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Proof validation and modification in secondary school geometry

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ABSTRACT

Proof validation is important in school mathematics because it can provide a basis upon which to critique mathematical arguments. While there has been some previous research on proof validation, the need for studies with school students is pressing. For this paper, we focus on proof validation and modification during secondary school geometry. For that purpose, we employ Lakatos' notion of local counterexample that rejects a specific step in a proof. By using Toulmin's framework to analyze data from a task-based questionnaire completed by 32 ninth-grade students in a class in Japan, we identify what attempts the students made in producing local counterexamples to their proofs and modifying their proofs to deal with local counterexamples. We found that student difficulties related to producing diagrams that satisfied the condition of the setproof problem and to generating acceptable warrants for claims. The classroom use of tasks thatentail student discovery of local counterexamples may help to improve students' learning of proof and proving.

1. Introduction

Proof and proving are recognized as playing a key role in shaping meaningful mathematical experiences for all students (e.g., Stylianides, Stylianides, & Weber, in press). One major area of proof research in mathematics education is research on the reading of proofs. As suggested by Weber (2015), empirical studies on the reading of proofs can be broadly classified into three categories. A first category encompasses studies in which different types of mathematical argument (visual, inductive, generic, or deductive) are presented to students and the students are asked to evaluate whether the arguments are personally convincing and whether the arguments can be considered to constitute proofs (e.g., Healy & Hoyles, 2000; Martin & Harel, 1989; Segal, 1999). A second category relates to students' understanding of given correct proofs. Based on models of proof comprehension (Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012; Yang & Lin, 2008), studies in this category seek to identify effective proof comprehension strategies and examine the effectiveness of specific training to improve students' proof comprehension (Hodds, Alcock, & Inglis, 2014; Samkoff & Weber, 2015). A third category involves proof validation; here, researchers show students purported deductive proofs and ask them to determine whether the proofs are valid or invalid (Alcock & Weber, 2005; Inglis & Alcock, 2012; Ko & Knuth, 2013; Selden & Selden, 2003; Weber, 2010).

This paper reports on a study that belongs to the third category, proof validation. We use the term proof validation to mean the reading of arguments constructed as proofs to check whether the arguments really constitute legitimate proofs; that is, whether the

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arguments can establish the truth of mathematical statements (see Section 3 for a more elaborate definition).

Although there have been studies of proof validation behavior, the participants in many of those studies consisted of undergraduates, trainee or in-service teachers, and/or mathematicians. The need for studies involving school students is pressing, especially given recent curriculum reforms in several countries indicating that proof validation should be introduced into secondary school mathematics. For example, the latest national curriculum in England declares that, as part of mathematical reasoning, "pupils should be taught to [...] assess the validity of an argument and the accuracy of a given way of presenting information" (Department for Education, 2014; pp. 5–6). In the United States, the Common Core State Standards Initiative (2010) also underlines the importance of constructing viable arguments and critiquing the reasoning of others and places these activities among the standards for mathematical practice. As Inglis and Alcock (2012) argue, "clearly, one way of critiquing explanations is by validating purported proofs" (p. 360). Moreover, Powers et al. (2010) show that explicit teaching of proof validation has the potential to improve students' capacity to construct valid proofs. Research into proof validation may offer an approach to addressing the difficulties frequently encountered by students attempting to construct valid proofs (e.g., Healy & Hoyles, 2000; Hoyles & Healy, 2007).

To introduce proof validation into secondary school mathematics, it is vital to explore the nature of the validation attempts made by secondary school students and their subsequent modifications to proofs. This is because if mathematics educators can deepen their understanding of students' behavior in this regard, such understanding can provide a basis for designing more effective teaching and for helping students become more effective at proof validation and modification. Hence, in this paper, we aim to explore this issue (more specific research questions are given in Section 3 below) using data from a task-based questionnaire completed by 32 ninth-grade students in a Japanese secondary school class.

The structure of the remainder of this paper is that in Section 2 we review the literature on proof validation. In Section 3, we define in detail the respective meanings of proof validation and proof modification, and specify the research questions addressed in this paper. We account for our research method in Section 4, and then present and discuss our analysis of the data with Toulmin's framework in Sections 5 and 6. We conclude in Section 7 by summarizing our study and indicating suggestions for future research.

2. Research on proof validation

To the best of our knowledge, Selden and Selden (1995, 2003) first introduced the term *proof validation* into mathematics education research (although Segal, 1999; and Knuth, 2002b; investigated relevant behavior in undergraduates and teachers without using the term). Selden and Selden (2003) conducted a study in which undergraduates were given four arguments, including both valid and invalid ones, and were asked to judge whether each of them could be regarded as a valid proof. The researchers found that although only about half of judgments made by the participants were correct at the initial stage, the rate of correct judgments improved after the interviewer's encouragement to reflect on their initial judgments. Another finding was that many students incorrectly accepted as a valid proof an argument showing the truth of the converse of a target proposition, indicating that they tended to judge the validity of arguments based on local details rather than global/structural points.

Since Selden and Selden's (2003) seminal work, several researchers have investigated how undergraduates and mathematicians work on proof validation. Alcock and Weber (2005) looked at line-by-line checking of an argument, observing that only a few undergraduates detected a hidden warrant in the step from a certain line of the argument to the subsequent line and recognized the invalidity of the warrant. Weber (2010) focused on failure reasons in proof validation by undergraduates, hypothesizing possible causes including students' tendency to overlook the inappropriateness of assumptions used in an argument, and limitations in students' own mathematical knowledge. Ko and Knuth (2013) considered arguments from various content domains (in algebra, analysis, geometry, and number theory) and observed that undergraduates' judgment on the validity of the arguments varied according to content domain. Ko and Knuth also dealt with ostensible counterexamples proposed to show the falsity of propositions. In contrast to these studies involving undergraduates, Weber (2008) and Inglis and Alcock (2012) explored how experts (mathematicians) engaged in proof validation and how their behavior was different from that of undergraduates.

Although these existing studies certainly provide multiple insights into proof validation behavior, there are two issues that remain open. First, that a majority of existing studies regarding proof validation involve undergraduates, trainee and in-service teachers, and/or mathematicians, means that the need for studies of proof validation in school mathematics is pressing. In doing so, we focus on the domain of geometry because it is a key area in which proofs and proving are learned and used in secondary school mathematics. While there have been numerous studies on proof and proving in secondary school geometry covering students' capabilities regarding proofs and their perspectives on proving in classes (Herbst & Brach, 2006; Hoyles & Healy, 2007; Senk, 1985), teachers' conceptions of proof (Knuth, 2002a, 2002b), textbook analysis (Fujita & Jones, 2014; Otten, Gilbertson, Males, & Clark, 2014), task design (Cirillo & Herbst, 2012; Komatsu, in press), and classroom-based research, including investigation of student-teacher interactions and teaching interventions to enhance student learning (Martin, McCrone, Bower, & Dindyal, 2005; Miyazaki, Fujita, & Jones, 2015), only a few studies have related to proof validation, mostly focusing on whether students can discern the invalidity of circular arguments (in which conclusions are used as suppositions). For instance, McCrone and Martin (2004) surveyed 18 American high school students and showed that only 22% of the students correctly judged a circular argument invalid. A similar result was obtained by Reiss et al. (2001), who administered the proof questionnaire by Healy and Hoyles (1998) to German secondary school students. More recently, Miyazaki, Fujita, and Jones (2017), using their own framework to capture students' understanding of the structure of proofs, analyzed a classroom episode where the invalidity of a circular argument was discussed among students; the researchers make reference to a component of their framework, namely hypothetical syllogism, to explain student inability to reject the circular argument. However, because proof validation does not only involve circular arguments, it remains necessary to address other types of proof validation.

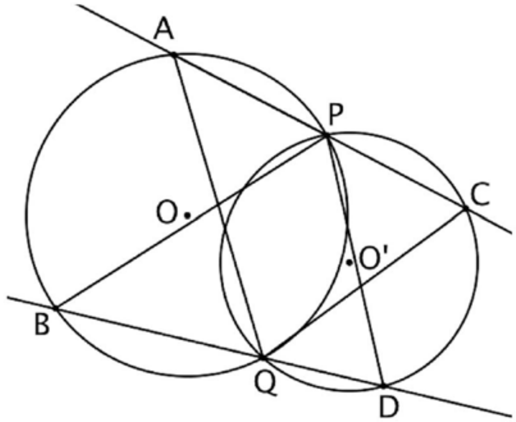

Fig. 1. Diagram attached to the proof problem.

