary to look at a learner's progress within a task and across tasks and the "factor of time is not easily represented on the standard mapping diagram" (Towers, 1998, p. 111). The present researcher therefore used a tabular layout similar to that of Towers (1998) as a first stage in recording the mapping points of the learners. It is important to note that these representations are for the purpose of analysis and did not in any way modify or affect the present researcher's own conception of the embeddedness of the layers within the Pirie-Kieren model.

Using the aforementioned approach explicated in Sections 5.2.3 and 5.2.4, the present researcher identified the DNB crossings from the research data as per Table 5-5. The instances of folding back associated with DNBs were also recorded.

Table 5-5 Number of instances of DNB crossings and folding back after crossing a DNB in the research data

Research participants	**Number of instances of:**				
	Crossing first DNB	**FB from the IH layer**		**Crossing second DNB**	**FB from the Formalising layer**
FIRST SOLUTION					
Harry	1	0		NA	NA
Jack	1	0		1	0
Rebecca	2	0		1	1
Mary	1	0		1	0
Gabriel	1	1		NA	NA
David	1	0		1	0
Karen	1	0		1	0
Kenny	0*	1		1	0
Samuel	0*	0		1	0
SUBSEQUENT SOLUTIONS					
Harry	2	4		NA	NA
Jack	NA	NA		NA	NA
Rebecca	2	2		NA	NA
Mary	1	0		1	0
Gabriel	9	8		NA	NA
David	4	2		2	2
Karen	6	4		NA	NA
Kenny	2	1		1	0
Samuel	2	2		1	0

NA: The child did not operate at the designated layer

*: Operated in the IH layer from T01

Following the identification of the DNB crossings, the present researcher examined each in detail and reviewed the contexts surrounding each crossing, by reviewing the associated transcripts and video recordings. This fine-grained analysis resulted in the findings presented in Chapter 7.

Chapter 6: Children's strategies for finding the fraction when solving partitive quotient problems

6.1 Overview

Chapter 6 focuses on answering the research question: 'What strategies do Year 5 children who have only been taught the part–whole fraction sub-construct use:

(i) to find the fraction associated with solving partitive quotient problems?
(ii) to find the fraction associated with solving partitive quotient problems in a sequence of problem-solving sessions?'

It is organised in three main sections. First, section 6.2 discusses the four main strategies used by the research participants of this study to find the fraction associated with solving the partitive quotient problems. This is followed by sections 6.3 and 6.4, which present findings that deal with strategy change within and across the task-based interviews, respectively.

6.2 Strategies children use to find the fraction associated with solving partitive quotient problems

Empirical research studies on different topics within the domain of mathematics education have consistently reported that children do not have just a single way of thinking about a particular class of problems. Instead, they typically utilise a whole collection of strategies (Siegler, 1991). Consistent with these findings, the present researcher identified that the children in this study used four main strategies to find the fraction associated with quantifying each person's share for the various partitive quotient problems. Each of these strategies is discussed in sections 6.2.1–6.2.4. Each section begins by presenting illustrative excerpts from the research data to exemplify each of the four strategies identified by the present researcher. Following this, the characteristics of each strategy, based on the exemplars from the data presented, are detailed. In all presented extracts of problem-solving interactions in this chapter, the assigned pseudonym represents the child's input and 'Researcher' is used to denote the researcher's questions. The strategies reported in this section include all strategies used by the children in solving the problems, whether the resulting fraction was correct or incorrect. Ginsburg *et al.* (1998) and Steffe and Olive (2010) agree that key insights into children's understandings can be gleaned from children's incorrect answers to problems. Different aspects of this particular research finding were presented at a British Society for Research into Learning Mathematics (BSRLM) research conference and at Spring School at the University of Wuerzburg in Germany, respectively (George, 2016a; 2016b).

6.2.1 Total number of pieces given to each person/Total number of pieces in all the items (TPPe/TPAI) strategy

The first strategy presented in this section is that of the 'Total number of pieces given to each person/Total number of pieces in all the items' or the TPPe/TPAI strategy. The six excerpts presented in this section, illustrate the different ways in which the research participants used this strategy in this study.

In the first two excerpts in this section, David and Kenny find the fraction that each person would get when five children share three cakes fairly. In each of the excerpts, the children have already partitioned each of the diagrams into five.

Excerpt 6-1

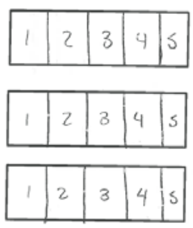
David: Now each child has gotten three pieces [from each diagram]. So it would, each child would get three-fifteenths, 'cause there are three cakes and I shared them equally into five parts and so five plus five plus five or five times three, it would be fifteen. And then each child got three pieces, so it'd be three-fifteenths.

Researcher: Put your answer, please.

David: [Writes 3/15.]

Excerpt 6-2

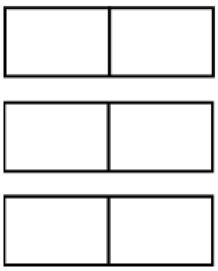
Kenny: So each child gets three pieces. And then the fraction it would be three, each child gets three out of twenty-three slices. No! Out of the fifteen pieces. Three out of the fifteen pieces. [Writes 3/15.]



From the two excerpts above, David and Kenny find the fraction, after having partitioned each of the items into the same number of partitions, by finding the numerator and denominator separately. The numerator of the fraction corresponds to the number of pieces that each person gets from each of the three diagrams. Children's verbalisations produce the evidence of this: 'Now each child has gotten three pieces' and 'So each child gets three pieces' and the subsequent placement of three as the numerator of the fraction for both David and Kenny. The children then find the total number of pieces in all the items. David's verbalisation, 'so five plus five plus five or five times three, it would be fifteen' confirms this. In the case of David, he states, 'Out of the fifteen pieces'. This 15 is then designated as the denominator of the fraction.

In the next excerpt, Gabriel quantifies the share associated with sharing three pizzas among six children.

Excerpt 6-3

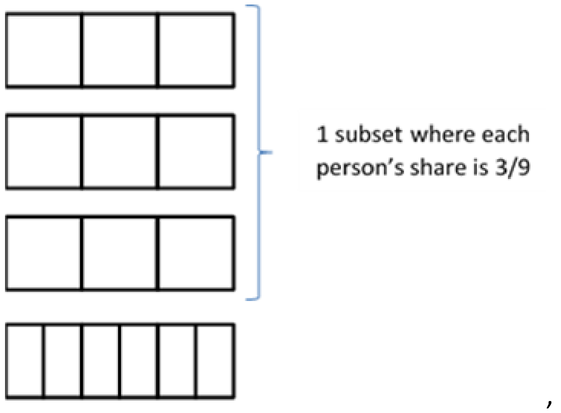
> **Gabriel:** Okay. So if there's six people and there's three [referring to pizzas], so if you cut them in half every, everybody would get enough, well –. Let me see if you cut them in half. [Partitions each rectangular diagram in two.]
>
> 
>
> Everybody will get one piece. Because [looks at partitioned diagrams], because, am, three is half of six and then three of these and there's six people so if you cut them in half each person will get one. And then the fraction would be... [looks at partitioned diagrams], one out of two. [Writes 1/2.] Half for one cake and out of three cakes it would be one out of six. [Writes 1/6.] One out of three because is three cakes. Wait! No, is six pieces. So it's six.

From the excerpt above, Gabriel finds the total number of pieces each person receives, verbalised by 'so if you cut them in half each person will get one'. This number is taken to be the numerator of the fraction amount associated with each person's share. He then writes his fraction as 1/6, where six represents the total number of pieces in all the items, based on the verbalisation ‘No, is six pieces. So it's six.’

In the next excerpt, Samuel finds the fraction associated with each person’s share when solving the problem of sharing four cakes among three children. He has already partitioned three of the diagrams into three and the fourth diagram into six pieces, as per the diagram in the excerpt.

Excerpt 6-4

> **Samuel:** So each child will get [Looks at diagrams and moves head slightly. Moves pencil slightly in the air] if, well if I count [taps pencil in various partitions in each of the three diagrams] in these three cakes how I divide them, I would count each piece separately. One, two, three, four, five, six, seven, eight, nine [taps pencil in each of the partitions in each diagram]. So if – when I count all the pieces together in these three cakes each child would get three over nine. [Writes 3/9.] Aaand also… One over– Two over, two, four, six. Two over six. [Writes 2/6 next to 3/9.]



From the excerpt above, the fraction amount that each person receives consists of two fractions. For the fraction of 3/9, Samuel finds the denominator of nine by counting ‘all the pieces together in these three cakes’. The numerator of three represents the number of pieces that each person receives.

In the following excerpt, Samuel finds the fraction associated with each person’s share when solving the problem of sharing three cakes among five children. He has already partitioned the diagrams as per Figure 6-1.

Excerpt 6-5

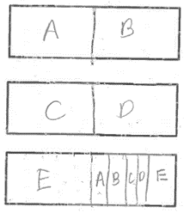


Figure 6-1 Samuel's partitioning for sharing three cakes among five children

> **Samuel:** So each child, if I count it [referring to the halves] altogether, each child would get, one out of five. [Writes 1/5.] And if I continue the pattern [partitions with very faint dotted lines the first half of the third diagram into 5], five, three, four. If I continue the pattern it would be two, well, one,... I would, one, two, three, four, five, six, seven, eight, nine, ten [writes 1/10]. But in this case it cannot be ten, because this five [points to the first half of the third diagram] has already been shared, so it would be one out of five. [Writes 1/5 above the 1/10]

From the excerpt above, the fraction amount that each person receives, similar to Excerpt 6-4, consists of two fractions. For the fraction 1/5, associated with the halves, the numerator of one refers to the number of pieces of cake each person receives. The denominator of five corresponds to the five halves in the three diagrams. This is evidenced by Samuel's verbalisation 'if I count it [referring to the halves] altogether, each child would get, one out of five.'

In Excerpt 6-6, Harry finds the fraction associated with each person's share when solving the problem of sharing two cakes among seven children. He has already partitioned the diagrams as per Figure 6-2 below.

Excerpt 6-6

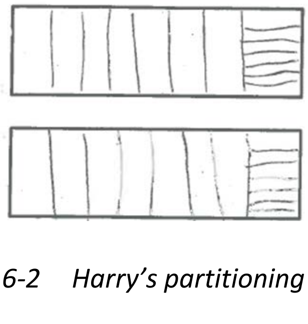


Figure 6-2 Harry's partitioning for sharing two cakes among seven children

> **Harry:** So each child would get two... [Looks at diagrams], two-sixteenths. 'Cause that and that [points to the each of the two diagrams in turn] is sixteen pieces. Two-sixteenths and two-fourteenths. [Writes 2/16   2/14.]

From Excerpt 6-6 above, similar to Excerpts 6-4 and 6-5, the fraction amount that each person receives consists of two fractions. For the fraction 2/16, the numerator of two refers to the number of pieces of cake that each person receives. The verbalisation 'So each child would get two' confirms this. The denominator of sixteen corresponds to the sum of the total number of partitions in each of the two diagrams, before re-partitioning the last partition in each into seven pieces. This is confirmed by Harry's verbalisation: 'Cause that and that [points to each of the two diagrams in turn] is sixteen pieces.' Harry finds the fraction of 2/14 in a similar way to the 2/16.

An examination of the six excerpts presented above shows that at least one of the fractions representing each person's share corresponds to: $\frac{Total\ number\ of\ pieces\ given\ to\ each\ person}{Total\ number\ of\ pieces\ in\ ALL\ items}$. An incorrect fraction corresponding to each person's share results from every application of this strategy. As it relates to this strategy, the excerpts above show that various children used this strategy regardless of the task characteristics, such as if the number of items were more or less than the number of people sharing. In addition, this strategy was used for different types of partitioning.

6.2.2 Total number of pieces given to each person/Total number of pieces in one item (TPPe/TPOI) strategy

Another strategy that some of the research participants in this study used for quantifying each person's share in the partitive quotient problems is the 'Total number of pieces given to each person/ Total number of pieces in one item' or the TPPe/TPOI strategy. The five excerpts presented in this section illustrate the different ways in which this strategy was used.

In the following excerpt, Samuel quantifies the share involving sharing two cakes among seven people.

Excerpt 6-7

> **Samuel:** If I divide each cake into twenty-one pieces, … in the first cake, child one or A will get three pieces because seven can go into twenty-one three times. So each child gets three over twenty-one. [Writes 3/21.] But there are two cakes which I am still going

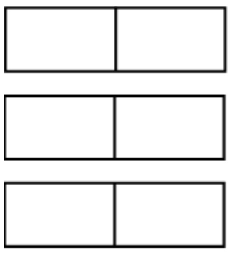
to share into twenty-one pieces and each child gets three over twenty-one. [Writes 3/21 next to the first 3/21.]

From the excerpt above, the fraction amount that each person receives consists of two identical fractions. The numerator of three corresponds to the number of pieces each person receives. This is confirmed when Samuel states 'in the first cake, child one or A will get three pieces'. The denominator of 21 represents the number of pieces that each diagram has been partitioned into. Samuel's verbalisation 'If I divide each cake into twenty-one pieces' and subsequent designation of 21 as the denominator of the fraction confirm this.

In the following excerpt, Harry quantifies the share involving sharing three pizzas among six people.

Excerpt 6-8

Harry: Miss is three pizzas and six child – six people. Then you can cut, share them in halves. [Partitions each diagram in two.]



And then each person would get one. One-half. [Writes ½.]

In the excerpt above, Harry does not explicitly separate numerator from denominator, but presents the quantification of each person's share as a fraction amount of one-half. The one corresponds to the number of pieces each person would get, confirmed by the verbalisation 'And then each person would get one'. The present researcher surmises that the denominator of two represents the number of partitions in a diagram, since there is no alternative explanation as to what the two could represent. In addition to this, Harry's problem solving for other tasks confirms the use of this strategy for quantifying each person's share.

In Excerpt 6-9, Jack quantifies the share involving sharing four cakes among three people.

Excerpt 6-9

Researcher: Sooo... how much cake does each child get?

Jack: [Long pause while looking intently at the diagrams.] Each child, each child gets fourrr... thirds of a cake. [Writes 4 3rd.]

………………………………. [There is an exchange, not relevant to the illustration, separating the two parts of the excerpt.]

Researcher: Could you tell me how you got that four-thirds?

Jack: Because one piece of cake is for one child, so that is one; another piece of cake for the same child [points at the first partition in the first and second diagrams followed by the first child, in turn] so that is two; another piece for that child, so that is three. Another piece for the same child, [points inside the unpartitioned third and fourth diagrams followed by the first child, in turn]. That is four of the ... threeee am pieces.

From the excerpt above, Jack distributes four pieces of cake to each person. He states, 'one piece of cake is for one child… another piece of cake for the same child,…, another piece for that child,… Another piece for the same child. That is four'. In the excerpt, Jack notes that there are three pieces in each cake by the verbalisation 'of the... threeee am pieces'. This corresponds to the part of his fraction that he labels as 3rd.

In the next excerpt, Karen quantifies the share related to sharing four cakes among three children.

Excerpt 6-10

Karen: Okay. I'd give them the same whole cake and this time [looks at the fourth diagram and places pencil inside the diagram] Hmmm [looks at the diagrams]. Each of them would get eight pieces [looks at the diagrams]..., twelve piec – [looks at the diagrams, then ahead and puts up three fingers in turn while lips move silently]... four pieces [looks up].

Researcher: How did you decide to give each four pieces?

Karen: For the bottom [referring to how the fourth diagram would be partitioned] I'd put twelve. And then that would be four times three so I'd give them four pieces out of the twelve.

Researcher: So what fraction would that–-.

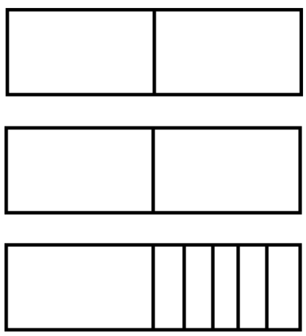
Karen: One out of... one out of [stares ahead] one and... one and four, four-twelfths. [Writes **1** 4/12.]

From the excerpt above, the fraction amount that each person receives is a mixed fraction. For the proper fraction, 4/12, of this mixed fraction, the numerator represents the number of pieces that Karen gives each person as per the verbalisation 'so I'd give them four pieces'. The denominator of 12 represents the total number of pieces in the fourth diagram as per the highlighted part of the verbalisation: 'so I'd give them four pieces out of the twelve'.

In the next excerpt, Harry finds the fraction associated with each person's share when solving the problem of sharing three cakes among five children. He has already partitioned the diagrams as per diagram at the start of the excerpt.

Excerpt 6-11

Harry:



Then each child would get one-half. And it would still remain one, one more [referring to one-half partition]. So you can share them in fifths. [Partitions one-half of a diagram into five.] And then each person would get a half and one-fifth. [Writes 1/2 1/5.]

From the excerpt above, the fraction amount that each person receives consists of two fractions. One of the fractions, 1/5 corresponds to the half that Harry partitioned into fifths. An inspection of the verbalisation suggests that one is the number of pieces each person receives and the five corresponds to the number of pieces in that section of the analog. This use of this strategy results in an incorrect fraction corresponding to each person's share.

From all of the excerpts above, at least one of the fractions which represents each person's share corresponds to: $\frac{Total\ number\ of\ pieces\ given\ to\ each\ person}{Total\ number\ of\ pieces\ in\ one\ item}$. This strategy in all but one instance (see Excerpt 6-11) resulted in a correct fraction corresponding to each person's share. For the fraction 1/5 obtained in Excerpt 6-11, although the total number of pieces corresponds to the partition that has been partitioned again and not to 'Total number of pieces in one item' as per the designation for this strategy, the researcher has chosen to include it within the TPPe/TPOI strategy. An inspection of the excerpt, as well as other instances where this approach is used, shows that Harry appears to treat the partition that has been partitioned again as a separate item and quantifies the share accordingly.

6.2.3 1/Total number of pieces given to each person from the items (1/TPPe) strategy

A third strategy that was used by one research participant for quantifying each person's share is the '1/Total number of pieces given to each person from the items' or 1/TPPe strategy. Excerpt 6-12 illustrates this strategy. In this excerpt, Rebecca solves the problem of sharing three cakes among five children by first partitioning each of the three diagrams into 10.

Excerpt 6-12

> **Researcher:** So in terms of fractions could you tell me how much cake each child would get?
>
> **Rebecca:** Each child would get a sixth.
>
> **Researcher:** Please, tell me how you came up with a sixth?
>
> **Rebecca:** Okay, well... I was thinking of ten for each and then – oh! Because if every child gets five, half, half of ten is five so if every child gets five plus the one from this half, it would be six. And in fraction is a sixth, a sixth. [Writes 1/6.]

From the excerpt above, for the denominator of six for the fraction 1/6, Rebecca states, 'five plus the one from this half, it would be six. And in fraction is a sixth, a sixth'. This six therefore appears to represent the total number of pieces that each person receives from all the diagrams. Rebecca assigns the numerator of the fraction as one.

This strategy was the only one used by this research participant on each of the eight task-based interviews. In every instance where this strategy was applied, an incorrect fraction corresponding to each person's share results.

6.2.4 Total number of pieces given to each person/Total number of items (TPPe/TI) strategy

The fourth strategy for quantifying each person's share is the 'Total number of pieces given to each person/Total number of items' or the TPPe/TI strategy. Similar to the '1/Total number of pieces given to each person from the items' strategy, this was used by one research participant and is illustrated in the next excerpt. In this excerpt, Jack explains how much or the fraction that each child would receive from sharing four cakes among six children. Before this exchange occurs, Jack partitions each of the four cakes into six.

Excerpt 6-13

> **Researcher:** So how much would this child [points to a picture of a child on the paper] for example get? How much cake?
>
> **Jack:** Miss he would get four-fourths of a cake Miss.
>
> **Researcher:** Could you show me on your diagram how you got this?
>
> **Jack:** Miss is one child so it – one piece for the same child, another piece for the same child, another piece for the same child, another piece for the same child [points to the first partition of each diagram in turn]. So that is four altogether. So the four, fours – the four-fourths is because – why I say that is because there are four rectangular pieces of cake, four rectangular full cakes there.

From the excerpt above, Jack obtains the four in the fraction amount of 'four-fourths' by finding the total number of pieces that each person receives. This is evidenced by the verbalisation: 'So that is four altogether. So the four.' The fourths appear to correspond to the total number of items being shared based on the verbalisation: 'fourths is because – why I say that is because there are four rectangular pieces of cake, four rectangular full cakes there.' In every instance when this strategy is applied, an incorrect fraction corresponding to each person's share results.

6.3 Strategy change within a given task-based interview

In addition to identifying the strategies children use to quantify each person's share in partitive quotient problems, this research is also interested in examining children's strategy use, intra- and inter-problem-solving sessions. The analysis in this regard revealed that the majority of the children used the same strategy for finding the fraction within a particular task-based interview. The next excerpt is an example of this, when David finds the fraction for each person's share on three different occasions when solving the task of sharing two cakes among three children fairly.

Excerpt 6-14

SOLUTION 1

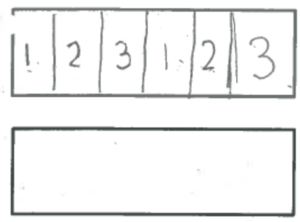
David: So then, each person would get [counts quietly pointing to each partition in turn with the pencil], one, two, three, four, five, six, so it would be two out of six. So two-sixths.

Researcher: Go ahead and write your fraction here.

David: [Writes 2/6.]

SOLUTION 2

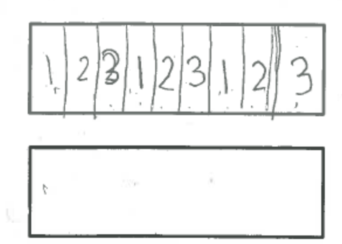
David: So, I'm going to do it with six this time.



Six,... um, let's say, each cake is going to, so I'm going to give one to the first kid, another one to the second kid, and another, and one to the third kid. Then I do it again. And so now each kid has two pieces of cake so far. And then the second cake, since there was two and the first cake, then there should be two in the second cake.

David: So then, they sh–, each cake ha–, gets fourrrr. Wait! No! twe–... Four-twelfths. [Writes 4/12.]

SOLUTION 3



David: And so each of them should [taps pencil on each of the '3's' in the first diagram] get eighteeeen. One, two, three, four, five wait!

One, two, three [taps pencil on each '3' in the first diagram], wait, not eighteen, not eighteen sorry.

One, two, three [taps pencil on each of the partitions with '1' in the first diagram]. One, two, three [taps pencil on each of the partitions with '1' in the second diagram.]

They should get six out of... eighteen [referring to the total number of partitions]. Six out of – six-eighteenths. [Writes 6/18.]

Further to this, the present researcher identified instances where children changed the strategy that they were using to find the fraction during a task. These instances were then categorised according to two categories of strategy change in a task-based interview: 'Transient strategy change'; and 'a regular pattern of strategy change'. For the domain of solving partitive quotient problems, the identification of these two strategy changes is a new and significant finding by the present research. Chapter 8 discusses this finding in detail. The remainder of this section presents each of the strategy changes identified, along with illustrative excerpts from the research data.

6.3.1 Transient strategy change

The first illustration of the transient strategy change involves Harry, engaged in two instances of quantifying each person's share for the problem of sharing two cakes among seven children. For the first solution, Harry partitions each of the two diagrams into seven.

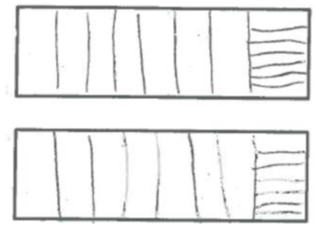
Excerpt 6-15

SOLUTION 1

Harry: Because there are seven children and then each child would get one piece. But it still remaining one more cake and you can still share it in seven. [Partitions the second diagram into seven.] Seven.

And each child would get two, two out of... one of our seven [Points to the first diagram], one out of seven, [Points to the second diagram... looks at the diagrams], so that'd be... two-sevenths. [Writes 2/7.]

SOLUTION 2

Harry: 

So each child would get two... [Looks at diagrams], two-sixteenths. Cause that and that [points to the each of the two diagrams in turn] is sixteen pieces. Two-sixteenths and two-fourteenths. [Writes 2/16   2/14.]

In the excerpt above, Harry uses the TPPe**/**TPOI strategy in Solution 1. Immediately following this solution, when quantifying each person's share in Solution 2, he uses the TPPe/TPAI strategy. Up until that point in the task-based interviews, he had never used the TPPe/TPAI strategy before. After this task-based interview, he never used it again. In previous task-based interviews, such as was presented in Excerpt 6-11, when Harry employed some version of this type of partitioning he used the TPPe**/**TPOI strategy.

Excerpt 6-16 presents another example of the transient strategy change. In this excerpt, Mary quantifies each person's share related to solving the problem of sharing two cakes among three children. For Solution 3, Mary partitions each of the two diagrams into six. Following this, she distributes the pieces and quantifies each person's share. For Solution 4, she partitions each diagram into 12.

Excerpt 6-16

SOLUTION 3

Mary: One, two, three, four [counts the first four partitions in the first diagram quietly].

One, two, three, four [counts the last two partitions in the first diagram then the first two in the second diagram quietly].

One, two, three, four [counts the last four partitions in the second diagram quietly]. So each one would get four-sixths. [Writes 4/6.]

SOLUTION 4

Mary: So, two plus – [Looks at the diagrams and moves pencil over different partitions.] That's twelve there. [Passes pencil over the second diagram.]

So, if I take out two [points to the first two partitions in the second diagram] for that [points to the first group of six partitions in the first diagram], that'll be eight plus and that would be eight [referring to the share of the second person] and that'll be one, two, three, four, five, six, seven [counts the last seven partitions in the second diagram]. So, that couldn't work.

One, two – [Places index finger on the fourth partition of the second diagram. [Note: the first four partitions of the second diagram represent the two each to be given to Person 1 and 2] then recounts the remaining partitions in the second diagram.] One, two, three, four, five, six, seven, eight [touches each partition while counting]. [Looks up.] So each person would get eight of twelve, of twenty-four. [Writes 8/24.]

From the excerpt above, from Solution 3 to Solution 4 Mary appears to change the method of finding the fraction for each person's share. In Solution 3 she uses the TPPe/TPOI strategy, where the four corresponds to the number of pieces that each person receives and the six to the number of partitions in each diagram. For Solution 4, she verbalises the following: 'So each person would get eight of twelve, of twenty-four'. From this verbalisation, she initially states the fraction as per the TPPe/TPOI strategy then, without a pause, changes it to the TPPe/TPAI strategy, where the twenty-four corresponds to the total number of partitions in the two diagrams. A close examination of the video recording shows that this shift occur with no overt acts of contemplation or evidence of cognitive dissonance. After this use of the TPPe**/**TPOI strategy, for problems of a similar type, where the number of items is less than the number of people sharing situation, the TPPe/TPAI strategy is used exclusively.

The third example exemplifying the transient strategy change is evident when Gabriel solves the problem of sharing three pizzas among six people. Excerpt 6-3 presented this data and so it is not repeated here. From the excerpt, Gabriel quantified each share as TPPe/TPAI as per the verbalisation, “out of three cakes it would be one out of six. [Writes 1/6]”. For a very brief period, however, Gabriel appeared to contemplate using the strategy of TPPe**/**TI. He quickly returns to the strategy he has used from the second task-based interview, when he states, “Wait! No is six pieces. So it's six”.

An inspection of the first two excerpts in this section shows that there was a change in the strategy used to quantify each person’s share from one solution offered to the next. For Gabriel, however, the strategy change seemed to occur within one solution. The strategy change has been labelled as transient because for the three children, the strategy was not used in another task-based interview. For the children who exhibited this strategy change, it is not clearly apparent what triggered the change.

