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Proof in Dynamic Geometry Contexts[1]

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Proof lies at the heart of mathematics yet we know from research in mathematics education that proof is an elusive concept for many mathematics students. The question that we are now asking is whether the introduction of dynamic geometry systems will improve the situation – or will it make the transition from informal to formal proof in mathematics even harder? How far will innovative teaching approaches with computers assist pupils in developing a conceptual framework for proof and in appropriating proof as a means to illuminate geometrical ideas or will computer use be seen to replace any need for proof?

1. Proof in School Curricula

Traditionally, in the school curriculum, proof has been taught largely in the context of Euclidean Geometry. It has tended to be presented as a formal confirmation of statements that pupils are told are true. This approach emphasises the precise formulation of a standardised linear deductive presentation of argument – form is often perceived as more important than content. Research evidence suggests that such an approach to proof is fraught with conceptual difficulties for pupils. We summarise the main issues:

Pupils fail to appreciate the crucial distinction between empirical and deductive arguments and, in general, show a preference for the use of empirical argument over deductive reasoning (Balacheff, 1988; Chazan, 1993; Finlow-Bates, 1994; Martin & Harel, 1989; Porteous, 1990; Williams, 1979). For many students, deductive proof provides no more than evidence (Chazan, 1993; Fischbein, 1982; Williams, 1979). Proof is not ‘used’ as part of problem solving and is widely regarded as an irrelevant, 'added-on' activity. Geometric proofs, for example and particularly the standard linear deductive style, frequently fail to ‘connect’ with learners who neither understand the purpose of the proof nor appreciate its role in mathematical activity. If formal proof is presented

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only as a way to demonstrate something that students are already convinced is true, proof is likely to remain a meaningless activity (e.g. Hanna and Jahnke, 1993; Tall, 1992; de Villiers, 1990).

The challenge for mathematics educators is to find ways in which geometric proof has communicatory, exploratory and explanatory functions alongside those of justification and verification. There are considerable differences between countries in the treatment of proof in the mathematics curriculum and we must be wary of making generalisations based on evidence from just a few countries. Additionally we must always attempt interpret student responses from a basis of what they are taught. what is stressed in their curriculum and what is ignored Clearly mathematical proof is unlikely to be appropriated without very careful teaching.

2. An Approach to Proof in the U.K. Mathematics Curriculum

In the U.K. a process-oriented approach to proof has been developed where students are encouraged to test and refine hypotheses in order to achieve personal conviction with little emphasis on generalisation and formal presentation of evidence of validity in the form of a proof. This approach to proving and proof is now enshrined in the Mathematics National Curriculum for England and Wales (see Chapter 6, section 1). There is a hierarchical sequence of progression whereby students first use inductive methods, move on to gain an appreciation of the difference between empirical evidence and mathematical argumentation before finally (only for those who reach the highest levels of attainment) constructing formal proofs. These changes to a ‘process-approach’ to proof are matched by a reduction in the study of geometry to small fragments of shape and space.

The disappearance of geometry is perhaps not surprising given the present emphasis in U.K on students generating their own empirical evidence – which, in Euclidean geometry with ruler/compass, would be difficult to achieve. In effect this means that the majority of students have little chance to appreciate the importance of logical argument and few opportunities to prove a geometry theorem. The implications of these curriculum changes on students’ (age 15 years) understanding of the multifaceted nature of justifying and proving in school mathematics have been investigated by Hoyles and Healy in a nationwide survey of over 2500 high-attaining students (see Hoyles, 1997). We can argue that the move towards a more process-oriented perspective represents an understandable attempt to move away from the meaningless routines that characterised geometrical proof in an earlier period. While some students managed to undertake the routines of Euclid correctly, far fewer understood more about geometry as a result. But in trying to remedy this problem, we have removed two important aspects of mathematics: first that it is not sufficient just to see a pattern, it is necessary to understand it and to look at it scientifically. Second, it is not the single elements of geometry that are important but the structures and relationships that bind them together. The question is, can the vital elements of proof and of geometry be salvaged without returning to the lifeless forms of Euclid which characterised the earlier mathematics curriculum?

