Cite as: Jones, K. (1998), Deductive and Intuitive Approaches to Solving Geometrical Problems. In: C. Mammana and V. Villani (eds), *Perspectives on the Teaching of Geometry for the 21st Century*. Dordrecht: Kluwer. pp78-83. ISBN: 0792349903

Deductive and Intuitive Approaches to Solving Geometrical Problems

Keith Jones
University of Southampton, UK

Approaches to the teaching and learning of a chosen topic in geometry can be located somewhere between what are sometimes perceived as two extremes. One such extreme is characterised as "intuitive"; the other as "formal" or "axiomatic". There seems to be a number of ways of looking at the relationship between these two positions. Piaget ([6] p 225), for instance, appears to suggest a hierarchy when he writes:

> Although effective at all stages and remaining fundamental from the point of view of invention, the cognitive role of intuition diminishes (in a relative sense)

during development. .... there then results an internal tendency towards formalisation which, without ever being able to cut itself off entirely from its intuitive roots, progressively limits the field of intuition (in the sense of non-formalised operational thought).

This perspective of a hierarchy, with a shift, be it sudden or gradual, from "intuitive" to "formal", is long-standing. In the UK, as long ago as 1923, the Mathematical Association were recommending three stages in the teaching of geometry [7]; briefly:

Stage A: intuitive, experimental work;

Stage B: 'Locally' deductive work in which formal symbolism and deductive reasoning is introduced, but where intuition and induction still have a place and will be used to bridge logically difficult gaps; and

Stage C: Globally rigorous work

This model, interestingly enough, has similarities to the van Hiele approach which has received some attention over recent years (see, for example, Fuys *et al* [4]). In the same vein, the US NCTM curriculum standards for school mathematics state that "the study of geometry in grades 5-8 links the informal explorations begun in grades K-4 to the more formalised processes studied in grades 9-12" (NCTM [8]p 112).

Nevertheless, other viewpoints have been expressed. Fischbein ([3] p 244), amongst others, for example, suggests either a plurality or a dialectic when he says that:

The interactions and conflicts between the formal, the algorithmic, and the intuitive components of a mathematical activity are very complex and usually not easily identified or understood.

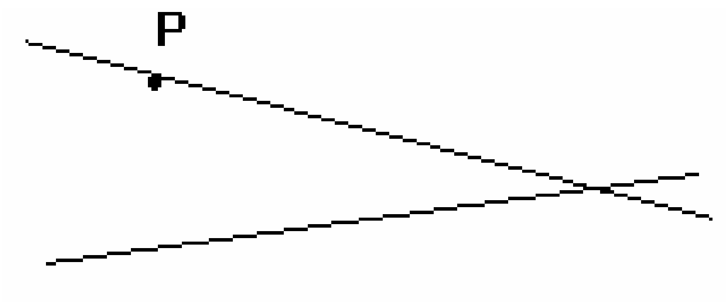
In this chapter I consider *why* people make the decisions that they do when solving geometrical problems. In doing so, I explore the role of geometrical intuition in geometrical problem solving and provide an example of the interplay between students' intuitive and formal (deductive) reasoning. The research I describe was designed to investigate the nature of the relationship between the formal and the intuitive components of mathematical activity as students were solving a series of geometrical problems (Jones [5]). The episode I relate involves two pairs of recent mathematics graduates tackling a well-known geometrical problem. The students were using the dynamic geometry package *Cabri-Géomètre.*

Schoenfeld has written extensively about his work with students solving (and not solving) the geometrical problem given below (for example, Schoenfeld [9] and [10]). There are four elements in Schoenfeld's framework for analysing mathematical problem solving and within each it is possible to suggest a role for intuition. The problem-solvers' *resources* include intuitive knowledge, *heuristics* involve knowing when to use which strategy, *control* focuses on major decisions about what to do in a problem and *belief systems* shape cognition, even when the problem-solver is not consciously aware of holding those beliefs. A conclusion Schoenfeld reaches is that , rather than being disjoint, "a deductive approach to mathematical discovery .... and an empirical intuitive approach .. are in fact mutually reinforcing".