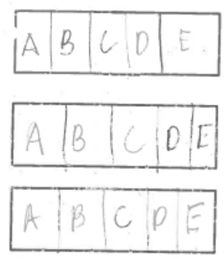
6.3.2 A regular pattern of strategy change

One research participant used the second type of strategy change that occurred within a given task-based interview. The next excerpt exemplifies this. In this excerpt, Samuel solves the problem of sharing three cakes among five children.

Excerpt 6-17

SOLUTION 2

Samuel:



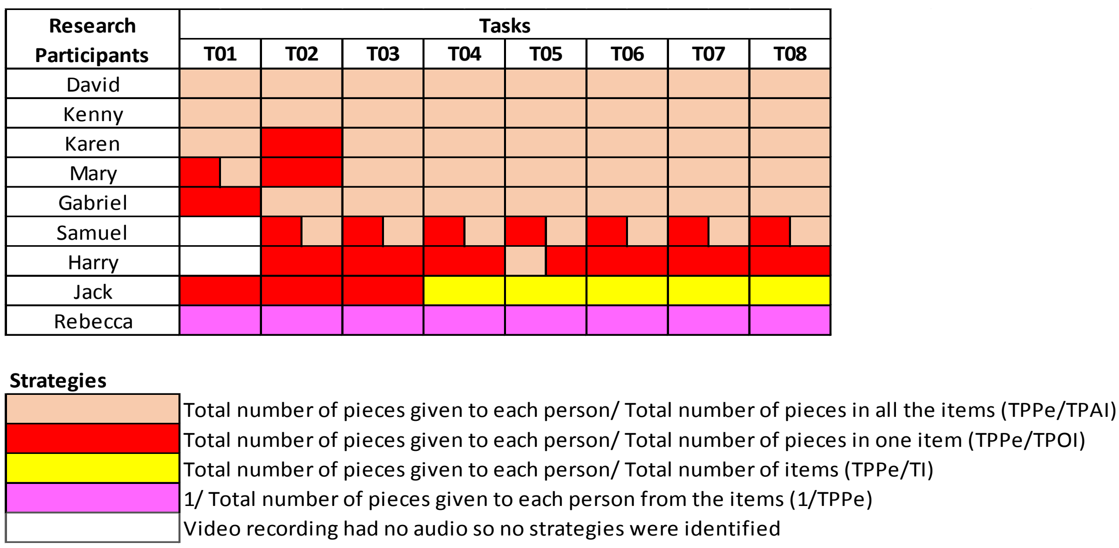
> So each child would get three over fifteen or [looks at diagrams] one [Places pencil in the last partition of the first diagram] out of five [Passes pencil over the entire first diagram], three times.
>
> So, it would be one, two, three [taps pencil on the last partition of each of the three diagrams in turn while speaking] out of one, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen fifteen [taps pencil on each partition of each of the three diagrams in turn while speaking] [writes 3/15] or one over five three times. [Writes 1/5 1/5 1/5.]

From the excerpt above, for the given solution Samuel finds his fraction in two ways. In one way, he utilises the TPPe/TPOI strategy for each diagram, verbalised as 'one over five three times'. In the second way, he utilises the TPPe/TPAI strategy. In seven of the eight tasks, Samuel uses these two strategies alongside each other to quantify the share that each person receives.

6.4 Strategy use across the eight task-based interviews

Table 6-1 shows the distribution of these four strategies across research participants and tasks. Each colour within a cell represents a different strategy, as in the key at the bottom of the table. When cells are of different colours within one task-based interview, such as T02 for Samuel, the child has used more than one strategy to quantify each person's share.

Table 6-1 Distribution of strategies across children and tasks



One finding of the present research relates to the use of strategies across the eight tasks. Over the eight tasks, six of the nine children used more than one strategy for quantifying each person's share in a partitive quotient problem but, by the third task, in general, the children in this study had settled into a regular pattern of strategy use. This finding is notable, because it situates the concept of the partitive quotient sub-construct within the strategy choice empirical literature for the first time, as far as the present researcher is aware.

6.4.1 The most common strategies and strategies of choice

The noteworthy aspect of children's strategy use across the eight tasks is not just that most of the children used multiple strategies, but the specific strategies that they used most often and the prevalent strategy of choice over time. As it relates to this, an examination of Table 6-1 shows that the research participants used the TPPe/TPAI strategy most often, followed by the TPPe/TPOI strategy. More specifically, seven of the research participants (David, Kenny, Karen, Mary, Gabriel. Samuel and Harry) used the TPPe/TPAI strategy on at least one occasion over the eight task-based interviews, whereas six children used the TPPe/TPOI strategy. While both the TPPe/TPAI and the TPPe/TPOI strategies were most often utilised, five children used the TPPe/TPAI strategy predominantly (for at least five of the eight tasks), while for the TPPe/TPOI strategy, it was one child (Harry). One child (Samuel) used both the TPPe/TPAI and TPPe/TPOI strategies for seven out of the eight tasks.

Further to this, all the children appeared to have settled on a strategy of choice after the third task-based interview. For five out of the nine children of this study, the strategy of choice was the TPPe/TPAI strategy. In addition to this, three (Karen, Mary, Gabriel) out of the five children who chose the TPPe/TPAI strategy as their strategy of choice, at some point in engaging with the tasks, also used the TPPe/TPOI strategy. Another aspect of the TPPe/TPOI strategy use is that after six children used some variation of this strategy over the eight tasks, only two of the children in this study chose this as their strategy of choice over time. One (Samuel) of these two children also used the TPPe/TPAI strategy alongside the TPPe/TPOI strategy in a combined approach of choice.

6.4.2 Pathways of strategy use

A scrutiny of the TPPe/TPAI strategy use for the five children who used this strategy primarily across the eight tasks reveals four pathways of use. First, two children (David and Kenny) used this strategy exclusively for finding the fraction amount across the eight tasks, regardless of the task characteristics.

The second pathway of strategy use across tasks is illustrated using one solution from the first three task-based interviews for Karen.

Excerpt 6-18

T01: SHARE TWO CAKES AMONG THREE CHILDREN

Karen: So, how much cake does a child get would be four. How much cake does a child get? Would be four.

Researcher: So in terms of a fraction, how much would that be?

Karen: [Looks at diagrams.] Four out of twelve.

T02: SHARE FOUR CAKES AMONG THREE CHILDREN

Karen: Okay. I'd give them the same whole cake and this time [looks at the fourth diagram and places pencil inside the diagram]. Hmmm [looks at the diagrams]. Each of them would get eight pieces [looks at the diagrams]..., twelve piec – [looks at the diagrams, then ahead and puts up three fingers in turn while lips move silently]... four pieces [looks up].

Researcher: How did you decide to give each four pieces?

Karen: For the bottom [referring to how the fourth diagram would be partitioned] I'd put twelve. And then that would be four times three so I'd give them four pieces out of the twelve.

Researcher: So what fraction would that–.

Karen: One out of... one out of [stares ahead] one and... one and four, four-twelfths. [Writes **1** 4/12.]

T03: SHARE THREE CAKES AMONG FIVE CHILDREN

Karen: Actually each child would get three pieces. One out of each cake [each cake is partitioned into five]. So the fraction would be three out of fifteen. [Writes 3/15.]

From the excerpts above, Karen uses the TPPe/TPAI strategy for T01, where the number of items is less than the number of people sharing. In T02, she does not employ this strategy when the number of items is less than the number of people sharing. Her approach appears to be guided by the new task features. For the third task, which is similar to the first task, she reverts to using the TPPe/TPAI strategy and continues to use this strategy for the rest of the tasks. Overall, Karen uses the TPPe/TPAI strategy for seven out of the eight tasks. Mary is another research participant who generally demonstrated a similar pathway to that of Karen across the eight tasks.

A sub-section of the third pathway is illustrated in Excerpt 6-19, which constitutes solutions from two tasks. In the first excerpt, Gabriel quantifies the share associated with solving the problem of sharing two cakes among three children. In this solution, he has partitioned each of the two items into three. For T02, Gabriel engages in solving the problem of sharing four cakes among three children.

Excerpt 6-19

T01: SHARE TWO CAKES AMONG THREE CHILDREN

Gabriel: The three children, they each get two and then they would be equally the same they would equally – they'll have the same amount. It would be equal.

Researcher: Okay. When you say two, what fraction would that represent?

Gabriel: That'd be one-third, um, each child gets two-thirds.

Researcher: Okay. Tell me how you got two-thirds.

Gabriel: Um… I got two-thirds from these three here [passes pencil over first diagram] and each child gets two. And, and then there's one left over [referring to one partition/piece remaining in the first diagram], but you still have like three [referring to the total number of partitions in the second diagram] more so you could take one more from that [pointing to the second diagram] and you give the child and there's two more [referring to the two partitions remaining from the second diagram] so you give the other child.

Researcher: Okay. So could you write down your fraction here please?

Gabriel: [Writes 2/3.]

T02: SHARE FOUR CAKES AMONG THREE CHILDREN

Gabriel: Because there are four and if I split it into three they will, um, one person can get [touches the first partition of the first diagram with a finger] and another person [touches the first partition of the second diagram with a finger] – they will all get four. So one person can get four here, here, here and here. [Points to the first partition of each of the four diagrams in turn.] And it will continue on. [Points to the second partition of each of the four diagrams in turn.] And then, it will it will be finished and each of them will have four and there will be none left over.

Researcher: How much cake would each child get?

Gabriel: Each child would get four-twelfths. [Writes 4/12.]

From the excerpts above, Gabriel quantified each person's share in T01 using the TPPe/TPOI strategy. In the second task, however, he switched to using the TPPe/TPAI strategy. For the rest of the task-based interviews, Gabriel continues to use this approach to quantify each person's share. This represents the third pathway for the children who primarily used the TPPe/TPAI strategy for quantifying each person's share. One child, Samuel, uses the fourth pathway. He uses both the TPPe/TPAI and the TPPe/TPOI strategies alongside each other for seven of the eight tasks.

While seven of the research participants used the TPPe/TPAI strategy on at least one occasion over the eight task-based interviews, two children, (Jack and Rebecca) did not use this strategy at all. Rebecca used the '1/Total number of pieces given to each person from the items' strategy for each of the eight task-based interviews. Jack's pathway across the eight tasks was unique. For the first three tasks, he used the TPPe/TPOI strategy, while from T04 he switched to using the 'Total number of pieces given to each person/Total number of items' strategy for the remainder of the tasks. The excerpts below illustrate a sub-section of Jack's pathway for strategy use from T03 to T04.

Excerpt 6-20

T03: SHARE THREE CAKES AMONG FIVE PEOPLE

Jack: Miss one child gets one piece, another child gets another piece, another child gets another piece, another child gets another piece, and another child gets another piece [points to each partition in the first diagram in turn], Miss.

Researcher: So how much cake would one child get?

Jack: Three-fifths Miss. [Writes 3 5th.]

T04: SHARE FOUR CAKES AMONG SIX PEOPLE

Researcher: So how much would this child [points to a picture of a child on the paper] for example get? How much cake?

Jack: Miss he would get four-fourths of a cake Miss.

Researcher: Could you show me on your diagram how you got this?

Jack: Miss is one child so it – one piece for the same child, another piece for the same child, another piece for the same child, another piece for the same child [points to the first partition of each diagram in turn]. So that is four altogether. So the four, fours – the four-fourths is because – why I say that is because there are four rectangular pieces of cake, four rectangular full cakes there.

6.5 Summary

Chapter 6 presented the main findings associated with the first research question of this thesis, as follows:

- The children in the sample used four main strategies for finding the fraction associated with quantifying each person's share for the various partitive quotient problems
- Most (six of the nine) children used more than one strategy for quantifying each person's share in a partitive quotient problem, over the eight tasks, with the TPPe/TPAI and the TPPe/TPOI strategies being the most commonly used
- By the third task, it appears that, in general, the children in this study had settled into a regular pattern of strategy use, with five of the nine children using the TPPe/TPAI strategy exclusively
- For the five children who used the TPPe/TPAI strategy primarily across the eight tasks four pathways of strategy use were identified
- Two categories of strategy change ('transient strategy change' and 'a regular pattern of strategy change') were identified within a task-based interview.

Chapter 7: Elaborating the Don't Need boundaries of the Pirie-Kieren theory

7.1 Overview

Chapter 7 focuses on answering the second research question of this thesis, which investigates: 'In what way(s) does evidence support or not support the Don't Need boundary (DNB) feature of the Pirie-Kieren theory for the growth of mathematical understanding?' In this regard, sections 7.2-7.4 present three main findings.

7.2 Uni-directional and bi-directional Don't Need boundary crossings

The first major finding of the present research focuses on presenting two types of DNB crossings identified by the present research, one of which incorporates folding back. Folding back is the process of mentally or physically returning to inner layers of the Pirie-Kieren model, from an outer layer, with the intention of illuminating some current outer level problem (see section 3.5.1). All of the nine children of the present study crossed the first DNB from the Image Making to the Image Having layer at some point in their engagement with the partitive problems. Table 5-5 in section 5.2.4 presents the specific number of DNB crossings identified by the researcher for each participant.

The present research also found several instances of learners who had crossed a DNB, folding back from the outer Image Having and Formalising layers, to work within the Image Making layer. Table 5-5 also presents a summary of this information for each DNB and research participant. The next three excerpts present instances of this for illustration purposes. Excerpt 7-1 shows Kenny engaged in sharing four cakes among three children. In this excerpt, for the first solution, he folds back from operating outside the first DNB in the Image Having layer to work within the Image Making layer of the Pirie-Kieren model. He subsequently returns to the outer Image Having layer.

Excerpt 7-1

IMAGE HAVING

> **Kenny:** So, I can share the cake into four pieces. Four cakes. [Partitions the first diagram into four.]

Researcher: Please tell me why you chose four?

FOLDS BACK TO IMAGE MAKING

Kenny: Because... [Looks at diagrams, touching the pencil to different parts of each diagram.]

Kenny: Not four.

Researcher: Not four? Why not four?

Kenny: Because when I, if I cut all the cakes into four pieces and I share them among the three people, one slice would be left over. And then, I would have to share, share one and – we must share all the cake so that each person gets the same amount and that no cake can be left over.

IMAGE HAVING

Kenny: Soooo... So I can share into six pieces.

Researcher: Any reason in particular why you chose six?

Kenny: Because six is a multiple of three and that you can – if you divide it equally, six times four is twenty-four. And if you divide like twenty-four by three you would get six.

At the start of Excerpt 7-1, Kenny appears to have an image for the number of partitions to share each diagram. This image/method for the number of partitions appears to be number of items (four) equals to the number of partitions (four). This is an example of a learner operating within the Image Having layer, but possessing and applying an incorrect image to the solving of a particular mathematical problem.

Kenny states his number of partitions and then he begins to partition the first diagram. After the researcher asks, 'Tell me why you chose four', Kenny seems to fold back to the Image Making layer, where he engages in activities such as inspecting the diagrams and his previously made partitioning. These actions appear to lead Kenny to reject the current image he possesses for the number of partitions to share each diagram. His verbalisation: 'Not four. Because when I, if I cut all the cakes into four pieces and I share them among the three people, one slice would be left over' supports this inference by the present researcher. Further, Kenny's folding back appears to be triggered by the question posed by the researcher, 'Tell me why you chose four?'

As part of Kenny's Image Making activities, he also touches his pencil to different parts of the diagrams. Following this, he appears to find a new image as to how to share the diagrams, which replaces his existing image. His verbalisation: 'So I can share into six pieces. Because six is a multiple of three and that you can – if you divide it equally, six times four is twenty-four. And if you divide like twenty-four by three you would get six' describes his new image. At this juncture, Kenny seemed to have returned to operating within the Image Having layer, thereby crossing the first DNB.

In the next excerpt, Karen solves the problem of sharing two cakes among three children. In the excerpt, Karen appears to move back and forth across the inner Image Making and outer Image Having layers of understanding of the Pirie-Kieren model.

Excerpt 7-2

IMAGE MAKING

Karen: Okay. [Long pause.]

Researcher: Tell me what you are thinking as you are/.

Karen: /I am thinking that you give one piece to each child and so one to him and one to him and one to her.

Researcher: Can you show me?

Karen: And then you put a child here. I count how much pieces. [Partitions the first diagram into 10 then counts the number of partitions.] One, two, three, four, five, six, seven, eight, nine, ten. Ten pieces.

Researcher: So how did you decide to choose ten? Why did you choose ten?

Karen: Because ten is an equal number, so, and there's three children so...

IMAGE HAVING

Karen: I decide to use twelve instead. Twelve.

Researcher: Twelve instead?

Karen: Yes.

Researcher: Tell me, why did you change your mind and decide to use twelve?

Karen: Because twelve, like three times... three times, three times four is twelve. So, like instead of that you can put, so each child like get four, four pieces... so.... That's how it works.

SOLUTION 4

Researcher: Can you show me another way that we could share the cakes, the two cakes among the three children? Do you think that there are other ways?

Karen: Well... Yes.

Researcher: Okay. You said yes with such confidence. Can you tell me and show me?

Karen: Well, this time we can put.... no, that's/ I'd try and say thirty and each child, each child would get ten pieces.

Researcher: Okay. Hmmm. And you said that without even having to draw or to use the diagrams. Tell me how you got that so quickly?

Karen: Because I know ten times three is thirty, so and there's three children so I just give each child ten pieces and that would add up to thirty. So that would be ten out of thirty.

SOLUTION 5

FOLDS BACK TO IMAGE MAKING

Karen: Umm.... [Long pause while looking at the diagrams.] I'd put.... [long pause while looking at the diagrams then staring ahead]. Let me try, umm, to try, sss, umm eight. Mmm.

Researcher: Tell me why did you choose eight?

Karen: I thought of eight because I saw is an even number so I tried to use most of the even numbers.

IMAGE HAVING

Karen: No I changing it, nine! [Smiles.]

Researcher: Oh you change it, okay. Nine? Okay.

Karen: Nine. One, two.

Researcher: Why did you change it to nine?

Karen: Because three times three is nine. [Partitions the first diagram into nine.] Then [looks at diagrams] so with nine, so then I put another nine. One, two, three, four, five, six, seven [partitions the second diagram into nine]. Then for each, three pieces, one, two, three [taps pencil on three partitions in the second diagram].

At the start of Excerpt 7-2, Karen appears to engage in some physical Image Making activities, such as partitioning and counting the partitions. She also appears to work mentally, but the present researcher is unable to state what this mental engagement entailed. After an extended time working within this layer, she appears to cross the first DNB from the Image Making layer to the Image Having layer when she states, 'I decide to use twelve instead… Twelve. Because twelve, like three times... three times, three times four is twelve'. In this verbalisation, she describes her image for the number of partitions even before she engages in partitioning. In the next solution, she uses this image to find 30 partitions in each diagram.

For Solution 5, Karen appears to fold back to the Image Making layer of understanding. She seems at that point, to have stopped applying the image from the two previous solutions, which is 'number of pieces given to one person multiplied by three'. The verbalisation: 'I know ten times three is thirty so and there's three children so I just give each child ten pieces' describes the image. For Solution 5, Karen does not appear to have an image for the number of partitions to use for the diagrams. Her folding back here appears to be associated with the task that she is engaged with.

Table 5-5 in section 5.2.4 suggests that, for most of the children in the present research, the crossing of a DNB is not necessarily a 'once and for all' phenomenon. Appendix K presents data for three more research participants, Samuel, Harry and Gabriel, exemplifying this observation.

The data analysis also showed another type of DNB crossing. While several research participants in the present study experienced the phenomenon of folding back at some point in their engagement with the partitive quotient problems, the research data also showed that some children did not fold back to work within an inner layer of understanding after having crossed a DNB. From Table 5-5, for the first solution, six children, on crossing the first DNB, remained working in the outer Image Having layer throughout the data collection period. For subsequent solutions, one child (Mary), on crossing the first DNB, remained working in the outer Image Having layer. For the second DNB, for the first and subsequent solutions respectively, six and three children remained working outside that DNB.

Excerpt 7-3 presents an illustration of this type of DNB crossing for Jack's engagement with the first solution for T01 to T08. After operating in the Image Making layer for the first solution for T01, Jack crosses the first DNB in T02, then subsequently the second DNB for the first solution in T03. Jack's operating is unique in that, although he sometimes struggled with the act of partitioning, he had an image as to the number of partitions he wanted to make. T02 and T03 exemplify this observation made by the present researcher. Appendix L presents four additional excerpts illustrating this way of operating for Mary, Harry, David and Karen, for first solutions to the partitive quotient problems.

Excerpt 7-3

T01: SHARE TWO CAKES AMONG THREE CHILDREN

IMAGE MAKING

> **Jack:** Am.... [Looks at the diagrams for an extended period of time.]
>
> **Researcher:** Tell me what you are thinking.
>
> **Jack:** I am thinking of how I have to set it down. I have to share among the three children. I have to share among the three children. [Long pause.]
>
> Miss... Each child gets one-third of a cake, Miss.
>
> **Researcher:** Could you show me on the diagram?
>
> **Jack:** Miss, there are three children... to share it equally you have to, like… He would get one-third [Uses the end of his pencil and traces a vertical partitioning line about a third of the first diagram], then there would be another one-third [uses the end of his pencil and traces a vertical partitioning line of about a third of the second diagram], so it would have no cake left over.

T02: SHARE FOUR CAKES AMONG THREE CHILDREN

IMAGE HAVING

> **Jack:** We cutting,... we are cutting one piece of cake [referring to the first diagram as one piece of cake]. [Partitions the first diagram into three.] So that is one piece of cake for one child [points to the second partition in the first diagram], another piece for another child [points to the first partition in the first diagram], and another piece for another child [points to the third partition in the first diagram].

……

Researcher: Why did you choose to share the cakes in three?

Jack: Because there are three children here.

TO3: SHARE THREE CAKES AMONG FIVE PEOPLE

FORMALISING [Jack appears to apply his image for number of partitions to all partitive quotient problems at this point.]

Jack: [Draws five vertical partitioning lines inside the first diagram.] [Pauses and looks intently at the diagrams.]

Researcher: Can you tell me a little bit about what you are doing?

Jack: Yes Miss. Miss, I am drawing lines to cut the, the rectangle sooo, I think I should do it a little bit bigger Miss. Miss do it on that one?

Researcher: Yes, you can just turn over. [Turns page over.] Tell me why you think you should do it a bit bigger?

Jack: Because when I do it small, it, each child would not get the same amount of cake.

Researcher: How many pieces are you trying to get in the rectangle?

Jack: Miss, five pieces.

Researcher: Could you tell me why five?

Jack: ’Cause there are five children.

T04: SHARE FOUR CAKES AMONG SIX PEOPLE

Jack: [Looks at paper.] I am going to put sixxx... like six straight lines down. That's one, two, three, four, five, Miss. [Draws five partitioning lines to form six partitions in the diagram.]

Researcher: Can you explain to me why you chose six?

Jack: Miss I chose six because there are six children, Miss.

T05: SHARE TWO CAKES AMONG SEVEN PEOPLE

Jack: Okay, Miss. [Partitions the first diagram into seven partitions.] Miss, one, two, three, four, five, six, seven. [Partitions the second diagram into seven.] One, two, three, four, five, six, seven.

………

Researcher: Why did you choose seven?

Jack: Miss because there are seven children.

T06: SHARE THREE PIZZAS AMONG SIX PEOPLE

Jack: One, two, three, four, five, six. [Partitions the first, second and third diagrams into six.]

T07: SHARE TWO PIZZAS AMONG FIVE PEOPLE

Jack: One, two, three, four, five Miss. [Partitions the first diagram into five.]

One, two, three, four and five, Miss. [Partitions the second diagram into six.]

……..

Researcher: So tell me why you chose five pieces.

Jack: Because there are five children Miss.

T08: SHARE THREE PIZZAS AMONG EIGHT PEOPLE

Jack: [Partitions each of the three pizzas into eight.]

……..

Researcher: Okay. Tell me why you are sharing each pizza into eight?

Jack: Miss, because there are eight children, Miss.

Taking into account the aforementioned observations from the empirical data of the present research, the first key finding for the second research question is that, as it relates to learners' crossing of DNBs in the Pirie-Kieren model, there are two possibilities. First, it is possible for a learner to cross a DNB and never return to an inner layer of understanding for one problem-solving session or over multiple encounters of solving a particular type of problem. The present researcher labels this type of DNB crossing as a 'uni-directional Don't Need boundary crossing', where a learner shifts to an outer layer of understanding, outside a DNB, and remains there over time.

Second, a learner's understanding may shift bi-directionally, one or multiple times, from inner to outer layers of understanding. This has been labelled as a 'bi-directional Don't Need boundary crossing' by the present researcher. Karen displayed a 'bi-directional Don't Need boundary crossing' in Excerpt 7-2 for subsequent solutions.

For the learning of a given mathematical concept/topic in one or more problem-solving situations, it appears that learners may make either or both types of crossings, as their mathematical understanding grows. Table 7-1 shows the distribution of uni-directional and bi-directional DNB crossings across the research data.

Table 7-1 Distribution of uni-directional and bi-directional DNB crossings

Research participants	**Number of instances of:**					**Type of DNB crossing**
	Crossing first DNB	**FB from the IH layer**		**Crossing second DNB**	**FB from the Formalising layer**	
FIRST SOLUTION						
Harry	1	0		NA	NA	UDNB
Jack	1	0		1	0	UDNB
Rebecca	2	0		1	1	BDNB
Mary	1	0		1	0	UDNB
Gabriel	1	1		NA	NA	BDNB
David	1	0		1	0	UDNB
Karen	1	0		1	0	UDNB
Kenny	0*	1		1	0	BDNB
Samuel	0*	0		1	0	UDNB
SUBSEQUENT SOLUTIONS						
Harry	2	4		NA	NA	BDNB
Jack	NA	NA		NA	NA	NA
Rebecca	2	2		NA	NA	BDNB

Research participants	Number of instances of:					Type of DNB crossing
	Crossing first DNB	FB from the IH layer		Crossing second DNB	FB from the Formalising layer	
Mary	1	0		1	0	UDNB
Gabriel	9	8		NA	NA	BDNB
David	4	2		2	2	BDNB
Karen	6	4		NA	NA	BDNB
Kenny	2	1		1	0	BDNB
Samuel	2	2		1	0	BDNB

Key :

FB : Folding Back

IH: Image Having

DNB: Don't Need boundary

UDNB: Uni-directional DNB crossing

BDNB: Bi-directional DNB crossing

Associated with this first finding, Table 7-1 shows that there are more bi-directional DNB crossings than uni-directional ones, and that six of the seven uni-directional DNB crossings were observed for the first solution of the partitive quotient task.

The first research finding is further expanded in the next sub-sections, 7.3.1 and 7.3.2, as the first and second DNB crossings by the research participants are explored more closely. Stemming from these two sub-sections, the second major finding for Research Question 2 is presented in sub-section 7.3.3.

7.3 Comparing the first two Don't Need boundaries

7.3.1 The first Don't Need boundary

After having crossed the first DNB to operate within the Image Having layer, two children (Gabriel and Kenny) folded back once to the Image Making layer. For subsequent solutions, however, after having crossed the first DNB to operate within the Image Having layer, seven of the nine children folded back at least once to the Image Making layer. Further to this, for the first solution, of the children who crossed the first DNB and then folded back to work in the Image Making layer, only one (Gabriel) remained operating in this layer at the end of the data collection period, whereas for subsequent solutions there were three children (Harry, Rebecca and Gabriel).