3. The Computer as a Context for Linking Empirical and Deductive Reasoning

It is our conjecture that if we accept that formal proof should have a central role in the curriculum, we need to design and begin to evaluate innovative activities which enable students to make links between empirical and deductive reasoning throughout that mathematical activity. We need to find new contexts through which to introduce the use of clearly formulated statements and definitions and agreed procedures of deduction which also allow opportunities for connections with empirical justifications. We suggest that the computer might offer just such a new context. This assertion must be treated with caution. We cannot assume that the introduction of the computer will bring about change, at least not change for the better. On the contrary, the challenge is to construct new pieces of learnable mathematics (based firmly on the old!) which are learnable precisely because they harness the potential of the new technology.

To date, work with computers in geometry education has largely been around the use of dynamic geometry software (e.g. *Geometric Supposer, Cabri Géomètre*) where basic screen objects – points, lines and circles – can be created and explored through direct manipulation. They provide a model of Euclidean geometry which offers feedback through 'dragging' as to whether constructions or theorems are 'correct'. Parameters can be varied so invariant relations are spotted, or lengths and angles measured so 'results' observed in the unchanging patterns in the measures. Thus students are able to generate ample empirical evidence for geometric theorems in ways which would have been difficult if not impossible before. Attention has tended not to be focused on proving and proof but rather on the software's potential in aiding the transition from particular to general cases – specific instances can be easily varied by direct manipulation or text-based commands and the results 'seen' on the computer screen (see, for example, diSessa, Hoyles & Noss, 1995; Laborde and Laborde, 1995).

The use of a dynamic geometry package such as Cabri-Géomètre may provide an opportunity for some students to consider the "why…." in addition to the "what if…." and the "what if not...". Such an approach demands the provision of appropriate tasks and in designing them there are been two (connected) issues. The first issue involves ensuring the students experience the necessity of geometrical facts that are true in Euclidean geometry. The second concerns providing the students with suitable experiences to allow them to explain why these geometrical facts are necessarily true.

In the following, we describes some work carried out with a lower secondary school mathematics class (pupils aged 12). The idea is to provide the students with opportunities to experience what could be called ‘proof as explanation’ (Hanna, 1989). The quality of the students, mathematical analysis suggests that the use of a dynamic geometry package such as *Cabri-Géomètre*, coupled with suitable tasks, may provide an opportunity for some students to develop the basis for a fuller appreciation of the nature and purpose of mathematical proof.

In developing dynamic geometry contexts the following factors are critical:

- encouraging the students to make conjectures focusing on the relationships between geometrical objects
- providing the means for the students to explain their actions and results

It is important here to distinguish between *drawing* and *figure*. Laborde (1993 p49) makes clear the distinction in the following way “*drawing refers to the material entity while figure refers to a theoretical object*”. In terms of a dynamic geometry package, a drawing can be a juxtaposition of geometrical objects resembling closely the intended construction. In contrast, a figure additionally captures the relationships between the objects in such a way that the figure is invariant when any basic object used in the construction is dragged. The ability to check a construction by dragging appears to be particularly important part of experiencing the necessity of relevant geometrical facts.

Healy *et al* (1994) suggest introducing students to the idea of “messing up” (or, more accurately, not messing up). They define “messing up” in the following way: “After a figure was drawn it could be dragged to see if it became unrecognisable, that is whether the different objects within the design moved together in a sensible way or not” (*op cit*). While messing up was an idea that was easily appropriated they found after a study of how students' understood the dependencies implicit in their constructions of some simple figures that few students had developed any appreciation of these – they did not example realise why they could not change the relationship of constructed objects (see Hölzl *et al* 1994) This study casts doubt on how far experimental work with Cabri can assist students in constructing systematic arguments – although much more work needs to be done to investigate this. Students need explicit mathematical goals if they were not going to use Cabri solely as a drawing tool rather than a tool for constructing geometrical figures.

A suitable context for work on “proof as explanation” is the analysis of static figures and their construction in the dynamic Cabri environment. In other words, asking the students to construct, given a geometrical figure drawn on paper, the corresponding geometric figure in Cabri such that the Cabri figure can not be “messed up”.