The second open issue is that in previous research, participants' activity has generally ended at proof validation, meaning that these researchers have not investigated how the participants might modify arguments after judging them to be invalid. Alcock and Weber (2005) is one exception, as they requested participating undergraduates to check an argument and revise it if needed; they found that participants did not zoom in effectively on a specific part that crucially affected the validity/invalidity of the argument, yet made only minor modifications on the whole. In this, Alcock and Weber did not focus on the modifications, treating them as a relatively minor topic. In our view, proof modification is an important educational activity because it provides students with opportunities to look back and improve mathematical knowledge, skills, and thinking. Accumulating such opportunities could foster reflective attitudes in students, which are essential for them to not only enhance the quality of their mathematical learning, but also become thoughtful citizens in general.

The need to address these two issues—proof validation and modification in secondary school mathematics—constitutes the rationale for our study.

3. The meanings of proof validation and proof modification

To clarify the meanings of proof validation and proof modification in this study, we borrow from Lakatos (1976) the notions of *global counterexamples* and *local counterexamples*. A global counterexample, which is a counterexample in the conventional sense, refutes a statement, while a local counterexample rejects only a step in a proof. There have been many mathematics education studies dealing with global counterexamples (in most cases, they simply use the term *counterexamples*, not *global counterexamples*). For instance, Balacheff (1991) analyzed 13–14 year-old students' responses to counterexamples, and indicated possible factors influencing these responses, such as types of conjectures, student knowledge of objects concerned in conjectures, their global conceptions of what mathematics consists of, and didactical contract perceived by students. Yopp (2015) obtained similar results from his study with prospective elementary teachers, and showed that their responses to false generalizations and counterexamples were influenced by several factors including knowledge of argumentation and knowledge and use of the mathematical practice of exception barring. In contrast to these studies, the characteristic of our study is in addressing local counterexamples, namely counterexamples to proofs, not to conjectures.

To introduce proof validation involving local counterexamples into secondary school geometry, this study uses a class of mathematical tasks named *proof problems with diagrams* (Komatsu, in press; Komatsu, Tsujiyama, Sakamaki, & Koike, 2014). These are tasks in which statements are described with reference to particular diagrams with labels, typically one diagram per problem. An example is as follows:

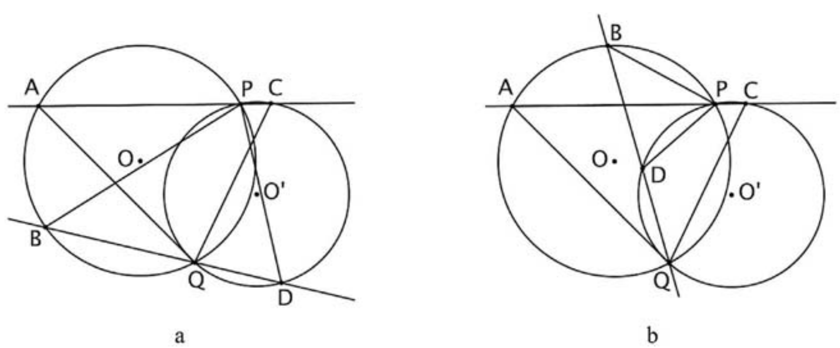

Fig. 2. Diagrams that satisfy the condition of the statement.

As shown in Fig. 1, two circles O and O′ intersect at points P and Q, and two points A and B are located on circle O. Draw line AP and let point C be the intersection point of the line and circle O′. Draw line BQ and let point D be the intersection point of the line and circle O′. Prove $\Delta AQC \sim \Delta BPD$.

That $\Delta AQC \sim \Delta BPD$ can be proved by showing $\angle QAC = \angle PBD$ and $\angle QCA = \angle PDB$, using the inscribed angle theorem.

There are two perspectives on tasks in the form of proof problems with diagrams. The first regards these tasks as only involving answering whether statements are true for the diagrams given in the tasks, while the second considers the truth of the statements for certain general classes to which the given diagrams belong. This study adopts the second interpretation. Thus, in the task above (which includes the diagram in Fig. 1), we consider that the statement claims that triangles AQC and BPD are mathematically similar regardless of the locations of points A and B on circle O (e.g., as illustrated in Fig. 2) as long as the stated condition is satisfied.1 This interpretation can elicit a proof validation where various diagrams are produced that satisfy the conditions of the statements and checks are made whether the constructed proofs are applicable to these diagrams. Here, the aforementioned proof for Fig. 1 is not applicable to the case shown in Fig. 2b because this case rejects the use of the inscribed angle theorem as a reason for showing $\angle QCA = \angle PDB$.[2] The proof thus needs to be modified by altering the theorem being utilized to the inscribed quadrilateral theorem that an interior angle is equivalent to the exterior angle of the opposite angle. The case shown in Fig. 2b represents one of local counterexamples in Lakatos' (1976) terminology because it rejects a specific part of the proof; it is not a global counterexample because the statement, $\Delta AQC \sim \Delta BPD$, is still true (further reference to local counterexamples in mathematics education research can be seen in Komatsu, 2012, 2016 and Larsen & Zandieh, 2008).

Based on this illustration, we define proof validation as *inspecting whether there are local counterexamples to proofs* and proof modification as *when noticing the existence of local counterexamples, constructing proofs that are valid for the local counterexamples*.

In each *proof problem with diagram*, a proof must take into account not only a given diagram but also a broader domain beyond it. That is, the given diagram is used as a *generic example*, defined as "an actual example, but one presented in such a way as to bring out its intended role as the carrier of the general" (Mason & Pimm, 1984; p. 287; for a recent review of generic examples, see Yopp & Ely, 2016). However, the proof may suffer from "one-case concreteness" (e.g., Dvora & Dreyfus, 2014; Presmeg, 1986; Yerushalmy & Chazan, 1990) in the sense that the proof based on the single concrete diagram may be applicable only to a subset of all cases considered in the statement.[3] The present study capitalizes on this ambiguity of *proof problems with diagrams* to create opportunity for secondary students to engage in proof validation and modification.

The meaning of *proof validation* in this study is distinct from that of existing research. Previous studies have involved participants spotting fallacies that make arguments invalid, such as the circularity of arguments, falsity of warrants used to deduce certain lines, mismatch of arguments resulting in proofs for the converses of target propositions, and inappropriateness of global counterexamples used to show the falsity of propositions (e.g., Alcock & Weber, 2005; Ko & Knuth, 2013; McCrone & Martin, 2004; Selden & Selden, 2003). In contrast, our study deals with not-completely-invalid proofs, or proofs that are valid in certain domains. In this sense, we treat proof validation in terms of in-between valid proofs and invalid arguments, and investigate whether secondary school students can recognize cases where proofs become invalid.

In an earlier related study (Komatsu, Ishikawa, & Narazaki, 2016), a task-based questionnaire was administered to 29 secondary school students to investigate whether they could successfully perform proof validation and modification. In this case, only the numbers of students who succeeded or failed at proof validation and modification were recorded and reported; in-depth analysis into the students' behaviors was not conducted. In this paper, we address this issue by providing more detailed data on a similar investigation, addressing the following research questions:

- What attempts are made by secondary school students in producing local counterexamples to their proofs?
- What attempts are made by secondary school students in modifying their proofs to cope with local counterexamples?