Excerpt 7-1 presented the data relating to Kenny working outside the first DNB and then folding back from the Image Having layer to the Image Making layer of understanding for the first solution. Excerpt 7-2 illustrated that, for subsequent solutions, Karen crossed the first DNB then folded back to the Image Making layer. In this excerpt, Karen solved the problem of sharing two cakes among three children. Excerpt 7-4 presents another excerpt illustrating a participant operating beyond the first DNB for the first solution and then folding back to the Image Making layer. In this excerpt, Rebecca engages in the seventh task-based interview, sharing two pizzas among five people. Prior to this, for the first solution, Rebecca operated in the Image Having layer from the third task-based interview.

Excerpt 7-4

> **Rebecca:** Okay. Fifteen. If you cut the cake in fifteen pieces then... you... you cut it in fifteen pieces.
>
> ………..
>
> **Researcher:** How did you choose fifteen?
>
> **Rebecca:** Fifteen?
>
> **Researcher:** Aha.
>
> **Rebecca:** I looked at five. So I was checking ten, let's see if is ten. So I looked at ten then I say to myself, I count in my head and I say ten cannot work so I go on to fifteen and then I would get fifteen.
>
> **Researcher:** Why do you think ten could not work?
>
> **Rebecca:** Well if you share it in ten pieces then half of ten is five. Five for one, five for another, then five for another and five for another, one child would stay without a piece.

At the start of Excerpt 7-4, Rebecca appears to have an image for the number of partitions for each diagram. Based on a comparison of first solutions for previous tasks that Rebecca engaged with, however, the present researcher noted that the choice of 15 for the number of partitions in each diagram deviated from Rebecca’s previous first solutions. Before this task, she multiplied the number of people sharing by two and used this answer as the number of partitions for each diagram for the first solution. Excerpt 7-5 illustrates this image.

Excerpt 7-5

> **Rebecca:** Fourteen. You have to share the cake in fourteen pieces [looks at the diagrams].
>
> **Researcher:** Why did you choose fourteen?
>
> **Rebecca:** ’Cause when you said two cakes among seven children, I was thinking in my mind two sevens a fourteen, so I choose fourteen. So you share the cake in fourteen pieces.

This deviation in working observed by the present researcher prompted her to ask Rebecca, ‘How did you choose fifteen?’ The explanation that follows suggests that Rebecca did in fact begin the excerpt in the Image Having layer with ten as her choice of number of partitions in each diagram. Her verbalisation: ‘I looked at five. So I was checking ten, let's see if is ten’ confirms this. Rebecca’s explanation as to her process of choosing 15 included considering ten, which fit her previous pattern for the number of partitions in each diagram. Following this, she appears briefly to fold back to engage in mental Image Making. Her verbalisation, ‘So I looked at ten then I say to myself, I count in my head and I say ten cannot work so I go on to fifteen’ encapsulates her Image Making activities. After this, she appears to return to work in the outer Image Having layer with her new image for the number of partitions in each diagram. This folding back and subsequent crossing of the first DNB again resulted in Rebecca developing a new image for the number of partitions to be made in each diagram for the partitive quotient problem.

A close examination of the excerpts presented thus far in this chapter showed that, although the research participants had crossed the first DNB to operate in the Image Having, there were several instances of folding back to the Image Making layer. An inspection of the excerpts presented and the other similar instances from the research data shows that the folding back that occurred was largely associated with hazy, incomplete and/or incorrect images that research participants held while working within the Image Having layer. In addition, while some learners operated in the Image Having layer of understanding with the aforementioned images, some question(s) posed by the researcher, for example from Excerpt 7-1, ‘Please tell me why you chose four? and ‘Would the children get the same amount of cake?’ from Appendix K.2 appeared to trigger the folding back that occurred. The participants' engagement with the task prompts (see Excerpt 7-2 and Appendix K.1) also appeared to be associated with folding back.

The Pirie-Kieren theory does not discuss the issue of folding back associated with DNB crossings, such as distinctions between learners operating immediately beyond a DNB versus when they are securely beyond the boundary, or outside the first, versus the second DNB. Consequently, this initial observation by the present researcher, that although the children in this study had crossed the first DNB, there were several instances of folding back, is potentially significant to the elaboration associated with the DNBs and forms part of the basis for the second research finding associated with the Pirie-Kieren theory.

The aforementioned finding also challenges the bold/solid character of the first DNB of the Pirie-Kieren model, thereby lending credence to the initial argument posed by the present researcher in the discussion of the literature associated with the DNBs in section 3.5.2. Further to this, it suggests that the first DNB is more porous in nature than previously presented in the empirical literature related to the Pirie-Kieren theory and so a revisiting of the Pirie-Kieren model may be in order.

7.3.2 The second Don't Need boundary

As it relates to crossing the second DNB, for subsequent solutions, four children (Mary, David, Kenny and Samuel) crossed the second DNB to work in the Formalising layer. Of the four children who operated within the outer Formalising layer for subsequent solutions, only one child, David, folded back twice from working in the Formalising layer to engage in Image Making activities. Excerpt 7-6 presents one of these instances of folding back in which David solves the problem of sharing three pizzas among eight people.

Excerpt 7-6

FORMALISING

> **David:** Okay. Well then I'm going, right now I'm just going to share it into eighths because there are eight children.

FOLDING BACK TO IMAGE MAKING

> **David:** Weelll... (?). I'm trying to see if I can use something else depending on the denominator. So [Looks at diagrams]... I can't really use seven because I think I will have uh,... not each child would get equal. One, two, three, four, five, six, seven. One, two,

> three, four, five, six, seven. One, two, three, four, five, six, seven. One, two, three, so I have three left and um...

FORMALISING

> **David:** So really you have to use other multiples. Like um twenty-four. You can use other multiples like that.

At the start of solving the problem shown in the excerpt above, David appears to have an image for the number of partitions to be made in each diagram. His verbalisation, ‘I’m just going to share it into eighths because there are eight children’ clearly describes this image as number of partitions equals to the number of people sharing. David consistently used this image and its extension, the number of partitions equals multiples of the number of people sharing, from the second task-based interview. In the excerpt above, David appears to fold back to the Image Making layer of understanding to see if other numbers of partitions could work. In this regard, he states, ‘I’m trying to see if I can use something else depending on the denominator’. Following the folding back, David states, ‘So really you have to use other multiples’. In so doing, he quickly returns to the outer Formalising layer. His second instance of folding back was for the same reason, to explore if there were other viable images for partitioning the diagrams. This excerpt shows that David’s purpose for folding back was to find other ways of solving the problem, and not because of a hazy, incomplete or impoverished image, as was the case for the folding back associated with Image Having to Image Making.

7.3.3 Comparing the first two Don't Need boundary crossings

Table 7-1 in section 7.2 shows that the number of instances of folding back after a learner crossed the second DNB to operate within the Formalising layer are fewer than the instances of folding back from the Image Having layer, having crossed the first DNB. This observation holds for both the first solution and subsequent solutions of the problems children engaged in solving. The data presented in sections 7.3.1 and 7.3.2 show that the images held by the children who had crossed the second DNB to operate in the Formalising layer of understanding appeared to be much more complete/stable/correct than those of children who had crossed the first DNB to operate in the Image Having layer.

Stemming from the aforementioned observation, the second major finding of the present research in relation to the Pirie-Kieren theory is that the first two DNB crossings to the Image Having and Formalising layers of the Pirie-Kieren model, respectively, appear to be different. This suggests therefore, that the first two DNBs in the Pirie-Kieren model are not identical, although the literature (see examples in section 3.5.2), examined by the present researcher, describing this feature of the Pirie-Kieren theory, suggests uniformity. This second finding from the second research question of this thesis is therefore significant, because it provides empirical evidence to refute the uniformity of the DNBs and, in so doing, provides a more nuanced picture of the DNB feature than previously presented in the literature concerning the Pirie-Kieren theory and associated model. Further to this, the present researcher suggests that the assertion by Pirie and Kieren (1994b) that outside the DNBs a learner can operate at a level of abstraction without needing to mentally or physically refer to specific images appears more consistent with the second DNB crossing than the first. This is because the participants operating at the Formalising layer of understanding seemed just to know the number of partitions to use for a given task, and did not appear to refer to specific previous inner layer images. The data presented in Excerpts 7-4 and 7-6, where the participants are operating within the Formalising layer of understanding for the first solution, is indicative of this.

7.4 Factors associated with Don't Need boundary crossings

The third key finding from the current empirical study relates to three factors that appear to be associated with learners' crossing the DNBs to operate in an outer layer of understanding, such as the Image Having or Formalising layers and remaining in the outer layer over time.

A first factor identified by the present researcher is the nature of the tasks that learners tackle. As discussed in section 5.2.2 of this thesis, the two parts of the partitive quotient tasks that children engaged with for the present research appear to involve different levels of cognitive demand/difficulty. The first part of the task, 'How can you share *x* items among *y* people fairly?' appeared to be less cognitively demanding than the second part, 'How else can you share *x* items among *y* people fairly?'

The data from Table 7-1 in section 7.2 shows that there are fewer instances of folding back associated with the first solution than with subsequent solutions. In other words, there were more instances in which the research participants remained working outside a DNB in an outer layer of understanding for the first solution than subsequent solutions. Based on this data, the present researcher suggests that learners are more likely to remain (with no folding back) working outside a DNB on simpler and easier mathematical problems than on those that are more cognitively demanding.

Another factor that appears to affect whether a learner crosses a DNB and remains working outside that boundary relates to the properties that s/he applies to the image(s) s/he possesses. Three excerpts are presented to illustrate this factor.

The first illustration of this involves Karen solving the problem of sharing two cakes among three children in Excerpt 7-2 in section 7.2. In this excerpt, Karen's verbalisation, 'I thought of eight because I saw is an even number so I tried to use most of the even numbers' points to a correct image (number of partitions = multiple of number of sharers = three) and an incorrect property (number of partitions are even) operating together. This incorrect property applied to the image for the number of partitions appears to prevent Karen from remaining in the outer layer of Image Having and appears to be associated with Karen's return to the Image Making layer.

Excerpts 7-7a and 7-7b also exemplify the aforementioned observation, but somewhat differently to Karen. In T03, for the first time, Mary began operating in the Image Having layer for the first solution. For subsequent solutions, however, Mary operated in the Image Making layer. For Mary's first solution, for T03, she found the total number of partitions to be 15. Using this image from Solution 1 (first solution), she finds a subsequent solution, but initially applied an incorrect property.

Excerpt 7-7a

IMAGE MAKING

T03: SHARE THREE CAKES AMONG FIVE PEOPLE

Mary: Three cakes among five children. [Looks ahead for an extended period.] I think eighteen.

Researcher: Tell me why you think eighteen.

Mary: I think eighteen because if I share the three cakes into eighteen pieces altogether, each child could get... four slices? I think? Mmmm.... but that couldn't work because I have three cakes, three pieces of cake. I have three cakes. Because eighteen is an even number it would not work. So I have to find an odd number so that I can share the cakes among the children.

In the excerpt above, Mary applies the property that the number of partitions should be odd, since the first number of partitions she found to be correct was 15, which is an odd number. As long as she held this property, she remained operating in the Image Making layer. In the excerpt above, she appears to be struggling to find a subsequent solution as she held on to this property. Later in the interview, Mary abandons this property and quickly moves to the Image Having, then the Formalisation layer of understanding. Excerpt 7-7b illustrates this.

Excerpt 7-7b

IMAGE HAVING

T03: SOLUTION 3

Mary: I think thirty.

Researcher: Tell me why you chose thirty.

Mary: Because since fifteen can work, fifteen times two is thirty.

T03: SOLUTION 4

Researcher: Do you think there are other ways that you could share the cake?

Mary: Well, if I had to share the cake again I would probably add fifteen again.

Researcher: Okay. You would add fifteen to what?

Mary: Thirty. Which would be forty-five.

T03: CONCLUSION

FORMALISING

Researcher: So if you were to explain to somebody how to share three cakes among five children what would you say to them?

Mary: I think I would tell them that they would have to try to find the first number [referring to the total number of partitions for the first solution] you would, you would get when you multiply three by five, because you share you are sharing three cakes among five children, so first you have to find out what number that you can actually share it in with the two numbers you have. 'Cause three fives a fifteen.

Excerpts 7-8a and 7-8b present two further examples where an incorrect property appears to hinder a DNB crossing from occurring. In Excerpt 7-8a, Gabriel engages in his fifth task of sharing two cakes among seven children and, in Excerpt 7-8b, he engages in the final task of sharing three pizzas among eight people.

Excerpt 7-8a

> **Gabriel:** So then the only thing you can do is cut each of them into seven pieces.
>
> **Researcher:** Could you tell me why seven do you think would work?
>
> **Gabriel:** Because I, I think since it's like am,... Am I don't know but the only thing I – it's an odd number? But um, the five... I don't really understand why. It's an odd number but it still worked. Am, but... I don't, don't know why but the seven just seems really odd.

Excerpt 7-8b

> **Gabriel:** I think you have to start going by eight because, I think it's because... there are three pizzas and that's – it's an odd number, so then you wouldn't be able to fit an even, an even number into, into the odd number, pizza.

For Excerpt 7-8a, Gabriel's number of partitions for each diagram is seven, while for Excerpt 7-8b the number of partitions in each diagram is eight. In the excerpts above, Gabriel appears to contend with the incorrect property that the number of partitions links in some way to an odd number. Although Gabriel appeared to be aware that this property was not completely correct, he did not have any correct property with which to replace it. Gabriel remained working in the Image Making layer for most of the first solution for the duration of the data collection.

A third factor that appears to affect whether a learner crosses a DNB and remains working beyond that DNB relates to the nature of the images that learners hold. Martin and LaCroix (2008) point out that the Pirie-Kieren theory posits that 'as a learner's mathematical understanding for a particular concept grows, he will make, hold, and extend particular images as he works on mathematical tasks' (p. 123). Pirie and Kieren (1994b) also state that the image a learner possesses for a particular mathematical concept/topic is not necessarily complete or correct. In particular, this third factor relates to the completeness of a learner's image(s) about a mathematical concept/topic and the applicability of these images to all problems of a particular type. Two children (Harry and Rebecca) used the image of halves in two different ways to engage with the partitioning aspect of the problem. While the application of this image resulted in a correct number of partitions for some problems, the present researcher found that there were boundaries within which the children could use this image and this appeared to affect their shifting to the Formalising layer of understanding. Excerpt 7-9 shows how the application of halving affected Rebecca's problem-solving relating to number of partitions.

Excerpt 7-9

Researcher: How else can you share the three cakes among five children?

Rebecca: Fifteen, fifteen, fifteen [points at each of the three diagrams in turn.] [Stares ahead intently and mouth moves.] Twenty, twenty, twenty [points at each of the three partitions in turn]. You have ten [points to the first half of the first diagram], ten [points to the second half of the first diagram], ten [points to the first half of the second diagram], ten [points to the second half of the second diagram], ten [points to the first half of the third diagram] and (?).

Researcher: I heard you talked about fifteen at first.

Rebecca: Yes.

Researcher: Then you changed it to twenty. Can you tell me why you thought fifteen would work?

Rebecca: Well because if I – I choose ten, now I choose fifteen, the times table. I thought fifteen would work, if I put fifteen, then fifteen, then fifteen [points to each of the three diagrams in turn]. Then, let's see [stares ahead]. Fifteen divided by two... [shakes head negatively]. Fifteen divi– you cannot get fifteen divided by two. So if you put fifteen, fifteen, fifteen [Points at each of the three diagrams in turn] then each child would have to get–

Researcher: Could you explain why you are dividing by two?

Rebecca: Because this cake, two child, two children would get, wait, if you cut this cake in equal length, two children would get from this cake and two from this cake and one from this cake, and then you would just give the rest of the children the other half of the cake.

From Excerpt 7-9 above, it appears that in addition to using multiplication tables based on the first solution to find the number of partitions, Rebecca is applying halving as well. The excerpt shows that this aspect of the image results in her rejection of a correct number of partitions, fifteen. Excerpt 7-4 in section 7.2, in which Rebecca shares two pizzas among five people, also illustrates how this image of halving constrained the number of partitions that Rebecca considered to be correct. In this excerpt, Rebecca first applies the halving approach to each of the two pizzas, which results in the creation of four sections. She then mentally partitions each half into five pieces. Rebecca's verbalisation: 'Well if you share it in ten pieces, then half of ten is five. Five for one, five for another, then five for another and five for another, one child would stay without a piece' indicates this. Figure 7.1 illustrates diagrammatically Rebecca's way of working, although during the excerpt she worked verbally and mentally.

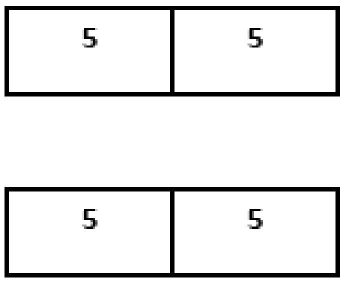


Figure 7-1 Illustration of Rebecca's sharing of two pizzas among five people

Note: '5' represents the number of partitions in the section.

Rebecca rejected sharing each of the two diagrams into ten partitions as a correct number of partitions in each diagram, although it was in fact correct. When each diagram is partitioned into ten, each person would receive four pieces each instead of five. The application of the halving approach prevented Rebecca from making this observation.

The next illustration of how particular images of a concept that a learner possesses could be limiting incorporates Harry's application of a halving approach, as well as other images for finding the number of partitions in the diagrams. In engaging with the partitive quotient problems, Harry utilised four different images for the number of partitions in the diagrams. These include:

- **Image 1:** Halving
- **Image 2:** Number of partitions in each diagram or multiple diagrams = Number of people sharing

- **Image 3:** Partitions each of the diagrams into a number of pieces e.g. ten and distribute evenly. If there is a remainder (one or more partitions), cut this into number of people (fifths).
- **Image 4:** Number of partitions in multiple diagrams = Number of people sharing

Excerpt 5-3 in section 5.2.2 illustrated Images 1 and 2, while Image 3 is exemplified in Excerpt 6-6 in section 6.2.1. The fourth image is illustrated as Harry shares two cakes among seven children. For easy reference and comparison, these three excerpts are presented in Appendix M.

From the excerpts, it appears that Harry's images for the number of partitions in the diagrams are distinctive. Additionally, it appears that the use of four different images may have hindered him from shifting to the Formalising layer in either part of the partitive quotient tasks due to the inability to notice similarities across number of partitions.

7.5 Summary

Chapter 7 presented three main research findings related to elaborating the DNB feature of the Pirie-Kieren theory. The current study found two types of boundary crossings. This includes a uni-directional DNB crossing, where a learner crosses a DNB and remains working outside that boundary over time, and a bi-directional DNB crossing, in which a learner, after having crossed a DNB folds back to work in inner layers of understanding.

A second major finding of the present research related to the Pirie-Kieren theory is that the first two DNBs appear to be qualitatively different. After children crossed the first DNB located between the Image Making and the Image Having layers of the Pirie-Kieren model, there was a much greater likelihood of folding back to an inner layer than when they crossed the second DNB.

The present research also found that the crossing of a DNB and remaining in an outer layer of understanding over time appears to be associated with three factors. These include the nature of the tasks students engage with, the correctness of the properties that learners glean from their images and the number, correctness and/or completeness of learners' images for a given concept/topic.

Chapter 8 discusses extensively the research findings that were presented in Chapters 6 and 7. In addition, the methodological consideration of the researcher role is examined.

Chapter 8: Discussion of the findings

8.1 Overview

Chapter 8 presents a discussion of the main findings presented in Chapters 6 and 7. In particular, section 8.2 discusses the findings related to children's strategies for finding the fraction associated with solving partitive quotient problems by first addressing the four strategies that children used for finding the fraction relating to quantifying each person's share in section 8.2.1. Following this, section 8.2.2 examines the TPPe/TPAI strategy more closely, in light of previous empirical research literature on children's engagement with partitive quotient problems, as well as mathematics education literature on the part-whole and partitive quotient fraction sub-constructs. Sections 8.2.3 and 8.2.4 delineate the choice of TPPe/TPAI strategy by children who have only been taught the part-whole meaning of fractions and the sample's pathways of engagement with the partitive quotient problems, respectively. An argument for generalisation of some of the findings related to Research Question 1 are presented in section 8.2.5.

Section 8.3 focuses on Research Question 2 of this thesis and discusses three research findings associated with the elaboration of the DNB feature of the Pirie-Kieren theory. Section 8.4 concludes the discussion of the findings by revisiting a key methodological consideration, relevant to this thesis, the role of the researcher, initially presented in Chapter 4.

8.2 Research Question 1

8.2.1 The strategies for finding the fraction associated with solving partitive quotient problems

The present research identified four main strategies for finding the fraction associated with quantifying each person's share for the various partitive quotient problems. Three of the four strategies that the children used to find the fraction resulted in incorrect answers. Only the use of the TPPe/TPOI strategy, excluding the quantification of the re-partitioning of an already partitioned section of a diagram (see Excerpt 6-11 in section 6.2.2), resulted in the correct quantification of a person's share.

The present research found that although many of the children in the sample produced an incorrect fraction to the partitive quotient problems, they were able to generate fair shares in their partitioning. This particular finding corroborates a finding from Toluk and Middleton (2001), who conducted a series of parallel individual teaching experiments with four children in the

United States to investigate the conceptual development of the partitive quotient. They state that, contrary to conclusions from previous empirical literature on fair-sharing problems, they found that the ability to partition quantities correctly into parts did not automatically indicate that children actually conceptualised the resulting fractions as quotients.

As it relates to how children who have only or predominantly been taught the part-whole fraction sub-construct solve partitive quotient problems, some previous empirical research (for example Yazgan, 2010; Charles and Nason, 2000) have dismissed some of the children's wrong answers as children making errors, being confused or having impoverished fractional knowledge. While this may be true in some instances, before one accepts this determination, children's problem-solving processes should be carefully examined, as has been done in this thesis. In discussing children's solving of some fraction problems, Steffe and Olive (2010) underscore that it is essential not to avoid the so-called 'incorrect' ways that children solve problems or to classify these ways of working as misconceptions. They suggest that, instead, teachers and researchers should aim to understand children's ways of working and in so doing help the children to modify their current ways and means of operating.

The '1/Total number of pieces given to each person from the items' (1/TPPe) strategy and the 'Total number of pieces given to each person/Total number of items' (TPPe/TI) strategy were two of the strategies used by two of the children, Rebecca and Jack, which resulted incorrect fractions. A close examination of the strategies reveals that, for at least one child, this may in fact be because of gaps in fraction knowledge. In the case of Rebecca, she assigned a numerator of one to every fraction she obtained, in every task. This way of working suggests that Rebecca may think that all fractions are unit fractions with a numerator of one and that it is the denominator that denotes 'number of parts shaded or number of parts each person receives'. Her incorrect quantification of each person's share, therefore, appears to be due to impoverished fraction knowledge. For Jack though, even after fine-grained analysis of his ways of working, for T04 to T08, the present researcher cannot discern a/the gap in his fraction knowledge. In quantifying a person's share for the first three tasks, he used the TPPe/TPOI strategy that produced a correct fraction in every case.

In solving mathematics problems, it is not uncommon for children to have gaps in their understanding or knowledge. In this regard, Martin and LaCroix (2008) state that, while learners can have powerful images of a task as it pertains to methods of solving a mathematical problem, they may lack the understandings of the mathematics involved to be able to do this. As it relates to solving partitive quotient problems in particular, Streefland (1991) and Middleton *et al.* (2015) agree that in solving partitive quotient problems children will invariably encounter various conflicts and take detours that divert, prolong or even stall their progress towards a mature understanding of the sub-construct. They add that, ultimately, children arrive at decisions regarding their ways of solving these problems, although these decisions are not always completely accurate. These statements are applicable to the solving of Rebecca and Jack. While the fine-grained examination of the use of the 1/TPPe strategy by Rebecca suggests that she had an impoverished fraction knowledge, for the TPPe/TPAI strategy this was not the case. In the next section, the TPPe/TPAI strategy is discussed in greater detail.

8.2.2 The TPPe/TPAI strategy

As presented in section 6.2, the present research found that children who had only been taught the part-whole fraction sub-construct used two main strategies for finding the fraction associated with solving partitive quotient problems. These are the TPPe/TPAI and the TPPe/TPOI strategies, which correspond to Total number of pieces given to each person/ Total number of pieces in all the items and Total number of pieces given to each person/ Total number of pieces in one item (TPPe/TPOI), respectively. Since the present study is the first empirical research study to focus exclusively on the quantification of each person's share associated with solving partitive quotient problems, as far as the present researcher is aware, it is an important finding that the two strategies most commonly used by the sample are the TPPe/TPAI and the TPPe/TPOI strategies. The significance of this finding is not only that two main strategies were identified, but also that most (five out of nine) children in this study, who had only been taught the part-whole meaning of fractions, ultimately chose the TPPe/TPAI strategy as their strategy of choice for quantifying each person's share in the problem contexts. Further to this, three out of the five children who chose the TPPe/TPAI strategy as their strategy of choice, at some point in engaging with the tasks, used the TPPe/TPOI strategy, but abandoned its use entirely at some point in the solving of the problems.

Two previous research studies, namely by Charles and Nason (2000) and Yazgan (2010), report that some of the children in their studies found that the fraction for quantifying each person's share in a partitive quotient situation corresponded to: Total number of pieces given to each person/Total number of pieces in all the items. While the present study classified this way of working as the TPPe/TPAI strategy, Charles and Nason (2000) described it as quantifying by or using the part-whole notion.

As discussed in section 2.3.1 of Chapter 2, the well-established explanation, based on empirical evidence (for example Streefland, 1991) for a child's answer of 3/12 for solving the problem of sharing three continuous items among four people has been the natural or whole number bias. A child who is under the influence of a whole number bias first finds the fraction amount that each person receives from the individual items: ¼, ¼, ¼. Then, acting exclusively on the fraction symbols, s/he adds all the numerators and denominators separately, as follows: ¼ + ¼ + ¼ = $\frac{1+1+1}{4+4+4}$, to obtain a fraction of 3/12. For the present research, similar to that of Charles and Nason (2000), the children's actions appeared be distinctively different from what has been described above. As illustrated in the excerpts in section 6.2.1, the children in the present study acted on and used the drawings that they had created to derive their fraction to quantify each person's share. They therefore were not considered to be under the influence of a whole number bias. The excerpt below illustrates evidence of this, for one child.

Excerpt 8-1

> **Researcher:** How else can you share the two pizzas among the five people?
>
> **Samuel:** I could use the number fifteen, also divisible by five. If I shared each tray of pizza into fifteen pizzas, now, and then I'm going to divide, am, well, distribute the cake.
>
> So first I could do five into fifteen which is three. So in the first cake each child will get three over fifteen. In the second cake, three over fifteen. If I do it together, each child will get six over thirty.
>
> **Researcher:** Okay. That's quite interesting. If you were to add three over fifteen plus three over fifteen, what answer do you think you would get?
>
> **Samuel:** Three over fifteen? Six over fifteen.

From the excerpt above, to check whether Samuel succumbed to whole number bias in general when performing addition of fractions, after Samuel had presented a particular solution the present researcher asked him what would be the sum of 3/15 and 3/15. Without hesitation, he stated 6/15. While this is the exchange for only one child in the study, the exchange suggests that whole number bias was not the correct explanation for Samuel's answer of 6/30 to the problem of sharing two pizzas among five people. The present researcher did not ask this question of the other children in the study because when she reflected on the aforementioned task-based interview, she felt this questioning could be potentially distracting or leading to the children. This conclusion cannot be extended to other children who used the TPPe/TPAI strategy. Based on the comparison of the actions and verbalisations of the children in the present study and that of Streefland (1991), Charles and Nason (2000) and Yazgan (2010), however, the present researcher initially concludes that the TPPe/TPAI strategy is consistent with quantifying by the part-whole notion, as designated by Charles and Nason (2000), not with whole number bias.