An important consideration is to utilise some of the students existing mathematical knowledge. A long-standing component of the primary geometry curriculum in many countries is the recognising and sorting of various simple plane shapes. This entails such matters as identifying circles and various specific triangles and quadrilaterals. In the secondary curriculum, students are expected to know and use the properties of quadrilaterals and be able to classify them on the basis of their properties. Given these factors, and the fact that there has been both discussion about the classification of quadrilaterals (for example, de Villiers, 1994) and research involving students, ability to classify them (for instance, Fuys *et al*, 1988), the 'family of quadrilaterals', provides a suitable choice of context for the Cabri tasks.

In constructing quadrilaterals, a key idea is the use of the circle as a length-carrier. This, in turn, depends on the locus definition of the circle. Just as the idea of functional dependency may not be easy for lower secondary school students to grasp, neither might the locus definition of the circle (or, at least, it contrasts with what is claimed by Bishop, 1983, to be the notion of the circle commonly-held by upper primary and lower secondary pupils: that is, that a circle is a disc). Some way is required to introduce the students to using the circle in the construction of other geometrical objects.

What might be a promising approach to this problem, illustrated by the work of one pair of pupils on a selection of the resulting Cabri tasks, is described in the next section.

4. Children Working on Dynamic Geometry Contexts

The students reported on here are 12 year olds with no previous experience of using a dynamic geometry package although they had all used various drawing packages and other IT resources. The class is an above-average mathematics class in a city comprehensive school whose results in mathematics at age 16 are at the national average. The mathematics teachers use a resource-based approach to teaching mathematics and the students usually work in pairs or small groups. The class has three 50-minute mathematics lessons per week.

Each of the classroom tasks requires the students, in pairs, to analyse a figure presented on paper. The students are asked to construct the figure using Cabri such that the figure is invariant when any basic point used in the construction is dragged. This means that the students have to focus on the relationship between the basic objects (points, lines and circles) necessary to construct the figure.

Students worked on a sequence of tasks, during many sessions which took place over a period of five months, with often several weeks between sessions, since pairs took it at turns at the computer. During that time, of course, the students continued with their regular mathematics programme. This included some work on certain aspects of 'shape and space', although none of it could be said to be directly relevant to the area of the mathematics curriculum that was the focus of the Cabri work. The tasks started with an exploratory session aimed at acquainting them with the software, progress through a series of 'starter tasks' involving the construction of patterns of circles and lines that could not be 'messed up', to a series of tasks which focus on the 'family of quadrilaterals'.

The next paragraph presents some aspects of students' work in one of the 'quadrilaterals' tasks in which the students are given the figure of a rhombus and its diagonals on paper, and they are asked to construct that figure with Cabri. When we analyze one pairs' work, we find that they tackle this with consummate ease, checking after each individual action that the figure does not mess up. They are expected not only to construct the figure but also to explain why the shape is a rhombus. Although the task refers to the shape as a rhombus, in part of their explanation, they refer to it as a diamond. This is what they say:

> The sides are all the same because, if the centre is in the right place, the sides are bound to be the same.
>
> The diagonals of the diamond cross in the middle, though they are different sizes [lengths]. The diagonals bisect each other. The angles [where the diagonals cross] are the same. They are right angles.
>
> The opposite angles [of the rhombus] are the same. Two are more than 90 degrees but less than 180 degrees, and the others are less than 90 degrees but more than zero degrees.
>
> This shape is a rhombus because the sides are the same, the diagonals bisect at right angles and the opposites have the same angles.

In order to write down their explanation, the two pupils needed appropriate support. The task on paper contained certain prompts. For instance, the task suggested that attention be paid to the sides, the angles, and the diagonals. The pupils are also prompted into focusing their attention on these attributes and are provided with technical language (such as bisect) when appropriate in order to assist the precision of their explanation.

The above provides some illustration of how the quality of mathematical analysis of such geometrical figures, done by these particular students, improved significantly and observably within three sessions using Cabri. Nevertheless, it has to be stressed that this experience is founded on the following factors: a well-established and carefully-nurtured classroom culture that values mathematical thinking, a sequence of carefully selected tasks, and a range of appropriate prompts from the researcher and teacher (and possibly other classmates). It might be that, without this combination of factors, the outcome would have been very different.