It is helpful, at this point, to give Fischbein's definition of *intuition* as a special type of cognition, characterised by self-evidence and immediacy, and with the following properties (Fischbein [2] p 43-56): *intrinsic certainty, perseverance, coerciveness, theory status, extrapolativeness, globality,* and *implicitness*. In Fischbein's view, intuitions are theories or coherent systems of beliefs. This conception has similarities to Cooney's [1] idea that the representation of an intuition is likened to a mini-theory, a model, which supports reaching a conclusion, with certainty, on the basis of incomplete information.

Problem

You are given two intersecting lines and a point P on one of them. Show how to construct a circle that is tangent to both lines and has P as its point of tangency to one of the lines.



Critical decisions in the solution of this problem are:

1. Constructing a perpendicular line through P
2. Constructing the angle bisector of the angle between the two intersecting lines **or** constructing a circle centred at the intersection and passing through P giving an intersection with the second line; a perpendicular line through this point intersects the perpendicular through P at the centre of the required circle.

One subject pair, both male and both with some experience of geometrical constructions, began by reproducing the problem diagram on the computer screen. Their first approach was to construct a circle with a centre chosen somewhere between the two intersecting lines, and with point P as the radius point. They then used the facility available with *Cabri* to drag the centre of the circle so that it appeared also to be tangential to the lower of the two intersecting lines. Though this gave them a solution, they were not happy with this and searched for a way of being 'absolutely sure'.

Subject CR says "Well, the tangent is perpendicular to the line of radius, isn't it?" so they constructed a perpendicular line through P and constrained the centre of the circle to lie on this perpendicular. Then subject CR suggested that they construct a perpendicular line to the lower of the two intersecting lines and move it into the correct place. At this point, TC wonders if the centre of the circle lies on the bisector of the angle between the two intersecting lines. With that the problem was solved.

Another subject pair, one female (KH) and one male (KJ), both with some experience of geometrical constructions, used a similar approach. They began by creating the diagram for problem 1 and then proceeded to construct two perpendiculars, one through P and a second perpendicular to the lower of the two intersecting lines. As was suggested but never implemented by the first pair, this second perpendicular line was then dragged into place. At this point, KH says "I tell you the other thing we could do and that's bisect that angle to find out where they should cross". With that they too had solved the problem.

For both pairs whose methods are described here, once they had solved the problem, they discussed their solution method. This resulted in them drawing up an argument that would properly serve as a proof. In this way, the solution of the problem suggested the structure of a deductive proof. None of the pairs studied used the alternative method, suggested above, of constructing a circle centred at the intersection and passing through P giving an intersection with the second line. Then a perpendicular line through this point intersects the perpendicular through P at the centre of the required circle.

.

For pair 1, the suggestion to draw the angle bisector was made quite tentatively towards the end of their problem-solving attempt:

TC: Yes ... Ah! Now would the centre of the circle lie .. I'm just thinking something slightly different now, because I'm just trying to think, there must be a way of securing the centre accurately .. and I'm thinking .. does the centre of the circle ..sit on the bisector of the angle that's made by those two lines ..

For pair 2 the student was more certain

KH: I tell you the other thing we could do and that's to bisect that angle to find out where they should cross.

This is how the students accounted for their actions as they watched a video-recording of their problem-solving attempt later the same day. In the case of pair 1:

TC: .. [long pause] .. well, partly previous knowledge. I wasn't .. completely sure. I wasn't saying `Oh, yes. This is what does happen'. I just had a sneaky feeling that we were missing something and I couldn't work out what it was, but I thought, well I'm sure the angle .. there must be some connection between the angle between the two lines and the centre [of the circle]. So, let's put the line in and see what happens.

It turned out to be right, but it was just a sort of stab .. well, it wasn't a stab in the dark completely ...

I can't think why, but I was sure we should be bisecting the angle.

In the case of pair 2:

KH: Ohhh! .. [laughs] .. That's quite interesting because, maybe, .. the fact that there's a cross there [where the two perpendicular lines intersect `opposite' where the original two lines intersect] actually encouraged me to think well, we need to know where the cross is going to be. Perhaps if we hadn't have drawn the other perpendicular it would not have come so quickly.

Looking at that picture now I think .. it's ..er .. er .. I mean just having that sort of cross there on the screen opposite the angle there, I mean, that just spells it out. I think perhaps that's why it just came so quickly.