In addition to a comparison with previous empirical literature on children solving partitive quotient problems, the present researcher also examines the TPPe/TPAI strategy from the perspective of mathematics education literature. The steps associated with conceptualising the part-whole sub-construct as put forward by English and Halford (1995) are as follows:

1. Recognise that the parts in the whole are equal
2. Identify the total number of parts (denominator)
3. Identify the number of shaded parts (numerator)
4. Coordinate the number of shaded parts and the total number of parts to obtain the fraction.

These steps presented by English and Halford (1995) relating to the part-whole sub-construct is in keeping with other literature from mathematics education such as Behr *et al.* (1993).

For the present research, based on the excerpts presented in section 6.2.1 and other occurrences where the TPPe/TPAI strategy was used by research participants in the research data, at least one of the fractions representing each person's share corresponds to: $\frac{Total\ number\ of\ pieces\ given\ to\ each\ person}{Total\ number\ of\ pieces\ in\ ALL\ items}$. In general, the steps taken by the children to obtain this fraction are:

1. Identify the number of pieces each person would receive and denote this as the numerator of the fraction
2. Identify the total number of pieces or parts in all the items being shared and denote this as the denominator of the fraction
3. Coordinate the number of identified/shaded parts and the total number of parts to obtain the fraction for each person's share.

An examination of the steps used by the children who used the TPPe/TPAI strategy in the present research and those put forward by English and Halford (1995) for conceptualising the part-whole sub-construct reveals that there is a complete match between the two. Based on this match and the previous empirical considerations, the present researcher argues that, for the present research, 'quantify by the part-whole notion' is an appropriate description for the actions of the children who used the TPPe/TPAI strategy.

8.2.3 The choice of TPPe/TPAI strategy by children who have only been taught the part-whole fraction sub-construct

Empirical research with primary school children as to how partitive quotient knowledge grows out of existing part-whole knowledge presents mixed findings. Some empirical research studies (for example Kerslake, 1986) have found that the part-whole meaning of fractions inhibits the learning of the other fraction sub-constructs. Alternatively, when some teaching accompanies the learning of the partitive quotient sub-construct, other research studies (for example Nunes, 2008) have reported that the part-whole and partitive quotient sub-constructs emerge alongside each other apparently seamlessly. In addressing the issue of diverse results on a particular research topic/subject, Sophian (1997) states that 'when studies conflict, often neither one alone tells the whole story and so our theoretical goal should be to integrate the findings rather than select among them' (p. 287). In keeping with Sophian's (1997) assertion, a key outcome of the present research has been to provide another dimension to the 'story' of children's engagements with the partitive quotient sub-construct. More specifically, the conclusion that the TPPe/TPAI strategy resembles the quantification by part-whole approach is significant in that it provides empirical evidence that for some children, previous part-whole knowledge affects the learning of the partitive quotient fraction sub-construct. In addition to this, while some previous research studies, such as Kerslake (1986) have asserted, quite broadly, that the part-whole meaning of fractions influences or inhibits the learning of other fraction sub-constructs, the present research study has been able to provide evidence as to one specific fraction sub-construct that it does affect, which is the partitive quotient. Another point that underscores the significance of this finding is that when Charles and Nason (2000) identified the quantification by part-whole strategy for finding the fraction associated with a person's share in a partitive quotient problem, they inform that it had not been reported before in the mathematics education literature. The finding of the present research, therefore, adds support to the findings of both Charles and Nason (2000) and Yazgan (2010) that could potentially serve as a springboard for further research into children's engagement with different fraction sub-constructs.

The notion that previous knowings and experiences influence new learning is well established in the domain of mathematics education (Mack, 1990, 2000; O'Toole, 2006). In this regard, McGowen and Tall (2010) proposed the notion of 'met-befores', which they define as 'all current knowledge that arises through previous experience, both positive and negative' (p. 171). They also provide a working definition for this notion, which is 'a mental structure that we have now as a result of experiences we have met-before' (p. 171). They point out that, where the acquisition of new knowledge is concerned, met-befores function in supportive and/or preventative roles, depending on the varying contexts. Taking into consideration the key role that previous knowings have on children's engagement with new knowledge, it is very plausible for children, whose fraction sub-construct learning has only been the part-whole meaning of fractions, to engage with a new fraction sub-construct through a part-whole lens. After all, this impact of previous learning on new learning is one of the underpinnings of the whole number bias. In this regard, Ni and Zhou (2005) state that while children are learning about the notion of fraction numbers, they are 'often overtaken by the aspects of fraction situations that appear to be consistent with what they have known about whole numbers' (p. 29). In most countries, globally, children are exposed for many years only to the part-whole fraction sub-construct, before they engage with other interpretations of fractions (Middleton *et al.*, 2015). The present researcher argues that it is therefore plausible that, based on this long-standing exposure, some children would apply their part-whole learning to new fraction situations involving a different fraction sub-construct, such as in a context involving partitive quotients. This is similar to how some children, based on their long-standing exposure to whole numbers, apply this knowledge to fraction situations.

From a theoretical point of view, Kieren (1993) linked the part-whole and partitive quotient sub-constructs. He revised his initial list of sub-constructs of fractions later to include only the ratio, quotient, measure and operator sub-constructs. He removed the part-whole sub-construct as a separate fraction sub-construct and opted rather to 'subsume it under the measure and quotient constructs as the dynamic comparison of a quantity to a dividable unit that allows for the generation of rational numbers as extensive quantities' (p. 57). Although this decision by Kieren (1993) is not analogous to providing evidence of a link between the quotient and part-whole constructs, it does lend support to the finding from the present research study that children may be using a part-whole lens to conceptualise partitive quotient problems.

As it relates to the link between the part-whole and partitive quotient sub-constructs, from an empirical research perspective, Charalambous and Pitta-Pantazi (2007) examined the associations among the different sub-constructs of fractions using a sample of 646 fifth and sixth graders in Cyprus. Citing results from their data analysis, they report that the partitive quotient task of: 'Three pizzas are evenly divided among four children. How much pizza will each child get?' was associated and related directly with the part-whole fraction sub-construct. They considered this reasonable, because, 'due to the numbers used in the problem, students could also use a part-whole representation to solve this problem' (p. 307). The present research found, consistent with Yazgan (2010), that regardless of the numbers used in the problem (number of items either greater than or less than the number people sharing), some children still used the TPPe/TPAI strategy.

Similar to Charalambous and Pitta-Pantazi (2007), Norton and Wilkins (2010) suggest that it is possible for children to use part-whole understandings in engaging with other sub-constructs.

Based on their research on the measure sub-construct (the other sub-construct Kieren (1993) posited was linked to the part-whole), they note that in solving fraction problems of the form 'if your stick is $\frac{1}{x}$ as long as the stick below, draw your stick', a successful response to the task might indicate only part-whole concepts. They explain this conclusion by stating that it is difficult to infer whether students have produced the specified fractional part, $\frac{m}{n}$, by iterating a unit fractional part m times (that is, using a measure interpretation) or by pulling *m* parts from *n* equal parts. These findings by Charalambous and Pitta-Pantazi (2007) and Norton and Wilkins (2010) not only provide empirical support to Kieren's (1993) conjecture that relates the part-whole sub-construct to the quotient and measure sub-constructs, but also provide further corroboration that the present researcher's conclusion is credible. Based on the discussion points in this section, the notion put forward by the present researcher, that the children who use the TPPe/TPAI strategy are actually conceptualising the partitive quotient problem as per the part-whole meaning of fractions, is deemed to be plausible and viable. Further to this, the present researcher contends that it is wholly plausible to suggest that some children, who have only been exposed to the part-whole fraction sub-construct learning in their schooling over the course of several years, may exhibit a 'part-whole bias' when introduced to the partitive quotient fraction sub-construct.

8.2.4 The pathways of strategy use for engaging with partitive quotient problems

As it relates to moving from one fraction sub-construct to another, Middleton *et al.* (2015) state that the transitional path is convoluted and does not follow a straightforward, one-dimensional path moving in an orderly, linear way from one sub-construct to another. While Middleton *et al.*'s (2015) assertion has been generally accepted, in principle, empirical evidence to support this has been lacking. In this regard, they state that a clear picture of how rational number knowledge develops over time, inclusive of the different sub-constructs, has yet to be discovered.

In keeping with Middleton *et al.*'s (2015) observation, another important finding of the present research, based on the data collected and analysed, relates to the possible ways that children who have only been taught the part-whole meaning of fractions engage with partitive quotient problems. For the present research, five of the nine children in this study utilised more than one approach to quantifying each person's share before achieving stability in terms of strategy use. Further to this, three of the five children who chose the TPPe/TPAI strategy as their strategy of choice, at some point in engaging with the tasks, also used the TPPe/TPOI strategy, which is a correct approach for quantifying a person's share for partitive quotient problems.

Shaffer and Kipp (2013) state that children of all ages have an array of strategies available to them from which they select when attempting to solve a problem. It is interesting and significant to note that, although six children (see Table 6-1 in section 6.4) used some variation of the TPPe/TPOI strategy over the eight tasks, only two of them chose this as their strategy of choice over time. One of these two children (Samuel) also used the TPPe/TPAI strategy alongside the TPPe/TPOI strategy, forming a combined approach of choice. Streefland (1991) observed this way of working by at least one child in his sample. He categorised it as having an absence of cognitive conflict. This means that the learner does not come into a cognitive conflict in situations where different results have been obtained to the same problem. A child accepts that both answers are correct, because different methods were involved.

The present researcher concedes that the pathways presented as part of the findings of this research do not constitute a general picture of how children, in general, who have primarily been exposed to the part-whole fraction sub-construct engage with the partitive quotient. She asserts that for a different sample of children it is possible that other pathways would emerge. The noteworthy feature of this finding is that the present research provides empirical evidence to show a nuanced picture of how the partitive quotient knowledge emerges alongside children's existing part-whole knowledge. This is in contrast to findings provided by researchers such as Lamon (2001), Naik and Subramaniam (2008), and Nunes (2008), who suggest that the part-whole and partitive quotient fraction sub-constructs develop and coexist seamlessly alongside each other in children's growing knowledge. Regarding this point, Lamon (2001) states that it is surprising that students who divide a unit into fourths and designate three of them, obtaining a value of 3/4, cannot make the leap to recognise that when dividing three pizzas among four people, each will receive 3/4 of a pizza. The present researcher contends that this taken-for-granted attitude by teachers and mathematics education researchers to how children should conceptualise mathematical concepts, versus how they are indeed conceptualising them, may potentially hinder progress in understanding children's learning and conceptualisation of fractions and/or their difficulties in developing new understandings from previous knowledge. The present researcher further posits that the teaching of fractions is also hampered if teachers and curriculum planners do not understand as fully as possible children's ways of working, and use it as a springboard to plan for children's instruction. In this regard, Sowder (2000) emphasises that most people underestimate the conceptual changes that occur as children progress in their study of different aspects of number, including fraction sub-constructs.

As it relates to the conceptual changes, as per the different aspects of number that children encounter, such as when moving from whole numbers to the part-whole meaning of fraction and then to the partitive quotient, Hiebert and Behr (1988) assert these result from associated changes in the nature of the unit or the whole. Charles and Nason (2000) allude to this when they state that children who quantified a person's share using the part-whole notion:

> regularly suffered 'loss of whole'. That is, when attempting to quantify each person's share, they were not thinking in terms of the number of pieces in each one whole unit, but rather in terms of: (i) the number of pieces in all the units. (p. 214)

They did not, however, link the change in the nature of the unit to the difference between the conceptualisations of the part-whole and partitive quotient fraction sub-constructs. The present researcher hopes that the following discussion fills this gap.

Behr and Post (1992) state that 'the concept of the whole underlies the concept of a fraction' (p. 13), but that the unit or whole differs for different aspects of number. The typical part-whole situation (see Figure 2-2 in section 2.2.2.1) that children encounter in primary school is that of a single continuous item partitioned into a number of equal parts, which represents one whole or unit. Rarely, if ever, do children encounter part-whole situations where the unit consists of multiple units, such as depicted in Figure 8-1 below.

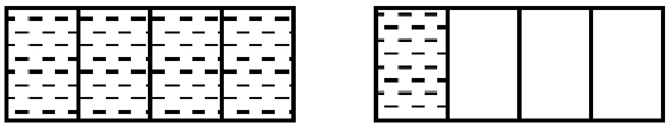


Figure 8-1 Part-whole situation where the unit consists of multiple units

For a similar representation as Figure 8-1, Mack (1990) found that all eight 11-year olds in her study categorised the depiction as 5/8. This is because a child tries to form one unit, which they are accustomed to in their engagement with the part-whole. They form this one unit by joining the two items together. In sharing situations where more than one continuous item is shared among a given number of people, this is the type of shift in the conceptualisations of a unit that children who have only been taught the part-whole are expected to make seamlessly and easily. Previous empirical research has shown that some children do make the conceptual leap from part-whole to partitive quotient conceptualisation. For at least one child in the study of Mack (1990) referenced above, when a similar representation as Figure 8-1 was presented in the context of describing an amount of pizza, a child stated 1 ¼. The present research and others such as Charles and Nason (2000) have shown that some children do not make the conceptual transition from the part-whole to the partitive quotient easily or smoothly. This finding is in keeping with Lamon (2012), who also highlights the 'qualitative leap in sophistication' and the 'very large conceptual jumps' (p. 14) that accompany the changes in the nature of units in the progressive study of number, from whole numbers to different fraction sub-constructs.

In early writing, Payne (1976) highlighted that some children have difficulty in determining the whole when there is more than one object in the whole. Hiebert and Behr (1988) addressed the issue more broadly from the perspective of changes in the nature of the unit, which they point out have far-reaching ramifications. They add that the changes in the nature of the unit from whole numbers, to the part-whole and then to other interpretations of fractions, is cognitively demanding and difficult for children. Kieren (1993) also suggested that the problem faced by the child in Mack's (1990) study could be attributed to confusion regarding unit identification. He added, though, that in relation to the development of one fraction sub-construct after another, 'this knowledge-building difficulty can have a different character' (p. 55). He presented the work of a 12-year-old girl to illustrate this statement. He states that the following task was given in an attempt to provide an environment that would lead to a transition from a partitioning/quotient-based knowing to focusing on fractions as measures: 'A truck with two fuel tanks was shown. A fuel gauge – a number line from 0 to 2 with each unit divided in sixteenths – was also shown with 0, 1 and 2 tanks of fuel labelled'. When the child was asked to circle the indicator that would represent ½ tank of gas remaining, she chose the mark corresponding to one and operated this way consistently, for other similar tasks. Kieren (1993) opines, similar to the present researcher, that 'this problem may have arisen from long, early experience with fractions as part of a static whole'. (p. 56)

In light of the aforementioned discussion, the veracity and the significance of the finding of the present research is further bolstered, that a significant number of children who have only been taught the part-whole fraction sub-construct use a quantification by part-whole approach to finding the fraction for partitive quotient problems.

Another reason why this finding from the present research is important is that it adds to the cross-sectional research findings from studies such as Charles and Nason (2000) and Empson *et al.* (2006) as to children's solving of partitive quotient problems. Lamon (1996) points out that cross-sectional studies, though useful in providing knowledge about performance, are unable to address the central issue of learning, although inferences can be made. Based on the data collected (see the strategy use across tasks for Gabriel, Jack and Mary in Table 6-1 in section 6.4, for example), if only one occurrence or session of children's solving of partitive quotient problems had been collected, the variableness in children's learning would have been missed. The microgenetic approach was therefore instrumental in capturing the participants' learning and growing understanding of the partitive quotient fraction sub-construct.

8.2.5 An argument for generalisation

As was previously discussed in section 4.2.1, the central goal of most qualitative studies, is to present a rich, contextualised understanding of an entity, through the intensive study of a small number of often purposively-selected research participants (Polit and Beck, 2010). In recent times, however, some qualitative researchers, have suggested that a generalisation argument can be made in relation to qualitative research findings (Lewis *et al.*, 2014). This section considers the research findings related to Research Question 1 more expansively by exploring whether the present research findings can be generalised beyond the research participants of the current study.

The present research found that children in the present study, who had only been taught the part-whole fraction sub-construct in formal teaching, used four main strategies to find the fraction associated with solving partitive quotient problems. In addition, one strategy appeared to be associated with the part-whole meaning of fractions. While the current study is unable to draw conclusions relating to the relative prevalence of these strategies among year 5 children (aged 8-10) from the Commonwealth of Dominica (CoD) or elsewhere, the present researcher suggests that these research findings represent a range of strategies that are likely be present in the parent population in the CoD, and in other localities as well. The present researcher makes this assertion because the discussion of the aforementioned research findings in this chapter shows that these research findings are strongly corroborated by previous empirical work. More specifically, three of the four strategies used by the children of the current study were previously reported in empirical research studies (see sections 2.3 and 5.1). Additionally, in two previous studies, in different localities, Yazgan (2010) and Charles and Nason (2000), reported that their research participants utilised the TPPe/TPAI strategy, which shows the specific interference of part-whole learning in the learning of the partitive quotient sub-construct. In addition, for the present research, children from two different sites/schools displayed the TPPe/TPAI strategy as they engaged in solving the partitive quotient problems. The present researcher, therefore, argues that despite the small number of research participants in the present study, it is plausible that the aforementioned findings would be evident for other children, in the CoD and in other localities, who had only been taught the part-whole fraction sub-construct.

Firestone (1993) supports the assertion that consistency of findings in different studies supports a generalisation argument and states, ‘when conditions vary, successful replication contributes to generalizability. Similar results under different conditions illustrate the robustness of the finding’ (p. 17). In this context, the present researcher describes replication from the stance that children in the various studies engaged in solving similar partitive quotient tasks, in which *x* items are shared among *y* people. Patton (2002), similar to Lincoln and Guba (1985), also agrees that some qualitative research findings are transferable or extrapolate from one context to another, despite differences in the characteristics of research settings. In this regard, he states that extrapolations are 'modest speculations on the likely applicability of the findings to other situations under similar, but not identical conditions. Extrapolations are logical, thoughtful and problem-oriented rather than statistical or probabilistic' (p. 584). Based on the aforementioned, the present researcher, in keeping with Patton's (2002) suggestion regarding making modest speculations, argues that the findings and conclusions for the first part of Research Question 1 of this thesis support inferential generalisation or transferability. This type of generalisation involves 'generalising from the context of the research study itself to other settings or contexts' (Lewis *et al.*, 2014, p. 351). Notwithstanding the previous assertion, the present researcher acknowledges that although the argument for generalisation appears robust, additional research, with larger numbers of research participants is needed to further confirm this.

The previous argument also appears to support a generalisation of the findings related to the first part of Research Question 1 to the parent population in the CoD. The present researcher, however, opts to err on the side of caution until the research findings have been validated, and does not argue for this type of generalisation. Furthermore, the finding that for some children there was a competition among strategies before they settled upon using one particular strategy is a new research finding, which has also not been validated by existing empirical research, although the use of various strategies for solving a mathematical problem has been found for other topics in mathematics such as addition. The present researcher, therefore, also does not extend the generalisation argument to this finding.

8.2.6 Summary

Section 8.2 discussed the findings related to the first research question. A more expansive summary of this is presented in section 8.5 at the end of this chapter. The next section discusses the three main findings (see section 7.5) relating to Research Question 2. This question aims to elaborate the DNB feature of the Pirie-Kieren theory and investigates in what way(s) does evidence support or not support the *Don't Need boundary* feature of the Pirie-Kieren theory for the growth of mathematical understanding.

8.3 Research Question 2

8.3.1 Bi-directional Don't Need boundary crossings

The present researcher posits that the finding that learners shift across DNBs bi-directionally is wholly reasonable from three perspectives. First, the notion of a bi-directional DNB crossing is consistent with Pirie-Kieren’s conceptualisation of growth of mathematical understanding as a non-linear process, whereby a learner’s growing mathematical understanding is encapsulated as back and forth shifts across the various layers of the Pirie-Kieren model.

Second, this finding is in keeping with previous empirical research centred on the Pirie-Kieren theory, for example, Towers (1998) and Martin (2008). While the focus of these studies was not the DNBs, their data showed instances of learners who had crossed either or both of the first and the second DNBs returning to inner layers of understanding by the process of folding back. In this way, the present research finding incorporates the notion of folding back, a central feature of the Pirie-Kieren theory (see section 3.5.1). While folding back deals with a learner’s shift from *any* layer of understanding in the Pirie-Kieren model to an inner layer of understanding (Martin, 2008), the folding back associated with bi-directional DNB crossings occurs from either the Image Having or Formalising layer of understanding, which are the layers found immediately outside the first two DNBs, respectively. This distinction sets this present finding apart from previous literature on folding back while also highlighting the compatibility of the DNBs and folding back, two central features of the Pirie-Kieren theory. In so doing, the cohesiveness of the theory is emphasised.

In addition to being consistent with the broad notion of folding back as per the Pirie-Kieren theory, bi-directional DNB crossings are also in keeping with some aspects of Martin’s (1999) elaboration of folding back. As previously stated in section 3.5.1, Martin (1999, 2008) reported that a learner’s shift from an outer to an inner layer of understanding appeared to be prompted by four sources: the teacher/researcher, another student(s), some curriculum material, for example, assigned task(s) or the learner. In accordance with Martin (2008), the present researcher found that folding back, which occurred after a learner had crossed a DNB, appeared, at times, to be triggered by questions posed by the researcher and by the task prompt given to the research participants, as illustrated in Excerpts 7-1 and 7-3 respectively. Considering that Martin’s (1999, 2008) findings related to the sources of folding back have not been validated in empirical literature, the present research finding is notable.

While the aforementioned is significant as per the continued development and validation of the Pirie-Kieren theory, it brings to the fore the issue of validity threats associated with conducting qualitative research. The observed threat in this instance is researcher bias and reactivity, whereby the researcher exerts some influence on the research participant being interviewed. These are delineated in more detail in section 8.4 of this chapter.

A third reason that supports the plausibility of the finding of bi-directional DNB crossings centres on the nature of the images that a learner can hold while operating outside of a DNB. As it relates to the first DNB crossing in particular, considering that learners can possess incomplete, incorrect and hazy images when operating within the Image Having layer, it is understandable that one or more instances of crossing bi-directionally between the Image Making and Image Having layers were observed in the empirical data for the present research. These bi-directional shifts across layers of understanding appeared to be necessary for children's images of the partitioning aspect of the partitive quotient problem to become sufficiently complete and correct, ultimately allowing them to remain working in an outer layer of understanding, such as the Image Having layer for a specific problem and/or partitive quotient problems in general. If, like Sfard (1991), learners could only progress to higher/outer layers of understandings if they had become skilful, skilled, proficient or capable with the mathematical entities at an inner/lower layer, then it is reasonable to expect that the likelihood of folding back would be much reduced. For the Pirie-Kieren theory, however, this is not the case.

This line of reasoning linking bi-directional DNB crossings and the nature of learners' images put forward by the present researcher is also compatible with academic literature not related to the Pirie-Kieren theory. According to Ryan and Williams (2007), errors that children make while engaged in solving mathematical problems are often linked to conceptual limitations. These, in many instances, comprise a 'conceptual structure that needs replacing, developing or otherwise fixing' (p. 14). Within the context of the Pirie-Kieren theory, the conceptual structure that Ryan and Williams (2007) refer to resembles a learner's images for a mathematical concept/topic. Consequently, while a learner may have an image for a particular concept/topic, an incomplete or incorrect image could prevent a learner's mathematical understanding from growing beyond the Image Having layer. Folding back triggered by various factors is therefore instrumental in facilitating the replacing, developing and fixing of these images/structures, thereby allowing a learner's understanding subsequently to shift outside one or both of the DNBs (Martin, 2008).

8.3.2 Uni-directional Don't Need boundary crossings

From the present research data, while bi-directional DNB crossings appear to be more common for learners, there were also several instances of uni-directional DNB crossings. The present researcher posits that, while some learners may have images or conceptual structures needing fixing or replacing, it is wholly possible for others to have developed a correct/complete image or conceptual structure for a given mathematical concept/topic. Ashlock (2002) confirms the present researcher's assertion and succinctly states that 'the mathematical ideas a student learns may be correct' (p. 14). This would allow learners' understanding to grow continually towards outward layers of understanding without the need to fold back.

Apart from being supported by the previously referenced mathematics education literature, the finding of uni-directional DNB crossings for this research is further corroborated by literature associated with children's solving of partitive quotient problems. In this regard, Empson *et al.* (2006) state that conceptualising partitioning in terms of numbers of parts appeared to be instinctive to the children in their study. Peck and Matassa (2016) support Empson *et al.*'s (2006) assertion when they state that sharing seems to come naturally to children. Further to this, according to Lamon (2012), concerning the specific images children develop and hold for the partitioning aspect of solving partitive quotient problems, children often divide each item being shared into the number of people involved in the sharing situation. Therefore, if a number of items is shared among four people, children will typically cut every item into four parts. This suggests that at least for the first solution of solving the given partitive quotient problems for the present research, it is highly plausible that some learners could develop and hold correct and complete images regarding the number of partitions, from initial engagement with the problem. Additionally, it is possible that some learners would be able to engage successfully in finding the first solutions for problems of this type over a sequence of sessions or, within the context of the Pirie-Kieren theory, move progressively to outer layers of understanding. This may explain why for the current study, more uni-directional DNB crossings were found for the first solution, than for subsequent solutions. This aspect, which highlights the link between the nature of tasks and DNB crossings is discussed further in section 8.3.3.

While the finding of uni-directional DNB crossings by learners is supported from the perspective of the partitive quotient task that the participants engaged with, the present researcher contends that this consideration in no way limits this finding only to the present research. She suggests that considering that in primary school mathematics, learners engage with different types of problems, with varying levels of difficulty, it is wholly likely that the finding of uni-directional DNB crossings would also be found for other topics/concepts in mathematics.

Considering that as per the Pirie-Kieren theory, 'mathematical understanding as observed using the model is a dynamic, non-monotonic process' and 'while rings grow outward towards the abstract, formal, general, rigorous and content-free, growth of understanding does not happen that way' (Kieren *et al.*, 1999, p. 218), the uni-directional DNB crossing may seem inconsistent with the theory. The latter part of the previous quote could potentially lead some to think erroneously that, in engaging with a particular concept/topic, folding back has to occur as a learner's understanding grows. The current elaboration of the DNB, consistent with the viewpoint that learners grow in their mathematical understanding differently, creates the space for both uni-directional and bi-directional DNB crossings to be observed in a given group of learners as they engage with a mathematical concept/topic. In this way, this elaboration provides a broader and potentially clearer depiction of how different learners' mathematical understanding may develop, while remaining consistent with the existing Pirie-Kieren theory.

8.3.3 Factors associated with Don't Need boundary crossings

Pirie and Kieren (1994a) state that 'much of the power of mathematics comes from one being able to… construct and work with its ideas in ways which are not dependent on physical contexts, are general and not local in nature' (p. 39). Consequently, a central objective in the learning of any mathematical concept/topic is to facilitate learners crossing DNBs, whereby they shift from working locally and concretely to working permanently at increasing levels of abstraction. This section, therefore, elaborates on three factors that appear to be associated with DNB crossings.