In spite of the limitations of this study, and the recognition that it took place in a rather particular context, we suggest that through working on tasks such as the one described above, the students gained some insight into the structure of plane geometry. They were able to explain the properties of a rhombus. When they came to constructing a square they not only knew that the shape had to be a square but they also knew why.

In this process, both of experiencing the necessity of a result (in this case, that the figure has to be a rhombus) and of coming to know why that particular result is obtained, the ability to check by dragging appears to be particularly important. This facility allows conjectures to be tested by focusing attention on the relationships between the geometrical objects that have been constructed. The commands "*delete object*" (and the object's dependents), together with "*undo*", have also shown themselves to be useful in a similar way. The argument is that experiencing the necessity of such geometrical facts involves perceiving (at some level) the axioms of the mathematical structure of plane geometry. Explaining why these geometrical facts are necessarily true involves constructing chains of reasoning. These are two of the essential components in a meaningful experience of proof.

From the evidence so far gathered, it may be that carefully chosen tasks using a dynamic geometry package provide a suitable context for some students to develop an appreciation of mathematical proof. In order to see how this view of proof as explanation develops further in the classroom, the process of acceptance within such a setting is a vital aspect. In due course, attention will need to be paid to that process and to how this contributes to developing in students the basis for a fuller appreciation of the nature and purpose of mathematical proof.

5. Discussion

The example serves to illustrate how it might be that in a curriculum that fosters exploratory activity students with dynamic software in a conjecturing atmosphere can develop a feel for proof as explanation – how certain 'inputs' lead to certain results. Yet explanation is only one aspect of proof (see, de Villiers, 1990). What about the rather crucial aspect of proof as a test of validity. After work with this sort of software is there any need for verification in a formal sense? Can 'a way to prove' be seen as part of rather the constructive process or simply added to it as was the case in paper and pencil contexts? There is a danger that the use of this software will foster a process approach to geometry not possible before, but in so doing – at least in UK – rather than taking a step forward we will simply replay the mistakes of the past, and limit the mathematical work of the majority to empirical argument and pattern-spotting.

Clearly, if proof is presented in the traditional way simply by replacing figures on paper with figures on the screen, we may expect to find way little improvement in pupils' conceptions. But can we bridge the gap between induction and deduction by (for example) stressing explicitly the steps of construction, reflecting on these steps by the use of the 'history' tool as a symbolic trace of the process or stressing the use of the 'check property' function? This approach will also aim to develop an explicit characterisation of the problem in the computer context which if carefully chosen so as to provide ‘the bones of a proof’. Students would conjecture about the relationships between geometrical objects, construct the objects for themselves, and prove the truth of their conjectures in various ways but not in a linear fashion but one which is spiral and iterative.

In this scenario, construction and proof would be we brought together in ways not possible without an appropriate technology. We envisage that, by constructing a ‘generic’ example with Cabri, students have to attend to the relationships they are setting up, which can provide a basis for proving together with a rationale for its necessity – provided that careful attention is paid to the primitive objects available, the activity structures promoted and the pedagogy on offer. In this way, we trust we can develop a flexible appreciation of the roles of proof to include illumination, discovery and communication alongside those of verification and rigour.

Clearly an enormous amount of work has to be done – to choose a construction process that might lend itself to a formal proof as well as to investigate the construction/proving cycle systematically. This is being undertaken in a research study conducted by Hoyles and Healy during 1995 to 1998 (see, Hoyles and Healy, 1995).

Every country has special challenges to face. In the U.K., students tend to work in a conjecturing and experimental atmosphere which surely must be a pre-requisite for productive work with software. Yet, there is already a noticeable trend to use powerful dynamic geometry tools in order to spot patterns, generate cases, measure lengths and angles simply to provide data. This data-driven approach could, if we are not careful, allow us to side-step all the important mathematical content which the geometrical domain is capable of offering.

It is certain that other countries with different traditions and cultures will have other problems in trying to incorporate computer work into geometry teaching and learning. What is required from all the community is deep thought about the role of the tools in relation to geometrical knowledge followed by careful and systematic evaluation of their use.

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