4. Methods

4.1. Context

In this paper, we use data from a task-based questionnaire conducted as one part of a larger research endeavor on curriculum development for explorative proving (Miyazaki et al., 2016). In our study, we designed and implemented a teaching intervention with the aim of developing task sequences that facilitate proof validation and modification in secondary school geometry. The intervention was designed in collaboration between researchers and teachers, and the third author implemented the intervention in his ninth-

[1] The conditions of the statement include the intersection of circles O and O′, the possibility of drawing lines AP and BQ, the existence of points C and D, and the existence of triangles AQC and BPD. Under these conditions, the statement claims the similarity of triangles AQC and BPD. Some special cases, such as the cases where circles O and O' coincide or do not intersect, and the cases where point A coincides with point P or Q, may be raised. However, these cases are logically not counterexamples but non-examples, because they do not satisfy the statement condition (Komatsu et al., 2014); therefore, these cases are precluded from the statement. Regarding the case where point A coincides with point P, however, if line AP is considered as a tangent to circle O, the statement is still true and can be proved with the alternate segment theorem. This also applies to the case where line AP is a tangent to circle O′ and the case where point B is moved. In another special case where points A and B coincide, the similarity of triangles AQC and BPD can be also proved in a different way (Komatsu et al., 2016).

[2] The case in Fig. 2b rejects neither the truth of the inscribed angle theorem nor the equality of angles QCA and PDB, because the theorem is a true proposition and the equality of these two angles can be proved with another theorem, as described.

[3] This is not true for all of proof problems with diagrams; in some cases, the constructed proofs are valid for all cases considered in the statements.

Table 1
Summary of the implemented lessons.

Lessons	Tasks	Purposes
Lesson 1	Prove ΔPAB ~ ΔPDC in Fig. 4a.	Introducing proof validation and modification, where students work on a local counterexample hinted at by the teacher.
Lesson 2	Prove ΔPAC ~ ΔPBD in Fig. 5a.	Providing another experience of proof validation and modification, where students discover local counterexamples by themselves.
Lesson 3	Prove ΔAQC ~ ΔBPD in Fig. 1.	Implementing the task-based questionnaire shown in Section 4.2.

grade (14–15-year-old students) classroom in a state school in Japan. After the intervention, the task-based questionnaire was administered in order to investigate how the students performed proof validation and modification. We used three (50-min) lessons for our research: two for the intervention and one for the task-based questionnaire. Table 1 represents the summary of the three lessons. The focus of our analysis in this paper is on the results of the questionnaire, not the intervention. Although the class had 33 students, one student's data were not analyzed because the student was absent from the intervention; thus, there were 32 participants.

We conducted our research during the class' regular teaching unit during which the properties of circles were addressed. Prior to this teaching unit, the students had learnt geometric proofs involving the congruence and similarity of triangles; they had become familiar with the inscribed angle theorem and the inscribed quadrilateral theorem. While both these theorems, as we explain below, are necessary for tackling the task-based questionnaire, the students' prior experience in using these theorems was solely in the context of problems requiring them to find unknown angles; they had not yet tackled proof problems that required these theorems. We expected that the teaching intervention, whose main purpose was introducing proof validation and modification, could also prepare the students to tackle such proof problems, as described further below. According to the third author, who has 25 years of teaching experiences across several secondary schools in Japan, the mathematical capabilities of the participating students were average for their age in Japan.

4.2. *Task-based questionnaire*

We employed *proof problems with diagrams* for our teaching intervention and for our task-based questionnaire. The task illustrated above, in Section 3, was employed for the task-based questionnaire. The individual questions are listed below, with the diagrams used in the questions shown in Fig. 3:

Q1. Answer the following questions:

Q1-1. If point A is on the place shown in Fig. 3a, your proof will be rejected. Which part of your proof will be rejected?

Q1-2. Modify your proof to show ΔAQC ~ ΔBPD even in the case of Fig. 3a.

Q2. Answer the following questions using Fig. 3b:

Q2-1. Place point A on various places on circle O other than as in the diagram in Q1, and find a case that rejects your proof.

Q2-2. Which part of your proof is rejected by the diagram you drew in Q2-1?

Q2-3. Modify your proof to show ΔAQC ~ ΔBPD even in the case of your diagram in Q2-1.

We conducted this task-based questionnaire during the latter part of the third lesson of our study; in the first half of the lesson, students engaged in proving the statement shown in Section 3, and in the second part, they worked individually on the task-based questionnaire. During the first part of the lesson, when some students were not able to construct proofs by themselves, the teacher asked students who had successfully done so to help those students. This was to ensure that all the students could write full proofs before attempting the task-based questionnaire.

Approximately 25 min of the third lesson were devoted to the task-based questionnaire. While working on it, students could see a poster on the wall of the classroom that summarized the inscribed angle theorem and the inscribed quadrilateral theorem.

This specific proof problem in the task-based questionnaire was chosen based on its suitability for the participating students, based on their status and progress in their regular mathematics class. The task requires students to apply knowledge of the properties of circles, which, as mentioned above, the participants were learning at that time. Hence, we considered that the questionnaire would

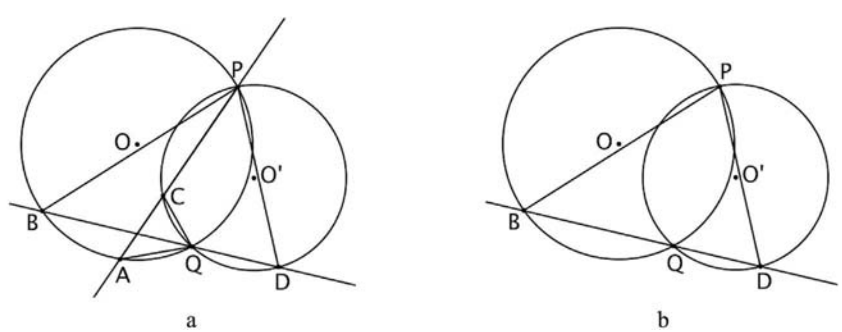

Fig. 3. Diagrams used in the task-based questionnaire.

not appear artificial to the students, but rather a natural extension to their regular class activity. In Section 6, we show that other tasks, such as common textbook tasks, can also be used for the form of proof validation and modification that is our focus in this paper.

In the task-based questionnaire, Q1 was prepared because our earlier study (Komatsu et al., 2016) showed that producing local counterexamples was challenging for participating students; thus, we thought some students might perform better if given a local counterexample as opposed to being expected to produce one on their own. More specifically, students given a local counterexample might be better able to recognize which parts of their proofs are rejected by the local counterexample and to modify the proofs to cope. While Q1 involves being presented with a diagram that can be regarded as a local counterexample, Q2 involves discovering a local counterexample. Q2-1 and Q2-2 entail the discovery of local counterexamples and, as such, are relevant to proof validation, whereas Q1-2 and Q2-3 address given/discovered local counterexamples and are thus relevant to proof modification.

4.3. Framework and data analysis

Following Pedemonte (2007) and Zazkis et al. (2016), we used the simplified version of Toulmin's (2003) framework to analyze students' attempts at proof validation and modification. This simplified framework consists of the claim (C) being argued, the data (D) used to demonstrate the claim, and the warrant (W) describing how the data support the claim. Although we acknowledge the importance of the remaining three components in Toulmin's framework (backing, modal qualifier, and rebuttal; Inglis, Mejia-Ramos, & Simpson, 2007; Jahnke, 2008), the simplified framework was sufficient for our data. What is more, we are reassured that Toulmin's framework can be used to analyze the sort of reasoning that Lakatos describes in which arguments may be faulty, but revisable, rather than being solely accepted or rejected (Aberdein, 2005; Pease & Aberdein, 2011).

Given that the main data analyzed in this study consisted of students' answers to the task-based questionnaire, analysis of students' responses was divided into three phases. In the first phase, we graded each student's answers to the questions in the questionnaire. The first author and fourth author carried out this grading independently. We then synthesized the results of our grading (the agreement rate was 92.5%), and any discrepancies were discussed until we reached a consensus.