8.3.3.1 The nature of the task

As previously mentioned, for the present research, learners appeared more likely to remain operating in outer layers of understanding after having crossed a DNB during engagement with the first part of the task, 'How can you share *x* items among *y* people fairly' than the second part, 'How else can you share *x* items among *y* people fairly'. For the first part of the task, the research participants generally made the link between the number of sharers and partitions to find a viable number of partitions for the items. Consistent with Empson *et al.* (2006), the first part of the task therefore resembled a one-step problem. For engaging with the second part of the task, which entailed finding subsequent solutions to each problem, however, the concepts of equivalence, multiples, divisibility, repeated addition, basic operations appeared to be utilised alongside the number of sharers. Based on children's engagement, this second part of the task appeared multi-step in nature.

Previous empirical work, for example by Quintero (1983) and Castro-Martínez and Frías-Zorilla (2013), seem to support the finding that the nature of the task is associated with whether a learner crosses a DNB and remains operating outside that boundary over time. Quintero (1983), in discussing mathematical problem solving, informs that 'the semantic aspect of the problem' or 'the concepts and relationships involved in the problem, are a strong determiner of problem difficulty' (p. 102). She suggests that it is likely that learners find multi-step problems more difficult than one-step problems, because there are more concepts and relationships to contend with in the former. Within the context of the Pirie-Kieren theory, folding back is associated with learners encountering difficulties and returning to inner levels of understanding to resolve these challenges. This suggests that the fewer the concepts/relationships/difficulties within a problem, the less likely that folding back occurs and the more probable that a learner will remain operating in an outer layer of understanding.

The aforementioned observation is also in keeping with Newman (1977) and Casey (1978), who developed hierarchies for learners' engagement with one-step and multi-step mathematical problems, respectively. Newman's (1977) hierarchy comprised several more steps than that of Casey (1978). This is consistent with more recent research by Gray and Tall (2007), who also represented one-step and multi-step mathematical problems diagrammatically. Further to this, Casey (1978) found that in solving a multi-step problem, a learner engaged with a number of sub-problems. Casey points out that as learners progress to the overall solution, they often return to lower stages in the hierarchy after each sub-problem has been solved, as well as while solving any particular sub-problem. Although Casey (1978) does not use the terminology of folding back, its essence is evident when he mentions that a learner may return to lower stages in the problem-solving process. Applying this line of reasoning to the context of the present research substantiates that a learner engaging with a less cognitively difficult task, such as a one-step problem, is more likely to remain working in an outer layer of understanding, over time, after having crossed a DNB. This is because, in line with Quintero (1983), for a one-step task, the problem-solving process is more straightforward and there are fewer concepts and relationships to grapple with.

This research finding also links to and extends Martin's (1999, 2008) finding that curriculum material, such as a mathematical task, is a source of folding back. While Martin (1999) found that mathematical tasks could trigger folding back, the present study refined this previous broad finding. More specifically, it found that the semantic aspect of a task, including the number and complexity of the concepts and relationships involved in a given mathematical problem are associated with whether a learner remains operating beyond a DNB over time or folds back to work in inner layers of understanding. Consequently, this finding elaborates not only the DNB feature of the Pirie-Kieren theory, but inadvertently the notion of folding back as well.

8.3.3.2 The properties applied to a learner's images(s)

Another factor that appears to affect whether a learner remains working outside a DNB, as opposed to folding back to an inner layer of understanding, on one or multiple occasions relates to the properties that the learner applies to the image(s) s/he possesses for a particular concept. Ashlock (2002), consistent with the Pirie-Kieren theory informs that during engagements with a mathematical concept/topic/process, learners look for and focus on commonalities or properties of the idea or image. Ashlock (2002) further states that this allows them to form an abstraction with certain common characteristics and their concept is formed. Within the context of the Pirie-Kieren theory, this description bears a striking resemblance to a learner operating outside of a DNB, which represents points of abstraction within the Pirie-Kieren model. Mitchelmore and White (2007) confirm Ashlock's (2002) assertion and add that the identification of key common features or properties of mathematical ideas is important because it allows learners to shift forward in their understanding, so that they can relate situations that were previously seen to be disconnected. In so doing, from the perspective of the Pirie-Kieren theory, learners shift from operating locally and concretely to more generally. Furthermore, Mitchelmore and White (2007) state that learners then form new ideas and 'are incapable of reverting to their previous state of innocence' (p. 332). The aforementioned description closely resembles the crossing of a DNB and remaining in the outer layer of understanding, where a 'previous state of innocence' is analogous to inner layers of understanding as in the Pirie-Kieren theory.

While the last paragraph highlighted that identifying key properties of an image facilitates abstraction, the present research found that if a learner applies a property with restricted applicability to a correct image that s/he holds, crossing of the DNBs permanently and the progressive growth of understanding to outer layers of the Pirie-Kieren model is unlikely to occur. At some point in the learner's engagement with the concept/topic, folding back is likely to occur. This assertion is consistent with Fennema and Romberg (1999) who state that, although the noting of relationships and/or properties is important, not all relationships are mathematically fruitful. In the case of Karen, Mary and Gabriel (see Excerpts 7-8 and 7-9), although they noted that some number of partitions were even or odd, these were not properties that were fruitful in extending their understanding of the number of partitions that each item should be shared into for the different partitive quotient problems. The use of incorrect properties as part of the problem-solving process by some of the learners in this study is also in line with Vergnaud (1987). He states that as learners engage in solving mathematical problems, they sometimes express local and non-coherent properties before they are able to solve the problem(s) proficiently. As was observed for Karen, Mary and Gabriel the property noted was tied to a specific solution that they had obtained and not to the partitive quotient problem being solved or partitive quotient problems, in general. Therefore, it is understandable that while this property was held, they were unable to either cross a DNB or operate outside of a DNB over time.

This particular finding, like others discussed in this section is significant to the elaboration of the DNB feature of the Pirie-Kieren theory but beyond it as well. This is because these findings have particular implications for teaching of the partitive quotient sub-construct. Lamon (2012) highlights this significance in a comment about her research where her participants, similar to the present research, were not given specific instruction on partitioning. She states that 'this type of student work is useful for helping us to understand what children do on their own accord, and hence, how instruction might have to play a role if we want something to happen differently' (p. 177). The specific educational implications arising from the current study's findings are discussed in section 9.4.

8.3.3.3 The nature of a learner's images for a mathematical concept/topic

A third factor that appears to be associated with whether a learner crosses a DNB and remains working outside the boundary relates to the completeness of a learner's image(s) about a mathematical concept/topic and the applicability of these images to all problems of a particular type. While this aspect was discussed in some depth in section 8.3.1, two additional considerations are expounded in this section.

Ashlock (2002) notes that in learning concepts and procedures in mathematics, learners are prone to form incomplete/incorrect images of these concepts/topics. He explicates that this occurs when learners overgeneralise and make a conclusion before there is adequate data at hand, after having engaged with only a few problems of a particular type. Alternatively, he opines, learners may overspecialise in the process of learning a concept and the resulting image has restricted applicability to a problems of a particular type, such as partitive quotient problems. Both these explanations appear to be applicable to Rebecca's image that the number of partitions related to halving and the number of people sharing (see Excerpt 7-9). From as early as the second task, Rebecca appeared to have noted and began applying the halving approach to each item being shared. While this image was appropriate for that task and several subsequent tasks, in Excerpt 7-4, the non-applicability of this approach to the current problem became evident to Rebecca. This resulted in folding back and amendments being made to the present images held.

Overgeneralising and overspecialising have been reported in previous research focused on children's solving of mathematical problems (Ashlock, 2002; Ryan and Williams, 2007), including where children often use their natural number knowledge to engage with fraction computation (Streefland, 1991; Mack, 1995). This overgeneralisation explanation, however, has not been previously applied to the partitioning aspect of partitive quotient problems. Gabriel *et al.* (2013), similar to Siemon (2003), point out that children are frequently exposed to the notion of half very early in life and formal schooling, and 'this limited vision of fractions seems to generate difficulties when it comes to generalization' (p. 9). Considering that Rebecca had been exposed to the half fraction from as early as Grade K (see Appendix A), it is understandable that she would apply this knowledge to the solving of partitive quotient problems, especially since the image resulted in a correct solution when initially applied to the tasks.

The present research also found that Harry used four different images for the partitioning and did not shift beyond the second DNB to work in the Formalising layer of understanding. Based on a close inspection of these images, the present researcher suggests that the differences among the images may not have facilitated Harry's development of an image for the number of partitions for all partitive quotient problems. Ashlock (2002) corroborates the aforementioned suggestion and points out that a critical component of abstraction, which allows a learner to gain an image for all problems of a particular type, is finding commonalities among the ideas/images a learner holds for a concept/topic. Vergnaud (1987) adds that students develop their knowledge in a fairly wide variety of situations whereby they notice the simplest properties and relationships first, then more complex ones, until they master the whole system. The present researcher posits that if the images held by a learner for a concept are too distinctive, key properties may go unobserved by a learner in a sequence of problem-solving sessions. The present researcher does not suggest that this is the only reason which may account for the observation presented, but that it is one plausible explanation for the data analysed.

8.3.4 Addressing contentions and inconsistencies as per the Don't Need boundaries

The three previous sub-sections discussed different aspects of each of the two DNB crossings found by the present research. This sub-section concludes this discussion by addressing a specific contention raised by Wright (2014) about the DNBs that was presented in section 1.3.3 of Chapter 1. It also delineates a potential inconsistency between the present finding and the current Pirie-Kieren theory. In so doing, this section lays the groundwork for a discussion of the second research finding related to the Pirie-Kieren theory in the next three sub-sections.

Kieren *et al.* (1999) state that 'the Don't Need boundaries… represent points of "abstraction" in the model, in that a child who has an image… can work with them *without need of* or recourse to the actions of Image Making' (p. 218). In keeping with Kieren *et al.* (1999), Charles and Nason (2000) state that abstraction is 'a lasting change that enables the identification of the same concepts, structures and relationships in many different, but structurally-similar tasks' (pp. 192–193). Based on the aforementioned descriptions related to abstraction, Wright's (2014) argument that, if a learner has reached the Image Having layer, s/he would hold readily-available images with which to engage related tasks without the need to return to the inner Image Making layer appears reasonable.

A consideration of the *process* of abstraction, however, from the perspectives of previous academic literature and the present research findings, reveals a more nuanced picture than the aforementioned. Dreyfus (2014) defines abstraction as a 'process by which learners try, succeed, or fail to reach a conceptualisation of the structure of a concept or a strategy or a procedure' (p.

5). From this description, learners succeeding in reaching a conceptualisation of a concept in a problem-solving setting closely resemble making a uni-directional DNB crossing, while the learners who try, or try and succeed, are akin to engaging in several bi-directional DNB crossings as they engage with a mathematical concept/topic. This suggests that, while some learners do operate as per Wright's (2014) assertion, some may also engage in folding back, once or on multiple occasions. From this expanded view of the notion of abstraction, the present research finding of two types of DNB crossings is supported and Wright's (2014) contention is addressed.

The final discussion point in this sub-section concerns Kieren *et al.*'s (1999) description of the DNBs presented earlier in this sub-section. The data presented in the first four excerpts in Chapter 7 of this thesis appears to contradict, partially, the aforementioned quote by Kieren *et al.* (1999). These excerpts suggest that while working at the Image Having layer, some children did in fact appear to need the inner layer activities to function optimally for solving the partitive quotient problems. Rebecca's working in Excerpt 7-4 provides an example of empirical data from the current study to support this observation by the present researcher. The excerpt shows that although Rebecca operated within the Image Having layer when she chose ten as the number of partitions to use in each diagram, it appears that the inner layer activities before and following the act of folding back, after that point, were both necessary for functioning mathematically for the topic of partitive quotients, at that time. These inner layer activities following folding back corrected, strengthened and challenged her existing images. The first point of abstraction or DNB in the Pirie-Kieren model is located after the Image Making layer. The layers (Image Making and Image Having layers) immediately associated with this boundary are both based on 'less formal, less sophisticated, less abstract and more local ways of acting' (Kieren *et al.*, 1999, p. 218). Consequently, the present researcher contends that it is plausible for a learner operating in the Image Having layer to refer to these specific previous inner layer images, especially when s/he initially begins operating at the Image Having layer of understanding or, in the case of Rebecca, when present images do not fit the present task.

8.3.5 The first two Don't Need boundaries are distinct

The second research finding related to the elaboration of the DNB feature of the Pirie-Kieren theory is that the first two DNBs appear to be distinct. This section discusses the plausibility of this finding from three perspectives. These encompass compatibility with the existing Pirie-Kieren theory, previous writing within the domain of mathematics education and some of the underpinning tenets of the Pirie-Kieren theory.

First, in the literature related to the Pirie-Kieren theory, Pirie and Kieren (1994b) discuss the Formalising layer of understanding in relation to inner layers, such as the Image Having layer located immediately outside the first DNB. They state, ‘the Formalising layer is the first formal mode of understanding’, in contrast to previous modes of understanding that ‘represent the core of context-dependent, local know how in mathematics’ (p. 39). Kieren *et al.* (1999) provide a further distinction between the two layers. On one hand, Kieren *et al.* (1999) describe the inner Image Having layer as involving ‘less formal, less sophisticated, less abstract and more local ways of acting’, while on the other hand they describe a learner’s engagement within the Formalising layer as ‘abstract, general, rigorous and content-free’ (p. 218). Learners who operate in the Formalising layer have an image *for all* problems of a particular kind, whereas children who operate in the Image Having layer have formed images that are connected to specific problems and/or contexts and may be hazy, incorrect or incomplete.

In light of the aforementioned delineation of a learner operating in layers immediately outside the first two DNBs, it is understandable that, for the present research, the images held by the children who had crossed the second DNB appeared to be much more complete/stable/correct than those of children who had crossed the first DNB. The aforementioned also wholly supports the data that shows that, after children crossed the first DNB, there was a much greater likelihood of folding back to an inner layer than when they crossed the second DNB. Stemming from these observations, the present researcher contends that the suggestion that the first two DNBs appears to be distinct is wholly in order.

Another line of reasoning that supports the second research finding about the Pirie-Kieren theory relates to previous writing within the domain of mathematics education associated with abstraction. In this regard, Hiebert and Lefevre (1986) suggest, in relation to abstraction, it is useful to differentiate:

> between two levels at which relationships between pieces of mathematical knowledge can be established. One level we will call primary. At this level the relationship connecting the information is constructed at the same level of abstractness (or at a less abstract level) than that at which the information itself is represented. That is, the relationship is no more abstract than the information it is connecting. The term abstract is used here to refer to the degree to which a unit of knowledge… is tied to specific contexts. Abstractness increases as knowledge becomes freed from specific contexts.
>
> Some relationships are constructed at a higher, more abstract level than the pieces of information they connect. We call this the reflective level. Relationships at this level are less tied to specific contexts. They often are created by recognising similar core features in pieces of information that are superficially different. The relationships transcend the

> level at which the knowledge currently is represented, pull out the common features of different-looking pieces of knowledge, and tie them together… It is at a higher level than the primary level, because from its vantage point the learner can see much more of the mathematical terrain. (pp. 4-5)

Although this long quote by Hiebert and Lefevre (1986) is independent of the context of the Pirie-Kieren theory, their conceptualisation of abstraction and its associated terms (for example, abstract, abstractness) appears to be in line with the Pirie-Kieren theory. In addition, the 'primary level' that they refer to bears a distinct resemblance to a learner working immediately outside the first DNB, within the Image Having layer, with images tied to specific contexts. The reflective level by contrast, appears to be associated with the second DNB crossing where a learner shifts to work within the Formalising layer of understanding. Table 8-1 shows three key commonalities between Hiebert and Lefevre's (1986) reflective level and Pirie-Kieren's (1994b) Formalising layer of understanding.

Table 8-1 Comparison between Hiebert and Lefevre's (1986) reflective level and Pirie-Kieren's Formalising layer of understanding

Formalising layer	**Reflective level**
Learners, after consciously thinking about the noted properties and abstracting commonalities, is able to express generalities associated with a mathematical concept without specific reference to a particular example, action or image	Learners operate in this level after they recognise similar core features or properties and tie them together
Learners working within this layer are not tied to specific, local contexts since the Formalising layer is the first formal mode of understanding	Learners are less tied to specific contexts
Learners have an image *for all* problems of a particular kind	Learners can see much more of the mathematical terrain

The distinction (primary versus reflective levels) made by Hiebert and Lefevre (1986) mirrors the distinction captured in the second research finding related to the Pirie-Kieren theory and further confirms the plausibility of this particular research finding. It is useful because, similar to Dreyfus (2007), the present researcher is aware that processes of learning and growth of mathematical understanding are complex for both learners and teachers of mathematics in any setting.

Although both boundaries represent the notion that, outside the boundary, a learner can work with his/her images *without need of* or recourse to the actions of Image Making (Kieren *et al.*, 1999), the enactment of this as learners engage in doing mathematics is distinct at each of the first two boundaries of the Pirie-Kieren model. The present researcher posits that capturing and bringing to the fore these nuances are potentially beneficial for both mathematics teachers and researchers who are interested in the Pirie-Kieren theory. This is because it, in effect, peels back the broad notion of the DNB to reveal more explicitly its underlying structure. Specific implications of the findings arising from this and other findings associated with the second research question of this thesis are presented in the next chapter in section 9.4.

The third consideration regarding the plausibility of the second research finding related to the Pirie-Kieren theory is associated with the finding's consistency with the tenets of recursion and level-stepping, which are two key underpinnings of the Pirie-Kieren theory put forward by Vitale (1989) (see sections 3.3.1 and 3.3.2). Pirie and Kieren (1989a, 1991), following Vitale (1989), characterised mathematical understanding as 'levelled', to portray that each level of understanding was not identical to the previous one. Although each level is in some ways defined in terms of itself (self-referencing, self-similar), each level is not the same as the previous level (level-stepping). The theory developers also applied the notion of recursion to their conceptualisation of mathematical understanding to capture the idea that a new layer transcends, but is compatible with previous layers of understanding.

The first two DNBs, if characterised as per the research finding under consideration, would also portray the two above-mentioned tenets as follows:

- **Levelled**: The first two DNBs are not identical.
- **Recursion and level-stepping**: The abstraction exemplified by crossing the first DNB appears to be less general and formal than that of the second DNB, which suggests that the second DNB transcends the first. Despite this transcendence, the two boundaries are wholly compatible.

This particular elaboration therefore remains entirely consistent with the original theory. In this regard, the plausibility of the present finding is further established.

In light of the second research finding related to the Pirie-Kieren theory, the next two sub-sections justify and suggest an amendment to the current Pirie-Kieren model, respectively.

8.3.6 Justifying an amendment to the Pirie-Kieren model

The present researcher suggests that an amendment to the DNB representation in the existing Pirie-Kieren model is in order, based on two considerations. First, the depiction of the DNBs as identical bold rings, in the present researcher's opinion does not capture, diagrammatically, the nuances of the DNB notion of the Pirie-Kieren theory as shown by the findings of the present research. Second, other theories of mathematical understanding (such as Sfard's 1991, theory of reification) have included the notion of abstraction, but the focus was on the abstraction associated with Pirie-Kieren's (1994b) Formalising layer and Hiebert and Lefevre's (1986) reflective level. Although the Pirie-Kieren model locates the DNBs at different places within the layers of understanding, the present researcher argues that the fact that the rings are pictured as identical is not particularly helpful in setting the Pirie-Kieren theory apart from other theories of understanding, where the notion of abstraction is concerned. Neither does it convey precisely what transpires as learners are engaged in doing mathematics, which is a key hallmark of the Pirie-Kieren theory.

8.3.7 Details of the amendment to the Pirie-Kieren model

Stemming from the discussion in sub-sections 8.3.5 and 8.3.6 up to this point, in order to portray the distinction between the first two DNBs of the Pirie-Kieren model, the present researcher suggests that a dotted/broken ring replace the solid ring as the first DNB, located between the Image Making and Image Having layers of understanding. The second DNB between the Property Noticing and Formalising layers could remain as a solid ring. This depicts that, beyond that ring, most learners operate at a more general, formal level of abstraction without the need mentally or physically to reference specific previous images. The dotted/broken ring, by contrast, suggests that the level of abstraction for a learner operating within the Image Having layer is not yet stable, and is more local, more context-bound, than a learner operating within the Formalising layer, hence the greater likelihood of learners operating within the Image Having layer to fold back to the inner Image Making layer. The distinction in the rings also adds clarity to Pirie and Kieren's (1994b) assertion that outside the DNB a learner can operate at a level of abstraction without needing mentally or physically to refer to specific inner layer understanding. Figure 8-2 depicts the amendment proposed by the present researcher.

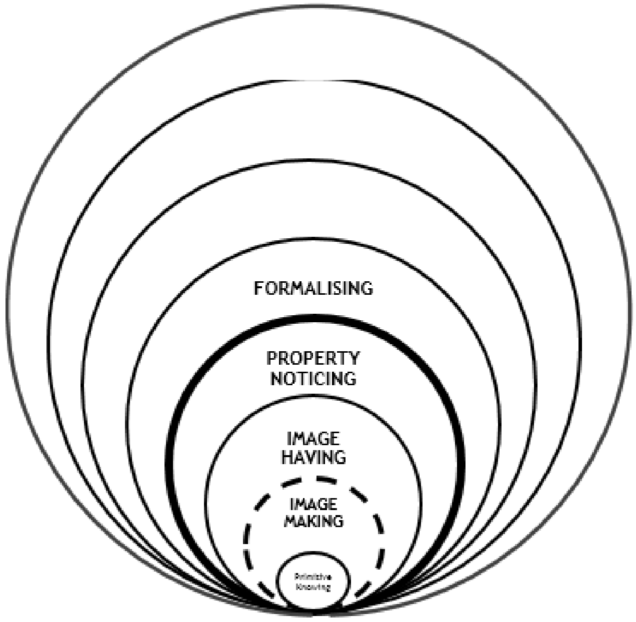


Figure 8-2 Amended Pirie-Kieren model for growth of mathematical understanding

While the present researcher concedes that additional empirical evidence is necessary in order to verify the aforementioned finding of the present research, the amended Pirie-Kieren model presented in Figure 8-2 provides a concrete start to expounding and clarifying the very important, but previously nebulous feature of the Pirie-Kieren model.

8.4 Methodological consideration: The researcher role

In empirical research, trustworthiness, both of the findings presented and conclusions made based on these findings, is a significant consideration. This section concludes the discussion of the research findings by revisiting the issue of validity in light of the empirical research conducted in order to confirm further the credibility of the report presented.

As previously stated in Chapter 4, in conducting qualitative research, particularly interviews, since the researcher is part of the world that s/he studies, the influence of the interviewer on the person being interviewed is inescapable (Maxwell, 2005), even when attempts to minimise the influence are made. Consequently, the observation that the present researcher unintentionally influenced some of the children's responses was not wholly surprising. More specifically, some of the question(s) posed by the researcher, which aimed at gaining insight into children's ways of working, appeared to trigger some children to fold back to an inner layer of understanding as per the Pirie-Kieren model (for example, see Excerpt 7-1). Martin (1999) reports a similar observation from his empirical research that he categorised as a source of folding back, labelled *unintentional* *teacher intervention.* He states that the while the teacher's aim was to check the two students' understanding and 'to validate an existing image for the topic of functions' (p. 75), her question unintentionally prompted one of the two students to fold back.

Considering the aforementioned, Maxwell (2005) suggests that researchers should focus on how they influenced what the research participant said and how this affected the validity of the inferences drawn from the data collected. In this regard, the present researcher found that although the data was collected through one-to-one interactions, the fact that some 'why' questions triggered folding back for some children, is in keeping with what would occur as a learner's understanding of a mathematical concept/topic grows as per Martin (2008) and based on her experience as a teacher of mathematics for over fifteen years. This therefore reinforces the richness and authenticity of the research data and findings, even as it confirms that the findings are relevant to actual children's learning, even though the data was collected outside a classroom setting in which formal learning typically occurs. This particular influence of the researcher brings to the fore the duality of the teacher/researcher relationship discussed in section 4.6.1.2.1. In section 4.6.1.2.1, a distinct tension between the two roles was presented. In the present consideration of the influence that the researcher exerts on his/her research participants; however, the experience of being a teacher has not created a tension, but has a definite advantage. This is because, from extended experience of working with learners in a classroom setting, the present researcher is able to contribute to the validation of the findings reported.

Additionally, since not all 'why' questions prompted folding back (see for example Solution 5 in Excerpt 7-2 and T02/T03 in Excerpt 7-3), the data collected captured a range of possibilities as to how learners' understandings grow and the effect of questioning on a child's solving, thereby adding to the breadth and richness of the data. Besides the aforementioned, the present researcher asserts that the aspect of researcher influence did not extend to the focal elements being investigated, such as the number of partitions children chose to use for each task or the strategies for finding the fraction as per Research Question 1. The researcher therefore posits that the validity threats encountered do not invalidate the findings or conclusions drawn from the data collected. Further to this, as has been discussed throughout this thesis, the study of the notion of mathematical understanding is difficult for various reasons, including its susceptibility to validity threats such as researcher bias or reflexivity. Existing research, for example by Pirie and Kieren (1994b) and Dreyfus (1991), have however, categorically emphasised and shown that it is tremendously important to investigate this type of phenomenon via empirical research. This is because the resulting research findings can potentially contribute to the learning and teaching of mathematics in general, and to difficult topics such as fractions in particular.

In addition to the aforementioned, the present researcher maintains that the validity of the inferences drawn from the data are robust because, for the present research study, the criteria for good standards for conducting qualitative research (see section 4.2.1.1) were rigorously followed. Some of these include that the research supervisors of this thesis served as auditors of this research study who thoroughly reviewed in an ongoing fashion, all aspects of the research, such as the research design, and the procedures for data collection and analysis. Furthermore, numerous thick descriptions, such as verbatim transcripts of interviews that allow others to make decisions regarding the validity of the findings, were used in presenting the findings of this research in Chapters 6 and 7. In addition, multiple participants from two primary schools, multiple tasks and task-based interviews were used to facilitate triangulation, a key aspect of establishing validity in qualitative research.

8.5 Summary

Chapter 8 focused on an extensive discussion of the research findings presented in Chapters 6 and 7. Section 8.2 discussed children's strategies for finding the fraction related to solving partitive quotient problems. In particular, the discussion showed that:

- the use of the 1/TPPe strategy appeared to be linked to gaps in a participant's fraction knowledge, whereas, the use of the TPPe/TPAI strategy by a significant number of children in this study seemed to have deeper conceptual roots;
- the TPPe/TPAI strategy was found to resemble the quantification by part-whole notion, which is consistent with the findings of two previous empirical studies;
- the present research provided empirical evidence to show that for some learners, the part-whole meaning of fractions affects the learning of the partitive quotient;
- the shift from part-whole knowing to that of partitive quotients required a corresponding shift in the conceptualisation of the unit of a fraction;
- partitive quotient knowledge does not emerge seamlessly alongside children's existing part-whole knowledge, in contrast to some previous empirical studies.

Section 8.3 discussed the findings related to elaborating the DNB feature of the Pirie-Kieren theory. The main points of this discussion are that:

- the plausibility of the findings of two types of DNB crossings and three factors associated with DNB crossings were examined and established as per the existing Pirie-Kieren theory and mathematics education academic literature;
- Wright's (2014) contention regarding the DNBs was found to have some merit, but did not consider learners' growth of mathematical understanding broadly enough;

- two sources of folding back: the teacher/researcher and the task as per Martin's (1999) elaboration of folding back were validated and elaborated, highlighting the compatibility and cohesiveness of two main features of the Pirie-Kieren theory;
- it is plausible that a learner operating in the Image Having layer refers to specific Image Making layer activities and images, in contradiction with the present Pirie-Kieren theory;
- the existing Pirie-Kieren theory, previous writing within the domain of mathematics education; and underpinning tenets of the Pirie-Kieren theory support the finding that the first two DNBs appear to be qualitatively different;
- an amendment to the DNBs whereby the first DNB is replaced with a broken ring to portray the greater likelihood of folding back at that point was suggested.