In both cases the students had some difficulty explaining their actions (a well-established methodological issue). Nevertheless, both previous experience and the visual image appeared to play a part in determining the course of action they were suggesting. In this context, Fischbein says, “Experience is a fundamental factor in shaping intuitions” (Fischbein [2] p 85). However, Fischbein (*ibid*) then goes on to say that “There is little systematic evidence available supporting that view, i.e. evidence demonstrating that new intuitions can be shaped by practice”. In terms of the visual image, Fischbein ([2] p. 103) claims that visualisation “is the main factor contributing to the production of the effect of immediacy”. Fischbein then goes on to relate visualisation to the domain of mental models. The evidence available from this study supports Fischbein’s views in the domain of solving geometrical problems.

Further analysis of the data from this study suggests that geometrical intuition has a role in the planning-implementation, and transition episodes of a problem-solving attempt (see Schoenfeld [9] p 292 for details of these episodes). In addition, it is possible to tentatively identify the following mechanisms as participating in the formation of the subject's geometrical intuitions: premature closure, primacy effect, factors of immediacy (particularly visualisation and anchoring), and factors of globality (see Fischbein [9] p 204-205 for an explanation of these mechanisms). However, because the analysis examined points of critical decision for the *successful* solution of the problem, instances of geometrical intuition may, inevitably, tend to form points of transition in the problem-solving process or occur during planning and implementing episodes. The analysis presented here does not consider how intuition may have led the subjects astray.

The framework Fischbein ([2]) proposes proved reasonably robust in this study. His problem-solving categories of intuition were identified, and a way suggested to differentiate between anticipatory and conclusive intuitions, in that the subjects' awareness of the critical nature of any decision they make appears to be associated with conclusive intuitions. Secondly, it is possible to tentatively discern the mechanisms that participated in the generation of these geometrical intuitions. The explanations supplied by the subjects in this study provides supporting evidence for these conclusions.

The study described here was designed to provide evidence of particular aspects of the nature and role of geometrical intuition in the process of solving geometrical problems, and of the possible mechanisms that participated in the generation of these geometrical intuitions. The students observed here used a mixture of a deductive approach in, for instance, drawing a perpendicular through point *P*, and an empirical intuitive approach provided for by *Cabri*: being able to 'drag' a second perpendicular into place(1). Once they had a solution, the ensuing discussion effectively provided a proof. This illustrates how a deductive and an intuitive approach can prove to be mutually reinforcing when solving geometrical problems.

REFERENCES

[1] Cooney, J. B. (1991), Reflections on the Origin of Mathematical Intuition and some Implications for Instruction. *Learning and Individual Differences* 3(1) 83-107.

[2] Fischbein, E (1987), *Intuition in Science and Mathematics: an educational approach*. Dordrecht: Reidel

[3] Fischbein, E (1994), The Interaction between the Formal, the Algorithmic and the Intuitive Components in a Mathematical Activity. In Biehler, R et al (Eds), *Didactics of Mathematics as a Scientific Discipline*. Dordrecht: Reidel.

[4] Fuys, D. *et al* (1988), *The Van Hiele Model of Thinking in Geometry among Adolescents*. Journal of Research in Mathematics Education Monograph 3. Reston, VA: NCTM.

[5] Jones, K. (1993), Researching Geometrical Intuition. *Proceedings of the British Society for Research into Learning Mathematics*. November 1993 pp 15 - 19.

[6] Piaget, J. (1966), General Psychological Problems of Logico-Mathematical Thought. In Beth, E. W. and Piaget, J. *Mathematical Epistemology and Psychology*. Dordrecht: Reidel.

[7] Mathematical Association (1923), *The Teaching of Geometry in Schools*. London: Bell.

[8] National Council of Teachers of Mathematics (1989), *Curriculum and Evaluation Standards for School Mathematics*. Reston, Va: NCTM.

[9] Schoenfeld, A. H. (1985), *Mathematical Problem Solving*. Orlando, Fl: Academic Press.

[10] Schoenfeld, A. H. (1986), On Having and Using Geometrical Knowledge. In Hiebert, J (Ed), *Conceptual and Procedural Knowledge: the case of mathematics*. Hillsdale, NJ: LEA.

---

1 1The mediating role of the computer is discussed in chapter 4.