In the second analysis phase, we classified students' attempts at proof validation and modification as successful or unsuccessful. For proof validation, students were judged as successful if they gave correct answers to both Q2-1 and Q2-2. Regarding Q2-1, students' answers were judged as correct if they produced one diagram that constituted a local counterexample; Q2-1 does not require the students to find all of the possible local counterexamples. Students who correctly answered Q2-1 but not Q2-2 were assessed as unsuccessful in proof validation because they had only shown themselves able to draw diagrams, not to recognize the diagrams as local counterexamples. Regarding proof modification, we considered both Q1-2 and Q2-3 because in both questions students are asked to modify proofs; the questions differ only in the origins of the local counterexamples (Q1-2 is related to the given local counterexample, while Q2-3 is related to local counterexamples discovered by the students). For Q1-2, we focused on students who correctly answered Q1-1. We did this because the necessity for proof modification would not arise if students could not recognize the given diagram as a local counterexample.[4] Students' attempts at proof modification were judged as successful if Q1-2 was correctly answered. Likewise for Q2-3, we focused on students who correctly answered Q2-1 and Q2-2; their attempts at proof modification were judged as successful if Q2-3 was correctly answered.

In the third phase of the analysis, we used Toulmin's framework to characterize, in more detail, students' successful and unsuccessful attempts at proof validation and modification. We began by categorizing unsuccessful attempts according to whether each of the data, claims, and warrants in the attempts was valid or not. After that, we looked into each of the categories and, when appropriate, made sub-categories based on the contents of the data, claims, and warrants. The same procedure was then applied to students' successful attempts.

Alongside the analysis of the students' answers to the task-based questionnaire, we conducted a brief analysis of the teaching intervention. We did this because we anticipated that the teaching intervention would influence the students' performance in the task-based questionnaire. The first author observed all of the lessons as a non-participant, and the video records and collected students' worksheets were used as data for this component of the analysis. As this aspect of the analysis is secondary in this paper, the lesson data are used only if they provide clarification and/or explanation of the findings from the main data (i.e., the task-based questionnaire). In other words, our focus is not on evaluating the intervention.

English translations of the tasks and the students' answers to the questionnaire are rendered from the original Japanese by the authors. All students' names used in this paper are pseudonyms.

4.4. The lessons

To help contextualize our analysis, and before presenting the analysis, we give a short account of the teaching intervention implemented immediately prior to the students completing the task-based questionnaire. In the first of the three lessons, the following task was used: "As shown in the Figure [this is Fig. 4a], there are four points A, B, C, and D on circle O. Draw lines AC and BD, and let point P be the intersection point of the lines. Prove $\Delta PAB \sim \Delta PDC$." The diagram in the students' worksheets consisted of only circle O and the four points. As such, the students first needed to complete the diagram by following the problem sentence (e.g.,

[4] We acknowledge that there are multiple reasons for revising proofs, for instance clarity or brevity. However, in this paper, we restrict the term proof modification to our definition involving local counterexamples.

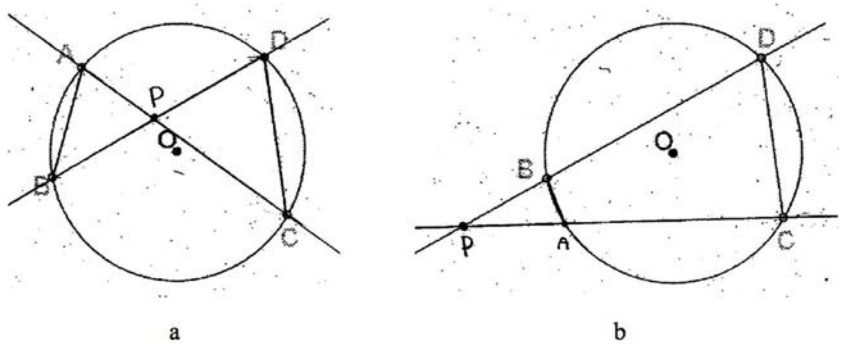

Fig. 4. Diagrams in the first lesson (from the students' worksheets).

drawing line AC). After the students proved the statement by showing ∠APB = ∠DPC and ∠BAP = ∠CDP, the teacher prompted the students to draw diagrams in which point A was on arc BC. The students recognized these diagrams (e.g., Fig. 4b) to be local counterexamples, and modified their proofs in a similar way to that described in Section 3.

In the second lesson, the following task was used: "As shown in the Figure [this is Fig. 5a], two circles O and O′ intersect at points P and Q, and two points A and B are located on circle O. Draw line AQ and let point C be the intersection point of the line and circle O′. Draw line BQ and let point D be the intersection point of the line and circle O′. Prove ΔPAC ~ ΔPBD." Like the first lesson, the students first completed the diagram (e.g., drawing line AQ and placing point C) and then proved the statement by showing ∠PAC= ∠PBD and ∠PCA = ∠PDB. Afterwards, in contrast to the first lesson where the position of point A was specified, the teacher said, "Let's place point A on various places on circle O, and search for cases where your proofs become invalid." The case of Fig. 5b, where point A is on arc PQ of circle O, was the local counterexample discovered by the most students, and the case of Fig. 5c, where point C is on arc PQ of circle O′, was also discovered by some students. The students then engaged in proof modification in a similar way to that described in Section 3.

As noted earlier, in the first part of the third lesson the students tackled the proof problem illustrated in Section 3, above. Similar to the first and second lessons, the students first completed the diagram on their worksheets (because the initial diagram included only part of the problem sentence); then they proved the statement. All the students' proofs were based on showing ∠QAC = ∠PBD and ∠ACQ = ∠BDP with the inscribed angle theorem. The following was a typical student proof.

In ΔAQC and ΔBPD, since inscribed angles corresponding to arc PQ of circle O are equal, ∠QAC = ∠PBD. ...(1)
Since inscribed angles corresponding to arc PQ of circle O′ are equal, ∠ACQ = ∠BDP. ...(2)
From (1) and (2), since two pairs of angles are equal, ΔAQC ~ ΔBPD.

As also noted earlier, the teacher provided the opportunity for students who could not construct proofs by themselves to obtain help from successful students. Following this, the students worked individually on the task-based questionnaire.

5. Results

5.1. Proof validation

Students' attempts at proof validation are summarized in Table 2. In this classification, we regard the claims (C) of all the attempts as "the proof is invalid in this case" because Q2-1 asked students to "find a case that rejects your proof." When referring to Toulmin's framework, we use superscript X to indicate that the relevant element is not valid; for example, W indicates that the stated warrant is valid, while W^X indicates that it is incorrect, inappropriate, or insufficient. We also include the case where a warrant is not expressed in W^X for the following reason. Suppose that a student draws Fig. 2b as a data, but does not express a warrant showing why this data supports the claim "the proof is invalid in this case." In this case, the student may implicitly hold a warrant, and the implicit warrant could be still correct even though it is not directly expressed. However, as shown in Section 5.1.3 and Komatsu (in press), it is also

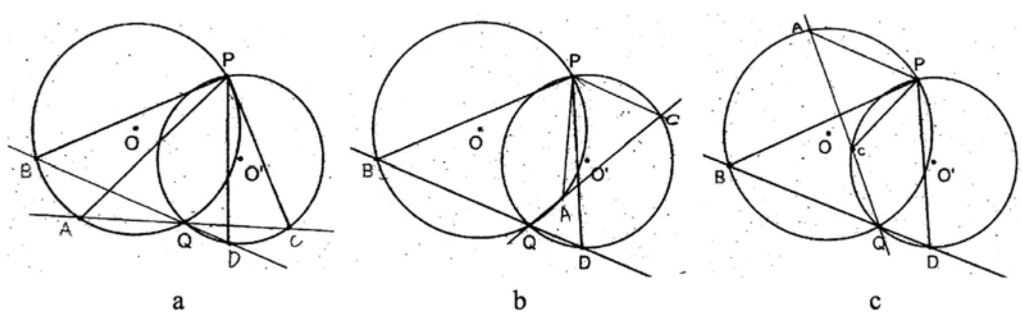

Fig. 5. Diagrams in the second lesson (from the students' worksheets).

Table 2
Classification of attempts at proof validation (D is data, W is warrant, C is claim).

# in category	Toulmin's model	Description of category	# in sub-category	Description of sub-category
18	D, W, C	Success in producing local counterexamples	17	Placing point A on arc PQ
			1	Placing point A on arc BP
9	D^X, W^X, C	Failure in producing diagram data that can be regarded as local counterexamples	6	Drawing diagrams that do not satisfy the problem condition
			3	Other or no diagram
5	D, W^X, C	Failure in showing the warrants for why produced diagrams invalidate proofs	3	Stating that valid parts of proofs are invalid
			2	No reference to the warrant

possible that the student's implicit warrant could be incorrect; he/she may wrongly consider that the valid part of the proof (i.e., the reason for ∠QAC = ∠PBD) is invalidated by this data. It is not possible to judge whether implicit warrants are valid from the written answers to the questionnaire. In this sense, non-expressed warrants are insufficient for providing complete descriptions of how the data support the claims, and thus we regard students' answers as W^X if they do not express warrants.