The chapter concluded by discussing the robustness of the research findings and conclusions, despite being exposed to the validity threat of researcher bias and reflexivity.

The next chapter completes this thesis by outlining the specific contributions made by this empirical research. The implications of this research for education and recommendations for future research are also detailed.

Chapter 9: Contributions, implications and recommendations

9.1 Overview

This thesis centres on fraction learning and mathematical understanding, two well-researched but still currently challenging domains of inquiry in mathematics education. While these two broad research areas are markedly distinct, the present research has bridged the gap between these foci to explore aspects of both entities in a complementary manner.

Chapter 9 concludes this thesis and in so doing links previous research to this study by considering the specific contributions that the current research has made to the academic literature, in light of the gaps in the empirical literature, presented in Chapter 1. At the same time, this chapter links to the future by presenting the specific implications arising from the findings and several suggestions for future empirical research inquiries stemming from the current study. While this research has many strong elements, this chapter, in an attempt to present a balanced view of this research, briefly outlines several limitations of this study. This chapter ends with a brief reflection on the empirical work undertaken by the researcher.

9.2 Contributions to the empirical literature

The first contribution of the present research to the empirical literature is that it has added to the small number of research studies which focus on how one fraction sub-construct develops from existing fraction knowledge and, more specifically, on the development of fraction knowledge/sub-constructs from existing part-whole knowledge. In this regard, the present research has provided empirical evidence to show that, for some children who have only been taught the part-whole meaning of fractions, this part-whole learning affects how they engage with the partitive quotient fraction sub-construct and quantify each person's share. Two key findings of the current research underpin this contribution. These include that one of the two main strategies used by children who have only been taught the part-whole fraction sub-construct appeared to resemble a part-whole conceptualisation for finding the fraction related to each person's share. In addition, by the third task, most children in this study exclusively used the aforementioned quantification by part-whole strategy for solving the partitive quotient problems.

Further to the aforementioned contribution, the present research has contributed to the mixed findings as to how partitive quotient knowledge develops from existing part-whole knowledge.

More specifically, the present research has provided empirical evidence to show that for some children who have only been taught the part-whole meaning of fractions, this part-whole learning affects how they engage with the partitive quotient fraction sub-construct and quantify each person's share. Consequently, the current study suggests that the part-whole and partitive quotient fraction sub-constructs do not develop alongside each other seamlessly, contrary to some previous empirical research. Two key findings of the current research underpin this contribution. These include that one of the two main strategies used by children who have only been taught the part-whole fraction sub-construct appeared to resemble a part-whole conceptualisation for finding the fraction related to each person's share. Furthermore, by the third task, most children in this study used the aforementioned quantification by part-whole strategy, exclusively, for solving the partitive quotient problems.

In addition to this, quite significantly, the current research study has provided a more nuanced picture of how partitive quotient knowledge develops from existing part-whole knowledge than currently exists in the empirical literature. One aspect of the present research findings that sets it apart from existing empirical research on this topic of inquiry is that it presented various portraits and/or pathways both within a given task-based interview and across a sequence of task-based interviews, as to how children's partitive quotient knowledge develops from their existing part-whole knowledge. In so doing, it has added to the existing empirical discourse about the impact of the part-whole on children's learning of the partitive quotient in a more meaningful and explicit way than just suggesting, like Kerslake (1986) and Charles and Nason (2000), that the part-whole serves as a possible inhibitor of the development of other interpretations of fractions.

Another contribution of the present research is that it has added to the small body of existing empirical research (for example by Charles and Nason, 2000) that puts forward the quantification by part-whole notion as another plausible or a rival explanation to that of whole number bias, for one of the results that children typically obtain when finding the fraction related to solving partitive quotient problems. The present researcher also suggested that some children who have only been taught the part-whole fraction sub-construct may exhibit a part-whole bias when engaging in solving partitive quotient problems. This part-whole bias is similar to the whole number bias that children who have only been taught whole number knowledge exhibit when first engaging with fractions. Sowder (2000) points out that there are immense changes in thinking and adaptations required for children to move from 'operating on whole numbers to… rational numbers' (p. 73). The present researcher adds, based on the findings of the present research, that perhaps the immensity of the changes may extend to moving from one fraction sub-construct to another.

A fourth significant contribution of the present research is that it has situated the concept of the partitive quotient sub-construct within the strategy choice empirical literature. The finding that most children used a variety/combination of strategies to find the fraction amount that each person receives from solving a partitive quotient problem, and that the use of multiple strategies was evident as children engaged in one task and over the sequence of eight tasks, provide the evidence of this contribution. In this regard, the present research found, consistent with the research on strategy choice for other topics in mathematics, that for some children of the present study there was a competition between strategies before they settled into the use of one main strategy for solving partitive quotient problems. The use of the microgenetic design facilitated these findings and so the present research has added to the growing body of empirical research that shows that this research design is particularly useful in investigating how people learn and understand a specific topic/concept.

The present research has also made a significant theoretical contribution to the Pirie-Kieren theory for growth of mathematical understanding by elaborating the DNB feature of this theory in several ways. As discussed in Chapter 1, a gap in the academic literature associated with Pirie-Kieren theory existed because the DNBs remained as one of its key features that had not been previously elaborated in empirical literature. Considering that the Pirie-Kieren theory for growth of mathematical understanding is a well-established theory in the domain of mathematics education, the continued refining and expansion of this theory is deemed a worthwhile undertaking by the present researcher, consistent with Sriraman and English (2010).

Using the description of the DNB feature and associated examples provided by the developers of the Pirie-Kieren theory as a starting point, the present research has provided greater insights into the DNBs and to how learners' understanding of a mathematical concept/topic, as conceptualised by the Pirie-Kieren theory, grows. Furthermore, this elaboration may afford future users of the Pirie-Kieren theory a greater understanding of the theory, in the same way that Martin's (2008) and Towers' (1998) work on folding back and teacher interventions, respectively, helped the present researcher, as a new user of the Pirie-Kieren theory, to understand the theory more fully.

The present research has also contributed to the mathematics education literature on abstraction, from the context of the Pirie-Kieren theory. The finding of the current research, that the Pirie-Kieren theory appears to support the notion of at least two layers of abstraction as a learner engages in learning a mathematical concept/topic, underpins this contribution. Abstraction is recognised as an important process (Dreyfus, 2007; Mitchelmore and White, 2007) and 'one of the most relevant features of mathematics from a cognitive viewpoint' (Ferrari, 2003, p. 1225). Within the domain of mathematics education, however, the term has a multiplicity of meanings and is fraught with complexities that many researchers would prefer to avoid

(Mitchelmore and White, 2007). Nevertheless, Pirie and Kieren (1994b), in discussing the Pirie-Kieren theory, point out that a key strong point of mathematics is the capacity to operate at a level without needing to reference basic concepts, which must form part of any theory of mathematical understanding. Delineating this aspect of the theory is therefore of particular significance to the Pirie-Kieren theory, in particular, and to mathematics education literature in general.

9.3 Limitations of the research

In conducting any empirical research, there are limitations. For the present study, the limitation related to researcher bias was addressed in section 8.4. The small number of research participants constituting the sample for this research is another limitation of this study. While this study is exploratory and qualitative in nature, and investigates entities that require fine-grained analysis, thereby justifying the small number of participants, a larger number of research participants could potentially strengthen the generalisation argument put forward in section 8.2.5. Section 9.5 outlines recommendations for future research related to this limitation.

Finally, Pirie and Kieren (1994b) have noted the limitation of any model that attempts to capture an individual's growth of understanding, since understanding is not visible, but can only be inferred by what an individual says and does. In this regard, the present researcher notes a limitation related to the use of the Pirie-Kieren theory and associated model. Notwithstanding the extensive empirical literature related to the theory available, she thought that the information outlined by the writing of the theory developers was somewhat inadequate in helping her identify learners' images for a particular topic in mathematics or use of the Pirie-Kieren model for assigning learners' verbalisations and actions at each layer of the model. The present researcher mitigated this particular limitation through direct communication, on several occasions, with Lyndon Martin, a researcher who has utilised the theory for the past two decades in his research work, elaborated the theory in several ways and worked directly with the theory developers.

9.4 Implications for teaching

With reference to the first research question of this thesis, four implications related to teaching are presented next. First, the present empirical study showed that previous part-whole learning affects children's development of the partitive quotient. Stemming from this finding is a natural consideration of the order in which fraction sub-constructs should be taught. The present researcher, however, does not consider that categorically stating which fraction sub-construct is most appropriate to teach first or next is of central importance. She posits that while a teacher may make varied decisions as to the order in which to teach the various fraction sub-constructs, knowing the different understandings/conceptualisations that students may have as they engage with partitive quotients, for example, having learnt the part-whole, is potentially useful to teachers. This point also brings to the fore questions to be considered by teachers and curriculum developers as they plan for teaching the partitive quotient after the part-whole in particular, if this is the path they choose to take, or more broadly for teaching the various fraction sub-constructs in general. Some of these questions include:

- Since previous fraction sub-construct learning can/may affect the learning of a new fraction sub-construct, in what ways does previous fraction sub-construct learning affect current fraction learning?
- How do teachers address students' errors and misconceptions stemming from previous fraction sub-construct learning effectively?
- How do teachers effectively transition from teaching one fraction sub-construct to another?

The present research has provided some concrete findings and new knowledge where the first question is concerned. Future research can address the two remaining questions.

A second implication related to Research Question 1 arises from the finding that the research participants used four approaches to find the fraction as they engaged in solving the partitive quotient problems. Although three of these strategies provided by the research participants resulted in an incorrect answer, the findings of the present research provide key insights relating to children's engagement with partitive quotients. In this regard, Empson and Jacobs (2008) comment that:

> In typical classrooms in most countries, when teachers ask questions, they listen for whether the learner knows what has been explained or can do what has been shown… Many simply may not realize that children have their own mathematical ideas and strategies, which can differ from teachers' own thinking about mathematics, and so they do not expect to hear these ideas and strategies. (p. 257)

As pointed out in Chapter 4 of this thesis, researchers, for example Ginsburg *et al.* (1998) and Steffe and Olive (2010), and teachers generally accept that key insights can be gleaned from children's incorrect answers to problems. Not only does this research provide teachers with children's approaches to finding the fraction for partitive quotient problems, it also proposes and discusses possible explanations for two of the three incorrect approaches used by the participants. Teachers may find this information helpful in two main ways as they prepare to teach partitive quotients. This knowledge may assist teachers in addressing misconceptions if or when they arise during teaching, or to structure teaching in a way that prevents a student from making particular errors.

A further implication for teaching stems from the finding that several research participants used multiple approaches in finding the fraction, either within a particular task or across a sequence of tasks. The approaches used often included a correct and incorrect approach, for example, the TPPe/TPAI and TPPe/TPOI strategies. This study has showed that some children engage in strategy selection as they consider their previous part-whole knowledge and the new developing partitive quotient knowledge. While the various pathways of learning that the children in this study traversed may not be the same pathways that other Year 5 children (aged 8-10) may take, this finding makes teachers aware that some children may engage in this competition between approaches. Since most children settled into using a particular approach to solving the problems, it also suggests that children need sufficient time, several lessons perhaps, to grapple with their varied conceptualisations until they arrive at a point of equilibrium.

The present research also found that it is plausible that children's use of a part-whole conceptualisation to engage with the partitive quotient fraction sub-construct may be linked to their understanding of the concept of a unit. A fourth implication for teaching stemming from this is that in teaching of fraction sub-constructs there should be a greater focus on the concept of a unit in the different fraction sub-construct contexts. This would potentially allow for more efficient transitioning from the teaching and learning of one fraction sub-construct to another.

Additionally, the present researcher suggests that teaching that explicitly highlights the differences among fraction sub-constructs may be useful in helping children to develop a complete understanding of the fraction concept, for example what constitutes the unit or whether the resulting fraction a/b is an extensive or intensive quantity. This is so that children are made more keenly aware that the same fraction symbol may mean different things in different contexts, in the same way that a tyre that is most commonly seen in a vehicle may be also used on the playground or in the home for planting flowers. Arising from this point, since it is presently generally accepted that children should be taught all the fraction sub-constructs in school when the topic of fractions is introduced, it should be made explicit to students that they will be encountering different meanings/sub-constructs of fractions later in their learning. This may better prepare them for the new knowledge as it emerges over their time in primary school and high-school.

Two practice-related implications of the research findings related to the Pirie-Kieren theory are presented in the remainder of this section. The present study found that three factors appeared to be associated with DNB crossings. As mentioned in various sections of this thesis (for example section 8.3.3), gaining insights into what may facilitate a learner in the process of abstraction is critical for any teacher of mathematics. One implication associated with teaching stemming from the aforementioned three factors is that teachers need to assist learners to develop as complete and accurate image as possible of a particular mathematical topic while operating within the Image Making layer. This requires teachers explicitly to think about what images/ideas a learner may have about mathematical topics as they plan for teaching. It also requires them to consider consciously which properties are essential to a particular image and which are not, and how to facilitate students' noting of these properties. Additionally, teachers may need to plan how they can find out more about these ideas/images during the course of instruction as learners' understandings grow over time. This implication is consistent with Dreyfus (1991), who asserts that research that focuses on the growth of mathematical thinking and/or understanding has the potential of making teachers of mathematics more conscious of what is occurring as a learner's understanding develops. This helps teachers to introduce such action explicitly in their classrooms or factor it into planning for teaching. This assertion also links to the next implication related to teaching.

A second implication relating to teaching stems from the notion that, within the context of the Pirie-Kieren theory, there are layers of abstraction. From the research data analysed, learners typically shift to operating within the Image Having layer, where understanding is still linked to local contexts, before moving to the Formalising layer, where a learner has a mental conception for a class of problems. The aforementioned highlights that a child's correct solving of related problems does not necessarily mean that s/he is operating within the Formalising layer of understanding. It also suggests that one of the aims of teachers' questioning should be to help them to ascertain whether a learner is operating at the first or second layer of abstraction as per the Pirie-Kieren theory since, based on the findings of the current research, this could impact a learner's propensity to fold back to inner layers of understanding.

9.5 Recommendations for future research and dissemination

The present research has shown that some children's engagement with the partitive quotient sub-construct is impacted on by the part-whole meaning of fractions. Since there are only a few empirical studies to have reported this way of working by children, future research could focus on investigating with a larger number of research participants, the prevalence of part-whole interference when children engage in solving partitive quotient problems.

Another recommendation is that future research could focus on exploring teachers' strategies for finding the fractions related to solving partitive quotient problems and their conceptualisations of these fractions. Previous empirical research focused on the teaching of various aspects of fractions has shown that teachers often struggle to understand fractions, which in turn makes the topic very challenging to teach. It would be interesting to compare and contrast the conceptualisations of students and teachers. Alternatively, further research could focus on addressing children's incorrect conceptualisations of the partitive quotient in a teaching experiment.

As it relates to the DNBs of the Pirie-Kieren theory, there are three recommendations for future research. First, further research also focused on elaborating the DNBs feature of the theory would be beneficial to validate the present research findings. A second recommendation is that further research could be conducted in a classroom setting where children work in small groups as compared to individually, which was the approach used for the present research. Third, the DNBs could also be investigated in relation to other aspects of the Pirie-Kieren theory, such as teacher interventions and, in so doing, continue to link the different parts of the theory together, as was shown for the present research.

The present research has put forward several important contributions to the professional knowledge of primary school teachers in particular and teachers of mathematics in general, as it relates to fraction learning. Ensuring that this knowledge is disseminated to both researchers and teachers is therefore a natural and significant consideration. In this regard, the present researcher proposes two main ways of dissemination. First, various findings of the present research could be presented at conferences, such as that of the Association of Teachers of Mathematics (ATM) and the British Society for Research into Learning Mathematics (BSRLM), that cater to both researchers and teachers of mathematics. Second, the research details and findings could be published in a variety of journals and/or other publication types, such as teacher periodicals, so that a wide cross-section of both professionals and lay(audience) could gain access to the present research.

9.6 Personal reflection and concluding remarks

Undeniably, undertaking of this empirical research was one of the most challenging, yet rewarding, journeys I have ever embarked on. While I recognise that this empirical research constitutes only a small part of the research associated with mathematics education in general and with fractions and mathematical understanding in particular, I consider it to be an important contribution to the domain of mathematics education and at a personal level as well.

At the start of this project, I thought that I had a somewhat extensive knowledge of teaching and learning of mathematics. This research, however, has challenged this initial view. While I am keenly aware of how markedly my understanding of learning, teaching and research has developed over the past four years, I recognise that there is much more learning and growing to be had. Consequently, I opt to conclude this thesis with a quote by Pirie (1988), previously presented in Chapter 3, of this thesis, that I have come to embrace fully:

> In all actuality, we can never fully comprehend 'understanding' itself… with each step that we take forward in order to bring us nearer to our goal, the goal itself recedes and the successive models that we create can be no more approximations, that can never reach the goal, which will always continue to possess undiscovered properties. What we can, however, do is attempt to categorise, partition and elaborate component facets of understanding in such a way as to give ourselves deeper insights into the thinking of children. (p. 2)

Appendix A Programme of study for mathematics, grades K-4, CoD

Programmes of Study for Mathematics of the Commonwealth of Dominica

GRADE K		GRADE 1	
Understand whole and a half	▪ Identify and discuss a whole and half of an object. ▪ Colour to show halves and wholes of given diagrams or objects. * ▪ Divide objects in different ways to show halves (e.g. cut, share, fold, colour)	**Understand whole, half and quarter**	▪ Identify and discuss a whole and parts of a whole. ▪ Identify and discuss one-half and one-quarter of a whole. ▪ Explain what one-half and one-quarter mean ▪ Represent one-half and one quarter of a whole. Read and write the fractions ½ and ¼.

GRADE 2

LEARNING OUTCOMES	SUCCESS CRITERIA
Demonstrate an understanding of number up to 100	▪ Count and make sets up to 100 objects in a variety of ways ▪ Count by 2's, 5's, and 10's to 100 and beyond. ▪ Count on from a given number. ▪ Play games to develop number sense (bingo, matching, jigsaws etc.) ▪ Identify, discuss, use and write numbers up to 100 and represent them in a variety of ways ▪ Compare and order sets of numbers in a variety of ways ▪ Use a calculator, pencil and paper procedures, or mental strategies to investigate number concepts.
Create and solve simple problems involving place value	▪ Use games and sorting activities to discuss and state the place value of any two digit number ▪ Discuss and write two digit numbers in expanded forms (e.g. 27 = 20 + 7) ▪ Create and solve problems involving place value
Create and solve real life problems involving addition and subtraction with numbers up to 100 and involving multiplication and division of one and two digit numbers.	▪ Discuss and use several strategies to recall the basic facts for addition and subtraction up to 20. ▪ Use several strategies to add a two-digit number to a one or two digit number, without and with regrouping, totals up to 100. ▪ Create and solve problems involving addition and subtraction of whole numbers with totals up to 100. ▪ Discuss and use several strategies to subtract a one or two-digit number from a two-digit number, without and with regrouping. ▪ Discuss and use several strategies (e.g., concrete objects, skip counting, properties of multiplication) to develop the multiplication basic facts for the 2, 5, and 10 times table. ▪ Create and solve simple problems involving multiplication and division using concrete objects
Use and write simple fractions in a variety of ways in real life situations	▪ Identify and compare simple fractions using concrete materials (halves, quarters, eighths) using games and puzzles ▪ State and write in numerals the proper fraction that corresponds to a pictorial or concrete representation of a fraction of a whole. ▪ Describe real life situations that involve fractions of a whole.

SUBJECT: MATHEMATICS	STRAND: 1 NUMBER	KEY STAGE 2		
ATTAINMENT TARGET: NUMBER				
The learner will be able to apply number operations and relationships with speed and accuracy to solve daily problems using a variety of strategies	GRADE 3		GRADE 4	
	Use and write fractions in a variety of ways in real life situations	▪ Identify, discuss and compare simple fractions using concrete materials (halves, thirds, quarters, eighths) ▪ Discuss and write, in words and numerals, the proper fraction that corresponds to a pictorial or concrete representation of a fraction of a whole. ▪ Discuss and describe real life situations that involve fractions of a whole. ▪ Calculate a fraction of a group of objects, using concrete objects, pictures/diagrams in real life settings. ▪ Create and solve problems involving simple fractions	**Solve simple problems involving elementary fractions**	▪ Identify unit and proper fraction of a whole or group of objects ▪ Represent unit and proper fraction of a whole or group of objects ▪ Compare halves, quarters, thirds, eighths and tenths using fraction pies in real life settings. ▪ Find simple equivalences between wholes, halves, quarters, thirds, eighths and tenths using fraction pies

Appendix B Consent forms

UNIVERSITY OF Southampton

CHILD'S CONSENT FORM (*v1.1*)

Study title: Exploring the growth of mathematical understanding of fractions: Main study

Researcher name: Lois Grace George

Study reference: GMUPQ_study

Ethics reference: 13550

Please put the first letter of your name and surname in the boxes if you agree with the statements:

I have read and understood the information sheet (10 January 2015/v1.1) and have had the opportunity to ask questions about the project.	☐
I agree to take part in the task-based interviews involving solving fraction problems and be video-recorded.	☐
I understand my participation is voluntary and that I may withdraw from the study at any time.	☐
I understand that my responses will be made anonymous in reports of the research.	☐

Data Protection

I understand that information collected during participation in this study will be stored on a password protected computer and that this information will only be used for the purpose of this study. All files containing any personal data will be made anonymous. All video data will be destroyed after the project end.

Name of child (print name)..

Signature of parent or guardian..

Date..

PARENTAL CONSENT FORM (*v1.1*)

Study title: Exploring the growth of mathematical understanding of the partitive quotient subconstruct of fractions in year 5 children: Main study

Researcher name: Lois Grace George

Study reference: GMUPQ study

Ethics reference: 13550

Please initial the boxes if you agree with the statements:

I have read and understood the information sheet (10 January 2015/v1.2) and have had the opportunity to ask questions about the study.	☐
I agree for my child to take part in the task-based interviews involving solving fraction problems and be video-recorded.	☐
I understand participation is voluntary and that I, or my child, may withdraw from the study at any time.	☐
I understand that my child's responses will be anonymised in reports of the research.	☐

Data Protection

I understand that information collected during participation in this study will be stored on a password protected computer and that this information will only be used for the purpose of this study. All files containing any personal data will be made anonymous. All video data will be destroyed after the project end.

Name of child (print name)...

Name of parent or guardian (print name) ....

Signature of parent or guardian...

Date...

Appendix C Letter to parents

Dear Parent/Guardian,

A researcher from the University of Southampton, Lois George, will be carrying out a research study on grade 5 children's understanding of fractions and your child has been chosen as a suitable participant. Your child was chosen because he/she is doing well in mathematics, seems to enjoy the subject and is likely to engage in discussion about his/her solving of mathematics problems. The purpose of the research is to learn about how Grade 5 children's mathematical understanding of sharing items among different numbers of people develops over time. Miss George is a trained mathematics teacher and is well known as a mathematics educator in the Commonwealth of Dominica.

The research will involve children taking part in task-based interviews where they will be engaged in solving fraction problems with Miss George, for six weeks – twice per week. Each session will be between 30 and 45 minutes long. The tasks are related to the maths curriculum and children who have already tried the activities have found them enjoyable. During the session, children will be video-recorded, so that their gestures and explanations can be analysed later. The video will only be seen by Miss George and two University supervisors (Dr C. Voutsina and Mr. K. Jones). The recordings will be securely stored on a password protected University computer at all times, and will be destroyed once the research project is finished.

Attached to this letter is a Parent Information Sheet (document: 10 January, 2015/v1.1), which contains full details about the research project. Please read this information sheet, and decide whether you would be happy for your child to take part in this research. Contact details are also provided, in case you would prefer to discuss the research before deciding. If you are willing for your child to take part, please sign and return the consent form (document: 10 January 2015/PCF_v1.1) attached.

Yours sincerely,

....

Lois George
Researcher, University of Southampton

10 January 2015 Letter Version 1.0 GMUPQ_study

UNIVERSITY OF Southampton

Parent Information Sheet

Study Title: Exploring the growth of mathematical understanding of the partitive quotient subconstruct of fractions in year 5 children: Main study

Researcher: Lois Grace George **Ethics number**: 13550

Please read this information carefully to decide whether your child may take part in this research.

What is the research about?
The purpose of the research is to learn more about how children's mathematical understanding of sharing items among a certain number of people develops over time.

Why has my child been chosen to take part?
The research is specifically investigating Grade 5 pupils, and pupils were chosen with the support of the class teacher. Pupils have been chosen who are doing well and enjoy mathematics, can do certain fraction tasks and so that they will find the research activities fun.

What does taking part involve?
If you give permission for your child to take part, and your child agrees, your child will take part in maths-activity sessions with Miss George for six weeks. Children are expected to attend sessions twice per week. Each session take place during lunch time and will be between 30-45 minutes long. In each session, your child will carry out maths tasks related to partitive quotients, and each session will be video-recorded.

Why do you need to video-record sessions, and who will see the video?
The purpose of the video is to allow analysis of children's written work, movements (e.g. pointing, cutting) and speech, to gain more information about how children's understanding of this concept develop as they complete the assigned tasks. The video-recording of your child will be analysed by the researcher and two supervisors at the University of Southampton (Dr Charis Voutsina and Mr. Keith Jones), and <u>not seen by anyone else</u>. The video recordings will be securely stored in accordance with University policy, on a password protected University computer at all times, and destroyed once the research project is finished.

Are there any benefits in my child taking part?
Taking part will give your child the opportunity to work on his/her fraction skills in new tasks, with one-to-one attention. Taking part will also benefit others, since it will add to the knowledge we have about children's learning, which can be used to improve and support maths education.

Are there any risks involved?
There are no risks associated with the study beyond the everyday risks of being in school.

Will my child's participation be confidential?
Yes. Data will be collected in accordance with the Data Protection Act and University policy. Data will be coded and kept on a password protected computer at all times. After the analysis, the video recording will be destroyed. Your child will not be identifiable from the research report, or by

UNIVERSITY OF Southampton

anybody outside the classroom. Your child's class teacher and fellow pupils will be aware that your child is taking part in this research, as the sessions will take place during school hours.

What happens if I change my mind?
You may withdraw your permission at any point. In addition, it will be explained to your child that they are free to stop participating at any point they choose.

What happens if something goes wrong?
In the unlikely case of concern or complaint, you should contact the University's Head of Research Governance, Dr Martina Prude. Dr Prude can be reached on 02380 595058, or by email at mad4@soton.ac.uk.

Where can I get more information?
If you have any questions about the research project or your child's participation, please contact me, Lois George, at lgg1g13@soton.ac.uk or at 1-767-4491360. I will be happy to answer your questions, either by email, phone, or by arranging to meet you before/after school if this is convenient. If you wish to leave a message with your child's class teacher or at the school's office, I can also be contacted this way.

UNIVERSITY OF Southampton

PARENTAL CONSENT FORM (*v1.1*)

Study title: Exploring the growth of mathematical understanding of the partitive quotient subconstruct of fractions in year 5 children: Main study

Researcher name: Lois Grace George

Study reference: GMUPQ_study

Ethics reference: 13550

Please initial the boxes if you agree with the statements:

I have read and understood the information sheet (10 January 2015/v1.2) and have had the opportunity to ask questions about the study.	☐
I agree for my child to take part in the task-based interviews involving solving fraction problems and be video-recorded.	☐
I understand participation is voluntary and that I, or my child, may withdraw from the study at any time.	☐
I understand that my child's responses will be anonymised in reports of the research.	☐

Data Protection

I understand that information collected during participation in this study will be stored on a password protected computer and that this information will only be used for the purpose of this study. All files containing any personal data will be made anonymous. All video data will be destroyed after the project end.

Name of child (print name)………………………………………………………………………

Name of parent or guardian (print name) …………………………………………………

Signature of parent or guardian……………………………………………………………

Date……………………………………………………………

10 January 2015 PCF Version 1.1 GMUPQ_study

Southampton UNIVERSITY OF

Children's Information Sheet

You have been chosen as someone suitable for my research project about year 5 children and mathematics. Your parents and [teacher's name] are happy for you to take part, but it's important to remember you don't have to if you don't want. If you do not want to attend a session at any time, just let me know and you will be excused from the session.

Do you have any questions you want to ask me? If not, I'll just tell you a bit about what will happen.

Who am I
My name is Miss George and I am a Mathematics teacher.

My job
I'm now studying at the University of Southampton in England and my job is to learn about how children in year 5 develop an understanding of one part of fractions that you have not done before in school.

What are we doing today?
We are going to solve some problems that you may have already met before. We are going to share some items equally among different numbers of people. You can use any of the material here (paper, pencil, crayons, markers) to solve the problems. As you are solving the problem I would like you to explain out loud what you are doing and why. Each session will be between 30 and 45 minutes long. I hope that we'll enjoy the tasks! While we work, this video camera [researcher points to it] will record what we say and do. Do you want to have a look at it, to see what it looks like?[1]

Why do you need to video it?
Looking at the video will help me to see how you do maths. It will record things like your questions, your explanations, and if you demonstrate something with your hands [use hands to give an example] which otherwise I might forget! The only people who will see the video are me, and two people I work with at the University.

What happens if I change my mind?
If you want to stop at any time, just let me know.

[1] In research with children in which video cameras are used, it is advised to give children the opportunity to become familiar with the technology. This helps to overcome the 'novelty value' as quickly as possible, as well as any uncertainties/anxieties they may have about the technology.

10 January 2015 CIS Version 1.1 GMUPQ_study

UNIVERSITY OF Southampton

CHILD'S CONSENT FORM (*v1.1*)

Study title: Exploring the growth of mathematical understanding of fractions: Main study

Researcher name: Lois Grace George

Study reference: GMUPO_study

Ethics reference: 13550

Please put the first letter of your name and surname in the boxes if you agree with the statements:

I have read and understood the information sheet (10 January 2015/v1.1) and have had the opportunity to ask questions about the project.	☐
I agree to take part in the task-based interviews involving solving fraction problems and be video-recorded.	☐
I understand my participation is voluntary and that I may withdraw from the study at any time.	☐
I understand that my responses will be made anonymous in reports of the research.	☐

Data Protection

I understand that information collected during participation in this study will be stored on a password protected computer and that this information will only be used for the purpose of this study. All files containing any personal data will be made anonymous. All video data will be destroyed after the project end.

Name of child (print name)..

Signature of parent or guardian...

Date..

10 January 2015 CCF Version 1.1 GMUPO_study

Appendix D Ethics approval documents

George L.G.

From: ERGO <ergo@soton.ac.uk>
Sent: 12 August 2014 14:53
To: George L.G.
Subject: Research Governance Feedback on your Ethics Submission (Ethics ID:10428)

Submission Number 10428:
Submission Title Exploring the growth of mathematical understanding of the partitive quotient subconstruct of fractions in year 5 children: Pilot study:
The Research Governance Office has reviewed and approved your submission

You can begin your research unless you are still awaiting specific Health and Safety approval (e.g. for a Genetic or Biological Materials Risk Assessment) or external ethics review (e.g. NRES).The following comments have been made:

"

Submission ID : 10428
Submission Name: Exploring the growth of mathematical understanding of the partitive quotient subconstruct of fractions in year 5 children: Pilot study
Date : 12 Aug 2014
Created by : Lois George

"

ERGO : Ethics and Research Governance Online
http://www.ergo.soton.ac.uk

DO NOT REPLY TO THIS EMAIL

3/4/2016 Your Ethics Submission (Ethics ID:13550) has been reviewed and approved

Your Ethics Submission (Ethics ID:13550) has been reviewed and approved

ERGO [ergo@soton.ac.uk]
Sent: 27 February 2015 17:49
To: George L.G.

Submission Number: 13550
Submission Name: Exploring the growth of mathematical understanding of the partitive quotient subconstruct of fractions in year 5 children: Main study
This is email is to let you know your submission was approved by the Ethics Committee.

Please note that you cannot begin your research before you have had positive approval from the University of Southampton Research Governance Office (RGO) and Insurance Services. You should receive this via email within two working weeks. If there is a delay please email rgoinfo@soton.ac.uk.

Comments
None
Click here to view your submission

ERGO : Ethics and Research Governance Online
http://www.ergo.soton.ac.uk

DO NOT REPLY TO THIS EMAIL

Appendix E Assessment of fraction knowledge

WHAT YOU KNOW ABOUT FRACTIONS (Part A)

Researcher: Good afternoon <child’s name>. Thank you for agreeing to take part in my research. This afternoon you are going to tell me and show me what you know about fractions by completing some fraction tasks. Do not worry about being right or wrong. I just want to know how much you know about fractions, so that I can prepare for our sessions next week. [Pause.]You can take as much time as you want to complete the tasks. [Pause.] Here are pencils, crayons, blank paper and cut-outs for you to use if you need them. [Points to the desk where these items have been placed.] **Researcher:** Do you have any questions?

Child: [Responds.]

Are you ready to begin?

Child: [Responds.]

****
**

Note: The group of fractions [$\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{8}$, $\frac{1}{5}$, $\frac{3}{6}$, $\frac{13}{7}$, $\frac{9}{8}$, $\mathbf{3}\ \frac{1}{4}$, $\frac{2}{10}$, $\frac{4}{16}$] will be placed on a large chart next to where the **Researcher** and child are seated and will be used for several items related to:

- basic fraction knowledge [types of fractions, naming fractions parts]
- finding equivalent fractions

Each fraction is taped on the chart and so can be removed, as needed.

Researcher will ask the following and the child will respond as needed verbally and/or with manipulation of the fractions.

<u>**Parts of fractions**</u>

Researcher: [Takes any fraction except the mixed numbers.] What is this called? [Points to the bottom number/denominator.]

Using the same fraction, the **Researcher** asks: ‘What is this called? [**Researcher** points to the top number/numerator.]

<u>Types of fractions</u>

Researcher: [Places a cut-out with the word 'improper fraction' written on it on the desk.] Have you ever seen this word before?

Child: [Responds.]

Researcher: Place any fraction from the circle which is this fraction under the label. [Points to where the child should place the fraction.]

Child: [Responds.]

Researcher: How did you choose which fraction to place here? [Points to the spot under the cut-out.] **This is to assess conceptual knowledge.** The same approach will be used for proper fraction and mixed number.

<u>Partitioning</u>

Researcher: We have two shapes here. I would like you to divide this shape [Presents the child with a medium sized rectangle cut-out], into 3 equal parts.

Child: [Responds.]

Researcher: How did you know where to place your separations/lines?

Child: [Responds.]

Researcher: I would like you to divide this shape into 3 equal parts. [Presents the child with a medium sized circle cut-out.]

Child: [Responds.] How did you know where to place your separations/lines?

Child: [Responds.]

Researcher: Last one. I would like you to divide this shape into 4 equal parts.

Child: [Responds.]

Researcher: How did you know where to place your separations/lines?

Child: [Responds.]

Which conception[s] of fraction does the child possess?/Part-whole

Researcher: Draw me some pictures to represent (i) $\frac{2}{4}$.

Child: [Responds.]

Note: If representations other than the part-whole are drawn, then appropriate follow up questions will follow, such as, ‘Can you tell me a little more about this drawing?’ or ‘Can you tell me a little more about what this drawing represents’. This response also helps to ascertain if the child only possesses part-whole fraction sub-construct knowledge.

Part-whole

Researcher: Shows the child a shape divided into seven parts with two shaded regions and asks the child state the fraction for the shaded part.

Child: [Responds.]

Note: Another task of this nature will be given where the shaded parts are not all next to each other.

Equivalent fractions

Researcher: Have you ever heard this term [Places a cut-out with equivalent fractions written on it] before?

Child: [Responds.]

Researcher: Do you see any fractions which are equivalent or represent the same amount on the chart?

Child: [Responds.]

Researcher: Please place these fractions here [**Researcher** points to the place where the cut-outs should be placed].

Child: [Responds.]

Researcher: How did you choose these fractions/How do you know these two fractions are equivalent or represent the same amount?

Child: [Responds.]

NOTE: AFTER 30 MINUTES, THE RESEARCHER WILL END THE SESSION, WHEREVER THE CHILD HAS REACHED. SINCE PART B IS SHORT, THEN THE REMAINDER OF PROTOCOL A CAN BE COMPLETED AT THAT TIME.

At the end of the session:

Researcher: Thank you for showing and telling me what you know about fractions!

WHAT YOU KNOW ABOUT FRACTIONS (Part B)

Researcher: Good afternoon <child’s name>. Thank you for agreeing to take part in my research. This afternoon you are going to tell me and show me what you know about fractions by completing some fraction tasks. Do not worry about being right or wrong. I just want to know how much you know about fractions, so that I can prepare for our sessions next week. [Pause.]You can take as much time as you want to complete the tasks. [Pause.] Here are pencils, crayons, blank paper and cut-outs for you to use if you need them. [Points to the desk where these items have been placed.]

Researcher: Do you have any questions at this point?

Child: [Responds.]

Researcher: Okay. Let’s begin.

<u>**Addition and Subtraction of fractions**</u>

Researcher: I would like you to work out these problems. Tell me how you are working each one out.

a) $\frac{2}{4} - \frac{1}{4}$ = **[Note: fractions with the same denominators]**

b) $\frac{1}{4} + \frac{1}{2}$ = **[Note: fractions whose denominators are multiples of each other]/Whole number bias present?**

c) $\frac{2}{4} + \frac{1}{3}$ = **[Note: fractions whose denominators are not multiples of each other]**

<u>**Comparing fractions**</u>

Researcher: [Places the cut-out ‘is greater than’ in the centre of the desk, then chooses two fractions $\frac{1}{3}$ and $\frac{1}{2}$ and hands them to the child.] Place these [points to the two fractions] so that the sentence is correct.

Child: [Responds.]

Researcher: Tell me how you decided to place the fractions.

Child: [Responds.]

Researcher: Now, choose two fractions of your own that would make this sentence true.

Child: [Responds.]

Researcher: Tell me how you decided to choose the fractions.

Finding unit fractions

Researcher: [Shows the child $\frac{3}{4} = \frac{1}{2} + \frac{1}{4}$ OR $\frac{1}{4} + \frac{1}{4} + \frac{1}{4}$, using a circle cut-out divided into four parts.] Now, it's your turn. We have $\frac{1}{2}$ [Shows the circular cut-out with half]. Can you tell me some fractions that would be equal to $\frac{1}{2}$?

Partitive quotient task

Researcher: We are going to share some pizza among some friends. For example, one pizza among three friends. Have you ever done this type of task in class before?

Child: [Responds.]

Researcher: We are going to share one pizza among some friends. Choose which shape you would like your pizza to be [Shows child rectangular and circular cut-outs].

Child: [Responds.]

Researcher: If I share 1 pizza among three friends [cuts outs of three persons will be shown] how much pizza will each person get if the pizza is equally shared? Please, explain to me what you are doing.

Child: [Responds.]

At the end of the session

Researcher: Thank you for showing and telling me what you know about fractions!

Appendix F Task details

F.1 Task-based interview protocol

<u>*Task-based interview 1 (T01)*</u>

INTRODUCTION TO THE RESEARCH

Researcher: Good afternoon.

Child:

Researcher: Are you okay to work on some fraction problems today?

Child:

Researcher: Today, you are going to solve problems where you will find how much cake/pizza each person would get if the cake/pizza is shared equally among the people and no (cake/pizza) is left over. When I ask you how much I am asking you to give me a fraction. [Pause.]

Child:

Researcher: We have pencils [points to pencils] for you to use and this task sheet [points on task sheet] or blank paper if you prefer. [Pause.]

Child: [Inspects the task sheet.]

Researcher: Remember we are not allowed to use erasers. If you make a mistake you can just go on to another set of diagrams [turns page to show the other set of diagrams being referred to]. [Pause.]

If you know the answer to the problem without having to use the diagrams of the rectangles, just tell me so. You do not have to use them if you feel that you do not need them. Use them if you think it will help you to solve the problem(s). [Pause.]

It is important that you tell me what you are doing as you solve the problem. Also tell me what is going on in your head, what you are doing and seeing in your mind/head as you solve each problem. [Pause.]

Remember to speak loudly and clearly so that the video can record all that you are saying and doing. [Pause.]

I will not tell you whether your answer is right or wrong. This is not important here. What I am interested in, is how you solve each problem. As you are working, I will ask you some questions about what you are doing. [Pause.] Are you ready to begin?

Child:

Researcher: Do you have any questions before we begin?

Child:

Researcher: For the next few weeks we will be solving fraction problems involving sharing some cake among friends. Grade 5 decided to have a bring-and-share snack day. Everyone decided that rectangular cakes would be brought and shared among the class. Children chose their groups and decided how much cake they could afford. Over the next few weeks, we are going to work on solving some **fraction** problems by sharing different numbers of cakes among different numbers of children, so that each person in a group gets the same amount of cake and no cake is left over.

PRESENTATION OF THE TASK FOR THE DAY

Here is our task for today. [**Researcher** reads from sheet with task.] Share two cakes among three children so that each child gets the same amount of cake. How much cake would each child get if each person gets the same amount of cake and no cake is left over?

Child: [Works on solving partitive quotient problem.]

FOLLOW-UP QUESTION TO BE ASKED EACH TIME AFTER THE CHILD HAS SOLVED THE PROBLEM TO ELICIT OTHER WAYS OF SHARING THE ITEMS

How else can you share the ……………… among …………. people?

Child: [Works on solving partitive quotient problem.]

CONCLUSION

Researcher: Thank you very much for participating in the session today.

<u>*Subsequent task-based interviews except task-based interview 6 where a new context is introduced*</u>

INTRODUCTION

Researcher: Good afternoon.

Child:

Researcher: Are you okay to work on some **fraction** problems today?

Child:

PRESENTATION OF THE TASK FOR THE DAY

Researcher: Here is our task for today. [**Researcher** reads from sheet with task.]

Share ……… cake/pizza among …… children/people so that each child gets the same amount of cake. How much cake would each child get if each person gets the same amount of cake and no cake is left over?

Child: [Works on solving partitive quotient problem.]

Follow-up question to be asked each time after the child has solved the problem.

How else can you share the ……………… among …………. people?

Child: [Works on solving partitive quotient problem.]

CONCLUSION

Researcher: Thank you very much for participating in the session today.

<u>Task-based interview 6 (T06)</u>

INTRODUCTION

Researcher: Good afternoon.

Child:

Researcher: Are you okay to work on some **fraction** problems today?

Child:

Researcher: Over the next few sessions we will be sharing pizzas with Harry's family and his friends. Harry lives with his mom, dad and three siblings. Harry's mom lent her circular pizza pan to a friend who has not yet returned it. She needs to make pizza for her family, and so decides to use a rectangular pan.

Here is our task for today. [**Researcher** reads from sheet with task.]

Child: [Works on solving partitive quotient problem.]

FOLLOW-UP QUESTION TO BE ASKED EACH TIME AFTER THE CHILD HAS SOLVED THE PROBLEM.

How else can you share the ……………… among …………. people?

Child: [Works on solving partitive quotient problem.]

CONCLUSION

Researcher: Thank you very much for participating in the session today.

<u>*POTENTIAL PROBES TO BE USED IN EACH TASK-BASED INTERVIEW*</u>

Go on.

Continue

Can you tell me a bit more about how you got that answer?

Can you show me what you mean?

Why do you think so?

F.2 Examples of partitive quotient tasks from previous empirical literature

Researcher	Task
Streefland (1991)	Divide three pizzas between four children (p. 49)
Lamon (2005)	I wish to divide three cakes among four people. How much cake will each person receive? (p. 89)
Mamede (2005)	You share a chocolate between two children. What fraction will each get? (p. 284)
Nunes (2008)	Four children will be sharing three chocolates. How would you share the chocolate? (The booklets contained a picture with three chocolate bars and four children and the children were asked to show how they would share the chocolate bars.) Write what fraction each one gets. (p. 34)

F.3 Description of the partitive quotient tasks for the pilot study

Task contexts	Task details
1: Sharing cake among friends	At CM primary school, students in Year 5 decide to have a bring-and-share snack day. The class was divided into four groups. Each group puts their money together to buy rectangular cakes to share equally among themselves. Share the cakes equally among the children so that each person gets the same amount and no cake is left over. How much cake does each child get?
	1A: Four cakes among six children
	1B: Three cakes, four children
	1C: Four cakes, three children
	1D: Three cakes, two children
2: Sharing pizza among Harry and his family	Over the next few weeks, we will be sharing circular pizzas with Harry's family and his friends. Harry lives with his mom, dad and three siblings
	2A: Harry's mom makes one pizza to share among her four children. How much pizza will each person receive if everyone gets the same amount and no pizza is left over?
	2B: The last time Harry's mom made pizza, the children complained that they did not get enough. The next week Harry's mom made two pizzas to share equally among her four children. How much pizza will each person receive if everyone gets the same amount and no pizza is left over?
	2C: At the end of the month Harry's mom invited four of their friends for pizza at her house. She made three pizzas. At the last minute, the four friends were unable to come. How much pizza will each of the four children receive if each person gets the same amount and no pizza is left over?

F.4 Example of a task

Research Participant: ....

Date: .... **Time: ....**

Week Number: .... **Session Number: ....**

Task 1A: Share 2 cakes among 3 children so that each child gets the same amount of cake. How much cake would each child get if each person gets the same amount of cake and no cake is left over?

How much cake does each child get: ....?

Appendix G Transcript with the time code and unit forming summary

03:23 David-T01--SOLUTION 1

David: So what I am thinking now is that I should separate them into quarters and stuff like that, to give each person [quickly moves pencil back and forth from person to diagram three times] and if it doesn't work out I will go on the next page.

Researcher: Can you show me?

David: [Looks at paper with the written task and diagrams and places pencil on the diagram of one child.] Wait! Actually, since there are three children and two cakes I'm going to separate the cakes into thirds, cause there are chi-, two children. [Looks at paper with the written task and diagrams while speaking.] The (Into?) thirds. [Partitions the first diagram into three using vertical lines.] And so then... make the thirds.

03:55

David: [Points to the picture of the first child.] This child gets one [writes 1 in the first partition of the first diagram], [points to the picture of the second child], this child gets one [writes 2 in the second partition], [points to the picture of the third child], the other child gets another one and then [points to the picture of the third child again] this child gets another one, [writes 3 in the third partition], gets one. [Looks at the partitioned diagram.] So now that I have shared [touches each of the three partitions in turn with pencil] the – first cake to ev–, everybody gets an equal, everybody gets equal. [Looks at partitioned diagram while speaking.]

04:14

David: So now I have to separate the second cake. [Points to second diagram with pencil.] So what I have, since the first cake was, am, separated equally, I'm going to do the same thing I did for the first cake, for the second cake. The rule is that I put this one into thirds as well [partitions the second diagram into three using vertical lines], and then each kid gets another piece. [Writes 1, 2, 3 in each of the three partitions while speaking.]

****

04:38 David-P1A-UNIT FORMING 1

Summary:

(1) Partitions each of the two cakes into three. Distributes one piece from each of the two diagrams to the three children.

(2) Quantifying each share = 2 out 6 = $\frac{1+1}{1+1+1+1+1+1}$, two-sixths or 2/6.

****

> **David:** So then, each person would get [counts quietly pointing to each partition in turn with the pencil], one, two, three, four, five, six, so it would be two out of six. So two-sixths.
>
> **Researcher:** [Points to line on paper.] Go ahead and write your fraction here.
>
> **David:** [Writes 2/6.]

G.1 Excerpt showing a strategy, never previously reported, for finding the fraction for solving a partitive quotient problem

Jack quantifies the share associated with sharing four cakes among six children.

Summary:

(1) Partitions each of the four cakes into six.

(2) Quantifying each share = four-fourths.

****

Excerpt 6-13

Researcher: So how much would this child [Points to a picture of a child on the paper] for example get? How much cake?

Jack: Miss he would get four-fourths of a cake Miss.

Researcher: Could you show me on your diagram how you got this?

Jack: Miss is one child so it – one piece for the same child, another piece for the same child, another piece for the same child, another piece for the same child [points to the first partition of each diagram in turn]. So that is four altogether. So the four, fours – the four-fourths is because – why I say that is because there are four rectangular pieces of cake, four rectangular full cakes there.

Appendix H Description and labels for codes for Research Question 1

Description	Label
$\dfrac{Total\ number\ of\ pieces\ given\ to\ each\ person}{Total\ number\ of\ pieces\ in\ ALL\ items}$	TPPe/TPAI
$\dfrac{Total\ number\ of\ pieces\ given\ to\ each\ person\ from\ the\ items}{Total\ number\ of\ pieces\ in\ ONE\ item}$	TPPe/TPOI
$\dfrac{1}{Total\ number\ of\ pieces\ given\ to\ each\ person\ from\ the\ items}$	1/TPPe
$\dfrac{Total\ number\ of\ pieces\ given\ to\ each\ person\ from\ the\ items}{Total\ number\ of\ items}$	TPPe/TI

Appendix I Images as per Kieren *et al.* (1999)

The instances of the use of image in Kieren *et al.* (1999) are given below. In each of these instances the term image and its associated terms such as Image Making and Image Having have been highlighted and underlined.

ACT I: UNDERSTANDING IN INTERACTION

Jo and Kay, two students enrolled in a mathematics teaching methodology course are involved in solving an arithmagon problem. Instances of the term 'image' associated with act I, scenes I and II are presented below.

Although they both found a solution, what is of interest is that these two students had **<u>very different images of what it meant to solve a system of linear</u>** (pp. 209-210).

Kay's image of solving a system of linear equations	Jo's image of solving a system of linear equations
Use of a formal method of applying row reduction procedures to find solutions to systems relating to the arithmagon.	Working with pairs of equations in a local non-systematic way.

- **<u>Jo's image of linear systems was one of working with equations in pairs</u>** (p. 210).
- Their disparate **<u>images of solving a system of equations</u>** made it difficult for them to work together on the arithmagon problem (p. 211).

ACT I: SCENE 2: A DYNAMICAL THEORY INTERPRETATION

- Perhaps because of their backgrounds in linear algebra, **<u>neither one of them engaged in much image making or playing with the problem in its own terms</u>**. They both appeared to have almost immediately developed an **<u>image of arithmagons that involved linear systems</u>** (p. 222).
- In contrast, Jo's understanding remained local and particular. She was trying to work with **<u>her limited image of linear systems</u>** (p. 223).

- She had *for the first time folded back* and was trying to use her formalised understanding of triangles **to gain an image of square arithmagons**. But **this new image of squares** as essentially 'joined triangles' did not work. (pp. 224-5).

- Although they both thought of arithmagons in terms of linear systems, their distinct modes of understanding (one formalising, **the other working with local images**) appeared to inhibit communication (p. 226).

- Jo and Kay appear to be working on problems that are much more closely related to one another. **They could be observed to have images that allowed them to have a common topic of concern** (p. 226).

- **While it is unlikely that the teacher and students will share the same image** with respect to some aspect of mathematics, nor will they ever share patterns or histories of understanding, the teacher can be aware that acting in particular ways can either facilitate or inhibit the understanding actions of students (pp. 228-9).

NARRATIVE II: STACEY TRIES SOMETHING

Stacey and Kerry are fourth-year university students involved in solving an arithmagon problem. Instances of the term 'image' associated with narrative II are presented below.

- **Except for a fold back to find out some actual local images** and relationships in the arithmagon led by Stacey (indicated by 2a, b, c in the figure below) Kerry and Stacey develop a pathway of changing understanding that is much like that of Kay in Act I earlier (p. 220).

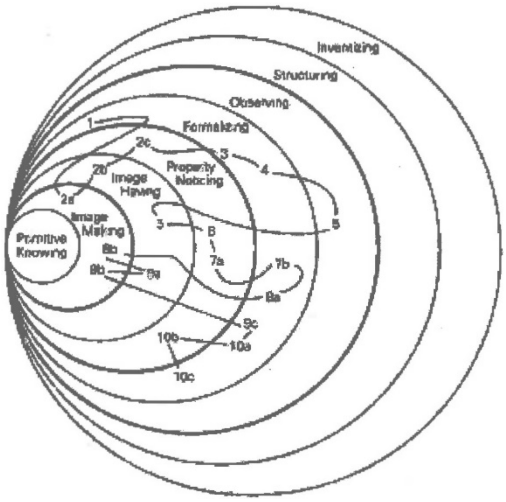


Figure Stacey and Kerry's pathways of understanding (Kieren et al., 1999, p. 120)

- This activity, as far as we could interpret it from the interview, **was all at the level of specific cases and local images and properties.** Yet it was impressive mathematics in its own right and the actions were recognised by Stacey and Kerry as showing understanding (pp. 221-2).

NARRATIVE I: KARA'S UNDERSTANDING OF FRACTIONS

Kara, an eight-year-old student is involved in using a half-fraction kit to form various fractions such as three-fourths. Instances of the term 'image' associated with narrative one are presented below.

- **Kara is observed to *have an image* of fractions** in that she can read simple fraction language, create fractional symbols, and can solve problems using halves and fourths (p. 229).

- Kara came to the fraction work described here **with an image of fractions**. She knows fraction words for halves, thirds, quarters, fifths and tenths and can use them in standard 'textbook' situations (p. 213).

- Kara has reflected on her own experience and generated another, deeper, broader **'folding' image of fractions** (p. 214).

- It is easy (and was easy for the teacher, the researcher and the observers in the settings) to think that at this point Kara has 'our' image of halves or sixteenths. But **her image of fractions, like those of many of her peers, was related to the action sequence that brought it about and not the fractional part or piece that was the product.** This is also reflected in her language. Although Kara had a **'folding' image** of fractions, which allowed her to envision, problem solve, and talk about 'half fractions' without having to actually do the folding, **this image** proved to be quite local in nature… Without acting on the large unit herself, **she could not 'see' how her image of 'half fractions' applied**. (p. 214).

- With only a couple of such **image making sessions, Kara and her classmates showed that they had extended their image of fractions**. Kara could, without reference to the 'kit', respond to questions such as, 'Write down or draw pictures of at least five things you know about three-fourths', and 'A fractional amount is missing. It is more than, or bigger than one-fourth; it is less than, or smaller than three-fourths. Can you find a missing fraction which fits the description?' **Kara now had an image of fractions as chunks or amounts or extensive quantities** (p. 215).

- Kara had to fold back again and build **<u>new images</u>** once more when fractions were introduced as linear measures through the folding of one-metre strips (p. 215).

- After engaging in the actions of image making and reflecting on or reviewing or noting and expressing (at least to oneself) the character of these actions, the child can replace these actions with a '**mental plan' of them (*the actions)* or their (*the actions')*** effects (p. 216).

Appendix J Mappings as per the Pirie-Kieren model

The following transcript takes us along a journey of three task-based interviews for Mary where she solves three problems and crosses two DNBs. The first example of a DNB crossing consists of two parts. In the first part, Mary solves the problem of sharing four cakes among three children so that each person gets the same amount of cake and no cake is left over. For this task, she offers three solutions. This is Mary's second task-based interview. In the second part of this transcript, Mary begins to engage in solving another similar partitive quotient problem. This problem involves the sharing of three cakes among five children.