While, in the first part of the third lesson, some four students could not write correct proofs despite obtaining help from successful students, the response of these four students to the task-based questionnaire are nevertheless included in Table 2 because their attempts at the questionnaire were observed. All their responses were assessed to be in the category of the combination of D^X, W^X, and C because the data and warrant they offered were invalid.

In the sections below, we present an analysis of the students' successful and unsuccessful attempts in terms of Toulmin's framework.

5.1.1. Successful attempts at producing local counterexamples

As shown in Table 2, 18 of 32 participating students (56%) succeeded in proof validation; that is, in discovering local counterexamples to their proofs. More concretely, they produced diagram data that constituted local counterexamples in Q2-1, and showed warrants identifying which parts of their proofs were rejected by the diagrams in Q2-2. All of these students except one placed point A on arc PQ in Q2-1. For instance, student Rena gave the data shown in Fig. 6 and expressed her warrant, "It is not possible to show ∠QAC = ∠PBD from inscribed angles corresponding to arc PQ of circle O." In other words, Rena pointed out that angle QAC was no longer an angle inscribed to circle O in Fig. 6; Rena's answer is depicted in Fig. 6. The only student who did not place point A on arc PQ drew a diagram in which point A was on arc BP, as in the original diagram in Fig. 1, but point C was on arc PQ.

From this we identified two possible reasons for the prevalence of local counterexamples in which point A was on arc PQ. The first relates to the diagrams given in the task-based questionnaire. Circle O in Fig. 3b can be broadly divided into arcs BP, BQ, and PQ. Point A is located on arc BP in the diagram in the original proof problem (Fig. 1) and on arc BQ in the diagram in Q1 (Fig. 3a). Hence, the students would (consciously or unconsciously) choose to place point A on the remaining arc PQ. The second reason relates to the influence of the implemented intervention. In the second lesson, the local counterexample discovered by the most students was the case where point A was on arc PQ (Fig. 5b). Thus, the students would likely draw similar diagrams in response to the questionnaire.

5.1.2. Unsuccessful attempts at producing diagram data that can be regarded as local counterexamples

As Table 2 shows, 14 students (44%) did not succeed in proof validation, specifically in the discovery of local counterexamples. We classify these unsuccessful attempts at proof validation into two types.

The first type consists of attempts in which students were not able to produce diagram data that could be regarded as local counterexamples. In particular, six students (19%) drew improper data that did *not* satisfy the condition of the problem. This number accounted for approximately half of the students who did not succeed in proof validation (6/14 = 43%). For instance, two students drew line AQ, not line AP. Nanami drew the data in Fig. 7 and gave no warrant in Q2-2. Another two students placed point C on the

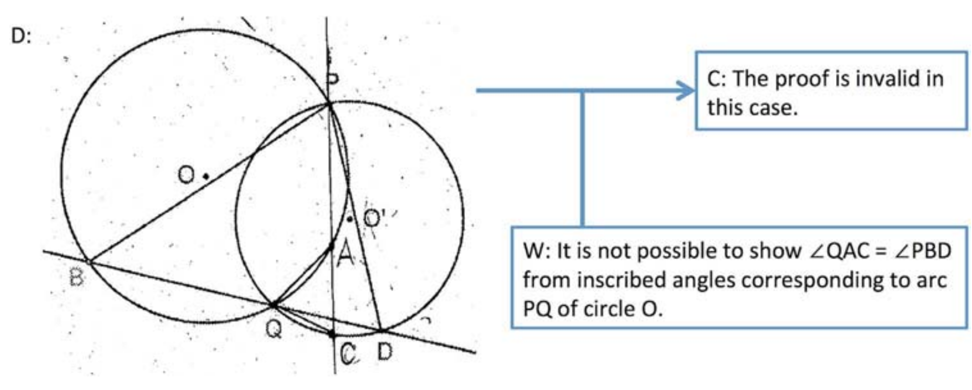

Fig. 6. Success in proof validation (diagram from student worksheet).

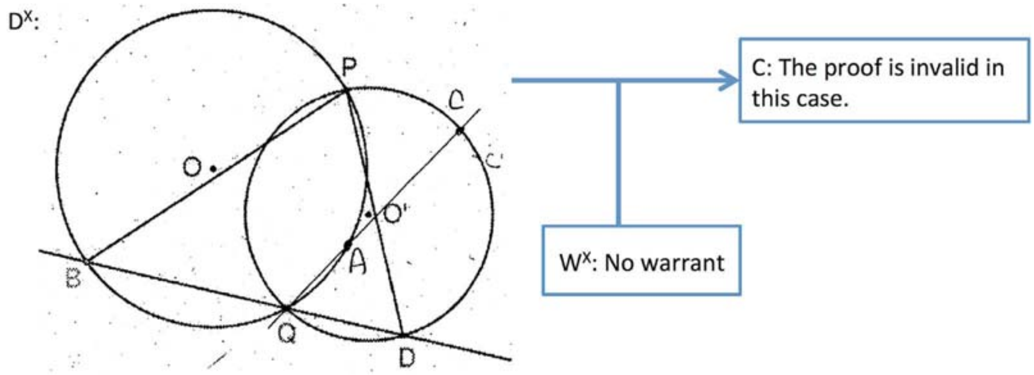

Fig. 7. Diagram that does not satisfy the problem condition (diagram from student worksheet).

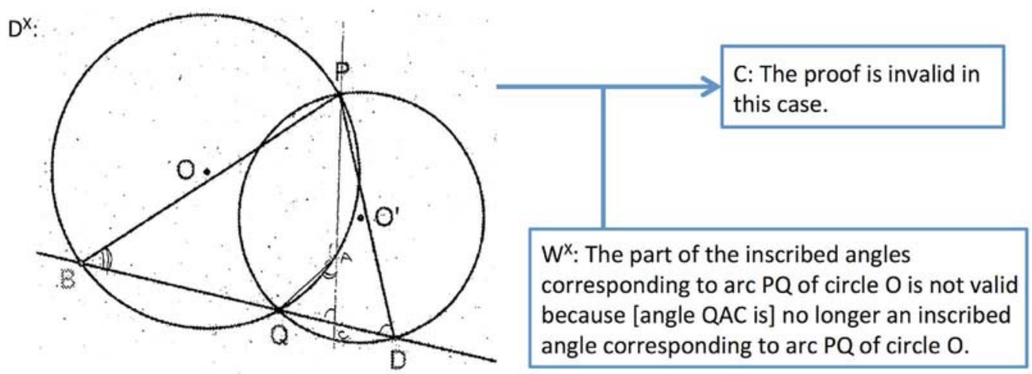

Fig. 8. Another diagram that does not satisfy the problem condition (diagram from student worksheet).

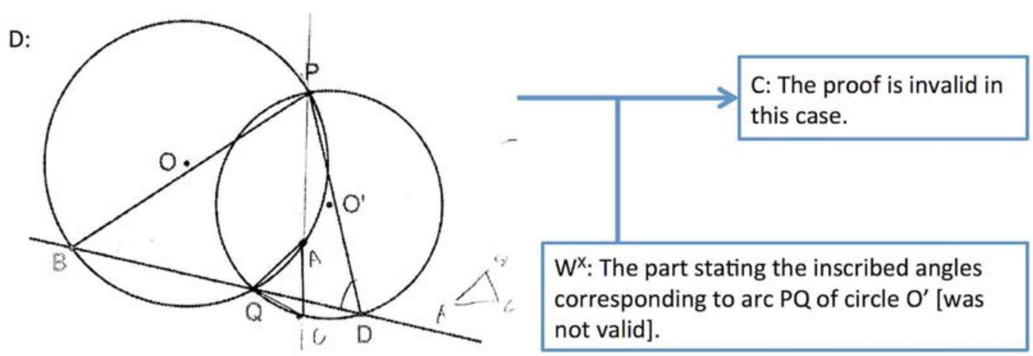

Fig. 9. Incorrect warrant stating that the valid part of a proof is invalid (diagram from student worksheet).

Table 3
Classification of attempts at proof modification.

# in category	Toulmin's model	Description of category	# in sub-category	Description of sub-category
26	D, W, C	Success in proof modification	14	Modifying proofs for local counterexamples produced themselves
			12	Modifying proofs for the given local counterexample
9	D, W^X, C^X	Failure to find another pair of equal angles	7	Not referring to a pair of angles
			2	Referring to unequal angles
6	D, W^X, C	Failure to show sufficient warrants for why referenced angles are equal		
1		Other type of failure in proof modification		

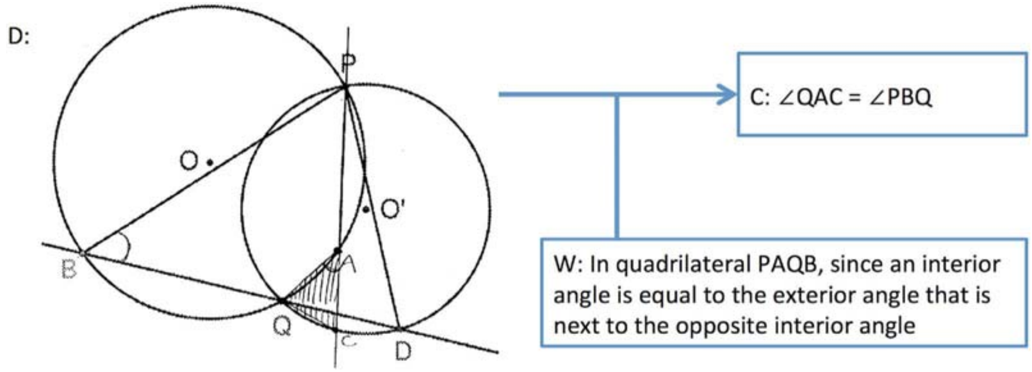

Fig. 10. Success in proof modification (diagram from student worksheet).

intersection point of lines AP and BD, rather than that of line AP and circle O′. Sakura gave the data in Fig. 8,[5] and described her warrant as follows: "the part of the inscribed angles corresponding to arc PQ of circle O is not valid because [angle QAC is] no longer an inscribed angle corresponding to arc PQ of circle O." The content of the stated warrant is true in itself; however, using the data that does not satisfy the problem condition is unacceptable for showing the invalidity of the proof. In this sense, the stated warrant is not appropriate. Nanami's case and Sakura's case can be summarized as Figs. 7 and 8, respectively.

Although these six students thought of cases that were different from the original case (Fig. 1) in terms of the location of point A, they did not understand the problem condition. As described in Section 4.4, each of the three implemented lessons began with drawing diagrams that satisfied the problem conditions; nevertheless, the responses of several students to the task-based questionnaire showed that they encountered difficulty producing diagrams satisfying the specified problem condition.

5.1.3. Unsuccessful attempts at showing the warrants for why produced diagrams invalidate proofs

The second type of unsuccessful attempt at proof validation, represented by five students, consisted of attempts that did extend to producing diagram data that could be regarded as local counterexamples but not to providing warrants showing which parts of the proofs were thereby invalidated. Three of these students stated warrants that incorrectly identified valid parts of their proofs as invalidated by their data. For instance, Yui drew the data shown in Fig. 9, which can be regarded as a local counterexample to her proof. For Q2-2, she wrote her warrant, "the part stating the inscribed angles corresponding to arc PQ of circle O′ [was not valid]." However, her data rejects the part of her proof related to circle O, not circle O′. More concretely, the angles inscribed to circle O′—angles ACQ and BDP—maintain this status in her data, and, as described above, angle QAC is no longer an angle inscribed to circle O. Hence, Yui's attempt can be depicted as in Fig. 9. Because these five students produced diagrams that could be regarded as local counterexamples, they were able to understand the problem condition and consider cases different from the original case; however, it was difficult for them to discern the parts of their proofs invalidated by their produced diagrams.

5.2. Proof modification

Some 24 students correctly answered Q1-1 of the task-based questionnaire and 18 students correctly answered Q2-1 and Q2-2. Thus, as mentioned in Section 4.3, 42 (=24 + 18) answers to Q1-2 and Q2-3 were available for analysis as attempts at proof modification. This number (42) is larger than the number of participants (32) because attempts by 17 students who correctly answered all of Q1-1, Q2-1, and Q2-2 were counted twice, namely their answers to Q1-2 and Q2-3.

Our classification of these 42 attempts at proof modification is shown in Table 3. Note that in all 42 attempts, the data were regarded as correct. This is because we considered both the diagram in Q1-2 of the questionnaire and the diagrams produced by students in Q2-1 as data. Since we analyzed only those students who produced proper diagrams, as mentioned earlier, all the data were correct.

In what follows, we first consider the successful attempts at proof modification, before then considering different forms of incorrect attempts.

5.2.1. Successful attempts at proof modification

As depicted in Table 3, 14 students successfully modified their proofs to cope with local counterexamples that they themselves had produced; this number accounts for 44% of the participants (14/32) and 78% of the students who produced local counterexamples themselves (14/18). Similarly, 12 students modified their proofs to address the local counterexample given in the questionnaire, accounting for 38% of the participants (12/32) and 50% of the students who recognized the given diagram as a local counterexample (12/24). Nine students (28% of the participants) achieved both types of proof modification.

For instance, Mirai drew the data shown in Fig. 10 as a local counterexample to her proof, and constructed the following proof for this case:

[5] Although the place of point C may not be clear in Fig. 8, the mark for angle ACQ in the figure tells us that the student let point C be the intersection point of lines AP and BD.

In ΔAQC and ΔBPD

Since inscribed angles corresponding to arc PQ of circle O′ are equal, ∠PCQ = ∠PDQ ... (1)

In quadrilateral PAQB, since an interior angle is equal to the exterior angle that is next to the opposite interior angle, ∠QAC = ∠PBQ ... (2)

From (1) and (2), since two pairs of angles are equal, ΔAQC ∼ ΔBPD

In her initial proof for Fig. 1, Mirai had used the inscribed angle theorem as a warrant for claiming ∠QAC = ∠PBQ. Then, in the above, she replaced her inscribed angle theorem warrant with an inscribed quadrilateral theorem warrant to justify the same claim. This can be regarded as "rewarranting," where an attempt is made "to find a new, more appropriate reason for a claim" (Zazkis et al., 2016Zazkis, Weber, & Mejia-Ramos, 2016). Thus, her proof modification can be summarized using Toulmin's framework as in Fig. 10. Regarding Q1-2, the students who succeeded in proof modification performed the same substitution of warrants in the reason for ∠ACQ = ∠BDP in Fig. 3a.

As seen in Mirai's answer, proof modification can be achieved by revising the part regarding a pair of equal angles invalidated by the local counterexamples. Hence, in what follows when we examine the unsuccessful attempts, we focus on the part of proof modification in which students addressed the invalidated parts of their proofs, rather than on the whole of proof modification.

5.2.2. *Unsuccessful attempts to find another pair of equal angles*

Students' unsuccessful attempts at proof modification can be broadly divided into two types. The first type was observed in nine attempts in which students failed in giving valid claims; more specifically, they were not able to find another pair of equal angles. For instance, Ayaka and Kotone answered Q1-2 as follows:

Ayaka:

In ΔAQC, ΔBPD

Since inscribed angles corresponding to arc PQ of circle O are equal

Kotone:

In ΔAQC and ΔBPD

Since inscribed angles corresponding to arc PQ of circle O are equal, ∠QAC = ∠QBP ... (1)

From the property of inscribed quadrilaterals, ∠PCQ = ∠PDQ ... (2)

From (1) and (2), since two pairs of angles are equal, ΔAQC ∼ ΔBPD.