Child's verbalisation and actions	Researcher's explanations	Layer of Pirie-Kieren model
Summary: *Mary engages in solving the problem of sharing four cakes among three children so that each person gets the same amount of cake and no cake is left over. For this task, she offers three solutions. This is Mary's second task-based interview.* **T02: SOLUTION 1** **Mary:** Okay. [Long pause.] **Mary:** Hmmm. What, what would go-... **Researcher:** Could you tell me what you are thinking? I see your eyes moving. Tell me what you are thinking. **Mary:** I'm thinking of sixteen but I don't really think that would work because three can go into ten three times and three and two more three would give fifteen only. So-... Am,... Hmmm. **Researcher:** What are you thinking of? **Mary:** I, I'm trying to picture four cakes in my mind and see if anything would work. Four cakes. Three children. One and a half. One and a half. One – so that	For T02, Mary is involved in activities such as physical and mental partitioning, trying different numbers of partitions, shading, inspecting the partitioned diagrams and labelling partitions to develop an image or method for the number of partitions/pieces that each cake should have. From even a cursory reading of the extract it is clear that in part one Mary does not have an existing partitioning strategy for the total number of partitions to be made in each diagram for the tasks. This is evidenced by how she worked and spoke slowly, haltingly even hesitantly. She appears to be figuring it out as she goes along. In addition, the length of time she spends looking at the diagrams, making small movements with her pencil before giving a response suggests that she does not have an existing partitioning strategy. She also explicitly verbalised that she was exploring with different	Image Making

Child's verbalisation and actions	Researcher's explanations	Layer of Pirie-Kieren model
wouldn't work. [Long pause while looking intently ahead.] **Mary:** One and a third. One and a third might work. [Partitions the fourth partition into three and leaves the other three diagrams unpartitioned]. One, two, three [counts each of the three partitions in the fourth diagram while partitioning] and three [touches the first diagram]. One [points on the first diagram] and one-third [shades the first partition in the fourth diagram which is partitioned into three and writes '1' above the shaded partition], that's one child. One [points on the second diagram] and a third [shades the second partition in the fourth diagram which is partitioned into three and writes 'two' above the shaded partition] that's two children. One [points on the third diagram] and a third, [shades the third partition in the fourth diagram which is partitioned into three and writes 'three' above the shaded partition] three children. **T02: SOLUTION 2** **Researcher:** How else can you share the four cakes among three children? But before you tell me that, why did you think a third would work. **Mary:** Another way, am... Hmmm... [Looks at diagrams]. **Researcher:** Please, tell me what you are thinking? **Mary:** I'm not really thinking of anything right now. I am trying to find it, any	numbers of partitions in each diagram when she states, 'Hmmm. What, what would go-...' and 'I'm not really thinking of anything right now. I am trying to find it'.	

Child's verbalisation and actions	Researcher's explanations	Layer of Pirie-Kieren model
possible way of getting the three out of four for three children. **Mary:** [Long pause.] **Mary:** Twelve could go... but I wonder... **Researcher:** Do you want to try it? **Mary:** I don't think it would work because I don't, really know how to share four cakes into the number I'm thinking of. Okay. Four cakes. Three children. One, two, three [partitions the first diagram into three]; one, two, three [partitions the second diagram into three]; one, two, three [Partitions the third diagram into three]; one, two, three. [Partitions the fourth diagram into three.] **Mary:** But if I did that I would have the same thing as one and a third. **T02: SOLUTION 3** **Researcher:** Are there any other ways of sharing four cakes among three children. **Mary:** I don't think so. Nines? I wonder. One, two, No. That, if I shared that in half, that in half, this in half, this in half, it would all add up to eight and each child has to get the same amount the exact amount of cake so nine would not work. Hmmm.... I already did twelve and I can't do six because that would not be even [speaks slowly]. **Researcher:** When you say twelve, you did twelve already, what did you mean? **Mary:** I said I did twelve because every single slice [Taps pencil in each of the		

Child's verbalisation and actions	Researcher's explanations	Layer of Pirie-Kieren model
twelve partitions in the three diagrams] adds up to twelve slices. Because three times four [referring to the number of items and people sharing] is twelve. Hmmm. [Long pause while staring ahead]. **Mary:** I don't think I know any other way to do, make am, do it.		
Summary: *Mary engages in solving the problem of sharing, fairly, three cakes among five children. The next excerpt is Mary's first solution to this problem.* **T03: SOLUTION 1** **Researcher:** Here's our task for today. Share three cakes among five children so that each person gets the same amount of cake. How much cake would each child get if each person gets the same amount of cake and no cake is left over? **Mary:** I think I could try fifteen [referring to the total number of partitions for all three diagrams]? Because three fives a fifteen. So each cake would have five.	For this extract, Mary, without any pause or physical partitioning of the diagrams, she states her approach: 'I think I could try fifteen'… So each cake would have five.' It appears that she now has an image as to the number of partitions for the first solution. She explains her image ''Because three fives a fifteen' which suggests that she is multiplying the number of items by the number of sharers to get a total number of partitions for all the items. This suggests she is operating within the Image Having layer.	Image Having
The next excerpt illustrates Mary's continued engagement with the problem of sharing, fairly, three cakes among five children. **T03: SOLUTION 2** **Researcher:** How else can we share the three cakes among the five children? **Mary:** Hmmm. [Looks ahead for an extended period.] Three cakes among five children. I can turn the page? **Researcher:** Of course. [Smiles.]	Similar reasoning as for T02 solutions 1, 2 and 3 stated above. Thus far, for subsequent solutions, Mary has worked in the Image Making layer for T01 and T02. While she appears to have developed an image for the first solution, thereby crossing the first DNB, for subsequent solutions for this solution, she is still operating in the Image Making layer and so has not crossed a DNB for this aspect of the problem solving.	Image Making

Child's verbalisation and actions	Researcher's explanations	Layer of Pirie-Kieren model
Mary: [Turns the page.] Hmmm. Three cakes among five children. [Looks ahead for an extended period.] I think eighteen. **Researcher:** Tell me why you think eighteen. **Mary:** I think eighteen because if I share the three cakes into eighteen pieces altogether, each child could get... four slices? I think? Mmmm.... but that couldn't work because I have three cakes, three pieces of cake. I have three cakes. Because eighteen is an even number it would not work. So I have to find an odd number so that I can share the cakes among the children. Mmmm. [Looks ahead for an extended time.] An odd number. I wonder how I could make twenty-five out of three cakes. Mmmm. [Looks ahead.] Twenty-five... Seven plus seven is fourteen and seven [adds seven to fourteen], twenty-one and seven [adds seven to twenty-one], twenty-eight. So that wouldn't work. Mmmm... [Looks ahead.] A way three can go into twenty-five. I don't think I know how. So I will have to skip that number. [Looks ahead]. Maybe thirty-five slices of cake altogether? But how would I share it? Eleven and eleven is twenty-two plus eleven is thirty-three. So that wouldn't work because if I put eleven on each or twelve on each it would add up into thirty-six and thirty-six cannot go – I don't think I can share thirty-six among five children. Hmmm [Looks ahead for an extended period.] I really wonder what would go (?). Hmmm. [Looks ahead for an extended		

Child's verbalisation and actions	Researcher's explanations	Layer of Pirie-Kieren model
period.] I don't think I can think of another way, because-. **Mary:** Hmmm. **Researcher:** Tell me what you are thinking. **Mary:** I don't even know what I'm thinking. **Researcher:** (?). **Mary:** Trying to find a different way of getting, five children a piece at least more than one slice of three cakes. [Long pause while staring ahead intently.]		
The next excerpt illustrates Mary's continued engagement with the problem of sharing, fairly, three cakes among five children. **T03: SOLUTION 3** **Mary:** I think thirty. **Researcher:** Tell me why you chose thirty. **Mary:** Because since fifteen can work, fifteen times two is thirty.	It appears that Mary now has an image as to the number of partitions for subsequent solutions. She verbalises her image, which is still tied to the local context as 'fifteen times two is thirty' or number of partitions from first solution times two. She appears to cross a DNB here.to operate within the Image Having layer.	Image Having
T03: SOLUTION 4 **Researcher:** Do you think there are other ways that you could share the cake? **Mary:** Well, if I had to share the cake again I would probably add fifteen again. **Researcher:** Okay. You would add fifteen to what? **Mary:** Thirty. Which would be forty-five.	This exchange continues to illustrate that Mary has a clear image (repeatedly the first number of partitions to obtain the number of partitions beyond the first solution. While this is similar to multiplying the first solution by two, mathematically speaking it is different) as to the number of partitions beyond the first solution. This suggests she is	Image Having

Child's verbalisation and actions	Researcher's explanations	Layer of Pirie-Kieren model
	operating within the Image Having layer.	
T03: CONCLUSION **Researcher:** So if you were to explain to somebody how to share three cakes among five children what would you say to them? **Mary:** I think I would tell them that they would have to try to find the first number [referring to the total number of partitions for the first solution], you would, you would get when you multiply three by five because you share you are sharing three cakes among five children so first you have to find out what number that you can actually share it in with the two numbers you have. 'Cause three fives a fifteen.	For the first solution, Mary appears to have crossed the second DNB to operate within the Formalising layer. Although she mentions the numbers in the problem that she has been working with her explanation has a more general tone when she states 'hey would have to try to find the first number'… 'so first you have to find out what number that you can actually share it in with the two numbers you have'. Before Mary begins operating in the Formalising layer of understanding she appears to notice that her images are connected (the first solution and subsequent solutions), thereby noting a key property.	Property Noticing (the first solution and subsequent solutions are linked) Formalising – First solution

Appendix K Excerpts illustrating bi-directional DNB crossings

K.1 Samuel's illustration

In this excerpt, Samuel engages in solving the problem of sharing four cakes among six children. In this excerpt, after working in the outer Image Having layer, he folds back to engage in extensive Image Making activities for Solutions 2 and 3. In Solution 4, he crosses the first DNB to return to operating within the outer Image Having layer again.

SOLUTION 1: IMAGE HAVING

> **Samuel:** I am going to cut, cut the first three cakes into half. And we still have a cake left over which I am going to divide into six pieces.
>
> ………………………………. [There is an exchange, not relevant to the illustration, separating the two parts of the excerpt.]

SOLUTION 2: FOLDS BACK TO IMAGE MAKING

> **Researcher:** How else can you share the same four cakes among the six children?
>
> **Samuel:** [Turns page over.]…. I am going – in this second diagram, I am going to divide each cake into two, no – three pieces. See how it works.
>
> **Researcher:** Are you sure that three would work or are you just exploring at this point?
>
> **Samuel:** I'm just exploring.
>
> ……..

SOLUTION 3

> **Researcher:** How else can you share the four cakes among six children?
>
> **Samuel:** Okay. In this second diagram I am going to... I am going to divide each cake into... nine pieces.
>
> **Researcher:** Why did you choose nine?
>
> **Samuel:** I'm, I don't know. I just exploring. One, two, three, four, five [works quietly]. [Partitions each of the four diagrams into nine.]

……..

SOLUTION 4

IMAGE HAVING

Researcher: How else can we share the four cakes six children?

Samuel: I can continue with eighteen... and 3, 6, 9, 12, 15, 18, 21, 24. Yeah, I can just go up and up.

At the start of Excerpt 7-2, Samuel appeared to be working in the Image Having layer because he appeared to have an existing image as to the number of partitions in each of the diagrams. In contrast to Kenny in Excerpt 7-1, for the first solution, Samuel appears to have a correct image for the number of partitions in each diagram. For the next solution, however, he clearly appears to fold back to work within the Image Making layer for an extended period. His verbalisations: 'I'm just exploring' and 'I'm, I don't know. I just exploring' corroborate the present researcher's conclusion that he does not appear to have an image for the number of partitions for subsequent solutions. Samuel's folding back, unlike Kenny's, appears to be prompted by the task prompt: 'How else can you share the four cakes among six children?'. Although the task given is in the form of a question posed by the researcher, the trigger for folding back is not a researcher question presented to Samuel as he engaged in solving the problem, as was evident for Kenny. A close examination of the trigger, which appears to be associated with Samuel's folding back from Solution 2 appears to be the task itself. It appears that his present image, although it is correct, is not sufficiently developed and therefore needs to be extended, in order to engage with the prompt.

In the last section of the excerpt (Solution 4), Samuel appears to cross the first DNB to operate in the Image Having layer with an image for the number of partitions. At this point, he does not refer to previous Image Making activities, but appears to operate apart from them. He states the number of partitions not just for the present solution, but also for an infinite number of solutions when he states, 'I can continue with eighteen... and three, six, nine, twelve, fifteen, eighteen, twenty-one, twenty-four. Yeah, I can just go up and up.'

K.2 Harry's illustration

Harry engages in solving the task: Share two cakes among seven children so that each person gets the same amount of cake and no cake is left over. In the following excerpt, Harry offers four subsequent solutions to the task, which followed from his first solution.

………

IMAGE HAVING

Harry: Seven children... and... you can share them in... if you share them in eighths you will still remain two more pieces [referring to one piece [one out of eight] from each of the two diagrams] and you can share the two pieces in one in fourths and one in... thirds.

Researcher: Would the children get the same amount of cake?

Harry: No Miss.

FOLDS BACK TO IMAGE MAKING

Harry: So then... You can share them [referring to the two remaining pieces] in... [looks at diagrams] sevenths. [Partitions the first diagram into eight.]

And then like that piece [points to the first partition in the first diagram] you give it out to seven children and you can share that one [referring to the last partition [out of eight partitions] in the first diagram] in sevenths. [Partitions one-eighth into seven using horizontal partitioning lines.]

IMAGE MAKING

Researcher: How else can you share the two cakes among seven children?

Harry: You can share them in tenths. But if is tenths then each child will get [looks at diagrams] one piece, two pieces. But then, [Looks at diagrams for an extended period] it would remain fourteen, it would remain… it would remain six more pieces. But it would not be enough. And you can [Looks at diagrams for an extended period] share – and you can give, and then you would have to share it in another piece... [twirls pencil above diagrams], in different fractions.

IMAGE HAVING

Researcher: Could you show me any other ways that you could share the two cakes among the seven children?

Harry: If you share, if you share… this one [taps on the first diagram] in… fourths and this one [taps on the second diagram] in fourths it would still remain one more piece [one piece refers to the amount remaining after each child receives one piece out of the eight partitions]. And then, you can share that piece in sevenths. [No physical partitioning occurs.]

K.3 Gabriel's illustration

Gabriel engages in solving the seventh task: Share two pizzas among five people so that everyone gets the same amount and no pizza is left over.

IMAGE MAKING

Gabriel: Okay. So, I, I think... you can't – if I do it, if I cut it in half, there won't be enough, but if I cut, if I cut it in like three slices it would be six slices and then that would be one too much. [Pause.]

IMAGE HAVING

Gabriel: So then I think I have to start cut, I, I have to cut it in five pieces, so each child would get two pieces. So like one, two, three, four, five [partitions the first diagram into five]. One, two, three, four, five [partitions the second diagram into 5].

………

And so if I cut them in ten pieces now.

One, two, three, four, five, six, seven, eight, nine, ten [partitions the first diagram in 10.]

One, two, three, four, five, six, seven, eight, nine, ten [partitions second diagram in 10.]

Researcher: How else can you share the same two pizzas among five children?

Gabriel: I want to do fifteen but then like some of it says, like some, like half of me tells, tells me that it's not going to work and half of me tells am it's going to work. But let me try fifteen.

FOLDS BACK TO IMAGE MAKING

Gabriel: One, two, three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen, fifteen [partitions the first diagram in 15 and counts aloud].

One, two [partitions the second diagram in 15, mostly silently]. [Counts the number of partitions silently, placing pencil in each partition in turn.] There are fifteen pieces.

………

Each child would get three pieces.

………

IMAGE HAVING

Researcher: Is there anything else you would like to add about solving this particular problem before we end our session today?

Gabriel: No. Because I think if you keep on, I'm, I'm pretty sure now that if you keep on going up by fives, you will, most likely be able tooo keep dividing into equal pieces.

Researcher: Okay. Why do you think five would work?

Gabriel: Well because you got, when you multiply – well, I still don't know why but I – like I – just I was exploring to get that answer and then it seemed to work so I, I went with it. I, I just, still don't know why.

Appendix L Excerpts illustrating uni-directional DNB crossings

L.1 Mary's excerpts for the first solution for T03-T08

After operating in the Image Making layer for the first solution for T01 and T02, Mary crosses the first then subsequently the second DNB for the first solution.

T03: SHARE THREE CAKES AMONG FIVE PEOPLE

Mary: I think I could try fifteen [referring to the total number of partitions for all three diagrams]? Because three fives a fifteen. So each cake would have five.

T04: SHARE FOUR CAKES AMONG SIX PEOPLE

Mary: I think each child would get, hmmm... four slices of cake out of twenty-four pieces of cake.

Researcher: Can you explain to me how you came up with that?

Mary: Because four times six is twenty-four, there are four cakes, six children.

T05: SHARE TWO CAKES AMONG SEVEN PEOPLE

Mary: Well, I think I would have to cut the piece of cake, the two cakes into fourteen pieces... and each child would get two pieces of cake.

Researcher: Why did you choose fourteen?

Mary: Because two times seven is fourteen, so I chose it.

T06: SHARE THREE PIZZAS AMONG SIX PEOPLE

Mary: Well, I got my answer already. I think each child would get three out of eighteen slices… of the pizza.

Researcher: Could you explain to me how you got – put your fraction here.

Mary: [Writes 3/18.]

Researcher: And then explain to me how you got that fraction.

Mary: Well I got the fraction by multiplying three by six which is eighteen. [No physical partitioning occurs but the three cakes are partitioned into eighteen.]

T07: SHARE TWO PIZZAS AMONG FIVE PEOPLE

Mary: Well already, I know how much each child would get. Each child would get two out of ten slices of cake. [Writes 2/10.] The reason why I chose ten was because two fives a ten and five.

T08: SHARE THREE PIZZAS AMONG EIGHT PEOPLE

Mary: I got my answer. Each child would get three out of twenty-four slices of cake.

Researcher: Go ahead and put your fraction and explain to me how you came up with that answer.

Mary: [Writes 3/24.] Since there are eight children and three pizzas I multiplied three by eight which is twenty-four.

L.2 Harry's excerpts for the first solution for T01-T08

After operating in the Image Making layer for the first solution for T01 and T02, Harry crosses the first DNB for the first solution in T03.

T01: SHARE TWO CAKES AMONG THREE CHILDREN

No audio

T02: SHARE FOUR CAKES AMONG THREE CHILDREN

IMAGE MAKING

Harry: Miss everyone got, am,... [Looks at diagrams for an extended period and moves pencil in the air over the diagrams.] A whole cake, because is four, and it would still remain one [looks at the diagrams]. And you can just share it in thirds.

TO3: SHARE THREE CAKES AMONG FIVE PEOPLE

SIMILAR APPROACH AS T02 SOLUTION 2

IMAGE HAVING

Harry: Okay Miss. There is three cakes and five children. Well, if we give everyone, each person one [referring to one whole cake] it would still remain two more people to get and it will have no more. So we can share them in halves. [Partitions each of the 3 diagrams in half.] Then each child would get one-half. And it would still remain one, one more [refers to half of a cake/diagram]. So you can share them in fifths. [Partitions one-half of a diagram into 5.]

T04: SHARE FOUR CAKES AMONG SIX PEOPLE

Harry: Okay Miss. There is four cakes and six children. You can share three of them in halves. [Partitions three diagrams in two.] And one in sixths. [Partitions one diagram into 6.]

T05: SHARE TWO CAKES AMONG SEVEN PEOPLE

Harry: I would share one cake in sevenths. [Partitions the first diagram into 7.] Sev–/ [/: Both persons in the interview speak at the same time.]

Researcher: /Could you tell me why sevenths?

Harry: Because there are seven children and then each child would get one piece**.** But it still remaining one more cake and you can still share it in seven. [Partitions the second diagram into 7.] Seven.

T06: SHARE THREE PIZZAS AMONG SIX PEOPLE

Harry: Yes, Miss.

Miss is three pizzas and six child – and six people. Then you can cut, share them in halves. [Partitions each diagram in 2.]

T07: SHARE TWO PIZZAS AMONG FIVE PEOPLE

Harry: Miss, you can share it in fifths.

Researcher: Please go ahead. Let me see what you mean.

Harry: [Partitions the two diagrams in 5.] Then each person would get two-fifths.

T08: SHARE THREE PIZZAS AMONG EIGHT PEOPLE

Harry: Miss, if you share, this one in fourths [taps on the first diagram]; this one in fourths [taps on the second diagram]; and this one in eighths [taps on the third diagram]. [Partitions two diagrams into four and one diagram into 8.]

L.3 David's excerpts for the first solution for T01-T08

After operating in the Image Making layer for part of the first solution for T01, David crosses the first then subsequently the second DNB for the first solution.

T01: SHARE TWO CAKES AMONG THREE CHILDREN

IMAGE MAKING

> **David:** So what I am thinking now is that I should separate them into quarters and stuff like that, to give each person [quickly moves pencil back and forth from person to diagram three times] and if it doesn't work out I will go on the next page.
>
> **Researcher:** Can you show me?
>
> **David:** [Looks at paper with the written task and diagrams and places pencil on the diagram of one child.]

IMAGE HAVING

> **David:** Wait! Actually, since there are three children and two cakes I'm going to separate the cakes into thirds, cause there are chi–, three children. [Looks at paper with the written task and diagrams while speaking.] The (Into?) thirds. [Partitions the first diagram into three using vertical lines]. And so then... make the thirds.

T02: SHARE FOUR CAKES AMONG THREE CHILDREN

> **David:** Okay. Like the last session we had, am, like yesterday, uuum... we have three children but this time we have four cakes. So right, like yesterday I would... am separate them into thirds? Separate into thirds.

TO3: SHARE THREE CAKES AMONG FIVE PEOPLE

FORMALISING

> **David:** Okay. So this time there are five children. So that means since... I, last time I shared it into threes, into thirds for the three children, now I'm gonna do it into fifths for the five children.
>
> **T04: SHARE FOUR CAKES AMONG SIX PEOPLE**
>
> **David:** So right now since there are six children this time, then now I'm going to separate it into six so each child gets one piece from one cake. So, I'm just gonna do it in my head. I'm not really going to use the diagram right now.

T05: SHARE TWO CAKES AMONG SEVEN PEOPLE

David: Okay, am, well, first I am going to share, aaam, these two cakes into seven. [Draws two partitioning lines in the first diagram.] I'm going to separate them into seven pieces.

Researcher: Please tell me why seven?

David: Because there's seven children so then I would give each person, so I can give each person one.

T06: SHARE THREE PIZZAS AMONG SIX PEOPLE

David: Okay. Sooo, first... I would just share it into, I would share it into six. 'Cause there's six people in the family. So I would share it into six.

T07: SHARE TWO PIZZAS AMONG FIVE PEOPLE

David: Since there are five children I guess I'll stick to the sharing the pizzas into fives.

T08: SHARE THREE PIZZAS AMONG EIGHT PEOPLE

David: Okay. Well then I'm going, right now I'm just going to share it into eighths because there are eight children.

L.4 Karen's excerpts for the first solution for T03-T08

For T01 and T02 Karen operates in the Image Making layer for the first solution. See Excerpt 7-2 for T01, Solution 1. From T03, Karen shifts to operating in the Image Having layer for the first solution, then in T05 shifts across the second DNB for the first solution.

TO3: SHARE THREE CAKES AMONG FIVE PEOPLE

> **Karen:** I'd give each – I'll start with five [referring to sharing each of the diagrams into five].
>
> **Researcher:** Could you tell me why did you choose five?
>
> **Karen:** Okay. I saw there are five children [smiles] so I'm going to start with five first [partitions the first diagram into 6]. One, two, three, four, five, six [counts the number of partitions in the first diagram aloud; Karen ignores this partitioning error and treats the number of partitions as 5]. Hmmm. [Partitions the second and third diagram into 5.]
>
> And I'd give each child one piece, one [passes pencil over the last partition in each of the 3 diagrams]. So each child would get one piece.

T04: SHARE FOUR CAKES AMONG SIX PEOPLE

> **Karen:** Okay. Well, I'm going to do four times six is twenty-four pieces and each child will get at least six pieces, because four – six children, so each child would get four pieces.

T05: SHARE TWO CAKES AMONG SEVEN PEOPLE

> **Karen:** [Smiles.] Fourteen. Fourteen for each cake.
>
> **Researcher:** When you say fourteen, what does fourteen represent?
>
> **Karen:** Am fourteen represents fourteen pieces in each cake. Then each child will get [looks at the diagrams] two pieces. So that'd be four pieces.
>
> ………
>
> **Researcher:** Why did you choose fourteen?
>
> **Karen:** Because I said two times seven. That would give you fourteen… so each – fourteen pieces and then you give two for each child.

T06: SHARE THREE PIZZAS AMONG SIX PEOPLE

Karen: Eighteen. Start with eighteen. Each child will get three pieces [refers to how much child receives from each cake.]

………

Researcher: Could you tell me why you chose eighteen to begin with?

Karen: Because three times six is eighteen.

T07: SHARE TWO PIZZAS AMONG FIVE PEOPLE

Karen: Okay. I will start by giving ten, ten – each child. Ten pieces in each cake and then each child will get two out of it.

……

Researcher: You came up with this very quickly. Could you tell me how did you decide to share it in ten?

Karen: Because is five children [It is not clear what word/phrase is said at that point.] and there'll be five. It'll be two times five will make the ten.

T08: SHARE THREE PIZZAS AMONG EIGHT PEOPLE

Karen: Okay, I'll start with… twenty-four in each cake–

Researcher: Pizza.

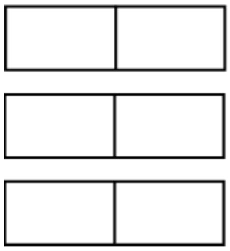
Karen: Pizza and each child will get three pieces.

Appendix M Excerpts illustrating Harry's partitioning images

Images: Halving and Number of partitions in each diagram or multiple diagrams = Number of people sharing

Excerpt 5-3

> **Harry:** Miss is three pizzas and six child – six people. Then you can cut, share them in halves. [Partitions each diagram in two.]

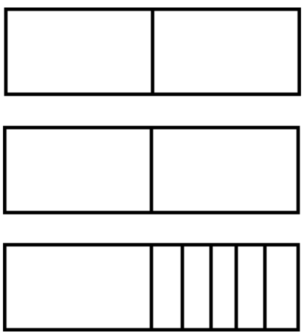


In the next excerpt, Harry finds the fraction associated with each person's share when solving the problem of sharing three cakes among five children. He has already partitioned the diagrams as per diagram at the start of the excerpt.

Images: Halving and Partitions each of the diagrams into a number of pieces e.g. ten and distribute evenly. If there is a remainder (one or more partitions), cut this into number of people (fifths).

Excerpt 6-11

> **Harry:**



Then each child would get one-half. And it would still remain one, one more [referring to one-half partition]. So you can share them in fifths. [Partitions one-half of a diagram into five.] And then each person would get a half and one-fifth. [Writes 1/2   1/5.]

Image: Number of partitions in multiple diagrams = Number of people sharing

Harry shares two cakes among seven children.

Harry: I would share one cake in sevenths. [Partitions the first diagram into seven.] Sev–-/ [/: Both persons in the interview speak at the same time.]

Researcher: /Could you tell me why sevenths?

Harry: Because there are seven children.

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