The data in Fig. 11 was taken from Ayaka's worksheet. In Q1-1, she recognized that ∠PAQ = ∠PBQ could be still shown with the inscribed angle theorem in circle O. In the above answer, she intended to write this again, but left it unfinished. Moreover, she gave no claim or warrant regarding another pair of angles. Thus, Ayaka's case can be depicted as shown in Fig. 11. As for Kotone, although she claimed ∠PCQ = ∠PDQ, these angles are not equal in Fig. 3a and so her claim cannot be warranted from the inscribed quadrilateral theorem. There were seven attempts like Ayaka's and two attempts like Kotone's.

5.2.3. *Unsuccessful attempts at giving sufficient warrants for why referenced angles are equal*

Another type of failure in proof modification was observed in six attempts in which although students referred to valid claims (i.e., a pair of angles that are mathematically equal), they provided insufficient warrants. It is noteworthy that all of these students changed which pair of angles was referenced from their original proofs. All six attempts were almost identical, so we take Moe's answer to illustrate them all. She drew the data shown in Fig. 12 and wrote the following argument:

In ΔAQC and ΔBPD

From the property of inscribed quadrilaterals, ∠AQC = ∠BPD ... (1)

From the inscribed angles corresponding to arc PQ of circle O′, ∠PCQ = ∠PDB ... (2)

From (1) and (2), since two pairs of angles are equal, ΔAQC ∼ ΔBPD

In her initial proof for Fig. 1, Moe had referred to the pair of angles CAQ and PBD; then, in (1) in the above answer, she changed her claim from this pair to the pair of angles AQC and BPD. However, her warrant is not sufficient, because ∠AQC = ∠BPD cannot be shown only from the inscribed quadrilateral theorem. This can be proved by using both the inscribed angle theorem and the inscribed quadrilateral theorem: ∠CQD = ∠CPD from the former theorem, ∠AQD = ∠BPA from the latter, and then ∠AQC = ∠AQD − ∠CQD = ∠BPA − ∠CPD = ∠BPD. Yet Moe did not present this reasoning; instead, she changed which pair of angles was referenced

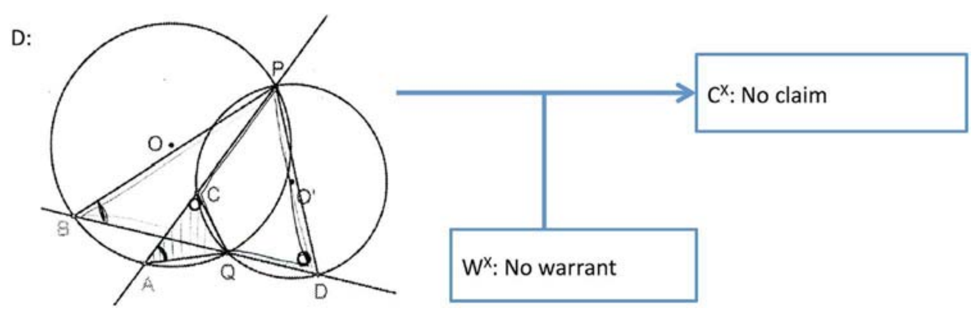

Fig. 11. No reference to a pair of angles (diagram from student worksheet).

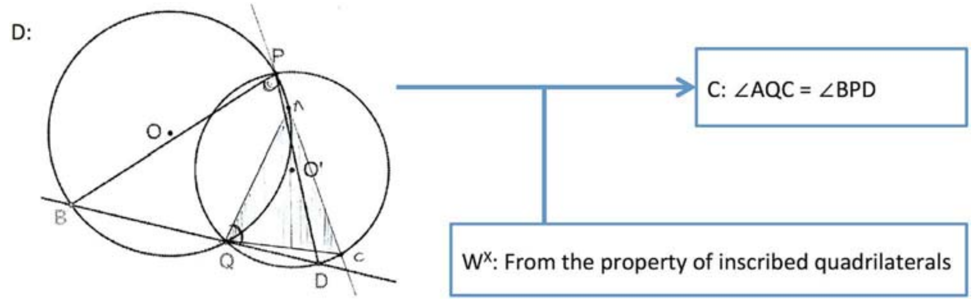

Fig. 12. Insufficient warrant for the equality of angles (diagram from student worksheet).

in a superficial way only. Thus, her attempt at proof modification can be depicted as shown in Fig. 12.

6. Discussion

In this paper, we have presented an analysis of the attempts that participating students made at proof validation and modification of a set problem. Some 18 of the 32 participating students (56%) succeeded in proof validation, while 14 students (44%) successfully modified their proofs to cope with local counterexamples that they themselves had produced. Regarding unsuccessful attempts, our analysis provides two explanations for why some participating students did not achieve proof validation and modification.

First, as shown in Section 5.1.2, there were six cases of proof validation in which students drew improper diagram data that did not satisfy the condition of the proof problem (part of the combination of D^X, W^X, and C). These six students accounted for 19% of the participating students and approximately half of the students who did not succeed in proof validation. Thus, the main cause of student failure in proof validation seems to have been the difficulty of producing diagrams that satisfied the problem condition. This phenomenon may be widely observed beyond this study if prevalent practice in schools is considered. Diagrams of proof problems are usually given by teachers and textbooks, and rarely produced by students themselves (Herbst & Arbor, 2004; Herbst & Brach, 2006). Under pressure to teach a large amount of content in a limited time, many teachers would devote their lessons to proof construction with given diagrams rather than starting with diagram drawing by students. In addition, if each student produces a diagram, then diverse diagrams appear in the classroom, some of which may be obscure or wrong in different ways and for different reasons. As such, these diagrams may hinder subsequent proof construction. Thus, a single, clear, given diagram that is shared amongst students helps teachers focus class attention on proof construction. As a result of this approach, however, students' opportunity independently to produce diagrams while considering problem conditions is limited, which would increase failure in proof validation.

Second, cases where students did not show acceptable warrants for their claims (i.e., the combination of D, W^X, and C) were observed in both proof validation and modification. In proof validation, the students' warrants were not correct because they answered that valid parts of their proofs were invalidated by the data that they had produced (Section 5.1.3). In proof modification, the warrants were not sufficient because it was not possible to show the claims from these warrants alone (Section 5.2.3). To overcome such difficulties, two types of actions can be taken. The first type is "rewarranting" (Zazkis et al., 2016). Rewarranting, as employed by Zazkis et al., is related to translation from graphical warrants to verbal-symbolic warrants; in our study, instead, it involves adding additional reasons in order to make sufficient warrants, as illustrated in Moe's case in Section 5.2.3. Rewarranting also involves replacing invalid warrants with valid ones. For instance, Yui's warrant shown in Section 5.1.3 can be corrected if her reference to circle O' is amended to the reference to circle O. The second type of action is to modify claims while keeping warrants, instead of changing the warrants. For example, in Moe's case in Section 5.2.3, if her claim is the equality of angles CAQ and PBD, it is possible to show this claim directly from her warrant. Thus, in summary, another cause of student failure at proof validation and modification was in the difficulty of showing acceptable warrants, but the actions of rewarranting or modifying claims based on warrants can help overcome this difficulty.

Proof validation as discussed in this paper—namely the discovery of local counterexamples by producing diagrams—is an important activity in secondary mathematics. Our study examines proofs valid in certain domains, focusing on the production of diagrams to find cases where the proofs become invalid. This activity enables students to experience an aspect of authentic mathematical practice that is a key to how mathematical knowledge is created in the discipline (Lampert, 1992). For example, De Villiers (2010) treats mathematical experimentation, which includes diagrammatic evaluation of conjectures and proofs, and points out the functions of experimentation in the discipline of mathematics. He points to the need "to explore authentic, exciting and meaningful ways of incorporating experimentation and proof in mathematics education, in order to provide students with a deeper, more holistic insight into the nature of our subject" (p. 220). Therefore, proof validation/modification as examined in this paper is significant in the sense that this activity can provide students with an opportunity to experience the integration of proof and experimentation that is essential for progress in mathematical practice.

Although the task-based questionnaire in this study was based on a single task, it is possible to create similar situations by using other tasks. The task used in this study is an example of a larger class of *proof problems with diagrams* (Komatsu, in press; Komatsu et al., 2014); the key characteristic of the task used here is that local counterexamples to a proof can be discovered by transforming the diagram given in the task. In other words, tasks with this characteristic can elicit the types of proof validation and modification

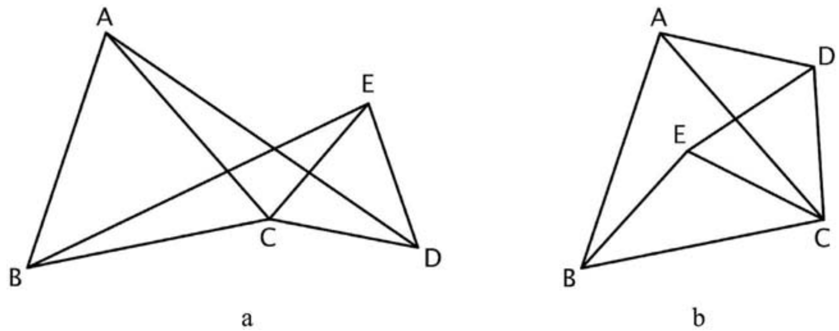

Fig. 13. Diagrams for another task involving triangles.

examined in this study. This kind of task can be found in school mathematics textbooks; in fact, in secondary school geometry, students are usually given proof problems for which diagrams and labels have already been prepared (Herbst & Brach, 2006), and in some of these problems, it is possible to find local counterexamples to proofs by transforming the given diagrams. The following task is a typical example (in which a summary of a proof is also shown):

Task. As shown in Fig. 13a, there are two equilateral triangles ABC and CDE that share point C. Prove AD = BE.

Proof (summary). AC = BC, CD = CE, and $\angle$ACD = 60° + $\angle$ACE = $\angle$BCE. Therefore, ΔACD ≡ ΔBCE and thus AD = BE.

This task and its variations are common in the sense that they are included in all eighth-grade mathematics textbooks in Japan. If the two equilateral triangles in Fig. 13a are rotated, it is possible to discover local counterexamples, such as Fig. 13b in which the part of the proof showing the equality of angles ACD and BCE becomes invalid and needs to be modified from "60 + $\angle$ACE" to "60 − $\angle$ACE." Thus, students have the opportunity to experience proof validation and modification if, after constructing a proof for this task, they are invited to transform the given diagram to inspect whether the proof is always valid.

Another option for eliciting proof validation and modification is to use universal geometrical propositions that are described in general terms without any diagrams or labels, but require case analysis for proofs to be constructed. An example is the inscribed angle theorem: the measure of an inscribed angle corresponding to an arc is half the measure of the central angle corresponding to that arc. Proving this theorem requires case analysis according to the positions of the inscribed angles; hence, students can experience proof validation and modification if, after showing the theorem in the case where an inscribed angle includes the center of the circle, they are invited to change the place of the inscribed angle (e.g., to the case where it does not include the center of the circle) in order to scrutinize whether their proofs are always valid (Komatsu, 2012). Furthermore, this activity can provide an opportunity for students to learn key mathematical content (the inscribed angle theorem) while engaging in the important mathematical practice of proof validation and modification (Dawkins & Karunakaran, 2016). We conjecture that some teachers may already practice this process in their classes.

The tasks used in this study, and the similar approaches just described, show that geometry can be a suitable content area for students to work on proof validation involving local counterexamples. Dawkins and Karunakaran (2016) contend that "research on proof-oriented mathematical behavior should attend to the role of particular mathematical content" (p. 72) because students' proving behavior is affected by their understanding of the mathematical content. This implies that whether certain activity can be successfully elicited depends on particular content areas. Given that geometry is the central area for proof learning at the secondary school level, the feasibility of proof validation in geometry leads to the possibility of more opportunities for students to experience this form of activity. Thus, our focus on geometry should be considered not as a limitation narrowing the study's generalizability to other content areas but rather a strength, showing that proof validation involving local counterexamples can be introduced in the main area for proof learning in secondary school.

This argument leads directly to an implication for teaching: teachers may profitably employ *proof problems with diagrams* to introduce proof validation and modification into secondary school geometry classes. Although scholars suggest that students' capability at proof comprehension and validation can be improved by relevant training (e.g., Alcock & Weber, 2005; Hodds et al., 2014), which kinds of tasks are useful for such training remains unclear. This study may offer relevant insight in that we illustrate how *proof problems with diagrams* can be used to teach proof validation entailing the discovery of local counterexamples. When using such tasks, however, teachers should consider the difficulties that the students in this study encountered. For example, it is likely to be crucial for teachers to help students produce diagrams that satisfy the conditions of proof problems. It is also likely to be important, during student proof validation and modification, for teachers to focus students' attention on whether their warrants sufficiently connect the data and their claims in terms of Toulmin's framework.

7. Conclusion

This study builds on the limited previous research by addressing proof validation and modification in secondary school mathematics. For that purpose, we employed the mathematical tasks called *proof problems with diagrams*. Using these tasks, this study considered proof validation as entailing the discovery of local counterexamples to proofs by producing diagrams, and proof modification as entailing the revision of the proofs to cope with these local counterexamples. This mathematical activity is meaningful at the secondary level because it enables students to work on critiquing arguments and engage in authentic mathematical practice. In this paper, we analyzed attempts at proof validation and modification by ninth-grade students in Japan, by means of their questionnaire answers. We showed that their failures in proof validation and modification arose mainly from difficulties drawing diagrams that satisfied the condition of the proof problem and/or from showing acceptable warrants for their claims.

Nevertheless, there exist limitations to our findings that should be kept in mind. In particular, we do not have information regarding the participating students' conceptions of proof, such as whether they recognize the limitations of empirical arguments and the generality of proofs. Students' conceptions of proof could influence their proof validation behavior, and thus it is necessary to investigate this influence. Additionally, the task-based questionnaire utilized for this study implied that there existed cases denying students' proofs and asked students to find these cases. More open-ended questions, such as whether proofs are valid or invalid, should lead to different kinds of attempts at proof validation and modification. Further, our analysis was based on the students' written answers to the task-based questionnaire; it would be worthwhile to look into student thinking by conducting additional interviews. For example, we included students' non-expressed warrants in the category of W^X, but, as mentioned in Section 5.1, the students may have implicitly held warrants, and the implicit warrants could still be correct even though they were not directly expressed. Additional interviews would be helpful for addressing this possibility.

Given that our analysis focused on the *product* of student answers to the questionnaire, task-based interviews would be also helpful for addressing the *process* of how students engage in proof validation and modification. In such cases, students may refer to the domains of the (in)validity of proofs, for analysis of which it is likely to be crucial to employ Toulmin's (2003) full framework, particularly the notions of modal qualifiers and rebuttals (Inglis et al., 2007; Jahnke, 2008). A final limitation is that the task-based questionnaire reported here involved a single task and was implemented in a single class in one Japanese secondary school. It is necessary to conduct larger-scale studies to examine the results of this study. We hope that future studies can address these limitations and that our study can serve as a stepping-stone to better understand these questions.

Aside from addressing these limitations, we have two additional suggestions for future research. First, the use of dynamic geometry software may be helpful for proof validation because the dragging function can help students more easily to transform diagrams. In relation to this, we have obtained a positive preliminary result in a task-based interview in which a triad of secondary school students engaged in mathematical activity related to proofs and refutations (Lakatos, 1976) by creating several types of diagrams in a dynamic geometry environment (Komatsu & Jones, in press). This preliminary result needs to be further scrutinized, however, as it is based on one interview only. Second, given the importance of proof validation at the university level (Alcock & Weber, 2005; Selden & Selden, 2003), it would be worthwhile to extend our research to this level as well. Aside from the choice of subjects, the features of our research do not distinctly reflect the characteristics of the secondary school level, and a similar approach would be adaptable to undergraduates. Of course, different results may be obtained. Given the relative maturity of undergraduates, it may be possible to implement the questionnaire among them without conducting a prior teaching intervention. Because proof validation is conceived in an innovative way in this study, as mentioned in Section 3, the extension of our research to the university level could give new insights into proof validation behavior among undergraduates and provide new teaching opportunities to engage them in proof validation and modification.

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